Mathematics success among students of Ethiopian origin in Israel (SEO)

(article begins on next page)

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TABLE OF CONTENTS VOLUME 4

Research Reports

Research Reports
Lei - Rob

EXEMPLIFYING DEFINITION: ON THE FIELD-RELATED CHARACTER OF TEACHERS' CONTENT KNOWLEDGE

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In this study we investigated whether teachers' content knowledge related to defining mathematical concepts is field-dependent. Our previous studies demonstrated that generating examples of definitions is an effective research tool for the investigation of teachers' knowledge. Hence, we examined teacher-generated examples of concept definitions in different areas of mathematics. We analysed individual and collective example spaces focusing on the correctness and richness of examples provided by the teachers. We demonstrated differences in teachers' knowledge associated with defining mathematical concepts in Geometry, Algebra and Calculus.

BACKGROUND

Research focused on proofs and proving reveals differences between learning and teaching mathematical proofs in geometry and other areas of mathematics, while proving tasks in school are usually associated with geometry (e.g., Harel & Sowder, 1998). We wonder whether such a difference exists in defining mathematical concepts. As a research tool in this study we use example generation.

Example generation as a research tool

This study stems from the position that teachers' ability to exemplify and define mathematical concepts is a fundamental component of their content knowledge. Watson and Mason (2005) discussed example-generation as an effective pedagogical tool. In our prior research (Zazkis & Leikin, 2007, 2008) we suggested that example generation is also an effective research tool that allows exploring both teachers' subject matter and pedagogical content knowledge. In Zazkis & Leikin (2007) we designed a framework for the analysis of teacher-generated examples of mathematical concept while in Zazkis & Leikin (2008) we applied this framework analysing teachers' examples for the definition of a square. We addressed teachers' ability to distinguish between necessary and sufficient conditions, their ability to apply appropriate mathematical terminology and, most importantly, their ideas about what a definition is. We argued that simply asking "what is a mathematical definition" could not have generated such an abundant source of data, whereas exemplifying definitions revealed one's 'answer in action' and illuminated deficiencies in this answer.

On definitions in teaching and learning mathematics

There is an agreement among mathematics educators on the importance of mathematical definitions in teaching and learning mathematics. A definition of a concept influences the teaching approach, the learning sequence, and the sets of

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theorems and proofs. The ways in which definitions are presented to students, shape the relationship between a concept image and a concept definition (Tall & Vinner, 1981). Moreover, teachers' knowledge of mathematical definitions affects their didactical decisions (De Villiers, 1998; Tall & Vinner, 1981; Zaslavsky & Shir, 2005, Zazkis & Leikin, 2008).

Based on the works of well-known mathematicians (e.g., Poincare (1909/1952); Solow, 1984; Vinner, 1991) Leikin and Winicki-Landman (2000) distinguished between mathematical and didactical characteristics of mathematical definitions. The mathematical fact that a definition establishes necessary and sufficient conditions for the concept is especially important for our study. While any definition introduces the name for a group of mathematical objects with common properties, conditions included in the definition determine the common set of properties of these objects. When asking participants to provide examples of definitions we trace their understanding of the concepts they define and their understanding of the logical structure of definitions.

The Framework

Following Watson and Mason (2005, p.76) we distinguish between different kinds of example spaces: *Personal (individual) example spaces* which are triggered by a task, cues and environment, *conventional example space* which are generally understood by mathematicians and as displayed in textbooks, and *collective example spaces,* local to a particular group at a particular time. Acknowledging these distinctions we developed a framework that serves as a tool for analysing teachers' personal and collective example spaces based on (a) accessibility and correctness, (b) richness, and (c) generality (Zazkis & Leikin, 2007; 2008), and in such allows for making inferences about their knowledge. In Zazkis & Leikin (2008) we demonstrated that 'correctness' and 'richness' of definitions are categories that provide a clear organizational lens, whereas 'accessibility' cannot be assessed within a written questionnaire. We also showed that generality is a less informative category when applied to written questionnaires. Thus, in this study we analyse correctness and richness of teacher-generated examples of definitions.

In our analysis of *correctness* we examine whether a provided definition includes a set of conditions for the concept that are necessary and sufficient. Incorrect examples, which are lacking either necessary or sufficient conditions, may relate to the lack of understanding of the specific concept and its critical features or lack of understanding of the concept of definition itself. In contrast, some examples may include unnecessary conditions and present specific cases only. Inability to distinguish between critical and non-critical features of the concept demonstrates the lack of understanding of the general notion of a definition.

In our analysis of *richness* of teachers' example spaces we consider diversity of concepts and topics within collective and individual example spaces, and whether the examples are situated in a particular context or are drawn from a variety of contexts.

Individual examples spaces also are examined with respect to conventionality and the presence of equivalent definitions of particular mathematical concepts.

THE STUDY

In order to explore field-related nature of teachers' knowledge associated with defining concepts we analysed differences between correctness and richness of example spaces generated by prospective mathematics teachers (PMTs) in Geometry, Algebra and Calculus. To this end, we posed the following research questions: How do examples generated by participants reveal their understanding of the mathematical concepts and the meta-mathematical concept of a definition? What differences can be found between the examples of definition in different fields?

The following **tasks** were presented to a group of 11 PMTs holding a BA in Mathematics during one of their final courses towards teaching certification:

Give as many examples as possible to mathematical definitions (1): ***in the field of Geometry,*** *(2)* ***in the field of Algebra****, and (3)* ***in the field of Calculus.***

Participants were invited to respond in writing with no time constraints for completing the tasks. Whole group discussion followed the written component.

FINDINGS

Overall 136 examples of definitions were provided by the PMTs to the 3 tasks. Of the 136 examples, 66 were provided for Geometry concepts, 43 for Algebra concepts, and 27 for Calculus concepts. This is in spite of the fact that Algebra and Calculus represent more than 70% of the school mathematics and the participants' exposure to Geometry in their BA program was rather limited. PMTs felt that providing examples of definitions in Algebra or Calculus was a challenging task, as exemplified in the following claims:

T1: You define and prove in geometry and you learn these things there but in algebra you solve and not prove… It is very difficult to find a definition… The problem is that … I could not tell what the difference between definition and theorem is. Axiom is also a definition or not?

T2: What is a definition? I think it has some properties of a figure. What are the properties in algebra? For example function as a mapping does not seem like a property.

Correctness

When examining correctness of the examples provided by the PMTs we distinguished between *appropriate* and *inappropriate* statements. As shown in Figure 1, among appropriate statements we identified (a) rigorous examples of definitions, and (b) appropriate but not rigorous examples of definitions. Inappropriate examples usually lacked necessary or sufficient conditions, and mostly represented specific instances of the concepts. Some of the inappropriate examples included listing concepts instead of defining them.

	(1)	**Appropriate rigorous examples of an definition**	
Geometry	G1.1	Sphere is a set of points with the constant distance from the given point in the space	T6
	G1.2	Isosceles triangle is a triangle with two equal sides	T1
Algebra	A1.1	The circle is all the points (x, y) in a coordinate plain that fulfil the formula $(x\text{-}a)^2+(y\text{-}b)^2=R^2$. (a, b) is the center of the circle and R is the radius	T5
	A1.2	Complex number is a number of the form a+ib when a and b are real and $i=\sqrt{-1}$ and it's a number composed of an imaginary part and a real part	T3
Calculus	C1.1	Inflection point of a function is a point at which convex function become concave (or vice versa).	T6
	C1.2	Interval of increase of a function is the range at which as the x values become bigger, so is the values of f(x) (become bigger)	T3
	C1.3	Circle: $c=\{x\times y/\ (x\text{-}a)^2+(y\text{-}b)^2=R^2\}$ in R2	T1
	(2)	Appropriate but not rigorous examples of definitions	
Geometry	G2.1	Circle is a set of points with the constant distance from the given point.	T2
Algebra	A2.1	$a^m= a\times a\times a\times\ldots..a$ (m times)	T3
	A2.2	Quadratic equation is a second degree equation formed of $ax^2+bx+c=0$	T2
Calculus	C2.1	Slope is the change in y unit as we increase the x by unit	T6
	(3)	Inappropriate examples of definitions	
Geometry	G3.1	The straight line is ∞ points crowded side by side, it has no beginning and no end.	T5
Algebra	A3.1	Function- there are many definitions such as $f(x)=y=ax+b$ which is: x- independent variable, y- dependent variable.	T10
	A3.2	In an arithmetic sequence the difference between every two consistent numbers is equal. Each number is bigger then the number before it	T8
Calculus	C3.1	Integral is an area	T2
	C3.2	sin and cos are trigonometric functions	T8
	C3.3	Derivative of a function is its slope	T6

Figure 1: Examples of definitions

Appropriate rigorous examples of definitions include necessary and sufficient conditions of the defined concept as well as accurate mathematical terminology and symbols and usually are minimal. Examples of this type are illustrated in Figure 1*(1)*.

Appropriate but not rigorous examples of definitions usually omit some constraint or use imprecise terminology. This may be due to the lack of PMTs attentiveness or due to the lack of rigor in mathematical language in a usual mathematics classroom. Figure 1*(2)* illustrates examples of this type.

Example G2.1 was provided by 6 PMTs. None of the 6 PMTs – including T6 who defined a sphere as a set of points in 3D space – mentioned that a circle is a locus in a plane. Additional not rigorous examples are illustrated by definitions A2.1 which lacks the restriction that '*m* is a natural number', and definition A2.2, where the missing restriction is '*a* differs from zero'. In example C2.1, T6 did not mention that this is a definition of a slope of a linear function.

Inappropriate examples: Contrary to the appropriate examples of definitions or non-rigorous examples that we consider as appropriate, inappropriate examples of definitions are not only lacking a particular constraint, but some of the conditions included in these definitions are neither necessary, nor sufficient. Figure 1*(3)* illustrates examples of definitions of this kind.

Some of the inappropriate examples of definitions demonstrate PMTs' misunderstanding of the mathematical concepts (e.g., G3.1, A3.2, C3.1, C3.3) these definitions include ill-defined conditions like: "no beginning no end", for a straight line; or unnecessary restrictions, such as "each number is bigger than the number before it", for an arithmetic sequence. Other inappropriate examples of definitions demonstrate that PMTs do not understand what a definition entails (e.g., A3.1, C3.2): instead of defining a concept they provided specific instances of the concepts.

Table 1: Appropriateness of the examples of definitions generated by PMTs

Examples of definitions \ Participants		T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	Total no. of	Percent within the field
Geometric	Appropriate	9	5	8	3	5	4	1	3	3	3	7	51	77%
	Inappropriate	0	1	0	0	1	2	5	1	2	3	0	15	23%
	Total	9	6	8	3	6	6	6	4	5	6	7	66	
Algebraic	Appropriate	0	3	3	0	3	1	0	1	0	2	2	15	35%
	Inappropriate	0	3	1	0	3	5	2	5	1	8	0	28	65%
	Total	0	6	4	0	6	6	2	6	1	10	2	43	
From Calculus	Appropriate	6	2	5	0	0	1	0	0	0	0	2	16	59%
	Inappropriate	0	2	0	0	2	2	0	2	1	2	0	11	41%
	Total	6	4	5	0	2	3	0	2	1	2	2	27	
Total	Appropriate	15	10	16	3	8	6	1	4	3	5	11	82	60%
	Inappropriate	0	6	1	0	6	9	7	8	4	13	0	54	40%
	Total	15	16	17	3	14	15	8	12	7	18	11	136	

Furthermore, among inappropriate examples in Algebra and Calculus PMTs provided names of concepts instead of defining them (e.g., "length", "unknown", "an equation of first degree", "minimal and maximal point"). We argue that naming was not due to the misunderstanding of the tasks, since each one of the PMTs provided several examples of definitions that were appropriate.

Table 1 depicts numbers of examples generated by each PMT to each task. It also demonstrates appropriateness of the examples and summarises distribution of the examples among the three fields. From Table 1 we learn about the differences between appropriate and inappropriate examples of definitions in collective example spaces generated for each field. In Geometry 51 of 66 (77%) examples are classified as appropriate, whereas in Algebra only 15 of 43 (35%) examples are found as appropriate. The number of examples in the field of Calculus is relatively small (overall 27 examples), of which the number of appropriate examples is almost the same as in Algebra: 16 examples (59%) were appropriate. PMTs who provided most of the examples in Calculus borrowed them from university courses, whereas Algebra examples were from school mathematics. Thirteen of these examples are generated by PMTs all of whose examples are appropriate.

From Table 1 we also note that (almost) all the examples in the individual example spaces of T1, T3, T4, and T11 are appropriate, whereas individual examples spaces of T6, T8 and T10 include more inappropriate examples than appropriate ones. T7 generated only 1 (of 8) appropriate example. We consider these differences among individual example spaces as indicators of individual differences within the PMTs' content knowledge.

Richness

We found that the numbers of topics to which appropriate examples of definitions belong are similar for Geometry, Algebra and Calculus (see Table 2). However geometric examples spaces appear to be richer: First, the total number of concepts in collective and individual example spaces was larger in Geometry. Second, only in Geometry PMTs provided examples of equivalent definition for the geometric concepts (identified ($\times n$)) in the table.

Analysis of the individual example spaces demonstrates that T1, T3 and T11 had the richest individual example spaces which provided the main contribution to the richness of the collective example space. The numbers of concepts addressed by these 3 PMTs were similar (15, 17, and 11 respectively - see Table 1), while the bigger number of examples of definitions given by T1 and T3 was due to equivalence of some definitions in their example spaces. Interestingly, T1 did not provide any examples for definitions in Algebra and all her examples in Calculus were from Analytic Geometry using symbolic representation of concepts (see C1.3 in Figure 1).

Table 2: Richness of the appropriate examples of definitions generated by PMTs

	Topics	Concepts	PMTs	No. of examples	
Geometry	Special quadrilaterals	Parallelogram(×4) Rhombus(×4) Rectangle(×2) Trapezoid	TT1(×3),3(×2),4,8,11 TT1(×3),2,3,4,5 T2, T4, T11 T11	18	50
	Triangles	Triangle(×2) Isosceles, equilateral, right tr. Height in a triangle Midline, Vertex	T10(×2) T1(3) T2, T11 T3(1), T9(2)	11	
	Circle/sphere	Circle Tangent line to a circle	TT2,3,5,6,8,10 T1, T6	9	
	Distance	Between a point and a line (×3) Between two points Height	T5(×3), T8 T3, T7 T2	7	
	Straight lines/angles	Parallel lines (×2) Intersecting lines Perpendicular lines	T3(×2) T11 T9	5	
	Sphere	Sphere	T6		
Algebra	Functions	Linear function Quadratic function	T2 , T10 T2, T8	4	15
	Equations	Quadratic equation True set	T2 T8	3	
	Analytic geometry	Straight line circle, canonic circle	T5, T6 T5(2)	4	
	Other algebraic definitions	Opposite numbers, Complex number Power Matrix	T3(2) T3, T11 T11	5	
Calculus	Function	Function, Image Inflection point Extreme point, Interval of increase, Polynomial of the second degree	T3(1), T11(2) T6 T3(3)	7	16
	Derivative	Slope, Asymptote	T2	2	
	Integral	A primitive function	T3	1	
	Analytical geometry	Circle, Ellipse, Hyperbola, Straight line, Plane, Angle between vectors	T1(6)	6	

CONCLUSION

In accord with previous studies (Zazkis & Leikin, 2007, 2008; Leikin & Levav-Waynberg, 2007), we demonstrated that exemplification is a powerful research tool for the exploring teachers' mathematical and meta-mathematical knowledge. This study revealed that PMTs' knowledge of definitions differs for different fields of mathematics. This finding reflects the nature of school mathematics textbooks and school curriculum.

The study also demonstrates the gap between mathematics learned at the university courses and school mathematics. T9 illustrates this argument:

T9: What does it mean definition in calculus. You have derivative and integral but we did not learn those definitions in school. Those are definitions

> from the university and I do not remember them. Since I passed the exam I do not remember the definitions.

We argue that defining activities in school should be incorporated in Algebra and Calculus. Reinforcing the findings of Zazkis & Leikin (2009) and Moreira & David (2008), we suggest that explicit connections between university mathematics and the school mathematics should be drawn in teacher education.

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MALAYSIAN EXPERIENCES OF TEACHING MATHEMATICS IN ENGLISH: POLITICAL DILEMMA VERSUS REALITY

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In 2003, Malaysia launched a controversial policy – known as PPSMI – of teaching mathematics and science in English in all schools beginning with Grade 1, Grade 9 and Grade 11. It is now six years after implementation, and a decision must be made about whether the policy should continue or whether the nation should revert to teaching in Malay language. This paper explores the complex relationship between the implementation of the PPSMI and the real situation of mathematics teaching in Malaysian classrooms. A brief description of the Malaysian school system and the historical development of the language policy in teaching mathematics are given.

INTRODUCTION

The Malaysian school system is made up of four levels: Primary (Grades 1 through 6), Lower Secondary (Grade 7 through 9), Upper Secondary (Grade 10 and 11) and Matriculation (Grades 12 and 13). At primary level, due to the multi-ethnic characteristics of Malaysia's people, three choices of primary schools are available depending upon the medium of instruction. These are (a) Malay-medium national schools; (b) Chinese-medium national-type schools; and (c) Tamil-medium national-type schools. At the secondary levels, all schools are conducted with a common medium of instruction –Malay language– the national language of Malaysia.

Mathematics is a compulsory subject in the Malaysian school curriculum, but in Grade 12 it becomes an elective. Prior to 1981, mathematics was taught in English up to Grade 11. In Malaysia between 1981 and 2003, the national language (Malay) was the medium of instruction in most mathematics classes from Grade 1 through Grade 11. However, in January 2003, the Malaysian government made a bold decision to change the medium of instruction for mathematics and science to English. This new policy was implemented in progressive phases, beginning with Grade 1, Grade 7 and Grade 11 in 2003, with the entire changeover complete for all levels in 2008.

The Rationale behind PPSMI

According to Rusnani (2003), the initial rationale that prompted the switch in the medium of instruction for mathematics teaching in Malaysia was an overall decline in students' English language proficiency. In view of the significant role played by the English language in meeting the challenges of globalization and the information explosion, the Ministry of Education wanted to improve students' command of English both at school and tertiary levels. To upgrade the students' English language proficiency, five possible strategies were proposed. These were to (a) revert to English-medium schools; (b) use the English language as a tool for learning; (c)

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enhance the teaching of the English language; (d) increase the time available in schools for teaching English; and (e) provide environments that support pupils' learning of English. After much debate, the second strategy was chosen.

Why then were mathematics and science chosen as the subjects for change? Rusnani (2003) and Choong (2004) argued that mathematics and science are the most dynamic and fastest changing fields of knowledge, both contributing significantly to national development. Most advances in mathematics and science are presented in reports written in English. Choong (2004) claimed that "teaching the subjects in the science disciplines in English would expedite acquisition of scientific knowledge in order to develop a scientifically literate nation by the year 2020" (p. 2). She further claimed that "projecting manpower needs in terms of qualifications and skills, the Cabinet made a decision to teach science and mathematics in English" (p. 2). Further, it had become far too challenging to translate the fast-growing literature on science and technological developments into the Malay language (Choong, 2004).

The rationale for PPSMI included the following four statements of need:

- to improve pupils' competence in using the English language, since it is the international language for knowledge acquisition and communication;
- to arrest the decline of English language proficiency levels among Malaysian students, both at school and at tertiary levels;
- to equip future generations with a language that would give them access to new developments and advances in science and technology in order to meet the challenges of globalization; and
- to overcome the increasingly challenging task of translating the latest technological developments into the Malay language.

THE DEBATE

Six years after its implementation, PPSMI remains controversial and is strenuously debated among educators, nationalists and politicians. There remain two strands of thought: (a) the teaching of mathematics and science in English does not and will not help to rescue the deteriorating standard of English; and (b) making English a tool of learning is the most effective way of ensuring that students are proficient in English as well as upgrading students' achievement in mathematics and science.

From the beginning, the policy was opposed by influential Chinese groups, including the United Chinese School Committees' Association (Dong Zhong) and the United Chinese Teachers' Association of Malaysia (Jiao Zhong). These groups gave several reasons for the opposition, including the fear of increasing the burden on pupils and changing the distinctive character of Chinese schools. They argued that mathematics is best taught in pupils' mother-tongue. In addition, they have been proud that research studies have shown that pupils in Chinese schools have consistently achieved better than their counterparts in Malay and Tamil schools. After much negotiation and mediation by the Chinese political parties, Chinese schools were allowed to teach mathematics and science bilingually following a formula of 4-2-2 for the upper primary and 2-4-3 for the lower primary pupils. The 4-2-2 formula refers to four periods of English language, two periods of mathematics in English and two periods of science in English while 2-4-3 refers to two periods of English language, four periods of mathematics in English and three periods of science in English. Hence, upper primary pupils in Chinese schools now have eight periods of mathematics per week – six in Chinese and two in English while the lower primary pupils have ten periods of mathematics per week – six in Chinese and four in English. Table 1 shows, in fact, that Chinese school pupils ended up with fewer periods in English, when compared with their Malay and Tamil counterparts.

Table 1:
Number of Periods/week in English/Mandarin for Primary Schools in Malaysia

	Subject	Chinese Schools		Nat./Tamil Schools
Medium of instruction		English	Mandarin	English
Level I (Grades 1-3)	Mathematics	4	6	7
	Science	3	3	3
	English	2	-	8
Level II (Grades 4-6)	Mathematics	2	6	7
	Science	2	3	5
	English	4	-	7

Other opposition came from those who were concerned that the standard of national language might deteriorate as a result of over-emphasis of English language. During 2008, the debate intensified, and even some ruling party politicians called for schools to revert to teaching science and mathematics in Malay.

Latest Developments Concerning Policy Implementation

According to a report in the press (*The Star On-line*, December 16, 2008), four roundtables were organized in July 2008 by the Education Ministry to gather feedback on the issue from a spectrum of stakeholders (politicians, academicians and representatives of non-governmental organisations and parent-teacher associations) The following seven proposals emerged from these roundtables:

1. Retain English as the medium of instruction for Mathematics and Science;
2. Revert to teaching in Malay language;
3. Let primary schools teach in the mother tongue but use English in secondary;
4. Allow each type of primary school to decide on its language of instruction;
5. Allow Mathematics and Science to be taught in Malay language and mother tongue for Grade 1 to 3, and in English from Grade 4 onwards;
6. Use a combination of mother tongue in the first three years, and a choice of the mother tongue or English after that;
7. Do not teach mathematics or science in Grades 1 through 3, but instead integrating them with other subjects.

On December 24, 2008, *The Star On-line* reported that Cabinet was unlikely to make any decision on the matter before January 5. According to *The Star On-line,* “Science and mathematics will continue to be taught in English in the new school term if there is no decision on the controversial policy before then.” At the time of writing this paper, the controversy remained unresolved.

THE CLASSROOM REALITY

To provide a glimpse of the real situation in schools, we examine the findings from two recent related studies. In the first study, Lim, Fatimah and Tang (2007) surveyed the views of school administrators, mathematics teachers and pupils on the implementation of the policy, while in the second study, Lim, Tan, Chew and Kor (2009) explored (through video-taped observations) the language used in the mathematics classrooms of three primary schools.

Conceptual Framework for the Studies

The word “language” can refer to “any system of formalized symbols, signs, sounds, gestures, or the like used or conceived as a means of communicating thought, emotion” (Random House, 2006). In this paper, language refers to oral and written language used by both teachers and pupils to communicate in mathematics classes.

Teaching and learning are social activities that involve teachers and pupils. In the process of teaching and learning, it is crucial that language is used effectively and efficiently. Language is not only a tool of communication but also a tool for reflection and thinking (Vygotsky, 1978). Part of learning mathematics involves learning words and terminology that are related to mathematics, and learning to communicate mathematically (Setati, 2005). Mathematical concepts are learned through communication and interaction between teachers and pupils. Encouraging children to *talk* about ideas helps them to discover gaps, inconsistencies, or lack of clarity in their thinking (Baroody, 1993). This implies the importance of ensuring that pupils are proficient in a language so that they are able to communicate clearly and confidently using that language. Moreover, there should be coherence between the language used by mathematics teachers and that used by their pupils.

Cummins (1981) postulated that there exists a minimal level of linguistic competence, a threshold that a pupil must attain to perform cognitively demanding academic tasks in mathematics and science effectively. Cummins acknowledged that the learning of pupils who speak more than one language, could be affected by the interplay between the different language codes. Thus “bilingual students who are not really fluent in either of the two languages that they use tend to experience difficulty in mathematics” (Ellerton & Clarkson, 1996, p. 1020). A similar argument can be applied to bilingual teachers who lack fluency in either language. Teaching mathematics in pupils’ second language poses various challenges both to teachers and pupils. Most studies (see e.g., Adler, 2001; Setati, 2005) on mathematics in bilingual or multilingual classrooms have argued that pupils can learn better when they are taught in their mother tongue. However, in countries such as South Africa and The Philippines, where school mathematics is taught in pupils' second language, code-switching by teachers is commonly used.

We shall now report data from two studies on the extent to which some Malaysian primary schools have adopted the policy of teaching mathematics in English after 5 or 6 years of implementation. The first author was the project leader of both studies.

Study 1: Perspectives of Primary School Administrators, Teachers and Pupils

The first study (Lim, Fatimah & Tang, 2007), conducted five years after the implementation of PPSMI, surveyed the opinions of a total of 443 primary mathematics and science teachers and 787 primary Grade 5 pupils as well as 20 primary school administrators. The sample of 20 schools was stratified by state (Penang, Kedah and Perak) and location (urban and rural). For each location, one Malay school and one Chinese school were chosen so that pupils were drawn from similar socioeconomic areas. Grade 5 pupils were targeted as they were in Grade 1 in 2003 when the PPSMI programme began. Two survey questionnaires were used: one for mathematics teachers and one for pupils. Teacher questionnaires had four sections – the background of respondents, teachers' self-assessed report of their language proficiency, issues faced, and teachers' perceptions towards the teaching of mathematics and science in English. Teachers' perceptions were measured on a five-point Likert scale, ranging from "strongly disagree" to "strongly agree."

Pupil questionnaires also contained four sections. The first asked for pupils' background information including self-rating of their English proficiency. The second concerned the extent of English language usage in the teaching of mathematics and science classes as well as issues faced in studying mathematics and science in English. The third section assessed pupil's feelings when learning mathematics and science, and the last section evaluated their views towards teaching and learning of mathematics and science in English. Since pupils from Chinese schools also learn mathematics and science in Mandarin, the questionnaires for these pupils included two additional questions which asked for their feelings towards learning mathematics and science in Mandarin. To minimize language barriers, pupils at Malay schools were given questionnaires in Malay, and pupils at Chinese schools were given questionnaires in Mandarin. All teachers' questionnaires were in English.

Analyses revealed that only 11.5% of the mathematics teachers in the study explained mathematical concepts entirely in English (Table 2). Thus almost 90% of teachers did not teach mathematics fully in English but resorted to using the pupils' mother tongue (either Malay or Mandarin) in their teaching. English was used exclusively in teaching more often in Malay schools than in Chinese schools, and more often in urban schools than rural schools (Table 2).

About 18% of the teachers rated themselves as "poor" in spoken and written English (Table 3). Most of the teachers were much more proficient in Mandarin or Malay language than in speaking and writing English. It is hardly surprising, then, that only a few teachers taught mathematics and science entirely in English.

Table 2:
Teaching of Mathematics Entirely in English in Malay and Chinese Schools (%)

		%
	Total Sample	11.5
Type of School	Malay Schools	17.6
	Chinese Schools	7.8
Location	Urban	14.4
	Rural	6.3

Table 3:
Teachers' Self-rated Proficiency in Malay, Mandarin and English (%)

Language	Oral			Written		
(N=443)	Fluent	Average	Poor	Good	Average	Poor
Malay (443)	58.5	41.5	0	48.0	51.6	0.5
English (443)	12.5	69.4	18.1	11.4	68.4	20.2
*Mandarin (308)	90.0	10.0	0.0	75.6	23.7	0.6

*Although the sample included the same number of Malay schools and Chinese schools, Chinese schools generally have a larger pupil and teacher population.

A much higher percentage of Chinese pupils (95-97%) than Malay pupils (75-82%) reported that their teachers did not use English entirely to teach mathematics and science. Could this be related to another finding that a slightly lower percentage of Malay pupils (41%) than Chinese pupils (45.5%) reported that they had problems learning mathematics and science in English? When pupils were asked to indicate their feelings towards learning mathematics and science in English, Malay pupils noted more positive feelings than Chinese pupils. Chinese pupils felt more confident and positive toward learning mathematics and science in Mandarin. In addition, proportionally 30 per cent more Malay pupils supported the policy than their Chinese counterparts. This could be explained by the findings that 40 per cent more Malay pupils found it easier to learn mathematics in English. Also, fewer Chinese pupils than Malay pupils had access to reference books in English (44% Chinese versus 80% Malay), and fewer Chinese pupils reported having family support (37% Chinese versus 75% Malay) when learning mathematics and science in English.

Because Chinese pupils lacked proficiency in English, their teachers felt that they themselves had to use Mandarin more often than English. Because of the reduced use of English by their mathematics and science teachers, the Chinese pupils became less confident and faced greater problems in learning these subjects in English.

Administrators' perspectives were solicited through in-depth interviews with 13 principals and seven senior assistants who had experience with the implementation of the PPSMI policy. The majority of the administrators interviewed agreed with the rationale behind the policy, and felt that it was necessary to prepare the future generation to be proficient in English and competent in mathematics and science.

However, full support for the policy was more forthcoming from Malay administrators than from their Chinese counterparts. Chinese administrators recognized four major concerns in their schools, reporting (i) too much content to be covered in too little time; (ii) a lack of home support in English; (iii) pupils not fluent in English, and (iv) teachers not confident in English. To resolve these problems, some suggestions made by the administrators were, (i) begin the policy at secondary school, (ii) increase the number of English periods to at least eight per week, (iii) allocate two periods per week for teaching science and mathematics terminology in English, and (iv) teachers should be encouraged to upgrade their language proficiency by attending specific English language courses.

Study 2: Language Used In Bilingual Primary Mathematics Classrooms

The second study (Lim, Tan, Chew & Kor, 2009) examined the language used and its role in teaching mathematics to primary classes of bilingual pupils. Data from video-taped lessons and interviews with teachers were collected. Three primary schools – a Malay-medium national schools; a Chinese-medium national-type schools; and a Tamil-medium national-type schools in Penang state participated. For each school, two mathematics teachers – an “expert” with more than 10 years’ teaching experience, and a “novice” with less than 5 years’ experience – were selected. Two mathematics lessons taught by each teacher were video-taped.

Verbal communication in mathematics classrooms usually involves: (a) questioning (b) explaining (c) representing (d) discussing and (e) conjecturing. Each video-taped mathematics lesson was analyzed quantitatively in terms of the kind of discourse and time taken for each kind of discourse, and the kind of language used for each kind of discourse. The video-taped lessons and interview transcripts were also analyzed to: (a) identify the roles and purposes of language used for each kind of discourse; (b) identify when and why there was a switch of language used (code-switching) for each kind of discourse. Finally, a cross-case analysis was carried out to compare the differences in language used between expert and novice mathematics teachers as well as between different types of schools. Analysis of video transcripts showed that the type of language used was influenced by class ability and teachers’ confidences in his/her own English language and that of their pupils. In higher-ability mathematics classes, more than 98% of discourse was conducted in English for the three types of schools. However, for lower-ability classes, the percentage varied between 46% and 98%. This trend was found regardless of whether the class was taught by an expert or by a novice teacher. For weak mathematics classes in Chinese schools, mathematics was taught bilingually, in English and Mandarin. Both teachers and pupils often used code switching and translated during classroom discourse. In one Malay school, one mathematics teacher seemed to code-switch in his mother tongue more often than his pupils. In a similar way, in one Tamil school, one of the teachers for a weak class seemed to code-switch and translated terminology into Tamil more often than her pupils. Interestingly, this result is consistent with the findings of the first study where both teachers and pupils in Chinese schools admitted no more than 50% of their mathematics lessons were taught using English.

CONCLUSIONS AND IMPLICATIONS

On the surface, it may appear that the implementation of PPSMI has been successful. However, the studies reported here indicate that there have been difficulties at the school level. In particular, given the finding that almost 90% of the 443 teachers in the study did not teach mathematics fully in English, greater attention needs to be given to developing teachers' confidence and fluency in using the English language. If the policy of teaching mathematics in English is to continue, and if it is to have the intended benefits, mathematics teachers urgently need support to enable them to enhance their English language proficiency, so that they can use English as the medium of instruction, with confidence, for a greater proportion of class time.

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FOSTERING FACILITATORS' LEARNING THROUGH CASE DISCUSSION

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The study examined the effect of case discussion on facilitators' knowledge of teaching and skills in leading discussions. Three facilitators participated four workshops and structured case discussion in schools. Transcriptions of case discussions, classroom observations, interviews, and reflective notes were the main data. The data were analyzed according to a coding schema emerged from data. Sources, contexts, materials, and motivating anticipated solutions related to the presentation of problems were reported here. Excepting concerning of learner's prior knowledge and instructional objectives, the facilitators motivated students' multiple methods and eliciting a specific anticipated solution in reconstrucing or revising a problem. Four issues of leading case discussions for facilitators were addressed.

INTRODUCTION

There has been increasing interest in developing and using cases for teacher education (Levin, 1999; Lin, 2002; 2005; Merseth, 1996; Stein et al., 2000), since the use of cases is useful for promoting critical reflection and for producing teacher to be a reflective practitioner. Studies conducted with experienced teachers suggest that case discussion can foster teachers' pedagogy and content knowledge (Barnett, 1998; Lin, 2002). Cases discussions also appear to have the potential to pre-service and beginning teachers to experience situations embedded in cases (Lin, 2005). Although a cohort of researchers is interested in the effects of cases on preserves and in-service teachers, there has been little empirical work to date on the impact of the facilitators on the case discussion (Levin, 1999).

The power of cases rests in not only the content of case but also case discussion. It means that what is discussed is as important as how it is discussed. Cases are contextualized narrative accounts of teaching and learning, including the problems, dilemmas, and complexity of teaching in some contexts. Cases are characterized as real, research-based, and potential to initiate critical discussion by users (Merseth, 1996). Case discussion is the key to decide whether the case contributes to users' critical thinking. A facilitator as a leader of case discussion can lead to clearer and elaborated understanding about the issues in a case. This indicates that the case discussion leader is important to what teachers learn from cases.

The roles of a facilitator include: selecting cases, convening the discussion, influencing how the learning communities develops, and influencing the discussion through questions asked and comments make, as well as how the discussion is structured (Levin, 1999). For these reasons it is important to provide a new experience and

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support for facilitators in leading a case discussion. Thus, the effect of case discussion on facilitators' pedagogical content knowledge and the difficulties the facilitators encountered in leading a case discussion become the purpose of the study.

THEORETICAL PERSPECTIVES

The theoretical perspectives of this study are based on the constructive perspectives as Piaget (1932) and Vygotsky (1978). These perspectives provide the rationale for why the case discussion is a crucial factor to consider in studying how facilitators' knowledge is constructed individually and socially. Piaget (1932) asserts that group interaction acts like a trigger for change in cognition because such interaction leads to learners to reflect individually on conflicting ideas that arise in social interaction. This assertion suggests that case discussion is potential to develop users' learning and development. The social interaction during case discussion has the potential for providing cognitive conflict, hence to trigger users' cognitive change.

Vygotsky (1978) claims that group interaction not only initiates change but also shapes the nature of the change. He states that what is learned in social interaction of the group is prerequisite to cognitive development. From this perspective, the social interaction and the quality of group discussion are essential to what learned from cases. In addition, according to Vygotsky's Zone of Proximal Development, experenced facilitators influence the thinking of others with less experience in the case discussion. Thus, studying the nature of the influences among the facilitators in case discussions was of interest in the study. The cases referred to in the study are characterized as: research-based, real, initiating critical discussion by users, and helping users to recognize salient aspects launched from a set of questions (Lin, 2006).

METHOD

Participants

Three primary teachers (F_1~F_3) participated in a teacher professional program that is designed to train teachers to be facilitators in case discussion. The teachers interested in mathematics instruction were recruited from those who were elected or assigned by schools as the leader of mathematical learning community in school. F_1, F_2, F_3 had 12, 8, and 6 years of teaching experience respectively. The participants played two roles: a fifth-grade teacher and a facilitator of case discussion when cases were implemented into school.

Case Materials

The casebook consisting of eight cases o fraction teaching from grade 3 to 6 constituted the materials for this study (Lin, 2006). Each case was constructured by the researcher colloborating with case-teachers developed in previous studies. These cases focus on the issues including a mismatch between instuctional objectives and activities, students' various solutions of a given problem, discourse of the case-teacher and students, and inappropriate tasks. For instance, students' multiple solutions embedded in several cases is intended to improve the users' knoweldge about anticipating students' potential solutions and misconceptions, recognizing the meaning of a specific solution, ordering the solutions from the easiest to the most difficut, and sorting them into different categories.

A set of "Discussion Questions" is the most crucial content in each case, because it is incorporated into the reflections of case-teachers and others who involved in constructing the cases. The questions in the cases are to stimulate users' rethinking about mathematical teaching and reflect to their practices. Through the cases, users were expected to deepen their understanding, expand their views of students' ways of thinking, and master the skill of building mathematical discourse. Besides, the skills of questioning, listening, and response were expected to develop for a facilitator.

Case Discussion Workshops

Four workshops were designed to increase participants' pedagogical knowledge and skills in facilitating case discussion, serving as a window into teachers' thinking. Besides, the workshops afforded the participants with new experience and supports while they carried out the case discussion in schools. Each of the four cases selected from the casebook was discussed in each workshop lasting for three hours. To structure the case discussion, the researcher, as the leader of case discussion in the workshops, played various roles in order to guide, probe, listen, and give feedback. Participants read cases at home and took notes on the cases in advance. Disagreement was encouraged in the case discussion.

Several questions with distinct purposes were asked in the case discussion. For instance, the researcher started case discussion with the questions: *What does the case say about? What is the major goal of the case to help you to learn?* To help the participants to understand the importance of the activities fit into instructional objectives, the questions were asked such as *Does each activity achieved by the case-teacher into instructional objective?* To increase the participants' pedagogical knowledge of fraction, they were asked to answer the questions: *How would you respond to each question in the "Discussion Questions"?* To help the participants to reflect on the role of facilitator, they were asked at the very beginning of the workshop: *How would you like to start if you are the leader of the case discussion?* At the very end of the discussion, the participants were asked to answer the questions: *What did you learn from the leader in the case discussion? What are the most salient issues in the case that you would bring back to share with your colleagues?*

Data Collection and Analysis

To understand what and how the participants put their understanding gained from workshops into classroom practices, they were observed three lessons before, during, and after the workshops. Lesson 2 was observed between the 2^{nd} and the 3^{rd} workshop. Individual participant was interviewed after and before the workshops. These interviews and lessons were audio-taped and video-taped.

To examine the effect of the workshops on skills in leading case discussions, each participant was encouraged to set up a case discussion group in school for carrying out the cases. The size of the discussion groups with 3 or 4 teachers was varied with the culture of each school. Each facilitator was free to structure the case discussion such as the selection and the number of cases. They were required to take notes on the work of leading case discussions. The discussions of each case in each school were audio taped. The transcriptions of case discussions gathered both from workshops and from case discussion group in each school and reflective notes were also the primary data collected for the study.

Analysis began by coding the facilitators' lessons associated with their responses to the questions and an inductive search for patterns. After reading and rereading the responses several times, a coding schema emerged from the data consisted of 3 themes with 36 categories was formed. Three themes were the ways of presenting the problems to be solved (16 categories), ordering students' solutions for discussion (10 categories), and interactions with students (16 categories). For instance, 16 categories in the theme of the ways of presenting the problems were: sources (5), contexts (3), materials (4), and motivating anticipated solutions (4). Each code was counted and frequencies were recorded. The data was coded by the researcher and four teachers who have master degree in mathematics education. We resolved any discrepancies through discussions.

RESULT

To help you understand coherently about the effect of the case discussions, the qualitative data presented here is merely related to ways of presenting the problems, even though ordering students' solutions for discussion and interactions with students were the focuses of the result. More results on ordering students' solutions and interacting with students are reported in longer paper.

Table 1 displays the frequencies accounting for each facilitator's presentation on the problems before, during, and after workshops. The number of problems given to students to solve in F_1's, F_2's, and F_3's three lessons were 5, 3, 3; 6, 4, 3; and 5, 4, 3.

Aware of the Need of Reconstructing the Sequence of Activities

Regarding the source of problems, the data of Table 1 shows that the number of given problems in a lesson was reduced into 3 (after workshops) from 5 or 6 (before the workshops). Before the workshops, all teachers highly relied on the textbook. They moved toward revising the problems given in textbook during the workshops and reconstructing the problems after the workshops. F_1 stated in interview that

"After reading teacher's guide, I don't think I have to revise or re-design the problems or activities in the lesson for students, since the activities in the textbook that is designed by authorities are perfect enough" ($F_{1,}$ Interview).

Through the case discussion, the participants recognized the importance of activities based on students' prior experience or knowledge on motivating students various solutions. For instance, Case 1 related to comparing two unlike fractions, the case-teacher reconstructed the sequence of the activities. To expect students coming up various strategies, she restructured the sequence of activities in the textbook. She claimed that the least common divisor is not allowed to be learned until ordering two unlike fractions has been learned.

Table 1: Frequencies of Each Participant's Performing on Problems Presented to Students

Categorie \ Participants / Frequencies		F_1			F_2			F_3		
		Lesson			Lesson			Lesson		
		1	2	3	1	2	3	1	2	3
Source	Textbook	5	1	0	6	1	0	5	0	0
	Revising from textbook	0	1	0	0	3	1	0	4	3
	Instructor	0	0	2	0	0	2	0	0	0
	Students	0	0	1	0	0	0	0	0	0
	Instructor and students	0	1	0	0	0	0	0	0	0
Context	Bared	2	2	0	2	2	0	2	2	0
	Telegram	3	3	0	4	4	0	2	2	0
	Real-world	0	0	3	0	0	3	1	1	3
Material	Verbal only	4	4	0	6	6	0	4	4	0
	Transparency or PPT	0	0	0	0	0	0	0	0	0
	Black- or white-board only	1	1	0	0	0	0	1	1	0
	Verbal+ Blackboard	0	0	3	6	0	3	0	0	3
Motivate anticipated solutions	Based on prior knowledge	4	3	3	6	4	3	4	3	3
	Meet to instructional objectives	4	3	3	5	4	3	4	3	3
	Elicit specific anticipated solution	0	1	3	0	0	3	0	0	3
	Motivate various solutions	0	1	3	2	2	3	0	1	3

Through reading and discussing the case, F_1 was reflected her past teaching on which the case-teacher's students comparing two unlike fractions with seven strategies. The seven strategies were: (1) using reference point 1/2; (2) compensate strategy by reference point 1 (3) finding a common denominator with reducing the numerals, (4) finding a common denominator with expanding the numerals, (5) finding a same numerator with reducing the numerals, (6) finding a same numerator with expanding the numerals, (7) finding the least common divisor.

Aware of the Roles of Problems Playing in Motivating Students' Multiple Strategies and a Specific Anticipated Solution

As shown in Table 1, four aspects were considered when the participants proposed the problems for motivating students' anticipated solutions. The data indicates that the teachers were used to taking students' prior knowledge and instructional objectives into account. However, motivating various solutions and eliciting a specific anticipated solution had never been concerned in their teaching.

They were often frustrated with motivating students' multiple methods to a given problem. For instance, the objective of "finding equivalent fraction by reducing numerals" can be easily achieved for most teachers by introducing the algorithm in the problem as "*A strip of paper has 8 meters long, what fractions are 4/8 of f the strip reduced into?*". The case-teacher claimed that the problem was inappropriate since students had not learned the term "reduced fraction" yet. Therefore, she revised it into "*A strip of paper has 8 meters long, what proportions of the strip have the same length* *with the 4/8 of the strip of paper?*". Students at this stage would learn better a fraction equivalent to another by comparing same quantity between them.

Through the use of cases, the participants were gradually efficiently proposing new problems for eliciting students' various solutions and a specific anticipated solution (3/3 for F_1, $F_{2,}$ & F_3). However, without enough participation, giving a problem for eliciting a specific anticipated solution was more challenged (1/3 for F_1, 0 for F_2 & F_3) than for motivating various solutions (1/3 for F_1, 2/4 for F_2, 1/4 for F_3). For instance, F_2 learned the importance of the problems playing in motivating students' multiple methods and eliciting a specific anticipated solution in the discussion. F_2 described in her reflective notes as follows.

"I have never noticed that the problems play such important role in encouraging students' multiple solutions. That could be the main reason why my students always gave me one solution only.……To elicit a solution produced by using reduced numerals strategy, the problem should be proposed as "*A strip of paper has 8 meters long, what proportions of the strip are same as the length of 1/2of the strip?*" To motivate multiple solutions by using both expanded and reduced numerals strategies, the problem can be proposed as "*A strip of paper has 8 meters long, what proportions of the strip have the same length with 2/4 of the strip?*" ($F_{2,}$ Reflective notes).

F_2 learned that the size of denominators and 8 meters in the problems have to be connected for motivating anticipated students' solutions. Following the case discussion, as we observed, F_2 confidently revised the problem from the textbook *"A box has 12 oranges. What fraction of 6 oranges is in the box?"* into *"A box has 12 oranges. What fractions of the oranges have the same amounts of 4 oranges in the box?"* The revision elicited students' multiple solutions, such as 4/12, 1/3, and 2/6.

Aware of the Importance of Complete Messages of Given Problem by Written associated with Verbal Presentation

To save teaching time, although the problems were connected to real world in textbook, they were presented in either bared (e.g. 4/8=( )/2) or like a telegram for simplifying a message, (e.g. "4/8box of the oranges = ( )/2 box of the oranges"). The evidence is depicted in Table 1. The frequencies of the problems presented with bared or telegram before and after workshops by F_1, $F_{2,}$ and F_3 are (5/5 *vs.* 0/3; 6/6 *vs.* 0/3; 4/5 *vs.* 0/3, respectively). The participants did not realize the presentations as an essential factor influencing students' understanding the meaning of the given problems until we discussed the Case 3. The case-teacher in Case 3 expressed the problems like a telegram message. The participants argued that the problem was not presented clearly. The over simplified messages were too brief to memorize it. As a result, students were frustrated with comprehension of the problem. After the discussion of the Case 3, as shown in Table 1, all problems posed by F_1, $F_{2,}$ and F_3 were presented with verbal associated with blackboard.

The Difficulties and Issues of Structuring Case Discussions in Schools

Each facilitator was free to structure the case discussion in schools with two purposes.

One was to examine how well the facilitators put their knowledge and skills they learned in case discussion into schools. The other was to disseminate the cases into schools to see what difficulties they encountered. The difficulties and issues of leading case discussions were summarized as follows.

The recruitment for facilitators convening discussion in school was much harder than that of the researcher. With lack of confidence and experience, they were merely able to invite beginning teachers or teaching experience with no more than 5 years to participate in case discussion group in schools. F_3 stated that the teachers who have willingness and good relationship with her were recruited easily and successfully. The number of teachers involving in F_1, $F_{2,}$ and F_3 were 4, 3, and 3 respectively. The participants in F_1's, F_2's schools were fifth-grade teachers, while the teachers in F_3's case discussion group taught at the fourth and fifth grade.

Each facilitator was frustrated with the time of discussion altogether. They complained that they were compulsory to involve in the workshops held by schools, even though they have already scheduled for the case discussions in advance. Accordingly, the facilitators only survived by using several times with half an hour for each. Each case was forced to be cut into several pieces and was discussed several times. This led to weaken the effect of the case discussion.

Four cases selected same as in researcher's case discussion group were used in each school. Two cases were new for F_3 but closely related to the lessons to be taught for fourth-graders. F_3 stated her needed supports from the researcher in the interview during discussing the two cases.

"To help my colleagues in improving their teaching through case discussion, I selected two cases related to fourth-grade lessons to discuss with them right before they teach the two lessons. I was uncomfortable with the discussions of these two cases, because I did not know how to ask key questions and follow-up questions for eliciting the teachers' deep understanding and thinking." ($F_{3,}$ Interview).

DISCUSSION

The case discussion created the opportunity of teachers' more concerns about the given problems for improving students' learning. Through the case discussions, the teachers realized that they need to scrutinize and criticize them whether the given problems in textbook are based on students' prior knowledge and are potential to achieve into instructional objectives. They moved toward motivating students' multiple solutions methods and eliciting a specific anticipated solution while revising or reconstructing the problems. From then on, to help students catch up the meaning of the problems, they gave up the bared or like a telegram problems. Instead, the problems were presented with complete messages by using blackboard with verbal expression. Motivating students' multiple solutions and eliciting a specific anticipated solution were more challenged than other aspects.

The teachers as facililators were supplied with new experience and needed support of case dicussion from the life participation of case discussion. They learned the roles of a facilitator in leading case discussions in which the manner is similar to that of the case discussion in schools. The set of discussion questions integrated with case-teachers' various perspectives are readily to initiate the teachers' reflection on practices and cause their cognitive conflicts of mathematical teaching, hence to trigger change. The case-teachers embedded in the cases play the significant role of more capable peers. Thus, the disscusstion initiated from the set of discussion questions seemed to be a catalyst for the teachers learnign to teach.

This study provided us insight about how the ways facilitators' structured case discussions affected potential outcomes. The case discussion to be succeeded should be considered four issues: 1) Participants' willingness, previous experiences, pedagogical content knowledge, and knowledge about case discussion were the essential factors influencing the quality of the discussion. 2) It is better for same grade teachers involving in case discussion, since same mathematical contents lent itself readily as a focal point, leading to in-depth discussions. 3) The better selection of the cases from a casebook is closely related to the lessons ready to be taught, since the content of the cases readily drew their attentions. 4) Facilitators need to be supported continually from experienced facilitators during the case discussion.

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HOW DOES THE TRANSFER OF LEARNING MATHEMATICS OCCUR?: AN ALTERNATIVE ACCOUNT OF TRANSFER PROCESSES INFORMED BY AN EMPIRICAL STUDY

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From the perspective of mainstream cognitive science, transfer occurs if the symbolic representations that people construct of initial learning and transfer situations overlap or if a mapping can be constructed that relates features of the two representations. We offer an alternative account of transfer processes via the focusing interactions framework, which we use to explain how social environments afford and constrain the generalization of learning. This research demonstrates that the ways in which students transfer their learning experiences depend upon students noticing particular mathematical features when multiple sources of information compete for their attention, which in turn depends jointly on students' and teachers' participation in classroom discursive practices.

INTRODUCTION

Transfer is a controversial construct, which faces a number of challenges regarding its conceptualization and the character of its underlying mechanisms. Numerous critiques of transfer (summarized in Beach, 1999) have contributed to a growing acknowledgment that "there is little agreement in the scholarly community about the nature of transfer, the extent to which it occurs, and the nature of its underlying mechanisms" (Barnett & Ceci, 2002, p. 612). Lobato (2006) argues that a major challenge facing the alternative perspectives of transfer which have emerged over the past decade, is that of articulating alternative transfer mechanisms. She posits a possible barrier to progress by noting that the construct of "mechanism" has been associated with a particular view of knowledge and causation which may no longer make sense within an alternative interpretative framework, especially one grounded in a situative or socio-cultural perspective.

One alternative transfer "mechanism" that has been offered in recent research is that of social *framing* (Engle, 2006). Framing involves bringing to bear a structure of expectations about a situation regarding the sense of "what is going on" in the situation and what are appropriate actions. Engle demonstrates two kinds of framing that are productive for transfer: a) when a classroom teacher frames learning activities as being temporally connected with other settings in which the students could use what they are learning, and b) when a teacher frames the students as contributing members of a larger community of people interested in what they were learning about. Like Engle, we offer an account of transfer

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processes, which pays attention to the classroom interactions that develop around the transfer of learning. We extend her work by additionally relating transfer to what learners "notice" mathematically in classrooms. Our goal is to offer an alternative account of transfer processes via what we call the *focusing interactions framework*. This framework is used to demonstrate that the differential ways in which students transfer their learning experiences are related to differences in what students noticing mathematically in the classroom. Furthermore, what students notice is socially organized and dependent upon classroom discursive practices.

THEORETICAL FRAMEWORK

The father of the traditional transfer approach, Thorndike (1906), situated transfer mechanisms in the environment. He asserted that transfer occurred to the extent to which situations share "identical elements" (typically conceived of as shared features of physical environments). With the "cognitive revolution," the notion of identical elements was reformulated as mental symbolic representations. That is, transfer occurs to the extent to which an individual's symbolic representations of initial learning situations and transfer situations are identical, overlap, or are related (Anderson, Corbett, Koedinger, & Pelletier, 1995). From these traditional perspectives, transfer mechanisms are factors that can be controlled in order to produce transfer (De Corte, 1999). van Oers (2004) criticized the traditional approaches for focusing exclusively on the conditions for transfer and defining transfer on the basis of result qualities. On the other hand, once the conceptual roots of transfer are questioned, the notion of a transfer mechanism doesn't make sense in the same deterministic way. We need a notion of mechanism that refers to an explanation of how social environments afford and constrain the generalization of learning; thus shifting the focus from external factors that can be controlled to conceiving of transfer as a constrained socially situated phenomenon. The focusing interactions framework offers one such approach.

Our work is grounded in an alternative perspective on transfer called the "actor-oriented transfer" (AOT) approach (Lobato, 2006, 2008). From this perspective, transfer is defined as the generalization of learning or more broadly as the influence of prior experiences on learners' activity in novel situations. The AOT perspective responds to the critique that traditional transfer experiments privilege the perspective of the observer and thus can become what Lave (1988) calls an "unnatural, laboratory game in which the task becomes to get the subject to match the experimenter's expectations," rather than an investigation of the "processes employed as people naturally bring their knowledge to bear on novel problems" (p. 20). Instead of predetermining what counts as transfer under models of expert performance, the AOT perspective seeks to understand the processes by which people generalize their learning experiences, regardless of whether or not these generalizations situations lead to correct or normative performance. Taking an actor-oriented approach to transfer often reveals idiosyncratic ways in which individual learners create relations of similarity (Lobato 2008). At first these idiosyncratic forms of transfer may seem random. However, our work on focusing interactions is demonstrating a basis by which the nature of individuals' generalizations (actor-oriented transfer) is constrained by socio-cultural practices.

METHODS

We created four after-school classes of 8-9 seventh graders per class and selected four teachers (one per class) using a screening interview so as to maximize the chances that different aspects of the same mathematical content would be emphasized in each class. During semi-structured interviews, conducted at the conclusion of each instructional session, students were asked to reason with several transfer tasks. The tasks were set in contexts that had not been addressed in any of the classrooms but covered common mathematical content. Because we wanted to be able to attribute differences in the ways students reasoned during the interviews to differences in the mathematical foci that emerged in class, the research design controlled for other potential explanatory sources, such as time-on-task, student ability, and mathematical content. To control for time-on-task, each class met for the same amount of time – 10 hours. To control for student ability level, students were assigned to classes using a blocked random assignment based on results from a screening test. To control for content, the teachers were provided with the same set of overarching content goals for a unit on slope and linear functions but were given freedom in terms of how to achieve those goals.

In the interest of space, we will present findings from the first two classes (referred to as Classes 1 and 2), even though we collected data in four classes. Both teachers used reform oriented curricular materials and a single context – speed in Class 1 and growing visual patterns in Class 2. Additionally, both instructional sessions moved from explorations of contextual problems at the beginning of the unit to conventional representations of linear functions such as tables, equations, and graphs. Analysis of the interview and classroom data followed the interpretive techniques of grounded theory in which the categories of meaning were induced from the data (Strauss & Corbin, 1990).

RESULTS

Interview Results: Transfer Differences

Qualitative analysis of the interview data indicated distinct differences in what students from Class 1 versus Class 2 noticed about tables and graphs of linear functions. One major difference arose when students were shown a graphical display of data involving the amount of water that had been pumped into a pool over time (see Figure 1) and were asked to find the slope of the line. The majority of students from Class 1 attended to the gallons and minutes amounts represented by the coordinate pairs. In contrast, nearly 90% of the Class 2 students appeared to see the graphs as constituting “boxes,” ignored the quantities, and consequently were unable to correctly identify and interpret the slope.

For example, Chanise demonstrates a typical response from Class 2. She initially talked about stairs, wrote “rise/run” on her paper, and then created a “stair step” between the points (0,0) and (3,6). She determined the rise as 2 and the run as 3 by counting boxes and ignoring the quantities represented by (3,6). She repeated this process by drawing a stair step between (3,6) and (5,10), again counting squares to arrive at a rise of 1 and run of 2. She concluded that the slope couldn't be found because 2/3 and ½ are not the same: "It doesn't work because it's different amount on the sides and the rise and the runs."

In contrast, none of the Class 1 students ignored quantities and counted boxes. Over half of the students treated the slope as 2 gal/min. Hector's reasoning typical. He focused on the quantities represented by the coordinate pairs rather than counting boxes. He determined correctly that the slope is 2 because "like the speed will be going by 2 because for every 1 minute it um, that'll be 2 gallons." In summary, while the majority of students from Class 1 appeared to notice the quantities of water and time represented in a graph, nearly all of the students in Class 2 ignored the quantities and counted boxes.

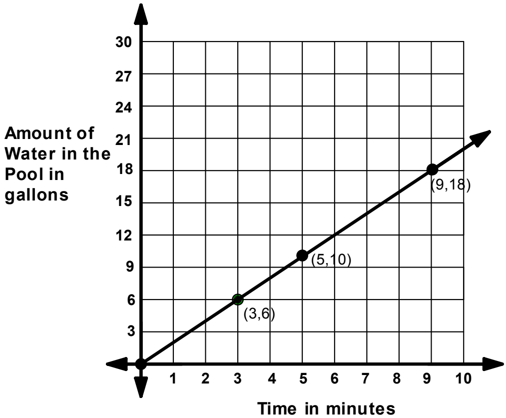

Figure 1. Graphical representation of water pumping data.

Classroom Analysis: The Focusing Interactions Framework

According to Goodwin (1994), the "ability to see a meaningful event is not a transparent, psychological process, but is instead a socially situated activity" (p. 606). We seek to explain how different ways of seeing emerged across the two classrooms by developing the focusing interactions framework. According to Goodwin, an event being seen – a relevant center of knowledge – emerges through the interplay between a domain of scrutiny and a set of discursive practices being deployed within a specific activity. Following Goodwin, we argue that the act of students' noticing a mathematical regularity (a *center of focus*) emerges through the interplay between *features of a mathematical task*, a set of discursive practices (*focusing interactions*) and engagement in particular types of *mathematical activity*. Thus there are four constructs in the focusing interactions framework: a) centers of focus; b) focusing interactions, c) features of mathematical tasks; and d) the nature of mathematical activity. Each will be briefly characterized and illustrated using data from the study.

Centers of focus refers to the mathematical features, regularities, or conceptual objects to which students attend. This construct captures what individual students notice mathematically during class. It also represents the individual, cognitive component of the framework. For a particular mathematical task or problem, there may be several centers of focus, and for any given center of focus there may be several students who seem to be attending to that particular center of focus.

Through analysis of the videotaped classroom data, distinct differences emerged in what students were attending to mathematically while they were graphing and finding the slopes of lines. In Class 1, the students treated the points to be graphed as measurable quantities. For example, in the speed context, the point (4, 10) was treated as the distance of 10 cm that was covered in 4 sec. In contrast, students from Class 2 spoke about points as physical locations. For example, the teacher in Class 2 introduced points as "a way of helping people with directions" to your house; for the point (1, 4) you go "right 1 street and then go up 4 streets." The students picked up on this language and plotted points by going "up and over" on the grid.

Additionally students from Class 1 treated a line as a collection of same speed values while students from Class 2 talked about lines as physical objects – mountains, slope, and ramps. We conjecture that this led to a focus in Class 1 on lines as *mathematical* objects and in Class 2 on lines as *physical* objects. Finally, the majority of students in Class 1 talked about slope as a relationship between quantities, e.g., as the speed of particular characters who were walking. In contrast, the students in Class 2 spoke about slope as a characteristic of a physical object, e.g., the steepness of a set of stairs.

Focusing interactions refer to the set of discursive practices (including gesture, diagrams, and talk) that give rise to particular centers of focus. It is through this construct that we account for the social organization of noticing. One such discursive practice is that of highlighting (Goodwin, 1994). Highlighting refers to visible operations upon external phenomena, such as labeling, marking, and annotating. Highlighting can shape the perceptions of others by making certain material prominent. Another type of discursive practice prominent in the Class 1 data was that of "quantitative dialog." Here the teacher presses for students to link numeric statements with what they refer to in the situation (in this case, the speed context).

As illustrated in the transcript below, the teacher in Class 1 introduced graphing by asking students to generate a distance and time pair that had the same speed as known pair. She then asked them to graph the distance and time pair (before introducing the numeric shorthand for representing coordinate pair). This directed attention to the measureable quantities. Consequently, when the first student graphed the pair, she looked to the vertical axis labelled "distance" to locate the 30 corresponding to the 30 cm. She highlighted the axis by gesturally sweeping her hand from the axis to the point. She repeated a similar highlighting gesture for the time quantity. Her activity contrasts with that of Class 2, where plotting points occurred in a grid with no reference to what was represented by the axes.

Teacher: What was one value that we know about that's the same speed as 10cm and 4 seconds?

Student: Um, 30 and 12 seconds

Teacher: Can anybody come up here and graph this [points to the annotation above the graph, "30 cm in 12 sec"] for me? Kalisha?

Manuel: Like a line graph or a bar?

Teacher: A point.

Kalisha: [sweeps hand right from the 30 on the vertical axes labelled "distance"; sweeps hand up from the 12 on the horizontal axis labelled "time"; marks a point; repeats sweeping gestures]

Teacher: How did she do it Ana?

Ana: She did it, because she went on the 30 and then she went all the way to the 12 line at the bottom, 12 um seconds

Teacher: You have to make sure that 30 is with the distance and then she went over here to the line that was 12 seconds, and we write that (12,30).

Mathematical tasks refer to the features of the problems and activities that contribute to the emergence of particular centers of focus. Specifically, we are interested in how affordances and constraints of the task context influence what students notice mathematically. For example, in Class 2, the scales on the axes were never labelled and were always treated as a scale of one. This likely influenced the students' subsequent attention to graphs as grids with little attention to how the axes were scaled.

The fourth component of our framework, *the nature of mathematical activity*, addresses the participatory organization of the classroom that establishes the roles and expectations governing students and teachers actions, which seem to contribute to the emergence of particular centers of focus. We are particularly interested in whether or not students are expected to develop their own ideas, share their strategies, justify their reasoning, and work with other students ideas. We are also interested in whether or not the teacher provides most of the mathematical content and how tightly the teacher guides class interactions.

In Class 1, students were encouraged to notice and share their discoveries regarding the points they had graphed on a line. For example, one student noticed that she could create points on a particular line ($y = 2.5x$) by starting with a given point and adding 2.5 cm and 1 sec over and over again to generate additional points on the line. Another student shared that instead of adding, he multiplied 2.5 by the seconds to get the distance amount. This contribution helped students shift their focus from iterative to multiplicative patterns, which subsequently aided in the construction of slope as a rate. In contrast, the interaction pattern in Class 2 was more tightly guided with of the new information coming from the teacher and with students providing short calculational responses.

DISCUSSION

In Class 1, points and lines appeared to be treated as mathematical objects and in Class 2, they appeared to be treated as physical objects and locations. This difference is significant. For example, if stairs are treated as a mathematical object, then one can form a ratio relationship between the height and tread of each step and see that the ratio is constant despite differences in size between the steps (as in the stairs on the left in Figure 2). However, if one treats stairs as a physical object (as Chanise appeared to do in the transfer task described previously), then each step must be identical in a given staircase, as is the case in the physical world (see the stairs on the right in Figure 2). Treating stairs as a physical rather than a mathematical object constrains one's ability to form a ratio between the heights and treads of the stair steps, which has ramifications for learning about slope. In the first case, slope can be formed as a ratio; in the second case, slope is a simply a pair of two whole numbers.

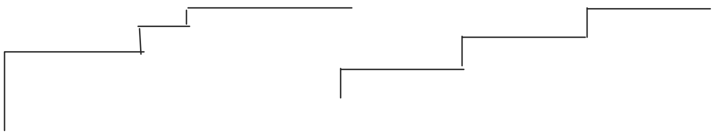

Figure 2. An illustration of stairs as a mathematical versus a physical object

In this study, the different centers of focus that emerged in each class were related to the ways in which students transferred their learning experiences. In reasoning with transfer tasks involving graphs of linear functions, students in Class 1 appeared to notice the quantities represented by the axes while students in Class 2 appeared to treat the graph as a grid of boxes and ignore the quantities. While students in Class 1 treated points and lines as mathematical objects representing quantities, students in Class 2 treated points and lines as physical objects and locations. Furthermore, what students noticed mathematically depended upon differences in the discursive practices, features of the tasks, and the nature of the mathematical activities in the two classes.

As new transfer processes are identified and elaborated, we can challenge current instructional approaches to transfer and development new ones. For example, it is a widespread belief that "knowledge that is taught in only a single context is less likely to support flexible transfer than knowledge that is taught in multiple contexts" (see a summary of research in Bransford et al., 2000, p. 78). However, this work on focusing interactions suggests that what is critical for the generalization of learning is not the number of contextual situations explored but the particular mathematical regularities and properties to which students' attention is drawn and that students notice.

Acknowledgment

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GENDER EFFECTS IN ORIENTATION ON PRIMARY STUDENTS' PERFORMANCE ON ITEMS RICH IN GRAPHICS

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This study investigated the longitudinal performance of 378 students who completed mathematics items rich in graphics. Specifically, this study explored student performance across axis (e.g., numbers lines), opposed-position (e.g., line and column graphs) and circular (e.g., pie charts) items over a three-year period (ages 9-11 years). The results of the study revealed significant performance differences in the favour of boys on graphics items that were represented in horizontal and vertical displays. There were no gender differences on items that were represented in a circular manner.

INTRODUCTION

The burgeoning information age has provided new and increased demands on our capacity to represent, manipulate and decode information in graphical forms. Increasingly, graphs are used to (re)present information and predict trends. Data can be transformed into detailed and dynamic graphic displays with increased sophistication (and ease), and consequently, the challenges faced by students decoding such graphics in school mathematics has changed. The purpose of this paper is to investigate the effect that the orientation of the graphic has on students' ability to decode various visual representations. In particular, we examine the performance of males and females on basic orientation items since there are gender differences (in favour of males) on map items (Diezmann & Lowrie, 2008).

A FRAMEWORK FOR VISUAL PROCESSING

Information Graphics

Visual representations, such as number lines, graphs, charts, and maps are part of the emerging field of information graphics found throughout current school curricula—with such graphics regularly used to represent mathematics content in standardized testing (Logan & Greenlees, 2008). Furthermore, the actual structure and composition of the graphic is generally treated in a single holistic static form rather than the actual elements contained in the graphic (Kosslyn, 2006). Recent studies have shown that the elements (including graph type and structure) contained within a graphics-rich item have a strong influence on decoding performance (see Lowrie & Diezmann, 2005). Consequently, the visual elements (e.g., line, position, slope, area) used in constructing a graphic have an impact on how well students understand and interpret the task, and select appropriate strategies and solution pathways.

Mathematical information can be contained in text, keys, legends, axes or labels (Kosslyn, 2006), as well as elements of density and saturation (Bertin, 1967/1983).

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This information is often represented in multiple forms in any graphic, and thus it is not surprising that young children may find it difficult simply moving between the text of a question and the information in the graphic (Hittleman, 1985). Even with much older college students, individuals tend to read and re-read graphs in order to keep track of the information in the axes and labels (Carpenter & Shah, 1989). With respect to map items, Bertin (1967/1983) argued that a decoder was required to make sense of the linear aspects of parallelism (specifically categories of horizontal and vertical orientation) and the variation of circular systems (e.g., pie charts).

In our previous research (Diezmann & Lowrie, 2008; Lowrie & Diezmann, 2005) we have found that a student's capacity to decode the information embedded in a graphic is demanding in its own right. This is particularly the case when students are required to decode items from standardized tests—with new forms of item representation (rich in graphics)—placing increased demands on cognitive and perceptual processing.

Gender differences on mathematics items

Although most gender differences are attributed to general experiences rather than neurological makeup (Halpern, 2000), males tend to outperform females on spatial tasks (e.g., Bosco, Longoni, & Vecchi, 2004) and particularly mapping tasks (Silverman & Choi, 2006). Diezmann and Lowrie (2008) have suggested that these performance differences are associated with confidence and attitudes toward mathematics and the everyday (out-of-school) experiences that students are exposed to—including increased exposure to technology-based entertainment games.

Saucier et al. (2002) suggested that males tend to utilise Euclidean-based strategies to describe directions and distance when decoding map items—in the sense that they use *directional language* (e.g., north, west, top). By contrast, females tended to use *landmark-based approaches* (e.g., left right, below) to make sense of visual information. In their study it was noteworthy that males outperformed females on tasks that were Euclidean in nature but there were no gender differences on tasks that were represented in a landmark-based form.

The present study goes beyond previous research by investigating basic elements of graphic design (Kosslyn, 2006) by analysing performance on horizontal, vertical and circular elements of graphics that combine to produce map items (Bertin 1967/1983). Moreover, we take note of Fennema and Leder's (1993) challenge to ensure that studies that consider gender differences in mathematics are focused and strategic. To isolate the horizontal, vertical and circular elements of graphics in our study, we selected graphics for investigation which predominately have specific structural properties (e.g., circulare orientation on a pie chart, see Appendix) rather than use map items which contain a multiplicity of orientations.

METHOD

This investigation is part of a 3-year longitudinal study which sought to interpret and describe primary students' capacity to decode information graphics that represent mathematics information. The aims of the study were to:

1. To document primary-aged students' knowledge of graphical items (e.g., number lines, graphs, pie charts) in relation to graphic orientation; and
2. To establish whether there are gender differences in students' decoding performance in relation to graphic orientation.

The Instrument and Items

The 15 orientation items (the five horizontal, vertical and circular-represented items) from the Graphical Languages in Mathematics [GLIM] Test were used in the analysis (for a description of the GLIM test see Diezmann & Lowrie, in press). The GLIM is a 36-item multiple-choice instrument developed to assess students' ability to interpret items from six graphical languages including number lines and graphs. The 15 items varied in complexity, required substantial levels of graphical interpretation and conformed to reliability and validity measures (Lowrie & Diezmann, 2005).

The GLIM orientation items were administered to students in a mass-testing situation annually for three consecutive years. The items were classified in relation to graphic structure. Horizontal items included single axis items (e.g., number lines) and opposed-position items (eg., column graphs) that were represented horizontally. Vertical items included axis and opposed-position items that were vertically orientated. Circular items included connection items (e.g., tree diagrams) and miscellaneous items (e.g., pie charts) which required students to decode information using topographical processing. The Appendix presents two of the items from each of the three orientation categories.

Participants

The participants comprised 378 students (M=204; F=174) from eight primary schools across two states in Australia. The cohort completed the 15 orientation items of the GLIM test each year for three years. The students were in Grade 4 or equivalent when first administered the test (aged 9 or 10). Students' socio-economic status was varied and less than 5% of the students had English as a second language.

RESULTS

Before analysis were undertaken, Guttman's (1950) unidimensional scaling technique was used to determine the suitability of the three orientation categories. The scale, which is validated prior to further data analysis, implies a development sequence for performance of items within a scale. On this scale, values greater than 0.9 are considered to indicate a highly predictive response pattern among items. Values over 0.6 are considered to indicate a scale that is unidimensional and cumulative. The coefficient of reproducibility for the three orientation categories were *horizontal* *orientation* (.86), *vertical orientation* (.84), and *circular orientation* (.91). The five items contained within each of the three orientation categories were included in the analysis given the strong coefficient measures—since the categories predicted high response patterns among these variables.

The two aims of the study were investigated through an analysis of the participants' responses to the 15 orientation-based items of the GLIM test. These items (the independent variable) were classified as either horizontal, vertical or circular graphical representations (see Appendix). A multivariate analysis of covariance (MANCOVA) was used to analyse mean scores across Grade and Gender dependent variables. A spatial reasoning measure (students' scores on Raven's Standard Progressive Matrices (1989)) was deemed to be an appropriate covariate [F(2, 1029=150.4, p<.01]. The MANCOVA revealed statistically significant differences between the mean scores of students across both *Grade* [F(6, 2052=24.38, p<.01] and *Gender* [F(3, 1025=10.85, p<.01] variables. There was no statistically significant interaction (*Grade* x *Gender)* [F(6,2052=1.12, p=.35]. Table 1 presents the means (and standard deviations) for grade and gender over the 3-year period.

	Grade 4			Grade 5			Grade 6		
	Total	M	F	Total	M	F	Total	M	F
Hor.	2.75	2.92	2.55	3.19	3.44	2.90	3.44	3.56	3.31
	(1.16)	*(1.11)*	*(1.15)*	*(1.23)*	*(1.20)*	*(1.21)*	*(1.14)*	*(1.14)*	*(1.12)*
Vert.	*3.45*	3.58	3.29	3.89	4.04	3.71	4.14	4.24	4.02
	(1.11)	*(1.07)*	*(1.15)*	*(1.05)*	*(0.96)*	*(1.11)*	*(0.93)*	*(0.91)*	*(0.94)*
Circ.	2.90	2.91	2.89	3.26	3.31	3.20	3.61	3.72	3.48
	(1.23)	*(1.29)*	*(1.17)*	*(1.21)*	*(1.28)*	*(1.21)*	*(1.12)*	*(1.03)*	*(1.20)*

Table 2: Means (and Standard Deviations) of Student Scores by Grade and Gender

Student performance increased between 12-16% from Grade 4 to Grade 5 across the three orientation categories. For both the horizontal and vertical categories the increases from Grade 5 to Grade 6 were 6-8%. By contrast the increase for the circular category from Grade 5 to Grade 6 was 11%. Subsequent ANOVA's revealed statistically significant differences in the performance of students across the three years of the study on both *horizontal-orientation* [F(2, 1036=36.38, p<.01] and *vertical-orientation* [F(2, 1036=52.92, p<.01] variables. Subsequent post-hoc analysis indicated that student improvement was significant across each grade level for each of the two orientation variables.

There were also statistically significant differences between the performance of boys and girls across two of the orientation variables: *horizontal-orientation* [F(1, 1034=24.23, p<.01] and *vertical-orientation* [F(1, 1034=14.26, p<.01]. For each variable, across each year of the study, the mean scores for the boys were higher than that of the girls. With respect to the vertical-orientation items means scores for boys were between 5%-9% higher than girls while they were between 6%-15% higher on horizontal-orientation items. By contrast there was no statistically significant difference between boys and girls on the *circular-orientation* variable [F(1, 1034=.56, p=.452].

DISCUSSION

Our study examined the effect orientation had on the performance of primary-aged students' capacity to decode items rich in graphics. Student performance increased significantly over the 3 year period for both the horizontal and vertical categories. When the graphics were represented in either a horizontal or vertical manner, boys outperformed girls in each of the three years of the investigation. In fact, the mean scores for the boys were approximately twelve months ahead of that of the girls. By contrast there was no statistical difference between the performance of boys and girls on items that were represented in a circular structure. These results go beyond Diezmann and Lowrie's (2008) earlier findings which highlighted gender differences, in favour of boys, on *map* items that required both horizontal and vertical decoding.

We suggest that the performance differences between boys and girls are associated with the way in which items are structured—graphical representations that require vertical or horizontal decoding are, in essence, Euclidean based. Our study has reduced these components to a more fundamental level by analysing the elements of graphical languages that in effect combine to produce maps, namely horizontal, vertical and circular elements. Significantly, there were no gender differences on items which did not contain the linear aspects of parallelism (Bertin, 1967/1983). As Silverman and Choi, (2006) found, females tend to use more holistic typographical approaches to solve graphics tasks, which are effectively employed in the circular items from the GLIM instrument.

CONCLUSIONS

The finding of gender differences in favour of boys on items that contained horizontal or vertical elements has four educational implications. First, everyday instruction in mathematics needs to provide opportunities for girls to become proficient in interpreting (and creating) visual elements in horizontal and/or vertical formats (e.g., graphs, maps and axis items). Such instruction should begin at an early age and the effectiveness of instruction should be monitored as gender differences in mathematics achievement increase over time (Winkelmann, van den Heuvel-Panhuizen, & Robitzsch, 2008). Second, there needs to be a shift in emphasis in the use of vertical or horizontal representations, such as number lines. The finding of gender differences in favour of boys suggest that initially girls need to learn about number lines rather than from number lines. Third, caution needs to be taken in interpreting or creating mathematics achievement tests. Given that items with graphics have a content dimension and a representational dimension, girls may be disadvantaged in a test where the types of graphics are more likely to be solved by boys than girls. Additionally, the content of an item may be masked by its representation. Finally, the literature on orientation and gender effects in mathematics typically focuses on dynamic orientation—a change in orientation. The findings of this study indicate that static orientation of visual elements in graphics is also a fruitful avenue for the exploration of gender differences in maps and other graphics typically used in mathematics.

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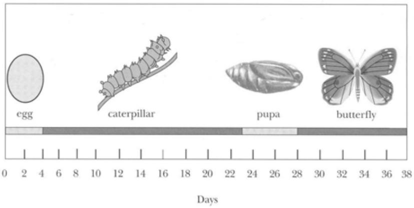
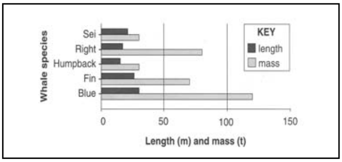
APPENDIX: ORIENTATION ITEMS FROM THE GLIM INSTRUMENT

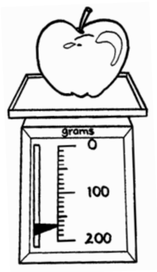
<table>
<tr><td>A. Horizontal Orientation Item[1]</td><td>B. Horizontal Orientation Item[2]</td></tr>
<tr><td>
The following graph shows the length of time taken for the four stages in the life of a butterfly.



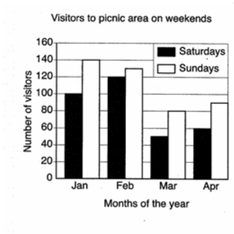
How many days are there in the caterpillar stage?
</td><td>
The graph compares the maximum length and mass to which some whales grow.



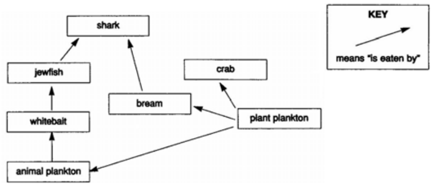
A fisherman reported that a whale 25 metres long and weighing approximately 80 tonnes had beached itself. Which species of whale could this be?
</td></tr>
<tr><td>A. Vertical Orientation Item[3]</td><td>B. Vertical Orientation Item[4]</td></tr>
<tr><td>
What is the mass of the apple?



</td><td>
This graph shows the number of visitors to the picnic area for Saturdays and Sundays.

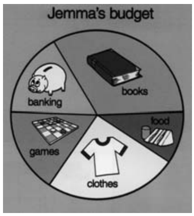


Which month had the most visitors on Sundays?
</td></tr>
<tr><td>A. Circular Orientation Item[5]</td><td>B. Circular Orientation Item[6]</td></tr>
<tr><td>
A simple food web



The animals in this food web eat only what is shown.

If all the animal plankton die which of the following will also die?
</td><td>
In 2003, Jemma budgeted $30 on clothes.
Approximately how much money did she get that year?



</td></tr>
</table>

THE 'VERBIFICATION' OF MATHEMATICS

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As part of a larger project focused on transforming mathematics education for Aboriginal students in Atlantic Canada, this paper reports on the role of the Mi'kmaw language in mathematics teaching. Examining how mathematical concepts are described the Mi'kmaw language gives insight into ways of thinking. A change in discourse patterns to reflect Mi'kmaw verb-based grammar structures, referred to as 'verbification', is described through the example of a grade 3 lesson on prisms and pyramids. 'Veribification' shows tremendous promise as a way to support Mi'kmaw learners as they negotiate their space between school-based mathematics and their own cultural ways of knowing and doing mathematics.

RATIONALE

The Mi'kmaq are the Aboriginal inhabitants of Atlantic Canada. Mi'kmaw communities in Nova Scotia have a unique jurisdictional agreement with the Government of Canada that gives them control over their education system and collective bargaining power. Disengagement from mathematics and science is a concern for many teachers in these schools as they grapple with the tensions between school-based mathematics and Mi'kmaw ways of reasoning about things seen as mathematical. Having taught in one of these schools for ten years I had experienced these tensions myself. I wanted my students to be successful learners of mathematics, yet I also suspected that the disengagement I sometimes witnessed in my classroom emerged in response to conflicting worldviews.

It has been argued that many Aboriginal students disengage from mathematics and science because of this discrepancy between their own culture and the cultural values embedded in school-based mathematics programs (Cajete, 1994; Ezeife, 2003; Aikenhead, 2002; NCTM, 2002). For some students the cost of participation means denying self and community to participate in the dominant view of mathematics. Often times these costs are seen as too great and children choose not to participate. Doolittle (2006) cautions that, in learning mathematics, "as something is gained, something might be lost too. We have some idea of the benefit, but do we know anything at all about the cost?" (p.19) If we are to meet the needs of Mi'kmaw students, and all Aboriginal learners, we must move towards a decolonized approach to education that allows for the inclusion of indigenous world views (Aikenhead, 2002; Orr, Paul & Paul, 2002; Tompkins, 2002; Battiste, 2000; Battiste, 1998). The journey of this research project has been an attempt to uncover key issues that must be attended to in transforming mathematics education for Mi'kmaw students.

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METHODOLOGY

Research for many Aboriginal people has been intimately connected with colonization and imperialism (Smith, 1999) and thus, any attempts to conduct research in aboriginal communities are often met with resistance and scepticism, and probably rightfully so. To ensure that work is accepted within the academy some indigenous researchers have attempted to bring indigenous values to traditional research paradigms (Wilson, 2003). Yet, such pasting of indigenous perspectives onto Western paradigms has not been proven effective in the decolonization of these paradigms and has not been effective in giving voice to the indigenous community (R. Bishop, 2005; Denzin, 2005; Smith, 1999). These practices, despite best intentions, through their demands for validity and generalisability have tended to essentialise the indigenous other.

As a response to this challenge, a new paradigm of decolonizing research or indigenist research has emerged (Denzin, 2005) and is seen as a way to "research back to power" (Smith, 2005, p. 90). The indigenist approach to research "is formed around the three principles of resistance, political integrity, and privileging indigenous voices" (Smith, 2005, p.89) and has a "purposeful agenda for transforming the institution of research, the deep underlying structures and taken-for-granted ways of organizing, conducting, and disseminating research and knowledge" (p.88). There is an underlying "commitment to moral praxis, to issues of self-determination, empowerment, healing, love, community solidarity, respect for the earth, and respect for elders" (Denzin, 2005, p.943). Such paradigms create space to privilege indigenous knowledge (Denzin, 2005; Smith, 2005) and acknowledge that knowledge production must happen in a relational context (Denzin, 2005).

In search of an appropriate indigenist paradigm, I sought the advice of many community elders. I searched for a way to describe the activity of people coming together to discuss an issue or solve a problem. During an informal conversation with one community leader, it was suggested that I use the word *mawikinutimatimk* which means 'coming together to learn together'. I checked with other community members who confirmed that this would be an appropriate word to describe the approach to research that I was seeking. It implies that everyone comes to the table with gifts and talents to share — everyone has something that they can learn. It conjures an image of a community of learners working in circle where all members are equally important and necessary. Each participant that joins in the circle has something unique to contribute. Thus *mawikinutimatimk* became the methodology for this project.

The project was conducted in two Mi'kmaw K to 6 schools. Times were arranged to meet with participant groups once or twice each month over a nine month period. Teachers, support staff, and elders were invited to participate. There were a total of 7 participants in one school group and 10 participants in the other school which was larger. Not all participants attended every session. Ten after-school sessions were held in one school and twelve in the other school. In addition to our conversations, I also frequently spent the day at each school and was often invited to work with teachers in their classrooms co-planning and co-teaching a lesson, or modelling a lesson. After-school conversations were recorded and transcribed. Classroom sessions were not recorded but field notes were kept and experiences from the classroom sessions were often discussed during our after-school sessions. Our conversations were often stimulated by inviting participants to simply notice and reflect on the tensions and challenges with mathematics for their students and share their thoughts with the group.

FINDINGS AND DISCUSSION

Through our conversations, four key areas of concern emerged as themes: 1) the need to learn from Mi'kmaw language, 2) the importance of attending to value differences between Mi'kmaw concepts of mathematics and school-based mathematics, 3) the importance of attending to ways of learning and knowing, and 4) the significance of making ethnomathematical connections for students. For this report, I will elaborate on the need to learn from Mi'kmaw language.

Learning from language

The important role of indigenous language in understanding mathematics was demonstrated by Denny (1981) who used a "learning from language' approach while working with a group of Inuit elders in Northern Canada to explore mathematical words in the Inuktitut language. Rather than developing curriculum and translating it into Inuktitut, they used the mathematical words to develop the curriculum and associated mathematics activities. More recently Bill Barton (2008) has shared the stories of his similar struggles in translating mathematics concepts into the Maori language. He has argued that mathematics evolves with language and as such claims that:

> A proper understanding of the link between language and mathematics may be the key to finally throwing off the shadow of imperialism and colonialisation that continues to haunt education for indigenous groups in a modern world of international languages and global curricula. (p.9).

During our *mawikinutimatimk* sessions, our conversations frequently turned to the need to learn from the Mi'kmaw language. Through asking questions such as "What's the word for…?" or "Is there a word for…?" the group began to gain new insight into the ways of thinking embedded in the language, something Mi'kmaw participants referred to as *Lnuitasi* (the ways of thinking of our people). For example, a resource teacher in one of the schools came to one session wondering about the Mi'kmaw word for 'middle'. She had concerns about an assessment report received on one student that stated he had been unable to point to the object in the middle. Knowing this child spoke mostly Mi'kmaq at home she wondered if this was a language issue. After an extensive conversation about many different words used in different contexts to describe being in the middle, the conclusion was that the word 'middle' does not really have a direct translation and as such is probably not a concept that is talked about at home very often, even when the conversation is happening in English. This awareness helped the teacher to take a different approach with the student, setting up opportunities for the child to experience the meaning of 'middle'. The student was then easily able to point to the object in the middle.

The Notion of Motion

As we continued to explore our translation questions, a key idea about the structure of the Mi'kmaw language and its potential impact on mathematics learning once again emerged for me. Mi'kmaq is a verb-based language. In Mi'kmaq, words for shapes and numbers act as verbs. Other indigenous languages including Maori share a similar grammatical structure (Barton, 2008). During one particular session in one of the two schools, Richard, a technology teacher and Mi'kmaw language expert shared with the group some ideas about the concept of 'straight'. He explained that the word *pekaq* means 'it goes straight'. There is a sense of motion embedded in the word. Similarly *paktaqtek* is a word to describe something that is straight such as a fence. He explained that here "is a sense of motion from here to the other end – *pektaqtek* [it goes straight]."

The role of using verbs in mathematics teaching is something I had become curious about prior to beginning this project. I had noted in my own teaching, a transition from asking noun-based questions such as "What is the slope?" to asking verb-based questions such as "How is the graph changing?" I am certain that I did this quite unconsciously initially although I am also sure that I was listening to the way students were talking and tried to model my language with similar grammar structures. It was only upon reflection that I realized I was changing my discourse to be more verb-based than noun-based. I found in my own experience that students often understood better when I used more verbs and when we talked about how things were changing, moving, and so on. I was excited to have the opportunity to explore this concept in our *mawikinutimatimk* sessions.

Nominalisation and 'Verbification'

Pimm and Wagner (2003) claim that a feature of written mathematical discourse is nominalisation – "actions and processes being turned into nouns" (p. 163). Mathematics as taught in most schools has a tendency toward noun phrases and turns even processes such as multiplication, addition, and square root into things (Schleppegrell, 2007). The dominance of English in school-based mathematics results in this objectifying tendency. 'We talk of mathematical objects because that is what the English language makes available for talking, but it is just a way of talking' (Barton, 2008, p.127). What would happen if we talked differently in mathematics? What would happen if we drew upon the grammar structures of Mi'kmaq instead of English?

Research relating to mathematical discourse suggests that there is a need to support students as they move from everyday language to more formal mathematical language (Schleppegrell, 2007). I would argue that it is not simply a matter of using everyday language; there is a need to go further and incorporate the grammatical structures of the students. It has been argued that mathematics could have developed differently and that ‘a non-objectifying mathematics is possible’ (Barton, 2008, p.127). I argue that mathematical discourse in the Mi'kmaw classroom should draw on the extensive use of verbs. I refer to this as the ‘verbification’ of mathematics. The following story from a classroom experience during the research project gives an example of how the ‘verbification’ of mathematics supported student understanding.

Prisms and Pyramids

Mary, a pre-service teacher at one of the schools, had asked me to help her with her lesson on prisms and pyramids in her grade three class. She was a bit uncomfortable with this unit because she was unfamiliar with the geometric concepts and worried about the quantity of vocabulary terms. I was excited to help her, so we planned the lesson activities together and I agreed to help her teach the class.

We began the class on the carpet and passed around some solids, inviting students to tell us something about them, we chose a cube and a square based pyramid. Each student was asked to say one thing about the solid when it came to them. Some counted vertices and reported how many corners; others counted faces but called them sides. One student offered that the cube was red while another was pleased to report that it felt soft as he rubbed it against his face. One young girl placed the prism on the floor and stated “It can sit still!” This was exciting.

We also used the carpet opportunity to re-introduce some vocabulary that the students would have learned in grade 2. They were talking about the sides and the corners and a few had counted edges although this vocabulary term did not seem to be shared by all. I took the lead on reviewing these terms. I asked the students if they knew a fancy name for side and I held the cube up next to my own face. “What is this?” I asked fanning my hand in front of my face. They all shouted “Face!” “That’s right,” I said. “I use my face to look at you and the cube can look at you with all six of his faces.” I rotated the cube a few times so that they could see each face looking at them in the same way I was looking at them. I then wanted them to get the word edge but I was determined not to tell them. “Does anyone know what we call these parts where the sides come together?” I asked running my fingers along the edges. Many of the students wanted to call them corners but I told them there was another word we use for these in mathematics. Then in a moment of inspiration I held up the cube and began to run my hand across the top face and as I moved toward the edge I said “I go over the…?” “Edge!” they all shouted. “Yes,” I said, “we go over the edge as we move across the top. These parts where the sides come together are called the edges.”

After the carpet activity we asked students to pair up and each group was given a geometric solid (ether a prism or a pyramid) to explore at their tables. We had planned which solids we would use for the lesson and we carefully chose which group received each solid varying the complexity of the task to some extent. They were given three tasks to do. Each group was asked to make footprints of each side in moon sand and record the shapes they made on a recording sheet. They were each also asked to report back on how many faces, vertices and edges their object had, and were also asked to add any other properties they felt were important. They were also asked to build the object with toothpicks and clay, and to report anything interesting they noticed while completing this task.

After their exploration, each group was asked to tell the class whether their solid was a pyramid or a prism and the responses were quite interesting. One pair of students declared that they had a pyramid because it looked like a pyramid. When prompted to explain what they meant by that they said "well it goes like, forming into a triangle." With this, they made a hand gesture showing how the sides were merging to a point. Another student also used a hand gesture to explain her declaration that her group had a prism "because it goes like this" and motioned her hands up and down in uniform fashion. A real challenge arose when it came time for the group with the triangular prism to report back. There was some debate about which category it belonged to.

"It kind of forms into a triangle" suggested one student but this seemed to be not enough to commit to it being a prism. "What if we look at it like this?" I asked as I rotated the picture card on the board so that it now appeared to be standing on its' triangular base. "Oh! It's a prism" a girl from the back offered, "Because it goes like this" and she motioned again with her hands up and down in a uniform manner. This seemed to convince her classmates who offered supporting arguments such as "Yeah, it's not coming to a point all around like the other ones." They all agreed that although it kind of looked like a pyramid in some ways, it was definitely a prism.

We then began to talk about the properties of these two types of solids based on how we had classified them on the board under the two headings. I asked students to tell me some things that all prisms had in common and some things that all pyramids had in common. We talked about some of the strategies they had been using earlier such as being the same thickness up and down or coming to a point. I asked students if they thought pyramids could stand on their heads and they all agreed that they could not because they come to a point. They did however believe that prisms could stand on their heads. This became an important way to distinguish between the two types of solids they had been exploring. I explained how these faces that we were referring to as feet and heads were known as bases and students were able to recognize that a prism had two congruent bases and a pyramid had only one base.

So where is the verbification in this classroom episode? The first moment of verb based discourse came from the student on the carpet who noticed that the cube could "sit still". We also spoke about looking with the face and going over the edge. Even the students' descriptions of the prisms as "going like this" indicated the motion embedded in their conceptual understandings. Talking about these properties with a sense of motion made them much easier to understand.

Perhaps the best example of verbification however came when we began to talk about the properties of prisms and pyramids. The children spoke about how the objects were forming. The students talked about the pyramids "coming to a point" or "forming into a triangle." It is interesting to note that the Mi'kmaw word '*kiniskwiaq'* means 'it comes to a point'. This connects to the sense of motion that is embedded in descriptions of shape in Mi'kmaq. This discourse led us to connect back to the ability to sit still that had been shared at the carpet. Could the pyramid stand on its head? No it could not.

This classroom episode gives just one example of how increasing the use of verb-based discourse patterns supports Mi'kmaw children's linguistically-structured way of understanding. In ensuing mawikinutimatimk sessions, Mary and I frequently referred to this lesson and shared our enthusiasm about the effects of our 'verbification' with the group who concluded that more investigation in this area was necessary.

CONCLUSIONS

This pervasiveness of nominalization in mathematics stands in direct contrast to ways of thinking about and doing mathematics in Mi'kmaq. Often in my own teaching career my Mi'kmaw students would tell me I was 'talking crazy talk' which I came to learn, often meant that I was using too many nouns. To these students it made no sense to talk about all of these static objects, there was no sense of motion, nothing was happening. There is perhaps a pervasive belief that mathematics is about objects and facts, things that can only be described as nouns. Could it be different? What does it mean to do mathematics? Byers (2007) argued that mathematics is a creative endeavour that is far more about the doing than the objects of mathematics. It is about observing change and puzzling over ambiguity. Turning mathematical processes into objects may provide some people with a way to talk about them in a more efficient manner but it also denies the journey of discovery from which the process emerged. It could be argued that turning processes into objects is useful as it allows us to then perform new processes on these objects; performing action on actions. Unfortunately, in school-based mathematics, much nominalisation ends there, and students are presented with these ideas as things to know rather than processes to use.

More work needs to be done in determining what can be learned from studying indigenous languages and their structures. There is a need to explore the ways in which language is used in mathematics classrooms and how it might be transformed to be more in line with Mi'kmaw and other grammar structures. As shown in the larger context of our *mawikinutimatim*k conversations, attention to language is even more helpful when connected to other issues at play in the local context. Even so, the issues relating to the structure of language alone, helps us to see potential tensions for Mi'kmaw students in mathematics and potential resolutions to these tensions. This shows tremendous promise as a way to support Mi'kmaw learners as they negotiate their space between school-based mathematics and their own cultural ways of knowing and doing mathematics. As part of the presentation, I will share the ideas arising from the conversations about language, and will also connect these ideas with the other three central themes discussed in the research conversations.

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CHARACTERIZING STUDENTS' ALGEBRAIC THINKING IN LINEAR PATTERN WITH PICTORIAL CONTENTS

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The purpose of this paper was to explore the relation among the abilities, investigative process, generality and algebraic thinking when the fifth and sixth graders solved a linear pattern with pictorial contents. The author developed an ability model of solving pictorial patterns according to related theorems (e.g., Herbert & Brown, 1997; Ma, 2008; Orton & Orton, 1999) and 40 students' problem solving. The model denotes that there are levels of 0, 1, 2, 3a, 3b, 4a, 4b, 4c. Students are from level 0, 1 (seeing of sensory) and level 2, 3a (pattern seeking), arithmetic thinking, to level 3b (pattern recognition), arithmetic-algebraic thinking transition with understanding how to approach generality, and finally to level 4 (generalization), algebraic thinking with ability in making a general rule. In addition the students at level 4 can achieve far generalization, while those at level 3b can achieve near generalization.

INTRODUCTION

Mathematics can be described as a science of pattern and order (Schoenfeld, 1992). Patterning activities (e.g., English & Warren, 1999; van De Walle, 2004) directly develop a sense of pattern and regularity and they provide students with practice in the skills of searching for patterns, extending patterns, and making generalizations. These processes will involve in variable and the concept of function. Currently curriculum document relative to patterning activities recommend that students investigate patterns in shape and number, formulate verbal rules, and then generalize the situation (Booth & Blane, 1992). As a result, Usiskin (1999) states the view that algebra is the language of patterns. Patterning activities play a significant role for primary graders to establish the algebra foundation (Herbert & Brown, 1997). Thus, patterns are greatly important for establishing a coherent research base in early algebra (Carraher & Schliemann, 2007).

The school practice involving generalization in algebra often starts from pictorial and numerical patterns. Orton, Orton, and Roper (1999) suggest that there are three purposes for setting pattern tasks within a pictorial context. The first one is for those students who think from a more geometrical approach. The second one is that pictorial content might be more elementary than purely symbolic content. The third one is just to vary the format to create more problems to be solved. Ma (2002) investigates 10-11-year-old students' preference for pictorial or numerical patterns. She finds that 84.4% students prefer pictorial patterns because of easiness, novelty and interest, or more hints. There are three methods of translating a picture to a numerical representation. 1. Count the objects (e.g., dots, sticks, etc.) for each shape presented in

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the task, then immediately converting them into numbers. 2. Look at how many more objects are required in each new shape. 3. See the shapes (Orton et al., 1999).

The investigative process to solve the patterns in numbers and shapes consists of three phases: 1. pattern seeking: extracting information, 2. pattern recognition: mathematical analysis, 3. generalization: interpreting and applying what was learned (Herbert & Brown, 1997). Expressing generality is described as one of four roots of algebra (Mason, Graham, Pimm, & Gowar, 1985). Stacey (1989) refers to two kinds of generalizations, that is, near generalizations (e.g., term 10 or 20) and far generalizations (e.g., term 100). The generalization problems require students to see, say, record, and test a pattern. Thus, the first stage of it for the learner is always "seeing" which refers to grasping mentally a pattern or relationship (Mason et al., 1985). Thus, the key to success seemed to be at the first stage of pattern perception (Lee, 1996). Based on the perspective above, the author extends Herbert and Brown's (1997) processes to a lower phase of seeing of sensory. The way of students to see for generalizing problem gave rise to a way of counting. The phase involves whether students are aware of the ignoring and stressing in the midst of seeing.

Ma (2008) combines and revises the stage and level of children's patterning abilities, suggested by Orton and Orton (1999). She establishes the levels about upper graders solving linear patterns with pictorial contents. They are 0, 1, 2, 3a, 3b, 4a, 4b, 4c. The procedure is adopted in which students' responses to what is noticed are placed at one of the levels. Level 0: no progress at all. Level 1: student notices some properties of the numbers, with perhaps partial patterns described. Level 2: student notices but not to describe a pattern, so the next number might not be described. Level 3: student knows how to obtain the next number using patterns extrapolated from the differences. 3a: recognize a relationship between successive terms, but only notice the answer; 3b: recognize a relationship between successive terms and the structure in relation to the shape. Level 4: student shows clear evidence of understanding the relationship, though an algebraic formula might not be expressed. 4a: a correct verbal statement; 4b: a worthy attempt at an algebraic expression; 4c: a correct algebraic representation.

The fifth and sixth graders who have not received the formal algebra instruction are in the arithmetic-algebra transition. They are the suitable candidates to exploit alternative strategies, which then provide them the potential for developing algebraic thinking. The activity of pictorial patterns is the best way to connect early algebra to concrete context (English & Warren, 1999). In textbooks, linear patterns are the type of patterns used most often. Linear patterns are those in which the difference between successive terms is constant. This paper would take a linear pattern with pictorial contexts as an example to investigated upper graders' performances about patterning abilities, the investigative process, generality, and algebraic thinking. The specific research questions are: 1. What are the relationships between students' patterning abilities and the investigative process? 2. What are the relationships between students' patterning abilities and algebraic thinking?

METHOD

Design of the study

The subjects carried out the patterning activities via an Internet discussion board. It will be a pedagogy innovation that mathematical activities are shifted to the Internet to carry out. Several pedagogical strategies may be used, including giving students sufficient time to generate and explore their ideas. An Internet discussion board, which function is a Bulletin Board Systems, is easily learned and used, and even fifth and sixth graders are familiar with the usage of it. Working on mathematics via the Internet can motivate students' interests and gain low-achievers' confidence in grades five and six content (Ma, 2004, 2005). It can be viewed as another channel for supporting upper graders to work on mathematics (Ma, 2005). Each participant had a specific account and password to enter the board.

Besides word typing and recording of an Internet discussion board, new functions such as basic summation and figures and tables pasting, enable students to express themselves more effectively. At the same time, students' data were conveniently collected due to the database functions of the board.

Participants

The subjects were forty upper-grade students in Taiwan. They were twenty-eight 11-year-old pupils of grade six in a rural school and twelve 10-year-old pupils of grade five in an urban school. There were 16 boys and 12 girls in grade six and 6 boys and 6 girls in grade five. They had basic computer skills and used the Internet regularly. Each subject was anonymous but had a fixed code, such as bm5, mbl1, mgm1, gh1. They were free to develop generalization under a non-threatening environment of Internet.

Instrument

Eight problems were given to the students to build patterns. These problems were revised by the researchers, who considered reference material such as numerical patterns of the Assessment of Performance Unit (undated), Orton and Orton (1999), and Hargreaves et al. (1999), as well as pictorial patterns of Orton et al. (1999). Among these eight problems, problems 1, 3, 5, and 7 were presented with pictorial contents, while problems 2, 4, 6, and 8 were presented with numerical contents.

Procedures

In the patterning activities, the teachers posed a problem on the Internet once two to three weeks. The students were asked to search for a pattern, extend the pattern, and develop a generalization for the pattern. They worked on the problems at lunch time or after school in their computer room. They were advised to solve the problems by themselves, because what they did had nothing to do with their academic achievement and their own methods were the best. The students were allowed to solve a problem based on their own perception over and over again. After having solved the problems on the Internet, students who had demonstrated unclear ideas or had much better performance on this activity were subsequently interviewed individually by their teacher. This methodology enabled the researchers to match the methods and processes students used when responding to the problems. Thus, data relating to students' understanding of patterns was collected in two forms: one was the written form to which students responded on the Internet discussion board, and the other was in oral form by interviewing students on a one-to-one basis.

ANALYSIS OF PROTOCOL

This article will take problem 5, a linear pattern with pictorial contexts, as an example. The pattern will be shown as Figure 1. The problem is as follows: How many dots will be used to make the 5th and the 20th shapes? How many in general? Ten protocols, three methods of translating a picture, will be analysed here. They direct quote from the Internet posting. Focusing on them is justified for three reasons. First, students needed to leave a rich trace of what they were thinking as they progressed. The trace included the written form and the interviews. Second, responses generated by these students needed to depend on their own perception. Third, students needed to find a rule associated with the pattern at least.

Figure 1: T shape

Count the objects, then immediately converting the shapes into numbers

24 of 40 students adopted this method. Two protocols are chosen as examples here.

1. bm5: The first is 5, the second is 8, and the third is 11. 5+3=8. 8+3=11. Thus, the 20th will be 20+3=23. The 100th will be 100+3=103. The n^{th} would be "n+3".

Student bm5 made a wrong description of the 20th, 100th, n^{th} (e.g., 20+3=23). Thus, he did not yet achieve a near generalizing task. His ability was at level 2, because he noticed a pattern but did not yet derive the next number. He extracted information from the pattern (i.e., +3), so his process showed the phase of pattern seeking.

2. mbl2: 5 dots, 8 dots, 11 dots. 5+3=8. 8+3=11. 11+3=14. 14+3=17. There are 17 dots in the fifth T. 17*3=51, 17+51=68. There would be 68 dots in the twentieth T.

Student mbl2 relied on a recursive approach and only produced a local rule (i.e., +3). He adopted a short-cut method (i.e., A_{20}=A_5 x 3+A_5; here A_n expresses the number of the n^{th}) .He did not achieve near generalization. He recognized a relationship between successive terms (i.e., +3), but only noticed the answer, so his ability was at level 3a. He could extract information (+3), so his process showed the phase of pattern seeking.

Look at how many more objects are required

10 of 40 students adopted this method. Four protocols are chosen as examples here.

3. gm3: Add a dot to the three vertexes of the T shape respectively. Keep going and you would get the answers. 5+3=8. 8+3=11. … 14+3=17….61+3=64. The twentieth would be 64.

All the processes gm3 solved the problem were similar with mbl2's. Thus, gm3's ability was at level 3a, and process showed the phase of pattern seeking. At last, she achieved a near generalizing task (i.e., the twentieth) based on a recursive approach.

4. bh3: The rule is "8-5=3", each T shape adds 3 dots." The first shape is 5. The second shape is 5+3. The third shape is 5+3+3. The fourth shape is 5+3+3+3. …The tenth shape is 5+3+3+3+3+3+3+3+3+3.

Student bh3 conceived of the adding processes he was using relative to the method of showing all the addends, not just the last result plus the new amount (e.g., 5+3+3+3). Thus, he recognized the structure in relation to the shape, and his ability was at level 3b. Also he made mathematical analysis (5+3+…+3), so his processes showed the phase of pattern recognition. Lastly he achieved near generalizing task (i.e., the tenth).

5. bh5: Totally add on three dots each time, because add a dot to the right, left, and under sides of T shape respectively. The fifth will be 11 plus 3 plus 3. The fifth will be 5 plus 3 plus 3 plus 3 plus 3, because 11 are 5 plus 3 plus 3. There would be 302 dots in the 100^{th} because 3 times 99 is 297 and 297 plus 5 is 302.

Student bh5 focused on the method itself. He made a worthy attempt at an algebraic expression "3 times 99 is 297 and 297 plus 5 is 302." He achieved a far generalizing task (i.e., the 100^{th}). Thus, his responses with showing clear evidence of understanding the relationship were at level 4b. His processes showed generalization phase.

6. mbm1: 5 dots in the first; what you have to do is plus 3 on each new shape. Shape 5 is 17, and shape 20 would be 5+19 groups of 3. Shape 100 would be 5+99 groups of 3. Shape *n* would be 5+ (*n*-1) groups of 3.

All the processes mbm1 solved the problem were same with bh5's except a correct algebraic representation (A_n=5+ (n-1) groups of 3). He achieved a far generalizing task (Shape 100). His ability was at level 4c. His processes showed generalization phase.

See the shapes

6 of 40 students adopted this method. Four protocols are chosen as examples here.

7. mgm1: There are 6 dots minus 1 dot in the first shape. There are 8 dots minus 1 dot in the second. There are 10 dots minus 1 dot in the third. …Keep going and you would get the hundredth. … There would be "x *n*" in the *n*th shape.

The individual interviews revealed that mgm1 viewed all T shapes as the structure of two lines with equal length. She was unaware of the incorrect content of seeing the picture (i.e., two lines with equal length), so her process only showed the phase of seeing of sensory. Her ability was at level 1, because she only noticed some properties of the pictures with partial patterns described (e.g., the structure of two lines).

8. mbh1: Add 2 on the horizontal and 1 on the vertical each time. Shape 3 is 7 on the horizontal and 4 on the vertical.…. Shape 5 is 11 on the horizontal and 6 on the

vertical. …Shape 20 is 41 on the horizontal and 21 on the vertical. …Finally 5 times 21 equals 105, and 5 times 41 equals 205. Shape 100 would be 105 on the vertical 100 and 205 on the horizontal.

In the interview, mbh1 said that 11 is the first "3" plus 2 plus 2 plus 2 plus 2 on horizontal 5, and 6 is the first "2" plus 1 plus 1 plus 1 plus 1 on vertical 5. He did near generalization (i.e., Shape 20). At last, he failed to achieve a far generalizing task because of adopting a short-cut method ($A_{100}=A_{20}$ x5). His ability was at level 3b and processes showed the phase of pattern recognition, like bh3's.

9. gh1: The first T is 3 on the up and 3 on the down. The second is 5 on the up and 4 on the down. The third is 7 on the up and 5 on the down. [3.3, 5.4, 7.5]. The first is 3 and keep plus 2 each time. I get 3, 5, 7, 9, 11,... The first is 3 and keep plus 1 each time. I get 3, 4, 5, 6, 7, … The tenth is 3+(2x9) =21 on the up and 3+(1x 9)=12 on the down. 21 plus 12 equals 33. … The hundredth is 3+(2x99)=201 on the up and 3+(1x99)=102 on the down. Then add 201 on 102.

Student gh1 focused on component parts of T shapes and on the method itself. She recognized the structure in relation to the shape and number. She made a worthy attempt at an algebraic expression and achieved a far generalizing task (i.e., the hundredth). Thus, her ability was at level 4b. Her processes showed generalization phase. Note that the author thought about students' method itself, not the numerical answer. Thus, it was not regarded that gh1 ignored T shape which overlaps in a common central dot (e.g., "201+102" should be"201+102-1").

10. mgh2: The first T is 5 dots; it is 1.1.3. The second is 8 dots; it is. 2.2.4. The third is 11; it is 3.3.5. … 4.4.6, 5.5.7. The first, second, and third number is the length of the left, right, and down of T shape respectively. The 5^{th} T would be 5.5.7 and total 17 dots. Thus, the 10^{th} would be 10.10.12 and total 32 dots; the 20^{th} would be 20.20.22 and total 62 dots. Keep going, and the 100^{th} would be 100.100.102 and total 302 dots. They will be the shape number times 3 plus 2.

Student mgh2 focused on component parts of T shape and on the method itself. She recognized the structure in relation to the shape and number. She made an attempt at an algebraic expression (the shape number times 3 plus 2) and achieved a far generalizing task. Thus, her ability was at level 4b. Her processes showed generalization phase.

CONCLUSION

1. The relationships between students' patterning abilities and the processes

(1) Students' abilities at level 0 and 1 are corresponding to the phase of seeing of sensory. Level 1: (7. mgm1). Her processes showed the phase of seeing of sensory. (2) Students' responses at level 2 and 3a are corresponding to the phase of pattern seeking. Level 2: (1. bm5). Level 3a: (2. mbl2) and (3. gm3). The processes of bm5, mbl2 and gm3 showed the phase of pattern seeking. (3) Students' responses at level 3b are corresponding to the phase of pattern recognition. Level 3b: (4. bh3) and (8. mbh1). The processes of bh3 and mbh1 showed the phase of pattern recognition. (4) Students'

responses at level 4 are corresponding to generalization phase. Level 4b: (5. bh5), (9. gh1), and (10. mgh2). Level 4c: (6. mbm1). The processes of bh5, gh1, mgh2 and mbm1 showed generalization phase.

2. The relationships between students' patterning abilities and algebraic thinking

(1) Students at level 0, 1, 2 and 3a engaged in arithmetic thinking, because they did not achieve near or far generalization. Note that students at level 3a, who could (e.g., 3. gm3) or could not (e.g., 2. mbl2) achieve the twentieth, did not guarantee to do a near generalizing task. (2) Students at level 3b might involve arithmetic-algebraic thinking transition, because their thinking about the methods of arithmetic will be a possible precursor to approach generality. Level 3b: (4. bh3) and (8. mbh1). Student bh3 or mbh1 might easily make progress from an additive approach (e.g., 5+3+3 or 3+2+2+2) to a multiplicative approach (5+3x2 or 3+2x3), while he could understand multiplication as repeated addition (e.g., 3+3=3x2). These multiplicative numerical expressions will be the foundation for algebraic expressions. (3) Students at level 4 engaged in algebraic thinking, because they had abilities in making a general rule. These results coincide with those of Ma's study (2008).

3. The author developed an ability model of solving pictorial patterns with pictorial contents according to related theorems (e.g., Herbert & Brown, 1997; Ma, 2008; Orton & Orton, 1999) and the findings of this study. It denotes that there are levels of 0, 1, 2, 3a, 3b, 4a, 4b, 4c. Students are from level 0, 1 (seeing of sensory) and level 2, 3a (pattern seeking), arithmetic thinking, to level 3b (pattern recognition), arithmetic-algebraic thinking transition with understanding how to approach generality, and finally to level 4 (generalization), algebraic thinking with ability in making a general rule. In addition students at level 4 can achieve far generalization, while those at level 3b can achieve near generalization. The model will be shown as Figure 2.

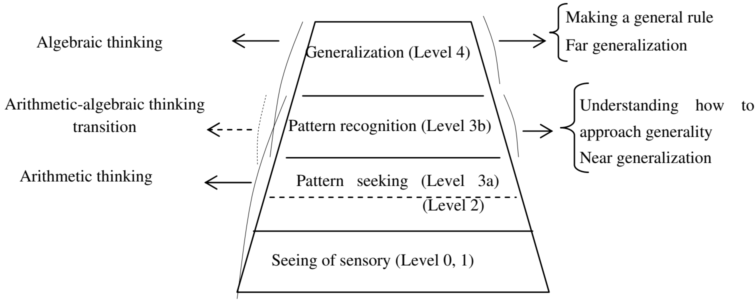

Figure 2: The ability model

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MITCHELMORE'S DEVELOPMENT STAGES OF THE RIGHT RECTANGULAR PRISMS OF ELEMENTARY SCHOOL STUDENTS IN TAIWAN

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The study presented in this report is part of a research project concerning elementary school students' developmental stages of representations. The purpose of this paper was to analyze the solid cuboids illustrated by elementary school students in Taiwan, and to compare the students' drawing results with Mitchelmore's (1978) stages of representation of regular solid figures. The conclusions were drawn as follows. (a). Based on 1,423 elementary students' drawings collected from this study, compared to the sample drawing given by Mitchelmore two decades ago, elementary school students in Taiwan seemed to have more representations of a prism. (b). The overall percentage that the students were assigned to each stage of representation of prism, from Stage 1 to Stage 4, were 18.73% (Stage 1), 21.76% (Stage 2), 18.73% (Stage 3A), 21.97% (Stage 3B), and 16.80% (Stage 4) respectively. It seemed that most of students were assigned to Stage 3B, followed by Stage 2.

INSTRUCTION

Mathematics is a powerful tool for solving practical problems and a highly creative field of study. One of the reasons is that ideas can be expressed with symbols, charts, graphs, and diagrams (Van de Walle, 2004). Symbols, graphs, and charts, as well as physical representations such as counters, fraction bars, and Cuisenaire rods are also powerful learning tools. Representations in the broadest sense, according to Kaput (1985), is something that indicates something else, and so must essentially involve some kind of relationship between symbol and referent, although each may itself be a complex entity. Moving from one representation to another is an important way to add understanding to an idea. In addition, different representations of mathematics ideas could be transformed with each others (Van de Walle, 2004). See Figure 1.

Representation has been one of the focuses of the mathematics education. It is one of five procedure standards in Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Representation refers to "the act of capturing a mathematical concept or relationship in some form and to form itself" (p. 67). Students should be able to "create and use representations to organize, record, and communicate mathematical ideas" (p.67), "select, apply, and translate among mathematical representations to solve problems" (p.69), and "use representations to model and interpret physical, social, and mathematical phenomena" (p. 70). It helps people to understand mathematical ideas, and to use these ideas. When students

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 57-64. Thessaloniki, Greece: PME.

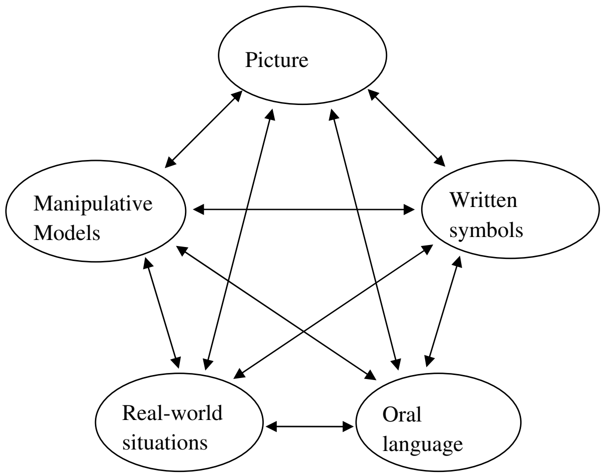

Figure 1. Five different representations of mathematical ideas (Van De Walle, 2004, p. 30)

develop flexibility with a variety of representations for mathematical ideas, not only do they add to their own understanding, but also obtain skill in applying mathematical ideas to new areas and communicating ideas to others. (Van de Walle, 2004).

Representation has also been one of the research interests (Goldin, 1987, 1998, 2000, 2003; Goldin, & Kaput, 1996; Lesh, Post, & Behr, 1987; Mitchelmore, 1973, 1974, 1975, 1980; Monk, 2003; Wu & Chen, 2004; Wu, Ma, & Chen, 2006; Wu, Ma, & Li, 2007). Among these research studies, Mitchelmore (1978, 1980) investigated 80 Jamaica elementary and middle school children regarding their 3D representations. According to Mitchelmore, children's representation of regular solid figures includes four stages: plane schematic, space schematic, prerealistic, and realistic stages. During the plane schematic, the child draws the solid figure as a singe face, or by a general outline. In the space schematic stage, more than one orthoscopical faces are shown, or hidden faces are included. The drawings in the prerealistic stage tried to represents the view from a single viewpoint and illustrate depth. During the last stages, parallel edges in spaces are represented by near-parallel lines.

Mitchelmore's set of development stages (1978, 1980) has been the focus of researchers interested in students' 3D drawings. However, this set of stages was developed based on Jamaica students' responses more than 20 years ago. Since then, the environment has been changed dramatically, and students today have many different ways to acquire information. It would be interesting to conduct a new investigation on students' drawings of solid figures.

PURPOSE OF THE STUDY

The purpose of this paper was to analyze the solid cuboids illustrated by elementary school students in Taiwan, and to compare the students' drawing results with Mitchelmore's (1978) stages of representation of regular solid figures.

METHODS

This study was conducted during spring semester, 2004. The participants were 1,423 elementary school students across grades 1 to 6, 708 males and 715 females. The numbers of students for each grade, from 1st to 6th, were 241, 242, 244, 240, 225, and 231, respectively. During an art class, students were assigned into 6 groups, sitting in a line. Each group was given a 30 cm by 18 cm by 18 cm cuboid, placing on their left front side. (See Figures 2 and 3.) All students could observe three planes of the cuboid, and each student would have the similar view of the cuboid as in Figure 3. Students were asked to draw that figure on paper. Students were also instructed not to copy others' images, or to discuss their thinking with others.

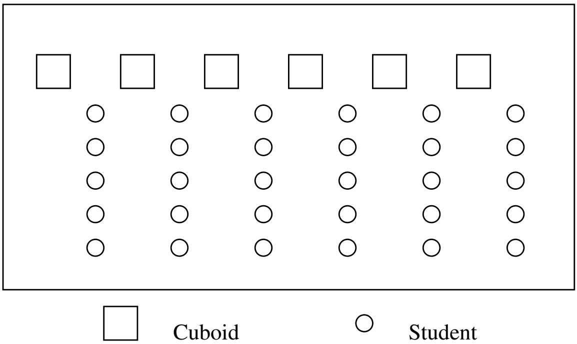

Figure 2. Seat arrangement

Figure 3. Cuboid from students' view

RESULTS AND CONCLUSIONS

The drawing results were arranged according to Mitchelmore's theory, and the differences were compared. The distributions of each stage and the examples of each stage were illustrated in Table 1 and Table 2 (Appendix), respectively.

Stage	1	2	3A	3B	4	Total
Frequency	272	316	272	319	244	1423
Percent	18.73%	21.76%	18.73%	21.97%	16.80%	100%

Table 1. The distributions of each stage

Stage 1 – Plane Schematic Stage

The results of this study indicate that there were 18.73% of participants in this category. Elements in the plane schematic stage remained the same. Students used a plane rectangle to represent a cuboid.

Stage 2 – Space Schematic Stage

Mitchelmore (1980) describes the drawing in this stage as three visible were shown, but the majority of people might think it is a 2-dimentional figure, rather than a 3-dimentional figure. However, eight types of drawing from this study were found. They are: (1) two visible faces only, (2) three visible faces incorrectly represented, (3) three visible faces, but could not be recognized as a cuboid, (4) three visible faces with a hidden one, (5) three visible faces with two hidden ones, (6) three visible faces, but more like a triangular prism, (7) two visible faces, and share a straight (top or bottom) line, and (8) two visible faces, and could be a 3-d representation from a different viewing angle. These drawing all indicated that these students had very few spatial concepts. Among these eight types of drawings, only the second ones fit Mitchelmore's description, and the rest did not. There were 21.76% of participants in this category.

Stage 3A –Prerealistic Stage A

According to Mitchelmore, in stage 3A, only 3 visible faces are shown, the edge of two adjacent faces is represented by a single line, and the faces are distorted from their actual shape to show depth. However, during stage 3A, not all elements will be shown. Participants in this study provided more detailed information, such as (1) only one element represented 3-dimentional, (2) three visible faces and a hidden one, some 3-dimentional elements were shown, (3) three visible faces, but the base were shown as a straight line, and the top did not have a pair of the parallel lines, (4) three visible faces, and the top illustrated as a pair of the parallel lines, but the base were shown as a straight line. All these drawings showed that students had been trying to represent the cuboid. These students had incomplete spatial perceptions. There were 18.73% of participants in this category.

Stage 3B –Prerealistic Stage B

Mitchelmore describes students in this stage should meet all three criteria in stage 3A. The differences between this stage and stage 4 were whether all parallel edges were illustrated by parallel lines. In this study, there were two types of representations, (1) the base line was straight, and the quadrilateral on top did not shown as a parallelogram or a trapezoid, and (2) the dash line for the hidden edges were shown incorrectly. There were 21.97% of participants in this category.

Stage 4 – Realistic Stage

According to Mitchelmore (1980), students in this stage illustrate all parallel edges by parallel lines, and represent a face orthoscopically only if it is in the frontal plane. It indicates that the student has established a Euclidean frame of reference to represent

3-dimentional spatial relations. In addition to the stage criteria described by Mitchelmore, the participant in this stage illustrated a cuboid with all perspective drawing correctly.

The drawing results of this study included three types: (1) all parallel edges illustrated by parallel lines, and only the frontal plane was shown orthoscopically, (2) all perspective drawing correctly, and use solid lines to illustrate the hidden edges, and (3) all perspective drawing correctly, and use dash lines to illustrate the hidden edges. Interestingly, some students' illustrated the front and back faces using two rectangles with different sizes. In those cases, not all parallel edges were illustrated by parallel lines as Mitchelmore described. There were 16.80% of participants in this category.

CONCLUSIONS

The conclusions were drawn as follows. (a). Based on 1,423 elementary students' drawings collected from this study, compared to the sample drawing given by Mitchelmore two decades ago, elementary school students in Taiwan seemed to have more representations of a prism. (b). The overall percentage that the students were assigned to each stage of representation of prism, from Stage 1 to Stage 4, were 18.73% (Stage 1), 21.76% (Stage 2), 18.73% (Stage 3A), 21.97% (Stage 3B), and 16.80% (Stage 4) respectively. It seemed that most of students were assigned to Stage 3B, followed by Stage 2.

This finding might provide educators a new direction of geometry curriculum development, and to plan lessons suitable for students' representation abilities. These finding will be analyzed by grades and gender in the near future, in order to study the stages for each grade by majority.

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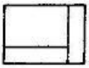
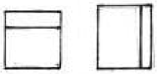
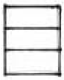
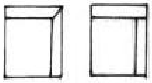
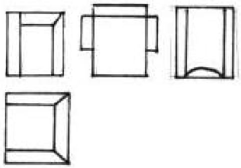
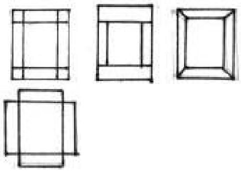
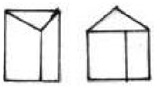
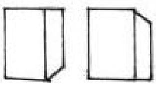
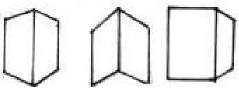
Appendix

Stages and Examples by Mitchelmore	Examples from this study		
Plane Schematic 			A plane rectangle
Space Schematic 	(1)		Two visible faces only
	(2)		Three visible faces incorrectly represented
	(3)		Three visible faces, but could not be recognized as a cuboid
	(4)		Three visible faces with a hidden one
	(5)		Three visible faces with two hidden ones
	(6)		Three visible faces, but more like a triangular prism
	(7)		Two visible faces, and share a straight line (top or bottom)
	(8)		Two visible faces, and could be a 3-D representation from a different viewing angle

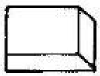
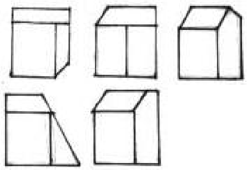
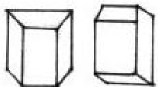
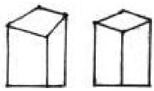
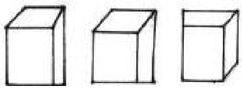
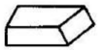
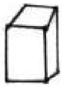
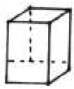
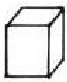
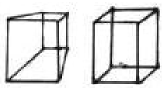
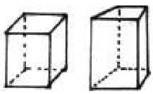
Stages and Examples by Mitchelmore	Examples from this study		
Prerealistic 3A 	(1)		Only one element represented 3-dimentional
	(2)		Three visible faces and a hidden one, some 3-dimentional elements were shown
	(3)		Three visible faces and a hidden one, some 3-dimentional elements were shown
	(4)		Three visible faces, and the top illustrated as a pair of the parallel lines, but the base were shown as a straight line
Prerealistic 3B 	(1)		The base line was straight, and the quadrilateral on top did not shown as a parallelogram or a trapezoid
	(2)		The dash line for the perspective drawing were not shown correctly
Realistic 	(1)		All parallel edges illustrated by parallel lines, and only the frontal plane was shown orthoscopically
	(2)		All perspective drawing correctly, and use solid lines to illustrate the hidden edges
	(3)		All perspective drawing correctly, and use dash lines to illustrate the hidden edges

Table 2 Different representations based on Mitchelmore's stages

EXPLOITING THE FEEDBACK OF THE APLUSIX CAS TO MEDIATE THE EQUIVALENCE BETWEEN ALGEBRAIC EXPRESSIONS

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This paper stems from the ReMath European project[1] that focuses on the role of representations in dynamic digital artefacts (DDAs), and on the role of theoretical frameworks with respect to their use in educational contexts. Framed within the Theory of Semiotic Mediation, the paper presents some results concerning how the potentialities of the Aplusix DDA, and in particular of the feedback component, can be exploited by the teacher in relation with the notion of equivalence between algebraic expressions. Through a semiotic analysis of excerpts from a classroom discussion, evidence of the semiotic process triggered by the teacher's interventions is provided.

INTRODUCTION

The key issue addressed by the ReMath European project is the role played by representations of mathematical objects offered by a Digital Dynamic Artefact (DDA) when used in educational contexts. Seven teams have been involved in the project, six DDAs (or new versions of already existent DDAs) have been developed, thirteen pedagogical plans (Earp & Pozzi, 2006) have been designed, many local experiments of them have been carried out in different countries. The methodology used in planning the whole experiment consists in a cross-experiment (Artigue & al., 2007) in which the different teams are supposed to experiment a DDA they are familiar with, since they have developed it, and a DDA they are not familiar with, that is, developed by another team. These are respectively called 'familiar experiment' and 'alien experiment'. In this contribution we report on an alien experiment based on the Aplusix DDA (Nicaud & al., 2006). The teaching experiment has been framed by the Theory of Semiotic Mediation, which is outlined in the next section.

THE THEORY OF SEMIOTIC MEDIATION

The Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) drawing from a Vygotskian paradigm considers learning processes deeply linked to teaching processes, in a social context. It states that the use of artefacts to accomplish a task leads the individual to the construction of *personal meanings*

[1] Representations in Mathematics with Digital Media, Project number IST4-26751, [http://remath.cti.gr/].

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 65-72. Thessaloniki, Greece: PME.

(Vygotsky, 1978), which are related to the actual use of the artefact. Under the guidance of an expert (typically the teacher), students' personal meanings may evolve towards *mathematical meanings*, i.e. meanings coherent with the mathematical theory.

> Thus any artefact will be referred to as *tool of semiotic mediation* as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention. (Bartolini Bussi & Mariotti, 2008, p. 754)

In this perspective, the function of semiotic mediation of an artefact is not automatically activated with the use of the artefact. In order to make meanings emerge it is crucial to identify the relationship, called the *semiotic potential* of the artefact (Bartolini Bussi & Mariotti, 2008), between the use of the artefact and the mathematical knowledge. Awareness of the semiotic potential of an artefact is a requisite for the teacher for developing suitable tasks for making meanings emerge, and for guiding the evolution of students' personal meanings towards mathematical meanings. Not only does the Theory of Semiotic Mediation provide a frame for designing the teaching sequence, but it also gives a frame for its analysis. In considering the process of teaching-learning in a semiotic perspective, it focuses on signs and on the process of transformation of such signs. In this context, the use of the term "sign" is deeply inspired by Peirce and is consistent with the claim of including different and more flexible kinds of signs (Radford et al., 2007, Arzarello et al., 2009). Transformation of signs can be considered an indication of learning intended as a change in the relationship between subject and knowledge: every intervention of the teacher with the goal of fostering or inhibiting such transformation can be considered a didactic action (teaching). As a consequence, the analysis of the educational process is to be centred around the description of semiotic processes as processes of evolution of signs. In order to accomplish such analysis a classification of signs, based on the level of connection with the artefact and its use, has been stated. Such a classification allows a description of the evolution process, characterized by a gradual detachment of signs from the reference context to contingent situations. It also gives the possibility of defining sequences of related signs which are called *semiotic chains*. Based on the original notion of chain of signification introduced by Walkerdine (1990, p.121), the notion of *semiotic chain* is consistent with the ones elaborated by Hall (2000) and Presmeg (2006), but holds a particular emphasis on the role of an artefact in triggering the semiotic process. In fact, a semiotic chain develops from what are called *artefact signs* (Bartolini Bussi & Mariotti, 2008, p. 756), strictly related to the use of the artefact, to the *mathematical signs*, that are the objective of the teaching-learning activity.

THE APLUSIX DDA: FEEDBACK-SIGNS

Aplusix is a computer algebra system which allows students to perform both arithmetical and literal calculations (Nicaud & al., 2006). One of the main characteristics of Aplusix is the presence of feedback that it provides when set up in *Training mode* (for a description of Aplusix feedback see Maffei & Mariotti, 2006). The feedback is based on the equivalence between two consequent boxes, each of them containing an expression. More specifically, different signs show whether the current expression is equivalent/not equivalent to the previous one, or whether it is not well-formed (Fig. 1). We called them *feedback-signs*.

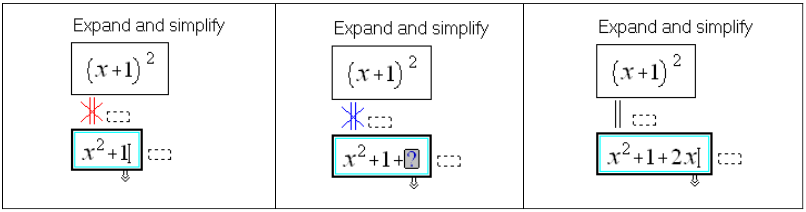

Figure 1. Three different feedbacks provided in Aplusix environment.

The black lines show that the first expression is equivalent to the second, the red crossed lines show that the first expression is not equivalent to the second, and the blue crossed lines show that the expression you are writing is not well-formed (i.e. a plus sign requires an argument).

THE SEMIOTIC POTENTIAL OF THE FEEDBACK-SIGNS

The feedback-signs provided by Aplusix have a twofold meaning. We can refer to them using the terms *primary interpretation* and *developed interpretation*. Let us, for example, consider the sign 'red crossed lines'. Its meaning is rooted in a social convention which can be reinterpreted in the school context; in fact, both the presence of the colour red and of the cross in the inscription refers to the sign of error which have the red colour and the cross in the set of its representations. While the primary interpretation could refer to common sense, the developed interpretation refers to a mathematical knowledge and for its nature is not immediate or immediately shared. Since the reaction of the machine is coherent with the mathematical knowledge at all times, the feedback-signs may become a possible instrument of semiotic mediation for the meaning of equivalence between algebraic expressions.

THE TEACHING EXPERIMENT

Educational and research features

According to the aims of the ReMath project, a pedagogical plan in which educational and research goals were strictly intertwined has been designed. The development of the pedagogical plan, centred on the use of the Aplusix DDA, has been guided by the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008). The semiotic potential of the artefact has been exploited to introduce students to the equivalence of algebraic expressions, which is one of the main goals of the pedagogical plan. The implementation of the pedagogical plan involved two 9th grade Italian classes, at the very beginning of high school, for about 20 hours. Data collected consists in log files of Aplusix, students' worksheets and written reports, audio and video-recordings, field notes by teachers and researchers. They were collected during the various phases of the teaching sequence, and analysed in an integrated way, providing the fabric for a semiotic analysis of which we will present an example in the next section.

The role of the DDA

One of the main objectives of our teaching experiment consisted in describing the role played by the DDA with respect to the stated educational goals. By analysing students' ongoing production through a semiotic lens, we have been able to identify key elements that provide evidence of the role played by Aplusix components, in particular by the feedback, both in students' learning processes, and in the teaching strategy. In the following, attention will be given to the interaction between the teacher and the students during a classroom discussion, aiming at identifying the emergence of the semiotic chains that link the emerged artefact-sign to the mathematical signs.

DATA ANALYSIS: TEXTURE OF MEANINGS IN THE SEMIOTIC CHAIN

We report on an excerpt from a collective discussion that followed the first activity with the DDA. Students are requested to work in pairs in Aplusix to accomplish a task of numerical calculation. As already said, when the students manipulate an expression, Aplusix constantly provides feedback related to the mathematical meaning of equivalence between expressions. The link between such kinds of feedback and its mathematical meaning is not automatic, but instead a matter of interpretation. It is just to stress their semiotic nature that we have introduced the term *feedback-signs*. The main goal of this first activity consists in making students interpret the three feedback-signs; students are therefore requested to observe the feedback given by Aplusix during calculation tasks. They are also asked to take note on a sheet of paper of how the signs change during the development of the calculation, providing a meaning for each of them. In the collective discussion following the activity with the artefact, the teacher aims at exploiting the semiotic potential of the feedback-signs and intends to make the students aware of the mathematical meanings of them. Here after an excerpt from the discussion.

1. Teacher: Have you all finished? There were four questions, the first asked to note down the signs appearing between a line (gesture as in Fig. 2 on the left) and the following one (gesture as in Fig. 2 on the right). What are the signs appearing between a line and the following one?

Figure 2. The teacher's hand as picking something at two different heights.

2. Mattia: The first sign appearing when you write in the second passage, is…two vertical lines with an x over them (gestures as in Fig.3)

Figure 3. Mattia draws two vertical lines top-down in the air indicating the feedback-sign "||".

3. Teacher: An x, two vertical lines with an x over them.

4. Davide: Parallel.

5. Mattia: And then, when you complete the passage, the x disappears.

6. Teacher: The x disappears. Is what he has said right? (The teacher draws on the blackboard the three signs) […]

19. Amalia: Well, there are three signs…well, those two vertical lines are when the passage is right and concluded […]

39. Teacher: What does it mean "to be right"?

40. Martina: That you didn't make any mistakes in the calculations.

41. Amalia: That you have not mistaken anything and you can go to the following passage […]

60. Teacher: And how can we do that not using the computer, understand that things are right without seeing the signs? Why are they right?

61. Ambra: Because if the calculation follows a logical thread, it is right.

62. Teacher: Because if the calculation follows a logical thread, it is right. What does it mean to follow a logical thread?

63. Martina: To do certain operations […]

66. Teacher: Why are passages right? What does it mean to have the passages right? Where does the logical thread lead? […]

67. Amalia: Because basically the last passage must give you the result of the first one.

68. Teacher: The last passage must give you the result of the first one: what does it mean?

69. Amalia: And yes because basically if you solve the first passage the result must be…equal to the second.

70. Teacher: Let's help her to say it well […]

73. Sabrina: Yes because finally the result is the simplification of the first, each passage has the same result.

74. Teacher: And so?

75. Amalia: Basically, if we have…I don't know…6/3 and we reduce it to the minimal terms it comes 2, doesn't it? (*The teacher writes on the blackboard 6/3 and 2, side each other*) So I tell that 2 is the result of the first passage […]

79. Teacher: [...] How do we say that the result of 6/3 is 2? In mathematics, when we speak, how can we say that the result of 6/3 is 2?

80. Cora: That the result of 6 divided by 3 gives 2.

81. Teacher: Yes, but...what do we say of these two (pointing to 6/3 and 2 with the two hands, Fig. 4) here?

82. Sabrina: That they are equivalent to each other

83. Teacher: That?

84. Sabrina: Yes, that they are equivalent to one another, they are equivalent.

Figure 4. The teacher pointing to 6/3 and 2.

85. Teacher: And what does it mean that they are equivalent?

86. Amalia: That they are equal...

87. Students: That they have the same value.

From the beginning of the discussion the teacher focuses on the interpretation of the feedback-signs of Aplusix (#1). As emerging from the discussion (#1-19), and confirmed by the collected written sheets, all the students' interpret the feedback-sign as "right passage" (see #19: “Two vertical lines [...] when the passage is right and concluded”). The personal meanings that students develop from the first activity with the artefact are consistent with the *primary interpretation* of the feedback. According to the classification provided by the Theory of Semiotic Mediation, the inscription “||" can be considered an *artefact-sign*, since its meaning is strictly related to the activity with the artefact. Under the guidance of the teacher it becomes the first element of a *semiotic chain* leading to the *mathematical sign,* referring to the notion of equivalence, Once it happened, the feedback-sign “||” has reached the level of the *developed interpretation*. In fact in the excerpt we can observe the following evolution for the interpretation of Aplusix feedback-sign “||”:

right / no mistakes (#19-41) becomes *passages with the same result (#67-73)* becomes *equivalence between passages (#82-84)*

This semiotic chain comes into existence thanks to a didactic strategy that, starting from the activity with the artefact, is focused on the students’ semiotic processes. This strategy uses, in a synergic way, different kinds of semiotic resources: speech, gestures (an example is in #1, Fig. 2, and the same kind of gesture-speech enactment is widespread in the whole protocol), and inscriptions on the blackboard (#81, Fig. 4). In particular, the teacher constantly stimulates the students to make the meanings of the signs involved explicit (‘what does it mean?’, #39, 62, 66, 68), to elaborate from the emerging contributions (‘Let's help her to say it well’, #70; ‘and so?’, #74), and to detach from the artefact (‘how can we do that not using the computer’, #60) to relate to mathematics domain (‘in mathematics, when we speak, how can we say that’, #79). Beyond the recurrent typical semiotic question “what does it mean”, the teacher’s strategy encompasses sentences and actions that have the functions of *echoing* and *amplifying* some students’ contributions to the whole classroom (#3, 6, 62, 68, 83), and generally *focusing attention* towards certain elements (see for instance the deictic gesture in Fig. 4). By repeating and re-formulating students' contributions on the one hand, and making explicit reference to mathematics domain on the other hand, the teacher fosters the weaving of a texture of meanings in which the meaning of equivalence comes to be sided and overlapped to that of the right passage. This double interpretation of the feedback-signs emerging from Aplusix is the core of the semiotic potential of this specific feature of the software in solving a given task. In the following excerpt, from a discussion that occurred a week later, we can see how this texture of meaning is correctly managed by the students and referred to the artefact-signs ('black equal' and 'red equal'):

1. Mattia: Aplusix uses some symbols, for instance when we write, and make a new passage: when we finish writing the passage, if the passage is equivalent to the previous one, and therefore it is right, we have a symbol telling us that it is right, whereas if the passage that we have written is wrong with respect to the previous passage, we have another symbol.
2. Teacher: So he is saying that if we have two different expressions that are equivalent then we have in Aplusix a symbol that is?
3. Davide: The black equal
4. Teacher: The black equal, two bars (gesture as in Fig. 5). If on the contrary these two expressions are not equivalent.
5. Davide: The red equal comes.

Figure 5. Two hands mimicking the two bars of Aplusix feedback-

As in many other cases in the protocols (see also above) we observe how the teacher uses, in a synergic way, different semiotic resources: in this case, the utterance is accompanied by a gesture that depicts the feedback-sign provided by the software. As it has been pointed out by many researches on the role of gestures in mathematics learning (see for instance Arzarello & al., 2009), a strict coordination among the various resources is found in the students as well (e.g. see # 2, Fig. 3).

CONCLUSIONS

We stress two issues that arise from the analysis of the classroom discussion. Firstly, the *pivotal role of the teacher* to develop the semiotic chain starting from the artefact signs and leading to a mathematical meaning. Indeed the teacher acts with a semiotic concern: she continuously asks questions to make the meanings of the signs emerge and develop. The nature of the semiotic process that she triggers can be analysed through the twofold interpretation of the feedback-signs as primary and developed. In this view, the teacher's action consists in making students shift from the primary interpretation to the developed interpretation. As a second result we highlight that accomplishing this transition requires that the students and the teacher interact through a *constant* *back and forth* between the primary interpretation and some attempts to reach the developed interpretation. The evolution doesn't follow a linear trend, from the primary interpretation to the developed one. On the contrary, the semiotic chain maintains a coexistence of both of them, in the process of unfolding the meaning of the Aplusix feedback-signs.

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A COMPARISON OF FOURTH AND SIXTH GRADE STUDENTS' REASONING IN SOLVING STRANDS OF OPEN-ENDED TASKS

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This paper reports on the forms of reasoning elicited as fourth grade students in a suburban district and sixth grade students in an urban district worked on similar tasks involving reasoning with the use of Cuisenaire rods. Analysis of the two data sets shows similarities in the reasoning used by both groups of students on specific tasks, and the tendency of a particular task to elicit all forms of reasoning in both groups of students. Attributes of that task and ways that those attributes can be replicated in other domains may have implications in the teaching of early reasoning.

INTRODUCTION

The NCTM states in its *Principles and Standards for School Mathematics* (2000) that a primary goal of mathematics education in grades K-12 are the development of reasoning (and proof). Further, the document points out that students must be exposed to different forms of reasoning and must learn to choose and use appropriate forms of reasoning, citing: "Students need to encounter and build proficiency in all these [e.g., reasoning by contradiction, cases, and direct deductive reasoning] forms with increasing sophistication as they move through the curriculum" (p. 59).

In the study described in this paper, we examined the forms of reasoning that were elicited by two different groups of students, fourth graders and sixth graders, investigating problems from strands of open-ended tasks and providing justification for their solutions. We identified the occurrence of direct reasoning, reasoning by contradiction, reasoning using upper and lower bounds, reasoning by cases, and generic reasoning (as defined in Table 1 below). The research questions guiding our work are: (1) what similarities or differences in forms of reasoning are exhibited by both groups of students as they worked on equivalent tasks? (2) Do certain tasks tend to elicit particular forms of reasoning? And, (3) if so, what are their characteristics?

THEORETICAL FRAMEWORK AND RELATED LITERATURE

The ability to reason is crucial for students to develop both a need and appreciation for making convincing arguments. It is also a basic requirement for supporting arguments in justification and proof making in the learning of mathematics. Several studies have documented the ability of elementary and middle school students to reason and provide justification for their reasoning as they work collaboratively in a supportive environment (Ball, 1991; Lampert, 1990; Maher & Martino, 1996; Mueller, 2007; Mueller & Maher, 2008; Steencken, 2001; Steencken & Maher, 2003; Yackel & Cobb, 1994).

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Psychologists have defined reasoning as the process of coordinating evidence, ideas, and beliefs to draw conclusions about what is true (Leighton, 2003). Mathematical reasoning has been described by Yackel and Hanna (2003) as both the use of induction, deduction, association, and inference to draw conclusions about quantity and structure, and a “communal activity in which learners participate as they interact with one another to solve mathematical problems” (p. 228). The second description situates mathematical reasoning in a context that allows for it to be elicited, while the first qualifies the forms of reasoning that are useful when doing mathematics. This study builds on and extends the research by examining patterns in the forms of reasoning elicited by elementary and middle school students as they worked on open ended tasks.

METHODOLOGY

This study draws from two data sets. The first is a longitudinal study of students’ mathematical thinking that was conducted by researchers in a fourth grade classroom of twenty-five students in a suburban school in New Jersey[1]. The second source of data is an informal after-school math program consisting of twenty-four sixth grade students that was conducted by researchers in an low socioeconomic, urban community in New Jersey, drawn from a school consisting of 99% Latino and African American students[2]. In the schools of this study, computation with fractions is introduced in grade 5. Hence, the fourth graders were not yet taught operations with fractions and related fraction ideas, while those in grade six had been taught procedures related to fraction ideas and operations. For both classes, the series of sessions were videotaped with at least two cameras. For this paper, we report data from the first seven 60 minute sessions from the fourth grade study and the first five 60-75 minute sessions from the sixth grade study[3]. The fourth graders began their work in pairs; while the sixth graders worked in groups of three or four. Both groups investigated tasks involving Cuisenaire rods. The strands of tasks were the same or very similar in both studies. Students were encouraged to provide justification for their solutions and to challenge and question the explanations of others.

Video recordings and transcripts were analyzed using the analytical model outlined by Powell, Francisco & Maher (2003). The transcripts were coded for forms of reasoning. For the purposes of this study, the forms of reasoning were defined as

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[2] The *Informal Mathematical Learning Project* (IML) directed by Carolyn A. Maher, Arthur B. Powell, and Keith Weber, was supported by a grant from the National Science Foundation (ROLE: REC0309062). The views expressed in this paper are those of the authors and not necessarily those of the funding agency.

[3] See Mueller 2007 and Mueller & Maher, 2008 for a detailed analysis of the second set of data

follows. First, *direct reasoning* was based on the assumption that "the hypothesis contains enough information to allow the construction of a series of logically connected steps leading to the conclusion" (Cupillari, 2005, p. 12). Second, *reasoning by contradiction* was based on the agreement that whenever a statement is true, its contrapositive is also true; or that a statement is equivalent to its contrapositive (Cupillari, 2005). Third, *reasoning by cases* involved students defending an argument by defending separate instances. Fourth, *reasoning using upper and lower bounds* was noted when the upper and/or lower bound of a subset S of some partially ordered set was defined, and an argument was then formed to justify a statement about the subset within the defined bounds (for example, that it is empty). Fifth, *generic reasoning* involved reasoning about a paradigmatic example whose properties can be applied to the set under discussion and lends insight into a more general truth, which in turn verifies the claim made about the particular example (Rowland, 2002)

Finally, the two data sets were compared, and similarities and differences were noted.

Task	Grade 4	Grade 6
If we call the dark green rod one, what number name would I give the light green rod?	Direct	Direct
Someone told me that the red rod is half as long as the yellow rod. What do you think?	Contradiction	Contradiction
If we call the blue rod one, which rod will have the number name one half?	All 5 forms noted	All 5 forms noted
If we call the orange rod one, what number name will I give the white rod?	Direct	Direct
If we call the orange rod one, what number name will I give the red rod?	Direct	Direct
Is 1/5 = 2/10?	Direct	Direct & Contradiction

Table 1: Sample Tasks and Forms of Reasoning that Emerged

RESULTS

Throughout the sessions, all students contributed to the discussion and direct and indirect forms of reasoning were elicited. Analysis of the data showed similarities between the forms of reasoning used by students in both grades. A representative sample of tasks that elicited similar forms of reasoning in both groups of students is listed in Table 1. Due to space limitations, this paper will discuss the similarities between the reasoning used by the fourth and sixth graders as they worked on the task: "If we call the blue rod one, which rod will have the number name one half?"

This task was posed during the second session of each study, and both groups of students worked to find the nonexistent rod that could be called one half of the blue rod (which is nine centimeters long). In both groups of students, the arguments were rich and varied, and correct lines of reasoning by contradiction, upper and lower bounds, cases, and generic reasoning were presented. In both groups, the students used direct reasoning to offer faulty solutions to the task.

The Fourth Grade

Reasoning Using Upper and Lower Bounds

In the fourth grade, David offered a solution and presented an argument using upper and lower bounds (c.f. Maher & Davis, 1995). He said, "I don't think that you can do that because if you put two yellows that'd be too big, but then if you put two purples … that'd be too short". When questioned by the researcher if there was any rod between the purple and the yellow, David replied, "I don't think there is anything."

David then presented his argument at the overhead projector. He showed that the length of the purple rod was one white rod shorter than the yellow rod, and then lined up the rods in a "staircase" pattern to show that, when ordered according to length, each successive rod was one white rod longer than the previous rod. He demonstrated that there is no rod that is shorter than the yellow rod and also longer than the purple rod.

Direct Reasoning (incorrect)

Erik incorrectly reasoned that the purple and yellow rod could each be called one half of the blue rod. Erik said, "See, I figured if you take a yellow and a purple it's equal [to the length of the blue rod]. They're not exactly the same, but they're both halves. Because the purple would be half of this even though the yellow is bigger because if you put the purple on the bottom and the yellow on top it's equal, so they're both halves, but only one's bigger than the other. So it equals up to the same thing."

Reasoning by Contradiction

Many students disagreed with Erik's suggestion. Alan and Jessica used the definition of one half to counter Erik's claim.

Alan: When you're dividing things into halves, both halves have to be equal – in order to be considered a half.

Jessica: This isn't a half. Those two aren't both even halves.

Alan made the definition of *one half* explicit. Jessica then showed the contradiction inherent in Erik's argument by saying that the purple and yellow rods aren't the same length, and therefore cannot be called one half of the blue rod. This counterargument is a sophisticated use of reasoning by contradiction, as it considers the faulty claim that Erik proposed and uses a definition to show the contradiction in the argument.

Reasoning by Cases

David then used an argument by cases to show that all rods can be classified as even or odd, and that their ability to be divided in half can be determined from that classification. He showed that the white, light green, yellow, black, and blue rods are "odd" because no rod exists that is one half their length. He then showed that the remaining rods are "even". To illustrate his point, he showed that two purple rods equal the length of the brown rod and two yellow rods are equal to the length of the orange rod, proving that the brown and yellow rods are even.

Generic Reasoning

David's argument by cases also contained an element of generic reasoning. By using the example of the blue rod, he showed that its properties can be applied to all "odd" rods in the set. After showing this general truth, he returned to the specific case of the blue rod, showing that since it belongs to the category of "odd" rods, there does not exist a rod that is half its length.

The Sixth Grade

In the sixth grade informal math session, similar arguments were used. There, direct reasoning, generic reasoning, and reasoning by cases, upper and lower bounds, and contradiction were used, and their arguments closely resembled those of the fourth grade students.

Direct Reasoning (incorrect)

Michael and Shirelle each proposed a solution similar to Erik's, suggesting that the yellow and purple rods are two halves of the blue rod. They showed that a purple and yellow train equals the blue rod, and called each of the rods half of the blue rod.

Reasoning by Contradiction

Chris used an argument by contradiction to show that there was no rod whose length was half of the blue rod. Building a model of nine white rods lined up next to the blue rod, he said, "Since there's nine white little rods you can't really divide that into a half so you can't really divide by two because you get a decimal or remainder so there is really no half, no half of blue because of the white rods."

Chris argued that since there were nine of the smallest size rods, and nine is not divisible by two, there was no rod that was one half of the blue rod. He used the fact that each rod was equal in length to a multiple of the white rods, and that if the length of the blue rod couldn't be divided into two equal groups of white rods, no rod that was half of the blue rod existed. Chris' use of decimals and remainders in his argument highlights one difference between the two groups of students. Due to their more extensive exposure to mathematics, the sixth graders included some mathematical ideas in their arguments that were not used by the fourth graders.

Reasoning Using Upper and Lower Bounds

Dante presented an argument using upper and lower bounds at the overhead projector. After repeating his argument by contradiction that the yellow is longer than the purple by one white rod and so each cannot be one half of the blue rod, he showed that there are no other rods that can be called one half of the blue rod. He said, "Usually for the blue piece, it would usually be purple or yellow but yellow would be one um one white piece over it and the pink would be, I mean purple would be one white piece under it."

Dante argued that either the purple or the yellow would be possible halves of the blue rod. However, he explained that the length of two yellow rods is one white rod longer than the blue rod, and that the length two purple rods is one white rod shorter than the blue rod. This argument showed that the yellow rod is an upper bound and the purple rod is a lower bound, and that there is no rod that is exactly one half of the blue rod.

Reasoning by Cases, Generic Reasoning

Justina then presented an argument by cases similar to David's. She called David's "odd" rods "singles", since those rods cannot be paired with a rod that is half its length. She then showed that the white, light green, yellow, black, and blue rods fall into this category. Similar to David's argument, her argument also contained an element of generic reasoning.

CONCLUSIONS AND IMPLICATIONS

In both studies, all forms of reasoning identified prior to the study were elicited, and varied arguments were presented to justify solutions that were offered. Many similarities in forms of reasoning used were noted, as can be seen from the sample results in Table 1. It can be concluded that careful task design can enable all students to reason effectively and learn to use different forms of arguments as they do mathematics. One significant difference that was noted between the forms of reasoning elicited for the different tasks. For the majority of tasks posed during the sessions in both studies, direct reasoning was the most common form used by the students. For example, of the six tasks analyzed in the table, four of the tasks elicited direct reasoning. When working on the task presented above, however, direct reasoning was only used to make incorrect claims, while reasoning by cases, contradiction, upper and lower bounds, and generic reasoning was used to provide correct solutions to the problem. This discrepancy highlights the tendency of this task to elicit forms of reasoning other than direct reasoning and the fact that different tasks tend to elicit different forms of reasoning consistently.

The discrepancy that was noted in the results pointed to a possible answer to the third and fourth research questions. We examined the task that elicited these different forms of reasoning to identify the characteristics of the task that may have been the cause of this difference. One possible reason for this difference is the structure of the task. Many other tasks required students to show the truth of a statement, and lent themselves to direct arguments. This task, however, encouraged students to show an argument by contradiction since, in the given set, there was no rod with number name one half, when blue was called one. This invited students to use other forms of reasoning, as called for in the NCTM standards, and resulted in the display of rich, varied arguments. The significant differences between the two groups under study, including their differences in age and socioeconomic status, as well as the different settings in which the tasks were posed, suggest that carefully designed tasks can provide the opportunity for all students to gain knowledge in using different forms of reasoning, meeting a primary goal of mathematics education. In order for students to practice using mathematical reasoning in all its forms, it might be helpful for teachers to introduce tasks that have the potential to elicit various forms of reasoning.

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OPTIMIZING INTUITION

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In this paper we establish a link between the Onto-Semiotic Approach to mathematics cognition and instruction (Godino, Batanero and Font, 2007) and the Cognitive Science of Mathematics (Lakoff and Núñez, 2000). We use an a priori characterization of "optimizing intuition" as the context for reflection.

INTRODUCTION

One characteristic of the research community in mathematics education is its large diversity of different theoretical perspectives and research paradigms. Although diversity is not considered a problem but a rich resource for grasping complex realities, we need strategies for connecting theories or research results obtained with different theoretical approaches. Each theoretical perspective tends to privilege some reality dimensions over others. Thus, it is not always an easy task to find links among research questions, descriptions, methodologies and conclusions that are elaborated within different paradigms. A specific research effort is needed in this direction. In this paper we establish a link between the Onto-Semiotic Approach to mathematics cognition and instruction and the Cognitive Science of Mathematics. We use an a priori characterization of "optimizing intuition" as the context for reflection. The confirmation of the existence (or not) of this type of intuition should be the result of a posteriori research.

The available literature on intuition is reviewed, the theoretical framework is presented and the constructs of the theoretical framework are used to explain what is meant by optimizing intuition in this work.

LITERATURE REVIEW

The relationships between intuition and rigor have been studied and debated in the field of Mathematical Education. Fischbein (1994) defines the notion of intuition and analyzes the essential role that it plays in students' mathematical and scientific processes. He classifies intuitions in two ways: according to its functions and according to its origins, although he warns that these distinctions should not be considered as absolute ones. Using Fischbein's work as a reference, Tirosh and Stavy (1999) developed the theory of the intuitive rules which allows the analysis of students' inappropriate answers to a wide variety of mathematical tasks.

One possible classification of types of intuition is to consider the mathematical content to which intuition is applied. This classification of intuitions strengthens our question about the existence of an "optimizing intuition".

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THEORETICAL FRAMEWORK

Below are summaries of the theoretical framework used in this work: *The Onto-semiotic Approach* and *the Cognitive Science of Mathematics*.

The Onto-Semiotic Approach (OSA)

In Figure 1 we represent some of the different theoretical notions of the Onto-Semiotic Approach to mathematics cognition and instruction (Godino, Batanero & Font, 2007; Font and Contreras, 2008). Here mathematical activity plays a central role and is modelled in terms of systems of operative and discursive practices. From these practices the different types of mathematical objects (language, arguments, concepts, propositions, procedures and problems) which are related, emerge building cognitive or epistemic configurations among them. (hexagon in Figure 1).

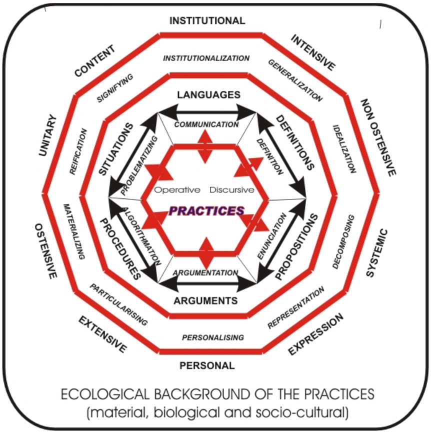

Figure 1. An onto-semiotic representation of mathematical knowledge

The problem-situations promote and contextualise the activity; languages (symbols, notations, graphics) represent the other entities and serve as tools for action; arguments justify the procedures and propositions that relate the concepts. Lastly, the objects that appear in mathematical practices and those emerging from these practices depend on the "language game" in which they participate, and might be considered from the five facets of dual dimensions (decagon in Figure 1): personal/institutional, elemental/systemic, expression/content, ostensive/non-ostensive and extensive/intensive. The dualities as well as objects can be analysed from a process-product perspective, a kind of analysis that leads us to the processes in Figure 1. In the OSA, instead of giving a general definition of process, it has been opted for a selection of a list of processes that are considered important in mathematical activity (those of Figure 1), without claiming that it includes all the processes implicit in mathematical activity because, among other reasons, some of the most important of them (for example, the solving of problems or modelling) are more than just processes and should be considered hyper or mega-processes.

Cognitive Science of Mathematics

Lakoff and Núñez (2000) state that the mathematical structures people build have to be looked for in daily cognitive processes such as the schemes of images and the metaphoric thinking. Such processes allow us to explain how the construction of mathematical objects is supported by the way in which our body interacts with the objects in everyday life. To reach abstract thinking, we need to use more basic schemes which are derived from the very immediate experience of our bodies. We use these basic schemes, called *image schemes*, to give sense, through *metaphorical mappings*, to our experiences in abstract domains. Lakoff and Núñez (2000) claim that the metaphors create a conceptual relationship between the source domain and the target domain. They distinguish two types of conceptual metaphors in relation to mathematics. A) Grounding metaphors: They relate a source domain out of mathematics with a target domain inside mathematics. B) Linking metaphors: They have their source and target domains in mathematics.

Lakoff and Núñez (2000) analyze four grounding metaphors whose target domain is arithmetic. In these four metaphors, we can find an approximation to the relationship of order, which is vital to the understanding of the concepts of maximum and minimum. On the other hand, these authors also consider that the graphics of the functions are structured through the metaphorical mapping of the "The Source-Path-Goal schema". Such a mapping conceptualizes the graphic of the function in terms of motion along a path - as when a function is described as "going up", "reaching" a maximum, and "going down" again. In this way, the idea of the ups and downs of a road is essential to the understanding of the concepts of maximum and minimum.

The link between these frameworks

In different research works conducted within the OSA framework, theoretical connections have been developed between this approach and the theory of Lakoff and Núñez. In Acevedo (2008) metaphorical processes are related to the 16 processes showed in Figure 1. To accomplish this task, the researchers use the graphic representation of functions as the context for reflection. In the present work, we are interested in commenting in detail the understanding of what metaphorical processes are. That understanding results from observing these processes from the "unitary – systemic" duality proposed in the OSA (Acevedo, 2008).

In Lakoff's and Núñez's works, the *unitary – systemic* duality has a central role. On the one hand, the metaphor is unitary (A is B). On the other hand, the metaphor allows us to generate a new system of practices (systemic perspective) as a result of our understanding of the target domain in terms of the source domain. Lakoff and Núñez develop the unitary–systemic duality for different metaphors. A good example can be the metaphor of the container, which according to Núñez (2000) is a metaphor used to structure the theory of classes. For this author, this metaphor is ontologic and unconscious and has its origin in every day life. (Núñez, 2000, p. 13):

Unitary: "Classes are containers"

Systemic:

Source Domain **Container Schemas**		Target Domain **Classes**
Interior of Container Schemas	→	Classes
Objects in Interiors	→	Class members
Being an Object of an Interior	→	The Membership Relation
An Interior of one Container Schema within a Larger One	→	A subclass in a Larger Class
The Overlap of the Interiors of Two Container Schemas	→	The Intersection of Two Classes
The Totality of the Interiors of Two Container Schemas	→	The Union of Two Classes
The Exterior of a Container Schemas	→	The Complement of a Class

Table 1. The metaphor "Classes are containers"

In fact, most research on metaphors has been mainly targeted at studying such a duality. In other words, given a metaphor, the source and the target domains are decomposed to determine what concepts, properties, relationships, etc. from the source domain are transferred to the target domain. The systemic vision of a metaphor leads us to understand it as a generator of new practices.

Because the OSA considers that on the one hand, among other aspects, an epistemic/cognitive configuration, depending on whether the adopted point of view is institutional or personal, has to be activated to do mathematical practices, and that on the other hand the systemic vision of the metaphor leads us to understand it as a generator of new practices, it is natural to ask ourselves the following question: *How is the metaphor related to the building components of epistemic/cognitive configurations?* The conclusion drawn by Acevedo (2008) on linking metaphors is that a linking metaphor projects an epistemic/cognitive configuration on another one.

The epistemic/cognitive configuration construct allows us to explain and precise the structure that is projected on the linking metaphors. There is a source domain that has the structure of an epistemic/cognitive configuration (whether the adopted point of view is institutional or personal) which projects itself on a target domain that also has the structure of an epistemic/cognitive configuration. This way of understanding the preservation of the metaphoric projection improves the explanation of such a preservation given by Lakoff and Núñez (2000), in which they just give a two-column table in which mainly properties and concepts are mixed. The reader can intuit that the properties are projected on properties and the concepts on concepts.

Still unresolved is the question of what structure is projected in the case of a grounding metaphor. We believe that unlike the linking metaphors, only some parts of the epistemic/cognitive configuration are projected. The specific study of each grounding metaphor will allow the identification of such parts.

THE OPTIMIZING INTUITION IN THIS RESEARCH

Now we explain what we mean by optimizing intuition in our research. To accomplish this goal, we will use the following question as a context for reflection: Why are there people who consider it evident that a graphic that looks like a parabola that is shown to them (Figure 2.A) has a maximum? To answer this question, we use three of the processes considered in Figure 1 (idealization, generalization, and argumentation), the image schemes, and the metaphorical mappings, in the way they were applied in Acevedo's doctoral thesis (2008).

Above all, intuition has to do with the process of idealization (Font and Contreras 2008). Let's suppose that the teacher draws on the blackboard the Figure 2(A) on the left and that he talks about it as if he were displaying the graphic of a parabola and simultaneously expecting that the students interpret such a figure similarly. The teacher and students talk about Figure 2(A) as if it were a parabola. If we look carefully at Figure 2(A), we observe that the graphic is not actually a parabola. It is clear that the teacher hopes the students will go through the same process of idealization of Figure 2(A) and draw it on the sheet of paper as he has done. That is to say, Figure 2(A) is an ideal figure, explicitly or implicitly, for the type of discourse the teacher and students make about it. Figure 2(A), drawn on the sheet of paper, is concrete and ostensive (in the sense that it is drawn with ink and is observable by anyone who is in the classroom) and, as a result of the process of idealization, one has a non-ostensive object (the parabola) in the sense that one supposes it is a mathematical object that cannot be presented directly. On the other hand, this non-ostensive object is particular. In the onto-semiotic approach, this type of "individualized" object is called an extensive object. Therefore, as a result of the process of idealization, we have moved from an ostensive, which was extensive, to a non-ostensive that continues to be an extensive object.

The process of idealization is a process that duplicates entities because besides the ostensive that is present in the world of human material experiences, it gives existence (at least in a virtual way) to an idealized non-ostensive. Font, Godino, Planas and Acevedo (in press) argue that the key notion of objectual metaphor is central to the understanding of how the teacher's discourse helps to develop the students' comprehension of the non ostensive mathematical objects as objects that have "existence". In fact, there are classic authors, such as Plato, that have considered intuition precisely as a bridge that allows the move between the space-temporal world of ostensives and the ideal world of non ostensives.

Intuition is also related to the generalization process because intuition can be considered as the process that allows us to see the general in the particular, a fact that is coherent with Fischbein's perspective (1994). In this case, there is also a generalization process according to which we consider this parabola as a particular case of any curve that has "a similar shape to that of a parabola". (In the OSA, this set is called an intensive object)

The relationship between intuition and generalization, for classic authors such as Descartes, is necessary to explain one of the basic characteristics of mathematical reasoning: the use of generic elements. Descartes proposes in his fifth meditation: it is necessary to consider a specific object for intuition, which cannot refer to itself but to particular objects, to be able to act. Intuition allows grasping what is general in what is particular (capturing the essence). It is not the goal of this work to deepen into the problem of the relationship between the generic element and intuition. We want just to highlight that any characterization of intuition should consider its relationship with generalization. Additionally, when intuition is related to the generalization derived from the use of generic elements, the complementary and dynamic relationship between rigor and intuition is highlighted. That is, intuition can be found in the intermediate steps of a proof or of the solution of a problem.

Given that intuition is usually considered as a clear and swift intellectual sensation of knowledge, of direct and immediate understanding, without using a conscious and explicit logical reasoning, we can consider that in intuition there is no explicit argumentation even though there is an implicit inference. In the case showed in Figure 2 the inference could be, for instance, "as in the curve there is first a part that goes up and then a part that goes down, there must be a point of maximum height".

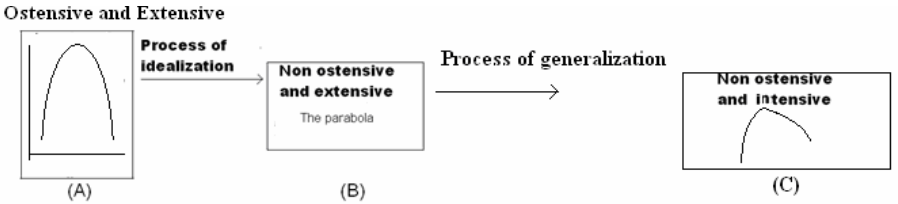

Figure 2. Idealization and generalization

The idea that intuition allows us to know the evident truth of specific mathematical propositions was a key element in what is known as the classic theory of truth, valid until the appearance of non-Euclidian geometries. Briefly, that theory states that: 1) A statement is mathematically true if and only if the statement can be deduced from intuitive axioms; 2) Deduction from intuitive axioms is a necessary and sufficient condition of mathematical truth; 3) People perceive certain mathematical properties (e.g. axioms) as truths evident in themselves.

We propose then the use of a vectorial metaphor in which the intuitive process is a vector with three components (Any of them could be "null" in some cases).

Intuition = (idealization, generalization, argumentation)

With this metaphor, it can be seen that intuition acts upon universal mathematical ideas (which are present through their associated ostensives) to get to results that are considered true without (or almost without) an explicit argumentation. In fact, the different ways in which intuition can be understood differ in the emphasis that they give to each one of the three components of the "intuition vector".

The task now is to find an explanation to the argumentation component in the specific case we are dealing with; that is, how we can explain that it is evident that "because it first goes up and then comes down, there must be a maximum". We claim there are reasons to assume that there is an *optimizing intuition* which makes this type of statements to be considered evident. This optimizing intuition has its origin in basically two types of everyday experiences. The first has to do with the fact that in everyday life we frequently have to face optimization problems such as when we try to find the best way to go from one place to another (not necessarily the shortest), also when we try to make the best buy, etc. This type of situations has an optimizing reasoning that seeks to find the best solution to a given situation.

The second type of experiences is related to the fact that we are subjects who experience how certain physical characteristics such as physical strength, health, etc vary as time goes by and go through critical moments (maxima and minima). In this second type of experiences we should consider also those related with the fact that we move frequently along roads which have ups and downs. We maintain that these two types of situations of everyday life allow us to make metaphorical mappings that contribute to the understanding of optimization problems. On the other hand, the very bodily experiences facilitate the appearance of the following optimizing image scheme (Figure 3), which can subsequently be projected in more abstract domains. We maintain that the metaphorical mapping of these domains of experience (preferences, consumer, etc) and of the optimizing scheme produce an understanding intuition of optimization problems, which is the one that allows us to answer intuitively the question with which we began this part of this section. The optimizing scheme derived from "The Source-Path-Goal schema".

Using Fischbein's classification as a reference, which distinguishes between primary and secondary intuitions, we believe that this intuition would be, in our opinion, of the primary type, which remains as stable acquisitions during the entire life, and that, as a consequence of the development of formal abilities, can gain in precision.

In terms of the epistemic/cognitive configuration, one of the characteristics of the primary optimizing intuition is that the epistemic/cognitive configurations of the solution of optimization problems solved with optimizing intuition do not present the argument that justifies why the solution obtained is the optimum, given that it is considered evident that the found value is the optimum. To get more details you can refer to Malaspina (2008).

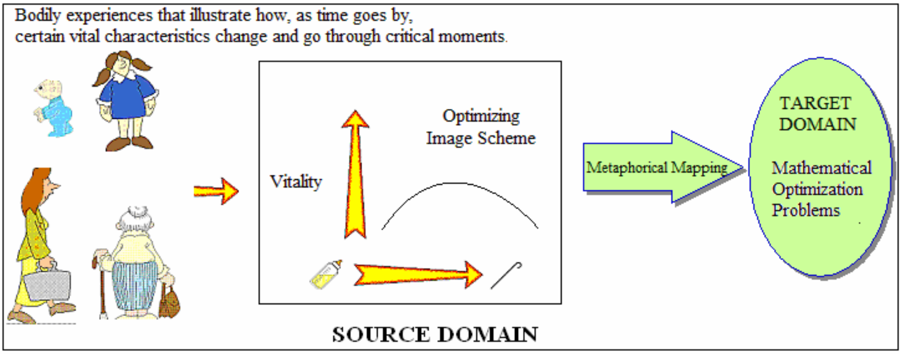

Figure 3. Metaphorical mapping of the optimizing scheme

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ENHANCEMENT OF STUDENTS' ARGUMENTATION THROUGH EXPOSURE TO OTHERS' APPROACHES

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In this paper, we discuss and illustrate the advantages of making available to students the work of their peers that yield a result in another form. Reflection on the structural differences inherent can give students a channel to strengthen the exposition that they originally gave

INTRODUCTION

An interest in Mathematics Education Literature is to encourage students to solve the same task in different ways. One obvious motive is to persuade students that there is no 'official' road to take in argumentation. Beyond this, the production of 'multiple solutions' (i.e. two or more solutions to the same task) may prompt students to reflect on their work, and ask why two (or more) approaches lead to the same result.

One special case arises when different legitimate approaches to a task yield different forms of the result. Such multiple answer tasks can be profitable in motivational terms; instead of just pondering why two approaches lead to the same statement, the students are engaged on the reconciliation of different guises of the result. Asking why one approach leads to one expression whereas a second leads to another might force students to gain structural insight.

In this paper, we are interested in particular in helping students that have reached a correct answer, but have some problems in writing down a 'satisfactory' presentation. The form of the help is to show each student the work of another whose result format is different. The student is asked to comment on whether the newly exposed material clarifies his/her original approach. The idea is that the structure evident in the approach used by the colleague may assist the student to advance his/her working. Then the student is guided by the researcher/teacher to obtain a particular mathematical construct that provides a common base for the student to express both approaches in 'acceptable' mathematical form. The construct then becomes a channel of reconciliation explaining the appearance of the differing results.

The explicit aims of this paper are to illustrate by an example

- How the exposition of two different forms of the result to the same task can lead to enhanced appreciation of the mathematical system supported by the task environment.
- How the environment of multiple answer tasks can prompt an improvement in quality of student proof production in one or more of the associated solutions.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 89-96. Thessaloniki, Greece: PME.

BACKGROUND MATERIAL

In the introduction, we refer to the term 'satisfactory' presentation. In the wide range of view concerning explanation, argumentation and proof found in the literature, it is apt to elaborate what we mean by it. First, we regard a presentation as a public artifact rather than a private one (Raman, 2003); its purpose is to communicate to the reader in an impersonal manner. From this prospective, convincing is not enough; full articulation is also required (Mamona-Downs & Downs, 2009b). Any mental argumentation occurring in the working has to be refined in order to create a proof, even though the presentation might well hide in its expression many facets of the thinking processes that were used to generate it (Haimo, 1995). A 'satisfactory' presentation is the laying out of an argument that is not only secure in an embodied sense, but also one that is free of suggestion; in particular, every implication made refers to explicitly defined objects and symbolism. This does not mean that formal demonstration is required (in fact the very notion of a strict proof is problematic, see e.g., Hanna & Jahnke, 1996; Thurston, 1995), but a degree of adherence to a mathematical language such as the one referred by Thurston and partially characterized in Downs & Mamona-Downs (2005) has to be maintained.

Quite often students have in their reach convincing argumentation, but they either do not realize that this does not qualify as a satisfactory presentation or are not able to affect the required transfer. We distinguish two categories. First, students can develop a mental argument, but are at loss to 'mathematisize' their thoughts. Second, students have found an appropriate mathematical framework in which to work in, but their personal thinking processes causes problems in translating them into the mathematical 'syntax'. The first category constitutes a behavior that has been noted often in the literature (e.g. Moore, 1994), the second is noted less. (The notions of procept and the proceptual divide due to Gray and Tall, 1994, are relevant to the second category, but the proceptual divide tends to be associated more to 'blind' symbolic operation rather than problems in regulating mathematical expression. The depiction of 'symbol sense' by Arcavi, 1994, perhaps sets a better balance.)

In this respect, we suggest a teaching device involving students comparing their own work with those of their peers. This general stratagem of course is commonly practiced in educational circles. However, we arrange that the material imported constitutes a different approach to that made by the receiving student. The student has to reconcile two solutions, making the task a multiple-solution connecting task, (see Leikin & Levav-Waynberg, 2007). Further if the two solutions lead to different results, this would be suggestive that the reconciliation has to combine two perspectives of the structure implicit in the task environment. Having available more than one way of 'seeing' a mathematical system is potent in both realizing and enhancing argumentation (Mason, 1989). By encouraging them to reflect on the differences in structure inherent in two separate approaches, students can be guided to craft a mathematical construct that acts as a common basis to expound both.

THE EXAMPLE

Method

The example was raised in project work taken by mathematics undergraduates attending a problem solving course, taught by the author. The course largely followed the ideas of Polya (1973) concerning strategy making and heuristics and of Schoenfeld (1985) concerning metacognition, especially executive control and accessing knowledge. Further, the conversion of mental argumentation into accepted mathematical frameworks was given some stress (Mamona-Downs & Downs, 2009a). The course was intended for the attending students to improve their own problem solving skills and mathematical literacy.

Half of the weight of the grading of the course was assigned to the students' contribution in project work. Each student was involved in one project. The students worked individually or in small groups of two or three. The time that the students had to complete their work was 3 weeks. The design of the projects were not particularly concerned about 'openness'. Instead, a sequence of tasks is given where the resolving of the later tasks likely requires either an indirect or direct reference to the solution of previous tasks. Usually, each project is assigned to two groups. After the project was handed back and appraised, I asked each group (as a body) for a semi-structured interview of 1-3/2 hours. If the working of the other group tackling the same project shows substantial differences to the work done by the group interviewed, this usually was made as a major point to be discussed in the interview. The example below is dealt in this context.

The task

The subject of the project at hand concerned the greatest power of one natural number that divides another given natural. Printed at the top of the assignment sheet was the following definition concerning symbolic convention.

"Given a natural number n, the symbol $2^r \| n$ means that the number r is the greatest whole number for which 2^r divides n".

This symbolism is commonly used in textbooks of number theory. Although its form can seem to be somehow convoluted, the students had past experience with it and showed no difficulty in interpreting it properly.

The particular task considered here is as follows:

Part(a): Let $n \in N$ (where N denotes the natural numbers). Find r_n satisfying

$$2^{r_n} \| 2^n!$$

Part (b): Let $m \in N$. Find s_m satisfying:

$$2^{s_m} \| m!$$

We shall discuss students' working on part (a) mostly; we shall refer to part (b) only in the context how the approaches appearing in part (a) extend naturally.

The two different forms of r_n raised.

The two groups of students assigned to this particular project will be denoted Group 1 and Group 2. In the project work, Group 1 attained the result $r_n = 2^n - 1$, whilst Group 2 gave their final answer as:

$$r_n = n + \sum_{i=1}^{n-1}\left(\frac{i.2^{n-i}}{2}\right)$$

The reasoning used.

For Group 1, the reasoning is presented below (translated into English from Greek):

"We know that from the numbers 1, 2, 3,…, 2^n, there are 2^{n-1} numbers which are divided by 2. We note that from the numbers 1, 2, 3,…, 2^{n-1}, there are 2^{n-2} numbers that are divided by 2. We note that from the numbers 1, 2, 3,…, 2^{n-2}, there are 2^{n-3} numbers that are divided by 2. Continuing to the end we have that $2^n! = 1.2.3\ldots2^n$ is divided by 2 raised to the power $2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1$. "

For Group 2, the approach used was basically following the lines of the argument below:

"Let S: $= \{1, 2, 3,\ldots, 2^n\}$. For i=0, 1, 2,…, n define $S_i = \{s \in S: 2^i \| s\}$. Then

$$r_n = \sum_{i=1}^{n} |S_i|.\, i \qquad (1)$$

To obtain $|S_i|$, consider $D_i := \{s \in S: s = k.2^i, k \in N\}$. Clearly, $|D_i| = 2^{n-i}$.

Now $S_i \subset D_i$ and $s \in S_i \cap D_i \Leftrightarrow s = k.2^i$ where k is odd. Therefore

$$|S_i| = \tfrac{1}{2}|D_i| = 2^{n-i-1} \quad when\ i \neq n \quad and \quad |S_i| = 1 \quad when\ i = n$$

Substituting into (1) we obtain:

$$r_n = n + \sum_{i=1}^{n-1} 2^{n-i-1}.\, i$$

However the above is somehow a 'cleaned up' version of the given in the script of Group 2. The main discrepancy is that Group 2 made their exposition in terms of equivalent classes, determined by the equivalent relation '~' on N defined by

$\alpha \sim \beta \Rightarrow \exists\, r \in N$ such that $(2^r \| \alpha) \wedge (2^r \| \beta)$.

Comments on the presentation of Group 1.

Basically the exposition put forward by Group 1 in their written response of part (a) simply states lists of numbers (from which the evens are calculated) without attempting to explain where these lists of numbers appear from. However, if a reader 'catches' the unexpressed central idea (on the mental level) motivating the production of the lists appearing in such a suggestive recursive manner, he or she would perceive the platform on which the approach is based on. Another reader might not 'catch on'. The reader has to interpret the written material, not to understand explicit argumentation. The students of Group 1 admitted in the interview that what they wrote down did not constitute a proof. They expressed difficulties to expand on their description of their 'method' in informal terms. Further, they considered that the attempts they had made in this respect were simply making the clarity of their exposition worse rather than better, so in the end they only wrote the rather minimal explanation that they state. The students clearly were frustrated in not being able to express their argument in a more complete manner. It should be noted that once they obtained the putative result 2n-1, they had the opportunity to prove their result by induction. By following induction, it had little importance how tentatively the induction hypothesis was obtained. The students of Group 1 did not undertake induction. This in a way was surprising because in general the participants of the course showed themselves adept in applying it. Likely, they had 'invested' a lot of effort to engineer their answer; the implementation of the technique of induction can have a psychological effect in that the original thinking would be left on the 'side-lines'.

Comments on the presentation of Group 2.

The use of the equivalence relation, as the students explained in the interview, helped them to organize their work. Their resorting to the equivalent relation rather than employing the sets S_i represents a particular way of analysis, where first the numbers 1, …, 2^n are taken separately as 'components' of $2^n!$, and then are classed according to having the same property or rank. The sets S_i are formed first by stipulating a property, and then determining which of the 'components' satisfies it. The difference in our case is only a cognitive one, which could be captured by contrasting the terms 'assigned to' and 'belonging'.

The written argument put forward by the students employing the equivalence relation for part (a) is essentially a valid proof. The symbolism associated to equivalent relations is more sophisticated to that of sets. The only shortcomings in their presentation appear in the form of symbol abuse; for example, there are cases of classes being related with elements, and some indexed symbols that are not explicitly defined. These flaws are minor, in the sense that the reader can easily read the intention beyond the logical liberties. However this is not true for part (b). Here, although the students were processing a sound mental argumentation and indeed obtained a legitimate result, their symbolic modeling of the argument brought in flaws and aspects of vagueness that ran right through their exposition. In this case, the reader only can surmise the students' intention, and this is done only with considerable effort. (Unfortunately space restrictions prevented us to reproduce the students' work here.)

The reconciliation

For the project work described above, both groups grasped a suitable argument to answer the task, but had difficulties in expressing their argument. One group suggests a recursive process but this is left implicit as to its basis. The other group firmly posits the argument on a mathematical framework, but the students have problems in controlling the associated symbolism. Both groups give appropriate results, but in differing manifestations. In the two interviews (one for each group), the interviewer (the author) was interested in exposing to each group the work of the other group. One motive in this was to see whether reviewing another approach would prompt the students to restyle their own for the sake of improved articulation. Due to space restrictions, no details or student protocols are given. Instead, we give a précis of what occurred in one interview, where the interviewees are the two students of Group 1.

In the interview of Group 1, the students realized that the equivalence classes used by Group 2 simply gather the elements of $\{1, 2, 3,\ldots,2^n\}$ that have a common greatest power of 2 as a divisor. After some time, they also understood the central core that supports the argumentation of Group 2; for each gathering, every member 'contributes' the same power of two, and so the total contribution is the number of members times the suitable power; the final result is gotten by the summation of the contributions from all the gatherings. They expressed that the presentation given by Group 2 in terms of equivalence relations seemed somewhat over-complicated, yet they did not propose an alternative mathematical framework in which the argument could be couched. (For instance, they did not explicitly form the system of sets S_i mentioned before). As far as comparing their own approach with the one given by Group 2, there was recognition by the students that their enumeration process cuts through rather than respects the gatherings. However they seemed at this point not to have the means to analyze this difference further.

At this point the interview came to an impasse. The interviewer decided to intervene. She suggested to the students to allocate a symbol M to the factorial of 2^n. M was the product of the natural numbers of 1 up to 2^n; the order in which these numbers are multiplied has no mathematical significance. The interviewer suggested that the students write down M in a resorted order that would reflect the approach of Group 2. After some discussion between the students and several aborted attempts they came up with the following:

$$M = \prod_{j\ odd} j \prod_{j/2\ odd} j \prod_{j/2^2\ odd} j \cdots \prod_{j/2^n\ odd} j$$

They appreciated in doing this that the j's under the separate product signs identified the gatherings implicit in the approach of Group 2 in 'blocks', and the result could be tackled by considering these blocks.

After, the interviewer reminded the students of their own presentation, and in particular asked them if they could explain what was signified by their first sentence

"We know that from the numbers 1, 2, 3,…, 2^n, there are 2^{n-1} numbers which are divided by 2" in terms of the expression above. In response, they wrote down:

$$M = 2^{n-1}\prod_{j\ odd} j \prod_{j/2\ odd} j/2 \prod_{j/2^2\ odd} j/2 \ \ldots \prod_{j/2^n\ odd} j/2$$

They made an action on the first expression that in essence comprised in dividing by two all the even numbers; to preserve the equality the suitable power of two has to be reinstalled before the product signs. Recursively, further similar actions are now obvious to perform:

$$M = (2^{n-1}+2^{n-2})\prod_{j\ odd} j \prod_{j/2\ odd} j/2 \prod_{j/2^2\ odd} j/2^2 \ \ldots \prod_{j/2^n\ odd} j/2^2$$

$$= \ldots$$

$$= \sum_{i=1}^{n} 2^{n-i} \left(\prod_{j\ odd} j \prod_{j/2\ odd} j/2 \prod_{j/2^2\ odd} j/2^2 \ \ldots \prod_{j/2^n\ odd} j/2^n\right)$$

What is left in the parenthesis is a multiplication of odd numbers, so in total is odd. Hence r_n is

$$\sum_{i=1}^{n} 2^{n-i}$$

In this way the students were able to convert their mental argumentation, up to now expressed only in a suggestive manner, within a concrete mathematical framework.

CONCLUSION

Multiple solution tasks are deemed by some mathematics educators to be a useful channel for encouraging students to make connections. Such connections may take many manifestations; for example, they may reveal various aspects of the interior structure of the task environment, or identify different areas of knowledge that can be resourced to obtain the diverse solutions of the task. In this paper, we examine a more practical question; can a student refine his/her argument through an examination of the work of peers that puts forward approaches at variance to the student's. We consider a special case of multiple solution tasks; those that have different solutions paths leading to different formulations of the result. The advantage in doing this is twofold. First, in terms of motivation, students are more likely to be interested how two approaches lead to differing 'results' rather than two that produce the same. Second, in terms of teacher guidance, the teacher can prompt a student to reflect on his/her work whilst discussing other students' output. In doing this, the sense of 'ownership' of the student for his/her own argument can be respected. In the paper, an illustration is given how a mathematical construct obtained by a group of students provided a framework allowing full articulation of the 'alternative' argument. The same construct was acted on by the students in such a way that their own line of thinking also became apparent; further, they could express their thinking far more concretely than before by referring to the construct. Thus the students were both improving the formulation of their argumentation and effecting reconciliation between two structural perspectives of the task environment.

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MATHEMATISATIONS WHILE NAVIGATING WITH A GEO-MATHEMATICAL MICROWORLD

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Educational Technology Lab / University of Athens

Twelve students of the 10th grade participated in a constructivist teaching experiment focusing on what kind of mathematical meanings students construct while using a navigational software called "Cruislet". It is a dynamic digital artefact designed to provide learners with the ability to be involved in mathematizing activities focusing on the use of vectors navigating in 3d large scale spaces. The analysis of the data reveals that students through a process of mathematization of geographical space construct meanings concerning the concepts of vectors, coordinates and functions.

Mathematics and Geography are very poorly connected in curricula all over the world. The mathematics of positioning, orientation and functional relationships represented as curves in space are tightly embedded in the context of Geographical spaces. In the quest to look for ways in which the generation of mathematical meaning for students may become richer in situations where they appreciate the utility of mathematical ideas (Ainley & Pratt, 2002), we designed a digital microworld embedding mathematical concepts in geographical space. In designing the microworld and the respective tasks we gave ourselves some distance from the traditional structure of the mathematics curriculum and looked for learnable mathematics concerning position and orientation in geographical three dimensional (3D) space, building on previous research on students' cartography (Yiannoutsou & Kynigos, 2004). In the design of the learning environment, we adopted the approach of students' gradual mathematization within game-like activities in problem situations that are experientially relevant to students (Gravenmeijer et al, 2000). Hence, our intention was to involve students in activities through which they would use symbols, make and verify hypotheses in order to solve a particular real problem in a rich learning environment. Quite some years ago, Hoyles et al (1989) identified four kinds of mathematical activity students engage in when working with microworlds, i.e. using mathematical ideas at the beginning without much clarity of their function and nature, then discriminating these amongst other concepts and features of the tool, generalising them beyond concrete cases and synthesizing between those embedded in the activity and those in other mathematical contexts. In our study we wanted to understand the nature of mathematizations students would lead to when engaged in activities such as experimenting, constructing classifications, making and verifying conjectures, generalisations and formalizations. The tools and tasks we gave them were designed as a half-baked (h-b) microworld (Kynigos, 2007), i.e. tools allowing them to navigate avatars in any way by making choices between

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 97-104. Thessaloniki, Greece: PME.

vectorial and Cartesian displacement controllers and to construct avatar trips by writing computer programs of sequential functional displacements. We called this 'the Cruislet environment' (Figure 3). H-b microworlds are designed to incorporate an interesting idea but at the same time to invite changes to their functionalities and are mediated to the targeted users as unfinished artefacts which need their input. In that sense, they invite constructionist activity (Kafai et al, 1996), they are *designed* for mathematizations through instrumentalization (Guin and Trouche, 1999). A digital medium becomes an instrument as it is internalised collaboratively by the students (Mariotti, 2002) while it is being changed often quite distinctly to what was designed by the researchers. The implication of this perspective is that students' expressions can gain mathematical legitimacy, even if they differ from and/or they are shaped and structured by the artefact in ways that lead them to diverge from curriculum mathematics. These kinds of constructionist environment provide dynamic visual means that support immediate visualization of multiple linked representations (Kaput, 1992). The key point here is that students can build their models into the medium that can act as a support for developing new meanings by investigating their hypothesis and argumentations.

THE CRUISLET ENVIRONMENT

The 'Cruislet' environment has been designed and developed within the ReMath project. It is a digital medium based on GIS (Geographic Information Systems) technology that incorporates a Logo programming language. It is designed for mathematically driven navigations in virtual 3D geographical spaces and is comprised of two interdependent representational systems for defining a displacement in 3D space, a spherical coordinate (vector) and a geographical coordinate system. The environment enables the user to explore spatial visualization and mathematical concepts by controlling and measuring the behaviours of avatars. The avatars can be airplanes and their displacement is represented by a vector. In this study we focus on what kind of mathematical meanings students construct while navigating in geographical space of Cruislet environment. Cruislet is primarily a navigational medium but it is also constructionist (Kafai & Resnick, 1996) since avatar trips can be constructed and visualised. It is designed to provide opportunities for learners to engage in expression of mathematical ideas through meaningful formalism (Kynigos and Psycharis, 2003) by means of programming and interdependent representations of Cartesian and Vector-differential geometrical systems.

METHODOLOGY

The research methodology is a constructivist teaching experiment along the same lines as described by Cobb, Yackel and Wood (1992).The researcher acts as a teacher interacting with the children aiming to investigate their thinking. The researcher, reflecting on these interactions, tries to interpret children's actions and finally forms models-assumptions concerning their conceptions. These assumptions are evaluated and consequently either verified or revised. Twenty four students of the 1st grade of upper high school, (aged 15-16 years old) participated in this experiment. Students worked in pairs in the PC lab. Each pair of students worked on the tasks using Cruislet software. To begin with, a set of tasks comprising simple avatar displacements were given including trips to specific places (cities) and back and trips described simply by distance covered. In the main tasks that were included in the teaching experiment students were encouraged to experiment with programs defining the relative displacements of two airplanes by varying the geographical coordinates of their new positions. Reflecting on their actions they encouraged to explore the rate of change of these positions and formulate the function that defines this dependent relationship. This function was hidden and the students had to guess it in the first phase of the activity based on repeated moves of aeroplane A and observations of the relative positions and moves of planes A and B. Initially, students were asked to study the relation between the two aeroplanes, the rate of change of their displacements and consequently find the linear function (decode the rule of the game). In order to decode "the rule of the game", they should give various values to coordinates (Lat, Long, Height) that define the position of the first plane. They were encouraged to communicate their observations about the position of the second plane to each other and form conjectures about the relationship between the positions of the two aeroplanes. In the second phase students were encouraged to build their own rules of the game by changing the function of the relative displacements of the two aeroplanes. The data consists of audio and screen recordings as well as students' activity sheets and notes. In our analysis we used students' verbal transcriptions as well as their interaction with the provided representations displayed on the computer screen.

VECTORS

While interacting with Cruislet environment, students defined the vector of displacement and through this activity they got involved with the notion of vector. As a result, several meanings emerged concerning vectors and their properties. In this session we present meanings regarding vectors in relation to geographical concepts.

Vectors' magnitude

Vectors' magnitude is represented by R in spherical coordinates, so it had to be defined when this system of reference was utilised. During their experimentation students realized that R was remaining constant for a displacement between two specific cities and additionally that was independent of the direction of the displacement. In the following episode students displace the airplane between two cities in their attempt to find their distance.

S1: This must be their distance. (Shows the vector created by airplane's displacement from Arta to Amfissa)

S2: Yes. But how can we find it?

S1: The R m. (Meaning R in spherical coordinates).

S2: No, it's not R m.Oh, you're right! Wait. (Displace the airplane from Amfissa to Arta and they watch R values in direction).

S1:You see? It's the same.

The interesting issue is that although they displaced the airplane towards one direction, they wanted to verify that the distance was remaining constant for the inverse displacement as well. If fact S1 used this as an evidence to persuade S2 that R represents the distance between the two cities. Our interpretation of S1's way of thinking is that perhaps he used his intuitions or pre-existed knowledge to apply a property of vectors' magnitude in this particular situation.

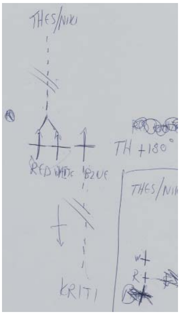
Addition of vectors

An interesting episode was that of a team that used intuitions to identify the resulting displacement if this is defined by multiple displacements. This was occurred while students were trying to construct the rules of a game for the other team. To be more specific, students' idea included the relative displacement of two airplanes, based on planes' coordinates. Here we focus only on the correlation of two planes' displacement (named red and blue by students), as they were moving relatively to theta angle and particularly their dependence can be represented as Thetablue = Theta white +180o. One of the preconditions of the game was also that the first (white) must go to a particular city (i.e. Thessaloniki) to end the first phase of the game. Initially students sketched their idea in order to explain it to the teacher, as shown in Figure 1. In the following excerpts, the students explain their drawing:

S2: As we go up, the other, the spy, will go down contrarily, towards Crete. […] Let's say, if we go 10 step upwards, he goes down 10 step downwards'.

S1: Blue is conversely commensurate. That is to say, we go 10 meters, he goes 10 meters above. When we get to Thessaloniki, he will get to Rethymno.

From their dialogue we can assume that they were thinking about multiple displacements, as specified by the length of each displacement (i.e. 10 meters). We see that S1 seems to think of the result of these displacements as he mentions the final destination of each airplane. The interesting thing is that he argues that when the first will be at a specific city, the other will be at a specific city as well, independently of the number of displacements, implying that he used his intuition to add the vectors of displacements and find the final destination of the 2nd plane. As the researcher was not sure if S1 used vectors' addition, she asked him to draw another figure and picture planes' position when the displacements would not be at the same line and asked him if the second airplane would be placed in the same city as in the first case. The student answered '*If we go to Thessaloniki, he'll be at Crete*' and draw the schema shown in figure 2. From his drawing we can see that although he hasn't added the vectors graphically he is thinking that the only thing that matters is the starting and the ending point.



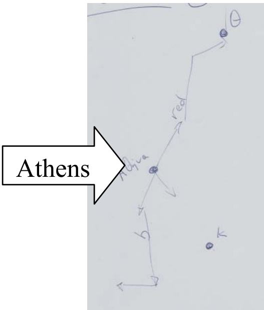


Figure 1: Drawings of airplane's displacements   Figure 2: Addition of vectors

So whatever the direction of vectors would be, the second plane would be placed in a specific city, taking into account that there is a dependent relationship between the two airplane. We find this episode interesting, due to the way students use their intuitions to express mathematical meanings without using vector's terms, that is to say without mathematical formalism.

COORDINATE SYSTEMS

Students didn't always choose one system of reference to navigate in space, but several times combined both to make a displacement. In this way they created links either between distributed coordinates (e.g. height of geographical and fi of spherical) or between all three of coordinates for the two systems of reference.

In their attempt to place the plane at a specific height, students used primiraly the height coordinate. However, there were some teams that were using spherical coordinates to carry out almost all displacements. Based on students actions on a team like that, students were trying to find a way to raise the airplane's height to a specific value, while utilizing the spherical coordinates. In fact one of them gave the idea to use the fi coordinate and raise the airplane by asking the other one: 'The height is fi?' and afterwards he edited the fi coordinate's value in order to raise the plane. This statement is interesting as the student endeavour to create meaning around the fi angle that represents airplane's perpendicular angle, in relation to the height that the plane will be placed.

Another episode where students create a link between coordinates is that of longitude and theta coordinates. In the following episode the students of a team argue about the system of reference that displace the airplane 'right – left'.

S2: It goes right and left. (referring to longitude)

E: Right and left.

S2: Yes.

S1: No. Theta is right and left.

S2: These are the degrees.

S1: Yes, the degrees it turns to the left or right.

S2: I'm saying to displace at the same time.

This episode is interesting as it depicts the way students verbally express the way they realize the displacement while using longitude or theta angle of spherical coordinates. In both cases they use the expression 'right – left' giving the displacement a sense of direction. However, S2 supports that longitude doesn't have to do only with turning like theta, but with displacing as well. The way he externalizes his thought demonstrates that he is aware of the interdependent relationship between longitude and theta.

FUNCTION AS COVARIATION

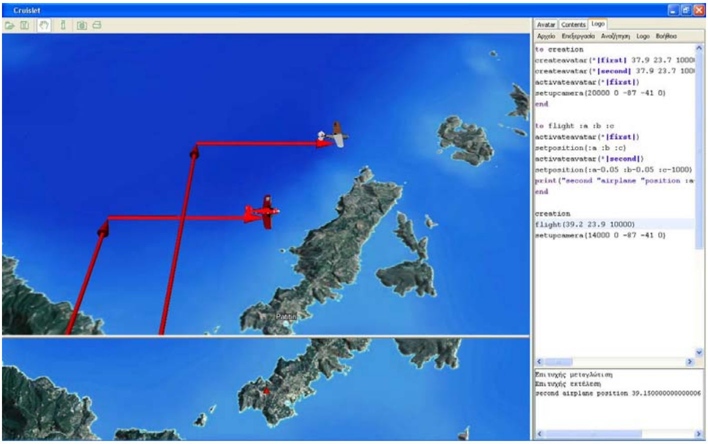


Figure 3: The Cruislet environment, (Logo procedure)

Students engaged with the notion of function, through their experimentation with the dependent relationship between two airplanes' positions, which was defined by a black – box Logo procedure (Figure 3). In their attempt to find out the hidden function, they were able to coordinate changes in the direction and the amount of change of the dependent variable in tandem with an imagined change of the independent variable. Our results indicate that students developed covariational reasoning abilities, resulting in viewing the function as covariation. Initially most of the students expressed the covariation of the airplanes' positions using verbal descriptions, such as behind, front, left, etc. as they were visualizing the result of the airplanes' displacements. In the following episode students express the dependent relationship while looking at the result displayed on the screen.

S1: So, he always wants to be close to us on our left.

R: Yes.

S1: And he is beneath, further down to us. Beneath.

S2: And behind.

Students experimented by giving several values to geographical coordinates in Logo and formed conjectures about the correlation between airplanes' positions. Through their interaction with the available representations, they successfully found the dependent relation of the function in each coordinate, resulting in their coming into contact with the concept of function as a local dependency.

As they were thinking the height coordinates had a proportional relationship, they suggested to carry out a division to find it.

S2: When we go up 1000, he goes up 1000.

R: Do you mean that if we go from 7000 to 8000 he goes from… let's say 2500 to 3500.

S2: He is at… 3000. No. Give me a moment. At 8000 he was at 5500. At 7000 he was at 4500. At 5000 he is as 2500. And then….

S1: We could do the division to see the rate.

It is interesting to mention that students separated latitude and longitude coordinates on the one hand and that of height on the other as they were trying to decode the hidden functional relationship between the airplanes' height coordinates. In particular, they didn't encounter difficulties in decoding latitude and longitude relationship in contrast to their attempts to find the height dependency. Although all three functions regarding coordinates were linear, students conceived the functional relationship between height mainly as proportional, in contrast to latitude and longitude that were comprehended as linear, from the beginning. In the following episode, students endeavor to apply the rate of change of the function to decode the height relationship.

CONCLUSIONS

The study revealed particular ways in which these students constructed mathematical meanings related to vectors, coordinates and functional relationships through a process of incorporating references to geographical places in an ad-hoc way. What was interesting to us was the way in which what initially seemed to the students to be a game – like activity with airplanes on 3D terrains gradually incorporated mathematizations which were perceived as functional tools to play the game or to solve a task. The students were interested in the game of constructing 'guess – my – flight' puzzles for other students by inserting hidden functional relationships between two airplanes' displacements. From a theoretical perspective we saw a helpful relevance in studying mathematizations in a constructionist environment as path towards clarifying the idea of *instrumentalization by design*. Guin and Trouche and other theorists have illuminated the idea that humans do not only generate instruments by leaving the artefacts untouched but change the functionalities of these artefacts as well in the process of instrumetalization. It seems worthwhile to us to try to find ways to further analyze the idea of intrumentalization in a pedagogical context, i.e. ways in which it is connected to the generation of meanings is specially designed learning environments.

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EXPLORING THE MATHEMATICAL MACHINES FOR GEOMETRICAL TRANSFORMATIONS: A COGNITIVE ANALYSIS

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In this article we present some results of a cognitive study on Mathematical Machines taken from the history of mathematics. In particular, we will propose a classification of the utilization schemes of pantographs for geometrical transformations. This classification, useful to describe the interaction between a subject and the machine, can be an efficient tool to analyse the exploration processes involved in the identification of the transformation and of the mathematical law incorporated in the machine.

INTRODUCTION

The study presented in this report is part of a wide research project concerning the teaching and learning of mathematics by means of instruments♣. The instruments involved in our study are working reconstruction of historical mathematical instruments, called Mathematical Machines. These machines belong to the collection of the Mathematical Machines Laboratory (MMLab: www.mmlab.unimore.it), a research centre at the Department of Mathematics in Modena (Italy). In the last twenty years, the MMLab researches has been mainly focused on the epistemological and pedagogical aspects involved in the activities with the Mathematical Machines (Bartolini Bussi & Pergola, 1996; Bartolini Bussi, 2000; Bartolini Bussi, 2005; Bartolini Bussi, Mariotti, & Ferri, 2005, Bartolini Bussi & Maschietto, 2008; Maschietto & Martignone, 2008).

In order to implement MMLab researches we carry out a cognitive analysis of Mathematical Machines exploration processes. In particular, we analyse pantographs for geometrical transformation, which establish a local correspondence between points of limited plane regions (connecting them physically by an articulated system) and incorporate some mathematical properties in such a way that allow the implementation of a geometrical transformation (i.e. axial symmetry, central symmetry, translation, homothety, rotation).

♣The study is realized within the project PRIN 2007B2M4EK ("Instruments and representations in the teaching and learning of mathematics: theory and practice") jointly funded by the Italian MIUR, the University of Modena and Reggio Emilia and the University of Siena.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 105-112. Thessaloniki, Greece: PME.

These pantographs are based on physical as well as on mathematical principles. For this reason the first step of our research has been the study of the interaction between subject and machine (Martignone & Antonini, to appear). In this paper, we present the second step of the research that concerns the analysis of the relationships between the manipulation and the exploration processes involved in the identification of the mathematical law incorporated in the machine.

THEORETICAL FRAMEWORK

A suitable tool to analyse the processes through which a subject interacts with a machine can be found within cognitive ergonomics, in particular in Rabardel's studies (Rabardel, 1995; Béguin & Rabardel, 2000). Rabardel conceived the instruments as psychological and social realities and studied the instrument-mediated activity: according to him an *instrument* (to be distinguished from the material -or symbolic- object, the *artefact*) is defined as a hybrid entity made up of both artefact-type components and schematic components that are called *utilization schemes*.

> "What we propose to call "utilization scheme" (Rabardel, 1995) is an active structure into which past experiences are incorporated and organized, in such a way that it becomes a reference for interpreting new data" (Béguin & Rabardel, 2000)

In the following section we describe how the utilization schemes emerge in the case of pantographs for geometrical transformations.

The pantographs utilization schemes

In literature there are not previous cognitive studies of this type on Mathematical Machines activities (a classification of utilization schemes of instruments of different nature is proposed in Arzarello et al., 2002). For this reason, the first step of our research has been the identification of Mathematical Machine utilization schemes: in particular we have proposed a first classification of pantograph for geometrical transformations utilization schemes (Martignone & Antonini, to appear).

The utilization schemes observed during pantographs exploration has been classified into two large families:

- the utilization schemes linked to the components of the articulated system: namely, the research of fixed points, movable points (with different degrees of freedom), plotter points and straight path, the measure of rods length, the research of geometric figures representing the articulated system or some part of it, the construction of geometric figures that extend the articulated system components, the research of relationships between the recognized geometric figures and the analysis of the machine drawings;
- the utilization schemes linked to the machine movements.

In the last quoted family we identified two main sub-families of utilization schemes (labelled: M-1 and M-2) summarized and described in the following tables (Table 1; Table 2).

M-1: The utilization schemes aimed at finding particular linkage configurations obtained stopping the action in specific moments.

Linkage Movement that stops in	Movements description
Generic Configurations	Movement that stops in a configuration which is considered representative of all configurations observed (that does not have "too special" features)
Particular Configurations	Movement that stops in a configuration that presents special features (i.e. right angles, rods positions…)
Limit Configurations	Movement that stops in configurations in which the geometric figures that represent the articulated system degenerate
Limit zones	Movement that stops in the machine limit zones (i.e. the reachable plane points)

Table 1: Utilization schemes linked to the machine movements, the sub-family M-1

M-2: The utilization schemes aimed at analysing invariants or changes during continuous movements:

Linkage Continuous movements	Movements description:
Wandering movement	Moving the articulated system randomly, without following a particular trajectory
Bounded movement	Moving the articulated system, blocking particular points or rods
Guided movement	Moving the articulated system, forcing a point to follow a line or a specific figure
Movement of a particular configuration	Moving the articulated system, maintaining a particular configuration
Movement between limit configurations	Moving the articulated system so that it can successively assume the different "limit configurations"
Movement of dependence	Moving (in a free, guided or bound way) a particular point and see what another particular point does
Movement in the action zones	Moving the articulated system so that all the possible parts of the plane are reached

Table 2: Utilization schemes linked to the machine movements, the sub-family M-2

In Martignone and Antonini (to appear), we have shown that this classification can be efficiently used to describe the interactions between subjects and Mathematical Machines.

In this paper, we shall see that the utilization schemes classification can be an interpretative tool for analysing exploration processes in activities with pantographs for geometrical transformations, where the term "exploration" refers to the process that leads to the formulation of a conjecture.

METHODOLOGY

The goal of our study has been to identify the machines utilization schemes and to investigate how these schemes are linked to the exploration processes. Because the focus was on the processes involved in the interaction between the subjects and the machines, in these first steps of the research we wanted to limit, as much as possible, the subjects' difficulties derived from the application of geometrical theory. For this reason, we selected subjects that were familiar with (Euclidean) geometry and with problem-solving but that did not have an a priori specific knowledge about these machines: the subjects are mathematicians (three pre-service teachers, two university students and one young researcher in mathematics) who were new in working with Mathematical Machines. This choice allowed us to collect observations about the generation of conjecture on the mathematical law implemented by the machine and, subsequently, argumentation and proof of mathematical statements that can explain the machine working.

The task given to the subject (who knows that the machine is an instrument that makes a geometric transformation) was the following one:

- To identify the mathematical law locally realized by the articulated system.
- In particular, to justify how the machine "forces a point to follow a trajectory or to be transformed according to a given law" and then to prove the existing relationship between the machine properties (structure, working…) and the mathematical law implemented.

The method used for investigation was the clinical interview. Subjects were asked to explore a machine and to think aloud during the exploration process. These interviews were videotaped and the analysis is based both on the transcripts of the interviews and on the manipulative activities.

A PROTOCOL ANALYSIS

In this paragraph we will show the opening phase of Scheiner's pantograph exploration. The pantograph of Scheiner (Fig.1-2) is a machine made by four bars pivoted so that they form a parallelogram (APCB) and a point pivoted on the plane (O). The points P, Q and O are in the same line and P and Q are corresponding in the homothetic transformation of centre O and ratio BO/AO.

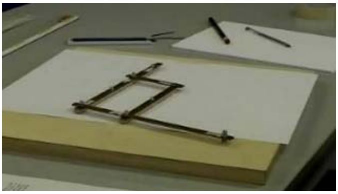

Fig.1: A photograph of Scheiner's pantograph

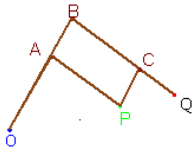

Fig.2: Scheiner's pantograph virtual image

The analysis focuses on the alternation between different utilization schemes associated to the exploration of static and dynamic machine features: i.e. the characteristics of the articulated system physical components and the machine movements, respectively. In particular we present two levels of analysis: a first one after each transcript sections (where we show the different utilization schemes) and a second one at the end of the protocol excerpt (where we analyse how these schemes developed, intertwined and grounded the exploration strategies).

First analysis level

The subject (a Math researcher that we will call Carlo), as requested by the given task, tries to understand how the machine works and what it does.

1 C: Then...(*he is looking at the machine*) the leads ... ah, those two points here (*he touches them: Fig. 3*)

After having read the task, he starts the exploration analysing the articulated system components: in particular he locates the plotter points (**research of plotter points**).

2 C: I'm looking how it is done (*he begins to move the linkage*) as far as I know, it could also be pivoted in other points ... (*he moves the linkage*)…therefore it is pivoted only here (*he touches the fixed point: Fig. 4*)

Carlo begins to move the articulated system (**wandering movement**) focusing on the possible degree of freedom of the linkage points; therefore he identifies what points are pivoted (**research of fixed points**).

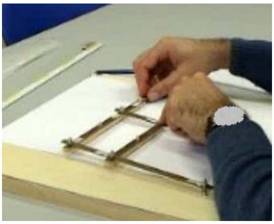

Fig.3: Carlo touches the plotter points

Fig.4: Carlo touches the fixed point

3 C: ... well, I can... I'm trying to see the movement I can do (*he moves the linkage by opening and closing the rods*) well, seeing the movements that I can do (*he is moving when he stops in a generic configuration*) …it is coming to my mind the angle bisector, but…no the bar should be pivoted in another point…in the middle…[…]

He is moving the articulated system in order to understand how should be the possible movements (**movements between limit configurations**) when he stops in a **generic configuration** because this reminds to him the geometrical construction of the angle bisector; this idea is just left because the linkage configuration does not fit the proprieties of that construction.

4 C: OK, then… I see that if here, for example, I do a circle (*without the lead, he moves the central plotter describing a circle*) it comes out a circle also on the other side ... and if I make a straight line here…it is very difficult to make a straight line, I have also a straight line on the other side (*he moves the linkage in silence always describing the line*) ... now what ... (*continues to move the linkage*) ... well ... (*he continues to move the linkage following different straight lines with different inclinations*) ... if I make a straight line here ... ...there are fixed straight lines (*he describes a straight line that seems to pass through the fixed point*)...

The machine configurations explored through the continuous movement do not seem to help him to understand the math law incorporated in the machine. For this reason he changes strategy and he tries to see what the plotter points do if they are forced to move along specific paths (**guided movements**). In particular he moves one of the plotters and observes how the other point behaves (he does not use the leads and so he can analyse only the movements). It is interesting to notice what kind of trajectory he chooses: circular and straight. These paths are not chosen randomly, in fact Carlo knows that these lines have different properties which could highlight the characteristics of the transformation implemented by the machine: specifically the machine seems to perform similar movements: he says that circles are transformed into circles and straight lines in straight lines (**analysis of "virtual" drawings**) but, it seems that he does not notice the changing of size.

5 C: (*he looks at the motionless machine in silence*) …no, it is not that …the fact that this (*he points to the small triangle PQC*) is smaller in this way… it comes to mind ... an homothety … could it be an homothety? A homothety … it comes to mind also because these three points are aligned, are aligned? [The protocol continued with the construction of an argumentation and a proof that the transformation is a homothety.]

Carlo leaves the previous exploration strategies linked to the production and analysis of "virtual" figures and he returns to the linkage structure analysis. In these few lines we can observe a concatenation of different utilization schemes (i.e. the **research of geometric figures in the linkage structure** combined with the **construction of geometric figures** and the following **individuation of relationships between the recognized geometric figures that extend the articulated system components**).

Second analysis level

The protocol analysis, carried out in the previous paragraph, can be considered the first step in the study of the machine exploratory process: in fact after the individuation of the utilization schemes, we can analyse the sequence of these schemes and their relationship during the exploration phase.

Carlo's protocol is interesting because shows an interlacement of different utilization schemes families. In fact, in the first phrases (1-3) Carlo, trying to discovered some of the machine key features, alternates **utilization schemes related to the exploration of the machine components** (research of plotter points and fixed points) and **utilization schemes linked to the movements** (wandering movement, movements between limit configurations that stop in a generic configuration). This interlacement continued also in the following part (phrases: 4-5) when he tries to determine which transformation the machine performs. At first, Carlo focuses on the **plotter points movements**, but, not using the leads, he does not see immediately that the machine implements an homothety: he is only aware of the fact that straight lines are transformed into straight lines and circles into circles (guided movements and analysis of "virtual" drawings). In order to recognize for certain the transformation, he needs to add to this information about the movements other data coming from the properties of the machine structure. Therefore he turns to **schemes that explore the machine structure** (research of geometric figures in the linkage structure, construction of geometric figures, individuation of relationships between the recognized geometric figures that extend the articulated system components).

The sequence of this utilization schemes collapses in the production of the conjecture about the transformation made by the machine. The conjecture is reinforced by the fact that the plotter points and the fixed point are aligned.

CONCLUDING REMARKS

This research report presents some results of a cognitive study on Mathematical Machines. This study has been developed through progressive construction of interpretative tools that allow the analysis of the processes involved in the exploration of these instruments. The first step of the research has been the identification and the consequent classification of utilization schemes of a particular family of Mathematical Machines, the pantograph for geometrical transformations (Table 1 and Table 2). In this paper we have shown that this classification can be efficiently used to observe, describe and analyse cognitive processes involved in the exploration of mathematical properties incorporated in the machines.

Further researches are needed in different directions. In particular the studies in progress concern the argumentative processes and their relationship with the utilization schemes and cultural factors during activities with the Mathematical Machines. These studies are founded on results that come from previous investigations (collected in Boero, 2007), carried out in different environments, concerning the study of the dynamic explorations and the transition between the conjecturing phase and the proof construction.

We think that the study on these processes will offer teachers tools that could be efficient to set up educational activities with the Mathematical Machines that can be productive for learning some geometrical topic but also for developing students' explorative and argumentative aptitudes.

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MODELING AND PROOF IN HIGH SCHOOL

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In this paper we propose a refinement of Chevallard's mathematical modeling process while still fully agreeing with his position that modeling is a key process in knowing mathematics. In doing this, based on our empirical study, we claim, 1) that there are at least two other stages in the mathematical modeling process; 2) that mathematical modeling is a non-linear process; and 3) there exists non-linearity at the interior of each modeling stage. We will illustrate each of these claims with episodes from our classroom intervention; one of the goals was to provide high school students the opportunity to use algebra as a modeling tool to prove.

INTRODUCTION

We fully agree with placing modeling at the center of knowing mathematics, following Chevallard (1985, 1989). While adopting this position, the goal of our paper is to work towards a refinement of Chevallard's mathematical modeling process. Chevallard (1989) proposed three stages in this process: (1) "*Identification of Variables and Parameters*;" (2) "*Establishing Relationships Among Variables and Parameters*;" and (3) "*Working the Model to Establish New Relationships*." While Chevallard provides the framework for our work, we will put forth three claims regarding a revised process: (1) there are at least two other stages in the mathematical modeling process: "*Interpretation of the Problem*" and "*Production of Competing Hypotheses*"; (2) mathematical modeling is a non-linear process; (3) the existence of non-linearity at the interior of each stage is evidenced by the presence of partial models. We will provide evidence for our proposed revision through an in-depth analysis of a triad of $9^{th}/10^{th}$ grade students (14-15 years of age approximately) who worked with the first author of this paper on a didactical sequence (the "Calendar Sequence") focused on students' use of algebra as a modeling tool to prove during fifteen one-hour-long lessons (see Martinez, 2008, in preparation; Martinez and Brizuela, under review).

DEFINITIONS OF MATHEMATICAL MODELING

Chevallard's definition of mathematical modeling

Chevallard (1989) described the three stages of the modeling process in the following way:

> (1) We define the system that we want to study by identifying the *pertinent* aspects in relation to the study of the system that we want to carry out, in other words, the set of *variables* through which we decide to cut off from reality the domain to be studied … (2) Now we build a model by establishing a certain number of relations R, R', R'', etc., among the variables chosen in the first stage, the model of the systems to study is the *set*

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 113-120. Thessaloniki, Greece: PME.

> *of these relations*. (3) We 'work' the model obtained through stages 1-2, with the goal of producing *knowledge* of the studied system, knowledge that manifests itself by new relations among the variables of the system. (p. 53. Emphasis in original)

In Chevallard's (1985, 1989) point of view, mathematical modeling plays a key role in knowing mathematics. His theoretical perspective has been valuable to conceptualize the stages underlying mathematical modeling, as described above. Another important aspect is his consideration of both extra-mathematical and intra-mathematical contexts resulting in a broadening of mathematical modeling for problem situations both *inside mathematics* (intra-mathematical context) as well and in contrast to *real world contexts* (extra-mathematical context). In addition, of special interest to us is Chevallard's understanding of algebra as a modeling tool.

Additional perspectives on mathematical modeling

Lesh and colleagues have largely studied modeling with an emphasis on "real world" contexts (e.g., Lesh, et al., 2003; Lesh & Doerr, 2003; Lesh, Lester, & Hjalmarson, 2003; Lesh & Zawojewski, 2007). Lesh and his colleagues understand the development of models as part of problem solving. Students begin their learning experience by developing conceptual systems (i.e., models) for making sense of real-life situations where it is necessary to create, revise, or adapt a mathematical way of thinking (i.e., a mathematical model; Lesh & Zawojewski, 2007). Given model-eliciting activities, students are expected to bring their own personal meaning to bear on a problem, and to test and revise their interpretation over a series of modeling cycles. Lesh describes the modeling process as consisting of four interacting processes that do not occur in any fixed order: 1) *description*: establishing a map from the model world to the real or imagined world; 2) *manipulation* of the model to generate predictions or actions related to the problem solving situation; 3) *prediction* (or translation), that involves carrying back results into the real or imagined world; and 4) *verification* of the usefulness of actions or predictions.

Hanna and Jahnke (2007) also investigate modeling within extra-mathematical contexts. They use arguments from physics as a method to build an explanatory proof. According to Hanna and Jahnke (2007), "modeling often has to do with creating a non-physical representation of a physical system" (p. 147) and relate their approach to "reality related proofs." However, of special importance for our study is that they view modeling and proof as being inextricably linked and as having complementary roles. Additionally, they relate their view of modeling to that held by applied scientists, as, "a circular or spiral process of setting up a model, drawing conclusions, modifying the model, drawing conclusions, and so on" (p. 150). We borrow from Lesh and his colleagues their view of modeling as consisting of interacting stages that do not occur in a particular order, and from Hanna and Jahnke their view regarding the connections between modeling and proof, and their view of modeling as being a circular or spiral process.

On the other hand, other researchers (e.g., Bolea, Bosch, & Gascón, 1999; Chevallard, 1985, 1989; Gascón, 1993-1994), working more closely to Chevallard's theoretical perspective, place a central focus on the study of modeling in and of itself as well as on the meaning and implications of conceiving algebra as a modeling tool in the school curriculum. Within this group of authors, Chevallard, Bosch, and Gascón (1997) claim that an essential characteristic of a mathematical activity consists of building a (mathematical) model about systems (within intra-mathematical or extra-mathematical contexts) to be studied, to use it, and to produce an interpretation of the obtained results. In others words, the mathematical activity can be characterized as making (mathematical) models of systems in (intra or extra-mathematical) contexts. The authors underline three aspects involved in building a mathematical model: the routine utilization of pre-existing mathematical models, the learning of models as well as the way of using them, and the creation of mathematical knowledge. From this group of colleagues we borrow their vision of the importance of modeling not just extra-mathematical contexts but also intra-mathematical contexts, as well as their vision regarding algebra as a modeling tool central to the mathematics curriculum.

METHODOLOGY

Participants

In this paper, we will describe and analyze the work of three $9^{th}/10^{th}$ grade students (Abbie, Desiree, and Grace) who participated in a teaching intervention, led by the first author of this paper, in which a total of nine $9^{th}/10^{th}$ graders took part, at a public charter school in the Boston area, Massachusetts, in the USA.

Procedure

Lessons: Fifteen one-hour lessons were held once a week. These lessons were part of the regular school schedule but not part of their regular mathematics classes. In this paper we will report on data collected during Lessons 1 and 2, in which students focused on Problem 1, which was implemented during the first half of the intervention during which students worked with variables, algebraic expressions, and equivalent algebraic expressions.

Problem 1

Part 1: Consider a square of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be

1	2
8	9

These squares will be called 2x2 calendar squares. Calculate the difference between the products of the numbers in the extremes of the diagonals. Find the 2x2 calendar square that gives the biggest outcome. You may use any month of any year that you want.

Part 2: Show and explain why the outcome is going to be -7 always.

Figure 1. Problem 1 from the Calendar Sequence.

In Problem 1 (Figure 1), students were asked to analyze the nature of the outcome of the described calculation (subtraction of the cross product). It was expected that students would anticipate some kind of variation in the outcome in relation to the set of days where the operator is applied and that students would find out, through exploration, that the same outcome is always obtained, no matter where they apply the operator. The challenge for the students was to find out why this happens, and whether this is “always” going to be the case. At this stage in the problem, from a mathematical point of view, algebra becomes a tool to solve the problem. Thus, one of the challenges in Problem 1 is to show the limitations of using a non-exhaustive finite set of examples to prove that proposition is true, and to encourage students to use algebra as *a tool* that allows them to express all cases using a unique expression.

FINDINGS

Claim 1: There are at least two other stages in the mathematical modeling process.

Interpretation of the Problem: Given that our interest in mathematical modeling, specifically the use of algebra as a modeling tool to prove, is educational, it is crucial to include a stage that accounts for students’ processes when they are mainly focused on understanding the statement of the problem, which does not exclude questions regarding the statement of the problem from re-appearing once students start to “solve” the problem. In Abbie, Desiree, and Grace’s group, the first five minutes of small group work were characterized by trying to understand the meaning of the statement of the problem. Among the issues that surfaced during this interpretation stage were: potential strategies to solve the problem; reaching an agreement regarding the operations involved; the nature of the numbers considered (negative, positive, absolute value); and the delimitation of the domain to study. After these initial minutes, the students reached an agreement regarding these issues and their main focus shifted towards the production of hypotheses regarding the outcome.

Production of Competing Hypotheses: This stage is characterized by students’ production of competing hypotheses as part of their investigation of the nature of the outcome. Students pondered how to obtain the largest number and how to characterize the nature of the dependence between the outcome and different aspects of the problem situations such as a square’s position within the month, across months and years, and what day of the week is the first day of the month. In Figure 2, we show the process by which our three students produced a variety of competing hypotheses and later agreed on a final hypothesis. Students employed a significant amount of time trying to understand the behaviour of the outcome and its dependence/independence to different elements of the context. In the example given, students constructed a variety of competing hypotheses (H1, H2, H3, and H4 in Figure 2) until they were relatively convinced about the truth-value of their final hypothesis (H5). *What* has to be proved is not obtained through a straightforward process, but rather through students' analysis of the likelihood of the truth-value of each of a succession of competing hypotheses. H1 linked the numeric outcome to the square's position at the beginning of the month, while H2 is a claim about the outcome's independence of the month that it was placed in. After abandoning H1 and H2, H3 was momentarily embraced: possibility of the outcome's dependence on the month where it is placed, which is in contradiction with H2. H4 seems to be a refinement of H3, given that it relates the potential variation of the outcome with what day of the week is the first day of the month. Students identify as the potential source for the variation in the outcome not only in what month the square is placed but also what day of the week is the first day in that month. The students hypothesize that a potential source of variation in the outcome is the relative arrangement of the days. To test their hypothesis, they tried months that differ in what day of the week is the first day of the month, always resulting in the same outcome, -7. Students seemed convinced given that they tried all possible unfavourable scenarios and they still got the same outcome, therefore they produced their concluding hypothesis (H5) stating the outcome is always -7 regardless of different arrangements of numbers, months, and first days of the month. We distinguish between two kinds of hypotheses: *exploratory* and *concluding*, the latter being the hypothesis to be proved. They differ in terms of the purpose and degree of certainty regarding their truth-value. An example of an *exploratory* hypothesis is provided by Desiree when she posed the following question to her group: "What happens if you start like at the beginning?" referring to the impact on the numeric outcome when placing the square at the beginning of the month. This is different from their concluding hypothesis as stated by Abbie: "They are always going to be the same." As discussed earlier, this concluding hypothesis was produced as the result of exploring various hypotheses, analyzing them using examples and properties of the relations involved, and getting a sense of the degree of certainty about its truth value.

Claim 2: Mathematical modeling is a non-linear process.

We claim that the mathematical modeling process is non-linear: stages can occur in *non-consecutive order*, and therefore does not necessarily follow the path as described by Chevallard (1985, 1989). As illustrated in Table 1, students already started analyzing some relations among variables (i.e., "*Establishing Relationships Among Variables*" stage) in the *Interpretation of the Problem* stage. In our analysis, it is possible to identify a main focus to students' work during a certain period of time. For instance, during the first five minutes, as the students were trying to understand the statement of the problem, elements from other modeling stages appeared. During "*Interpretation of the Problem*," Abbie noticed the relationships between elements in the corner of the diagonal of the square (i.e., "*Establishing Relationships Among Variables*"). At that moment, her group did not elaborate on her comment until later when they were mainly producing hypotheses; more specifically, until the moment of writing the expression to prove their concluding hypothesis, illustrating that stages do not necessarily happen in a fixed order.

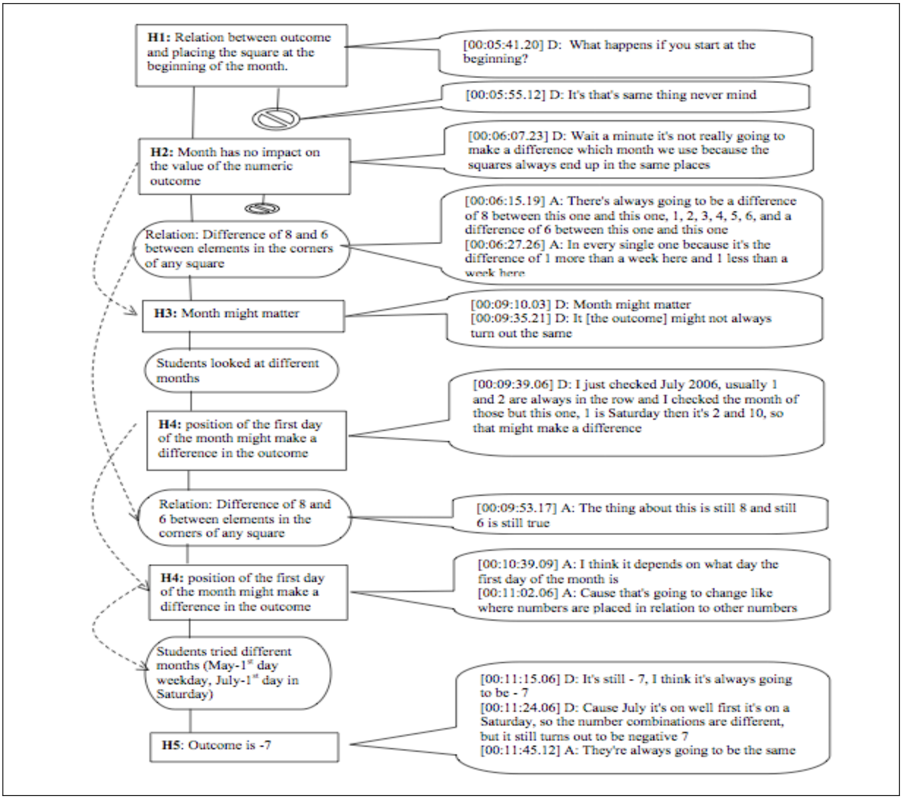

Figure 2. Proposed Production of Competing Hypotheses Stage. Hypothesis are represented on the left in order of appearance. On the right, excerpts of students' discussion along with the time code and the student initial.

Claim 3: Partial models within stages of the mathematical modeling process.

Even though Chevallard's mathematical modeling process has proved useful, it does not account for the complexity involved at the interior of each of the modeling stages. Within Chevallard's stages of *Identification of Variables and Parameters* (i.e., first stage) and *Establishing Relationships Among Variables* and Parameters (i.e., third stage), we have found that students produce *partial models* of the situation under consideration. A *partial model* is a model that includes some (but not all) variables, parameters, or relations among them. Thus, a *complete model* is what we would traditionally call model, and it includes *all* variables, parameters, and relations that are relevant to study the situation under consideration. For instance, building on the relations identified when they first encountered the problem, Abbie proposed to write with “algebra” the following: “All right so this is what I have 'x' times 'x' minus 8,

Transcript	Stage	Significance
[00:05:41.20] and [00:06:07.23] Production of H1 and H2.	Production of Competing Hypotheses	New proposed stage.
[00:06:15.19] A: There's always going to be a difference of 8 between this one and this one, 1, 2, 3, 4, 5, 6, and a difference of 6 between this one and this one.	Establishing Relationships Among Variables	Chevallard proposed this as the final stage in the modeling process.
[00:09:10.03] Production of H3.	Production of Competing Hypotheses	New proposed stage.
[00:09:53.17] A: The thing about this is still 8 and still 6 is still true.	Establishing Relationships Among Variables	Chevallard proposed this as the final stage in the modeling process.
[00:09:39.06] Production of H4.	Production of Competing Hypotheses	New proposed stage.

Table 1. Illustration of non-linearity of the modeling process.

minus 'y' times 'y' minus 6 that's basically what we're doing, yes.” At this moment they had produced a partial model of the situation since they were not aware of the relation between *x* and *y*. After producing that expression, they continued to check whether it was correct or incorrect; in order to do that, they were going to replace the letters with numbers. In the process of doing so, Abbie realized that y and x are in fact related: “'y' and 'x' aren't like random numbers … so we can say everything in terms of this number right here.” This is when a complete mathematical model was produced, including one independent variable *x* and three dependent variables *x-1*, *x-8*, and *(x-1)-6* with explicit relationships among the different variables. In this case, students’ goal was to write down the expression representing the model that they thought captured all necessary information, in order to work on it with the ultimate goal of proving their concluding hypothesis. However, they came to realize that there were two relationships that they had overlooked. Therefore they “went back” to re-write these “unnoticed” relationships.

CONCLUDING REMARKS

In this paper, we have proposed a refinement of Chevallard’s mathematical modeling process. We proposed the inclusion of at least two others stages namely: “Interpretation of the Problem” and “Production of Competing Hypotheses”. In addition, we provided evidence illustrating the non-linearity (stages do not happen in a fixed order) of the mathematical modeling process. Lastly, we showed students' partial models as an indication of the complexity at the interior of the stages in the modeling process.

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STUDYING TEACHERS' PEDAGOGICAL ARGUMENTATION

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We present a case study analysis of the arguments used by an experienced high-school teacher. We employ the model of argumentation schemes in adjunction with Toulmin's scheme. We focus on the content and the structure of the arguments used, and examine the different aspects of teacher knowledge that emerge. There is evidence of a structured example space which instantiates through the use of pedagogical examples as a conclusive and integral part of teacher's argumentation. This example space is based on a sense of deep pedagogical intuition framed by teacher's craft knowledge.

INTRODUCTION

There is a growing amount of research in mathematics education literature focusing on mathematical arguments (structures of inference with mathematical meaning) produced by students and teachers of mathematics. The main methodological tools for the analysis of these arguments are primarily Toulmin's model (Toulmin, 2003). Some researchers use Toulmin's model as a lens through which they document students' learning progresses in a classroom (Krummheuer 1995), while other researchers study the quality of a certain mathematical argument (Pedemonte, 2007). Nevertheless not much has been done in the direction of analysing the pedagogical arguments of teachers. In this paper we use Toulmin's model coupled with argumentation scheme analysis (Walton and Reed, 2005; Walton, Reed & Macagno, 2008) to dissect the structure of argumentation of mathematics teachers when they interpret and comment on students' answers and design teaching interventions to help students overcome emerged difficulties. By examining the kind of arguments they employ and studying their structure we seek to investigate some aspects of teacher knowledge (Schulman, 1987; Ball, Thames & Phelps, 2007) that is lying beneath each argumentative scheme. So we examine the following questions: (a) what is the kind of arguments teachers use, (b) what is the structure of these arguments and (c) what can be inferred about teachers' knowledge.

THEORETICAL PERSPECTIVE

Toulmin's (2003) model asserts that most arguments consist of six basic parts, each of which plays a different role in an argument. The claim (C) is the position or claim being argued for. The data (D) are the foundation or supporting evidence on which the argument is based. The warrant (W) is the principle, provision or chain of reasoning that connects the data to the claim. Warrants operate at a higher level of generality than a claim or reason, and they are not normally explicit. The Backing (B)

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provides the support, justification or reasons to back up the warrant by presenting further evidence. The modal qualifier (Q) represents the verbalization of the relative strength of an argument and the rebuttal (R) consists of exceptions to the claim stating the conditions under which it would not hold. Since Krummheuer (1995) a lot of research has been taken place using Toulmin's scheme. Nevertheless there are some notable difficulties. The notion of warrant has proved notoriously difficult to interpret, while due to the elliptic form of human argumentation not all the premises are explicitly stated. In order to bypass such turns, we turn to a useful tool in argumentation theory: the theory of argumentation schemes. These are forms of argument that represent structures of common types of arguments used in everyday discourse as well as in special contexts like those of scientific and legal argumentation . Their flexibility to accommodate deductive, inductive and abductive (or defeasible) forms of arguments has led to a recent paradigm shift in logic, artificial intelligence and cognitive science. Recent work in analysis of presumptive argumentation and argumentation schemes (Walton et al.; 2008) has led to an extensive compendium of argumentation schemes on which we base our work in the classification of the arguments used by mathematics teachers. The structure and the content of argumentation schemes can reveal facets of teachers' pedagogical content knowledge (PCK). We use the term PCK in the sense of Schulman (1987) as refined by Ball et al (2007) who proposed a further classification of content knowledge for teaching. Two basic components of this classification are i) PCK, which contains the subdomains of knowledge of content and students (KCS, is knowledge that combines knowing about students and knowing about mathematics) and knowledge of content and teaching (KCT, is knowledge that combines knowing about teaching and knowing about mathematics), and ii) specialized content knowledge (SCK) which is the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work and belongs to the general domain of content knowledge. Finding an example to make a specific mathematical point is one of the mathematical tasks of teaching that characterize SCK and mirror a teacher's conceptions of mathematical objects involved in an example generation task, his pedagogical repertoire, his difficulties and possible inadequacies in his perceptions (Zazkis & Leikin, 2007). We examine the examples that teachers provide in order to support an argument, as an indicator of their mathematical and pedagogical knowledge. We employ the notion of example space by Watson and Mason (2005) which is a collection of examples that fulfill a specific function influenced by individual's experience and memory, as well as by the specific requirements of an example generating task. In our case several argumentation schemes include as support examples of pedagogical issues. In order to differentiate them from mere mathematical examples or empirical examples (that are used as data of an argument) and to emphasize the metalevel they belong to relatively to the ground level of the didactic episode they talk about, we adopt the term metaexamples (Metaxas, 2008).

METHODOLOGY

This study is part of a larger research project investigating PCK and SCK that high school mathematics teachers have as well as the evolution of their knowledge bases. The participants were 18 high school mathematics teachers with a degree in mathematics and were enrolled in a 2-year master's program in mathematics education. In partial fulfilment of their master's degree, they had to attend thirteen 2-hour classes of a semester long graduate course in Didactics of Calculus and this study was based on this course.

Process: The process in tentative chronological order was as follows: i) Observation and video recording of the lessons of the course. The lessons were based on a number of tasks which contained hypothetical didactic situations (Biza, Nardi & Zachariades, 2007) and they triggered off extensive discussions on mathematical, pedagogical and didactical issues, while the role of the instructor (and member of the research project) was that of a facilitator. The class was scheduled to adhere to the basic complex science principles (Davis & Simmt, 2006) of decentralized control (a complex form is bottom-up; its emergence does not depend on central organizers or governing structures), neighbour interactions (the neighbours that must 'bump' against one another are warrants, ideas, and other parts of argumentation) and enabling constraints (The participants, for example, expect the topics of discussion to be appropriate to their work etc.). ii) A half an hour tutoring audio-taped session with a high school student. The student was given a worksheet with some calculus tasks designed by each participant teacher and in the tutoring session the teacher discussed with the student his/her responses. This happened during the first week of the classes and the same was repeated during last week. iii) Two semi-structured interviews with six of the participants. The interviews were based on teachers' answers to the tasks and on the analysis of their videotaped classes and audio taped tutoring sessions. The two interviews, approximately 90 minutes each, were held during the first and last week of the semester. The purpose of the questions addressed to the teachers was to make them elaborate on their written answers and oral arguments they used in class and unveil their argumentation base. iv) Transcription and analysis of all video and audio tapes. v) Triangulation of the initial results obtained, by means of a third interview with every one of the six teachers where they commented on our own interpretations.

Data and data analysis: Data consisted of the: i) teachers' responses on the tasks used in the class, ii) videotapes of all classes, iii) two audio-taped half hour tutoring sessions of each participant with a high school student of his choice, iv) two semi-structured audio-taped interviews with each participant. We analysed line by line the dialogues, coded the parts of every argument as D,C,W,B,R or Q (the elements of Toulmin's model) and categorized them using a levelling structure similar to analysis by Chinn and Anderson (1998) : we consider wherever applicable, a Backing as simply a Datum in a second argument frame whose Claim is the Warrant (either explicit or entailed) from the first argument frame. This presupposes the existence of a conceptual continuity among the arguments, which allows including them in the same unit, which we call it, a chain. A chain is a series of argumentation schemes that are linked with each other by a claim, a warrant, a backing or a rebuttal and share a common idea or concept in a gradually expanding web of interlocking argument frames. Each argumentation scheme is characterized by the level of depth it has in the chain (first level for the first scheme in order of a chain etc.). So for example, D/3 means a Datum in the third argumentation scheme of a certain chain. In order to study the kind and structure of inferences used, we characterized the arguments of each argumentation scheme according to a compendium of schemes (Walton et al., 2008). At the same time, we categorized each such scheme according to the kind of teachers' knowledge it exhibited (PCK, SCK, CKS, CKT). In order to check any discrepancy among our records we included a comparison of data collected in the video and audio tapes with the data from written records. We also allowed for a grounded approach: trying to take notice of any emerging pattern, either in the content or in the structure. Due to space limitations, we will present here the case of one of the teachers-participants, John (pseudo-name), with 20 years teaching experience of teaching calculus in high school. John was in the third semester of his graduate studies.

RESULTS

We will present some of the results based on the two interviews with John while including a characteristic extract from his first interview.

Types of arguments used

As it is clear from table 1 below, John mostly used five types of arguments while only in 4 out of 44 total cases he relied on a theoretical (pedagogical) argument with no support from experience. The main types of argumentation schemes (Walton et al., 2008) that John used were: *argument from illustration* (Premise 1: usually if x has property F the x has property G, Premise 2: in this case x has property F and G, Conclusion : the rule is valid), *argument from analogy* (Similarity Premise : generally case C is similar to case F, Base Premise : A is true in case C, Conclusion : A is true in case F), *argument from classification* (Premise 1: k has property F, Premise 2: for all x, if x has property F then x can be classified as having property G, Conclusion : k has property G), *abductive argument from effect to cause* (F is a set of facts in the form of an event that has occurred, E is a satisfactory explanation of F, therefore E is plausible as an hypothesis for the cause of F) and *argument from opposite* (the opposite of S has property P, therefore S has property not-P). The first three of the above kinds of arguments are basically inductive in their structure with one of their premises to be personal experience. This would probably mean that John was still relying heavily on intuitive knowledge he gained through his experience either as a teacher or as a high school student even after three semesters experience at graduate level. His extensive use of metaexamples (15 out of 19 argumentative chains contain metaexamples) concurs with the empirical backing that the above kind of arguments usually have and help reducing the "cost" for understanding an argument (in the sense of Besnard and Hunter, 2008).

Number of arguments	Arguments from opposite	Arguments from analogy	Arguments from classification	Arguments from illustration	Arguments from effect to cause	General pedagogical arguments
46	2	22	6	10	2	4

Table 1: Classification of arguments used in two interviews

Structure of argumentation

The pattern we see also in the extract below, namely the conclusion of an argumentative chain by the means of a metaexample, is indicative of a more general trend : 13 out of 19 argumentative chains have a metaexample as a backing in the last argument of the chain. It signifies probably the conclusive character that his experience had for John, as he himself stated: "all the teaching episodes that I have experienced as a teacher or as a student, have strongly shaped my views and I regard them as a definite and valuable asset". On another aspect, considering the appearance of the modal qualifiers (Q and R) of the Toulmin's scheme, we note that 13 out of 14 of their total appearances are realized in arguments that are not backed by empirical data.

J(ohn)	What I do is writing certain examples on the board, so	D/1
	I regard that a general solution method can be inferred from concrete	
	examples….	C/1
	For example, to check the continuity of a function I give some specific	
	functions and discuss how to prove their continuity	B/1
	Obviously, you can not see all the cases ….	R/1 & D/2
	but still this teaching approach has better results	C/2
	Otherwise, I think that describing to the students a general method from	
	the very beginning you narrow their thinking….	Q/2 & W/2
	the students can not think if constantly follow someone else's instructions	B/2 & D/3
	Only after the examples we can generalize the method	C/3
	….I think if the student discovers some rules on his own he understand	
	better the mathematics behind them.	W/3
	For example, in finding the number of the roots of an equation using	
	Bolzano's theorem the student may realize the meaning of "at least one	
	root" in the theorem statement	B/3 & D/4
R	Could this example be confusing for the not-so-good students?	
J	It depends on the classroom environment.	C/4

If classroom norms encourage the dialogue, a weak student won't hesitate
to express his confusion whenever he feels so. W/4
For example, I could ask them to construct a first order equation with no
roots. Any student could feel free to ask for clarification. B/4

In particular in the extract we cite, we see that in the first scheme John was stating his didactic method which he claimed it adheres to general pedagogical principle (line 2). An explicit warrant is missing, but he supplied as backing an example of a case in calculus (metaexample) (line 4) to illustrate his method. It's an argumentation scheme from analogy (Walton et al., 2008) with a premise based on his experience (so basically inductive). His rebuttal (line 6), which actually is an undercut of the claim (Besnard and Huntler, 2008) means that some uncertainty remains about the universality of his claim but (turning the previous undercut to a datum for a second level argument) he warranted his preferred method by stating another pedagogical argument and a qualifier is used to express the uncertainty of the validity of this argument (lines 8, 9). His 2nd level backing (line 10) now serves as a datum to a 3rd level argument. The warrants at levels 3 & 4 (lines 12,13 and 19,20) are again general pedagogical (student's initiated search and classroom norms) which are backed by metaexamples of teaching a certain calculus paragraph (lines 14-16 and 21-22). Both arguments are arguments from illustration (at level 3 it is coupled with a defeasible form of modus tollens).

Argumentation and teachers' knowledge

This extensive use of metaexamples in his argumentation schemes indicates the existence of a rich example space (in the sense of Watson and Mason, 2005) which is an indicator of familiarity with pedagogical issues like students' misconceptions, teaching methods etc. We concur with the view of Watson and Mason (2005) that "to understand mathematics means, among other things, to be familiar with conventional example spaces" and we note that in our case John's example space is pedagogical in nature and thus extends the above mentioned notion to the direction of pedagogical content knowledge. It is of significant importance that all the (meta)examples provided by John, were given by him spontaneously and as an integral part of his argumentation schemes. This means they are experienced as members of a structured space. Furthermore, the existence of a structured example space on the level of pedagogical knowledge shows a certain level of knowledge of the types of KCS and KCT. The conclusive character that his experience had for John as noted before, demonstrates a considerable degree of confidence in his empirical intuitions. This agrees with Fischbein's remark (1987) that overconfidence is related to the degree of intuitiveness of the various items considered and experience plays a fundamental role in shaping these intuitions either primary or secondary. The overconfidence is also indicated by the function of the modal quantifiers that we encounter. It augments the above mentioned note regarding metaexamples and empirical foundation and surely casts a shadow on the strength of theoretical pedagogical knowledge that teachers - students are learning. This deep belief in his personal pedagogical experience without an equivalent abstract foundation is reminiscent of "deep intuition" of mathematics (Semadeni, 2008) that sometimes students may have without backing it with a sound theoretical knowledge. We could in a sense talk also here about a deep intuition of pedagogy (either as KCS or KCT) that doesn't rely on a deep theoretical knowledge. This deep intuition is a common characteristic among all the teachers that participated in this project.

Conclusion

In this paper, we have presented our preliminary findings emerging from our work on the study of argumentation of high-school math teachers. In particular, we analyzed the structure and the pedagogical content of their arguments. Methodologically, we used the theoretical framework of argumentation schemes extending Toulmin's model in order to obtain a more precise characterization of the arguments involved while with the notion of argumentation chain we took accountability of the intertwined nature of teachers' reasoning. In the case study of John, most of the arguments he used belonged to the "applying rules to cases" category (Walton et al., 2008) which are arguments that relate to a situation in which some sort of general rule is applied to the specifics of a given case. His backings in these argumentation schemes were mainly metaexamples while whenever he used modal qualifiers his premise was a general pedagogical rule, remark etc. It seems that a certain pattern emerges: he is overconfident about his personal empirical knowledge while he is quite reserved about general pedagogical statements. Furthermore, his accessibility to a wide range of metaexamples and his readiness to use them for supporting his arguments, indicates a strong pedagogical content knowledge (KCS and KCT) and a structured pedagogical personal example space that entail a deep pedagogical intuition. Also the place of them in the last part of an argument, shows again a belief in the concluding power of empirical knowledge. The suggested framework is a guideline, it is neither comprehensive nor complete but it offers a way to pose new questions that are related to teacher reasoning and knowledge.

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PRE-SERVICE ELEMENTARY TEACHERS' UTILIZATION OF AN EQUIPARTITIONING LEARNING TRAJECTORY TO BUILD MODELS OF STUDENT THINKING

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In this design study, we investigate how 29 pre-service elementary teachers used an equipartitioning learning trajectory to build models of student thinking. Results indicate that teachers used the processes of describing, comparing, inferring, and restructuring in constructing models of student thinking.

INTRODUCTION

In the past two decades, research on learning has focused on understanding how student thinking changes and evolves over time. Some researchers have verified consistent findings relating to these constructs, which they have articulated in the form of learning trajectories. While this has contributed greatly to the knowledge base of how students learn, the field has just begun to explore the extent to which learning trajectories can be integrated into the practice of teaching or in the preparation of pre-service teachers (PSTs).

RELATED LITERATURE

Different terminology and definitions have been used to describe learning trajectories in the literature. According to Clements, Wilson, and Samara (2004), a learning trajectory is comprised of a mathematical goal, domain-specific developmental progressions that children advance through, and activities that correspond with these distinct levels of progression. Catley, Lehrer, and Reiser (2005) suggest learning should be viewed as the process of developing key conceptual structures (Case & Griffin, 1990), or *big ideas*, which coordinate and integrate isolated conceptual components, indicating that instruction can be viewed as an orientation towards core ideas that direct teaching and assessment around foundational concepts. They suggest that teaching should trace a prospective *developmental corridor* (Brown & Campione, 1996), or a *conceptual corridor* (Confrey, 2006), that spans grades and ages, with central concepts introduced early in the school experience and progressively refined, elaborated, and extended (Catley et al., 2005).

Learning trajectories play an important role in instructional design. Designing instruction around central concepts, or "big ideas," results in greater coherence and alignment between teaching and learning (Catley et al., 2005). Simon (1995) indicates that a *hypothetical learning trajectory*, a teacher's anticipation of the progression of the learning path, provides a rationale for designing instruction, taking

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into account the learning goal that defines the direction, learning activities, and the teacher's prediction of the potential reasoning and learning of students. Gravemeijer (2004) asserts that teachers need a framework for exemplary instructional activities, along the learning path, that can be used as a catalyst for students to reinvent mathematics, which are articulated in *local instruction theories*.

A common theme among the various terminologies is that knowledge progresses from less sophisticated to more sophisticated levels of understanding in a relatively predictable way. Building on the work of others in the fields of mathematics education, science education, and the learning sciences, Confrey, Maloney, Nguyen, Wilson, and Mojica (2008) define a *learning trajectory*, as:

> a researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.

We view a learning trajectory as a tool that can be utilized by PSTs to inform key instructional activities, such as planning, teaching, and assessing. While student understanding cannot be observed directly, learning trajectories seek to identify and describe key items, constructs, and behaviors, which can be observed. From these, we will investigate the extent to which an equipartitioning learning trajectory can mediate elementary PSTs' ability to construct models of student thinking.

THEORETICAL FRAMEWORK

According to Cobb and Steffe (1983), students construct models of mathematical concepts, while teachers or researchers build models of students' thinking. Hollebrands, Wilson, and Lee (in review) identified four distinct processes that PSTs employ in creating such models: Describing, Comparing, Inferring, and Restructuring. *Describing* is characterized by PSTs' explicit attention to students' actions and words, written or verbal, in making decisions about students' thinking. When PSTs construct models of students' reasoning, by relating the students' work to their own, they are *comparin*g, making either explicit or implicit comparisons between their own work to that of students'. In *inferring*, PSTs analyze students' work and build models of student thinking by making inferences about how students reason, using students' work as evidence. *Restructuring* is characterized by PSTs' use of models of student thinking in their own practice of teaching. Models of student thinking inform their own thinking and regulate instructional decisions. These four distinct behaviors can serve as a framework with which to examine the effects of learning trajectories on the building of models of student thinking.

METHODS

Participants

This paper reports findings from a larger on-going study involving 57 PSTs (51 juniors and 6 sophomores) enrolled in one of two sections of a mathematics methods course within the elementary education department at a large southeastern U. S. university. This one-semester course met for 75 minutes twice weekly. The PSTs also interned weekly in K-2 classrooms. This paper focuses on participants from one section of this course (26 juniors and 3 sophomores).

Design

This design study involves "engineering particular forms of learning and systematically studying those forms of learning with the context defined by the means of supporting them" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p.9). The study took place during eight weeks of the methods course, targeting the teaching and learning of equipartitioning. The first author of the paper was the instructor of both sections of the course.

One goal of a design study is to create instructional activities or tasks for classroom use (Cobb, 2000). Thus, a series of instructional activities, or interventions, were designed and implemented to investigate the extent to which PSTs use an equipartitioning learning trajectory to build models of student thinking. These interventions included the following:

- engagement with equipartitioning tasks to develop PST content knowledge;
- an introduction to the articulation of an equipartitioning learning trajectory;
- instruction in the conduct of clinical interviews with different types equipartitioning tasks; and,
- instruction in the analysis of video and student work.

Confrey et al. (2008) conducted a synthesis of the literature on equipartitioning and other areas of rational number reasoning, where they articulate a learning trajectory for rational number reasoning concepts and organize children's reasoning of equipartitioning into four cases: (A) sharing a discrete collection, (B) sharing a continuous object, and sharing multiple continuous objects between (C) more people than objects and (D) more objects than people. Confrey et al. (2008) has built a progress variable for equipartitioning (see Table 1), describing the behaviors and verbalizations of different levels of understanding of equipartitioning. Within each level (i.e., 1.1, 1.2, etc.), the Confrey at al. (2008) describes another level of the progression of knowledge: methods, multiple methods, justification, naming, reversibility, and properties.

During the course, PSTs were first introduced to the construct of a learning trajectory. Next, they were introduced to the equipartitioning learning trajectory for rational number reasoning (Confrey et al., 2008) that situates equipartitioning within this realm of rational number reasoning, to help PSTs recognize the foundations of equipartitioning in developing a more robust understanding of a rational number than is currently enacted in United States classrooms (Confrey et al., 2008). Over the eight-week period, PSTs were exposed to parts of the equipartitioning learning trajectory and progress variable one case at a time.

Case	Equipartitioning Progress Variable
D	1.8 m objects shared among p people, $m > p$
C	1.7 m objects shared among p people, $p > m$
B	1.6 Splitting a continuous whole object into odd # of parts ($n > 3$)
B	1.5 Splitting a continuous whole object among $2n$ people, $n > 2$, & $2n \neq 2^i$
B	1.4 Splitting continuous whole objects into three parts
B	1.3 Splitting continuous whole objects into 2^n shares, with $n > 1$
A	1.2 Dealing discrete items among p = 3 - 5 people, with no remainder; mn objects, n = 3, 4, or 5
A, B	1.1 Partitioning using 2-split (continuous and discrete quantities)

Table 1: Equipartitioning Progress Variable and Cases A - D.

As each case was introduced, instructional activities initially focused on equipartitioning tasks to assess and support the development of PSTs' content knowledge of equipartitioning. The equipartitioning/splitting construct (Confrey et al., 2008) was new to all of the PSTs. After the PSTs had engaged in equipartitioning tasks and discussed their own solutions, as well as the underlying mathematical structures of the tasks, they were introduced to the components of the learning trajectory and progress variable. When these components were introduced, video exemplars of K-2 students, engaged in working with equipartitioning tasks, were presented. Next, instructional activities focused on analyzing other video exemplars of K-2 students. The video exemplars illustrated a range of students' verbalizations and activity as they participated in clinical interviews on the same equipartitioning tasks with which PSTs had previously engaged. Class discussion focused on analyzing student thinking with respect to the equipartitioning learning trajectory. Student work samples were also incorporated into class discussions. Lastly, PSTs implemented equipartitioning tasks with students in their K-2 classrooms.

During the final week of the study, PSTs engaged in an individual video analysis of student thinking, during a regular class meeting (approximately 75 minutes). PSTs viewed three video clips of a five-year old kindergarten student, Emma, who had engaged in several equipartitioning tasks during a clinical interview. Prior to viewing the video clips, PSTs were asked to solve each of the tasks on their own and to anticipate how a K-2 student might solve each task. After these two components were completed, they viewed the first video clip and responded to questions such as: a) What does Emma understand? Explain. b) What does Emma have difficulty with?

Explain. PSTs then viewed the second and third clips, each time reflecting on their own interpretation of Emma's understanding before moving on to the next clip. Their written responses to the video analysis were coded for evidence of using the processes of *describing*, *comparing*, *inferring*, and *restructuring*.

Data Collection and Analysis

Data collection was designed to take advantage of PSTs' experiences in K-2 classrooms to gather evidence about how they engage in teaching and assessing students while using an equipartitioning learning trajectory. Data included video recordings of each class meeting, audio recordings of small group discussions, and the researcher's field notes of observations of PSTs' work with K-2 students, pre- and post-tests for assessing PSTs' knowledge of equipartitioning and teaching, all coursework completed, including an individual video analysis and clinical interviews conducted by PSTs, their own analysis of student thinking, and their reflections.

RESULTS

Results from the analysis of PSTs' individual video analyses will be reported. Two video clips that PSTs examined will be summarized. In Clip 1, Emma has just been asked to share 24 pieces of pirate treasure between four pirates, and is not told the total number of pieces of treasure. Emma creates four 3 x 2 arrays with the coins, saying each pirate's share is "six cents." Emma is then told that one of the pirates has left on a ship and is asked to share the pirate treasure fairly among the three remaining pirates. Emma pulls all the treasure back together in one pile. Then, she builds three 4 x 2 arrays. She says, "the magic number is eight." A transcript summarizing the remaining part of the clip follows:

Interviewer: How do you know they each get the same amount?

Emma: Last time it was six. Now, you just added two more [points to two coins on top row of one 4 x 2 array]. Cause he had six [points to the location of the fourth pirate's share from previous task]. They added two more to each one, which makes six. One, two, [points to two coins on the top row of one 4 x 2 array] three, four, [points to two coins on the top row of the second 4 x 2 array] five, six [points to two coins on the top row of third 4 x 2 array].

In Clip 2, Emma has been asked to share a circular birthday cake fairly between two pirates. She draws a line through the center of the circle and says, "it's halves" as she points to each half. When asked how she knows, Emma shrugs and says she just does. She is asked again to justify her solution. Emma creates the following story.

Interviewer: How do you know that this pirate's share is the same as this pirate's?

Emma: This is a pirate. They're like, 'do you want to share this cake?' 'Ok.' And, they are like, 'how will we split it?' 'We could cut it like that [folds the circular piece of paper into two distinctly noncongruent parts]. 'And, I'll have the big and you could have the tiny.' 'No, that won't work. We need

> the same amount so it will be fair.' 'Ok. Why don't we cut it in the middle?' 'Ok.' [folds the circular piece of paper into two congruent halves down the center]. See, that is exactly how I folded it [holds up folded circle]. [Then she puts the folded circle on top of her first circle and redraws the center line.]

Describing

Some PSTs used the process of *describing*. They engaged in this process by identifying student verbalizations and activity. Almost all PSTs focused on the verbalizations and behaviors identified in the equipartitioning learning trajectory and progress variable, rather than focusing on other constructs, like counting. For example, some PSTs, like Bonnie, made general observations of Clip 1 such as, "She understands how to share the coins fairly with the three pirates." Other PSTs described a specific action that they observed when responding to their interpretation of what Emma understands. Wendy described Emma's actions by stating, "She split up larger numbers into smaller groups." Yet others focused on the constructs of equipartitioning and counting. Marianne stated, "She counts the piles at the end to make sure each one is even." Marianne referred to the arrays as "piles," and she described the behavior of counting as a verification to make sure that the shares are "even." Throughout the design study, Marianne frequently used the word "even" to mean "equal." Using counting to verify that shares are the same is a justification relating to this specific task within the progress variable.

Comparing

Very few PSTs engaged in the process of *comparing* student activity and verbalizations to their own. No direct comparisons by PSTs were observed. None of the PSTs made explicit comparisons to their own solutions in building models of Emma's thinking. A few implicit comparisons were made. For example, regarding Clip 2, Maria stated, "She drew a line horizontally through the middle of the circle. Then, she folded another circle in half, and placed the folded half on a portion of the drawn circle." Next to her statement, Maria drew a circle with a horizontal line down the center. In Maria's own solution before she viewed Clip 2, she drew a circle with a vertical line through the center and wrote, "Divide the cake down the middle vertically." Maria only noticed behaviors that were similar to her own and made no inferences about how such behaviors relate to what Emma might understand.

Inferring or Restructuring

It was difficult to distinguish between *inferring* and *restructuring*; thus, we categorized these responses together. PSTs engaged in the processes of *inferring* and *restructuring* by making inferences about student activity and verbalizations and by restructuring their own knowledge. These processes were identified more than any others. With respect to Clip 1, many PSTs responded that Emma understood that she needed to redistribute the fourth pirate's share to the other three pirates or that Emma had some understanding of compensation. Susan stated

> Emma knows that each pirate needs to have an equal share. She seemed to understand that after a pirate left and there were only three, you could divide the treasure up and distribute two pieces of that pirate's treasure to the pirates that were left.

Similarly, Sophia described Emma's understanding as

> She understands the very basics of division and compensation because she was able to see that she needed to disperse two more coins among the three remaining groups without having to re-count and deal out the chips. She clearly knows how to compensate and quickly alter the groups when the situation changes.

Many PSTs, like Kelly, responded that they would want to ask Emma, "Why does each of the three pirates get two of the pirate's coins that left?" and "How did you know that eight was the magic number?" These PSTs were able to identify specific questions to help them better assess Emma's understanding from Clip 1. Kelly's response is representative of most of the questions created by the PSTs.

DISCUSSION

In a study involving 18 PSTs, Hollebrands et al. (in review) found that PSTs engaged in distinctive processes while examining videocases of students engaged in statistical tasks as they used technology. While we were able to categorize most PST responses into the categories of *describing*, *comparing*, *inferring*, and *restructuring*, we found that it was difficult in many cases to distinguish between the processes of *inferring* and *restructuring*. Since the processes of *describing* and *comparing* were so distinct, further research should be conducted to explore whether *inferring* is a special case of *restructuring* and whether other processes exist.

We found that many students engaged in the processes of *inferring* or *restructuring*, while almost no PSTs utilized the process of *comparing* in their model building process. Further research should also examine the role of the learning trajectory in these processes. In other words, does using a learning trajectory influence the types of processes that PSTs engage in? Would the absence of a learning trajectory in the same domain-specific areas result in the same model building processes? No claims can yet be made with regard to these questions. A research agenda focused on these issues could help us better understand how to use learning trajectories to help teachers build more robust models of student thinking as they engage in instructional activities, such as planning, teaching, and assessing.

Hollebrands et al. suggest that the categories of *describing*, *comparing*, *inferring*, and *restructuring* are a valuable framework for considering how PSTs tap into and create their own understanding. We found that this framework was useful in articulating PSTs process of building models based on the examination of student's verbalizations, activity, and work samples. Similar to Hollebrands et al., we believe that this framework can be a powerful lens for understanding how PSTs construct and restructure their own mathematical knowledge.

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KINDERGARTEN STUDENTS' UNDERSTANDING OF PROBABILITY CONCEPTS

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This study explored kindergarten students' intuitive strategies and understandings in probabilities. The paper aims to provide an in depth insight into the levels of probability understanding across four constructs, as proposed by Jones (1997), for kindergarten students. Qualitative evidence from two students revealed that even before instruction pupils have a good capacity of predicting most and least likely events, of distinguishing fair probability situations from unfair ones, of comparing the probability of an event in two sample spaces, and of recognizing conditional probability events. These results contribute to the growing evidence on kindergarten students' intuitive probabilistic reasoning. The potential of this study for improving the learning of probability, as well as suggestions for further research, are discussed.

INTRODUCTION AND THEORETICAL FRAMEWORK

The importance of having all students develop a sound awareness of probability concepts and appropriately use these concepts in solving problems has been recognized in recent curriculum documents (e.g., National Council of Teachers of Mathematics, 2000). These recommendations adopt the position that young students, even at the kindergarten level, need to explore the processes of probability (NCTM, 2000). The teaching of probability is, however, not an easy task (Fischbein & Schnarch, 1997; Langrall & Mooney, 2005). As argued by Shaughnessy (1992), modeling probabilistic situations is complex and the teaching of probability concepts is often hindered by students' primitive intuitions and alternative conceptions. Following recommendations for early introduction of probability concepts in school curricula and for students to exhibit probabilistic thinking, there is a need for students to understand probability concepts that are multifaceted and develop over time (Jones, Langrall, Thornton, & Mogill, 1997). Although there has been substantial research on young children's probabilistic thinking (e.g., Fischbein, 1975; Fischbein, & Schnarch, 1997; Piaget & Inhelder, 1975; Shaughnessy, 1992), little recent research has been done in the field of teaching and learning probabilities to young learners and further on how young learners' intuitive models and strategies on probability concepts are incorporated into solving problems related to probability.

Fischbein (1975) reported that 'probability matching', "the expression of … the intuition of relative frequency" (p.58), had been observed and generally well established in pre school children. Although the concept of ratio appears to be crucial to the development of probabilistic reasoning (Piaget & Inhelder, 1951) and therefore the concept of chance cannot be obtained before proportional reasoning is mastered

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(Greer, 2001), the intuitive foundations of pre-school students can serve for the development of probabilistic knowledge. As primary intuitions of chance and the concept of change certainly exist in pre-school students (Greer, 2001; Langrall & Mooney, 2005), it is important to take these intuitions into consideration in designing and implementing problem-solving activities in probability. Moreover, it is generally agreed that even before formal instruction in probability, children already acquire an elementary understanding of probability and are able to compare the probability of two situations in a qualitative way (e.g., English, 1993; Fischbein, 1975; Fischbein & Gazit, 1984; Sharma, 2005).

For the purposes of the present study we used the cognitive framework proposed by Jones and colleagues (1997, 1999), which can be used to describe and predict students' probabilistic thinking. In line with previous research, the proposed framework assumes that probabilistic thinking is multifaceted and develops slowly over time. Four key constructs are incorporated in the framework, to satisfactorily capture the manifold nature of probabilistic thinking and its interconnections. These constructs are sample space, probability of an event, probability comparisons, and conditional probability. Furthermore, young children's probabilistic thinking is described across four levels for each of the four constructs: the subjective level, the transitional level, the informal quantitative level, and the numerical level (Jones et al., 1997, 1999).

Since the present study focuses on exploring and identifying young learners' probabilistic thinking, students' actions at the subjective and transitional level are presented next. At the subjective level, children can list an incomplete set of outcomes for a one-stage experiment, predict most/least likely events partially based on subjective judgments, and recognize certain and impossible events. Children can also compare the probability of the same event in two different sample spaces, cannot distinguish "fair" probability situations from "unfair" ones, and recognize when certain and impossible events arise in a non-replacement situation (Jones et al., 1997, p.111). At the transitional level, the children list a complete set of outcomes for a one-stage experiment and sometimes list a complete set of outcomes for a two-stage experiment using limited and unsystematic strategies. Children can predict most/least likely events based on quantitative judgments (but sometimes may revert to subjective judgments), and make probability comparisons based on quantitative judgments (may not quantify correctly and may have limitations when non-contiguous events are involved). At the transitional level children begin to distinguish "fair" probability situations from "unfair" ones, recognize that the probability of some events changes in a non-replacement situation. Recognition is, however, incomplete and is usually restricted only to events that have previously occurred (Jones et al., 1997, p.111).

The aim of the present study was to investigate kindergarten students' intuitive probabilistic strategies and understandings in solving problems related to probabilities. For this purpose, the framework developed by Jones and colleagues

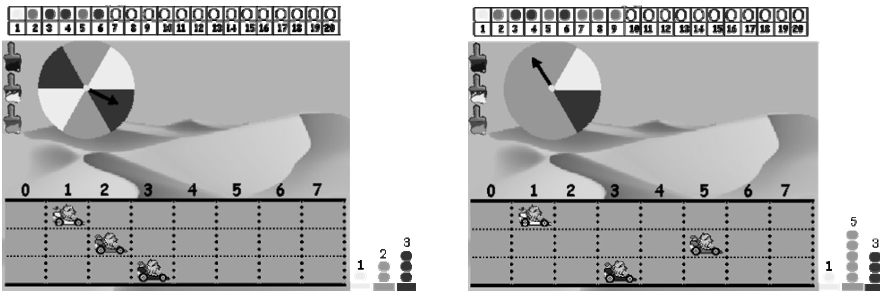
(1997) was used as a basis for identifying, exploring, and providing an in depth analysis of kindergarten students' thinking strategies.

DESCRIPTION OF THE STUDY

Participants and Procedures

Students in a large rural kindergarten school formed the population for this study. Four classes of the school are currently participating in a 2-year longitudinal study of students' probabilistic thinking and mathematical modeling. The school population is representative of a broad spectrum of multicultural and socioeconomic backgrounds. Twelve students, six from each of the two grade levels (one grade for 3-4 year olds and one for 5-6 year olds) were randomly selected and served as case studies. Prior to the start of this study, none of the students had been exposed to probability instruction. Due to space limitations, the interview of one pair of students (one from each grade level) is presented in this paper, namely Alex, 4 years and 3 months and Chris, 6 years and 1 month. It should be noted that both students are ranked (by their teachers) among the best in their classes.

The data reported here are from the first year of the respective longitudinal study and are drawn from one of the problem activities the children completed during the first year. The *Car Racing* problem (see Figure 1a and 1b) is a math applet, developed in Scratch (http://scratch.mit.edu), a freeware visual programming software, that can directly run from the Web. The problem presented a spinner (see Figure 1a for initial colours), three cars and a number of different representations related to the car racing.



(a) (b)

Figure 1: The Car Racing Activity.

These included the position of each car, a bar chart for the three colours and a "pattern style" representation for the different trials. Additionally, the applet gave students and teacher the opportunity to recolour the spinner (see Figure 1b for an example).

Data Collection and Instrumentation

A semi-structured interview protocol based on the framework proposed by Jones and colleagues (1997) was administered by the authors. The interview assessment comprised tasks related to the Car Racing problem. The tasks were associated with *sample space*, with *probability of an event,* with *probability comparisons*, and with *conditional probability* (see selected tasks, Table 1). The tasks enabled the researchers to explore students' probabilistic thinking across the two levels of the framework. The data sources included video-tapes of students' responses to the interview questions and our own field notes. The two students worked together. Some questions, however, were directed to one of them, while in other questions students were asked to first discuss the question between them and then answer.

Sample Space	Probability of an Event	Probability Comparisons	Conditional Probability
What colour will you get if you spin the spinner again and again? Is that all? How do you know?	Which colour has the least chance to appear? (1/2 was yellow, 1/3 was blue and 1/6 was green)	Colour the spinner in a way that you will have the best chance to win, using at least two colours.	What colour has the best chance of getting? Why? (no yellow in last four trials and all colours were 1/3)

Table 1: Selected Tasks from the Interview.

The transcripts were reviewed by the authors and data were analysed using interpretative techniques (Miles & Huberman, 1994) to explore and identify developments in students' probabilistic thinking with respect to: (a) the four key constructs of the proposed framework (sample space, probability of an event, probability comparisons and conditional probability), and (b) the two levels of probabilistic thinking (subjective and transitional).

RESULTS AND DISCUSSION

We report here on the students' understanding of probability concepts in terms of the two levels of probabilistic thinking as reported by Jones and colleagues (1997) and discuss possible further enhancements of the proposed framework, based on the results of the study. The individual responses and discussions between the two students were analyzed, and summaries and exemplars were produced to illuminate a number of the probabilistic thinking strategies outlined in the proposed framework and to suggest new thinking strategies. None of the students tended to generate the same level of probabilistic thinking for all four constructs. We therefore decided to present their results are follows: First we focus on students' probabilistic thinking strategies that are related to Level 1 (Subjective), and then we focus on their strategies that appear to be linked to Level 2 (Transitional).

Level 1 Probabilistic Thinking Strategies

Alex, the younger child exhibited both level 1 and level 2 probability thinking strategies. It should be noted, however, that he did not provide correct answers for all questions and problem situations related to the four constructs at level 1. Consequently, he provided fewer correct responses to problems corresponding to level 2. Chris, the older child successfully answer all questions related to all four level 1 constructs.

An explicit difference in the two students' responses was the absence of any subjective beliefs in Chris' judgements. He totally based his answers and comments on his probabilistic related intuitions and on his understandings on other mathematical constructs. On the contrary, Alex quite frequently based his comments on subjective beliefs. However, he did not consistently use subjective knowledge, but he rather used it when he felt that he could not use any of his prior mathematical or other understandings. On sample space related questions, he easily listed all possible outcomes when, for example, colours had equal probabilities. Sometimes, in questions that colour probabilities were not equal, he only listed his favourite colour or the colour that was more likely to happen. On a task, for example, where 5/6 of the spinner was shaded yellow and 1/6 blue, he reported that it was not fair because green was missing. He responded that only yellow would appear, since blue was too small compared to yellow. Somehow contradictory to what Jones (1997) reported, sometimes Alex spontaneously listed all expected outcomes. He could even recolor the spinner in a number of ways as to match a predefined list of outcomes. So, for example, when he was prompted to recolor the spinner in a way that only green and blue were the possible outcomes, he coloured 4/6 green and 2/6 blue. When asked if that was the only solution, he coloured one green slice into blue. His two solutions are presented in Figure 2.

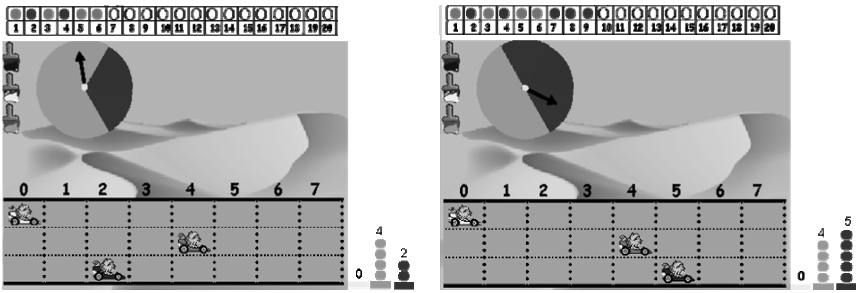


Figure 2: Alex's solutions.

A typical thinking strategy of Alex in probability comparisons, which was consistent in almost all his actions and responses, was his tendency to believe that the number of slices was more important than the size of them. When he was presented with a task where 1/3 was yellow, 2/6 was green and 2/6 was blue, he reported that it was not fair for yellow. He said, “Green is best because it has two slices and it is my favourite colour. Blue is the same …has more than yellow”.

On conditional probability tasks, both children experienced difficulties. Our problem setting did not include any tasks related to item replacement (or not). Alternatively, we used the pattern related representation that appear on the top of the applet screen and that presented a history of the game results. In a task where 1/3 was yellow, 1/3 was green and 1/3 was blue, in the first five attempts the spinner returned blue, blue, green, blue and green. When asked what colour had the best chance of getting, both students identified yellow as the best for the next spinning, since according to them “it has not appeared yet” and “it is time now for yellow”.

Level 2 Probabilistic Thinking Strategies

Quite impressive, Chris, the 6-year old pupil, reported typical level 2 probabilistic thinking strategies, in almost all four constructs. This was impressive not only because of his age, but also because of the absence of any formal instruction. Chris consistently identified a complete set of outcomes. We do not claim here that he used a generative strategy, since there are not enough data to support this claim. Consequently, in Chris' answers, similar to Alex's, there was quite frequently a tendency to overlook outcomes, rather than consider sample space and probability in combination. Chris exemplified quantitative reasoning in comparing probabilities. Similar to what Jones (1997) reported, Chris always correctly used the “more of” the target colour strategy. In stark contrast to Jones' proposed framework, Chris tended to recognize the effect of conditional probability of related events. When asked, for example, how he could increase the probability of green without using the green painter in a setting where 1/2 was green and 1/2 was blue, he reported that he could use the yellow painter to paint one or more blue slices.

Another difference from Jones' second level of probabilistic cognitive framework was the absence of any subjective reasoning in Chris' answers. No doubt, Chris is not a level 3 pupil in any of the four constructs and he is probably not a level 2 pupil in all constructs. He tried his best to employ quantitative reasoning on all items relating to the probability of an event. Since his knowledge of fractions was very limited, he used the number of slices for each colour as the basis for his quantitative reasoning. When presented, for example, with a task where 4/6 was green, 1/6 was yellow and 1/6 was blue, he reported that “the probability of green was four times bigger than the probability of blue”. Quite interesting, in a consecutive task where 3/6 was green, 1/6 was yellow and 2/6 was blue, when asked to compare probabilities of the different colours, he replied that “probability of green was 3 times bigger than the probability of yellow…I can not compare green and blue…it is two times…no…I do not know”.

CONCLUDING POINTS

Although there has been substantial research on the probability constructs investigated in this study (e.g., English, 1993; Fischbein, & Schnarch, 1997; Piaget &

Inhelder, 1975), we claim that the present study provides some interesting insights into kindergarten students' probabilistic thinking, insights that are needed to guide classroom instruction and assessment. Although the purpose of the study was not to validate the framework proposed by Jones and colleagues (1997) at the kindergarten level, the results of the study revealed that students at the kindergarten school and before any formal instruction on probabilities hold and successfully employ in problem solving a number of probabilistic concepts. Even at the age of four, the case study student's probabilistic thinking across all four constructs appeared to be consistent. Further, the six-year-old student not only did not use any subjective knowledge in his work, but he also further realised the appropriateness of the quantitative reasoning in comparing probabilities and in calculating the probability of events, without any formal instruction on fractions. This kind of knowledge on students' probabilistic thinking should enhance information available to curriculum designers and teachers.

In accord with the framework of Jones and colleagues (1997, 1999), we claim that even at the age of four, without any formal instruction and based on their intuitive strategies, students start developing strategies for some of the four constructs at level 1 of the proposed framework. Further, the results showed that the six-year-old who participated in the study started developing successful quantitative and qualitative strategies for all four constructs at both levels. Further, even problem posing was not part of the tasks, the older student managed to pose correct probability problems for the younger student in order to exemplify his thinking during their discussion on several interview tasks. We do not claim that this is the case for all or for the majority of students and we are aware that very often students, especially at this age level are often distracted and misled by subjective knowledge, contradictory intuitions and other irrelevant aspects of the problems presented to them (English, 1993; Langrall & Mooney, 2005). However, the results provide some evidence that probability concepts should be introduced to students at the kindergarten level and teaching needs to consider all aspects related to students' prior intuitive strategies and cognitive models related to probability and number sense.

The results of the study illuminate the framework constructs by identifying more in-depth insights into students' probabilistic thinking. We need to address here the contribution of the software applet in framing the context of the problem situation presented to students and in providing fundamentally new representational resources (Greer, 2001). Clearly substantial more research is needed to identify the extent to which the car race scenario, the different representations (spinner, bar-chart like graph, pattern style), and the active manipulation of the spinner (changing colours at the beginning and during an experiment) contributed in enhancing student's probabilistic thinking.

The small sample size and given that both students were high achievers may limit the extent to which conclusions about the probabilistic thinking strategies students hold at the kindergarten level can be drawn. Further studies are needed to investigate in depth the probabilistic thinking of young students, covering a broad spectrum of multicultural and socioeconomic backgrounds. Clearly, more research is needed to examine the extent to which instructional programs influence the development of probabilistic thinking and to identify the critical steps in students' development of probability concepts. Such research would result in a more pervasive description of students' probabilistic thinking and could be even more useful in informing instruction in kindergarten and elementary school.

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STYLES AND STRATEGIES IN EXAM-TYPE QUESTIONS

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This paper focuses on the links between the 'thinking styles' (Sternberg, 1999) of students following a BSc in Mathematics and the strategies they employ when they deal with exam-type questions. The students' strategies were identified according to the 'A-B-Δ classification', a classification that builds on Weber's (2005) 'semantic', 'syntactic' and 'procedural'. The students' 'Initial Strategies' seem to be linked with the students' thinking styles, whereas the students' 'Back-Up Strategies' seem to be linked with the nature of the exam-type questions. The identified links between styles and strategies are discussed, drawing from Skemp's (1979) views about reality (inner and social) and survival (respectively, internal consistency and social survival).

INTRODUCTION

Mathematical thinking is one of the main fields of interest of mathematics education research (Gutiérrez & Boero, 2006). It is posited that mathematical thinking occurs at the interaction of *specificity* (the students' *actual* thinking about mathematics) and *generality* (the students' *preferred way* to think about mathematics). The existence of 'specificity' is not questioned, as students do think about mathematics, but 'generality' has been a matter of debate, although cognitive consistencies have been found in the students' mathematical thinking (Gray & Pitta-Pantazi, 2006; Pinto & Tall, 1999). Could such consistencies go beyond the scope of mathematics and become a general cognitive preference?

In the general educational and psychological research, Zhang & Sternberg (2006) suggest that various researchers argue for the existence of such general cognitive preferences, usually described by the construct of *style*. In this study, Sternberg's (1999) *thinking styles* are considered, in order to identify the implications, if any, of students having a specific thinking style profile in their dealing with university mathematics. At issue is the question: *What is the nature of the relationship(s) between the students' thinking style profile and their strategy choice(s) when they deal with exam-type questions?*

THEORETICAL FRAMEWORK

Thinking styles are defined as the "preferred way[s] of using the ability one has" (Sternberg, 1999, p. 8). Thinking styles are conceptualised as being relatively stable over time and context and they are *value-differentiated*: the students' stylistic profile

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may help them to deal with some tasks, while it may ‘obstruct’ their efforts to successfully survive other tasks. The purpose of this paper is to consider an element of a wider study into mathematics undergraduates thinking styles (Moutsios-Rentzos, submitted) and to examine the role of the students’ thinking styles in undergraduate mathematics exam-type questions.

Although some mathematics educators have worked on the wider notion of ‘cognitive styles’ (Duffin & Simpson, 2006), it was decided that Sternberg’s more narrowly defined ‘thinking styles’ were more suitable for the study as they derive from a coherent theory: the *Theory of Mental Self-Government (MSG)*. MSG is based on a metaphor between the way that the individuals organise their thinking and the way that society is governed (Sternberg, 1999). Thirteen thinking styles are identified and organised in five dimensions[2]. Moreover, Zhang and Sternberg (2006) identified three *types* of thinking styles: a) *Type I* (linked with “low degrees of structure, cognitive complexity, nonconformity, and autonomy”, *ibid*, p. 164), b) *Type II* (opposite to Type I preferences), and c) *Type III* (linked with either Type I or Type II depending on the task and the "individual’s level of interest in the task”, *ibid*, p. 167).

Kirby (1988) identified as a s*trategy* the “combination of tactics, or a choice of tactics, that forms a coherent plan to solve a problem” (p. 230-231). Strategies are heavily dependent on the characteristics of the question. For example, in exam-type questions, the students are expected to present a mathematically acceptable proof. Weber (2005) identified three strategies that the students employ in a proof construction: *procedural* (the student attempts “to locate a proof of a statement that is similar in form and use this existing proof as a template for producing a new one”; *ibid*, p. 353), *syntactic* (the student “logically manipulating mathematical statements without referring to intuitive representations of mathematical concepts”; *ibid*, p. 355) and *semantic* (the student “uses the informal representations to guide the formal work that one produces”; *ibid*, p. 356). In the current study, *Initial Strategy* (the students’ first strategy choice) was differentiated from *Back-Up Strategy* (a *different* strategy to the Initial Strategy; not re-attacking the question with the same kind of strategy). Bergqvist (2007) found that about 70% of the questions included in university mathematics examinations could be successfully dealt with *imitative reasoning* (Lithner, 2008). ‘Imitative reasoning’ can be linked with Weber’s ‘procedural’ and/or ‘syntactic’ strategies, thus it is reasonable to conjecture that students may employ such strategies when they deal with exam-type questions. Moreover, it seems reasonable to expect that students with a preference for Type II thinking styles

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[2]Succinctly: 1) *functions* (‘legislative’, preference for creativity; ‘executive’, preference for implementing rules, ‘judicial’, preference for judging), 2) *forms* (‘monarchic’, preference for focusing on only one goal; ‘hierarchic’, preference for having multiple prioritised objectives, ‘oligarchic’, preference for having multiple equally important targets, ‘anarchic’, preference for flexibility), 3) *levels* (‘local’, preference for details and the concrete; ‘global’, preference for the general and the abstract), 4) *leanings* (‘liberal’, preference for originality; ‘conservative’, preference for conformity), and 5) *scope* (‘internal’, preference for working alone; ‘external’, preference for working in a group).

(linked with conformity) would choose more ‘procedural’ or ‘syntactic’ strategies when dealing with exam-type questions (Bergqvist, 2007) than the students with a preference for Type I thinking styles (linked with creativity). Finally, drawing from previous studies (for example, Zhang & Sternberg, 2006), students’ attainment and the nature of task were also considered in this study.

THE STERNBERG-WAGNER THINKING STYLES INVENTORY (TSI)

A translated to Greek version of the Sternberg-Wagner Thinking Styles Inventory (TSI; Sternberg, 1999) was used for the identification of the students’ thinking styles. TSI is a self-report, paper-and-pencil test, consisting of 104 seven-scale Likert type items (8 for each style) asking students to indicate a range of individual preferences, for example: “I like problems, where I can try my own way of solving them” (‘legislative’; Sternberg, 1999, p. 28), “I like projects that have a clear structure and a set plan and goal” (‘executive’; *ibid*, p. 33). Each participant’s preference for a style is labelled (six labels ranging from ‘very low’ to ‘very high’) according to the norms developed by Sternberg’s research. Although TSI has shown good cross-cultural validity (Zhang & Sternberg, 2006), there does not appear to be an existing norm for the Greek population. Therefore, in a previous study (Moutsios-Rentzos & Simpson, 2005), the participants’ scores were also labelled according to ‘adjusted norm’ produced from the data of this population following Sternberg’s (1999) process. The latter norm serves as a ‘tighter lens’, which helps in spotting intra-population differences. In this study, style identification was based on the adjusted norms (as they did not contradict with Sternberg’s norms).

METHODOLOGY

The design of the study included questionnaires and interviews whilst, although the focus was on the 2nd year students (99 students: 45 males and 54 females) following a BSc in Mathematics in a Greek University, the questionnaires were also administered to a broader sample of undergraduates (224 students: 112 males and 112 females), in order to validate the translated to Greek TSI (‘t-TSI’) and the ‘adjusted’ norms. 15 second year students were interviewed in order to determine their strategies. In sum, the design of the study included the identification: 1) of the students’ thinking styles, 2) of clusters of students with similar style profiles, 3) of suitable representatives of these clusters, and 4) of the strategy choice(s) of those representatives.

The questionnaires included the t-TSI and questions about the students’ age, gender, year group and attainment. The interviews –two for each interviewee– were designed to last from 40 to 60 minutes each and were video recorded. Six questions in exam-type format were included (see Table 1) from courses in analysis and algebra. The first four questions in Table 1 were *common* questions that students may encounter in exams, while the last two were *non-common*. The interviewees were asked to ‘think aloud’ and to provide exam-type answers, whilst it was stressed that the researcher’s focus was on the strategies they used to solve the questions. Drawing from Weber’s

First Interview	Let a, b, c $\in$Z and (a,b)=1 and a|bc. Prove that a|c.
	Let a sequence $(a_n) \in$R, n$\in$N. Prove that if (a_n) is convergent, then (a_n) is bounded.
	Let A, B non-empty subsets of the real numbers R and A, B are bounded. Does the sup(A$\cup$B) exist? If yes, find it. Justify in full your answer.
Second Interview	Let G = <a>, cyclic, finite group, rank n. Prove that a^κ, $\kappa\in$Z, is generator of G, if and only if $(\kappa, n)=1$.
	Let a$\in$N. Prove that a is divisible by 9, if and only if the sum of its digits is divisible by 9.
	Let f: R$\to$R and f periodic with period T>0. If $\lim_{x\to\infty} f(x) = b\in$R, then prove that f is constant.

Table 1: The exam-type questions.

(2001) technique, the interviewees were provided with any mathematical knowledge they asked for, but not the answer. Thus, the focus was on examining the students' *manner of accessing* mathematical knowledge, since it was hypothesised that this would help in revealing the students' strategy choices. The students' strategies were *initially* identified according to Weber (2005). The purpose was to examine whether or not this classification could sufficiently describe the students' strategies and, if not, to identify any additional 'missing' categories.

THE TRANSLATED TSI AND STYLE CLUSTERS

The data analysis suggested the sufficient validity and reliability of t-TSI[3].

As a result of data analysis two style *Cores* (akin to the Types found by Zhang & Sternberg, 2006) were identified: *Core I* thinking styles (creative, original, critical and non-prioritised thinking) and *Core II* thinking styles (procedural, already tested and prioritised thinking). From these, two clusters were identified: *Cluster* 1_C2_C (high Core I and/or low Core II) and *Cluster* 3_C4_C (high Core II and/or low Core I). Interviews were carried out with those students who were 'closer' to the centre of the cluster to which they were assigned.

THE A-B-Δ STRATEGY CLASSIFICATION

The data analysis suggested that the students' strategies differed in whether the students explored the 'truth' of the statement they were asked to prove (*truth development*) or not (*proof development*). Following this differentiation, the *A-B-Δ strategy classification*, was introduced, expanding on Weber's (2005) 'semantic', 'syntactic' and procedural'. Overall, the students appeared to employ five strategies when they dealt with an exam type question (see Table 2): a) the *Alpha strategy* ('A'; akin to 'semantic'), b) the *Beta strategy* ('B'; akin to 'syntactic'), c) the *Delta-Beta* strategy ('Δ_B'; akin to 'procedural'), d) the *Delta-Alpha* strategy ('Δ_A'; different arguments employed for 'ascertaining' and for 'persuading'; in the sense of Harel &

[3] The alpha coefficients for 8 of the 13 measured styles were above 0.7. Three style scales were less reliable (but over 0.53), which, still, is in accordance with previous studies (Zhang & Sternberg, 2006). Principal axis factoring (oblimin with Kaiser normalisation) led to a 3-factor solution (63.8% of variance). The first factor is related to creative, original, critical and non-prioritised thinking, the second factor is linked to procedural, already tested and prioritised thinking and the third factor embodies the 'scope' dimension of MSG ('internal'-'external').

	Strategy Types		
	α-type ‘truth’	β-type ‘memory’	δ-type ‘flexibility’
A-B-Δ	A ‘semantic’		
		B ‘procedural’	
	Δ_A -	Δ_B ‘syntactic’	Δ_Δ -

Table 2: The A-B-Δ classification

Sowder (1998), and e) the *Delta-Delta* strategy (‘Δ_Δ’;΄flexible). The A-B-Δ strategies can be grouped into three *Strategy Types* depending on their links with *truth*, *memory* or *flexibility* (see Table 2): *α-type* (‘truth’; Alpha & Δ_A), *β-type* (‘memory’; Beta & Δ_B), and *δ-type* (‘flexibility’; Δ_Δ). Finally, the students’ strategies were independently identified according to the A-B-Δ classification by a colleague. The inter-rater reliability analysis suggests the ‘almost perfect’ agreement (Kappa=0.924, p<.001) between the two raters.

LINKING STYLE PROFILES WITH STRATEGY CHOICES

The typical representative of Cluster 1_C2_C (↑Core I/↓Core II) preferred α-type Initial Strategies, rarely resorted to Back-Up Strategies, and chose β-type strategies for ‘known’ questions. In contrast, the typical representative of Cluster 3_C4_C (↑Core II/↓Core I) preferred β-type Initial Strategies, resorted more frequently to Back-Up Strategies, and had a low preference for α-type Initial Strategies.

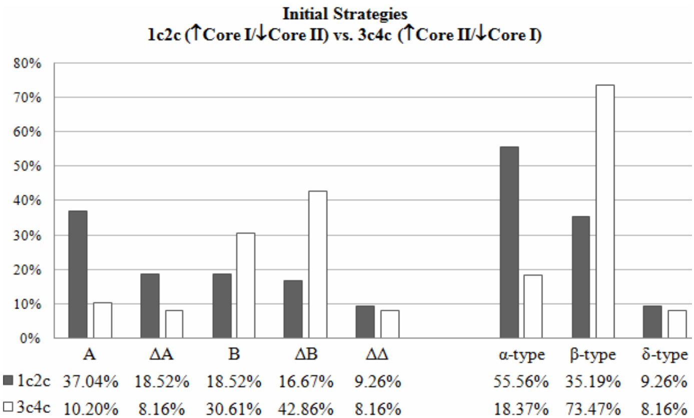

Figure 1: Inter-Cluster Initial Strategy Comparison

More than half (55.56%) of the Initial Strategies employed by the members of 1_C2_C were α-type (see Figure 1), whereas almost three quarters (73.47%) of the Initial Strategies employed by the members of 3_C4_C were β-type. The analysis (Mann-Whitney) suggested that the members of 1_C2_C employed significantly more α-type strategies (U = 6.5, p<0.05, r = -.66) and significantly less β-type strategies (U = 2, p<0.05, r = -.79) than the members of 3_C4_C. Moreover, the members of 1_C2_C in comparison with the members of 3_C4_C chose significantly more (U = 7.5, p<0.05, r = -.63) Alpha strategies (37.04% vs. 10.20%) and significantly less (U = 6.5, p<0.05, r = -.55) Δ_B strategies (42.86% vs. 16.67%).

Although no significant differences were found in the use of Back-Up Strategies, the students assigned to 3_C4_C appeared to employ more than twice (17 vs. 7) Back-Up Strategies than those assigned to 1_C2_C.

In the context of the nature of task, the statistical analysis suggested that the members of Cluster 1_C2_C appeared to employ significantly different (p<0.05, Fisher's exact test) Initial Strategies (more α-type and less β-type) to those used by the members of Cluster 3_C4_C in the two 'non-common' questions: the 'divisible by 9' and the 'periodic-constant'. Furthermore, some questions appeared to attract specific strategies (e.g. the 'convergent-bounded' question attracted more β-type strategies).

It seems that there is a link between the Initial Strategies that the students employ and the cluster to which they are assigned: the members of Clusters 1_C2_C seemed to prefer more α-type and less β-type Initial Strategies than the members of Cluster 3_C4_C (see Figure 2). For the 'common' tasks, this link is 'skewed' depending on whether the task is expected to favour α-type or β-type strategies. For 'non-common' tasks, this link is amplified (which explains the statistically significant identified strategy differences). It is conjectured that the fact that these tasks are 'non-common' reduces the exam (and/or university) effect on the students' way of thinking, allowing for their thinking styles to be more dominant, leading to the amplification of the contrast between the strategy choices made by the members of the two style clusters.

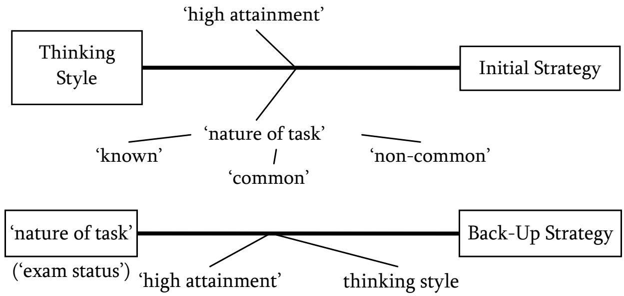

Figure 2: Linking thinking styles with strategies.

It appears that the effect of style on Back-Up Strategy is minimised, mainly affecting the frequency of the students' resorting to a Back-Up Strategy (for example, 3_C4_C is linked with higher preference for Back-Up Strategy). It is conjectured that the students' need to successfully survive the exam type situation overrides their Initial Strategy preference and they resort to β-type or δ-type strategies, which seem more 'appropriate' for exams (Bergqvist, 2007). Furthermore, although the members of both clusters choose similar Back-Up Strategies, the 'high' attaining members of 3_C4_C choose twice as many Back-Up Strategies. Consequently, it is argued that stylistic preferences and 'high' attainment, appear to regulate a link between nature of task and Back-Up Strategy (see Figure 2), rather than forming a style-strategy link (as in the case of Initial Strategy).

CONCLUSION: VIEWING THE RESULTS THROUGH SKEMP'S THEORY

According to Skemp (1979), students' survival is realised in three modes: *actual*, *social* and *internal*. In this study, Skemp's *social survival* and *internal consistency* highlight the duality of the students' goal setting and goal achieving when they deal with a mathematical question, thus helping to explain the links between style, Initial Strategy and Back-Up Strategy as outlined in Figure 2.

For Cluster 1_C2_C (↑Core I/↓Core II), students' need for internal consistency leads them to choose an Initial Strategy that can be linked to creative, original, critical and non-prioritised thinking. In an exam type question, this appears to translate into a strategy that incorporates the exploration of whether or not the given statement is 'true', whether or not it 'makes sense' and thus the selection of α-type strategies. Such a choice allows the students to be critical of the validity of the given statement, which can also help them in being creative and original (in their search for an ascertaining argument). On the other hand, the students' need to socially survive leads them to choose a persuading argument that is either the ascertaining argument presented in a mathematically acceptable manner (Alpha strategy) or a completely new argument (Δ_A strategy).

In contrast, for Cluster 3_C4_C (↑Core I/↓Core II), students' need for internal consistency leads them to choose an Initial Strategy that can be linked to procedural, already tested and prioritised thinking. In an exam type question, this appears to translate into a strategy that draws from memory, either in the form of reproduction of an answer or in the form of remembering certain techniques that help in answering exam type questions. Furthermore, this choice is also in accordance with the social survival of an exam type question. Therefore, these students appear to prefer a β-type Initial Strategy as it incorporates these elements, thus allowing those students to be consistent with both their inner and social reality.

The situation radically changes when the students search for a Back-Up strategy. The failure of the students' Initial Strategy leads them to re-evaluate the task itself, thus changing the realities in which the students have to survive. When employing a Back-Up Strategy, the students view the question 'stripped' of its multiple dimensions, being projected only on the 'exam-type status' space, affecting the students' inner reality: internal consistency is now mainly linked to the students' perception of an exam-type question, thus minimising the role of style. Therefore, the students would search for β-type or δ-type strategies, as they are considered to be more 'suitable' for such questions (Bergqvist, 2007). Therefore, for both clusters, the students choose β-type and/or δ-type Back-Up Strategies, in order to satisfy their need for internal consistency and social survival. Overall, Skemp's theory suggests that the students choose different strategies, because they survive different (perceived) situations.

In conclusion, 'thinking styles' seem to be useful for the identification of students who employ qualitatively different strategies when they deal with exam-type questions. It is argued that these students survive different realities and, therefore, they need different pedagogical treatment. Further research could focus on designing appropriate pedagogies.

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CHALLENGING "THE LAWS OF MATH"

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This paper reports on the strategies chosen by a group of sixth grade students in an urban informal learning program as they worked to solve an open-ended, non-routine task. In particular, the paper focuses on the ability of these students to rise above their previous, procedure-based misconceptions and arrive at a mathematically reasonable solution. Factors in the problem task and the problem-solving environment are analyzed to determine the conditions that encouraged students to approach mathematics as a logical, meaningful, sense-making activity.

INTRODUCTION

The Principles and Standards for School Mathematics (NCTM, 2000) stress the role of reasoning and proof in the curriculum. However, students' misconceptions often hinder their ability to reason. When these misconceptions prevent students from reasoning correctly, students must be open to adapt their schemas to accommodate their new understandings. With support, students can overcome these mental roadblocks by building alternative representations and by sharing and discussing their new ways of thinking. In this paper, we share an episode from an after-school mathematics program where a group of students were prompted to rethink what they know about fraction "rules". They did this by building their own evidence and convincing themselves and others to believe in their power to reason.

THEORETICAL FRAMEWORK

Davis (1992) asserts that, given opportunities, students will create their own ways of understanding and build representations and understanding based on their previous knowledge and experiences. However, Davis points out that what students learn is built upon this foundation of understanding and therefore future learning may be limited by previous understanding.

Often, the mathematical instruction in schools does not validate children's natural, experience-based understandings; instead, it requires students to adapt their reasoning styles to fit those valued by schools (Malloy, 1999). The traditional approach to teaching mathematical concepts emphasizes students' memorization of rules and procedures and manipulation of symbols. Many of these rules may seem meaningless to children, having been learned by rote methods (Davis 1994). Erlwanger (1973) reports the case of Benny, a twelve-year-old boy in the sixth grade using Individually Prescribed Instruction (IPI), who learned that mathematics consists of different rules for different problems that were invented at one time but work like magic. In Benny's

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eyes, mathematics was not a rational and logical subject where one has to reason, analyse, seek relationships, make generalizations, and verify answers; rather, it was a game where one discovers the rules and uses them to solve problems (Erlwanger, 1973). Benny created his own "rules" for adding fractions based on what he perceived as random procedures. Kamii and Diminck (1998) argue that teaching rules and conventions can be harmful because they cause children to relinquish their own ideas and disconnect the content from the concepts. When exposed to this kind of instruction, students often remember erroneous rules and procedures, as was seen with Benny. Kamii and Warrington (1999) propose that the focus of instruction should shift from teaching that emphasizes physical and social knowledge to that which values and encourages children's own reasoning.

Yackel and Hanna (2003) concur and argue the view of mathematics as reasoning can be contrasted with the view of mathematics as a rule-oriented activity. Other researchers support the fact that the sole teaching of algorithms can be detrimental and counterproductive to the development of children's numerical reasoning. Mack (1990) came to this conclusion after finding that algorithms often keep students from even trying to use their own reasoning. Through her work with eight sixth-grade students, she also found that students often remember erroneous algorithms and have more faith in these rules than in their own thinking.

According to Skemp (1971) "…to understand something is to assimilate it into an appropriate schema" (p. 45). Therefore, a student's level of understanding is dependent upon the schema he or she has created during instruction. Understanding develops as students form connection between new and old knowledge and create appropriate schemas to make sense of new knowledge. These schemas are built on previous understanding as students make connections between schemas. Often, students run into roadblocks that they must overcome through building alternative representations and with the sharing of ideas. Students exhibit logical understanding when afforded the opportunity to justify their reasoning in a community of learners and thus are able to adapt previous [mis]understandings/beliefs.

METHODOLOGY AND DATA SOURCE

This research is a component of a larger ongoing longitudinal study, Informal Mathematics Learning Project, (IML)[1] that was conducted as part of an after-school partnership between a state university and a school district located in an economically depressed, urban area. The district's student population consists of 98 percent African

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American and Latino students. Our study focuses on the development of reasoning of middle-school students. We report on the first cohort of students, 24 sixth-graders, who, over five, 60-75 minute sessions, worked on fraction tasks, interacted with peers, and had ample time to explore, discuss and explain their ideas. Cuisenaire rods were made available to the students as they worked on the tasks. The students worked in groups of four and participated in whole class discussions. They were invited to collaborate and discuss their ideas with one another, were encouraged to justify and make sense of their solutions, and were challenged to convince one another of the validity of their reasoning.

Video recordings and transcripts were analysed using the analytical model outlined by Powell, Francisco & Maher (2003). The video data were described at frequent intervals; critical events (episodes of reasoning) were identified and transcribed, codes were developed for flagging for solutions offered by students and the justifications given to support these solutions. Arguments and justifications were coded according to the form of reasoning being used, direct or indirect, and as valid or invalid, based on whether or not the argument started with appropriate premises and the deductions within the argument were a valid consequence of previous assertions. Students' construction of solutions and their subsequent justifications were then traced across the data in an effort to document and analyse their journey to mathematical understanding.

RESULTS

During the third session of the after-school program, the blue rod was given the number name one and students were asked to give a number name for the white and red rods. The task was then revisited at the beginning of the fourth session. After students worked in small groups and then shared results with the larger community, they were asked to give a number name to the yellow rod, when blue was named one.

A Group of Three: Chanel, Dante and Michael

Chanel lined up five white rods next to the yellow rod and used direct reasoning to name the yellow rod five-ninths. She then initiated the task of naming all of the rods using the staircase model (see Figure 1). She named the remainder of the rods, using direct reasoning based on the incremental increase of one white rod or one-ninth and used the staircase model as a guide and named the rods until she arrived at the orange rod. As she was working she said the names of all of the rods, "One-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths, ten..– wow, oh, I gotta think about that one, nine-tenths". Chanel showed Dante her strategy of using the staircase to name the rods and explained the dilemma of naming the orange rod to Dante, "See this is One-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths - what's this one?". Dante replied, "That would be ten-ninths. Actually that should be one. That would start the new one". He initially named the orange rod ten-ninths but corrected himself and said that the orange rod would "start a new cycle"; and named it one-tenth. Michael named the orange rod a whole and explained that it was equivalent to ten white rods and Chanel agreed.

Chanel: It should be called a whole.

Dante: This is one, this is nine-ninths also known as one. This should be blue and this would start the new one – would be one-tenth.

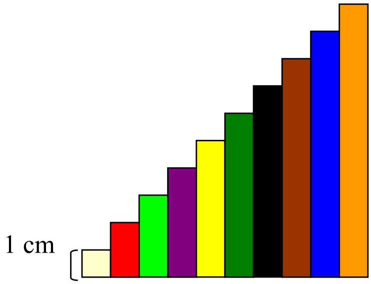

Figure 1. “Staircase” Model of Cuisenaire Rods.

After students worked for about five minutes drawing rod models, Dante told the group that that he heard another group calling the orange rod ten-ninths.

Dante: Why are they calling it ten-ninths and [it] ends at ninths?

Michael: Not the orange one. The orange one’s a whole.

Dante: But I’m hearing from the other group from over here, they calling it ten-ninths.

Michael: Don’t listen to them! The orange one is a whole because it takes ten of these to make one.

Dante: I’m hearing it because they speaking out loud. They’re calling it ten-ninths

Michael: They might be wrong! …

Chanel: Let me tell you something, how can they call it ten-ninths if the denominator is smaller than the numerator?

Dante: Yeah how is the numerator bigger than the denominator? It ends at the denominator and starts a new one. See you making me lose my brain.

As the students were working a researcher joined their group. Dante shared his conjecture, “It’s the end of it and it starts the new one to one-tenth because the blue ends it and so the orange starts a new one just like - pretend there were smaller ones than just a white. So this would be considered like blue, a one”. The researcher reminded him that the white rod was named one-ninth and that this fact could not change. Again she asked him for the name of the blue rod and he stated, “It would probably be ten-ninths”. When prompted, Dante explained that the length of ten white rods was equivalent to the length of an orange rod. The researcher asked Dante to persuade his partners.

Chanel:	No, because I don’t believe you because –
Michael:	I thought it was a whole.
Dante:	But how can the numerator be bigger than the denominator?
R1:	It can. It is. This is an example of where the numerator is bigger than the denominator.
Chanel:	But the denominator can’t be bigger than the numerator, I thought.
Michael:	That’s the law of facts.
R1:	Who told you that?
Chanel:	My teacher.
Dante:	One of our teachers
Michael:	That’s the law of math.

In the above dialogue, we see that even though Dante named the orange rod ten-ninths, using previous knowledge (of the name of the white rod) and a concrete model, he still questioned his answer. His prior understanding of the “rule” was so strong that he questioned himself even after building a concrete model and explaining the concept.

The Whole Class

At the end of the session groups were asked to share their results with the class. One of the students shared the groups’ solution with the class: “We had a challenge that says if we call the blue rod one, what do we call the orange rod?” Students were asked to share their results. Malika and Lorrin named the orange rod ten-ninths and reported that they initially thought the numerator could not be larger than the denominator.

Lorrin	Because, before, we thought that because we knew that the numerator would be larger than the denominator and we thought that the denominator always had to be larger but we found out that that was not true. Because two yellow rods equal five-ninths, and five-ninths plus five-ninths equal ten-ninths
Malika	We found out the denominator doesn’t have to be larger than the numerator because we found out that two yellows equal five-ninths so five-ninths plus five-ninths equals ten-ninths.

Kia-Lyn and Kori explained that the blue rod had two number names.

Kia-Lynn	We found that the blue rod has two number names and the orange one has two number names. So because the orange one and the blue one – I

thought that – our group had found out - that the orange is bigger than the blue one but when you add a one-ninth, a white rod, to the blue top it kind of matches. It kind of matches and we found out that you can also call the blue rod one and one-ninth and the orange one, without the one-ninth, without the white rod, is also called one-ninth, too.

Kori So we were saying that if this [orange] is called one -

Kia-Lynn It's also called one – um ten-ninths as Malika and Lorrin had said. But if you have one…white rod and you add it to the blue, it's one-ninth plus one is one and one-ninth and so if the blue rod and one white [they are using overhead rods to show a train of the blue rod and a white rod lined up next to an orange rod]. If you put them together then this means that it's ten-ninths also known as one and one-ninth.

Finally, Dante presented his strategy:

Dante Well all I did was start from the beginning – start from the white – and you and all the way to the orange and what – like Kia-Lynn's group just said - I had found a different way to do it. Because all I - I had used an orange, two purples, and a red and since these two are purple and this is supposed to be purple but I had purple and I used a red since four and four are eight so which will make it eight-ninths right here and then plus two to make it ten-ninths. [He builds this model on the OH] That's what I made.

R2 So it's another way of showing that orange is equivalent to ten-ninths?

Dante Um hum. And then I just did it in order – then the one I did right here – I just did it in order of whites by doing ten whites. [he shows the model lined up next to ten white rods]

DISCUSSION AND IMPLICATIONS

Specific factors in the after-school session enabled the students to challenge and revise their ways of thinking about mathematics. These factors include, but are not limited to, the following: challenging, open-ended tasks that invite students to extend their learning as they build and justify solutions, the promotion of student collaboration in small groups and the opportunity to share ideas in the whole class forum, the portrayal of student as determinant of what makes sense, strategic teacher questioning, and the opportunity to build models using concrete materials.

The tasks were open-ended such that students could expand on a given task, as Chanel worked to name all the rods by using her staircase model. In addition, the students were provided tools to build models and therefore they could conceptualize the fraction relationships. The Cuisenaire rods offered a concrete, visual model of ten-ninths and the students were thereby provided the means to show, using concrete evidence, that this fraction did indeed exist. Further, the physical environment promoted student collaboration and it was further encouraged by the researchers asking students to listen to each other's ideas and to judge the merit of each others justifications.

Importantly, the researcher's careful questioning prompted students to explain their reasoning and invite their classmates to evaluate their thinking. When Chanel first grappled with naming the orange rod the researcher suggested she share her dilemma with Dante. After being afforded more time to think about the task, Dante was asked to explain his thinking. Rather than correcting Dante, the researcher reminded him of the facts that were already established (the white rod was named one-ninth). This subtle prompt enabled him to revise his thinking through the use of his own reasoning. Dante was then asked by the researcher to explain his thinking and convince his partners that his reasoning was correct. By working to convince his partners, Dante was able to reaffirm his reasoning and further convince himself of its validity.

Dante was further encouraged to have confidence in his own thinking during the second phase of the activity. After students were provided the opportunity to explain their thinking and discuss their ideas in their small groups, they participated in a whole class discussion, providing the opportunity for them to validate their ways of reasoning about the problem. Further, the arguments presented by others introduced them to alternative models and justifications. Although, with his partners Dante used the staircase model to incrementally increase the names of the rods by one-ninth, in his presentation he chose a different representation. After viewing the other presenters and listening to their presentations his thinking was validated and thus he expressed confidence in his solution. This confidence led him to show two alternative models for naming the orange rod.

Malika and Lorrin shared that they also previously believed that the numerator of a fraction could not be larger than the denominator; however, their reasoning and concrete evidence to show that five-ninths plus five-ninths is equivalent to ten-ninths was a stronger influence on their ultimate decision. In an environment that encourages reasoning, these students learned to trust their own logical ability and were thereby able to challenge and rethink their earlier understanding.

The nature of the tasks and the time allotted for exploration allowed students to work at their own pace and readiness level. As the other students grappled with convincing themselves that ten-ninths was a viable number name, Kia-Lyn and Kori took the task to the next level and showed that ten-ninths is equivalent to one and one-ninth. In an environment that allowed students to act as teachers and present new concepts to their peers, the students were exposed to new ideas in a manner that was conducive to their assimilation.

When students are encouraged to reason and become members of a community of engaged, active learners, they are able to exhibit understanding and build trust in their own thinking. Implementation of conditions of learning similar to those described here may be the critical approach that can enable all students to reason and to build the true understanding and reasoning that is the goal of all mathematics learning.

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INVESTIGATING TEACHERS' USE OF QUESTIONS IN THE MATHEMATICS CLASSROOM

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As part of a study investigating effective numeracy practices, the types of questions asked by teachers during a series of mathematics lessons were examined. The findings indicated that the types of questions teachers asked influenced the nature of the students' responses and that probing questions were particularly utilised to encourage student explanations.

BACKGROUND TO THE STUDY

A significant amount of Australian mathematical research in the past decade has focused on teachers and their classroom activity, with the most common theme being characteristics of effective teachers (Groves, Mouseley & Forgasz, 2006). Extensive research in this area has also been conducted in the UK (e.g., Askew, Brown, Rhodes, Johnson & Wiliam, 1997) and New Zealand (Anthony & Walshaw, 2007). Among other factors, these studies all identified the importance of teachers' questioning and the expectation that students explain and justify their answers. Although some of the reports of these studies have involved illustrative examples of some of the exchanges that occurred between teachers and students, there have been surprisingly few research studies into the issues of classroom discussion and questioning (Groves, et al., 2006). This paper addresses the gap in the literature in this area through providing qualitative examples of the nature of teachers' questioning in three Tasmanian upper primary classrooms and their relative effectiveness in eliciting explanations and justifications from students.

Probing and challenging students' thinking

There is widespread agreement in the literature that effective teachers consistently require students to explain their mathematical thinking and ideas (e.g., Askew et al., 1997; Clarke et al., 2002; Groves et al., 2006; Reynolds & Muijs, 1999) and some studies have examined the types of questions asked. For example, Hardman, Smith, Mroz & Wall (2003) documented the occurrence of probing questions (where the teacher stayed with the same student to ask further questions) and uptake questions (where the teacher incorporated a student's answer into a subsequent question). Their findings indicated that although the use of these questions had the potential to facilitate purposeful discussion, the questions used by the teachers in their study rarely went beyond the recall and clarification of information, indicating that it is the quality of the follow-up move made by the teacher, rather than the questions themselves that facilitates a more interactive learning environment (Kyriacou &

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 161-168. Thessaloniki, Greece: PME.

Issitt, 2008). Similarly, Tanner, Jones, Kennewell, and Beauchamp (2005) found that the use of questioning to scaffold students' learning is under-exploited and that only about 25% of questions asked by the teachers in their study actually encouraged students to think more deeply about their ideas.

Types of questions

In addition to probe and uptake questions, distinction has been made in the literature between the use of open-ended and closed questions (Sullivan & Lilburn, 2004). Although closed questions simply require an answer or response, usually given from memory, open-ended questions tend to require a student to think more deeply and to give a response that involves more than just recalling a fact or reproducing a skill.

Student responses

In one of the few studies that documented student responses, Wood (2002) found that differences in students' thinking and reasoning could be attributed to the type of questions asked, and that the detail of students' descriptions varied depending on the extent to which the teachers demanded comprehensiveness and clarity through their questioning. A lack of comparable research, however, indicates that while there has been a focus on 'good questions', it is perhaps more difficult to define what constitutes good answers to these questions. Still less research has been conducted into documenting extended exchanges used by teachers. Over 30 years ago, Gall (1970) argued that follow-up questioning of the student's initial response, such as through the use of probing and uptake questions (Hardman et al., 2003) has substantial impact on student learning and that more research needs to be undertaken in this area.

METHODOLOGY

Participants and procedure

Three teachers were chosen for the study using purposive and opportune sampling (Burns, 2000). The teachers all taught upper primary grades and in primary schools that were geographically similar and had classes of similar size. The teachers selected were all highly regarded by their principals as being 'good practitioners' in terms of having established positive relationships with their students and effective behaviour management and organisational practices (not necessarily particularly effective teachers of mathematics).

The researcher observed between four and seven mathematics lessons for each of the three teachers, with a total of 17 lessons being observed overall. The introductory and plenary sessions of the lessons where exchanges occurred between the teacher and the students as a whole class were videotaped, field notes were taken and student work samples collected. Each case study teacher conducted a sequence of lessons on a particular mathematics topic and all data were collected for one teacher before the next set of observations occurred. The video footage was fully transcribed within hours of observation and the footage was viewed collaboratively with the individual teachers. This allowed for discussion to occur about the lesson and for the researcher to clarify observations made (these discussions were audio-taped and were integral to a second part of the study which focused on teachers' self reflection, but which is beyond the scope of this paper).

Data analysis

Data analysis commenced during the data collection process and units of analysis were created through ascribing codes to the data (Miles & Huberman, 1984). The transcript of the lessons were analysed and every verbal exchange was assigned a code; frequency counts of each type of question and response were recorded. Table 1 shows the categories that were used to classify the questions asked by the teachers. Although some probing questions could be further classified as either open or closed, if they were asked of the same student, then they were counted as probes only. The categories were derived from recommendations found in the literature (e.g., Gall, 1970; Sullivan & Lilburn, 2004) regarding the effective use of particular types of questions.

Type of question	Explanation	Illustrative example
Open 1	Requires explanation (may begin with how/why?)	"How will that raise us money?" [John, lesson 2]
Open 2	Requires justification, generalisation or seeking of alternatives (may begin with what if?	"What if the number didn't fit evenly? What would we do then?" [John, lesson 1]
Closed	Allows for only one acceptable answer	"Who can tell me what a half would be as a percentage?" [Ronald, lesson 2]
Non-mathematical	Generic question	"Who got a similar answer?"
Probe	Teacher stays with the same child to ask further questions	"Why do you think there are two combinations Abbie?" [Sue, lesson 1]
Uptake	Teacher incorporates student's answer into a subsequent question	(Asks class) "Do you think there would be more than 45?" [Sue, lesson 3]

Table 1: Classification of questions

Students' responses to the questions asked by the teachers were also analysed and the following categories were used to classify their responses: explanation, sharing, justification, question, challenge, answer/response. Explanations differed from sharing in that students were required to explain their answer or strategy and justifications referred to instances where students were required to elaborate on their explanation and usually occurred in response to a probing question. The challenge category was derived in response to situations whereby students questioned or challenged the answer or method proposed. For example:

> Lauren: If you weren't saying how many you surveyed, it would still be the same wouldn't it? Like 50% could have been out of 10 000 [people] or out of 100 [people].

The answer/response category referred to situations whereby students provided brief short responses or answers and were typically three words or less.

RESULTS

Types of questions asked

For every part of the lessons that were videotaped for each teacher a summary of the types of questions asked and student responses was recorded. Table 2 provides an overview of this data in relation to open, closed and probe questions and the frequency of students' explanations or use of answer/responses. The table shows that the majority of questions asked by Sue and John were classified as probes, whereas Ronald asked more closed questions than other question types. In contrast to Hardman et al.'s (2003) findings that teachers used probing questions in only 11% of questioning exchanges, the table shows more frequent use of this type of question. The teachers in this study also used more open questions than the 10% recorded in Hardman et al.'s (2003) study.

Teacher	Question types			Student responses	
	Open	*Closed*	*Probe*	*Explanation*	*Response/Answer*
Sue	27%	28%	45%	44%	34%
John	23%	31%	46%	19%	54%
Ronald	16%	48%	36%	18%	72%

Table 2: Percentage of question types asked and student responses received by each teacher

In order to determine the types of questions that were likely to produce explanations, the data were further analysed. Table 3 shows that open questions produced more explanations than any other type of question, supporting the finding that open questions require more than recalling a fact or reproducing a skill (e.g., Sullivan & Lilburn, 2004). Ronald's frequent use of closed questions resulted in only 18% of student answers being classified as explanations. From the data it would appear that open questions have the potential to avoid the tendency of brief student exchanges with fewer than three words (Smith et al., 2004, as cited in Tanner et al., 2005) and to thereby facilitate purposeful discussion.

Teacher	Question type			
	Open	**Probe (open)**	**Closed**	**Probe (closed)**
Sue	57	37	3	3
John	36	41	9	14
Ronald	45	29	16	10

Table 3: Question types that produced explanations

Probing questions

All three teachers made use of probing questions in their exchanges with students. The frequency of this varied between teachers, but it was notable that the use of probing questions increased when the teachers conducted their ‘whole class’ discussions with small groups. For example, in the fifth lesson observed for John, nearly 70% of all questions asked by him were classified as probes. This lesson was characterised by two separate teaching groups that provided an enhanced opportunity to conduct extended exchanges with individuals.

An examination of the probing exchanges conducted by Sue, John and Ronald revealed that overall, probing questions were used in four main ways: to probe further into students’ thinking, to encourage students to move beyond explanation to justification, as a scaffold to facilitate students’ conceptual understanding and as a means to help other students understand or follow a particular student’s strategy. For example, Sue often used probes in response to students who provided one-word answers or needed encouragement to expand on their answers. Probing questions such as “Why do you think 10 Randall?” and “OK, you got 7. Would you like to explain what you did?”, encouraged students to articulate their thinking and assisted in promoting purposeful discussion. The following exchange, which occurred in a discussion about surveys and sample size, illustrates how probes were also used to encourage students to move beyond explanations to justifications:

> Teacher: If I asked 10 people, 100 people … 1000 people or 10 000 people, which one is going to be the best survey do you think?
>
> Brad: 100
>
> Teacher: Why do you think that?
>
> Brad: Because it seems like the percentage is out of 100, so it kind of sounds like out of a hundred
>
> Sebastian: 10 000
>
> Teacher: Why do you think 10 000?
>
> Sebastian: Because if you ask 100 people, you could have asked people that just liked one thing; the more people you ask the more even it’s going to be. You might 100 people

> who all like KFC, but the rest of the world don't like KFC, so the survey of 100 might say 40%, but it's really more like 3% who like KFC.

Ronald in particular made use of probing questions to expose other students to different ways of thinking. These exchanges were mainly conducted with Sebastian, a very capable student who frequently demonstrated higher order thinking and was adept at explaining the mathematics he used:

> Teacher: OK, is there anything else we could have converted those into?
>
> Sebastian: Fractions, but that would have been very hard.
>
> Teacher: Fractions, why would that have been hard?
>
> Sebastian: Because once you've converted them, they're still different, like a third, and a quarter, and you actually want them the same, so it would be easier with decimals.
>
> Teacher: Yes, so it would be easier to convert them back to decimals wouldn't it?
>
> Sebastian: Yes because they're pretty much the same as percentages, like .75 is 75%.

When probing questions were used as a scaffold to facilitate students' understanding they assumed more of a closed nature and resulted in primarily answer/responses. This was often appropriate, however, in the context of the lesson and served to maintain the focus on the particular mathematical concept being taught. The following excerpt taken from one of John's lessons on volume demonstrates how probing questions were used as a scaffold to further one student's understanding of how the formula for volume is derived:

> Teacher: Explain yours to me would you Cameron?
>
> Cameron: There's 2 rows of 15 which equals 30 cubic centimetres.
>
> Teacher: So how long is your rectangular prism?
>
> Cameron: 15 centimetres.
>
> Teacher: 15 centimetres long. How wide?
>
> Cameron: 2 centimetres.
>
> Teacher: 2 centimetres wide. How high?
>
> Cameron: 1 centimetre.
>
> Teacher: 1 centimetre – remember what we said last week – if we multiply the length by the width by the height we would get how many were in the shape.
>
> Cameron: 2 times 15 is 30.
>
> Teacher: 15 times 2 is 30. Times 1?
>
> Cameron: Is 30.

The above exchange illustrates how probing questions, even when they are essentially closed in nature, can be used to scaffold a student's understanding. It also demonstrates the importance of using qualitative data to further inform findings derived from quantitative data. In this instance a high frequency count of closed questions did not necessarily indicate that purposeful discussion and valuable communication did not occur.

DISCUSSION AND CONCLUSIONS

One feature that was evident from all three teachers' questioning practice was the promotion of a positive classroom environment. The teachers regularly used positive reinforcement and encouraged students to participate in the discussions. In all three classrooms the students appeared comfortable with being asked to explain their thinking and willing to share their ideas. The teachers' practices reflected that at least part of their numeracy teaching was "based on dialogue between teachers and pupils to explore understandings" (Askew et al., 1997, p. 32).

Further analysis, however, of the nature of the explanations extracted from students revealed that the discussion was often dominated by social, rather than cognitive discourse. The following example from Sue illustrates this tendency:

> Teacher: What about you Cam? What did you do?
>
> Cam: I just did one and timesed [sic] it by two for the first one
>
> Teacher: OK, so you worked the first one out and timesed it by two. OK, how did you go Amelia?

Although Sue provided the opportunity for all children to contribute, she did not attempt to link their contributions to each other. In the above excerpt, for example, she could have asked students if they had completed it a similar way, or used a variation of the numbers in the problem to try Cam's method to solve it or simply asked for feedback on the logic of Cam's method. This may have resulted in more purposeful discussion and moved the discourse from a social function that was helpful to learning, to one which was fundamental to learning (Alexander, 2000, as cited in Pratt, 2006).

While the results indicated that the teachers were willing to have extended exchanges with students, the exchanges were often similar in nature to those identified by Hardman et al. (2003) in that they were often limited to recall or clarification of information. This suggests that the teachers may have been aware of reform practices that recommended that students should be encouraged to communicate with others and explain their answers, but had not extended this to stimulate more purposeful discussion. While it was encouraging that the teachers were willing to use probing questions in a variety of ways, there were few examples of student-student exchange and questions that resulted in students' justifying their answers were rare. The teachers provided the opportunity for discussion to occur, but often did not capitalise on utilising explanations to maximise conceptual understanding, supporting the contention that is the quality of the follow-up move made by the teacher, rather than the questions themselves, that facilitates a more interactive learning environment (Kyriacou & Issitt, 2008).

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MATHEMATICS SUCCESS AMONG STUDENTS OF ETHIOPIAN ORIGIN IN ISRAEL (SEO): A CASE STUDY

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In this study we explored success stories of five students of Ethiopian origin (SEO) who enrolled in a pre-academic program at a prestigious technological university in Israel. Our goal was to understand how these students perceive and interpret their experiences and achievements as learners in advanced level mathematics classes, where SEO as a distinct ethnic group are significantly underrepresented. Using qualitative case study methodology, we identified perceived personal variables such as: a positive mathematics and academic identity, self-regulated learning (e.g. 'lone learning), personal attribution for success and failure in mathematics, and ethnic identification enhanced with a strong sense of commitment to the ethnic group as playing key roles contributing to achieving and maintaining success against the odds.

INTRODUCTION

Many studies have investigated the economical, social, and educational difficulties encountered by most Ethiopian Jews since their extensive immigration to Israel, which began in the early 1980s. More specifically, Ethiopian Jews have encountered many barriers and stereotypes from different sectors of society, mainly because of their unique cultural and historical background and their distinctive black skin in a predominantly white society (Kaplan & Solomon, 1998; BenEzer, 2002). The vast majority of this ethnic minority live in poor and depressed neighbourhoods in peripheral cities and they remain as a separate and segregated group. This is in stark contrast to their previous expectations and their continuous strong desire to be identified with and integrated into the broader Jewish society (BenEzer, 2002). The academic achievements of the students of Ethiopian origin (SEO), half of whom are second-generation immigrants, lag significantly behind the national average, despite the numerous intervention and remedial programs. Many studies have documented the overall academic underachievement, particularly in mathematics, the relatively high dropout rates, and the high representation of SEO in special education programs (Wagaw, 1993; Lipshiz, Noam & Habib, 1998 (in Hebrew); Tzruiel & Kaufman, 1999; Levin, Shohami & Sapoliski, 2003 (in Hebrew); Wolde-Tsadick, 2007; Rosenblum, Goldblatt, & Moin, 2008). Many SEO are considered ‘under prepared’, ‘low achievers’ or ‘unmotivated’ and are consequently placed in the lower-level mathematics matriculation tracks in the senior high schools. For example, during the years 1999-2003, among all students who were eligible for the 'Bagrut' in mathematics (the matriculation exam taken at the end of grade twelve in different

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 169-176. Thessaloniki, Greece: PME.

subjects), only 2% were in the advanced track[1], compared with 17% of the entire student population[2]. This placement of a disproportionately large number of SEO in the lower tracks have undermined many SEO's ability to pursue careers in scientific and technological fields as well as their access to study at prestigious academic institutions (Wolde-Tsadick, 2007).

Accumulating research has revealed that many immigrant children from low socioeconomic backgrounds are at risk of academic failure in their host country. Thus, Ethiopian Jews, who on the whole, have arrived with economic and educational disadvantages, would be at high risk. In general, immigrant and ethnic (racial) minorities often encounter social, economical, and cultural disadvantages, as well as barriers related to their racial or ethnic background that create situations in which their academic achievements are at risk. However, several studies revealed that there are differential achievements between the various groups of minorities and among individuals within the minority groups, implying that these factors do not fully account for minority students' success or failure (e.g., Ogbu, 1991; Ajose, 1995; Martin, 2000, 2003; OECD, 2006). Moreover, it is argued that overemphasizing the importance of factors, often uncontrollable by educators, can become an excuse for not striving towards improvement within schools (Stigler & Hiebert, 1999). Within learning contexts, many studies link personal and social/environmental factors, providing insights on the reasons why individuals choose to engage or disengage in different learning activities and contexts and how these affect their academic (mathematics) achievements.

It is claimed that although contextual factors (socioeconomic and educational backgrounds, limited access to resources, as well as school quality, race, ethnicity, and language) are informative, they are also deterministic and ignore crucial factors such as individual agency, identity, and motivation (Martin, 2000; Schoenfeld, 2002). Consequently several studies highlight the 'identity' construct as an important concept to consider in mathematics education research, because it draws together a range of interrelated elements such as beliefs, attitudes, emotions, cognitive capacity, and life histories that are integral to understanding students mathematics achievement (Steele, 1997; Martin, 2000; Boaler, William, & Zevenbergen, 2000; Nasir, 2000; Sfard & Prusak, 2005).

Understanding the characteristics and circumstances of students' underachievement may contribute to the development of strategies that minimize or inhibit the harmful consequences of the risk factors. Yet, studying success is claimed to be a

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[1] In the Israeli education system not all students are eligible for matriculation; eligibility is determined according to the students' prior achievements. Those eligible have three levels: basic (3 units), intermediate (4 units), and advanced (5 units).

2 Data-source – Israeli Ministry of Education, 2004, Jerusalem

complementary promising approach to research, since studies of failure regardless of their academic depth and seriousness, will not necessarily account for why and how some students succeed or can succeed (Garmezy, 1991; Martin, 2003). It is further argued that, understanding success would not only contribute to the understanding of the phenomena, but it would also support the efforts "to uncover a range of solutions focused on what works, where, when, and why, rather than trying to lump all students together and applying one-size-fits-all interventions" (Martin, 2003, p.18).

Previous studies didn't reveal much regarding differential achievements in mathematics *within* SEO, and especially little is known about those who, in spite of their circumstances, study in the advanced level matriculation tracks. In this study, we focused on successful SEO in mathematics and explored how they perceive and interpret their experiences as mathematics learners in the context of being a student of Ethiopian origin, while many of their SEO peers with comparable capabilities, socioeconomic backgrounds, home environments, and school resources have been significantly underachieving. We sought to answer the question: what perceived personal variables (beliefs, values, behavior, etc.) among successful SEO, play key roles contributing to achieving and maintaining success in advanced level mathematics?

Since a disproportionately high number of SEO have an increased risk for mathematics underachievement, the search for alterable factors that foster academic trajectories against the different barriers is particularly important. Ultimately the understandings we may gain from successful students may contribute to both the theoretical knowledge about mathematics learning by students from disadvantaged ethnic groups, and to the development of appropriate intervention programs designed to meet the needs of underachieving SEO in mathematics.

METHODOLOGY

We used a qualitative collective case study research design (Strauss and Corbin, 1990; Stake, 1995; Shkedi, 2005) for our study. The qualitative methodology is suggested as suitable for capturing and understanding individual perspectives, lived experiences, behaviors, and feelings (Stake, 1995; Strauss & Corbin, 1990). It is also claimed to be ideal for the understanding of an understudied group of people or phenomenon in order to gain more in-depth information that may be difficult to convey otherwise (Strauss & Corbin, 1990).The main sources of our data consist of students' self reports, self-reflections on and self-constructions of their experiences as students of Ethiopian origin learning mathematics. Additional data sources for triangulation included a number of mathematics classroom observations and many informal conversations.

Sample: five SEO (18-19 years old), who enrolled (a year after completing high school) in a special pre-academic program in a selective technological university, participated in this study. Qualitative research, which stresses in-depth investigation such as this study, uses purposeful sampling (as opposed to random sampling), with the goal of becoming "saturated" with information on the topic (Shkedi, 2005). As Merriam (1998) explains, purposeful sampling is used when an investigator wants to "discover, understand and gain insight and therefore must select a sample from which the most can be learned" (p. 61).

Following a pilot study to locate sites and candidates, this sample consisting of three males and two females were recruited to participate in this study. They were selected among a total of thirteen SEO who were studying in a special pre-academic program (sponsored by a national project for academic excellence) at a very prestigious technological university. Selection criteria for this site and the participants in this program were: availability of a relatively large number of potential candidates in one institute; agreement of the respective authorities to conduct the study in the site, and students' willingness to participate as well as their accessibility and availability upon request for the purpose of data collection. These participants constituted a diverse group regarding: a) the high school they attended (type, resources, and location) c) their parents' education and employment, and d) their immigration status.

Data collection and analysis: the data reported in this study were collected through semi-structured interviews held by the first author at the university, and lasted, on average, 1½ hours each. They included leading questions such as, how would you describe your schooling? What do you think of yourself as a student/as a mathematics student? To whom/or what do you attribute your success? How do you learn, at school or at home? What do you think/know about mathematics? Etc. In these interviews, the interviewer collaborated with and probed the participants in a variety of ways in order to unpack their knowledge and the nature of their experiences as mathematics learners and as students of Ethiopian origin. The interviews were conducted (in Hebrew) with extreme care and in a conversational and emergent manner. The interviewer knowledge of the cultural codes of communication and general condition of the ethnic community, being herself a member of the same ethnic group, were instrumental in obtaining rich first-hand information from these students without fear of misunderstanding, and in gaining the students' trust and openness to share their stories.

The transcribed interview data generated from the cases were analyzed at two levels: a detailed analysis of each case, followed by formulating themes within the case, and then more broadly, a thematic analysis across the cases (Strauss and Corbin, 1990; Shkedi, 2005). Data analysis constituted three coding phases (Strauss & Corbin, 1990): open, axial and selective coding, using the constant comparison method at all phases in both levels of analyses: within the case and across the cases.

FINDINGS

The personal variables that emerged from the data, which were perceived to play prominent roles in the pursuit of the students' success in mathematics, were as follows: a positive mathematical and academic identity, personal attributions for success and failure in mathematics, and fostered use of self regulated learning, 'lone' learning being prominent and ethnic identification enhanced with a strong sense of personal agency and commitment to the ethnic group as playing key roles contributing to achieving and maintaining success against the odds. These variables were found to be closely interrelated and dependent on each other, as will be described below.

Most SEO begin their school career with disadvantages as first- or second-generation immigrants, and must work hard and persist to catch up with their non-SEO classmates. Students' awareness of the importance of general academic success in future life, a belief that they share with their parents, and the instrumentality of mathematics to achieve this end were generally strong motivating factors to succeed. Thus, their effort to maintain success in advanced level mathematics was part of preparing themselves to attain their academic goals. However, a stronger source of success was their mathematical identity (Martin 2000, 2003), which refers to the students' self-efficacy beliefs, beliefs in their capability to perform in mathematical contexts (Bandura, 1997), the perceived importance of mathematics, a growing interest and enjoyment in meaningful mathematics learning, and seeing mathematics as playing as critical role to achieve their academic and socio-economic goals as well as the perceived centrality of mathematics in their developing identity.

The students attributed their success also to factors that are malleable and controllable by them: effort (investing much time, persisting, doing more than what is expected), and fostered self-regulation. Their occasional failures were attributed to the same factors: a lack of effort and poor self-regulation. Such attribution beliefs were important factors that motivated the students to persist in the face of difficulties. Self-regulation includes holding self-efficacy beliefs, useful attributions for success and failure, and choosing and shaping a productive work environment (Zimmermann, 1989). Students learned to regulate their cognitive actions, mathematics related motivational beliefs (mathematics identity) and behavior in order to maintain their success. Students' self-regulation strategies were efficient, enabling them to succeed according to the schools' expectations. They reported that to succeed in mathematics at school and to fulfil the teachers' expectations, one needs to learn well the procedures, as they were taught in the classroom. Thus, as hard-working students who wanted to succeed, these students 'played the game' very well, but with hindsight, they were dissatisfied about not being challenged intellectually with problems that required them to think.

'Lone' learning, which was the preferred strategy of these students, is an example of students' enacting self-regulating strategies. One of the things these students seemed to fear most were external distractions. To these students, collaborative learning with others was perceived as a potential source of distraction, which would make their time-on-task inefficient. Thus, it seems that 'lone' learning was a convenient strategy for them to fulfil school requirements for deep understanding of the mathematical idea and to achieve good grades. This finding however is in contrast to some studies that claim 'lone' learning as a non-effective strategy for meaningful mathematics learning. For example, Triesman (1992) revealed that social and academic isolation were sources of the main difficulties that affected African-American students' achievement in a calculus course at a prestigious university, whereas the social and academic interactions between the Asian–American peers promoted their achievements.

In addition, students' ethnic identity (Phinney, 1990), becomes salient through their 'solo status' experience in the advanced mathematics classes, as the only members of a distinct group. Students' awareness of the existing stereotypes, as well as social and cultural barriers unique to their group could have undermined their self-efficacy beliefs and personal agency (i.e., the belief in one's power to attain goals through actions), and eventually their intellectual functioning (Steele, 1997) and success in mathematics. Yet, for these students, the challenges they faced was the basis of their belief that they are on a mission, representing their ethnic group as successful members. We found two wide social goals strongly integrated with the personal (social) identities of these students. One goal faces outwards and it refers to their self-imposed role of creating a positive image of Ethiopians within Israeli society at large, and the other goal faces inwards and is driven by the belief that their success would influence their peers (especially the younger generation) by establishing strong role models. Success particularly in mathematics was thus perceived by the participants as crucial and instrumental to this end. Although this sense of responsibility strengthened their ethnic identification (Phinney, 1990), their self-concept, their academic motivation, and persistence, it was an additional burden unique to them. This is in line with Martinez and Dukes (1997), who argued that a strong ethnic identity reduces the impact of negative stereotypes and social denigration on individuals.

From our findings, the existing negative stereotypes concerning the competence of SEO in advanced mathematics, which the participants are also usually aware of, did not influence them negatively. These students held personal agency, a sense of control and confidence in their ability to deal with problematic situations (Bandura, 1997). The barriers strengthened their mathematical identity, their ethnic identity, as well as their self-efficacy beliefs, which are considered to be one of the strongest predictors of mathematics performance among students (OECD, 2006). The students' sense of confidence and motivation was also enhanced with the perceived motivational support and high expectations that they received from their parents.

In sum, beyond the theoretical importance of these findings, there are some important practical implications. Programs based on promoting these desirable characteristics among SEO could potentially increase the success stories among underachieving SEO. Furthermore, since studies on differential mathematics/academic achievements within SEO are scarce, more research on the issue will be valuable.

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COGNITIVE PROCESSES ASSOCIATED WITH THE PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

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This paper proposes a model of professional development which highlights the teacher's cognitive processes, based on Sfard's stages of interiorisation, condensation and reification. The model is applied to the case study of a primary teacher participating in a collaborative project for professional development. This adaptation of Sfard's stages proves to be of especial value when interpreting the process of the teacher's professional development from a cognitive perspective.

TOWARDS A MODEL OF PROFESSIONAL DEVELOPMENT

In studies of professional development and teacher training, the teacher has been considered from various perspectives. Our interest lies in the processes involved in generating teachers' knowledge, and builds on Brown and Borko's (1992) description of professional development in which the teacher is seen as apprentice. These researchers take "a view of the teacher as an adult learner whose development results from changes in cognitive structures; these cognitive structures … are the thinking patterns by which a person relates to the environment" (ibid., p. 227).

From this perspective, we understand that the teacher learns in contact with their peers through a consensual process involving their personal conceptual schema, beliefs and motivations, in which language and communication play a fundamental role. In this respect we coincide with the considerations of ontology, epistemology and learning theory characteristic of social constructivism (Ernest, 1996). We share the view that the individual and the group are interconnected and knowledge is built as part of the social process (Carrillo et al, 2008). In this paper, however, discussion is focused on a model which takes account of the cognitive processes implemented by the teacher in the course of professional development, highlighting the influence of the group on those processes, for which purpose we refer to Sfard (1991).

Sfard states that acquisition of new mathematical notions usually begins with an operative conception of the notion, and that the transition from computational operations to abstract objects is a difficult process requiring three stages: interiorisation, condensation and reification.

Sfard uses 'conception' to refer to the internal representations and associations evoked by a concept, reserving this latter term (synonym of 'notion') for a mathematical idea expressed in its conventional form.

Although it would be a mistake to identify the nature of mathematical understanding with that of professional understanding, or the process of abstracting a mathematical

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notion from operative to structural conception with the process of acquiring professional knowledge, we believe the process of professional development does share some degree of parallelism with the process of abstraction which is involved in moving from low level work (interiorisation) to higher level work (reification) (from a cognitive point of view). Our aim, however, is to adapt Sfard's (1991) stages to the professional development of teachers, and to then particularise them to the case of a collaborative context (Feldman, 1993).

It is not possible to maintain the differentiation between the conception of the notion (in this case the specific professional issues) as process and as object. Extending the definitions above to the case of professional development, the key to moving from one stage to another would lie in the maturity (understood as the increasing complexity of reflection upon the phenomenon of education, or as the assimilation of elements which deepen professional knowledge) with which the issue in question was approached. Hence, as will be seen in the example in section two, the interiorisation stage is characterised by familiarity with the issue, and is most probably initiated with the analysis of similar situations. It is characterised by the teacher mulling over something that does not seem quite right or that they feel could be improved. This may be accompanied by specific solutions (to these particular circumstances, that is, without making generalisations about the underlying issue or perceiving its wider dimensions), which themselves may or may not be put into practice. The condensation stage involves freeing oneself from the particular and seeing the issue as something more general, which means introducing a new variable into mapping the terrain of professional practice. It can be seen not only when a teacher reflects on their practice prospectively (that is, when planning) or retrospectively, but also while activities are being put into practice and decisions are being taken about them. The reification stage would add to the above an understanding of the issue in its complexity (relative to the level at which the teacher is operating, one cannot speak of an 'absolute' understanding), along with its relations and derivations.

We can thus imagine the teacher progressing along a kind of professional development helix, where the role of the teacher trainer becomes that of providing learning contexts which will further the progress. The contents of the helix have been thoroughly studied and range from teachers' everyday activities to professional knowledge (Shulman, 1986, Hill et al, 2008). This professional knowledge can partly be seen deployed in situations directly related to the classroom (eg planning). The helix also involves direction, defined by the three stages of interiorisation, condensation and reification. Finally, the helix has a specific shape, given to it by reflection. Reflection is both content and generatrix of the helix – content in that we have included it within professional knowledge, taking reflexive practice as reference; generatrix in that it is the means by which the helix is created. Reflection likewise allows progress through the three stages mentioned above, and in its turn leaves a trace in the contents of professional knowledge.

Julia's development viewed through the lens of the continuous helix model

In the study and promotion of professional development, collaborative contexts (involving teachers and researchers) have shown themselves to be especially appropriate (Llinares and Krainer, 2006). Climent (2005) and Climent and Carrillo (2003) analyse the professional development of primary teachers participating in a collaborative research project (PIC: 'Proyecto de Investigación Colaborativa'), in terms of both reflection on practice (for which the PIC has proved a fruitful context) and teacher training.

The PIC started in 1999 and is made up of two experienced primary teachers, a novice teacher (Julia), a novice and two experienced researchers. In an atmosphere of co-operation in which discussion and reflection play a vital role, its work pays special attention, among other issues, to the participants' reflections on their conceptions of their pedagogical content knowledge concerning school mathematics.

Julia showed great interest in forming part of the project, as she considered it an opportunity to continue her training with other teaching professionals. We collected information during two years, in which Julia acted for the first time as group tutor (to 6-year-old pupils). We used a wide variety of data collection techniques in two contexts: Julia's classes and the PIC. Chief amongst these were her classroom diaries, interviews, classroom recordings and recordings of the PIC sessions.

In this paper we focus on a single aspect of her practice: planning. It was highly significant that initially Julia considered each day's plan as a rigid document which had to be followed to the letter, independently of local factors (the tiredness of the students, particular difficulties, etc). Due in large part to her individual reflection and the group reflection in the PIC, Julia began to consider the planning stage in a more flexible light and was able to foresee such difficulties at this stage and to incorporate ideas to deal with them. Below we briefly describe this process, highlighting the key role that reflection played as a force for moulding her professional development.

Interiorisation

Julia approached her teaching in conformity with the culture at her school: teaching was traditional, with minimal use made of manual exercises and a high degree of reliance on the textbook. From the start, Julia used it as the principal source of her teaching material, showing reluctance to leave a section half done, even when her pupils showed clear signs of fatigue, When she started the second teaching unit she began to keep a class diary, a practice she continued into the following unit.

Analysing the diary entries for the instances where her pedagogical sensibility was overridden by her desire to complete a particular section, we noted that her personal reflections showed little potential for change. They did not seek to prioritise and select key sections from the textbook, or to modify this in any way, but remained at the level of sequencing the activities in terms of difficulty or conceptual demands so as to improve the pupils' chances of getting through them. In short, the diary entries only allowed her to become aware of her difficulties and to consider ideas for improvements which never materialised. However, they do indicate that such issues were beginning to stir in Julia, awakening in her certain dissatisfaction.

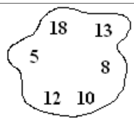
It was the collective reflection in one of the group sessions analysing a video of her teaching which contributed a new outlook. At the beginning of the lesson, Julia asked the pupils, on the spur of the moment, to accurately define a rectangle. Such were the demands of this task that she spent nearly an hour trying to get the pupils to deduce from a series of examples the defining features of a rectangle – such as parallel or perpendicular sides – even though they were clearly feeling extremely tired. In her subsequent reflection, in both her diary and the PIC, Julia acknowledged the pupils' intuitive understanding of the concept, but she didn't consider leaving the activity unfinished, because she wanted them '*to be a little bit more precise …because I think that they generalise too much"*. The response of one of the experienced teachers in the PIC to this rationale was especially interesting:

> Inés: You don't think it was because you wanted to get somewhere and you saw that they weren't getting there? […] / Julia: There was a moment when I saw it was too much for them…/ I: Why do they have to get to that point? / J: Because it doesn't seem right to me not to finish things […] / I: […] No, what happens, Julia, is that sometimes we get involved in something and we have to [… know] how to go back over something, in the sense of saying, 'OK, for whatever reason, this isn't working out, and it's OK if this doesn't work out,' and you say, 'OK, we'll have another go at this tomorrow.' […] because […] the one who goes away with the sense that things haven't been finished is you, but not the children.

Reflection with other professionals helped Julia become more fully aware of what her decisions were aiming to achieve, and of the importance of noting the pupils' reactions and responding to them. Her improved understanding of the situation provided Julia with new elements of judgement which would later be useful for facing situations from another perspective. In the case described above, in which the pupils' difficulties condition the course of the planning, we can say that Julia was undergoing an interiorisation phase, in that it was now that she was becoming aware that such situations could arise in the classroom and that there were various ways of responding to them.

Condensation

At the end of the first year, Julia conducted four problem-solving activities on number decomposition, which had been designed in the PIC. Below we describe the implementation of the fourth of these, called 'The Same, Bigger, or Smaller', and for which Julia had no teaching notes to follow (table 1).

Select the numbers which add up to: a) 27 b) 36 c) 43	

A) Is the addition of 23 and 15 greater than 20? Than 30? Than 40? 50? (Repeat with 'less than'). B) Is the addition of 23 and 32 greater than 20? Than 30? Than 40? 50? (The same with less). C) Is the addition of 19 and 32 greater than 20? Than 30? Than 40? 50? (The same with less)

Table 1. 'The Same, Bigger, or Smaller'

At the preparation stage, Julia predicted various difficulties that could arise, stemming principally from students' limited familiarity with making estimations: "*...there will be some who, without meaning to, or thinking about it, will try to add up all the combinations of numbers and this makes it a really laborious task for them*". Once in class, Julia looked very doubtful, foreseeing the difficulties. She started the explanation without making overt mention of specific numbers and writing symbols instead. Then, she decided to introduce some numerical examples. By means of leading questions, she focused the pupils' attention on the numbers which were grouped together, with the aim that these became meaningful to them and they realised which were the smallest and the largest. She also drew their attention to cases in which it was not possible to express them as the sum of three others, such as 15 (which can be obtained only with two numbers: 10 and 5). Next, taking three numbers to be added together from the group (12, 10 and 13), she gave 35 as an example, and explained the strategy of estimation which they could follow in order to find out the numbers required to be added together and so solve the problem.

At first, the work was done individually. It was noticeable that many pupils found it hard to concentrate. In the feedback phase, she was interested not only in the results, but in the reasoning followed. She tended to select the more able pupils. In the second phase of the activity, the whole class worked together, with the pupils having to do mental calculations so as to estimate the result of the addition of two numbers. She frequently had to ask pupils to pay attention because they seemed tired (the previous activity was 40 minutes long) and she tried to keep them participating, but in the end it became very difficult to continue. Finally, Julia worked through all three estimations, but went through the last two more quickly.

Although she managed to do the three estimations, her impression was that of having left the activity half way: *"I see that the problem is that they normally solve almost everything successfully, and this time it began to be a bit frustrating ... and we had to finish earlier than expected." [PIC session]*

This episode is representative of the condensation stage. Her decision not to continue in the way that had been foreseen would seem to indicate that Julia was more inclined to take into account the pupils' learning difficulties and to adapt her original plan to meet them. Also significant is that this decision was made while the class was in progress (favoured to a certain extent by her previous reflection), whereas at the interiorisation level Julia only arrived at such a realisation after the event.

She was beginning to view her planning with a certain degree of flexibility, and to be able to explain her decisions openly in the PIC. It seems the joint reflection had a significant influence, providing her with the necessary pedagogic support to take decisions like these.

Reification

One year later, the possibility of Julia repeating the decomposition activities was discussed in the PIC. Taking into account her experience and her pupils' current knowledge, she decided that they would be appropriate, albeit with various modifications. With regard to the activity 'The Same, Bigger or Smaller', she omitted it and selected another from an activity bank created in the PIC. Julia took this decision not only because of the feelings of failure experienced the previous year, but also as a result of questioning the rationale of the activity itself, considering that the difficulty lay in the degree of abstraction demanded, by a mental task involving numbers without reference to a specific context. Nevertheless, when she was planning the lesson, she considered the possibility of including a modified version:

> "I could make the numbers easier…But it was going to look like the one we did yesterday,… about estimation... Maybe I'd have been wrong and I'd have got a pleasant surprise, but as last year was far from being productive, I thought, 'Where's the point of wasting an hour by repeating the experience?" [PIC session]

She choose an alternative activity, which consisted in completing four dominoes laid out in a square so that the total represented along each of the four sides always came to be the same (10) (figure 1).

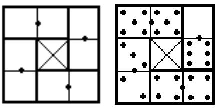


Fig. 1: Example of the activity 'Domino Squares'

In fact, this activity is very similar to the first part of 'The Same, Bigger or Smaller' in that there is still a fixed amount (10 in the example) which is to be obtained by adding the other three numbers which are also given (from 0 to 6). The concepts involved in 'The Same, Bigger, or Smaller' were focused on separately in two of the activities ultimately included in her plan. It can be noted that what had been modified was the mode of presenting these concepts, but not the concepts themselves.

We identify her reasoning with the development of the reification stage because Julia now showed herself able to consider the pupils' learning difficulties at the actual time of planning. She makes use of her knowledge of the features of the activities and their corresponding cognitive demands to look for an alternative means of working on the mathematical contents. She had a degree of professional knowledge, with respect to activities and learners, which was also available to her during her lesson planning (and not merely an awareness of this variable). Previously, she had been making use of an activity designed by the group and acted more as a guide for what had been decided elsewhere (with a stake in the debate, certainly, but with an understandably diminished sense of her own authority given that it had been her first year as teacher and member of the group). Her greater professional understanding, then, made possible and was closely linked to the increased flexibility of her lesson planning with respect to the pupils' difficulties.

She no longer became aware of such difficulties only if they arose, either afterwards upon reflection (as in the interiorisation stage), or during the class itself (as in the condensation stage). This behaviour led her to make suggestions for improvements to subsequent lessons (although not always put into practice) in the former case, and to make slight adjustments as she went, in the second. However, at the reification stage she no longer took a reactive attitude to difficulties, but rather anticipated them in the planning process, so giving a fuller pedagogical treatment to the content. In other words, Julia successfully converted consideration of pupils' difficulties into an element to be treated independently of the circumstances in which they arose.

3. Final reflections

Each stage is defined by an "advance" in Julia's cognitive and/or teaching dimension. The interiorisation stage supposes the deployment of the idea that the pupils' difficulties require consideration once the class plan is in operation. It is accompanied in this case by suggestions for treatment which for the moment are not put into action. The condensation stage treats pupils' difficulties cognitively as one more variable in the analysis of what happens in the classroom. This is reflected in the decisions made during the course of the lesson. In this case the progress is less cognitive, or unconnected to action, as in the previous stage. It is knowledge in action (Schön, 1983). In the reification stage, the potential pupils' problems are taken into account from the planning onwards, giving less importance to externally established pedagogic treatment. So it is also a case of cognition in action (at the planning phase).

We are aware that various factors influence this process of abstraction, but we highlight the context of the PIC because it has shown itself to be a meeting point which promotes and enriches reflection on classroom practice, and encourages the consideration of other variables in lesson planning (Ticha y Hospesová, 2006).

From the analysis of teachers' learning systems (Krainer, 2004), we can note that, through joint reflection with others (reflection-networking) and acting and reflecting on her own practice (action-autonomy-reflection), Julia gained additional competence and self-confidence in autonomous planning and interaction (autonomy and action) and in her ability to reflect on mathematical teaching practice and to reflect and communicate with colleagues and take advantage of their ideas (reflection and networking). There is no doubt that Julia's participation in the PIC, providing resources and principles on which she could base her decisions, was a source of valuable support, but credit must also be given to her individual reflection.

We can say that in her process of professional development, Julia completed a preliminary cycle of interiorisation-condensation-reification, with respect to one element of her teaching, that of planning. Once 'reified', we consider that this new conceptualisation of planning itself undergoes a process of development, in which she would begin to take account of more and more considerations in her lesson planning and to be able to foresee more alternatives for dealing with particular elements.

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INTERACTIVE DIAGRAMS: ALTERNATIVE PRACTICES FOR THE DESIGN OF ALGEBRA INQUIRY

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We investigate whether and how printed diagrams vs. interactive diagrams, video clips vs. interactive animations, create different contexts for mathematics learning. The present paper analyzes an experiment in which two algebra activities are presented to students in a task-based interview. One activity, describing a motion situation, is presented in a video clip and in an interactive diagram (ID). A second activity, requiring the description of a linear function, is illustrated by a printed diagram and by an ID. Analysis of the problem-solving processes of the two activities that include IDs indicates that the process of concept construction occurred as a result of the students' decision to change the representation of the data in the activity, build a focused collection of data, expand the given representations, or build new ones. Both activities shed light on the ways in which problem solvers use sketchy IDs designed to encourage the problem solver to interact with the diagrams in a way that transforms sketchy information into an important component of conceptual learning.

ILLUSTRATING INTERACTIVE DIAGRAMS

The domain of digital interactive mathematics textbooks is new and largely unexplored. We seek to identify practices associated with the design of this type of textbook and to focus on a few functions of interactive visual representations, mainly on ways to design activities and learn with interactive diagrams. By *diagram* we mean a drawing, plan, scheme, or other method of clarifying or demonstrating a concept, an object, and etc. An interactive diagram (ID) is a relatively small and simple software application (an applet) built around a pre-constructed visual example. IDs should not be assumed to be transparent: they call for interactive work, and the tools needed form an integral part of the diagram. Creating such a setting that requires action and participation from the student has been the ultimate mission of the curriculum reform movement and of the development of microworlds over the last two decades (e.g., Hoyles & Noss, 2003, Kaput & Hegedus, 2002, Schwartz, 1999).

There are profound differences between the traditional page in math textbooks that appears on paper and the new page that derives its principles of design and organization from the screen and the affordances of technology. Based on visual semiotic investigation by Yerushalmy (2005), *illustrating diagrams* are the most frequent ones in paper math textbooks. In general, *interactive illustrating diagrams* are simply operated unsophisticated representations, and most dynamic diagrams found as applets on the Web are of this type. They are intended to orient the student’s

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 185-192. Thessaloniki, Greece: PME.

thinking to the structure and objectives of the activity by usually offering a single graphical representation and relatively simple actions. *Elaborating IDs* provide the means students may need to engage in activities that lead to the formulation of a solution and to operate at a meta-cognitive level. *Narrative IDs* are designed to call for action in a specific way that supports the construction of the principal ideas of the activity and may serve to balance open-ended explorations and support autonomous inquiry. The present paper studies how the designed artifacts are used and analyzes the potential and constraints of the design functions of the diagram. We analyze the work of 13-year-old students in task-based interviews focused on two algebra activities adopted from Yerushalmy, Katriel, & Shternberg *(2002/4)*. One activity deals with a temporal phenomenon and was first illustrated by a video clip that students were asked to describe. Subsequently a similar motion situation was presented as an illustrating animation. On both the clip and the animation, users could watch at all times locations on the runway, continue the run, or initialize the motion (Figure 1.1). The second activity, an analytical one, requires writing a symbolic expression to describe a given linear function graph. It was first illustrated by a paper diagram (Figure 1.2 left) and then by an ID (right). Students could read Cartesian numerical values by the marks on the paper or by interactively pointing with the mouse.

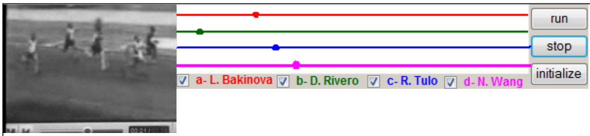


Figure 1.1

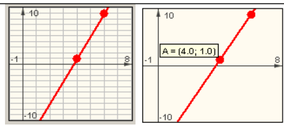


Figure 1.2

According to design functions, both IDs are *illustrating* IDs because they allow only the viewing of the given examples and a limited degree of intervention by activation of controls in the animation and the graph. While analyzing the problem-solving processes we asked how the similar design of the two diagrams in each of the two domains (temporal and analytic) is reflected in the students' problem solving processes. Ten 8th-grade students were interviewed. All interviews were videotaped and transcribed. We illustrate the general observations with one interview in each of the activities.

Video clip and animated simulation of motion

Unlike static diagrams, video clips and animations are intended to reduce verbal explanations and be more efficient than signs and symbols in describing dynamic processes (Jones & Scaife, 2000). But because research on animation (Hegarty, Kriz, & Cate, 2003) suggests that learning by watching a motion picture is very limited, we embedded tools in both in the video clip and the animation that allow students to stop, continue, and restart the motion. Students played the video clip several times, following the motion without pausing. They used spatial terms, such as left and right, to identify the runners. They invested time in analyzing the runners' bodily motions, trying to reach a conclusion about the effect of bodily motion on the running procedure. Finally, they concentrated on the runners' actions at the start of the run and summarized the final results of the race:

Dan: When they ran, they moved their body a little bit back and their feet a little bit forward and… this maybe gave them, I think, more acceleration. And in the end the one that was on the left won, sort of. They all made almost the same movements; just that there were some that started running and some that jumped out later and some that jumped a little sooner. The left part was a bit sooner… and… until… but until… until a couple of seconds, and the right side won, covered more distance between them. It almost came to the same position. Here, and you can see here (*on the screen a picture of the relative position of the runners at the finish line*) that the right and the left side [runners] came about second, and these two people and the last are about at the same line.

In an attempt to reach a diagrammatic description the interviewer asked Dan to think about additional ways of describing the motion, and Dan suggested building a series of "critical pictures" that describe the change in the relative positions of the runners:

Dan: Ah, you show the pictures of the people from the start line, and then you show their pictures when they jump of, you show their pictures in the middle and during this process, to see… and you show the **most critical pictures**, the moments someone passes someone, or someone that… and in the end you show the picture… the ending picture that shows all the people that... who won first place, second, third, fourth… yes, the important points… the **critical points**, let's say, if they're at the same distance, in that distance it's not critical, you don't need to shoot every time like in a movie the same picture… you **need to picture the points that matter, that will prove that he passed this and that**.

While working with an activity in the ID format, Dan used colors on the animated ID to identify the runners. He, as well as the other interviewees, was intrigued by the relative slowness of one of the runners in the beginning of the movement and by his victory in the end despite the faster movement of the second runner, who did not win.

Dan: The man on the pink line starts out fast, later he begins to lose pace, and the blue passes him. The black stays in the back and in the end he came first, even though at the beginning he jumped off to late and was slow, in the end, he gained acceleration and passed all of them. The red continued at about the same difference, like, almost the same distance he was between them. No, not the same distance, I mean, his running pace was fixed.

Interviewer: How do you know he runs at a fixed pace?

To demonstrate his conclusion, Dan reactivated the animation, described the run again, and toyed for a while with an idea of describing the motion using a graph or table:

Dan: And there is the thing of the table that you can show how much he runs to a… km per second… You can do it in a graph form. He actually started picking up speed; you can see it in a graphic shape, let's say the graph rises when he has 20.

But Dan pointed out that he could not really do this because "I need data," and returned to the idea of describing the motion by building a series of "critical pictures" that describe the change in the relative positions of the runners that he brought up while working with the clip. He started the animation, paused in locations that he viewed as important, and described the situation. Each time he stopped he drew a diagram on paper to represent the captured moment:

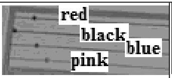
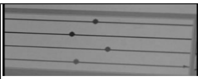
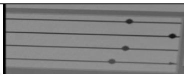
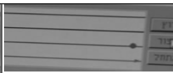
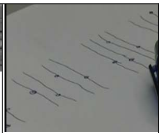
				

Figure 2.1	Figure 2.2	Figure 2.3	Figure 2.4	Figure 2.5

Dan: Ok, this is also an important critical point (Figure 2.1): the black still stays in the back, while they are already beginning. …the pink continues to lead and the black is still stuck at the back. Then (Figure 2.2) actually the blue now is ahead of him, the red continues sort of... continues in the same pace. The pink begins to slow down. The red passes already… the red passes… the red passes the pink. The black soon, here (Figure 2.3) it shows that he begins to gain acceleration and that the pink stays a bit behind. In a situation like this, here, when everyone begins to slow down, and here the black passes. In the end, the pink that started stays in… came in last. Ah, and the end (Figure 2.4) it's the blue came last. Wait a second. This is actually the point where the black is no longer in the picture, because he won. In the end the blue stays last.

At this point Dan's event sequence of static diagrams (Figure 2.5) prompted him to mentally recreate and describe the entire race, pointing to the runners' changes of speed in correlation with changes in their relative positions.

Dan: So in the end it shows that the pink that at first advanced, that began to gain acceleration, and the blue that passed him later, who also began to be the fastest; actually the black accelerated and passed him. The blue and the red started slowing a bit and the red continued at the same pace, and in the end passed the blue and came third. The black was first. The pink came second and the red came in third place. And the blue stayed… came last, even though in the middle he started leading in the distances, and then the black began to pass everyone, and won actually.

Observing Dan and the rest of the interviewees it became apparent that although the two interactive representations provided similar interactive control tools they generated different descriptions. The bodily movements and gestures of all the runners were important details in the motion analysis based on the video clip, the interactive diagram based on color coded animation promoted comparative descriptions. Interviewees differentiated between the runners, addressing each one and the relative progress of each, and were able to describe the chain of events in a schematic way that served as a static model from which they were able to reconstruct the dynamic process.

Paper diagram and interactive illustrating diagram of linear function

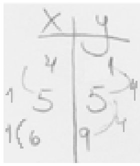
The second, analytical, activity required writing a symbolic expression describing a given linear function graph. It was first given as a paper diagram on which students could read Cartesian numerical values by the marks on the paper, compute the slope of the line, and use as many point values as they needed to write the function's expression. All the students studied linear functions in graphic, numeric, and symbolic representations and were familiar with the symbolic representation $f(x)=ax+b$. The challenge of the activity was to find the function on a diagram on which the y-intercept point and the slope were not shown.

With the paper diagram Roni found the coordinates of the marked points but was not able to write the symbolic expression of the function.

Roni: It won't help me much, to do it without anything, like that, on paper, I can't do it alone.

Of the six interviews we analyzed two students were able to solve the problem given as a paper diagram. The four students who asked for help and were unable to complete the task then requested to work with the interactive version of the diagram. After Roni read an activity in the ID format she moved with the mouse over the graph of the function and checked the coordinates of the points on the graph. She lingered over the x-intercept (probably because the y-intercept did not exist on the given diagram) and decided to create a table of values:

Roni: I will try to construct a table of values. I'm trying to find points of integers. *(She moved the mouse up and down the line several times, monitored the changes of the coordinate values on the line, picked coordinates to write in the table, and wrote differences between the values.)*



Interviewer: Would you explain what are you doing?

Roni: I'm checking the intervals for the slope. The software does not really help me, aside from finding points, at the moment. The slope is four, *4x*. *(writing 4x on the page)*.

Interviewer: How do you know that the slope is four?

Roni: Because these are the intervals here, and here it is one and it comes out four divided by one. And *y*… I guess that the free term would be about minus 13 *(Roni extends the line with the mouse outside the borders of the diagram until she envisions that it "meets with the y-axis")*. I can't really see it here, because I can't increase the scale, but by the slope and that it's going to be minus 10 and it goes a little below… I don't exactly know the free term, I also can't exactly write the function accurately. It is about *4x-15*.

Interviewer: Can you check whether this is true?

Roni: Yes. I substituted 5… after the substitution, when Y equals 5 then X equals 5 also, 5 multiplied by 4, 20, 20 minus 15 really equals 5… so this is the symbolic expression.

Roni followed the changes of the coordinates along the line, tracked the coordinates on the graph, and organized values of consecutive integers in a table. Two of the three points she chose to treat were the marked points on the graph. She calculated the differences between the values in the table and the ratio between the differences to find the slope, concluded that the slope was 4, and wrote in the expression of the function as *4x*. To obtain to the constant term she extended the line, using mouse movement, over the borders of the system, until the imaginary intersection with the *y*-axis. She estimated where the line would cross the *y*-axis and suggested that the function is approximately *4x-15*. To check, she substituted one coordinate in the function and obtained the expected correct result. Working with ID, Roni was able to solve the task. ID served as a scaffold for the activity: watching the coordinates resulted in a table on a page and the calculation of the slope and the expression $f(x)=ax+b$. An intriguing question is what kept her from completing the activity when working with the paper diagram. It was possible to read the slope by the rise of 8 in an interval of 2 between the marked points: (4, 1) and (6, 9). We think that one important reason is that the ID turned the static sketch into a detailed graph. The option to read any point on the line led Roni to create a familiar representation on request. As Timna (2008) found in a recent study comparing student conceptions of line and point when presented on paper and on screen, the technological environment that allows seeing the coordinates of the points increases the diversity of student attitudes toward the concepts of "point" and "straight line" and toward the relationship between them, and produces a sensory experience that is different from the experience of working with pencil and paper. The dynamics of mouse tracing have led Roni to imagine the undrawn part of the line, enabling her to reach the missing information about the *y*-intercept.

DISCUSSION

Both activities shed light on the ways in which problem solvers use sketchy IDs designed to encourage the problem solver to interact with the diagrams in a way that transforms sketchy information into an important component of conceptual learning. The animated simulation designed as a dynamic sketch of a race and the interactive diagram of the graph include fewer details than their corresponding static diagram. By contrast, the video clip and the paper graph diagram were both detailed specific examples. We observed the interviewees' language and gestures to understand how sketchy information and simple interactive features are being used in problem solving. The two versions of the motion problem share the same interactive features, with run, stop, and restart options. These tools proved to be necessary in our experiment and helped students focus on events during the process. Although both the clip and the animation presented a similar motion episode, the work followed a different path. In the clip the emphasis was on getting the story right, which required attending to details such as the runners' body motion. As a result, documenting a sequence of events became a complicated task. Speiser & Walter (1996), who describe the "catwalk" pictures used to learn calculus of motion, described the students' decisions about where to watch and what to describe as governing the narrative. Although "catwalk" pictures or video clips represent an important stage in the modeling process, they sometimes keep learners too close to the situation and prevent them from thinking in the abstract. The diagrammatic nature of the ID presentation made it easier to distinguish between the runners, to address each one using colors, and to identify their relative progress. Moreover, the two races (in the clip and in the animation) were different in the given example. The clip showed a close race, whereas the animated ID was designed as a generic example with an exceptional case that captured the students' attention and became a pivot in the description of the race. As a result, it was easier to document and chart a sequence of events that students deemed important, and then mentally replay the sequence, turning it into a purified motion episode. Comparing the students' work on paper and their work with the interactive linear function graph, attention to and awareness of details in the given diagram, and the personal choices students make in the construction of additional details are important considerations. Although the two pictures (the printed picture and the one on screen) were similar, the ID made it possible to address the given graph as a sketch that reveals the "big picture": a line with a positive slope that intersects "somewhere below." The terms used by the students reflect this concentration on the sketchy description of the object, but at the same time the sketch can be interactively unfolded into a detailed numeric diagram, which caused students to change their focus from data testing to choosing the necessary data. Analysis of the problem-solving processes of the two activities that include IDs indicates that the process of concept construction occurred as a result of the students' decision to change the representation of the data in the activity, build a focused collection of data, expand the given representations, or build new ones.

IMPLICATIONS

An important role of research in mathematics education about the new digital culture in school mathematics is to inform teachers about new processes of knowing and about the stability of known processes. The present study of interactive diagrams identifies such processes in the domain of school algebra – a domain that in the last decade has undergone major changes, including several innovative uses of technology. We hope that our work contributes to the ways in which teachers, curriculum developers, and designers of digital books design activities.

Acknowledgement

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INTRODUCING THE CONCEPT OF INFINITE SERIES: THE ROLE OF VISUALISATION AND EXEMPLIFICATION

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In this paper we report a study that aims to analyse curriculum content, pedagogical practice and student perceptions of the often counter-intuitive but significant concept of infinite series. Our analyses of texts used to introduce the concept to students in the UK and in Canada highlight that the presentation of the concept is largely a-historical and decontextualised, with few graphical representations and even fewer applications or intra-mathematical references to the concept's significance and relevance. We also draw on interviews with university lecturers to discuss how pedagogical practice can assist students' overcoming of persistent perceptions, such as 'if terms of the sequence become smaller then the series converges' through uses of evocative images and key examples of divergence and convergence.

LEARNING AND TEACHING THE CONCEPT OF INFINITE SERIES

The work we report in this paper is the first, self-contained, phase of a study that investigates the learning and teaching of a complex, often counter-intuitive but significant mathematical concept, the concept of *infinite series*. The applications of infinite series in mathematics and science are wide ranging and crucial (González-Martín & Nardi, 2007). In mathematics, for example, the concept of infinite series is a fundamental element of the *Riemann Integral*, the calculation of the area under a curve. Infinite numbers, such as ⅓, are expressed, and can therefore be studied, as infinite series (here the sum of $0.3 + 0.03 + 0.003 + \ldots$). In Medicine and Biology infinite series provide ways of modelling situations such as the distribution of medications or poisons. Overall infinite series are central to the mathematical education of a wide range of scientists and professionals. It is therefore quite surprising that the studies of its learning and teaching are rather few.

Students' difficulties with the concept of infinite series have been reported mostly indirectly in the works that study the concept of convergence (e.g. Robert, 1982) – often in the context of the infinite series underlying some mathematical situations such as integration (e.g. Fay & Webster, 1985). These studies suggest that early misunderstandings of the concept may originate in perceptions of infinity, such as that the sum of infinitely many quantities is always infinitely great, and may result in some of the difficulties with understanding the concept of Riemann integral and, particularly, improper integral (e.g. González-Martín, 2006). These studies also suggest that the absence of visual understanding (e.g. Alcock & Simpson, 2004) associated with the concept of infinite series poses severe limitations in students' understanding and application of the concept (e.g. Mamona, 1990).

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In sum previous research suggests that students appear to have little understanding of what the concept actually means, have no visual imagery associated with it and see little or no relevance to it in mathematical and other situations. As is often the case with the teaching of complex mathematical topics at upper-secondary and university levels (e.g. Artigue, Batanero & Kent, 2007), teaching, through reduction to an algorithmic approach – e.g. exercises that require an often blind application of formulae; static use of graphical representations; absence of a connection to other crucial concepts; no attempt to alter related misconceptions about infinity etc. – may evade addressing students' difficulties. Our study aims to explore whether this is the case with curriculum content and pedagogical practice and, if so, to propose appropriate modifications.

A STUDY OF CURRICULUM CONTENT AND PEDAGOGICAL PRACTICE

The work we report here is part of a study currently in progress in the UK and Canada which aims to investigate the teaching and learning of infinite series through:

I. Study of the student learning experience with regard to:
- **a.** Analysis of curriculum content and pedagogical practice;
- **b.** Analysis of students' perceptions

II. Design, implementation and evaluation of a pedagogical intervention that addresses student needs as emerging from I.

Here we report analyses regarding *Ia*. Our work concerning curriculum content within *Ia* was launched with a preliminary analysis, conducted by the Canadian team led by the third author (González-Martín, 2008) of eight recent (1993-2008) upper-secondary texts used towards students' first encounter with the concept. We note that, as books are not the only, and not always the dominant, resource students use, we are also examining lecture notes, exercise sheets etc. where the concept is introduced, even informally, for the first time. Analogously to the three dimensions described by Artigue (1992) this preliminary analysis aimed to address the following questions:

- *Epistemological*: what are the mathematical ideas these texts aim to convey, particularly in the light of the concept's history?
- *Cognitive*: what student learning issues do these texts aim to address?
- *Didactical*: what teaching are these texts conducive to, particularly considering the institutional context in which the concept is taught?

This preliminary analysis suggested that, even though the concept enjoys substantial coverage in most texts (an average of 15% of total number of main text pages), its presentation is largely a-historical and decontextualised, almost exclusively in the algebraic register (Duval, 1995) and with few graphical representations (average of about 0.1 picture per page, slightly increasing from older to newer texts) and applications. Particularly with regard to applications only two of the texts offer real-life applications of the concept and only three present applications of the concept in other disciplines (including Medicine, Economics etc.); and it's not always the more recent texts that present more applications. Finally, there are hardly any historical references in these texts – with the exception of a few, but usually out of context, references to Zeno's paradox – and no attempt is made to present the concept's evolution in the history of mathematics. In sum the texts studied by the Canadian team do not appear to take into account research results regarding the pedagogical potential of introducing concepts via multiple representations (e.g. Duval, 1995) and with a historical-epistemological perspective. The algebraic register and representations are privileged and, overall, the introduction to this concept remains quite formal.

The discussion we present in what follows is based on the analysis of seven texts[1] by the UK team (first two authors). We identified these texts with the help of lecturers teaching the concept to undergraduates in mathematics, science and engineering in the UK. We also draw on interviews with a small number of these lecturers. The texts we discuss here are amongst those mostly recommended by the lecturers to students in university or foundation courses of applied and pure mathematics. Our interviews and text analysis address questions that have emerged from the literature and González-Martín's preliminary analysis.

These questions include:

- Do the text and pedagogical practice support – and how – students' overcoming of key misconceptions tantalising the learning of the concept of infinite series, such as 'infinitely many addends, infinitely great sum', for example, through reference to this concept's epistemology and history?
- Does the text and pedagogical practice use – and how – visual representations in order to enrich students' understanding of the concept?
- In what order does the concept appear in the text and lectures (for example, in relation to the appearance of the notion of numerical sequence of which it is a logical precedent). And does this order – and how – take into consideration the fundamental differences between a mathematically 'appropriate' order and the ways in which students acquire a new concept?
- Do the text and pedagogical practice instil – and how – an algorithmic and mechanical approach to the concept (despite recent research and policy advice to the contrary)?
- Do the text and pedagogical practice contextualise – and how – the concept in terms of its *raison-d'être* in mathematics and its applications in mathematical and other situations?

Here we address some of these questions through drawing on our first-level analysis of these seven texts and the interview with one lecturer (with about sixteen years of teaching experience, in applied and pure mathematics as well as other disciplines, and affiliated with a well-regarded mathematics department). Our overall aim is to explore how texts and – through the perspective of the lecturer – pedagogical practice address certain student needs with regard to the learning of infinite series.

In accordance with the questions listed above we have recorded in a spreadsheet the following information on each text – see also (Nardi, Biza & González-Martín, 2008):

- The number of pages dedicated to the concept of infinite series.
- The number and type of figural representations (e.g. graphs, drawings etc.) and the ratio of representation per page.
- The number and type of applications of the concept of series (e.g. real life applications, applications in other disciplines, problem solving, modelling etc.) and the ratio of application per page.
- The number and type of historical references (e.g. simple references to events, integration of history in teaching etc.) and the ratio of references per page.

Regarding historical references, we found none and so we do not discuss them further even though the discussion of their pedagogical potential remains significant.

Regarding applications, we found three: two in Bostock and Chandler (0.10 per page) and one in Gilbert and Jordan (0.13 per page). One provided a context for a materially-based calculation of $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}$ (a piece of string cut in half, and then in half etc.). The other two made a rather decorative reference to formulae relevant to Economics and Physics and proceeded with the usual application of the mathematical processes. We note that in the Spivak text, although we found no extra-mathematical applications (namely applications outside the field of mathematics), we identified a tendency for intra-mathematical connections (namely connections not necessarily between different disciplines but between mathematical topics). So, for example, in an exercise on p.411, the non-convergence of $\sum_{n=1}^{\infty}\frac{1}{n}$, the *harmonic series*, is connected to the discussion of the infinite number of positive rational numbers.

We allocate the rest of the discussion in this paper to the use of visual representations in the texts as well as in lectures and lecture notes.

VISUALISATION AND EXEMPLIFICATION IN TEXTS AND TEACHING

Regarding visual representations, we found eleven figures related to series in three texts: three in Kreyszig, five in Spivak and three in Stephenson (0.23, 0.19 and 0.12 figures per page respectively). These figures are used mainly for the visual representation of the series terms or the partial sums as points on the number line (Figures 1, 2 and 3) or areas of rectangles (Figures 4 and 5).

In particular, Figure 1 features a neighbourhood of $s(x_1)$, a visual expression for the inequality $\left|s(x_1)-s_n(x_1)\right|<\varepsilon$. Figure 2 features the partial sums s_1, s_2, s_3, s_4 of the series $x_1-x_2+x_3-x_4+\ldots$, where $\{x_n\}$ is a monotonic decreasing to zero sequence.

According to the Leibniz Theorem this series converges and the illustration of Figure 2 supports the claim in the proof that:

$$s_2 \le s_4 \le s_6 \le \ldots,\ \ s_1 \ge s_3 \ge s_5 \ge \ldots \text{ and } s_k \le s_l \text{, if } k \text{ is even and } l \text{ is odd}$$

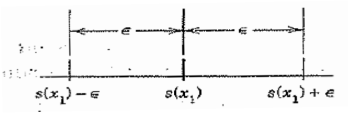

Figure 1. Kreyszig, p. 172

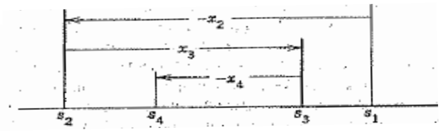

Figure 2. Kreyszig, p. A70

Figure 3 features the terms and the partial sums of the series: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=1$. Through this picture, not only the order of the terms is illustrated but the convergence of the series is evident and as Spivak suggests this is "an infinite sum which can always be remembered from the picture" (Spivak, 1967, p. 391).

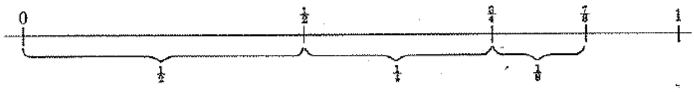

Figure 3. Spivak, p. 391

Figure 4 visualises the symbolic expression: $f(n+1) < \int_n^{n+1} f < f(n)$ for monotonic functions, whereas Figure 5 features "[…] a graphical argument. Each term of the series represents the area of the rectangle with base equal to the unity and height equal to the magnitude of the term" (Stephenson 1973, 72)

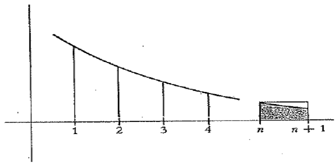

Figure 4. Spivak, p. 396

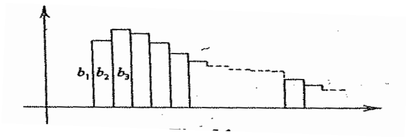

Figure 5. Stephenson, p. 72

In some of the texts we found the following non-figural but rather evocative representation of the proof of the divergence of the harmonic series (via grouping of the terms) – we return to this in our discussion of the lecturer interview data:

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\ge\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\ge\frac{1}{2}}+\underbrace{\frac{1}{9}+\ldots+\frac{1}{16}}_{\ge\frac{1}{2}}+\ldots$$

Figure 6. A non-figural representation of the divergence of $\sum_{n=1}^{\infty}\frac{1}{n}$

The scarcity as well as the often decorative, inconsequential nature of the figural representations used in the texts we examined led us to the preliminary conclusion that the texts do not seem to encourage the – largely supported by the relevant research (e.g Alcock & Simpson, 2004) – dynamic interplay between the algebraic and the graphic registers in their introductions to the concept of series.

Using some of the representations we exemplified above, as a trigger for discussion, in the interviews we explored how pedagogical practice may rise above merely replicating the texts' almost exclusive emphasis on the algebraic register. Below we quote from one interview for this purpose.

The lecturer agrees with our preliminary conclusion about the scarcity of visualisation and applications in books and attributes this partly to 'the rightful fear', originating mostly in the pure mathematics community, that 'when we are drawing one diagram we are showing one example whereas of course, we are trying to show a general argument'. So even though he 'like[s] to draw as much as [he] can' he also acknowledges that he risks his drawing being seen as 'a mere toy'. Despite these risks he insists that diagrams are 'a good heuristic' and offers – see below – several examples of this in the context of series. To stress the value of visual representations to the students he describes his practice of acknowledging their presence in student written work both with encouraging verbal commentary and 'certainly some credit, though not full, some marks'.

At the heart of the lecturer's argument is that, above all, visual representations can be 'persuasive' and can help students overcome limited or misleading perceptions concerning series. One such perception, that is very common amongst newcomers to the concept and needs to be overcome 'before all else', is that 'all I need is for the terms to get smaller and then I have convergence' (another one is confusing the series $\sum x_n$ with the sequence $\{x_n\}$, particularly as students need to understand partial sums S_n of the terms of $\{x_n\}$ as a sequence itself). The lecturer offers two explanations for the origins of the student belief that if the terms of the sequence become smaller then the series converges. First, their previous experience with series may consist entirely of working with the geometric series $\sum_{n=0}^{\infty} x^n$ (which, for $0<x<1$, is convergent and the terms of the sequence indeed become smaller). Second, their previous experience and practice is often limited to a very small number of examples, almost always of convergent series: convergence offers the opportunity to formulate questions with 'an answer to arrive at, a number' and, at least in school, this closure is seen as an opportunity for a 'satisfying' student experience. As a result students may arrive at university with divergence being something 'they have never been confronted with'.

The role of key examples (Mason & Watson, 2005) in altering student beliefs such as the above can be instrumental. The interviewee elaborates the harmonic series as one of these key examples: under the influence of above mentioned belief students start off absolutely convinced of its convergence. But the series is divergent.

When we showed the lecturer Figure 6 he recalled a figural-concretised version of this (Figure 7) that he uses often. He also stressed the effectiveness, in his experience, of offering students 'visual portrayals' of divergence and convergence and mentioned the following two: 'squares and rectangles' (e.g. for the geometric series for powers of ½) and 'staircase' (with its 'capacity to be of a particular height' equal to the sum of the infinite series this is a 'good demonstration' of convergence; significantly, with a slight modification in the size of its 'steps' it becomes a 'dramatic' demonstration of divergence).

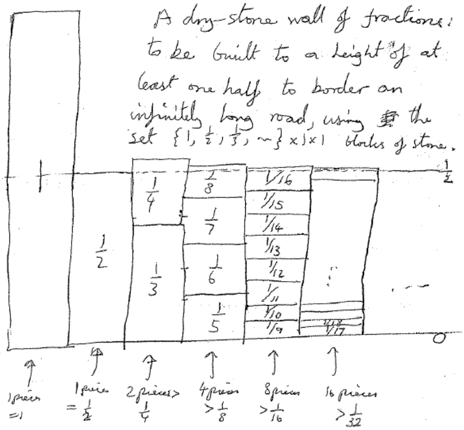

Figure 7. Harmonic series, divergence

In the above we discuss pedagogical practice concerning the introduction to the concept of series in terms of how this practice can address urgent cognitive needs of the students. In this case these needs relate to key perceptions of the concept (e.g. confusing series with sequences; the belief that if the terms of the sequence become smaller then the series converges). The pedagogical practices for addressing these needs include the use of key examples (e.g. the harmonic series) and highly evocative representations (such as the ones in Figures 3, 6 and 7) that may generate productive conflict with these perceptions.

CONCLUDING REMARKS AND FURTHER STEPS

In this paper we presented analyses of texts used in the introduction of the concept of series to upper secondary and university level students in order to explore whether, and how, texts address student cognitive needs with regard to the learning of key concepts. We also began to address how pedagogical practice attempts to address those needs through reference to lecturer interview data. Our analysis of the texts concluded that the presentation of the concept, while substantial in its extent, is largely a-historical and decontextualised, with few graphical representations and even fewer applications or intra-mathematical references to the concept's significance and relevance. Furthemore, bearing in mind the recommendations made in the relevant literature and the interview data, we proposed that privileging, almost exclusively, the formal-algebraic register denies students the insight that can be gained from engaging with visual representations (Duval, 1995), key examples (Watson & Mason, 2005) and an epistemologico-historical perspective on the concept (González-Martín & Nardi, 2007). We will explore the conjectures emerging from this proposition while completing phase *Ia* and launching phases *Ib* and *II* of the study.

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CHARACTERIZATION OF BUS CONDUCTORS' WORKPLACE MATHEMATICS – AN EXTENSION TO SAXE'S FOUR PARAMETER MODEL

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The line of research on everyday mathematics has pointed out the importance of situations that evoke superior performance in quantitative reasoning in everyday settings, and researchers have called for further investigation of everyday practices that involve mental mathematics. The general aim of this study is to develop a better understanding of the mathematics used in everyday situations. In particular, this study focuses on investigating the mental mathematics involved in bus conductors' work in Chennai, India. In this paper, we discuss the extent to which different goal-related activities and the dynamics of their workplaces contribute to their use of mental computation.

INTRODUCTION

In the last two decades several researchers have analyzed and documented the mathematics practices of adults as well as children, which take place outside the school settings (e.g., Carraher, Carraher, & Schliemann, 1987; D' Ambrosio, 1985; Gerdes, 1996; Saxe, 1991). This association has given rise to the recognition of different forms of mathematics such as situated cognition, ethnomathematics, and everyday cognition. The essential principle guiding such studies is the acknowledgement of the fact that people in several walks of life perform mathematical activities out of school, at home, and at work. One area of study that has stemmed from the research field of everyday cognition is concerned with investigating the mathematical practices of adults in various workplaces. This line of research gives some insight into how people conceptualize the role of mathematics in their work. More recent research in workplace mathematics has attempted to uncover the mathematical practices of specific groups such as nurses (Noss, Hoyles, & Pozzi, 2000), automobile workers (Magajna & Monaghan, 2003), and carpet layers (Masingila, 1994). Researchers have investigated the nature of mathematical knowledge used in workplaces and examined how it is similar to and different from mathematics learned in school.

Although over the past 15 years, mathematics education research has begun to explore the nature of the mathematics used in different workplaces, very few studies have investigated the nature of workplace mathematics in India. Guided by the desire to add to mathematics education research in India, the general aim of this study is to develop a better understanding of the mathematics used in everyday situations. In particular, the research purpose associated with this study is to observe, understand, describe, and analyze the mental mathematical practices of bus conductors in their

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workplace and examine what this knowledge can add to the study of everyday mathematics.

THEORETICAL FRAMEWORK

Starting with a broad theoretical field of everyday cognition, we narrowed our focus to concentrate on workplace mathematics. The assumptions underlying the present study are that conductors' workplace mathematical practices are influenced by their working conditions and that their practice-linked goals emerge and change as individual conductors participate in this practice. The three component analytical framework and four-parameter model developed by Saxe (1991) are used to explore the research purposes of this study. Saxe's framework consists of three analytic components that are concerned with goals, forms and functions, and the interplay among various cognitive forms. The first component pertains to the analysis of practice-linked goals that emerge and keep changing as individuals participate in their cultural activities, whatever they might be. The second component analyses how "cultural forms" influence the practice of the participants and how these forms shift in their functions with increased participation. The third component focuses on the interplay between cognitive processes of individuals who participate in distinct practices.

The purpose of this study was to gain insight into the use of mental computation in the bus conductors' workplaces. The research reported in this paper is part of a larger project that investigated the workplace mathematical practices of bus conductors in Chennai, India. In this paper, we discuss the extent to which different goal-related activities and the dynamics of bus conductors' workplace contribute to their use of mental computation. In particular, we address the following research question and propose an extension to Saxe's four-parameter model to suit the findings of this study: In what ways do the bus conductors' goal-directed activities influence their mental computational activities?

METHODOLOGY

Research methods

Addressing the research question required understanding, identification, and analysis of bus conductors' work, work-related goals, goal-oriented activities, and work-related, goal-oriented mental mathematical activities. An instrumental case study approach was employed to carry out this study. The bus conductors are employees of the government organization, Metropolitan Transport Corporation (MTC). A convenience sampling method was used to determine two bus depots from which to select the conductors. Once the two bus depots were chosen, a purposive sampling was employed and five participants were carefully and appropriately chosen based on participants' years of service, educational qualifications, service records, and their willingness to participate in the study. The five conductors who participated in this study are called Mr. Alpha, Mr. Beta, Mr. Gamma, Mr. Delta, and Mrs. Omega.

Data collected included official documents, field notes, summary of observations and informal conversations, transcriptions of formal and semi structured interviews and personal reflections. Documents that were examined and used for this study included a conductor' manual, route information manual, conductors' service records, administrative documents, newspaper articles, and web sources. Over a period of three months, the primary researcher accompanied and observed each bus conductor during their work shifts four-six times, before, after, and during their bus trips. Observation sessions were always accompanied by short informal and semi structured interviews, which took place whenever an opportunity arose or at break times. Informal interviews were aimed at obtaining information about participants' perceptions of mathematics, their views about the role of mathematics in their work-related activities, and their opinions about formal school taught mathematics. Semi structured interview sessions dealt with questions related to the on-site observations and their work-related documents. Using copies of work-related documents the conductors used during the observation sessions, the primary researcher identified specific mathematical activities in which they engaged. The participants were asked to give verbal explanations for some of their actions, to reflect on their actions, and provide insights into their mathematical understanding. They were probed regarding their choice of solution strategies, use of alternate strategies, and use of school taught strategies.

Analysis

With data gathered from the study, it is now possible to conceive of bus conductors' work as consisting of a three-phase structure.

Sign-on time ⟶	Spread over time ⟶	Sign-off time
(Before bus trips)	(During bus trips)	(End of bus trips)

In the first phase, conductors obtained information regarding their bus routes, bus services and bus drivers and collected the required artifacts before they entered into the second phase of their work – the bus trips. During this phase, bus conductors commuted several times on a bus from point A to point B along different routes. They picked up and dropped off commuters en route and regulated their entry into and exit from the bus. Bus conductors' duties included issuing a ticket to a commuter based on the entry and exit point, tendering the exact change back to the commuter when the commuter gave more money than the required amount, keeping a record of the number of tickets sold, calculating the daily allowance based on the day's collection, and submitting the trip earnings to the supervisor at the end of a shift. After completing all of their scheduled bus trips, conductors entered into the third phase of their work-shift. During this phase they remitted their overall earnings to the accountant and collected their daily allowance.

Overall Goals Related to Mathematical Activities

Based on the data collected from observations, field visits, and interviews with bus conductors, we identified specific goals and goal-directed, work-related activities that required conductors to do mental computation. Every day, as bus conductors reported to work, they aimed to complete all of their work-related duties. To help them complete their duties, conductors set overall work-related goals. These overall goals were fixed and determined the general plan of approach that the conductors followed to complete their goals. To execute their plan of approach, conductors carried out goal-oriented activities. Attached to some of the work-related, goal-oriented activities were mental mathematical activities. In table 1, we have presented bus conductors' overall goals, goal-oriented activities, and related mental computational activities.

Overall Goal	Goal-Oriented activities	Mental Computational Activities
Authenticate every passenger's travel in the bus (spread-over time).	Approach passengers and gather information. Issue tickets quickly and efficiently.	Solve mental computational problems associated with ticket transactions.
Complete official documents (spread-over time).	Complete Traffic Return (official document) after each fare stage. Complete *waybill* (official document) to determine overall earnings.	Solve mental computational problems associated with determination of (a) total number of tickets sold in each denomination, (b) collection amounts for each ticket denomination, and (c) total collection amount.
Submit daily earnings (sign-off time).	Complete *waybill* using information from a TR and ticket bundles. Calculate daily earnings using *waybill* information. Use overall daily earnings to calculate *batta* (daily allowance).	Solve mental computational problems associated with calculation of *batta* using daily collection information.

Table 1: Description of work-related and mental computational activities.

Use of mental computation

During a work shift, conductors used mental computation to complete ticket transactions, to complete a *waybill* (an official document used to keep track of the number of tickets sold in each ticket denomination) and to determine their *batta*. To carry out a ticket transaction, conductors determined the number of fare stages between the entry point and the exit point, calculated ticket fares for single and multiple passengers, and determined the balance amount due to passengers, if any. To complete a *waybill*, they calculated the ticket collection amount in each denomination and the overall collection amount. They used the overall collection amount to determine their *batta*.

Determination of total earnings: Conductors used the *waybill* to record the number of tickets sold in each denomination and used this information to calculate overall earnings. In table 2, we present information adapted from Mrs. Omega's *waybill* contents (official record).

Ticket denomination (In rupees and paise)	Total number of tickets sold	Total amount collected
2.00	424	848.00
3.00	349	1047.00
3.50	123	430.50
4.00	79	316.00
4.50	43	193.50
Token ticket	1	30.00

Table 2: Total earnings by ticket denomination.

Mrs. Omega was asked to explain her thinking as she completed the *waybill* entries corresponding to each ticket denomination and the total amount corresponding to each ticket denomination. At work, she said that she resorted to conductors' mathematics to determine ticket collection amounts. She explained the way in which she mentally calculated the total ticket earnings for each denomination.

> I have sold 123 tickets corresponding to 3.50 ticket denomination. Our (conductors') technique is to calculate ticket amount for every 100 tickets. Thus the collection amount is Rs. 350 for the first 100 tickets. I should now account for the remaining 23 tickets. I first calculate ticket amount for the first 10 tickets, which is Rs. 35.00. I double it to get Rs. 70.00 for 20 tickets. I now add Rs. 350 and Rs. 70 to get Rs. 420. I then calculate collection amount for 3 more tickets, which is Rs. 10.50. Now I add this to Rs. 420 to get Rs. 430.50. I do it all in my mind.

Using the above technique, Mrs. Omega was able to quickly determine daily earnings. Further, she said that she successfully used this form of mathematics to work on new bus routes with varying ticket denominations.

Calculation of *batta*: Here we use data from Mr. Gamma's *waybill* (official record) to demonstrate his use of mental computation as he determined his *batta*. Mr. Gamma determined his daily allowance at the end of his work shift. On a certain day, the *waybill* records indicated that Mr. Gamma claimed Rs. 89 as the *batta* and remitted Rs. 2894 to the accountant (field note). I asked Mr. Gamma to explain how he arrived at these figures. I present Mr. Gamma's *batta* calculation techniques below in his own words.

Batta is the sum of fixed and variable allowances. In an ordinary service, for every 100 rupees collected, the variable allowance is Rs. 2.35. Total amount collected: 1000 + 1000 + 953 = 2953.

Variable *batta* is 23.50 + 23.50 + 23.50 = 70.50

I also sold three token tickets that day; for each ticket I will get 10 paise each. Thus the total variable *batta* is 71 after rounding to the nearest rupee. The fixed allowance is 18.50. If this amount is added to Rs. 71.00, I get the *batta* as Rs. 89.50. In order to subtract this amount from 2953.00, I first subtracted 100, to get 2853.I then compensated by adding the extra amount that I took away by adding 10.50 to it. Thus the total amount after deducting *batta* is 2863.50. To this amount, I added the advance amount that I received during sign-on time (Rs. 30.00) to get the total amount as 2893.50. This is what I owe the accountant.

DISCUSSION

Characterization of conductors' mathematics

Conductors' workplace mathematics has certain unique characteristics that are shaped by the context and the tools specific to their workplace. The first characteristic concerns their understanding about the specifics of their workplace activities. This includes knowledge about different bus depots and routes of the MTC, fare stages along different routes, and knowledge about ticket fares. When conductors completed ticket transactions, they drew upon their understanding of work-related notions such as determination of fare stages, fare stage numbers, and ticket prices associated with fare stages. This understanding, which we term *work-specific knowledge*, helped conductors determine the ticket fares and the balance amount due to the passengers and thus execute ticket transactions smoothly. Second, certain observed mathematical activities of bus conductors have close connection to school-learnt mathematical concepts. All conductors acknowledged that the mathematical ideas that they learnt at school helped them solve problems at work. They pointed out that they used school-learnt arithmetic concepts to solve problems that arose out of ticket transactions and *waybill* calculations. Unlike participants in certain other workplace investigations (e.g. those involving candy sellers and newspaper vendors) who did not have access to school education, all of this study's participants possessed varying levels of school education. This fact, juxtaposed with the conductors' belief that they used school-learnt arithmetic concepts at work, leads us to conclude that bus conductors integrated school-taught mathematical ideas with other features specific to their workplace to solve work-related mental mathematical problems. Third, the use of monetary units played a significant role in shaping their mental mathematical ideas. When describing the factors that helped them compute mentally efficiently, conductors were quick to single out the use of monetary units. They said that they completed mental mathematical problems that involved whole numbers, decimals, and fractions by treating the whole number part as the rupee equivalent and the decimal part as the paise equivalent. Mrs. Omega's comments on this topic highlighted the advantage of the use of monetary units.

Mrs. Omega: We used to have 1.75, 2.35 and such types of ticket denominations before. Calculations could become very confusing. I agree with what other conductors said about the use of money. It is very easy to do calculations. There is also another benefit. How can you be sure you are correct? This is where money can help. You can check if you are doing correct calculations by looking at the money in hand.

Although the conductors noted that they knew how to solve such problems mentally (without the use of monetary units), the availability of these units helped them complete the problems more quickly. In summary, conductors' workplace mathematics is the result of integration of work-specific knowledge, formal school-learnt mathematical ideas, and the knowledge about the currency system.

Emergent Goals Related to Mental Computational Activities

Elsewhere we have described in detail, using Saxe's four-parameter model, bus conductor's and our perceptions of emergent goals using Saxe's four context-related parameters: *activity structures, conventions and artifacts, social interactions, and prior understandin*gs (Naresh & Presmeg, 2008). Bus conductors' emergent goals were related to activities such as issuing tickets, carrying out transactions, maintaining a record of transactions and so on. The activity structures, artifacts, and their prior understandings of the practice influenced the emergent goals. We claimed that the differing perceptions of bus conductor's work-related goals were complementary, and completely consonant with Saxe's four-parameter model.

Extension of Saxe's Four-Parameter Model

In the original model proposed by Saxe, the structure of the emergent mathematical goals is explored in terms of four context-related parameters – *activity structures, prior understandings, social interactions, and conventions and artifacts*. In order to suit the findings of this study, it was necessary to slightly modify Saxe's original model. Figure 1 presents a connection between bus conductors' overall goals, goal-directed activities, and the emergent goals associated with the mathematical activities. In Figure 1, the double arrows should be interpreted as "gives rise to". The emergent goals connected with mathematical activities were influenced by four context related parameters – activity cycles, social interactions, work-related artifacts and tools, and prior knowledge and skills. In Figure 1, the single arrows connected to the emergent goals indicate the "influence of" context related parameters on the emergent goals. For example, the emergent goal associated with a ticket transaction could be "I issued 5 tickets to a passenger who travelled from Adyar signal (entry point) to Purasawalkam (exit point)". Here the conductor obtained information regarding the number of tickets and the entry and the exit points when the ticket transaction process was initiated. Thus, the goal associated with this ticket transactional activity surfaced and disappeared at those instants when the related activity was initiated and completed. Similar goals (associated with ticket transactions and *waybill* and *batta* calculations) arose and faded with the initiation and completion of related activities. Another facet to the emergent goals is provided by our interpretation of these goals. When we described and analyzed the emergent goals associated with bus conductors' mathematical activities, we focused on bringing out the mathematical problems associated with these activities, which were expressed directly by the conductors.

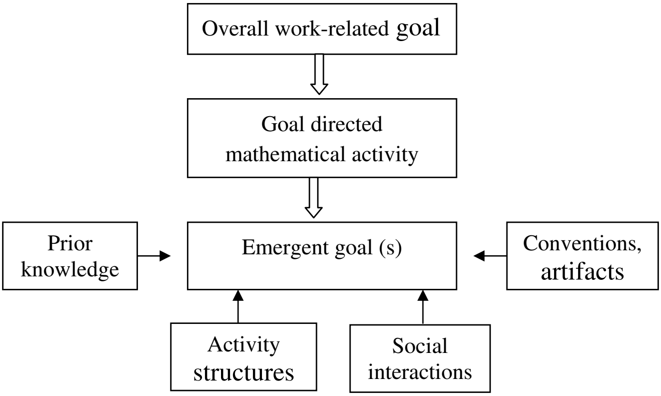

Figure 1: Extended four-parameter model

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SECONDARY SCHOOL STUDENTS' CONCEPT OF INFINITY: PRIMARY AND SECONDARY INTUITIONS

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The concept of infinity has an important place in secondary and higher education curricula. The aim of this study is to investigate primary and secondary intuitions of Turkish primary school students in relation to the concept of infinity as well as to determine to what extent schooling is successful in the attainment of the concept of infinity. Qualitative research techniques were used in this study. Data collection included open-ended questionnaires conducted with 131 primary school students aged 13-14 and semi-structured interviews with ten of these students. The data were analysed by categorisation. Results indicated that students' personal experiences mainly determined their concept of infinity and that formal education had minimal effects. Some misconceptions were also found to exist.

INTRODUCTION

The concept of infinity is intuitive *per se*. The history of mathematics indicates that certain concepts are not immediately internalised. It can even take decades or centuries for the mathematics world to accept such concepts (Fischbein, 1987). Unwillingness to regard infinite sets as a mathematics object could be extended to the time of Aristo, who argues that infinity is only a potential rather than a reality (Tirosh, 1999). The concept of infinity had caused several controversies emerging from its own nature. Galileo and Gauss concluded that real infinity could not be included in rational and consistent reasoning. Gauss emphasized in 1831 that an infinite multitude can never be allowed to be used as a complete quantity. Kant, on the other hand, referred to the infinity of space and time and argued that human mind cannot comprehend either the finiteness or infinity of the world in terms of both space and time. (Fischbein, 2001)

Philosophers and mathematicians distinguished real and potential infinity. Real infinity is a concept difficult and even impossible to grasp by human intelligence as in the examples of "*the infinity of the world, the infinity of the points on a line*". Controversies start to emerge when real infinity is studied (ibid.). For example, the definition of infinite sets by Cantor constitutes a crucial perceptual handicap in relation to the idea of infinity as it includes the concept of "equivalence of a set with one of its prime sub-sets". The acceptance of this equilibrium requires perceptual effort because it necessitates the acceptance of the idea that "the whole is bigger than its parts" may not be valid for all sets. Therefore it is a small probability that the definition of infinite sets be naturally used by students who have yet learned the theory of sets (Tirosh, 1999).

Since Cantor the concept of infinity which includes aforementioned difficulties has been widely studied. Fiscbein *et al*. (1979,4-5), although indirectly, proposed the work

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 209-216. Thessaloniki, Greece: PME.

of Piaget and Inhelder (1956, p.125-149) to be the beginning of studies on children's understanding of infinity (Monaghan, 2001). Since this study on the repetitive division of geometric figures research on the concept of infinity have advanced. Research on intuitive methods students' use to determine the finiteness or the infinity of a set (Falk *et al*., 1986; Tsamir, 1994) showed that eight year-olds were able to understand that the infinity of the set of natural numbers. 11-12 year-old children then discover the non-dimensionality of points and subsequently argue that line segments can be infinitely divided. In these studies, children were asked whether some processes will come to an end or not. Children who argued that the process would not end were accepted by the researchers to comprehend that the obtained set is infinite (Tirosh, 1999). Infinity intuitions do not change and remain stable after the age of twelve (Fischbein *et al*.,1979) which confirms the researches conducted with older children and reported that students had difficulties in comprehending the Cantorian set theory (Narli and Baser, 2008; Tsamir and Tirosh, 1992,1994)

Infinity is an intuitive knowledge which is naturally comprehended and its accuracy absolutely accepted. However, this process does not emerge on its own. Intuitive knowledge is either acquired through educational intervention and not natural experience (secondary intuition, Fischbein 1987, p. 71) or developed by the individual independent of a systematic education as a result of personal experiences (primary intuition, ibid., p. 202). In terms of the relationship between the two on the same concept, such as infinity, secondary intuition could be less consistent than primary intuition. Primary and secondary intuitions on the concept of infinity play an important role in mathematics especially in the full comprehension of numbers. This study aimed to determine students' primary and secondary intuitions in relation to infinity and to investigate to what extent schooling effects students' idea of infinity.............. There is no "infinity" topic in Turkish curricula of primary school mathematics on its own. Moreover there is no relation or expressions in the objective of "Numbers" relevant to infinity. Although similar studies exist in literature (Singer and Voica, 2003; Tirosh, 1999), there is almost none in Turkey. A study in Turkish context would contribute to literature of infinity.

METHODS

This research has interpretive approaches (Cohen et al., 2000, p.22). Case study is used as a research strategy to make an in-depth examination of students' intuition of infinity concept in this study (ibid., p.181-182). Qualitative techniques were conducted to collect data; the open-ended questionnaire and semi-structured interview. Content validity of the data collection instruments was obtained by a detailed consideration of the scope of research by four tutors in Department of Mathematics Education. In order to ensure reliability qualitative data were categorised and coded (Miles and Huberman, 1984:23)**.** Compatibility rates among these categories were then calculated. The coding revealed a compatibility rate of %92.

Purposeful sampling technique (Patton, 1990) of non-probability sampling methods, which accept individuals or events as they are, was used for the selection of the sample

(Cohen *et al.*, 2000). The sample of this study was selected from eighth grade students of a primary school of the Ministry of National Education. Open-ended questionnaires were administered with 131 students selected from two primary schools in Izmir, one in the centre and one in a province. The interviews were then conducted with ten of these students. Table 1 presents the sample distribution of the study.

RESULTS

The analysis of open-ended questionnaires and semi-structured interviews was considered to indicate the source of students' intuitions of infinity.

The open-ended questionnaire

The open-ended questionnaire used in the study was composed of 7 questions. Three of these questions were analysed in this article.

Students' ideas in relation to infinity; The first question addressed to the students in the open-ended questionnaire was "What occurs to your mind when you think of infinity and how would you define infinity in your own words?" The answers to this question were categorised in six items (Table 1).

CATEGORIES	Number of Students	Percentages
1.Endlessness and continuity	91	69%
3.Spatial	44	33%
5.Countable and Operational	30	23%
2.Emotions and Beliefs	27	20%
6.Other	23	17%
4.Relating to life	18	13%

Table 1. Students' ideas in relation to infinity

As presented in Table 1, ***endlessness and continuity*** category has the highest frequency. Students tend to explain infinity with continuous and endless concepts. *"a long thing without an end such as sea. For example you can stay at a point; it is a length whose end cannot be seen for example natural numbers are infinite. They go to infinity."* Some of the explanations in this group accept the existence of a beginning but not an end to infinity: *"Things without an end, I mean I think of things with a beginning but without an end."* To a lesser extent the explanations in this group implied neither a beginning nor an end: "*Some things are innumerable, it is used for things which do not have a definite beginning and end – they don't have a beginning nor an end*". There are also statements which accept both types of infinity *"A concept with a beginning but without an end or without a beginning and an end"*. However, there were still a small number of infinity statements in relation to formal education: "*Natural numbers are infinite, thus I think of natural numbers*".

emotions and beliefs category includes answers which students deduced from their emotions of infinity and from social orientations (religious beliefs). Some students in this group preferred to define infinity based on the effects of infinity on their inner worlds: "*It's loneliness", "It's immortality", "It's freedom", "When I think of infinity I'm somewhat scared, I feel as if I'll be lost in infinity"*. It is interesting that students explain infinity with extreme feelings. Culture which is the source of higher order cognitive processes affect students' views (Vygotsky, 1978). A group of students reflected society's beliefs and religious views in their definitions of infinity: *"Infinity are goods which will never come to end, which will never run out, for example: after life is a never ending infinite life. An infinite life means that we will live there throughout our lifetime without dying"*.

A group of students tried to define infinity influenced by ***visual*** stimulus. They generally thought of infinity as a place which cannot be restricted or which is endless*: "An endless tunnel and not thinking of anything in that tunnel"*, *"I think of space"*. Some students relate infinity to ***life***: "*Infinity for me means to live long*, *life without an end"*. Eventually some students identify infinity with immortality. This resulted in some students to even deny infinity: "*If there was infinity plants would not die they would all be still alive"*.

Countable and operational category includes students' formal statements of infinity: "*I first think of numbers without an end which last forever, Natural numbers are infinite, thus I think of natural numbers"*. Moreover some students make a distinction: "*There are two types. One is the endlessness of the world and universe, and the other is the infinity of natural numbers in mathematics, an empty set"*. Although definitions of infinity in relation to students' knowledge gained through formal schooling were elicited, formal education seems to be insufficient in teaching students the idea of infinity (Table 1). Moreover students may be mistaken by their identification of being uncountable and infinite: "*It means that some things are innumerous, it is used for things without a certain beginning or end"*.

There were also ***other*** definitions beyond these categories. "*I remember unlimited internet and msn"*, *"I don't remember anything"*, "*They are things I can't do"*

Students' First Encounters with the Concept of Infinity; Table 2 summarises the answers elicited in response to the question "*When (at what age), where and how did you first encounter the concept of infinity?*" from the open-ended questionnaire. Almost half of the students were found to encounter the concept of infinity at school. While some students said "At school – at preschool – at secondary school – at primary school", others replied in terms of lesson and topic: *"at Mathematics lesson – I learnt that universe is infinite in the planets topic – I encountered it when we were learning space-in the topic of sets-in numbers-in rational numbers –in natural numbers –in citizenship lesson (thought it was the concept of humanity)- by asking how many numbers there are"*. Some students stated that they first encountered the concept of infinity within their family: *"One day I was trying to count the stars. Then my mum said "you can't count stars because they are infinite" -at home- within my family- while I was talking to mum.* While some students mentioned that they encountered the concept of infinity at very early ages (below 10), some mentioned later ages such as 13-14. Individual responses were grouped as "other"; *"for as long as I know myself"*.

CATEGORIES	Number of Students	Percentages
1.Age; (1-10)	52	39%
(10-12)	9	6%
(12-14)	5	3%
2.At school, during the lessons, from my teacher	72	55%
3.At home, from the family	22	16%
4.Other	45	34%

Table 2. Students' views about when they first encountered the concept of infinity

Students' encounters with the concept of infinity at school; Table 3 summarises students' responses on whether they had encountered the concept of infinity in formal education. The number of students who expressed that they didn't encounter the concept of infinity at school is almost nonexistent. Almost half of the students stated that they were introduced to the concept of infinity at school. The highest frequency in terms of lesson and unit was observed in the mathematics lesson.

CATEGORIES	Number of Students	Percentages
I have	69	%52
I haven't	7	%5
I don't know	2	%1
Lessons		
In mathematics lesson	74	%56
In other lessons (Turkish, Sciences, History, Arts, Knowledge of Life, Religion, Geography, Social)	46	%35
Units		
In the units related to numbers such as Integers	57	%42
Other (Sets, Geometry, Space, Lines, Pressure	23	%17
Other	16	%12
Everywhere-I had some information in this concept-While we were learning about Atatürk "Atatürk has an infinite place in our hearts, he never died and will never die" etc…		

Table 3. Students' views on how at school they encountered the concept of infinity

Data obtained from semi-structured interviews

The interview answers were observed to largely overlap with the categories obtained from the open-ended questionnaires. During the interviews the students did not seem to have a certain idea of infinity and they explicitly stated that: *"... infinity, erm, how can I say, without an end. I mean even thinking about it doesn't have an end*", *"...something endless, unreachable"*,*"...endlessness, freedom"*. One of the interesting points emerged during the interviews was that students initially tended to state that infinity would have a beginning but not an end. However, these responses were later observed to shift. There were also students who right away stated that infinity could be without a beginning or an end. Some students initially objected to infinity having a beginning and an end, though later accepted that it could have an end. When prompted to provide an example for infinity with or without a beginning or end, they tended to look for examples in things they could see. This could cause delusions. When the subjects were asked about the infinity of numbers, they oriented towards finding a beginning and end to numbers and thus were inclined to provide an answer. When the students were asked about when they first encountered the concept of infinity, despite a clear answer the responses included at school *"... in sciences they say space and the like are infinite and in mathematics they say numbers are infinite"* or out of school *"...my mother's advice is infinite*". About the opposite of infinity that is finiteness students usually have much clear expressions "*you start doing something and finish it either this way or that, it's something that ends, with limits, with a certain ending, something restricted, the person who prevents you*".

DISCUSSION

Students' definitions of infinity with their own words were generally related to their primary intuitions except "Countable and Operational" category. Students mostly used informal expression in their definitions of infinity. Their expressions were consistent with the definitions of Singer and Vocia (2003). Additionally, this study also included a "related to life" category which is different than Singer and Voica. This research indicated a similarity between primary intuitions of young students and teacher trainees.

Findings indicated that students were inclined to explain infinity using concepts of endlessness and continuity. Students also related infinity to extreme emotional statements such as "loneliness, death" etc. Such emotions might seem endless during the time they are experienced and thus could be related to infinity. Statements in relation to beliefs in this category were mainly elicited from students who lived in the suburbs. Social structure could be argued to affect primary intuitions especially higher influence of religion in the suburbs was considered. ....That might show the influence of cultural difference on intuitions in relation to infinity. Therefore comparative studies may be done between cultures and countries. Conducting a study in Turkey is good opportunity to compare with other countries and culture

Students who used spatial-visual expressions in relation to infinity always described unrestricted places. This could imply that students do not think about limitedness and infinity together. Problems in students understanding of the infinity of limited real intervals could thus emerge. Literature reports the existence of primary school students who accept real spaces to be infinite by also accepting finite sets as limited sets as well as infinite sets as unlimited sets (Singer and Voica, 2003).

As some students were observed to relate infinity to life, students could be argued to create links between the concept of life and infinity. Thus, future research into intuitions for life, could further our understanding of students' intuitions of infinity in relation to this category.

Most of the formal statements used by the students that were categorised under "Countable and Operational" were about mathematics and number sets. It is thus possible to conclude that secondary intuitions in relation to infinity are mainly influenced by number sets. It was interesting to elicit responses such as "*There are two types. First is the endlessness of the world and universe. Second is the infinity of natural numbers in mathematics*". This could be primal evidence for the existence of primary school students who can comprehend real and potential infinity.

Although students first encountered the concept of infinity at school their initial statements in relation to the concept of infinity were generally informal. This could indicate that schooling was not influential enough to substantially change students' ideas of infinity. Moreover, students expressed that they rather came across infinity in mathematics lesson at school. In terms of units, number sets are foregrounded. Thus, the concept of infinity could be related to the teaching of numbers.

Students generally used statements which referred to their life out of school. These statements included cultural elements such as religion. This could indicate that society and personal experiences influence primary intuitions in relation to infinity. Students' formal definitions of infinity were mainly about numbers. Furthermore these statements were not clear and primary intuitions were of little support to these statements.

Internalisation processes for the concept of infinity, which is difficult to comprehend and could only be expressed by intuition, should be enriched by considering students' personal conceptualisation skills and daily life, numbers, geometry and the like. Despite difficulties in defining the concept of the infinity, conceptual images could be enriched by experiences, activities and projects (Vinner**,** 1991).

CONCLUSION and SUGGESTIONS

The findings indicate that students did not have a clear idea of infinity and that their ideas of infinity were not learned through formal schooling. Students mainly defined infinity related to their experiences out of school. This could indicate that schooling was not sufficiently beneficial to the development of ideas regarding infinity. Thus, being a crucial element of intuitional understanding the concept of infinity should be attached sufficient importance in the primary school curriculum. Activities and projects could be designed accordingly for this purpose. An example for such an activity could be to calculate when a ball, thrown from a certain height, would stop by bouncing half its original height each time or placing an object in the middle of a rectangular prism with all interior walls except the top surface covered in mirrors and see infinite points.

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INVESTIGATING INDONESIAN ELEMENTARY TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING GEOMETRY

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This paper reports on an exploratory study investigating Indonesian elementary school teachers' mathematical knowledge for teaching (MKT) geometry to determine factors that influence this particular knowledge. A survey was administered to 167 elementary school teachers to collect data on their educational background, and an adapted version of the U.S. based MKT instrument was used to measure subjects' MKT. Analyses of variances on subjects' MKT revealed significantly differences based on the number of years of teaching experience, educational level attained, school type, and grade range taught. A multiple regression model was developed and showed that educational level, school type, grade range taught were significant predictors of elementary teachers' mathematical knowledge for teaching geometry.

INTRODUCTION

Improving teachers' content knowledge has been central to improving student achievement in mathematics (Hill, Rowan, & Ball, 2005). Elementary teachers in the United States and many developing countries are typically in the bottom one-third of high school graduates (National Center on Education and the Economy, 2007). Moreover, these teachers are trained to be generalists and may not have extensive knowledge of mathematics. In the case of geometry, prospective teachers study geometry once as students themselves when in secondary school, and then typically encounter geometric concepts only once more in a college course before they are certified to teach (Grover & Connor, 2000). Therefore, it is not surprising that studies show that pre-service and in-service elementary teachers' content knowledge of geometry is particularly poor (Jones, Mooney, & Harries; Mooney, Fletcher, & Jones, 2003; Fujita & Jones, 2006).

With the growing attention to subject matter knowledge that is situated in the classroom, researchers have argued that teachers need a specialized content knowledge (Ball, 1999). This knowledge, termed *mathematical knowledge for teaching* (MKT), represents an idiosyncratic type of professional knowledge necessary for the various aspects of teaching mathematics (Ball, Hill, & Bass, 2005). Measures have been developed to capture this type of knowledge (Learning Mathematics for Teaching, 2006), and have been used to evaluate professional development programs (Hill & Ball, 2004). The MKT measures have also been used to study the relationship between mathematical knowledge and student achievement (Hill, Rowan, & Ball, 2005), and to study the relationship between mathematical knowledge and the mathematical quality of instruction (Hill, Blunk, Charalambous,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 217-224. Thessaloniki, Greece: PME.

Lewis, Phelps. Sleep, & Ball, 2008). However, the MKT construct and measures were based on teaching in the United States, and hence, may or may not be applicable to other cultural settings since evidence suggests that teaching is a cultural activity (Stigler & Hiebert, 1999). Studies to examine construct equivalence and the validity of the measures in several countries are under way (e.g. Delaney, Ball, Hill, Schilling, & Zopf, 2008) and will provide different cultural perspectives on what teachers need to know.

This study, built on this area of research on mathematical knowledge for teaching, had two purposes. The first purpose was to examine how well the U.S.-based construct performed in a developing country. In other words, can the measures be used to discriminate among elementary teachers based on their mathematical knowledge for teaching? The second purpose was to investigate factors related to Indonesian teachers' mathematical knowledge for teaching geometry.

METHODOLOGY

Subjects in this study consisted of 167 elementary teachers enrolled in professional development programs in Indonesia. Two instruments were used. The first instrument consisted of a survey that required subjects to provide background information on number of years of teaching experience, grade levels taught, educational level attained, school type (public or private), number of hours of professional development completed, number of college level geometry courses taken, and instructional practices. The instructional practices scale consisted of eight questions, each with a 6-point Likert-scaled response, ranging from *never* to *every day*. These questions were classified into two subscales: traditional instructional practice and reform instructional practice. The second instrument was the mathematical knowledge for teaching (MKT) geometry measure developed by the Learning Mathematics for Teaching project (LMT, 2006) which was translated into Indonesian and adapted for cultural suitability. This measure consisted of 19 multiple-choice questions (based on third through eighth grade geometry content) with a range of 3 to 7 possible solutions. This measure was also administered to the same sample of subjects mentioned earlier. Subjects' responses were recoded, 0 for incorrect answers and 1 for correct answers. Raw scores were obtained and then converted to MKT scores, which are linear and can be expressed in standard deviation units.

To determine how well the MKT measure distinguished one individual from another, the reliability of the measure using Cronbach's alphas was calculated. Analyses of variance were conducted to examine differences between the teachers based on the background variables. To explore factors that might be contributing to teachers' MKT, a multiple linear regression model using a backward elimination selection process was conducted to determine which factors were significant predictors of teachers' mathematical knowledge for teaching geometry.

ANALYSES

The Cronbach's alpha for the MKT measures was 0.634, which measures the amount of observed individual differences attributable to true variance in subjects' mathematical knowledge for teaching. A good measure will have a reliability of at least 0.7. However, Figure 1 shows that the distribution of the subjects based on their mathematical knowledge for teaching was close to a normal distribution. This result shows that the measures were able to discriminate the subjects based on their mathematical knowledge for teaching.

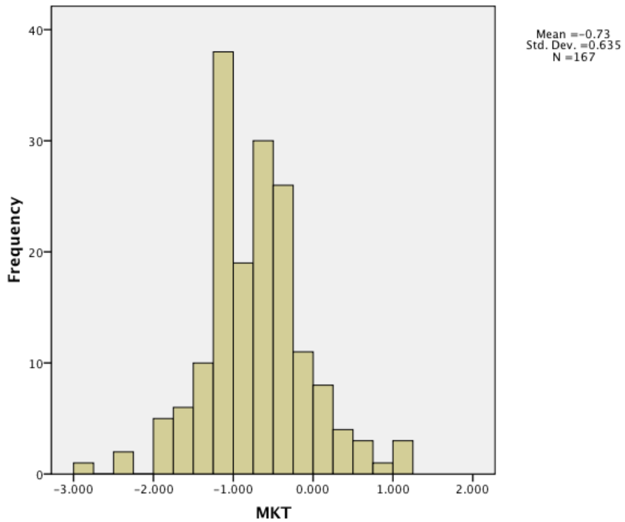

Figure 1: Distribution of Mathematical Knowledge for Teaching (MKT) Scores.

Analysis of variances revealed that there were significant differences between groups of subjects based on their years of experience in teaching ($p<0.05$) as shown in Table 1. The relationship between years of teaching experience and subjects' MKT score was not linear. However, the number of subjects who had taught zero to one year (N=8) and the number of subjects who had taught two to four years (N=18) were relatively smaller compared to the other groups. Ignoring these two groups resulted in a more linear relationship between number of years of teaching experience and the mean MKT score, but the relationship was an inverse one; subjects who had taught for a longer period of time tended to have lower MKT scores. The result of this study contradicts Hill's (2007) study on middle school teachers in the United States, where teachers with more experience were found to have better mathematical knowledge for teaching. The discrepancy may be attributed to difference in the sample; Hill's study consisted of middle school teachers who were better prepared in the terms of mathematics content compared to elementary teachers who were generalists. Possible reasons for Indonesian elementary teachers' weaker knowledge over years of experience include the lack of requirements for Indonesian teachers to continue learning content throughout their careers (Bjork, 2005), limited opportunities for teachers to access resources (Saito, Imansyah, Kuboki, & Hendayana, 2007), and the minimal number of high quality professional development programs in Indonesia (Bjork, 2005; Joni, 2000).

Years of Teaching Experience	N	IRT Mean (SD)	p-value
0-1 year	8	-0.99 (0.37)	0.033
2-4 years	18	-0.64 (0.74)	
5-9 years	31	-0.48 (0.67)	
10-15 years	35	-0.71 (0.67)	
16-20 years	31	-0.69 (0.51)	
> 21 years	44	-0.95 (0.60)	

Table 1: Mathematical Knowledge for Teaching (MKT) Score Based on Number of Years of Experience (N = 167, significant at 0.05 level).

There is a possibility that the relationship between subjects' number of years of teaching experience and their IRT score is indeed not linear. Subjects who had taught only for one year were considered to be novices and had lower MKT scores because they were still in the process of adjusting to the profession, even if they had completed a high-quality pre-service program that prepared them with the necessary content knowledge for teaching. On the other hand, subjects with more experience in teaching might have completed a pre-service training programs that was not very rigorous or mathematics focused, since the requirement for prospective elementary teachers in Indonesia has changed significantly over the past 60 years. These subjects mathematical content knowledge for teaching might be weak because they only completed the minimal teacher education requirements. Although these subjects had more experiences in the classroom which exposed them to specialized content knowledge for teaching, they did not have a strong content knowledge base on which to build their content knowledge for teaching.

Higher education levels, as expected, contributed to teachers having better mathematical knowledge for teaching (Table 2). Subjects who had four years of training scored significantly higher on the MKT measures than those who had two years of training, and subsequently they did better than those with only a high school diploma. It is unclear, however, whether this pattern will continue or will taper off beyond four years of higher education. None of the subjects in this study had earned a degree beyond a bachelor's degree.

Educational Level	N	IRT Mean (SD)	p-value
High School Diploma	23	-1.01 (0.57)	<0.001
Diploma Degree	78	-0.85 (0.55)	
Bachelor Degree	66	-0.50 (0.69)	

Table 2: Mathematical Knowledge for Teaching (MKT) Score Based on Education Level (N = 167, significant at 0.001 level).

This study showed that teachers' mathematical knowledge for teaching did not differ between those teaching lower elementary level and upper elementary level grades. However, the range of grade levels teachers have taught contributed to their mathematical knowledge for teaching (Table 3). Teachers who had taught a wider range of grades did significantly better on the MKT measures than those who had taught only one or two grades even after controlling for number of years of teaching experience.

School Type	N	IRT Mean (SD)	p-value
Grade Range Taught			0.038*
1-2 grades	55	-0.88 (0.62)	
3-4 grades	63	-0.74 (0.65)	
5-6 grades	49	-0.56 (0.60)	

Table 3: Mathematical Knowledge for Teaching (MKT) Score Based on Grade Range Taught (N =167, significant at 0.05 level).

The regression model of teachers' overall score on their background information indicated that there is a relationship between educational background, school type, grade range taught, and reform instructional practice and mathematical knowledge for teaching geometry even when holding other factors constant (Table 4). One exception is years of teaching experience. Although there were differences in subjects' mathematical knowledge for teaching scores, based on their teaching experiences as shown from the analysis of variance mentioned previously, the regression model did not identify experience in teaching to be a significant predictor of teachers' knowledge for teaching. Educational background, school type, grade range taught, and reform instructional practice were predictors of higher levels of mathematical knowledge for teaching geometry. Having a diploma was associated with about 0.16 standard deviations on the MKT score; a bachelor degree increases the score by 0.375 standard deviations. Teachers who taught a range of three to four grades performed almost a tenth of an extra point on the MKT measures; teachers who taught a range of five to six grades had an increase of 0.31 standard deviations. Teaching in private schools was associated with an increase of 0.32 standard deviations compared to teaching in public schools. Finally, each additional frequency of using reform instructional practice was associated with nearly 0.02 standard deviations, which is considerable given that the average teacher scored 16.12 points on their reform instructional practice scale.

Predictor	Slope	p-value
Intercept	-0.49	0.008
Grade Range Taught		0.024
1-2 grades span	-0.31	(0.008)
3-4 grades span	-0.22	(0.047)
5-6 grades span	0	-
Educational Background	0	0.017
High School Diploma	-0.375	(0.009)
Diploma Degree	-0.216	(0.037)
Bachelor Degree	0	-
School Type		0.001
Public	-0.32	(0.001)
Private	0	-
Reform Instructional Practice	0.018	0.041

Table 4: Multiple Regression Model for Item Response Theory Score on Mathematical Knowledge for Teaching.

CONCLUSION

This exploratory study showed that the MKT measures were relatively useful in discriminating between subjects based on their mathematical knowledge for teaching. In this study, the number of years of teaching experience was not a significant predictor of mathematical knowledge for teaching. Unless continuous opportunity for teacher learning is supported throughout their career, teachers may not improve their knowledge solely by teaching for an extended period of time. Implications from this study suggest that one way teachers develop their mathematical knowledge for teaching is by teaching across a range of grade levels. Schools may want to implement this practice as an option for teacher development. More studies, however, are necessary to examine what aspects of teaching across grade levels contribute to teachers' MKT.

This study provides support for educational policy initiatives that require higher education standards for pre-service teachers in developing countries. Prospective elementary teachers appear to benefit from at least four years of college in terms of their mathematical knowledge for teaching. Many countries, including Indonesia, require elementary school teachers to earn at least a four year college degree. However, in practice this requirement is waived in many developing countries due to teacher shortages.

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DESIGNING PROBLEMS: WHAT KINDS OF OPEN-ENDED PROBLEMS DO PRESERVICE TEACHERS POSE?

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This paper describes preservice teachers' reported experience of problem posing based on self-selected original digital images. The 176 participants from Australia and Canada designed open-ended problems as part of their mathematics education course. Their 444 problems and accompanying photos have been analysed to explore the types of problems posed and the focus of the mathematical connections. Findings indicate that preservice teachers are challenged when posing open-ended problems however, this process enables them to develop strategies for problem posing and to become more critically aware of the mathematical potential within their environment.

INTRODUCTION

Learning to pose mathematical problems to students is a significant aspect of mathematics teaching. Teachers select problems to assess their students' understanding of mathematics. They decide on appropriate problems as examples to illustrate a mathematical concept. And they select, adapt and extend mathematical problems to provide a context for learning mathematical skills, concepts and inquiry. Deciding on what counts as an appropriate problem or worthwhile problem to pose is a complex and important task. It is a significant aspect of planning. Problems or tasks selected give students implicit images about what counts as mathematical inquiry or what it means to do mathematics (Schoenfeld, 1992). Problems contextualize, provide possibilities for inquiry, and can pedagogically frame students' attention toward noticing mathematical ideas. Some problems more than others may be better-quality exemplars for learning specific concepts (Watson & Mason, 2005). Other problems and how they are varied might be better at inviting abstraction and generalization or help students in seeing mathematical ideas (Marton & Booth, 1997; Marton & Tsai, 2001).

How teachers use mathematical problems and tasks in the classroom is receiving increased attention. Stein, Grover, and Henningsen (1996) reported that it is extremely difficult for teachers to maintain with students the high cognitive demand of potentially high-level tasks that were initially research-informed. Teachers adapt tasks based on what they know about their students, their understanding of the mathematical topic, teaching goals, and classroom environment. How and why teachers change and adapt tasks was the topic of a research forum at the 2008 International Group for the Psychology of Mathematics Education meeting. Herbst (2008) examined the stakes for teachers of investing class time on certain tasks and how accountability, management and institutional obligations might play into teachers' decisions to change a task while teaching. Herbst stated that the teacher "is responsible for the task as a representation of the *mathematics* to be learned and for

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 225-232. Thessaloniki, Greece: PME.

the task as an opportunity to *study* and *learn* mathematics" (2008, p. 126). How might teachers learn to use tasks in this way?

Sullivan (2008) offered a research-based model for developing task-based lessons particularly to address barriers to mathematics learning for some students. The model includes: a) teachers selecting tasks and deciding on their sequence, b) enabling prompts to support students experiencing difficulty, c) extending prompts for those who complete the initial task readily, d) making implicit teaching strategies more explicit so that all students have access the intended goals and expectations, and e) developing a learning community. Sullivan's model is important for providing possibilities for how to support teachers incorporating designed tasks into their teaching. Yet, in this case the task is given. Although the model emphasizes making the pedagogical practices explicit it leaves hidden the task design practices. Watson (2008) instead suggested that "[a]nother way to engage teachers with tasks is to involve them in the design process" (p. 152). Given that a task is both a representation of mathematics to learn and an opportunity to learn Watson further stated "it makes sense, therefore, to work with teachers on task design rather than only on task implementation" (2008, p. 153).

Over the past few years we have been working in the spirit of Watson's call of task design but in our case with preservice teachers. In this paper we build on our previous research in which we examined preservice teachers' responses to the experience of posing mathematical problems (Bragg & Nicol, 2008; Nicol, 2006; Nicol & Bragg, forthcoming). Specifically, we examine the types of problems preservice teachers create, what they notice and attend to and the challenges they experience when designing mathematical problems within the context of a teacher education course.

THEORETICAL CONSIDERATIONS

Our current research is informed by a theory of variation (Marton & Booth, 1997; Marton & Tsui, 2004; Runesson, 2006) and conceptualization of learning and awareness (Watson & Mason, 2006). A theory of variation posits that learning involves the development of a capability to discern or notice critical aspects of a phenomenon while at the same time being focally aware of these aspects. It is assumed that learners only discern that which varies and so discerning requires experiencing variation. Thus the critical features of a phenomenon are brought to the fore of our awareness when we experience variation in those features and are at the same time able to compare the current instance with our past experience of the feature. Watson and Mason (2006) argued that awareness of discernment is more likely if it is experienced against a background of relative invariance. For example, if students have only experienced addition number sentences of the form $a + b = c$ then it is less likely that they will be aware that $c = a + b$ is a different way of writing the same equation. Comparing these two situations and systematically varying the placement of the equals sign, or the number of terms to be added can help direct students' attention to critical features of a number sentence or algebraic equation.

Thus experiencing a phenomenon in a new or different way can change students' awareness of its structure.

Marton and Booth (1997) referred to the something or phenomenon to be learned as *the object of learning*. The object of learning has, according to Marton and Booth (1997) and Marton and Tsui (2004), different characteristics depending on the perspective of different actors throughout the teaching and learning experience. From the teacher's viewpoint the object is referred to as the *intended object of learning*, from the researcher's perspective it is the *enacted object of learning*, and from the student's perspective it is the *lived object of learning*. The use of variation theory for developing pedagogical problems and using these problems with students in mathematics classroom situations is documented in various recent studies. The effectiveness of a pattern of simultaneous variation was demonstrated in studies by Pang, Linder and Fraser (2006) where economic principles of supply and demand were simultaneously varied. In addition, Al-Marani (2007) documented how deliberate and systematic use of dimensions of variation had some influence on students' learning of algebra concepts. Our study adds to this research and focuses on the experiences of preservice teachers learning to pose open-ended mathematical problems within the context of a mathematics teacher education course for elementary teachers.

Variation theory was used to inform and develop adaptations to the task posed to preservice teachers. Our intended object of learning was to help preservice teachers broaden the common ground or space of learning between themselves and their future students by learning to pose mathematical problems that were open-ended inspired by a set of digital images collected by preservice teachers. As with other studies using variation theory, this study could be described as action research - Author B is one of the researchers and also the teacher educator. Our study explored the lived object of learning of preservice teachers: the kinds of open-ended problems they posed, what they noticed, and what they found challenging in the process.

METHODOLOGY

The participants were given the task of creating a set of Problem Pictures during their mathematics education course taught by Author B. The task required that the students capture four original photos and develop a set of 3 to 4 accompanying open-ended problems for each photo. The photos and problems were to have some connection to and be suitable for elementary aged students. Sullivan and Lilburn's (2004) definition of a "good" problem was employed to assist the preservice teachers in developing open-ended questions. The three main features of a good question are; 1) it requires more than remembering a fact of reproducing a skill, 2) students learn by doing the task and teachers learn from the students' attempts, and 3) there are several acceptable answers (p. 2). The following is an example of an open-ended problem given by Anna a participant in this study with an accompanying photo of Dakota her dog; "Dakota has gained weight recently. The vet recommends that everyday Dakota walk 10% further than she did the day before. What are some possible distances that Dakota could walk for 8 days? Show your work".

The participants in this research were from one Australian and one Canadian university. The data were collected from three cohorts over two years. The two Canadian cohorts (C2007 n=33 and C2008 n=23) were engaged in a 13 week mathematics education course as part of a post-graduate teacher education program. The mathematics education course was in the first semester of their teacher education program. They engaged in one day a week teaching practicum experience running concurrently with the course. The Australian cohort (A2008 n=120) were in the final semester of a four year under-graduate Bachelor of education program. They had accrued 90 days of teaching practicum and completed two mathematics education courses prior to this final 10 week mathematics education course.

The data collected consisted of students' work samples in the form of the Problem Pictures they had developed (as described above), researcher field notes, and a written response survey completed by participants at the conclusion of the course. For the purpose of this paper we draw upon the students' work samples and their written survey responses.

A written survey of 15 open response questions was developed to understand the creation of Problem Pictures from the preservice teachers' perspective and was administered through an online survey program (SurveyMonkey). This paper specifically explores the participants' responses to the strategies the students employed and the challenges faced in the creation of open-ended questions based on original photos.

The researchers met on several occasions to develop and cross check a coding system for the student work samples and survey responses. The data were coded independently and the researchers met again to cross check for consistency and themes that arose from the data. A statistical computer programme, SPSS, was employed to collate and analyse the data gathered from the student work samples. The statistical methods employed were an examination of frequency and percentage of the open-endedness of the problems, the focus and the appropriateness of the mathematics to the problems, and the use of the photos. These data are presented in the form of tables in this paper. A qualitative computer program, Nvivo, was employed for analysis of the online survey data. The survey data are presented in a narrative form and are typical of the views articulated by the many of the participants.

RESULTS AND DISCUSSION

This section presents work samples and survey data to illustrate the types of Problem Pictures preservice teachers design. At the time of writing this paper, an analysis of the data from the Canadian 2007 cohort was completed and is presented. The survey data suggested that the preservice teachers found designing problems of an open-ended nature difficult. Their experience as problem solvers was in finding one correct answer and the Problem Pictures task was their first formal experience in creating problems. Andrea's experience was typical of those in this group,

> "For most [of the problems], I thought of a question for the picture, and then tried to turn it into an open ended question. It worked sometimes, but wasn't the most efficient method. However, it was difficult to think open-endedly as so much of what we learn is about exact answers etc. I think this open-ended theory is something that needs to be further explored in the classroom."

Despite their recent induction to this process, an analysis of the 444 problems revealed that 97% of the problems were open-ended in nature. It was clear from the work samples that the process of creating open-ended problems was achievable for novice teachers despite the initial uncertainty and challenge of the task.

The local curriculum standards had an impact on the preservice teachers' selection of the intended object of learning in their problems. As noted by Alice, "I used the IRPs [curriculum document] as a guide and tried to cover a variety of the Prescribed Learning Outcomes with the questions". For coding purposes the mathematical focus of the problems were categorised in line with the provincial curriculum document for (see table 1 below). The data indicated that the mathematical focus of the Problem Pictures (n=444) were: Shape and Space (38%); Number (37%); Pattern (12%), and; Statistics and Probability (12%). Whilst the traditional preferred focus of Number in mathematical problems is popular in the context of these Problem Pictures, a high percentage of problems focused on Shape and Space. It is possible that the context of the photo appeared to lend itself more towards shape and space type problems, it was noted that a relatively high number of photos of buildings were featured. A common shape related question was, "Indentify 3 symmetrical shapes in this photo. Draw these shapes and show the lines of symmetry." Most of the Statistics and Probability questions were focused on data analysis in the form of creating surveys or plotting charts based on data from the photos. For example, "Which fruit do you think is the most popular in your classroom? Create a survey, record and chart your results." Pattern problems were strongly linked to lower grade levels in repeating or extending patterns in the photo rather than linking with more algebraic related problems.

Accompanying each problem was a statement of the intended object of learning to clarify the mathematical focus for the reader. The mathematical statement was assessed for its strength of relationship to the problem by the researchers. A three point scale was devised for coding; 0 = no link, 1 = partial link, 2 = strong link. The data indicated that the strength of the relationship was: No link (32%); Partial link (42%), and; Strong link (26%). It appears that linking the intended object of learning to the problem was a more challenging task than creating an open-ended question for these preservice teachers. However, this was not articulated in the survey responses. The result is not surprising given the minimal knowledge these participants had with the local curriculum framework and limited classroom experience. However, with nearly a third of the problems not linked to the stated intended object of learning it is an important consideration when assisting preservice teachers with problem posing to ensure that meaningful links are made to the intended object of learning.

The nature of the use of the photo was explored to determine the relationship with the content of the problem. A problem was coded as *Interactive* if the photo was necessary to complete the problem and *Illustrative* if the photo was a motivational device or visual enhancement to the problem but unnecessary for solving the task. Figure 1 depicts a photo with an interactive and illustrative problem. A larger proportion of the problems were considered Interactive (59%) versus Illustrative (41%). The preservice teachers attempted to engage the students with the context of the photo in a meaningful way. The preservice teachers stated that designing interactive questions was extremely challenging for all questions. However, their awareness of the potential for mathematics in the environment had elevated as reported in Authors (2008).

Interactive What types of patterns do you see in this picture? Describe and draw different patterns you see.

Illustrative The perimeter of the chain link fence of Richardson Elementary School is 300m. What different shapes of the schoolyard can you make with a perimeter of 300m? Construct your new schoolyard fence, label sides and show your new formula for perimeter. Share and compare with a partner.

Figure 1. Problem Picture with Interactive and Illustrative Problems.

Survey results indicate that preservice teachers appreciated the opportunity to explore mathematics through taking and analysing digital images. Most preservice teachers (95%) stated that the task was challenging. Sarah's comment is representative of others when she stated: "It took me more than an hour to generate one question." Others found it easier to develop a single question for an image but found it extremely difficult to create more for that same image. They did, however develop different strategies that helped them design open-ended problems. These included: 1) thinking of a closed question then removing some information; 2) looking at the photo and thinking about major math topic areas; 3) forming questions around the curricular topics then fitting these with the photo; 4) imagining themselves as young children; 5) playing with the language of the problem to make it more open. Of those surveyed only one preservice teacher stated her main strategy for creating a problem was asking herself if the problem made sense.

CONCLUSION

Our study explores elementary preservice teachers' experiences and the types of problem they posed during a mathematics teacher education course. Our results indicate that preservice teachers can pose open-ended mathematics problems and that posing these problems within the context of collecting digital images broadens their awareness of what is possible in mathematics teaching and learning. Nonetheless preservice teachers indicate that posing open-ended problems inspired by the world around them is a challenging task. Posing one open-ended problem from an image was challenging but achievable, posing more than one moved preservice teachers to develop strategies for creating open-ended problems that could be used across images. Thus keeping the photo invariant, that is, requiring that preservice teachers pose more than one problem for each photo increased their awareness of problem posing practices. They developed strategies for creating open-ended problems that were then used across the various images and they compared these to the process of designing closed problems. Opportunities to develop mathematical problems with images increased their awareness of what counts as a possible mathematics problem.

Our results also indicate that in developing open-ended problems inspired by images, preservice teaches were concerned with attending to the appropriateness of the problem for children related to the intended object of learning and to what they thought would be an interesting context for students. At the same time few preservice teachers mentioned creating math problems as exemplars of big mathematical ideas or as problems they personally were inspired to solve. The problems preservice teachers posed were thus created from a pedagogical perspective (for students to solve) rather than a personal perspective (for them to solve).

The challenge preservice teachers experienced in posing open-ended problems is shared with practicing teachers. Gerofsky's (2004) analysis of teachers' use and development of word problems indicated that even experienced teachers who may see the world with "calculus eyes" may have difficulty seeing the world with other concepts such as fractions. Teachers and preservice teachers could be supported with strategies for creating and adapting problems. The work of Prestage and Perks (2007) provided such support for adapting and extending math problems given an initial task. What might these strategies look like in the context of developing problems from collected images? Might these strategies help preservice teachers shift their attention to explore mathematics for themselves or to create questions that encourage their students to generalize? Might they provide more explicit opportunities for preservice teachers to observe variation or regularities in creating problems and thus become more familiar and experienced with the practice of problem posing? These questions are important to consider as we continue to explore pedagogical strategies for developing a space of learning that supports preservice teachers in learning to pose good problems that may contribute to their future students' mathematical sense-making.

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THE RELATIONSHIP BETWEEN READING COMPREHENSION AND NUMERACY AMONG NORWEGIAN GRADE 8 STUDENTS

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In the autumn of 2007, Norwegian grade 8 students sat national tests in reading and numeracy. Tests scores on the item level have been linked for a national sample of students, allowing for a correlation analysis of the relationship between reading and numeracy. The overall tendency shows that good readers achieve high numeracy scores while struggling readers also have low numeracy scores. The correlation between reading comprehension and numeracy reaches as high as 0.714. Correlation is stronger for items within the content area of number than for other content areas. Gender and reading comprehension level explain 54 % of the variance in numeracy sum scores. Item format is of less importance.

INTRODUCTION

Mathematics textbooks, national tests and exams all draw heavily on word problems. One concern among many mathematics teachers is that, even though reading mathematical text is considered part of the core aims of schooling, word problems disadvantage both bilingual and weak students. Prior research has well documented strong relationships between reading and numeracy (see for instance Roe & Taube, 2006); the relationship between the two, however, is not easy to understand.

Reading and numeracy were declared core basic skills for the Norwegian curricula implemented in the autumn of 2006 and test constructs for the national tests where defined accordingly (NDET[i], 2006a; 2006b). In the autumn of 2007, Norwegian grade 8 students sat national tests in reading comprehension and numeracy. Test scores have been linked for a national sample of students, allowing for a correlation analysis of the relationship between reading and numeracy. The aim of this paper is to present and discuss some of the results from this analysis.

PRIOR RESEARCH

The connection between reading comprehension and doing mathematics—as in solving routine problems and problem solving—has intrigued research for a long time. Knifong and Holtan (1977) for instance, investigated erred word problems and found little evidence that poor reading abilities where the cause for student errors. Their investigation contrasts the many studies of the connection between reading comprehension and problem solving where the main hypothesises is that reading comprehension underlies success in solving word problems (see for instance Reed, 1999; Verschaffel, de Corte, & Greer, 2000; Österholm, 2005). Cummins, Kintsch, Reusser and Wiemer (1988), unlike Knifong and Holtan, suggest a connection

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 233-240. Thessaloniki, Greece: PME.

between error patterns and comprehension strategies, and that erred problems often are the correct solution of the problem as students comprehend it.

To understand or to comprehend a (word) problem is to form a mental representation that allows the student to progress towards producing a solution to the given problem (Nortvedt, 2008; Thevenot, Devidal, Barrouillet, & Fayol, 2007; Verschaffel et al., 2000). To arrive at such a mental representation, the student first needs to read the problem. Roe and Taube (2006) found that literacy measures for reading and mathematics correlated 0.57 for Norwegian and Swedish students in an analysis of Programme for International Student Assessment (PISA) 2003 data. Their analysis revealed that proficiency in reading is positively correlated to solving mathematics items in general. They also found that proficiency in reading plays a more important role in solving some items than others (change and relationship items). The crucial item aspect is not text length but rather content, format and difficulty (ibid., p. 138). Österholm considers not "mathematics in itself (…) the most dominant aspect affecting the reading comprehension process, but the use of symbols in the text is the more relevant factor" (Österholm, 2005, p. 325). Students use a wide range of strategies to identify relevant information in word problems (Cook, 2006); while struggling students can be characterized as using surface-level strategies, proficient students apply deep-level strategies to discriminate between textual elements.

The literature of the field gives the general emerging picture that reading comprehension play a crucial role, even though causes of student errors can be found elsewhere, as in failure to master algorithms. Vilenius-Tuohimaa, Aunola and Nurmi (2008), who control for decoding, also support this and find that although technical reading increases performance on solving word problems, comprehension still has a significant correlation of 0.38 to solving the same problems.

DESIGN AND METHODS

National tests in reading comprehension and numeracy for grade 8[ii] comprise the tests used for this research. Students sit both tests in the last weeks of September. Expert groups at Norwegian universities develop the tests on behalf of NDET. Teachers score their own students' tests and report all student scores on a national web site (NDET, 2008). The Ministry of Education have granted access to the results of the students in the national sample for the present research project, allowing for data at the item level.

Sample

Of a national, representative sample of 1360 students from 26 schools throughout Norway, 1265 students (M = 631, F = 633) participated in both tests. Norwegian classrooms are inclusive, and, hence, all ability levels are represented in the sample.

Analysis

As the aim of this study is to analyse the influence of reading comprehension to solving a numeracy test, only results for students present for both tests are included in the analysis. Tests are linked at the student level, allowing for correlation analysis. Some analysis is performed at the student level on the linked database. However, data at the item level resulting from the analysis at the student level and from a classical item analysis of the tests are used to construct a second database. This database consists of data for each of the 76 items in the numeracy tests allowing for correlation analysis comparing item characteristics.

Students have been assigned to three proficiency groups based on their scores on the two tests. Students scoring below the 20th percentile on both tests have been assigned to the low-performing group and students scoring above the 80th percentile on both tests have been assigned to the high-achieving group. Each group consists of approximately 12 % of the sample of 1264 students.

The numeracy test

The test construct for the numeracy test is given in a national framework, developed by NDET on behalf of the Norwegian Ministry of Education (NDET, 2006b). The purpose of the test is to assess students' numeracy skills, i.e., their knowledge of numbers, measurement and basic statistics. A majority of the problems should be applied or given some everyday context (ibid.). All items have at least one line of text. Seven items have short texts like "find the sum", two of the seven are multiple choice items.

Every item in the test is assigned to one of the content areas based on content analysis by the group developing the test and in accordance with reviews from content experts (Ravlo, 2008). The numeracy test consists of 71 exercises giving a total of 76 items. Each item can receive one score point. Multiple choice items make up the majority of items (51). The rest are open-response formats where students have to construct an answer in an appropriate format, mainly a number. For a few items, students will finish a diagram or make a drawing. Test reliability is high: 0.947 measured with Cronbach's Alpha. Table 1 shows other reliability measures for the numeracy test.

Content area/Item format	Number of items	Cronbach's Alpha
Number	46	0.928
Measurement	19	0.782
Statistics	11	0.679
MC items	51	0.926
OR items	25	0.862

Table 1: Numeracy test reliability measures

The reading test

The framework developed by NDET for the reading test is similar to that of the PISA Study, defining the sub-competencies of reading comprehension as the capacity to retrieve, interpret and reflect on text content (OECD, 1999; NDET, 2006b). The test contains continuous as well as non-continuous texts. A few items are connected to each text. Each item is assigned to one of the construct areas by the research group developing the test (ILS/UiO, 2008). The reading test consists of 43 items, mainly multiple choice items (30). Each item gives a maximum of one point, no partial points are assigned. Students must give a short written response for open-response items, with one or two lines of space provided. The reliability of the reading test is 0.887 (Cronbach's Alpha). Table 2 reveals other reliability measures for this test.

Sub-competencies/ Text format	Number of items	Cronbach's Alpha
Retrieve	18	0.784
Interpret	17	0.709
Reflect	8	0.720
Continuous	22	0.821
Non-continuous	21	0.789

Table 2: Reading comprehension test reliability measures

TEST RESULTS

On average, students score 39.4 points or 52% of a total of 76 (SE = 0.448, SD = 15.915) for the numeracy test. Boys score significantly higher than girls, having averages of 40.95 and 37.86, respectively. The difference arises with boys outperforming girls on multiple choice items, where the difference in favour of boys is 2.76 score points ($p = .001$). When comparing the three content areas, boys score significantly better than girls in number and measurement ($p = .001$). In number, the difference is 2.15 score points.

The average sum score for the reading test is 26.62 score points (62 %) out of 43 (SE = 0.232, SD = 8.236). Girls outperform boys, average scores are 27.6 and 25.63, respectively ($p = .001$). Notable differences in favour of girls can be found for all sub-constructs as well as for all the item and text formats. Differences are significant at the .05 level; however, some are small measured in score points.

CORRELATING DIFFERENT ASPECTS OF READING AND NUMERACY

To sum up, while boys outperform girls on the numeracy test, girls outperform boys in reading. Measured in standard deviations, the differences are 0.19 and 0.24, respectively. In Figure 1 the relationship between reading and numeracy is displayed. The size of the Pearson correlation[iii] between reading and numeracy is 0.714 ($p < .001$); a strong connection exists between how students score on the reading test and the numeracy test.

Even given the differences between boys and girls on reading and numeracy, gender and reading comprehension level can explain 54 % of the variance in the numeracy sum score, $F(2, 1261) = 750$, $p = .001$. Reading alone can explain 46 % of the variance, F(1,1262) = 1062, p = .001. Splitting the sum score on the numeracy test into a sum score for multiple choice items and a sum score for open-response items gives correlations with reading at 0.676 and 0.714, respectively (p < .001 for both). Gender and reading comprehension level explain about the same amount of variance in the sum scores for the two item formats; for multiple choice items 50 % of the variance, F(2,1261) = 633, p = .001; for open-response format 52 % of the variance, F(2,1261) = 690, p = .001.

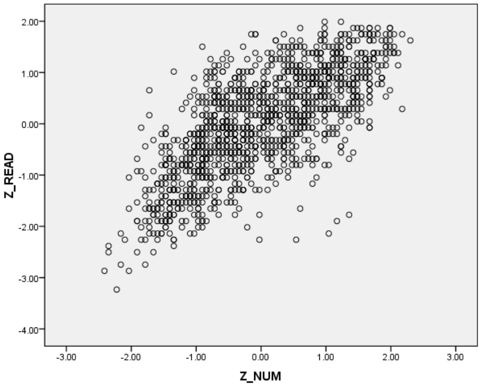

Fig. 1. Z-scores in reading and numeracy

Strong correlations can be found between the different aspects of reading comprehension and numeracy. Of the different reading aspects, retrieving information has the highest correlation to the numeracy aspects as well as to numeracy overall. Of all the part-part correlations, the highest is between retrieve information and number. However, the correlations between the different aspects of numeracy are higher than between the same aspect and the different reading constructs.

	Reading sum score	Retrieve	Interpret	Reflect	Cont. text	Non-cont. text
Number	0.690	0.677	0.577	0.555	0.581	0.698
Measurement	0.624	0.628	0.516	0.494	0.514	0.642
Statistics	0.628	0.626	0.536	0.543	0.555	0.648

Table 3. Pearson correlation: All significant at the .001 level (2-tailed)

The correlation for non-continuous texts (0.734) is higher than the correlation for continuous texts (0.600) to numeracy overall (p = .001 for both), and also to the different numeracy constructs. As many numeracy items contain tables or figures and are non-continuous texts themselves, this result might not be surprising.

CORRELATION ANALYSIS ON THE ITEM LEVEL

All items on the numeracy test have been correlated to the sum scores in reading and numeracy before plotting the correlations in Figure 2. All correlations are significant at the .05 level. As could be expected, given the strong relationship between reading and numeracy scores on the student level, the correlation between the correlation measures are also high: 0.871 ($p < .001$).

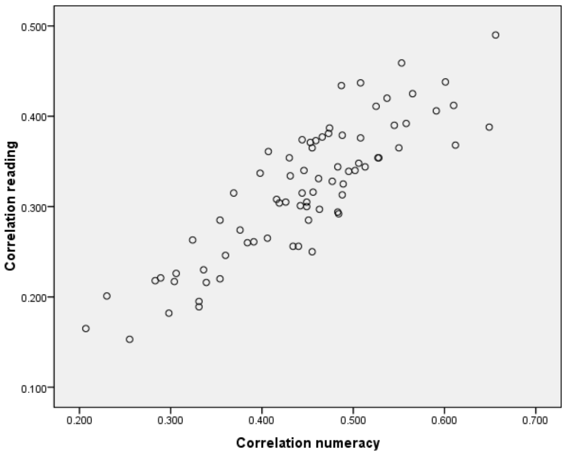

Fig. 2. Item correlation to reading and numeracy sum scores

Next all numeracy test items have been sorted according to their correlation to the reading sum score. Two groups of items are of special interest: the 10 items with the highest correlation to reading and the 10 items with the lowest. The 10 items with the highest correlation all have a correlation above 0.400 to reading. The correlation to numeracy is higher. Of the 10 items, 8 appear in the first half of the test. P-values are between 35% and 74%, on average 51%. All but two items are within the content area number. The last two have been assigned to the statistics area. Seven of the items are open-response format. The 10 items with the lowest correlation to reading all have a correlation below 0.218. The correlation to the numeracy sum score is higher and has a wider range as does the distribution in difficulty level; p-values are between 20 % and 86 %, on average 44 %. Only two of these items are open-response items. Items are evenly distributed across the three content areas and considering the item's placement in the test.

The majority of the high-correlation items are of an open-response format. For the whole test, approximately one third of the items have this format. Open-response items seemingly correlate higher to reading than multiple choice items do; the average correlation is higher. However, the correlation between item format and reading is not significant. A regression analysis reveals that content area, item format and p-value together can explain 14.3% of the variance in item correlation to reading comprehension, $F(3, 72) = 4.010$, $p = .011$. This relationship is less straightforward to interpret and understand. An investigation of the two groups of low- and high-achieving students and the items with the lowest and highest correlation to reading reveals a pattern regarding average sum score for the two groups of 10 items. High-achievers appear disadvantaged by items with low correlation to reading, and vice versa for low-achievers. The difference in difficulty level for the two groups of items is relatively small, but to the low-achievers especially, the difference is large.

	Partial sum score High-correlation items	Partial sum score Low-correlation items
Low-achievers	1.01	2.72
High-achievers	8.97	6.71
Full sample	5.08	4.37

Table 4. Partial sum scores for low and high achievers

Extending this analysis to other groups of 10 items modifies the picture slightly, but low-achievers remain particularly disadvantaged by items with high correlation to reading and the high-achievers likewise by items with low correlation to reading. Within groups, boys and girls display the same pattern. Among the low-achieving students, the pattern is slightly stronger for boys.

CONCLUDING REMARKS

As in prior research, the relationship between reading comprehension and numeracy proves rather strong (Roe & Taube, 2006; Vilenius-Tuohimaa et al., 2008). Text formats that resemble the mathematical format, as in non-continuous text, better predict numeracy scores. Being able to retrieve information has a higher correlation to numeracy sum scores than the other two reading sub-competencies. This might be because students need to retrieve information from all numeracy items, but only some of items demand students to reflect on and interpret the text in order to be able to determine the nature of the mathematical relationships in the text (Cook, 2006; Cummins et al, 1988; Reed, 1999). Number correlates stronger to reading than the other two content areas, unlike in the Roe and Taube study (2006). Item format, p-value and content area explain some of the variation in correlation to reading comprehension for single items. Item format less than in Roe and Taube (ibid.) More puzzling, however, is how low- and high-achievers are more advantaged for items with low or high correlation to reading. One possible explanation to results on the items with high correlation to reading comprehension is that struggling readers more often use surface-level strategies such as searching for key words, while proficient readers master deep-level reading like using the problem question to guide discrimination between text elements (Cook, 2006). To discriminate and to recognize mathematical relationships in the text, students draw on prior knowledge (Roe &

Taube, 2006), which high-achievers possess to a much larger extent. Explanations to the reversed pattern, that students are more 'equal' when it comes to solving items with a low correlation to reading, reminds to be found.

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[i] Norwegian Directorate for Education and Training, in Norwegian: Utdanningsdirektoratet

[ii] Grade 8 is the first year of lower secondary school (grade 8 – 10). Students are 13 years old.

[iii] Both sum scores are normally distributed; all correlation coefficients are Pearson correlation coefficients.

MODELLING PARTICIPATION IN PRE-COLLEGE MATHEMATICS EDUCATION

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University of Nottingham

Concerns about declining participation in pre-college (Advanced level) mathematics education have been growing in England in recent years. In this paper I develop a statistical model for exploring participation at Advanced level with a particular focus on examining the between-school differences in completion rates. After accounting for learners' prior attainment, social background and the school mix there remains considerable variation in completion of pre-college mathematics. Much more of this is attributable to the post-compulsory school (16+) than on experiences up to age 16.

INTRODUCTION

International concerns about the declining numbers of students following science, technology, engineering and mathematics (STEM) courses are well documented (Committee on Science, Engineering and Public Policy, 2007; Gago, 2004; Roberts, 2002). The Royal Society's recent 'state of the nation' report (2008) offers a detailed overview of the patterns of STEM attainment and participation in the UK. The underlying motivation for these concerns is the securing of future economic productivity, the argument being that in order to be positioned at the forefront of the global economy advanced nations need to be leaders in the production and application of scientific knowledge. Therefore, high levels of engagement in STEM education are essential and substantial sums of money are being committed to a range of policies to increase participation.

Matthews and Pepper's (2007) analysis for the Qualifications and Curriculum Authority (QCA) looks in considerable detail at the particular case of mathematics in England. The QCA are engaged in a programme of reform of 14-19 mathematics qualifications. These include the piloting of new qualifications and developments of existing qualifications in order to develop a range of mathematics learning pathways for the full range of students. The express aim of this programme is to improve the quality of mathematics education in schools and colleges, thereby widening and increasing participation, particularly at Advanced level.

However, in all of these studies and policy recommendations, although there is a great deal of analysis of trends across cohorts by social variables, there is little in the way of attempt to measure between-school variation in completion of pre-college mathematics. How much difference do schools make in steering students towards or away from further mathematical study; in retaining or losing them to mathematics? The danger in exploring such questions is that tools developed to explore school effects might then being used uncritically by policymakers as a managerial technology of the standards agenda. Nonetheless, the analysis is worthwhile.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 241-248. Thessaloniki, Greece: PME.

There is some research in science (e.g. Cleaves, 2005) and mathematics (Brown, Brown, & Bibby, 2008; Hernandez-Martinez et al., 2008; Mendick, 2005) about student choices beyond compulsory schooling but often this accounts for individual choice rather than structural effects. As 'choice' is a problematic concept (Ball, Davies, David, & Reay, 2002) I want to explore the extent to which schools effect completion of pre-college or A level mathematics. From the studies on participation in mathematics we know that enjoyment and success are key factors in students deciding to continue. We would expect, therefore that schools would have some effect on these and other measures associated with likely participation but research to date has not tried to quantify the between-school variation in participation.

Given the interest in widening and increasing participation in pre-college mathematics this paper explores variation between schools. Firstly I am interested in whether or not there is any significant difference linking the school one attends (both up to 16 and from 16) to one's chances of completing a pre-college (A level) mathematics qualification. It might also be useful to have some estimate of the level of this effect. Secondly, we need to understand what the causes of such differences might be. This is a more complex question and beyond the scope of this paper although I will explore some possible avenues for enquiry.

The dataset

In order to explore this first question I make use of England's National Pupil Database (NPD) and Pupil Level Annual School Census (PLASC). These extensive datasets include students' qualification records throughout their educational careers as well as a range of social measures (e.g. gender, ethnicity, eligibility for free school meals, etc). The dataset records the schools attended by each student which allows us to explore the nested, multilevel structure of the data. We might expect any between-school variation to be unstable at the school level, in other words, year on year variations within a school might be considerable but it is not the aim of this paper to explore this (see Gray, Goldstein, & Thomas, 2001, for an example of a related analysis on year-on-year A level attainment).

The dataset used here is the 2005 cohort of 16 year olds completing their GCSE (General Certificate of Secondary Education) qualifications in the East and West Midlands (Government Office Regions) of England who then completed any Advanced level qualification over the following two years (36696 students). Several decisions have been made in preparing this data for multilevel modelling. I have explained these elsewhere in more detail (Noyes, 2008) but the key points for this analysis are:

- Only students completing one or more A level courses are included in the dataset, i.e. I am concerned only with those students who have chosen some A levels, and might have included mathematics amongst these;
- I have only included students who obtained a GCSE grade C in mathematics as this is the official eligibility criteria for entry to A Level mathematics.

However, this presents a significant problem since entrance criteria vary between schools;

- Only those students from mainstream state secondary schools are included here (around 90% of the cohort)

Learner trajectories do not all fit into this two year cycle (i.e. 2005-7) but it is generally applicable. This analysis accounts for student qualifications in the two years following GCSE awards in 2005. When modelling completion, we are unable to tell from the dates of awards whether an AS (the first half of the full A level) in 2007 took one or two years to complete. The model considers whether a student has gained at least this AS qualification.

Another limitation of using the NPD/PLASC data is that it only reports results (and therefore entries) and so doesn't give the full picture about participation and attrition. Survey data from another strand of the larger project (Noyes & Sealey, 2008) indicates that approximately 10% of Year 12 students who start mathematics do not complete. This is one of the highest attrition rates for Advanced level subjects and a different methodology is required to explore that aspect of participation.

Modelling school effects on the completion of any Advanced level mathematics

The modelling in this analysis consists of three level, cross-classified binary response models. Students (level 1) are nested within schools at Key Stage 4 (level 2) and either the same or a different school from 16-18 (level 3). The majority of these students (58%) stay in the same school but since there is movement at 16 both into and out of many schools, levels 2 and 3 of the model are cross-classified. A dummy variable is included to account for changing schools at 16. Models are run initially using predictive quasi-likelihood (PQL) estimation and these coefficients then act as prior estimates for the Markov Chain Monte Carlo (MCMC) estimation which a) gives more reliable estimates of the size of effect attributable to a range of factors and b) is required due to the cross-classified data structure.

The modelling is developed from a single level logistic regression model in which the binary response (0,1) (whether or not they completed any A level mathematics between 2005-7) for the ith student with prior attainment x_i is y_i. Denoting as π_i the probability that $y_i = 1$ gives the general model:

$$f(\pi_i) = \beta_0 + \beta_1 x_i + e_i$$

There are a number of possible link functions $f(\pi_i)$ which can be used in such logistic regression models but here I adopt the logit link function (Rasbash, Steele, Browne, & Prosser, 2005) where $f(\pi_i) = \log(\pi_i/(1-\pi_i))$. The following model is developed for the ith student in the jth school for GCSE (up to 16) and the kth school for A level mathematics (post-16):

$$\text{logit}(\pi_{ijk}) = \beta_{0jk} + \beta_1 x_{ijk} + e_{ijk}$$

$$\beta_{0jk} = \beta_0 + v_{0k} + u_{0j}\ ,\quad v_{0k} \sim \text{N}(0, \sigma_v^2)\ ,\ u_{0j} \sim \text{N}(0, \sigma_u^2)\ ,\ e_{ijk} \sim \text{N}(0, \sigma_e^2)$$

The model is run in MLwiN. Due to the size and complexity of the model a burn in period of 5000 was used and 200000 iterations of the model were run in order for the effective sample size to be sufficiently high (>1000). The resulting parameter estimates are shown in Table 1.

Fixed Part	
Constant	-5.764 (0.155)
GCSE mathematics grade (ref. grade C)	
Grade B	1.755 (0.067)
Grade A	3.432 (0.074)
Grade A*	4.630 (0.096)
Female	-0.824 (0.037)
Difference of GCSE mathematics and English grades	0.486 (0.027)
Difference of GCSE mathematics and average grade	0.283 (0.041)
Number of A level entries	0.658 (0.036)
IDACI score	0.654 (0.150)
Ethnicity (ref. White British. Only statistically significant categories included here)	
Any Other Asian Background	0.950 (0.191)
Indian	0.946 (0.075)
Pakistani	0.802 (0.119)
African	1.151 (0.233)
Bangladeshi	0.691 (0.224)
Chinese	1.167 (0.193)
Post_16 School s.d. of number of A level entries	-0.128 (0.042)
Random Part	
Post-16 between-school variance	0.569 (0.075)
Pre-16 between-school variance	0.252 (0.038)
Number of post-16 centres	509
Number of pre-16 centres	634

Table 1 Parameter estimates for the three-level, cross-classified model of Advanced level mathematics completion 2005-7

Notes: Free School Meals (FSM), Special Educational Needs (SEN), English as an Additional Language (EAL) and changing school were not significant predictors. Centre variables (at Level 3 of the model) are potentially misleading as this dataset only contains A level students; a large college would have many non-Advanced level students too and so such centre level measures would no doubt have different effect. IDACI (Income Deprivation Affecting Children Index) here is left on the 0 (low deprivation) to 1 (high deprivation) scale.

A number of things are worth pointing out from the above model. Firstly, consider the between-school variance in completion of some A level mathematics. The variance participation coefficient (Goldstein, Browne, & Rasbash, 2002) is the total amount of residual variance attributable to levels 2 and 3 in the model and can be estimated in more than one way. Here I use the following linear threshold model:

$$VPC = \sigma_u^2/(\sigma_u^2 + 3.29)$$

Using this I calculate estimates for the variances as 0.569/(0.569+3.29) = 0.147 at level 3, i.e. the A level centres, and 0.252/(0.252+3.29) = 0.071 at level 2; the GCSE centres. So around 15% of the residual variance in completion of any Advanced level mathematics is attributable to which school you attend after 16. Schools attended for GCSE contribute half as much variation again. Together, the schools attended account for over 20% of the variation of completion of some pre-college mathematics, after accounting for prior attainment, social background and school mix. This is substantial and much greater than the typical between-school variances of secondary school contextual value added modelling. Survey data arising from a different strand of this project supports this result, showing that take-up, withdrawal from AS and continuation to A2 all vary considerably between schools/colleges (Noyes & Sealey, 2008).

The most significant predictor of completion of A level mathematics is, unsurprisingly, prior attainment. I could calculate predicted probabilities for GCSE grades A*, A, B and C. However, this is actually not that helpful since the other performance and social measures also combine to make significant differences in these probability estimates as I will show below.

A positive difference between GCSE mathematics grade and students English and mean GCSE grades increases the likelihood of them completing some A level mathematics. This seems sensible and relates to evidence that self-efficacy influences likelihood of continued study. It is also reasonable that completing a greater number of A levels increases the chances of having some mathematics included in one's portfolio of qualifications. From interviews with students and teachers it is clear that different schools and colleges have different policies on A level entries. Having explored the potential significance of this by including school level measures (mean and standard deviation of the number of subjects awarded) only one measure was significant. The negative influence of 'K5centre_s.d. of number of Advanced level entries' suggests that heterogeneity of intake has some small detrimental effect upon likely completion of some mathematics. However, caution needs to be exercised here as we don't know the true mix of the centres from this data as we have only included students on A level pathways. That said, if this measure of heterogeneity were important then it would only become more so if the full range of college students were included in the model.

Turning to the social variables we can see, as anticipated from the research literature, that gender has a significant impact on participation with girls being less likely to complete some mathematics. The IDACI score shows that students from more deprived backgrounds are actually more likely to study some mathematics, when all other factors have been taken into account. I have shown elsewhere (Noyes, 2008) that GCSE mathematics performance is associated with social class. So any 'classed' pattern of post-compulsory mathematics participation was shaped earlier in the education system. It should also not be a surprise that the impact of ethnicity is very variable with Chinese/Indian/Pakistani/African students having a much increased predicted probability of completing some mathematics compared to the White British base category.

Having looked at the effect of these background variables we can make probability estimates for different students. For example, let us consider GCSE grade A mathematics students taking 3 A levels, remaining in the same school for A levels, with a very low (i.e. 0) IDACI score:

	White British	Chinese
Male	0.41	0.69
Female	0.23	0.50

Table 2: Predicted probabilities of completing pre-college mathematics course

The differences here are striking and reflect a far more complex patterning of participation than can be explored using only GCSE maths grades or gender, which are the typical units of analysis in England.

An examination of the level 3 residuals suggests that around 10% of schools are significantly different (at 5% level) from the typical school and I am interested in understanding why these differences exist. Are they temporary, annual or short term fluctuations, which are now out of date since this cohort of students finished some time ago? Or, on the other hand, is there something about the school, the recruitment policy, the teaching and learning, etc., which is contributing to these between-school differences. Further modelling might help to answer the first of these questions but the second one requires different research tools. One challenge is that the cohorts of interest (schools significantly different from mean) cannot be identified until it is too late. If, on the other hand, the school effect is more stable over time then school case study work can be used to explore these issues further. Another strand of the project is working in a number of case study schools in order to explore how curriculum, school and departmental cultures, teaching and learning approaches might help to create these significant different schools

DISCUSSION

One of the motivations for my modelling of A level mathematics completion is to problematise the notion of school effectiveness, looking at it from a different direction. Recently the press has reported that England has done well in the latest TIMSS study: we have 'gone up' in mathematics and science. Not reported as loudly was that students appear to be enjoying their work less. This issue is at the heart of how schools effect participation in pre-college mathematics. Are schools which are very successful in maximising contextual value added (CVA, for mathematics) also good at encouraging these students to continue their study of mathematics? If we could model mathematics CVA for 11-16 year olds and compare these models with models for A level mathematics for the same cohort how much correlation would there be? This paper is one part of that analysis.

The model above suggests that the amount of between-school variance in completion of any Advanced level mathematics is in excess of 20%, with the amount attributable to the post-16 schools about double that attributable to the pre-16 schools. Why this should be is not clear although the variable retention rate in A level centres referred to earlier probably has an effect.

We know that gender has a significant impact on participation and this is shown in the models although prior attainment has a far greater impact. Raising the GCSE attainment of girls would probably be the best way of increasing their (and overall) participation in pre-college mathematics. Although the impact of gender is significant it is only similar in size to some of the ethnicity categories. However, these ethnic groups are relatively small so the overall *effect size* (Schagen & Elliot, 2004) of gender is bigger.

The number of A levels taken is clearly important. If a student takes more A levels (the maximum in this dataset is 7.5) then the chances of mathematics being included among them is increased. The parameter estimate for an extra A level (0.7) is not dissimilar, but opposite in effect, to that of being female (-0.8). This factor is included as it would seem sensible that school policies on the number of entries would have an impact on completion and therefore school residuals. For example, a policy that all students must complete 4 A level courses in year 12 (compared to another school in which one need only do 2 as a minimum) would have quite a difference. Although teachers have pointed out these policies in schools in which we are working the best way to access this was to include centre level variables. In the models I used mean and standard deviation of A level entries per candidate and of mean GCSE score. These were largely insignificant (at the 5% level). Unsurprisingly higher average prior attainment of the year 12/13 cohort has a positive effect on completion of any A level mathematics. More unexpected is the small negative effect of increased school mix, i.e. wider range of number of A levels entered. Changing school between GCSE and A level seems to have a small negative effect but this is not statistically significant in this model.

So, finally, this analysis is part of a larger study exploring the trajectories of learners of mathematics. Throughout the paper I have suggested further analyses that need to build on these models. These include within-centre cohort models that will identify trends over time; comparisons of school effects from 11-16 and 16-18 (for 11-18 schools); contextual value added models for A level mathematics.

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PROCESS AND MEANS OF REINTERPRETING TACIT PROPERTIES IN UNDERSTANDING THE INCLUSION RELATIONS BETWEEN QUADLIRATERALS

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This study investigates the process and means of understanding inclusion relations between quadrilaterals. We argue that the main difficulty in comprehending inclusion relations comes from the pre-existing tacit properties children have in their minds, such as "all angles are not 90 degrees in a parallelogram". We conducted a teaching experiment focusing on how these properties can be changed or reinterpreted to be more accurate. Results showed that a process of tautologous evolution based on the equilibration between positive and negative aspects was a natural phenomenon in children's understanding of inclusion relations. Moreover, we identified the acts of correctly interpreting the language used and the use of analogies with other, more easily grasped inclusion relations as effective means.

INTRODUCTION

This paper is part of a study that explores how elementary school students can improve their interpretation of geometric figures towards deductive geometry at the secondary level. Specifically, we investigate the process and means of fifth graders' understanding of quadrilateral inclusion relations through a teaching experiment. As van Hiele (1985) notes, inclusion relations are recognized when the definitions of figures come into play at the third level (We use 1-5 numeration model); similarly, we assume that students' understanding of inclusion relations develops along with their recognition of figure definitions (Silfverberg and Matsuo, 2008).

However, previous studies have shown that a majority of students still find it difficult to understand these concepts even after learning proofs in secondary geometry (Senk, 1989; Okazaki and Fujita, 2007). In particular, one difficult problem is that children naturally develop strong tacit models in their mind that they have abstracted from daily experience and the typical examples continually provided by their teachers (Wilson, 1986; Fischbein, 1989; Hershkowitz, 1990). The tacit models can influence students' interpretations of geometric figures, such that they often fail to classify some figures inclusively; further, regression phenomena may occur once they come to understand the correct inclusion relations, that is, they return to their previous exclusive classifications when the tacit models are activated (Okazaki, 1995). Thus our teaching experiment is focused on clarifying how children can overcome their tacit models in understanding the inclusion relations between geometric figures.

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THEORETICAL BACKGROUND

Several researchers have reported that difficulties in learning mathematics often stem from tendencies of students to rely on concept images rather than concept definitions (Vinner, 1991). Fischbein (1993) introduced a figural concept that has both conceptual and figural aspects, noting that figural aspects are typically dominant.

For this study, we follow Piaget's (1985) concept of equilibration between the positive and negative aspects of things: "the mind spontaneously centers on affirmations, or the positive characteristics of objects, actions, and operations. Negations are neglected or are constructed only secondarily and laboriously" (p.13). Namely, Piaget states it is necessary to construct new aspects called 'negations' and coordinate them with affirmations (positive aspects) to equilibrate schemes in disequilibrium, which is a result of the primacy of positive aspects. We believe that in parallelograms, for example, the positive aspects often considered by students are its properties such as 'parallel opposite sides' that they have learned explicitly in school, while the negative aspects correspond to the variability of each side and angle (Wilson, 1986). This variability, however, goes relatively unstressed in the teaching process. Children's schemes may then be exposed as tacit models once they have given noncritical meanings to a parallelogram's attributes based on its visual images (Fischbein, 1989; Hershkowitz, 1990). In actuality, students are likely to tacitly add properties such as 'unequal adjacent angles' and 'unequal adjacent sides' to a parallelogram in addition to its true properties (Okazaki, 1995):

Initial state of a student's conception of 'parallelogram' = $p_1 \wedge p_2 \wedge \cdots \wedge \neg q_1 \wedge \neg q_2$
p_1 : Opposite sides are parallel, p_2 : Opposite sides are parallel, ...
$\neg q_1$: The sizes of adjacent sides are not equal.
$\neg q_2$: The sizes of adjacent angles are not equal.

These tacit properties are robust (Fischbein, 1989). Okazaki (1995) found that even regression phenomena can occur. For example, when a child first understands the inclusion relation between rhombuses and parallelograms, because a rhombus has all the properties of a parallelogram, the child may revert to their previous exclusive classifications as soon as they become aware of these 'tacit properties' of parallelograms (We use the term 'tacit property' hereafter). In this way, tacit properties may appear like a "ghost" in the child's mind until they can be replaced.

One way to eliminate these tacit properties has been suggested by Okazaki (1995). The method is not to remove the property $\neg q$, but rather to add q (e.g., there are cases in which the lengths of adjacent sides of a parallelogram are equal). This method can be described as follows:

$$p \wedge \neg q \longrightarrow p \wedge (\neg q \vee q) \longrightarrow p \quad \text{(tautologous evolution)}$$

It shows that if a tautology is made, then $\neg q$ is subsequently eliminated (Murakami, 2002). Although mathematically this is the same as the simple removal of $\neg q$, we believe it is different psychologically. Simple removal would imply that the teacher has denied the child's tacit properties. However, in such teaching, the child may keep having the tacit properties in their mind. In the alternative method of tautologous evolution, however, the child's existing concept ($\neg q$) is protected, and for this reason it becomes necessary to newly construct (q). Below, we examine such a process.

Moreover, we think that the meanings of both linguistic and functional activities are worthy of consideration, that is, "correctly interpreting the language used for class inclusions" and "understanding why such class inclusions are more useful than partition classifications" (de Villiers, 1994), respectively. We believe that these activities play important roles in the transition to deductive geometry.

METHODOLOGY

We distributed a preliminary questionnaire (see Okazaki and Fujita, 2007) to fifth graders in a public school, and chose 14 children who could provide correct answers to questions related to the properties of figures but did not soundly understand their inclusion relations. For instance, the students well know "The lengths of the opposite sides of parallelograms are equal", but they were unsure of the statement "There is a parallelogram which has all equal angles".

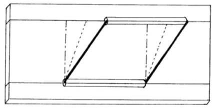

Figure 1. Operative material

We conducted teaching experiments (Steffe and Thompson, 2000) for the paired children. First, the teacher (author) asks what ideas and images they have of a "parallelogram", and checks what they consider to be its intensions and extensions. Next, the teacher encourages the children to see parallelograms dynamically using the operative material (Fig. 1), in which a parallelogram can be made to transform continuously into various shapes of parallelograms, two of which are a rhombus and rectangle. Then, the teacher instructs the children to write down what changes and what does not change.

Then, the teacher asks them about two inclusion relations between quadrilaterals: (1) rhombus and parallelogram and (2) rectangle and parallelogram. The teacher may also try to evoke cognitive conflicts in the students' cognition by reminding them that a rhombus and rectangle appeared in the parallelogram transformation, or by asking why one inclusion relation may be true while the other may be false. Once they agree with the inclusion relations, the teacher suggests the existence of tacit properties to check whether the regression phenomenon may happen. When they return to the exclusive classification, he encourages them to reflect on the notes they wrote down in Step 2. If they understand the inclusion relations, the teacher checks their final conception of "parallelogram".

All exchanges were recorded on video camera, and transcripts were made of the video data. We analyzed them qualitatively by noting when and how the students changed or further developed their views and why such changes happened. Through these analyses, we attempted to construct a model of the children's understanding of the inclusion relations of geometric figures.

RESULTS

We found as results of our analysis that 13 of the 14 students could comprehend two inclusion relations. Specifically, we could identify three types of understanding: (1) tautologous evolution (5 pairs), (2) analogies to equivalent inclusion relations (1 pair), and (3) a search for the relations of diagonals (1 pair). Here, we examine (1) and (2) due to space limitations.

(1) Understanding the process of tautologous evolution

First, the students Hiro and Nao (aliases) gave the following answers as the properties of a parallelogram: 'parallel opposite sides', 'equal opposite angles', and 'equal opposite sides'. Next, they observed whether these properties were maintained during the continual transformations of the operative material (Fig. 1), while also noting that the angle and side sizes of the figure could change as long as it remained a parallelogram. They then summarized these observations on paper.

Next, the teacher asked them whether a rhombus is a parallelogram. They responded affirmatively even when the teacher pointed out the fact that the four sides were equal. Thus we conclude that the students' concept of "parallelogram" did not include the tacit properties related to the sides. The situation was different, however, in the case of a rectangle and parallelogram. They did not agree with the inclusion relation.

Nao: It is different when it is 90 degrees. (He gestures for Hiro to agree.)

Hiro: Yes, he's right.

Nao: Because the parallelogram is... The opposite angles are equal, and the opposite sides are parallel, but only the opposite angles are equal.

T: In a parallelogram, are the adjacent angles not equal?

Hiro & Nao: No, they aren't equal.

We can see from this data that the children could at least recognize a parallelogram $p_1 \wedge p_2 \wedge p_3 \wedge \neg q$ (p_1: parallel opposite sides, p_2: equal opposite angles, p_3: equal opposite sides, $\neg q$: unequal adjacent angles).

Next, when the teacher asked the children to check whether the properties of a parallelogram are maintained in the case of a rectangle, they entered a state of disequilibrium. After some reflection, Hiro said 'I have decided to include it as a parallelogram, because it has all the properties of a parallelogram', and Nao agreed with Hiro. Although they could state the inclusion relations in terms of common properties, the teacher decided to bring up the tacit property again because the children still seemed to have $\neg q$ as part of their conception of parallelograms.

T: But you said the angles are different in a parallelogram, right?

Hiro: (After a short discussion in a small voice) They are different things.

Nao: Yes, these are different.

T: So you don't think a rectangle is a parallelogram?

Hiro & Nao: No, we don't.

We subsequently confirmed that this regression phenomenon occurred, and indeed it occurred in all the pairs of children. The following shows how the children were made to reflect on the variability of the angles in a parallelogram.

T: What do you think when you see this sheet? You wrote that the size of the angles and the length of the sides can change.

Nao: Oh, then it is included! Earlier we said that the size of the angles is changeable. So, we can also include it as a "parallelogram" when the angles are 90 degrees.

T: So, is it a member or not?

Hiro: No. Because the size of the angles in the rectangle changed, but they can't be changed.

Although Nao could identify the inclusion relation by observing the changes in the angles, Hiro could not. Although Hiro incorrectly interpreted the transformation, we believe his intention was to basically differentiate a rectangle (90 degree angles) from the other figures (parallelograms with angles other than 90 degrees). Next is an example of the teacher using an analogy with the rhombus-parallelogram relation with which Hiro had already agreed.

Teacher: A while ago, you agreed that a rhombus is a special type of parallelogram. This time, however, the problem is whether a rectangle is a special type of parallelogram.

Nao: Yes, it is. We say yes.

Hiro: No, we cannot say so.

Nao: Because it's special, it's special... even if the angles change in a parallelogram, it remains a parallelogram. A rectangle is a special type of parallelogram.

Hiro: Oh, I see.

Hiro was then convinced of, and agreed with, the rectangle-parallelogram relation because he could correctly interpret the word 'special' that Nao had stressed.

Last, the teacher asked them to describe exactly what a parallelogram was as a geometric figure. They then stated "The opposite sides are parallel, the opposite angles are equal, but the sizes of the angles are changeable; the lengths of opposite sides are equal, but the lengths of sides are changeable". Hiro and Nao were the only two children who included negative aspects in their definition. However, the other children could also describe a parallelogram by overcoming their tacit properties.

(2) Understanding by analogy with equivalent inclusion relations

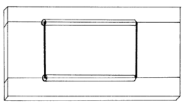

Figure 2.

Similar to Hiro and Nao, Mike and Aki also agreed with the rhombus-parallelogram relation, but they refused the rectangle-parallelogram relation, which was stated as follows: 'In a parallelogram opposite angles are equal, but all the angles are not equal.' Mike and Aki's tacit properties had developed so strongly that they continued to deny the correct rectangle-parallelogram relation. For them, 90 degrees was an inherent property that only rectangles (and squares) possessed, and that also served as a way to differentiate rectangles from other shapes. Moreover, the methods of 'reflecting on negative aspects' and 'analogy with the rhombus-parallelogram relation' was not effective for them. The following is an episode in which the teacher asked Mike and Aki to define a parallelogram to make them realize that their long definition was not economic (de Villiers, 1994).

Teacher: Well, what would you say if asked to describe a parallelogram?

Mike & Aki: The lengths of opposite sides are equal. The sides of opposite angles are equal, and opposite sides are parallel.

Teacher: These characteristics hold for a rectangle, too, right?

Mike & Aki: (After a reflection) The angles are different... It's okay if the angles are different... hmm... there is true for a parallelogram, but not for a rectangle...

They eventually could not describe a parallelogram in such a way that it would exclude rectangles. Thus we found that even if a child has the concept of a tacit property strongly, the property is not always expressed precisely through language.

Next, the experiment shifted to a discussion of square-rectangle and square-rhombus relations. The children easily recognized the square-rhombus relation even when the teacher pointed out the 90 degree angles characteristic of a square, but they denied the square-rectangle relation strongly. Okazaki and Fujita (2007) found that this phenomenon was widespread among Japanese students. The following is an episode in which the teacher suggested that the square-rhombus relation was similar to the rectangle-parallelogram relation.

T: Well, a rhombus is usually shaped like this, but you agreed that it was a rhombus when it had 90-degree angles. Shall we compare this with the case of rectangle and parallelogram? The magnitudes of the angles of a parallelogram are usually different, but when they are all 90 degrees... Do you think they are similar?

Mike & Aki: (Looking surprised.) Yes!

T: If you say that a rhombus remains a rhombus when the angles are 90 degrees...

Mike: It's a parallelogram... When the angles are 90 degrees, we can say it's a rhombus. So, even though it is a rectangle, we can say it's a type of parallelogram.

Aki: Yes. We can include it because it has the same qualities of a square when it has 90-degree angles.

The analogy with the square-rhombus relation was effective for them. We consider that it is more related to mathematical attitude where they tried to comprehend the various relations consistently, rather than the logic of the inclusion, because they had already had the logic at least for the square-rhombus relation.

DISCUSSION

We believe that for children to understand inclusion relations, they need to be able to grasp a figure dynamically and confirm which properties are maintained during the figure's dynamic transformation (Leung, 2008). Indeed, all of the children could attempt to recognize the inclusion relations in our study when the related properties were stressed. However, we found that the recognition of the geometric properties alone was not always sufficient for the understanding of the inclusion relations. Namely, the children often reverted back to their previous, exclusive classifications as soon as they became aware of the tacit properties that had pre-existed in their minds. As mentioned earlier, we propose that the process of tautologous evolution is a natural way for children to overcome their pre-existing tacit properties.

The understanding of the children in this study included the tacit property of 'the interior angles of a parallelogram are not 90 degrees'. Our hypothesis was that the children's recognition of the variability of sides and angles (negative aspects) would serve as an important component for constructing the understanding that there are cases in which adjacent angles are of equal size. Here, it should be noted that because such negative aspects are 'constructed only secondarily and laboriously' (Piaget, 1985), their cognitive "power" is relatively weak, and as a result sometimes the tacit property is superior (Okazaki, 1995). Thus in this study we instructed the children to write down these aspects for the purpose of reflection at later stages of the experiment. Results showed that 10 of the 14 children could successfully reflect on the language used in the descriptions, construct tautologies, and thus eventually replace their tacit properties with more accurate concepts. To summarize, the basic process and means may be described as follows.

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q$$

A. Dynamic transformation of the figure and recognition of property conservation

B. Understanding the variability of the sides and angles, and associating them with the properties of a parallelogram

$$\downarrow$$

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge (\neg q \vee q)$$

$$\downarrow$$

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n$$

As additional means, we found that the correct understanding of the word 'special' played a crucial role in properly understanding the inclusion relations. We believe that this, along with reflection on the notes each student wrote down regarding the negative aspects in the above description, corresponded to the linguistic meaning of the activity (de Villiers, 1994).

We found that analogical approaches were also effective. Japanese students tend to have an accurate understanding of rhombus-parallelogram and square-rhombus relations (Okazaki and Fujita, 2007), and these two relations functioned well as analogies for those who were not at first able to recognize the correct rectangle-parallelogram relation. Nonetheless, we believe these approaches should be used carefully because the effect can be the opposite if the base relation is not stable.

We conclude that after properly understanding the inclusion relations, the children were able to see parallelograms as a set of properties that are not influenced visually. Furthermore, we feel that they could develop a broader understanding of the general relational characteristics that connect different geometric figures (van Hiele, 1986). In this sense we believe the children's level of understanding went beyond the second van Hiele level. We nevertheless feel it is still premature to declare they had reached the third level, because they still did not know the role of definitions in proofs (de Villiers, 1994) and that properties can be ordered (van Hiele, 1985). Therefore we conclude that the children who successfully accomplished the learning of inclusion relations attained a level of comprehension somewhere between the second and the third van Hiele levels. Our future task will thus be to clarify further steps of learning that permit children to accelerate their transition towards deductive geometry.

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THEORIES OF EPISTEMOLOGICAL BELIEFS AND COMMUNICATION: A UNIFYING ATTEMPT

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In order to develop more detailed knowledge about possible effects of beliefs in mathematics education, it is suggested that we look more in-depth at more general types of theories. In particular, the study of relations between epistemological beliefs and communication is put forward as a good starting point in this endeavor. Theories of the constructs of epistemological beliefs and communication are analyzed in order to try to create a coherent theoretical foundation for the study of relations between the two constructs. Although some contradictions between theories are found, a type of unification is suggested, building on the theories of epistemological resources and discursive psychology.

INTRODUCTION

Regarding the study of beliefs, educational research has somewhat neglected theoretical aspects (Op't Eynde, De Corte, & Verschaffel, 2002; Thompson, 1992). Within cognitive psychology some more attention has been given to the development of theoretical models or frameworks, in particular regarding epistemological beliefs (see Hofer & Pintrich, 2002). Many studies about beliefs in mathematics education have been descriptive, by focusing on what different types of beliefs exist among students or teachers, where beliefs are seldom more directly related to other factors, such as students' learning (De Corte, Op't Eynde, & Verschaffel, 2002). By using theoretical frameworks from cognitive psychology, connections between epistemological beliefs and several different aspects of learning have been studied more in-depth, showing many connections (Pintrich, 2002). Mathematics education research could therefore benefit from relating to and utilizing more general types of research about epistemological beliefs, in particular regarding theory and relations between beliefs and aspects of learning.

A starting point for research about beliefs in mathematics education focused on students' learning and problem solving (Schoenfeld, 1983), but there seems to have been a shift in focus from students to teachers, and to the relation between teachers' beliefs and teaching practice. A general problem with this kind of research is the focus on such a "large" construct as teaching practice, since many factors can influence the decisions a teacher makes during lessons (Skott, 2005). Thus, there is a need to study possible influences of beliefs at a more detailed level. Cognitive psycholo-

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gist have done such more detailed studies of relations between beliefs and other aspects of cognition, but almost all these results come from studies of correlations and there is a need for more theoretical work on explaining how and why these connections exist (Pintrich, 2002).

Communication is central to processes of teaching and learning, as it is to all social situations:

> Language is so central to all social activities it is easy to take for granted. Its very familiarity sometimes makes it transparent to us. […] Moreover, language is not just a code for communication. It is inseparably involved with processes of thinking and reasoning. (Potter & Wetherell, 1987, p. 9)

The last part of the quote stresses the close connection between the use of language and thinking. One way to study relations between beliefs and aspects of cognition and behavior in more detail can therefore be to examine relations between beliefs and communication. Such relations can include how beliefs can affect, or be affected by, communication, for example how one expresses oneself or how one interprets something expressed by someone else (in writing or orally). Another perspective on the relations between beliefs and communication is to not see them as two separate "objects" that can affect each other, but as more integrated aspects of cognition and/or behavior.

So far I have discussed both beliefs in general and also epistemological beliefs in particular. From here on I will limit myself to epistemological beliefs, for several reasons: A more focused theme is thereby created while at the same time it is general enough to be relevant for the study of many educational situations and phenomena, and also there exist elaborate theoretical frameworks for this type of belief.

The first step in the project of studying relations between epistemological beliefs and communication has been to focus on theoretical aspects of the two constructs; beliefs and communication. So far I have studied the notion of beliefs by examining existing types of definitions in mathematics education research literature (Österholm, 2009). It is common to define beliefs through the distinction between beliefs and knowledge, but for educational research the analysis of existing definitions shows that this distinction is problematic since it tends to create "an idealized picture of knowledge, as something pure and not 'contaminated' with affect or context" (Österholm, 2009, p. 6). Instead of focusing on this distinction one can utilize the notion of a person's conceptions (as also suggested by Thompson, 1992) and focus on what a certain conception is about, such as epistemology. Thus, it is not important whether we label something as epistemological *belief* or epistemological *knowledge*. This conclusion is somewhat consistent with research in cognitive psychology, where the general notion of *personal epistemology* is used, which will be discussed later.

Purpose and structure of paper

In order to study relations between epistemological beliefs and communication, there is a need to have a theoretical framework that in a meaningful and coherent way defines both these two constructs and describes possible ways to study them. This paper constitutes the starting point in the process of creating such a framework. Existing theories of the two main constructs will be analyzed and related to each other in order to examine if and how such theories can be used in the creation of the needed type of framework. The present paper will discuss (1) theoretical frameworks of each construct separately, including how the theory of one construct views the other construct, and (2) possibilities to unify frameworks of both constructs.

THEORIES OF EPISTEMOLOGICAL BELIEFS

Focus is here on theories of personal epistemology (Hofer & Pintrich, 2002), since this area of research is where a most comprehensive theoretical treatment of epistemological beliefs exists. However, even if all theories within this area of research agree on the basic focus of the research; "an individual's cognition about knowledge and knowing" (Pintrich, 2002, p. 390), personal epistemology is not a unitary theoretical framework. Pintrich notes that differences between theories of personal epistemology can be related to more fundamentally different ways of viewing human cognition; developmental, cognitive and contextual approaches. An analysis of the different types of theories reveals that the developmental and cognitive approaches to personal epistemology cannot readily explain the empirical results showing a context dependence of epistemological beliefs (Louca, Elby, Hammer, & Kagey, 2004). Actually, many research methods used within these two approaches (implicitly) *presuppose* that epistemological beliefs are independent of context, for example by using questionnaires to simply ask a person about his or her beliefs (Hammer & Elby, 2002).

Instead of defining beliefs as a property of individuals' mental representations, Hammer and colleagues (Hammer & Elby, 2002; Louca et al., 2004) define beliefs by referring to more fine-grained parts of mental representations called epistemological resources. These resources are directly related to specific individual experiences and are not more abstract types of epistemological theories that are applied in different situations. In a certain situation, different epistemological resources can be activated and thereby utilized in the activity at hand. For example, the resource 'knowledge as propagated stuff' can be seen when children determine that they know something because one of their parents has told them so (Hammer & Elby, 2002, p. 178). Beliefs can then be defined as a property of the whole of all resources that have been activated in a specific situation.

The view of communication

Since all theories of personal epistemology focus on (the activation of) certain parts of mental representations, communication is not part of such a theory but becomes a relevant topic when discussing how to study these mental representations. That is, communication becomes a mean for "identifying which cognitive structures should be attributed to individual minds" (Hammer & Elby, 2002, p. 184).

THEORIES OF COMMUNICATION

One way to define communication is to describe it as the transmission of information from a sender to a receiver, for example when one person says something and another person hears and interprets this. This quite simple model of communication has received much criticism for being too simplistic, for example since (1) the model describes language as a static system but language do change, even during one specific conversation, (2) language is not only referential but also constitutive, and (3) language is important for doing things (e.g., greeting and denying) (Taylor, 2001, pp. 6-7). Thus, instead of viewing the use of language in communication as a mere medium by which information is encoded and transmitted, a more constructive view of language can be adopted: The use of language is seen as "a medium *of* action" (Potter & te Molder, 2005, p. 3, emphasis added), and not solely a medium *for* transmission of information or *for* anything else. Such a view is fundamental in certain types of discourse analysis (Taylor, 2001) and analysis of natural interaction (Potter & te Molder, 2005), both which include a broad range of different types of research and which have many overlaps between them. However, this type of research has mostly focused on sociology and "questions of psychology have rarely been explicitly addressed" but that "discursive psychology is the perspective that has addressed cognition in the context of interaction most systematically in a psychological context" (Potter & te Molder, 2005, pp. 18-19). Thus, for the purpose of the present paper, when focusing on cognitive aspects of beliefs, discursive psychology is a good candidate for a suitable, more detailed perspective on communication.

As for all discourse analytic approaches, discursive psychology focuses on the language itself, and how it is used in natural occurring situations. Thus, instead of viewing psychological vocabulary as referring to some mental states, "these words are *themselves* an autonomous part of particular social practices" (Potter & Wetherell, 1987, p. 179). More specifically for discursive psychology, this perspective works in three ways (Edwards & Potter, 2005, pp. 241-242): (a) By exploring situated, rhetorical uses of psychological terms such as angry, know or believe, (b) by examining how psychological themes are handled without necessarily being expressed explicitly, for example to explore how intent, doubt or belief is constructed and made public indirectly through descriptions of events, objects, persons etc., and (c) by respecifying standard psychological topics in terms of discourse practices, for example regarding the theory and measurement of attitude or of causal attributions, where often a criticism towards existing theories and measures in cognitive psychology is a central aspect.

The view of epistemological beliefs

Communication is here seen as the central site for psychology and not as a window to something else, where a researcher should not try "to see through their [speakers or writers] words to some underlying meaning, or to uncover attitudes or beliefs of which the speakers themselves are unaware" (Taylor, 2001, p. 19). Thus, as also noted above, discursive psychology defines cognitive notions (such as belief) as parts of discourse practices instead of as parts of mental representations/processes.

THEORETICAL UNIFICATIONS

A general difference between cognitive psychology and discourse analytic approaches is that the object of study is either mental representations/processes or the use of language. This difference is not only an empirical matter but is also theoretical, since either the mental or the discourse is seen as 'where the action is'. An analysis of theoretical possibilities for studying relations between epistemological beliefs and communication is here done in the following way: Discussions of potential limitations and developments within the theories of epistemological resources and discursive psychology will be used as a foundation for a discussion of the possibility to integrate the two theories.

Epistemological resources as a starting point

Within this theoretical approach, two aspects need improvement, for the study of relations between epistemological beliefs and communication: First, Pintrich (2002, p. 394) points to the general need within research about personal epistemology to develop models of how epistemological thinking may be represented cognitively. Second, communication needs to be problemized more in-depth, where a person's statements cannot be seen as a direct reflection of mental representations. This second aspect refers both to a methodological problem (i.e., how to get a good 'picture' as possible of mental representations) and also to a theoretical problem (i.e., how to model communication and interaction). For the focus of the present paper, theories need to be of the kind that somehow relates communication to cognitive representations and processes. To include at least some aspects of communication within such a theory, Kintsch's (1998) theory of 'comprehension' could be possible to use. This theory includes detailed models of mental representations (using associative networks) and of the process of interpreting/comprehending something 'external' (such as a text or an oral statement), and have proven useful when for example describing and predicting readers' comprehension of texts.

The descriptions of epistemological resources as a fine-grained cognitive network and the utilization of these resources as the activation of different resources depending on the context (see Louca et al., 2004) also fit within Kintsch's (1998) comprehension framework: The activation of resources can be modeled through the associative property of mental representations, which is a driving force in the activation of prior knowledge (resources) in the process of comprehension, in contrast to the activation of some abstract or general type of belief.

Discursive psychology as a starting point

From this theoretical perspective, one can see a need for redefining 'beliefs' in general and 'epistemological resources' in particular, in terms of discourse practices (however, for reasons discussed in the introduction, I am not primarily interested in the notion of beliefs but on notions of epistemology). Such a respecification can be achieved by an analysis of the discourse in situations where a treatment of knowledge or learning occurs, or by an analysis of how people explicitly use and refer to epistemological notions. These types of analyses have some similarity with suggestions by Hammer and Elby (2002) of how to find possible epistemological resources by examining how people reason and draw conclusions in situations where they need to make some kind of judgment of knowledge. This kind of overlap between the different theories can be taken as a starting point in a possible unification, for example by seeing the activity, the discourse, as the site where epistemological beliefs come to existence, through explicit or implicit references to prior experiences (epistemological resources).

One limitation in discursive psychology is that thinking as an individual and silent activity is not included, since all psychological terms are specified in terms of discourse practices. Sfard (2008) suggests a type of expansion of a discourse analytic approach by defining thinking as the individual version of communicating. This suggestion somewhat removes the distinction between the mental/individual and the public/discourse. Therefore, such a theory can be seen as a possibility for joining the theory of epistemological resources (a theory about mental representations/processes) and the theory of discursive psychology (a theory about discourse practices).

A mixture of epistemological resources and discursive psychology

The two theories have so far often been described in a contrasting manner, in particular regarding how they see the existence or relevance of mental representations and processes. However, to focus a discussion on this issue may result in

> fruitless debates about the reality or non-reality of mental entities, which can easily end in the kind of linguistic imperialism which denies all significance to cognitive processes (Potter & Wetherell, 1987, p. 180).

Therefore, in order to study aspects of epistemological beliefs and communication under the same theoretical framework, it is suggested that we both see the relevance of mental representations and processes and also adopt a constructive view of language (i.e., highlight the constitutive property). This suggestion is of course not new in a broader perspective (e.g., see the description of different authors' positions by Potter & te Molder, 2005, p. 5), but most relevant for the present paper is to discuss possible ways to draw on this suggestion specifically for theories of epistemological resources and discursive psychology, which is done in the following.

Epistemological resources are here not seen as necessarily consisting of mental entities that directly say something about epistemology, but that the resources consist of prior experiences that can be used for some kind of judgment of knowledge. Although not described in this manner by Hammer and colleagues (Hammer & Elby, 2002; Louca et al., 2004), this description seems to be in line with their description of looking at examples of children's behavior where they make some type of judgment of knowledge as an origin of a resource that can be used in other situations.

The utilization of prior experiences can be described in more detail by using a model of the structure of mental representations (i.e., memory of prior experiences) and a model of how these representations can be activated and utilized (i.e., the mental process), such as the theory of 'comprehension' mentioned earlier (Kintsch, 1998). Such models can then be used as a foundation for describing and explaining how and why certain prior experiences (resources) are used, and others are not, in a specific situation. This placement of a situation within a context, through references to and utilization of prior experiences, can also be seen as central to discourse analytic approaches, in particular since prior experiences is not limited to events long prior to a specific situation but can also include for example earlier statements in one conversation.

Epistemological belief is not seen as a property of mental representations but as a property of the activity in a certain situation, which is dependent on prior experiences (epistemological resources). However, beliefs are not *determined* by existing resources since the discourse is constitutive and not only a reference to mental representations. Thus, beliefs are being constructed in a specific situation. In this way, epistemological beliefs can be seen as different ways of thinking, where the processes of utilizing prior experiences and of participating in a discursive practice are of fundamental importance.

CONCLUSIONS

The present analysis has revealed different theoretical possibilities for future studies of relations between epistemological beliefs and communication. In particular, it seems possible to join the theories of epistemological resources and discursive psychology, through some additions or clarifications: (1) to include a model of the structure and utilization of mental representations, (2) that mental representations are primarily seen as describing the memory of prior experiences, and (3) that the utilization of prior experiences is seen as a central aspect of the contextualization of discourses. This suggested unification of theories is therefore seen as a good starting point for a continued development of theory and for future empirical studies.

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DEVELOPMENT OF PEDAGOGICAL CONTENT KNOWLEDGE WITH REGARD TO INTERSUBJECTIVITY AND ALTERITY*

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In this paper we focus on the development of pedagogical content knowledge (PCK) and examine this development in the light of the notions of intersubjectivity and alterity. We base our examination on data obtained from a teacher preparation program in which 20 teacher candidates take part. On the basis of the analysis of the data we argue that development of PCK is a dialogical process and such development could be described in terms of increasing intersubjectivity amongst the participants and that alterity accounts for the different approaches adopted to teach a topic at hand in such a way that makes it comprehensible to the others.

INTRODUCTION

The notion of pedagogical content knowledge (PCK) has long been on the agenda of mathematics educators. The term was described as the "subject matter for teaching" by Shulman (1986, p.9) while referring to the knowledge required teaching a particular subject in such a way that makes it comprehensible to the others. Shulman considers knowledge of student understanding, and teaching strategies and representations as the two main components of PCK. Later exploration of PCK developed this notion further and different models with additional components (e.g. knowledge of assessment and of curriculum) were articulated (Park and Oliver, 2008). For our study, we examined these components with the purpose of designing a course aimed at developing PCK for the teacher candidates. To this end, we designed a course as part of teacher preparation program for the mathematics teacher candidates. After the course, we observed development in the participant candidates' PCK. In our attempt to make sense of the development of teacher candidates' PCK, we found the notions of intersubjectivity and alterity useful. In this paper we will argue that these notions provide a useful framework in explaining the nature and dimensions of the development of PCK and deepen our understanding of the complexity of the dynamics involved in such a development.

In what follows we first attend to the notions of intersubjectivity and alterity as the framework of this paper. Later we detail the course design based on the works of Magnusson et al. (1999). Then data obtained from the participant teacher candidates are provided and analysed in the light of the intersubjectivity and alterity.

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THEORETICAL FRAMEWORK

Theoretical framework of this paper is centered on two opposing tendencies which might be viewed as characterizing any dialogue: intersubjectivity and alterity. The importance of these two tendencies may vary depending on the specific conditions of the communication; yet both are always at work in any given dialogue (Wertsch, 1991). Dialogue here is used in the sense of Bakhtin (1981) who considers it as a constant interaction between meanings which potentially affect others. Following Bakhtin our account of dialogue is very general, not limited to face-to-face verbal interaction, but concerns verbal communication of any type whatsoever; in this respect even a book (i.e. "verbal performance in print") is dialogic (see Wertsch, 1991).

The problem of intersubjectivity is, generally speaking, related to the conditions under which participants in a dialogue achieve a coherent and viable interaction. In this respect Uhlenbeck (1978) draws attention to the necessity of shared understanding of what has gone before in order to understand the speaker's intention and meaning. Schegoff (1991) views shared understanding as a crucial ingredient for the coherence and viability of an interaction. For otherwise there occur divergences in understandings, which, according to Schegoff (1991, p.158), embody 'breakdowns' of intersubjectivity, that is, 'trouble' "in socially shared cognition of the talk and conduct in the interaction." To achieve the coherence and viability of interaction, authors, such as Clark (1996), insist on the establishment of a common ground on which participants of a dialogue can interact successfully, can share their understandings with one another. In the light of these considerations, intersubjectivity is concerned with, broadly speaking, the extent to which different aspects of an activity is shared amongst the participants and/or held in common such as perspectives, understandings and assumptions (see also Wertsch, 1998).

Although in theory it is possible to talk about 'pure' intersubjectivity, in practice it rarely happens. The basic problem with pure intersubjectivity is that it treats the message as transmitting an unaltered meaning and ignores the differences in, for instance, interpretations, perceptions and perspectives on the talked-about reality. This brings us to the issue of alterity. The term 'alterity' is derived from the writings of Bakhtin who rejects the transmission of meaning through language (Bakhtin, 1986). He ascribes the other (as opposed to the self) a focal position in the creation of meaning of an event, of the talked-about reality due to the differences that the other brings with his/her own world view and due to the fact that one sees the talked-about reality from his/her own perspective and through his/her own conceptual horizon (see Wertsch, 1991). To sum up, alterity is concerned with the differences in perspectives, conceptual horizons and world views as well as with the differences and the changes occurring in perceptions, understandings and interpretations in the course of dialogue.

We believe that any development of PCK whether it be gained through a preparation program or material experiences of teaching in actual classrooms comes about via dialogue (in the broadest sense) because teachers (candidates) interact with their own peers, students, books, curriculum scripts and so on and these interactions contribute to their PCK. Limiting our considerations in this paper to the development of PCK during a course designed for prospective teachers, we argue that such development can be viewed in terms of increasing intersubjectivity among the participants and that alterity accounts for the differences in participants interpretations of how to make the subject at hand comprehensible to the others. Next we attend to the context of the study and methodology.

CONTEXT OF THE STUDY AND METHODOLOGY

This paper stems from a research project for which a course was designed for mathematics teacher candidates to develop PCK. In designing the course, based on the work of Magnusson et al. (1999) we drew on five components of PCK: knowledge of multiple representations, of student difficulties, of instructional strategies for teaching, of assessment and of mathematics curriculum. All these components guided the design of the course. One of the authors conducted workshops to the teacher candidates on each of these components by focusing on the mathematical topic of derivative. Twenty teacher candidates participated in the course during which they read relevant texts such as student difficulties and misconceptions in different mathematical topics and assessment (e.g. Ozmantar et al., 2008), prepared lesson plans, examined curriculum scripts, designed and conducted microteachings, and followed their peers during microteaching sessions.

The data on which this paper reports came from teacher candidates’ lesson plans, interviews on the preparation of lesson plans, candidates’ detailed teaching notes, and video-record of microteachings. At the beginning of the course, every candidate was asked to prepare a lesson plan to introduce the concept of derivative at a point. The aim was to see how they would go about introducing the topic and to figure out their prior knowledge on each of the abovementioned PCK components. We also gave a set of questions on derivative to identify, if they have, their difficulties with the topic. We conducted semi-structured interviews with ten of the candidates. After the workshops, the candidates were asked to prepare the lesson plan on the same topic once again by drawing on what they learnt during the workshops and also asked to evaluate their initial lesson plans. Having collected the second lesson plans we interviewed the candidates to obtain their reflections on their initial plans and to find out whether they considered the components of PCK in their second lesson plans. Following these interviews ten of them, based on their plans, did microteachings.

In our analysis of the data we scrutinised the microteachings along with the teaching notes and critically evaluated the teacher candidates’ approaches to introduction of the derivative. We also examined and compared the lesson plans prepared before and after the workshops together with interview transcripts. In our analyses we realised a telling difference in their lesson plans before and after the workshops. We also recognized that before the workshops there was a huge gap between our understanding of the components of PCK and those of the teacher candidates.

However following the workshops, as is clear from the interviews, candidates developed certain understandings which created a common ground. Hence we focused on the issue of intersubjectivity in making sense of the candidates' development. Our analysis of microteaching records along with second lesson plans brought the issue of alterity into our attention as there were quite different, yet in our view successful, approaches adopted by the candidates in introducing the concept of derivative despite the fact they all were part of the same course, followed the same program and received the same content. In the following section, we provide data and analyse them from the lenses of intersubjectivity and alterity.

THE DATA

In this section we provide excerpts from the interviews with teacher candidates on their first and second (before and after workshops on PCK) lesson plans along with our analyses of their microteaching records. During the semi-structured interviews we asked questions about the five components of PCK. Analysis of the interviews on the first lesson plans reveals that before the course their understandings of the five components were rather different than those of the researchers. For example with regard to the multiple-representation component we asked candidates what kind of representations could be used while introducing derivative. Answers of two candidates were as follows (RP and AG stands for the initials of the candidates):

> RP: While giving the geometrical interpretation of derivative, graph might be used. Apart from this something colourful, if possible, particularly while explaining instantaneous velocity, pictures of automobiles, or radar pictures or maybe a driver got caught to the radar, or it might even be a dialogue between the police and a driver or that kind of things. That would be good.

> AG: Representations regarding the purpose of using derivative in real life; this can be found in velocity for instance…then the transition from the graph of velocity to the acceleration and after giving the meaning of these, they come from slope…in different ways derivative could be expressed but I can't really answer this.

As is apparent from the answers, students' understandings of multiple representations of derivative was inappropriate or at least wanting. Clearly they were not able to make sense of the talked-about reality that derivative can be represented in different ways, for example, graphically, algebraically or by means of table of values. We do not think this stems from deficiency of conceptual knowledge as derivative test given these teacher candidates suggest that they can read and interpret different representations. Surely knowledge and awareness of different representations of a particular topic is indispensable component of PCK for otherwise it is not possible for a teacher to select the appropriate forms of content representation (see Shulman, 1986). Hence there was a great alterity not only amongst the teacher candidates themselves but also between the candidates and the course designers as to multiple representations of derivative. During the interviews on the first lesson plans, we also asked what kind of difficulties students might have in understanding the derivative. They answered as follows:

RP: Introducing the concept of derivative…they might have difficulties if they don't know limit and continuity but I don't think they would have much trouble, it's just… I mean if you explain well… but later they could have difficulty.

AG: Student may ask why we find derivative, anyway students generally have "why we learn maths" kind of questions "we don't use maths" they say. I mean derivative…they may have difficulty in understanding the purpose.

The literature is replete with the student difficulties in understanding three different aspects of derivative as rate of change, as the slope of the tangent line and as the limit of difference quotient (see Bingolbali, 2008 for an extensive review). However the teacher candidates were not aware of these difficulties and in this sense there were divergences among the teacher candidates and the researchers in their understandings and perspectives. The situation with the other components of PCK was not much different but due to space limitations we suffice to provide these excerpts which, we believe, give the reader an idea about the extent to which different aspects of the PCK is shared among the participants (i.e. teacher candidates and the course designers). After the workshops on the five components of PCK, there were dramatic changes in the perspectives of the candidates. In order for readers to better appreciate the importance of the changes, it will be useful to briefly summarise RP's approach to introducing derivative in her first and second lesson plans. In her first lesson plan, RP immediately starts with providing algebraic (limit) definition of derivative and then illustrates the application of the definition with an example. Next she explains the physical interpretation of derivative as instantaneous velocity and gives a problem in which instantaneous velocity is calculated with the limit definition of derivative. Later she moves to geometrical interpretation noting that derivative at a point equals to the slope of tangent line at that point and gives an example to illustrate this.

In her second lesson plan, she begins with a problem to find the average velocity in intervals. She fills a table of values by finding average velocity in different intervals in the neighbourhood of a point. She uses this to get students to predict the instantaneous velocity which she later finds via limiting process. She relates this to the rate of change and notes that derivative at a point is the rate of change at that point and gives the limit definition of derivative. She then interprets this definition on a graph and shows that limit definition refers to the limit of secant lines which is the slope of the tangent line at a point. She returns back to table of values and relate geometrical interpretations to this table and to the rate of change. Finally she returns to instantaneous velocity and connect this to the slope of tangent line at a point.

The development and the change from RP's first to second lesson plan is all too apparent. In the first plan, her lack of knowledge of five aspects of PCK, for example multiple representations, clearly affects her planning and she prefers a definition-application kind of structure without really relating different aspects of derivative with one another. However in her second approach she adopts a chain composed of problem-discovery-hypothesis-corroboration-generalising-interconnecting (MEB, 2005). Such a dramatic change reflects her development during the PCK course. This development in fact leads to the establishment of a common ground between the teacher candidates and the course designers. To illustrate this let us turn to RP's interview on her second plan. With regard to the multiple representations, for example, RP was not only able to explain them but also able to devise ways in her lesson plan in connecting these representations to three main aspects of derivative:

> RP: Triple representations, as we call it multiple representations: physical, geometrical and algebraic. I used them three. I mean I gave table of values. With the help of numerical table, after taking the limit of rates of changes, I [wanted] students to generalise this limiting to derivative… only after the limiting process of rates of change, that I gave the algebraic definition of derivative. After showing the relation between rate of change and the slope of the tangent, I [had students] found the slope of tangent line as equivalent to derivative and did so through the instantaneous velocity…

When asked about student difficulties regarding the introduction of derivative, she noted the student difficulties in connecting different aspects and added:

> RP: [students] memorise the algebraic definition and just say derivative is the slope without making much sense of it…they believe that equation of the tangent line equals to the derivative at that point…they don't know why the limit of the rate of change gives us the derivative [at a point].

RP's awareness of student difficulties shaped her second lesson plan. She paid much more attention to connecting different representations and relating them to the different aspects of derivative. We have observed similar developments in other teacher candidates as to five components of PCK. However, the development does not necessarily bring about the same approach to introducing derivative; there was a great diversity amongst the approaches. For instance, our interviews with AG clearly show the similar development as in RP but AG's approach was rather different. To succinctly summarise, AG started with composing table of values in different intervals for a function and later depicted these values on a graph. Then she brought in the idea of rate of change and stressed slope of secant lines. Next she related rate of change with the slope of tangent line and the instantaneous velocity. Only after all these she gave the limit definition of derivative and connected them all together. Such diversity points to the existence of alterity in the developmental path of candidates.

DISCUSSION

In our analysis it becomes evident that teacher candidates taking part in our course displayed remarkable development with regard to the five components of PCK. This observation can be easily corroborated on the basis of lesson plans prepared before and after the course, interview transcripts and microteachings as exemplified in the previous section. We are convinced that intersubjectivity and alterity can be viewed as characterizing this development. At the beginning of the preparation program, there was not a common ground neither amongst the teacher candidates themselves nor between them and the course designers. However during the course, the participants developed a shared understanding of the issues with regard to the five components of PCK and hence in the second interviews one can sense an established common ground amongst the participants. This surely indicates the teacher candidates' development of PCK. On this basis it can be argued that the development of PCK can be viewed in terms of increasing intersubjectivity.

Development of this intersubjectivity achieved through the dialogue. When viewed from a Bakhtinian perspective, the dialogue was not just between the teacher candidates and the course designers. The candidates were also in interaction with the meanings that others brought into the program such as the writers of the texts on different components of PCK and curriculum scripts that they were reading and examining as part of the program. Further to this, during the program candidates worked in small groups with their peers in understanding the multiple representations, students' difficulties, how to draw on these difficulties in overcoming them, what kind of assessment could be used to evaluate student comprehension of the topic taught and so on. All these interactions contribute to the development of their PCK and hence increasing intersubjectivity.

However, this development cannot be best understood in terms of intersubjectivity alone. Alterity was equally important for such development to come about. It is true that before attending the course due to high alterity participants were not able to make sense of the talked-about reality (the five components of PCK) and there were differences among their perspectives which initially embodied the breakdowns of intersubjectivity (Schegoff, 1991). Yet these differences played an engine role in achieving intersubjectivity in that the candidates were struggling to make sense of the components of PCK and it was through this struggle that they come to terms with the issues such as multiple representations or student difficulties. The changes in their understandings and interpretations were all too apparent as evidenced by the second interviews and lesson plans in which they, as exemplified in our analysis section, paid attention to, for instance, the different representations and devised ways to make the connections among them comprehensible to the others. During the course, the teacher candidates were working with one of the course designers who had dramatically different perspectives on the components of PCK. Yet it was this accompany with the different that led candidates to develop. Hence we share Bakhtin's view that meaning of an event, which in our case is the components of PCK, is created by virtue of the co-existence or co-being with the different.

Alterity did exist not only before the candidates' participation of the course but also after their participation. Despite the fact that the candidates were the part of the same course, working and discussing collaboratively and led by the same lecturer, read the same texts and examined the same curriculum scripts, their interpretation of how to make the introduction of derivative at a point comprehensible to the others varied from person to person. As explained in data analysis section, RP and AG's approaches to introducing derivative were quite different, though successful, and in this sense alterity was still existent after the course. This, we think, is quite understandable when we realise that each person was looking at the issue from his/her own perspectives, conceptual horizon and world views (Wertsch, 1991). Existence of such differences resulting from alterity suggests the plurality of PCK for any particular topic and that, we believe, is a source of valuable enrichment. Hence we take Bakhtin's view that alterity should be accepted and defined positively rather than associated with some kind of insufficiency.

As a final point we wish to note here that as our analysis and discussion hitherto implies in the course of development of PCK intersubjectivity and alterity are often co-existent at different degrees and with relative importance. Therefore, development of PCK as a dialogic process cannot be best understood in terms of one or the other in isolation but rather through a consideration of both of these tendencies.

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AFFECTIVE AND COGNITIVE FACTORS ON THE USE OF REPRSENTATIONS IN THE LEARNING OF FRACTIONS AND DECIMALS

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Cognitive development of any concept is related with affective development. The present study investigates the structure of students' beliefs about the use of different types of representation and their respective self-efficacy beliefs in relation to their cognitive performance on the concepts of fractions and decimals. The interest is concentrated on differences between student's structure at primary and secondary education and on differences at the interrelations between cognitive and affective factors. Results revealed that multiple-representation flexibility, ability on solving problems with various modes of representation, beliefs about use of representations and self-efficacy beliefs about using them constructed an integrated model with strong interrelations in different educational levels.

INTRODUCTION

The relationship between cognition and affect has attracted increased interest on the part of mathematics educators, particularly in the search for causal relationship between affect and achievement in mathematics (Zan, Brown, Evans & Hannula, 2006). This is due to the fact that the mathematical activity is marked out by a strong interaction between cognitive and emotional aspect. The affective domain is a complex structural system consisting of four main dimensions or components: emotions, attitudes, values and beliefs (Goldin, 2001). At the present study we focus on students' beliefs and their self-efficacy beliefs about using different types of representations in mathematics learning and understanding. We concentrated our attention on the notion of fractions and decimals. Fractions are among the most essential (Harrison & Greer, 1993), but complex mathematical concepts that children meet in school mathematics (Charalambous & Pitta-Pantazi, 2007).

Mathematics is a specialized language with its own contexts, metaphors, symbol systems and purposes (Pimm, 1995). One's behavior and choices, when confronted with a task, are determined by her/his beliefs and personal theories, rather than her/his knowledge of the specifics of the task. Numerous studies on the use of representations have attempted to explain their contribution to learning concepts and to efficiency in problem solving. Among the many strategies that have been suggested to improve efficacy in solving math word problems, using diagrams and more generally external representations has been described as one of the most effective (Uesaka, Manalo & Ichikawa, 2007).

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 273-280. Thessaloniki, Greece: PME.

Beliefs is a multifaceted construct, which can be described as one's subjective "understandings, premises, or propositions about the world" (Philipp, 2007, p. 259). The construct of self-efficacy beliefs signifies a person's perceived ability or capability to successfully perform a given task or behaviour. Bandura (1997) defines self-efficacy as one's perceived ability to plan and execute tasks to achieve specific goals. Self-efficacy beliefs have received increasing attention in educational research, primarily in studies for academic motivation and self-regulation (Pintrich & Schunk, 1995). It was found that self-efficacy is a major determinant of the choices that individuals make, the effort they expend, the perseverance they exert in the face of difficulties, and the thought patterns and emotional reactions they experience (Bandura, 1986).

Recognizing the same concept in multiple systems of representations, the ability to manipulate the concept within these representations as well as the ability to convert flexibly the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh, Post, & Behr, 1987) and allow students to see rich relationships (Even, 1998). The different types of external representations in teaching and learning mathematics are widely acknowledged by the mathematics education community (NCTM, 2000). Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2006). The necessity of using a variety of representations or models in supporting and assessing students' constructions of fractions is stressed by a number of studies (Lamon, 2001).

An issue that has received major attention from the education community over the last years refers to the students' difficulties when moving from primary to secondary school and to the discontinuities in the curriculum requirements, the use of teaching approaches, aids and methods. Pajares and Graham (1999) investigated the extent to which mathematics self-beliefs change during the first year of middle school. By the end of the academic year, students described mathematics as less valuable, and they reported decreased effort and persistence. The findings of the Deliyianni, Elia, Panaoura and Gagatsis's (2007) study suggest that there is a noteworthy difference between primary and secondary education in Cyprus concerning the representations used in mathematics textbooks on fractions. There are also differences in the functions the various representations in the school textbooks fulfil.

Most mathematics textbooks today make use of a variety of representations more extensively than every before in order to promote understanding (Elia, Gagatsis & Demetriou, 2007). Much more research is needed for the students' beliefs about the role of those representations in relation to their self-efficacy beliefs for using them as a tool for the better understanding of the mathematical concepts (Patterson & Norwood, 2004). The present study investigated students' beliefs at primary and secondary education about the use of different representations for the learning of the fractions and decimals and their self-efficacy beliefs about the use of different forms of representations. It examines the confirmation of the structure of cognitive and affective factors concerning students' beliefs and self-efficacy beliefs about the use of multiple representations and their performance on fractions and decimals. The interest concentrated on the interrelations among the abovementioned cognitive and affective factors and the respective differences on those dimensions at primary and secondary education.

METHOD

The study was conducted among 1701 students of 10 to 14 year of age who were randomly selected from urban and rural schools in Cyprus. Specifically, students belonging to 83 classrooms of primary (Grade 5 and 6) and secondary (Grade 7 and 8) schools were tested.

A questionnaire was developed for measuring students' beliefs about the use of different types of representations for understanding the concept of fractions. The questionnaire comprised of 27 Likert type items of five points (1=strongly disagree, 5=strongly agree). The reliability of the whole questionnaire was very high (Cronbach's alpha was 0.88). For example there were items such as "I can easily find the diagram that corresponds to an equation of fractions" and "When I solve a problem with fractions, I use the number line for executing the operations". At the same time a test was developed for measuring students' ability on multiple representation flexibility as far as fraction addition and decimal number addition is concerned. There were treatment, recognition, conversion, diagrammatic problem-solving and verbal problem-solving tasks (further details for the tasks can be found at the paper of Deliyianni et al., 2008).

The tests and the questionnaire were administered to the students by their teachers at the end of the school year in usual classroom conditions. Right and wrong or no answers were scored as 1 and 0, respectively. Solutions in treatment, recognition and translation tasks were assessed as correct if the appropriate answer, diagram, equation or shading were given respectively, while a solution in the problems was assessed as correct if the right answer was given.

RESULTS

In order to confirm the structure of students' cognitive and affective abilities in the concepts of fractions and decimals at primary and secondary education, a CFA (Confirmatory Factor Analysis) model was constructed by using the Bentler's (1995) EQS programme. The tenability of a model can be determined by using the following measures of goodness of fit: $x^2/df < 1.95$, CFI>0.9 and RMSEA<0.06.

Figure 1 presents the results of the elaborated model that fits the data reasonably well for both the levels of education (primary education: $x^2/2713= 1.42$, CFI=0.911 and RMSEA=0.026,secondary education: $x^2/2707=1.49$, CFI=0.913 and RMSEA=0.027). The second order model which is considered appropriate for interpreting students' beliefs and abilities involves 15 first order factors and 4 second order factors. The five first order factors (F1 to F5) express the multiple representation flexibility

(MRF) for the concepts of fractions and decimals. Specifically, F1: recognition tasks with the same number of digits, F2: recognition tasks with different number of digits, F3: treatment tasks, F4: conversion tasks in which the initial representation is symbolic and the target one is a diagram, F5: conversion tasks from a diagrammatic to a symbolic representation. Those five first order factors regressed on a second order factor concerning students' multiple representation flexibility. The next second order factor express problem solving ability. It is consisted of two first order factors (F6: problem solving ability on problems with a diagrammatic representation and F7: problem solving performance on verbal problems). The third second order factor express students' self-efficacy beliefs about the use of representations. It consisted of three first order factors (F8: self-efficacy beliefs about the conversion from one type of representation to another, F9: general self-efficacy beliefs about mathematics, F10: self-efficacy beliefs about the use of verbal representations). The fourth second order factor explains students' beliefs about the use and the role of representations on learning and understanding. It is consisted of three first order factors (F11: beliefs about the use of number line, F12: beliefs about the use of models, materials and representations, F13: beliefs about the use of diagrams in problem solving).

The interest concentrated on the interrelations between the second-order factors, as indications of the impact of affective factors on cognitive performance and vice-versa. The highest statistically significant ($p<0.05$) interrelation is between the multiple representation flexibility and problem solving performance (primary .915, secondary .954), indicating that students who are efficient in using different types of representations have higher performance on problem solving tasks with representations. As it was expected, very high is the relation of students' beliefs about the use of representations and their self-efficacy beliefs (primary .877, secondary .816). Students' with high self-efficacy beliefs about their ability to use representations, express positive beliefs about the use of representations on teaching and learning. The relation is lower in secondary education where teachers use fewer representations and consequently students have less positive beliefs about their usefulness and less positive self-efficacy beliefs due to the lack of recent experiences. The relations between the self-efficacy beliefs about the use of representations with the problem-solving ability (primary: .572, secondary .626) and the multiple-representation flexibility are higher in secondary education (primary .590, secondary .621), indicating that students have more precise self-representation about their cognitive and affective performance. There are no statistically significant interrelations of beliefs about the use of representations with the multiple-representation flexibility. That means that the students encounter conversion and recognition tasks as exercises which have no contribution on the constructing of positive beliefs for representations' value as teaching tools on the learning procedure.

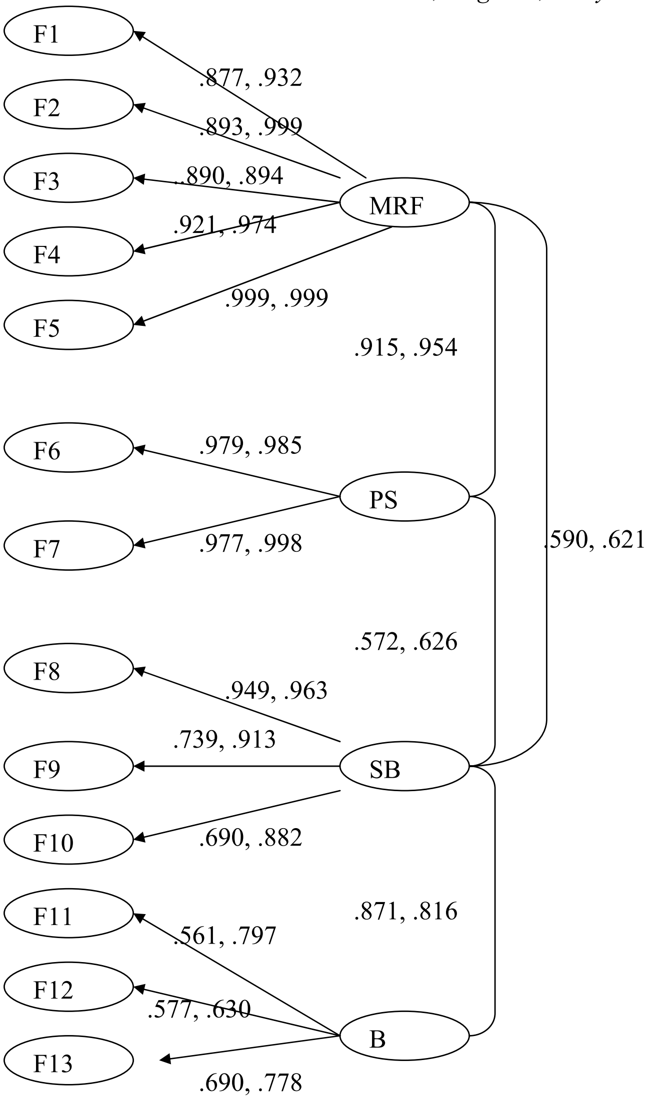

Figure 1: The CFA model of students' cognitive and affective abilities in fractions and decimals at primary and secondary education

Note: 1. MRF= Multiple representation flexibility, PS= Problem solving Ability, SB= Self-efficacy beliefs, B= Beliefs, 2. The first and second coefficients of each factor stand for the application of the model at primary and secondary education

The loadings of the whole model are higher in the case of secondary education. Therefore the particular cognitive structure is an integrated model on cognitive and affective factors concerning the use of representations for the concepts of decimals and fractions which becomes more stable across the educational levels, as a result of the continuous experiences in the teaching procedure and the more precise self-representation about the cognitive and affective performance.

DISCUSSION

Learning involves information that is represented in different forms. Given that each representation has different advantages and limitations, using multiple representations is important of mathematical understanding (Duval, 2006). The main emphasis of the present study was on investigating the structure of students' beliefs and self-efficacy beliefs about the use of representations and their cognitive performance (multiple representation-flexibility and problem-solving ability tasks) on the concepts of fractions and decimals. Results confirmed that multiple-representation flexibility, ability on solving problems with representations, beliefs about use of representations and self-efficacy beliefs about using them constructed an integrated model with strong interrelations in different educational levels. All the abovementioned cognitive and affective dimensions have impact on decimal and fraction addition understanding. The results indicated the important role of the multiple-representation flexibility and problem-solving ability in primary and secondary school students' fraction and decimal number addition understanding and the important role of beliefs about the use of different representations for the specific concepts and their respective self-efficacy beliefs. The invariance across primary and secondary education on the structure of the model underlines the need to develop curriculum and teaching methods which have a continuity from primary to secondary education.

Although the conversion and recognition tasks have not important contribution on developing positive beliefs about the use of representations, there is an indirect impact, as far as tasks for developing multiple-representation flexibility seems to be a presupposition for developing the ability to solve problems with multiple representations. Problem-solving ability correlates with self-efficacy beliefs and those with beliefs about the use of representations. Recent experiences and success in solving tasks with representations affect the development of self-efficacy beliefs and as a consequence construct positive beliefs about the use of representations. The lower relations in secondary education than in primary education indicate that students need more experiences of using a variety of representations at this level of education in order to stable their beliefs on the specific domain. On the one hand, the difference can be explained by the fact that students face difficulties in multiple-representation tasks which are increased in secondary education since no emphasis is placed on learning with multiple representations. On the other hand, the lower relation can be explained by the construction of a more precise self-image when students become older (Demetriou & Panaoura, 2006), while younger students tend to overestimate their performance and have very high self-efficacy beliefs.

The significant interrelation of students' self-efficacy beliefs with their multiple-representation flexibility and the problem-solving ability confirm that students with lower performance on mathematics have at the same time negative self-efficacy beliefs about their ability to use representations because they cannot use them fluently and flexibly as a tool to overcome obstacle on understanding the concepts of fractions and decimals. This is an important indication for teachers, curriculum designers and researchers in order to improve students' self-efficacy beliefs and beliefs about the use of representations and mainly their understanding of difficult mathematical concepts which are presented in different representational forms, such as the concepts of fractions and decimals. It would be interesting and useful in future to examine the effects of intervention programs aiming to develop students' cognitive performance concerning fraction and decimal numbers by improving dimensions of affective factors such as beliefs and self-efficacy beliefs and vice versa.

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MAPPING EXPERIENCE OF DIMENSION

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This study explored how individuals experience and think about dimension. Through an analysis of the literature, followed by a phenomenographic study of how individuals experience dimension, we present a characterisation of dimensional thinking, which can inform future research and pedagogic practice.

INTRODUCTION

Dimension is implicit in many geometric ideas though little research has explicitly focussed on dimension. This study reports on a phenomenographic study reporting on how dimension is experienced. The mathematics curriculum in schools treats geometry as independent from reality and as a consequence, independent from students' prior-experiences and knowledge. As Glenn (1979) argues:

> The child comes to school with a good practical working knowledge of this world, but instead of building on it we tend to force all subsequent learning into the two-dimensional abstraction (p.21).

It is globally accepted that both three-dimensional and two-dimensional geometry are of great importance to children. On the one hand, three-dimensional shapes are important because they correspond to the everyday lives of the students, while two-dimensional shapes are presented in books, in child's drawings, and arguably in some everyday phenomena such as shadows.

In the UK, official materials (DfEE, 1999) present quite separately 2D and 3D shapes and their properties and therefore it seems likely that students' may not be encouraged to make the connections between the two types of geometry. In addition, the limited time offered in the curriculum for geometry, the attitudes of teachers' towards this 'frightening' area of mathematics together with the avoidance of including geometry in the students' examinations act as excuses for ignoring the teaching and learning of geometry compared to the rest of the curriculum and thus the teaching and learning of dimension.

WHERE IS DIMENSION EXPERIENCED?

Dimension is experienced in different ways across a variety of settings.

In everyday life: Whereas 3D objects are encountered and manipulated every day, two-dimensional geometry is less explicit, restricted to phenomena such as shadows and idealisations of surfaces.

Formally: More formally, dimension can be considered to be a parameter or measurement required to specify the size of an object. In Geography, dimensions are

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 281-288. Thessaloniki, Greece: PME.

used for locating a place on a 2-dimensional map or even locating a person on earth while in Sciences, dimensions are considered as degrees of freedom. In mathematics, n linearly independent vectors are required to describe any location in n-dimensional space. However no definition of dimension adequately captures the concept in all situations and so mathematicians have formulated numerous definitions of dimension for different types of spaces. All mathematical definitions on dimension, however, are inspired by the notion of the dimension of Euclidean n-space E^n (see Figure 1).

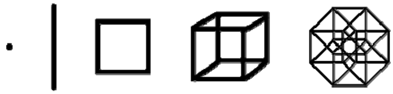

Figure 1: 2-dimensional renderings (i.e. flat drawings)

In school: Geometry and dimension in particular have a dual nature: the theory and the reality. Consequently, many educators have suggested that school geometry should promote the relationship between reality and abstraction (Battista & Clements, 1988; Fujita & Jones, 2002; Jones & Mooney, 2003; Pritchard, 2003; Usiskin, 1982).

In dynamic geometry settings: Some progress has been made in supporting those connections. The invention of dynamic geometry software, for example, has led to the notion of “figure” as a bridge between unrestrained drawing and the mental geometric ideal (Laborde, 1995). More recently, such technological developments have moved into the third dimension (see Figure 2).

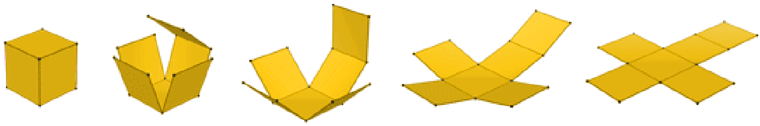

Figure 2: A cube in Cabri 3D

The aim of this study was to map experiences of dimension across these settings through a phenomenographic study. In order to interpret accounts of such experiences across these settings, we paid particular attention to three aspects.

First, we considered the heuristics people use to make sense of dimension, which will point to intuitions (Fischbein, 1987) perhaps expressed in situated terms. The question raised now is what kind of intuitions people have regarding dimension, a focus directly relevant to practicing teachers and educators.

Second, we noted how people’s accounts of experiencing dimension incorporate visualisations. While one of the aims of the UK school curriculum is to develop in students the ability to visualise (Royal Society/JMC 11-19 report, 2001; Usiskin, 1982), Guttierez (1996) points out that little research has been conducted regarding visualisation in 3-dimensional geometry. What is available tends to show the limitation in young students’ visualisation skills, such as making volume judgements based on one or two dimensions only (Piaget, 1968; Piaget, Inhelder, & Szeminska, 1960; Raghubir & Krishna, 1999).

Third, we attended to how individual's primitive knowledge emerges in people's accounts. Ogden (1937), for example, talks about a special type of geometry called 'naive geometry', which includes all those self-evident geometric forms, referred to as prototypes by Üstün and Ubuz (2004) and others. In the case of geometry, the creation of these prototypes is unavoidable and begins from the child's first experiences of the real world.

A PHENOMENOGRAPHIC APPROACH

Phenomenography provides a research methodology to study the "ways in which people experience, perceive, apprehend, understand and conceptualise various phenomena in and aspects of the world around us" (Marton, 1994). Students experience a specific phenomenon from various perspectives, and therefore they create multiple understandings of that phenomenon.

Semi-structured interviews were designed around questions regarding the exploration of the individuals' dimensional thinking. This study did not aim to represent the wider population, an aim beyond the available resources, but instead identifies instances of experience highlighting variation in how dimension is experienced. We focus on (i) two pairs of UK students, referred to as (S) in the protocols, 10 years old and regarded as upper-middle ability by their teachers, and (ii) eight teachers, four from the UK and four from Cyprus, primary (P) and secondary school teachers of mathematics (M) and physics (Ph).

The purpose of the questions used was the exploration of individuals' experiences of dimension, including accounts of how they thought about dimension, inside and outside the school environment. Although some of these questions might have been examined through lessons in class (e.g. What is the difference between 2D and 3D? How many dimensions does a line have?), most of them were questions that the individuals might have never thought about (e.g. Can you think of examples in the real world which are not 3-dimensional? How many dimensions does a reflection in the mirror have? How many dimensions does a shadow have?). Teachers' interviews also included questions exploring their views on the curriculum (e.g. How do you think the notion of dimension should progress in the mathematics curriculum? What experiences do you think should students have that might help them to progress?). Below, we present thematically the findings as they arose from these interviews.

FINDINGS

The findings showed that people experience and think about dimension in a variety of ways as set out below.

Dimension as Action

Dimension was frequently experienced as the outcome of an action:

Paige (S): Maybe you could get like a flat shape and stick it onto a cube or something these would be like two dimensions because one part it is 2D and the other part it is 3D

In the above case, the action provided a means for connecting 2D and 3D. At other times, distinctions between 2D and 3D were articulated. Kristina (P), Michalis (P) and Irene (M) referred to depth and height when talking about the difference between 2D and 3D. Dimensions could also be added. Nicholas (S) and Chelsea (S) argued that, if a point has one dimension, then the square which has 4 points should be a 4-dimensional object. Michalis (P) and Elena (P), when asked what dimension meant to them, referred to locating and measuring:

Michalis (P): It's the placement in space.

Elena (P): It is a measurement of distances on a plane that might exist.

The actions that are required in this perspective on dimension seem then to embrace connecting, distinguishing, adding, locating and measuring, and include both the expression of dimension as an act and as an outcome of an action.

Dimension as State

Seeing dimension as a state was articulated by Kai (S):

Kai (S): Our planet is on a round surface and if we didn't have the third dimension we would have been 2D or 1D or something like that, but if we were 1D we would be unseeable (not visible) and if we were 2D we would be really flat and we couldn't really survive in there.

According to Michalis (P), the place where the object is located was considered a very important element for defining the number of dimensions the object has:

Michalis (P): It depends where the point is at. If it is on a piece of paper it would have 1-2 dimensions, but if it is in space then it would be 3-dimensional.

Similarly, Themis (Ph) pointed out that a point cannot exist by itself and it has to belong somewhere. Expressing the plane and space as an object was common among the teachers. For example, Themis (Ph) proposed that a plane is an example of a 2D object while 3D corresponds to Space.

Material Dimension

Students' naive thinking was typically situated and expressed heuristically. We noted many claims by students, as indicated by the following characterisations: 'the smaller the dimension, the more flexible and bending the object is', 'the smaller dimension a shape has, the easier it is to be transformed', 'the more dimension, the thicker the object', 'the thicker a line, the greater its dimension', 'the smaller is more difficult to see the shape', 'the more vertices or edges an object has, the greater its dimension'. Each of these heuristics related the notion of dimension to everyday materialistic experiences. Teachers also sometimes discussed dimension as if it were a tangible object. Thus, when asked whether something can have 0 dimensions, Michalis (P) responded, "No, because it would have no matter." For Michalis (P), dimension was a property, dependent on the object it represents.

Abstract Dimension

Even the teachers were at times inconsistent, especially when it came to discuss dimension in non-materialistic ways:

Researcher: Can something have 0 dimensions?

Elena (M): I don't know... the point? *[after a while*]

Researcher: How many dimensions does a point have?

Elena (M): I think one.

Similarly, when Michalis (P) was asked if there is something with 1 dimension, he answered negatively pointing out that everything should have at least two dimensions but, when he was later asked for the number the dimensions of a line, he replied that it is zero. On the other hand, Griff (Ph) spoke of 1D and 0D as being idealisations that do not exist in the real world.

Dimension as prototype

Students often made use of prototypes in their responses, for example *'flat shapes'* when describing 2D, and *'standing up shapes'* or *'shapes that pop up'* when talking about 3D. It was also noted that even teachers, such as Kristina (P) answered some questions with automatic prototypes:

Researcher: Can you think of examples in the real world which are not 3D?

Kristina (P): Anything flat that doesn't have depth.

Such statements seems to be ritual echoes of experiences probably based in school. Some prototypes however seemed to be connected with everyday experiences. Kristina (P) often referred to dimension as something in outer space, most likely influenced by science fiction films and the media.

Dimension as hierarchy

There was also discussion about the pedagogy of dimension and a suitable curriculum. Michalis (P) and Themis (Ph) talked of the importance of teaching a notion first by using simple terms and then using the abstract ones while Irene (M) also made the distinction between the use of physical objects in the teaching of dimension in the primary school and the abstract terms to be introduced in the secondary school. Talking about secondary education, Irene (M) suggested that the teaching of 1D should come first and then the higher dimensions to follow:

Irene (M): Well, I would definitely start from the 1st dimension and then explain to them what a dot means, but as I explain the 2nd and 3rd dimension I would definitely move back and constantly compare it with 1 dimension and 2 dimensions if we are talking about 3 dimensions.

On the contrary, Griff (Ph) proposed that aspects connected to the real world should be taught first before moving on to the idealisations of 1D and 2D. As for the start of teaching 3D geometry, the teachers' answers varied from proposing that it should start from early years through play to Griff's (Ph) opinion that the teaching of 3D is not of a great importance because everything is now mapped onto 2D surfaces.

CHARACTERISING DIMENSIONAL THINKING

In this section, we discuss the findings in the phenomenographic study and propose a series of characterisations of how dimension is experienced, leading us towards a definition of dimensional thinking.

Location

Seeing dimension as a state involves locating a point (for example) on a line, a plane or in space. It is possible for dimensional thinking to focus on the object itself (for example the point) or the domain of that object. Dimensional thinking involves the gradual separating of these two aspects of location, clarifying questions such as: '*What is the relationship between the object itself and the domain it is in?'* and '*How does the domain influence the object?*'

Measuring

The phenomenographic study showed that individuals can often perceive dimension as measuring or as a measurement. Indeed, The UK National Numeracy Strategy (DfEE, 1999) presents dimension under the topic of Shape, Space and Measures, where metric units are taught and measurement is a central focus. Actions such as adding, which are appropriate for measurements, are not however appropriate for dimension itself. Dimensional thinking seems to involve the gradual abstraction of a notion that stands outside of measurement, in the sense that it is not subject to the same actions, even though dimension is inevitably rooted in measuring.

Abstracting dimension

Students' responses revealed that their thinking is restricted to a materialistic perspective. Teachers did discuss both material and abstracted aspects of dimension but they often tended not to separate them. Reaching more sophisticated meanings for dimension will inevitably involve passing through naïve states of thinking where the actions associated with measuring are confused with those that might be appropriate for dimension itself. More sophisticated thinking about dimension would require a clear distinguishing of the two and an appreciation of the relationship between them in order to reach the abstract perspective of geometry, which includes all these points and lines that are nothing more than approximations of 'ideal objects' in an 'ideal world'(Atiyah, cited in Pritchard, 2003).

Representing dimension

As discussed above, dimension can be experienced materialistically or abstractly. At a naïve level, this distinction is not made and in fact naive thoughts were often based on vision, touch and thickness, expressed as materially-oriented heuristics (see § *Material Dimension*). Measuring is an action that enables the specification of dimension and leads to the association of perceptual qualities with the notion of dimension. There is after all an element of truth in the heuristic, 'the more dimension, the thicker the object', but ultimately such representations of dimension have to evolve towards a sense of dimension that is not perceptually related.

Visualising dimension

When dimension is visualised, there is typically a dependence on prototypical cases. These prototypes are certainly useful tools or resources for thinking about and communicating geometric ideas. For example, students were able to imagine a square moving into a 3rd dimension and "becoming" a cube. We believe that even sophisticated dimensional thinking will draw upon such prototypes but we would expect it to be less restricted to those prototypes.

Relationships across dimensions

To gain an expert-like sense of dimension requires an appreciation of what is different about 0, 1, 2 and 3 (and more) dimensions and for these distinctions to be consistent within an abstracted concept of dimension. The example the previous section shows how these students were able in some prototypical cases to move their thinking between 2 and 3 dimensions. It is reasonable to speculate that experiences that cut across differing numbers of dimensions might support the construction of such relationships that begin to articulate what is invariant and what changes as you move across dimensions. Such tasks may be intractable in the material world but may be feasible in the mental or virtual worlds.

FUTURE WORK

The findings gathered from the phenomenographic study formed a characterisation of dimensional thinking, based on identifying the ways in which some individuals think about dimension. Other ways of thinking about dimension are not excluded and the aspects of dimensional thinking noted were not all of the same frequency.

A pervasive aspect of the responses is the inconsistency in people's thinking about dimension. The map created from the phenomenographic study provides some sense of the variety of experience but it does not reflect this inconsistency in a satisfactory way. The interviews acted as 'a camera, grabbing snap-shots' of people's experience. Developing that metaphor further, we would like to 'capture video' of how dimensional thinking changes during activity. The phenomenographic study has proposed in its characterisations a sense of what more sophisticated dimensional thinking might look like. Further study might focus on cross-dimensional tasks set in virtual worlds to report on the micro-evolution of dimensional thinking through carefully designed tasks. Such a study would seek first to identify naïve ideas already held but then to perturb such thinking in order to explore limitations and potentials.

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6[TH] GRADE STUDENTS' STRATEGIES IN RATIO PROBLEMS

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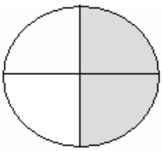
This study is one part of a research project which focuses on students' performance in a number of tasks concerning the concept of ratio. Sixteen 6th grade students were clinically interviewed in order to categorize their responses to real life tasks which include all the aspects of the concept of ratio. From the analysis of the data a number of issues about their strategies emerged. It seems that students' reasoning does not follow a predetermined path, and varies across the tasks. Moreover, the impact of the qualitative elements of the tasks seems to affect their reasoning. A qualititative – proportional approach is adopted in the cases where the proportionality is not adequately formed.

The concepts of ratio and proportion have been widely studied. A number of research studies have focused on the performances and strategies of sixth – eighth graders solving proportion problems (Karplus, Pulos and Stage, 1983; Tourniaire, 1986; Adjiage and Pluvinage, 2007). Lachance and Confrey (2002) in their study aimed at building a path of understanding from ratio and proportion to decimal notation, focus on the multiplicative notion of ratio. Under this perspective, ratio is considered to be the multiplicative comparison of two quantities and it is connected to the operations of multiplication and division. In the proposed alternative model for developing mathematical understanding, children's experiences with the multiplicative notion of ratio are the basis for their understanding of fractions, decimals and percents. Furthermore, fractions, decimal and percents are considered to be subsets of ratios. A ratio is an ordered pair of quantities that includes a comparison between their magnitudes. The ratios could be distinguished based on the kind of quantities that are included in the comparison. More particularly, ratios that involve the comparison of a part of the quantity to the whole quantity are called "Part-to-Whole Ratios", while the comparison of one part of a whole to another part of the same whole refers to a "Part-to-Part" ratio. Figure 1 could be used to illustrate an example of each of the two different kinds of ratio.



- The two grey parts are compared to the whole 2:4. (Part-to-Whole ratio)
- The two grey parts are compared to the other two white parts of the same whole 2:2. (Part-to-Part ratio)

Figure 1

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 289-296. Thessaloniki, Greece: PME.

Fractions and percentages can be considered as Part-to-Whole ratios and consequently are subsets of the concept of ratios. Ratios that involve comparisons of unalike quantities with different measures are called Rates. Such rates that are considered to be fundamental principles of physics and chemistry are speed (u=d/t), which is defined as the ratio of the distance that an object covers in a period of time to this period, density (d=m/v) as well as the price of a product (€/product).

STUDENTS' STRATEGIES

According to the NCTM Curriculum and Evaluation Standards (1989), 'the ability to reason proportionally develops in students throughout grades 5–8. It is of such great importance that it merits however much time and effort that must be expended to assure its careful development' (p. 82). (Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998). It must also be pointed out that previous research has shown that young students (6-8 year-olds) seem to deal with similarity as an operative equivalence, which is evident in their understanding of, and reasoning about, ratios (Van denBrink and Streefland, 1979). It must be outlined that Steinthorsdottir (2006) reviewing the literature on proportional reasoning indicates that the development of student's strategies follows a four level path. That path was first evinced by Piaget and Inhelder (1958). Initially, students' responses on ratio problems are incomplete. That means that students are not able to relate the data of the task and support what they believe. Afterwards students use qualitative approaches to ratio and proportion tasks in terms of focusing on the qualitive relations of the ratios. They then proceed to additive ones which are the first attempt at quantifying the relationships but ratio and proportion are proved to be based on multiplicative reasoning. Finally, students arrive at proportional reasoning and multiplicative strategies. After additive compensations and before proportional reasoning there is what Piaget calls preportionality (Piaget et al., 1968). Strategies on proportion problems can be categorized according to a strategy scale developed by Karplus et al., (1983). This scale has four levels: *incomplete, qualitative, additive, and proportional.*

Using "the lemonade problem" we will try to exemplify the four levels of students' strategies in proportion problems. The task is as follows: "If you want to make some tasty lemonade you must mix one lemon and two spoonfuls of sugar in a glass of water. How many ingredients are needed in order to make a jug of lemonade? The jug is equal to five glasses of water". Below we illustrate representative students' responses in each category:

1. ***Incomplete Strategy***
   - 5 lemons and 5 spoonfuls of sugar, because there are 5 glasses of lemonade.
   - I can not know how many lemons or spoonfuls of sugar are needed.
2. ***Qualitative Strategy***
   - 8 lemons and 8 spoonfuls of sugar, because there is much more lemonade, so there should be much more lemons as well as sugar.

3. ***Additive Strategy***
   - \- 6 lemons and 7 spoonfuls of sugar, because there should be 5 more lemons and spoonfuls of sugar than water.
4. ***Proportional Strategies***
   - \- 5 lemons and 10 spoonfuls of sugar, because there should be twice as much sugar as lemons.
   - \- 5 lemons and 10 spoonfuls of sugar, because there should be five times as many lemons as sugar.

In this study we investigate students' strategies to a number of ratio – proportion problems and try to classify their responses using the above categorization.

RATIOS TASKS

The four problems, below, were categorized into the three groups according to what the ratio represented.

1. Frogs are champion jumpers. A 7,62 cm frog can hop 152,40 cm. That means the frog is jumping 20 times its body's length. Let's measure your height. If you, like a frog, could hop 20 times your body length (your height), how far could you go?
2. Your father had bought you a chocolate. While he was coming home he met a child who lived in the same neighborhood, cut the chocolate and gave him two of the five pieces. The piece left is drawn below (Figure 2). How much chocolate is left for you? Could you draw the whole chocolate?

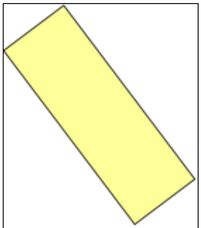


Figure 2

3. The children of a classroom are separated into two groups A and B. The first group has 15 children and group B has five children. Then we give 8 rolls and ask them to share them in order that no group complains.
4. If you want to make some tasty lemonade you must mix one lemon and two spoonfuls of sugar in a glass of water. How many ingredients are needed in order to make a jug of lemonade? The jug we have is equal to five glasses of water.

In the "chocolate" problem the ratio used in the task represents a part of the chocolate to the whole chocolate. In the "rolls" problem the ratio used expresses a comparison between parts of different quantities. In the "lemonade" problem the ratio compares dissimilar quantities (rate). Finally, the problem "if you hopped like a frog" introduces proportional reasoning to the children through a correlation of fun and science, imagination and reality.

METHODOLOGY

The research methodology is *clinical interview.* The clinical research methods are based on the principles of constructivism and are aimed at the investigation of children's conceptions. The researcher acts as a teacher interacting with the children aiming to investigate their thinking. The researcher, by reflecting on these interactions, tries to interpret children's actions and finally forms models-assumptions concerning their conceptions. These assumptions are evaluated and consequently either verified or revised. (Bell, 1993; Hunting, 1997).

Participants: Sixteen children of the 6^{th} grade participated in this experiment. The sixth grade class was chosen because in this particular grade ratio and proportion first appear in the Greek school curriculum. This class was a mixed ability class, with a disproportionate number of females and males (5 females, 11 males). Students were divided into 7 groups (five pairs and two groups of three students). Each group of students was interviewed for about an hour on the tasks concerning ratios. The students were asked to solve the four ratio - proportional problems. The researchers provided them with equipment such as ruler, scissors, paper - pencil for optional use. The interviews were audio taped.

Analysis of the data: The data consists of the 7 transcribed audio recordings and researcher's field notes. Initially, we attempt to analyse the transcribed teaching experiments by coding the strategies that were developed by each child. Then, by scrutinizing the data line by line we identified children's conceptions and we formed categories that describe children's thinking about ratios.

ANALYSIS

We organized the results separately according to what strategy each child used. In table 1 the categorization of the strategies that developed each student in each task are presented. It must be clarified that "I" means that children used incomplete strategy in order to solve the task, "Q" qualitative strategy, "A" additive and "P" means proportional. However there were students whose responses could not be categorized at any level of the scale. These students seem to approach the task proportionally but they could not support their thoughts. Their strategies fall between the proportional and the qualitative level. For these strategies we introduce a new category, the P-Q one.

First of all it is of great importance that students do not confront each task in the same way, as is easily observed in Table 1. The strategy that each student uses in order to solve the first task may not be the same as for the second, the third or the fourth. For instance, student S_1 seems to think proportionally in the first and the fourth task but his strategy on the second and the third task is characterized as incomplete. There is no one student who followed the same route in dealing with the four tasks. Obviously there may be some factors that have a great influence on children's choice of solution strategy which vary from task to task while many of them seem to be able to reasoning proportionally.

Tasks/ Students	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
If you hopped like frog	P	P	P	P	Q	P	P	P	P	P	P	Q	P	P	I	I
The chocolate	I	I	A	A	A	A	A	Q	Q	Q	Q	Q	P-Q	Q	Q	Q
The rolls	I	Q	P-Q	A	P-Q	P-Q	P-Q	P-Q	Q	Q	A	Q	P-Q	A	Q	A
The Lemonade	P	Q	Q	P	P	P	P	P	P	P	P	I	P	P	I	I

Table 1: Students' strategies in ratios problems

Incomplete strategies

There were eight responses that are characterized as incomplete. These students did not manage to support their thought appropriately and they were not able to reach a certain level of reasoning. For example, student S_1 in his trial to find a solution to the rolls task could not understand how to share the rolls and his response was:

S_1: We do not know if the rolls are from Thessalonica where rolls are too big.

S_3: Of course. …

S_1: We should take more rolls in order for all the children to eat.

Qualitative strategies

Qualitative approach is one of the interesting strategies that students use because it allows children to reason about relations between combinations, increases and decreases and comparisons but without quantification. (Singer, Kohn & Resnick, 1997) Such a strategy was used by student S_{12} on the "if you hopped like a frog" task. His strategy was based on empirical, intuitive data. Interacting with his classmate S_{13} they assumed that jumping like a frog results in a 33 meters jump through the multiplication of their height by 20 times.

S_{12}: I think that 33 meters that we found from the multiplication is too high.

S_{12}: I think that 15 meters is more logical.

Researcher: Do not forget that you could jump like a frog.

S_{12}: 33 meters is too much.

Researcher: So what do you think?

S_{12}: 15 meters.

Additive strategies

Nine of the sixty four responses belong to the additive category. Ratios and proportions require appreciation of multiplicative relations between numbers. On the contrary, some students applied their well-practiced knowledge of the additive properties of whole numbers to ratio tasks. An instant example of this situation is the dialogue between S_{10} and S_{11}.

S_{10}: We should give five rolls to the fifteen students and three rolls to the five students because the first group is larger than the second and the rolls are not enough for all.

S_{11}: In this way the five children would take too many rolls.

Researcher: Why do they seem too many to you?

S_{11}: If we give the 5 children three rolls there will be two less rolls than children. If we give the ten children five rolls there will be five less rolls than children. This is not fair, they will complain.

Proportional strategies

The majority of the students used proportional strategies in order to confront the tasks as is clarified by Table 1. Twenty three (23) of their responses were based on proportional thought. Specifically for the first task "if you hopped like a frog" student S_4 was sure about his view and supported it whereas his classmate insisted on disagreeing with him.

S_4 : Frogs can jump 20 times the length of their body. So if we were frogs we could jump 20 times our height. We must multiply our height by 20.

Another representative type of proportional strategy used by students was S_{10} answer. The researcher interrupted the students' conversation in order to understand why they multiplied.

Researcher: Why did you multiply?

S_{10}: Because the problem mentions that a frog can jump 20 times the length of its body. If we jump like a frog we can jump 20 times our height.

"The Lemonade" task was faced proportionally by most of the students. Student S_{14} response comprises a definite sample of proportional thought.

S_{14}: If one glass of water needs one lemon, five glasses will need five lemons and ten spoonfuls of sugar.

Researcher: How did you find that?

S_{14}: I multiply 5 times one lemon and 5 times the two spoonfuls.

Proportional – qualitative strategies

However there were seven (7) responses which could not be coded in any of the levels of the scale. These answers were between qualitative and proportional strategy.

As is already mentioned after additive compensations and before proportional reasoning there is what Piaget calls preportionality (Piaget et al., 1968). We cannot range the answers to preportionality but, to an intermediary stage. Students who are thinking in such a way can understand the multiplicative relationships between the variables as well as the qualitative relations between the amounts. It must be emphasized that they can make numerical judgements about combinations, changes and comparisons but they cannot reach the exact proportional reasoning. Consequently, they seem not to be sure about their strategy and they doubt about outcomes. For the "rolls" task S_6 and S_7 were two such students who were trying to make multiplicative judgements but the outcome and their intuitions did not allow them to reason proportionally.

S7: We should divide 20 children by 8 rolls

Researcher: Why do you believe you should perform this division?

S7: In order to find how many rolls each child will take.

They performed the division and found 2,5.

S_8: They would take 2 rolls and a half of a roll.

S_7: No, no we do not have enough rolls. We should divide 15 by 8 and 5 by 8.

Researcher: Why?

S_8: To find how many rolls the fifteen children will take and how many the five but if we add them we will find 2,5?

S_7: We'll see.

As we can see S_7 divided in order to find out how many rolls each child would take. However he observes that if each child takes two rolls and a half, the rolls will not be enough for all. Consequently, he suggests dividing fifteen by eight and five by 8 in order to find how many rolls the fifteen and the five children would take correspondingly. It seems that S_7 wants to follow a proportional strategy but, thinking qualitatively, doubts the outcome and his intuition does not allow him to develop such a strategy.

CONCLUSIONS

This article focused on the strategies used by the students in order to deal with problems with ratio and proportion. Through our research and decoding of the children's interviews the internal thinking concerning their reasoning was revealed. However, the already developed scale that literature has imposed seemed to be inefficient. Some children's thoughts could not be categorized according to the levels of this scale. Piaget had already mentioned a stage between additive and proportional reasoning which he called preportionality, showing in this way that children's reasoning development does not follow a predetermined path. More specifically, through our experiments, the specially formed problems gave the children the opportunity to use strategies similar to those they would use in real life. As a consequence, the children expressed their opinions using their intuition and experience. This study suggests that some of the student's strategies which can not be categorized into the fourth level scale comprise another separate stage, the P-Q. In addition it seems that children, although able to reason proportionally follow different strategies in order to solve each task, probably affected by some factors such as the elements of the task. On-going research in this domain should reveal the factors that influence student's strategies in ratio tasks as well as the stages that a student follows in order to confront such problems.

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IDENTIFYING ENDOGENOUS AND EXOGENOUS FACTORS THAT INFLUENCE STUDENTS' MATHEMATICAL PERFORMANCE

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This paper presents some results of a larger study that investigates endogenous (students' motivational constructs) and exogenous factors (teachers' practices) that influence students' mathematical performance. The participants of the study were 321 sixth grade students and their 15 teachers. Students' data were collected through a questionnaire comprised of six Likert-type scales measuring motivational constructs and a test measuring students' performance in fractions, while teachers' practices were collected via an observational protocol. Findings revealed the importance of multi-level modelling in the analysis of the endogenous and exogenous factors that promote students' performance in mathematics.

THEORETICAL BACKGROUND AND AIMS

Motivation is treated in Mathematics education as a desirable outcome and a means to enhance understanding (Stipek et al., 1998). Motivation researchers propose a model of achievement goal theory in which students' achievement goals are embedded in multiple sociocultural contexts and are a product of prior and current experiences in those contexts (Friedel et al., 2007). Moreover they provide substantial evidences of instructional practices that foster students' motivation (Anderman et al., 2002). These instructional practices are alike the ones developed by mathematics educators to achieve both learning and motivational outcomes (Stipek et al., 1998).

Building on the results of previous research (Pantziara & Philippou, 2007) the present study investigates endogenous (students' motivation) and exogenous factors (deviations in instructional practices) that may influence students' mathematics performance.

Students' learning and motivation

In the realm of Mathematics education, the socio-constructivist perspective on learning is characterized both by its focus on the situatedness of learning and by the recognition of the close interactions between (meta)cognitive, motivational and affective factors in students' learning. Clearly is perceived that students' motivation as well as other motivational constructs are an integral part of students' learning (Op't Eynde et al., 2006).

Hannula (2006) states that motivation cannot directly be observed but it can be noticeable only by its interaction with affect, cognition and behaviour. We define motivation as the inclination to do certain things and avoid doing some others

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(Hannula, 2006). Achievement motivation literature focuses on the prediction and explanation of competence-relevant behaviour. One of the various approaches to achievement motivation is Achievement goal theory. This theory was developed within a social-cognitive framework and it has studied in depth many variables which are considered as antecedents of students' motivation constructs. Some of these variables are students' competence based variables, such as need of achievement or fear of failure, self-based variables, such as self efficacy beliefs, demographic variables, e.g. gender and also the environment (Elliot et al., 2005).

Achievement goals are conceptualized as the competence-relevant aims that individuals strive for in achievement settings, and these different aims are posited to lead to differential performance outcomes. Two distinct goals have been emphasized in the literature, namely mastery goals that focus on learning and understanding, and performance goals that focus on the demonstration of competence. Recently, there has been a theoretical and empirical differentiation between performance-approach goals, where students focus on how to outperform others, and performance-avoidance goals, where students aim to avoid looking inferior or incompetent in relation to others (Elliot et al., 2005). These goals have been related consistently to different patterns of achievement-related affect, cognition and behaviour (Anderman et al., 2002; Elliot & Church, 1997).

Mastery goals are characterized as a challenge-based form of regulation that evokes a host of positive processes (effort expenditure, persistence, task absorption) and facilitate many positive outcomes like achievement and interest (Elliot & Church, 1997). Performance-approach goals are viewed as evoking many of the same positive processes evoked by mastery goals (e.g., effort expenditure, persistence), as both goals represent approach forms of regulation developed by challenge appraisals. However, the focus on the demonstration of competence in these goals is not correlated with some processes and outcomes (e.g., deep processing, intrinsic interest), but it is presumed to enable these goals to facilitate performance in a broader range of situations (Elliot et al, 2005). Performance-avoidance goals entail regulating according to a negative normative possibility that is posited to evoke a host of negative processes (distraction, anxiety, and avoidance of help-seeking) that undermine performance in most achievement settings (Elliot & Church, 1997; Elliot et al., 2005).

Instructional practices and students' learning

Goal theorists posit that children are sensitive to the emphasis teachers place on different types of achievement goals, as expressed through instructional practices and the ways in which teachers respond to children's accomplishments or shortcomings (Friedel et al., 2007). These instructional practices are presumed to influence students' achievement through the adoption of certain achievement goals by students.

Thus, earlier studies on achievement goals specify various classroom instructional practices as contributing to the development of different types of goals and consequently, eliciting different patterns of motivation and achievement outcomes (e.g. Anderman et al., 2002). Goal orientation theorists lying on a large literature on classroom motivational environments focus on six categories that contribute to the classroom motivational environment. The categories, represented by the acronym TARGET refer to task, authority, recognition, grouping, evaluation and time. Task refers to specific activities, such as problem solving or routine algorithm, open or closed questions in which students are engaged in; Authority refers to students' level of autonomy in the classroom; Recognition refers to whether the teacher values the progress or the final outcome of students' performance and how the teacher treats students' mistakes (as a part of the learning process or as cause for punishment); Grouping refers to whether students work with different or similar ability peers; Evaluation refers to how the teacher treats assessment, giving publicly grades and test scores, or focusing on feedback as a means for improvement and mastery; Time refers to whether the schedule of the activities is rigid or flexible.

This framework has been adapted and developed by goal theory researchers working within classroom context (Anderman et al., 2002). Using classroom observations and qualitative analysis, they found that instructional practises in classrooms in where students adopted mastery goals differed from instructional practises in classroom characterized by students' low mastery goals or high performance goals listing specific instructional practices that developed either mastery or performance goals.

In mathematics education domain, Stipek et al. (1998) in a relevant study referring to instructional practices and their effect on students' learning and motivation found that affective climate was a powerful predictor of students' motivation and mastery orientation. Students in classrooms in which teachers emphasized effort, pressed students for understanding, treating students' misconception and in which autonomy was encouraged, reported more positive emotions while doing fractions work and enjoying mathematics relatively more than other students.

While much progress has been made toward understanding how the elements of the classroom context relate to children's goal orientations and although there are numerous studies investigating the relationships between achievement goals and specific motivational constructs or achievement (Elliot et al., 2005), relatively few studies have investigated students' achievement goals in the realm of mathematics education. Most importantly, to the best of our knowledge none of these studies used multilevel analysis to identify endogenous (students' motivational constructs) and exogenous factors (instructional practices) that may influence students' performance in mathematics. In this respect the aims of this study were:

1. To test the validity of the measures for the six motivational factors: fear of failure, self-efficacy, interest, mastery goals, performance-approach goals and performance-avoidance goals, in a specific social context.

2. To construct and test the validity of an observational protocol that includes convergent variables referring to instructional practices in the classroom from the mathematics education domain and the achievement motivation one.
3. To identify students' motivational constructs and instructional practices suggested by achievement motivation theory and mathematics education theory that affect students' performance in the mathematics classroom.

METHOD

Participants were 321 sixth grade students, 136 males and 185 females from 15 intact classes and their 15 mathematics teachers. All students-participants completed a questionnaire concerning their motivation in mathematics and a test for achievement in the mid of the second semester of the school year.

The motivation questionnaire comprised of six sub-scales measuring: a) mastery goals (five items), b) performance goals (five items), c) performance avoidance goals (four items), d) self-efficacy (five items), e) fear of failure (nine items), and f) interest (seven items). The whole questionnaire comprised 35 Likert-type, 5-point items (1-strongly disagree, and 5 strongly agree). The first four subscales were adopted from the Patterns of Adaptive Learning Scales (PALS) (Midgley et al., 2000); respective sample items in each of these four subscales were, "one of my goals in mathematics is to learn as much as I can" (Mastery goal), "One of my goals is to look smart in comparison to the other students in my class." (Performance goal), "One of my goals is to keep others from thinking I'm not smart in class." (Performance-avoidance goal), and "I'm certain I can master the skills taught in mathematics this year" (efficacy beliefs). Students' fear of failure was assessed using nine items adopted from the Herman's fear of failure scale (Elliot & Church, 1997); a specimen item was "I often avoid a task because I am afraid that I will make mistakes". Finally, we used Elliot and Church (1997) seven-item scale to measure students' interest in achievement tasks; a specimen item was, "I found mathematics interesting". These 35 items were randomly spread throughout the questionnaire, to avoid the formation of possible reaction patterns.

Regarding students' performance in mathematics we developed a test measuring students' understanding of fractions. The tasks comprising the test were adopted from published research and specifically concerned students' understanding of fraction as part of a whole, as measurement, equivalent fractions, fraction comparison and addition of fractions with common and non common denominators (e.g. Stipek et al, 1998).

To examine teachers' instructional practices we used a specially developed protocol to code teachers' mathematics instruction in the 15 classes during two 40-minutes periods each. The protocol was based on the convergence between instructional practices described by Achievement goal theory and the Mathematics education reform literature. Specifically, we developed a list of codes around six structures, based on previous literature (Anderman et al., 2002; Stipek et al., 1998), which were found to influence students' motivation and achievement. These structures were: *task, visual aids, practices towards the task, affective sensitivity, messages to students, and recognition*. The structure *task* included algorithms, problem solving, teaching self-regulation strategies, open-ended questions, closed questions, constructing the new concept on an acquired one, generalizing and conjecturing. We checked whether teachers made use of *visual aids* during their lesson. *Practices towards the task* included the teacher giving direct instructions to students, asking for justification, asking for multiple ways for the solution of problems, pressing for understanding by asking questions, dealing with students' misconceptions, or seeking only for the correct response, helping students and rewording the question posed. *Affective sensitivity* included behaviour such as teachers' anger, using sarcasm, being sensible to students, having high expectations for the students, teachers' interest towards mathematics or fear for mathematics. *Messages* to students included learning through active engagement, reference to the interest and value of mathematics tasks, students' mistakes being part of the learning process or being forbidden, and learning being receiving information and following directions. Finally, *recognition* referred to praise for students' achievement, effort, behaviour and the use of external rewards by the teachers. During the two classroom observations lasted for 40 minutes for each teacher, we searched for incidences of occurrence of each code in each structure.

FINDINGS

Regarding the first aim of the study, we conducted confirmatory factor analysis using EQS (Hu & Bentler, 1999) in order to examine whether the factor structure yields the six motivational constructs expected by the theory. For a detailed description of the process followed for the identification of the six factors see Pantziara and Philippou (2007). From the analysis the factor performance-avoidance goals failed to be confirmed. In line with motivation theory, a five-factor model was tested (see fig. 1). Items from each scale are hypothesized to load only on their respective latent variables. The fit of this model was found to be quite satisfactory (the indices found x^2 =691.104, df= 208, p<0.000; CFI=0.770 and RMSEA=0.086). After the addition of correlations among the five factors the measuring model has been further improved (x^2 =343.487, df= 198, p<0.000; CFI=0.931 and RMSEA=0.049).

Concerning the second aim of the study, the analysis of the observations involved estimating the mean score of each code for the two 40-minutes observations using the SPSS and creating a matrix display of all the frequencies of the coded data from each classroom. Each cell of data corresponded to a coding structure.

From a first glance, the observational protocol succeeded in detecting differences in teachers' practises during the mathematics lessons (Pantziara and Philippou, 2007). For instance, regarding the structure *task*, teachers 4, 9, 13, and 15 used more algorithmic tasks than the others, while teachers 2, 4, and 7 used more problem solving activities than their other colleagues. From the category *practices towards the* *task* justification of students' answers were asked from almost all teachers except from teachers 2, 3, 10, and 13. Regarding teachers' *affective sensitivity*, teacher 1 expressed anger while teacher 7 showed great sensitivity to students. Concerning the structure *messages* all teachers apart from teachers 1 and 15 treated students' erroneous responses as part of the learning process, while regarding the structure *recognition*, teachers 1 and 7 rewarded students for their performance.

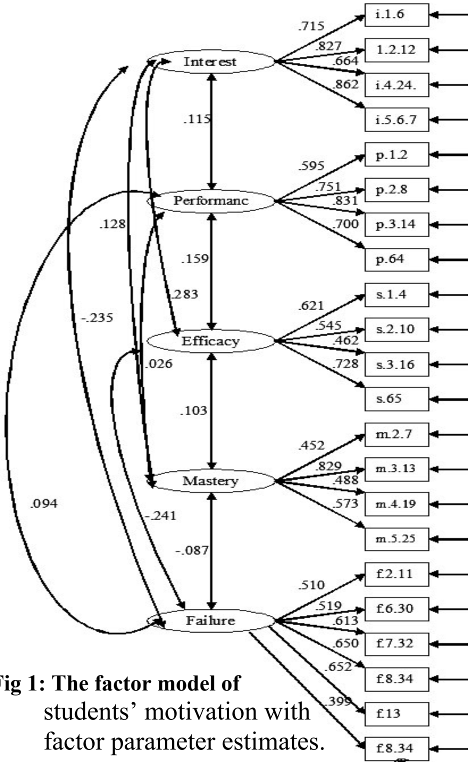

Fig 1: The factor model of students' motivation with factor parameter estimates.

With regard to the third aim of the study, the identification of students' motivational constructs and instructional practices suggested by achievement motivation theory and mathematics education that affect students' performance in mathematics we applied Multilevel analysis using the program MLwin (Opdenakker & Van Damme, 2006). Multilevel analysis is a methodology for the analysis of data with complex patterns of variability, with a focus on nested sources of variability: e.g. students in classes, classes in schools, etc. The main statistical model of multilevel analysis is the hierarchical linear model, an extension of the multiple linear regression model to a model that includes nested random effects. Multilevel statistical models are always needed if a multi-stage sampling design has been employed (a sample of students withing classes in our case) because the clustering of the data should be taken into consideration avoiding the drawing of wrong conclusions (Opdenakker & Van Damme, 2006). The simplest case of this model is the random effects analysis model (null model). This model allows the estimation of variance at each distinguished level (e.g. students and teachers). The null model can be expanded by the inclusion of explanatory variables at all levels. In our case a two level model was employed with students' performance as the depended variable and students' motivational constructs and teachers' practices as the exploratory variables.

The null model's analysis showed that the amount of variation at the students' level was 92% and at the teachers' level was 8%. Following the procedure described by MLwin analysis (Opdenakker & Van Damme, 2006) we then added in the model students' demographic variables (father's and mother's educational background). Mother's educational background had statistically significant effect on students' performance. This variable explained 10% of the total variance, 8% at students' level and 2% at teachers' level. Next we added to the model students' motivational constructs (mastery and performance goals, self-efficacy, and fear of failure). Mastery goals had statistically significant positive effect on students' performance while fear of failure had statistically significant negative effect. Self-efficacy beliefs and performance goals did not have statistically significant effect on students' performance. This model explained the 21% of the total variance, 19% at the students' level and 2% at the teachers' level.

Next we added to the model teachers' practices concerning the structure *task*. The teachers use of open-ended questions had statistically significant effect on students' performance. This model explained 23% of the total variance solely at teachers' level. Then we added teacher's practices concerning the category *practices towards the task* but none of these practices had significant effect on students' performance. Unexpectedly, none of the practices of the other categories had significant effect on students' performance.

Next, we followed Stipek et al. (1998) process grouping positive and negative instructional practices in each of the six categories regarding the observational protocol. The subcategory positive teachers' practices from the structure task (problem solving, self-regulation strategies, open-ended questions, constructing the new concept on an acquired one, generalizing and conjecturing), had statistically significant effect on students' performance. Figure 2 presents the results of the multilevel analysis in identifying exploratory variables that affect students' performance in mathematics.

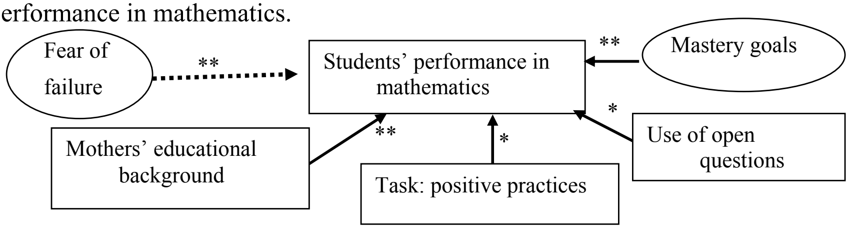

Fig 2: Results of the Multilevel analysis on students' performance.

DISCUSSION

Building on earlier results (Pantziara & Philippou, 2007), the current study examined the effect of endogenous and exogenous factors on students' mathematics performance using the multilevel analysis. In line with the results of other studies (e.g. Elliot & Church, 1997) fear of failure was found to have statistically significant negative effect on students' performance. In addition mastery goals were found to have statistically significant effect on students' performance contrary to the results of other studies (Elliot et al., 2005) which did not traced any effect.

Regarding the environmental factors, in support to the socio-constructivist perspective (Op't Eynde et al., 2006), mother's education influences students' performance. In line with Stipek's et al. (1998) findings, the use of open-ended questions and the grouped instructional practices labelled as positive under the category *task* had significant effect on students' performance. Worth mentioning, was the fact that most effect on student's performance found to have students' variables. This may be due to the new analytical tools used (multilevel analysis) or to the small number of teachers involved in the study. Thus, further research is needed using multilevel analysis in domains regarding achievement goals and mathematics education for the identification of endogenous and exogenous factors that promote students' motivation and achievement in mathematics.

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ESTIMATING AREAS AND VERIFYING CALCULATIONS IN THE TRADITIONAL AND COMPUTATIONAL ENVIRONMENT

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We present summarized strategies used by 5th and 6th graders estimating the area of irregular shapes and verifying their results, using the paper-and-pencil environment or the computational one. We also present our conclusions from the analysis of the collected data.

INTRODUCTION

Mathematics in general and especially geometry teaching and learning seem to be supported by the introduction of the so-called Information and Communication Technology (ICT). On the one hand, there exists a lot of software that could be used for math teaching at all educational levels. On the other hand, there is equally a great number of research studies focusing on the usage of ICT in Mathematics Education. However, it is not satisfactorily known the way ICT changes the landscape in Mathematics Education: change in time-management in the classroom, change in didactic contract, in students' problem solving strategies and possibly in students' conceptions and mistakes. In this paper we present an inclusive view of a project concerning primary school students which lasted three years. The aim of the project was to explore and enhance students' comprehension of the concept of area with an emphasis on problem solving techniques for the estimation of the area of irregular plane figures. We present summarized and systematically strategies used by the students either for estimating the area of irregular shapes or for verifying their results using the "classical technology" of paper-and-pencil or the computational one (i.e., Dynamic Geometry Software or other environments appropriate for Geometry). We consider that the verification process is equally important to the process of the estimation of the area since verification is an important part of the problem solving process itself but unfortunately it is not extensively explored. In the next section we briefly present research findings concerning area teaching and usage of technology in math education. After that, we describe our study and then follows the section of the results. We end with conclusions and some indications for future research.

BRIEF LITERATURE REVIEW

The concept of area and its parameters constitute a strong motivation for research in the domain of mathematics education. Problem solving involving the area of irregular shapes enable students to develop techniques for the estimation of the area using specific problem solving strategies (Rickard, 1996). The pupils use the cut-and-paste technique in order to compare areas of irregular shapes, which indicates an

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understanding of the preservation of area through transformation (Baturo & Nasons, 1996). They also use the square grid as a measurement device. Additionally since, in these shapes, the coverage with whole measurement units is not attainable, the students have to develop various methods of approximate calculations based on the usage of sub-units or partial units (Clements & Stephan, 2004; Reynolds & Wheatley, 1996). This problem-solving landscape has recently been influenced to a great extent by the introduction of new technologies. For example, they affect the way students explore properties of mathematical entities (functions, shapes) in addition to the way calculations are made. Students can use the computer as a device for exploring various solution paths and decision making; they can try out ideas and strategies and simultaneously receive feedback on those ideas and strategies (Clements 2000; Balachef & Kaput, 1997). However it seems that the technological tools not only develop students' problem solving skills in general, but they also facilitate the verification process (Papadopoulos & Dagdilelis, in press).

DESCRIPTION OF THE STUDY

The study was accomplished by 52 students of the 5th grade (18 in the computer environment and 34 in the traditional) and 46 students of the 6th grade (18 and 28 students respectively) of a primary school in an urban area of Greece. It is part of a larger research project that aimed to explore students' comprehension of the concept of area, emphasizing on problem solving techniques for the estimation of the area of irregular shapes. Through their regular classes in mathematics the students had already been taught the concept of area and the formulas for the calculation of the area of known shapes (i.e., triangle, square, rectangle, etc). The project took part in parallel with the normal teaching. Emphasis was given to the development and usage of various tools enabling them to calculate areas of irregular shapes like the usage of grids in a geoboard, the subdivision of an area unit into subunits, the cut and paste method. The tasks were non-standard in the sense that they could not be solved by merely relying on known formulas or procedures (see some examples in Appendix A). Problems like these are not included in the official textbooks. In the computer environment three different applications were used in order to undertake the tasks: MSPaint (the well known program of painting), GeoComputer (an electronic geoboard) and Cabri Geometer (Dynamic Geometry Software). The students worked individually on the same problems trying to solve a problem per session without interventions from the researchers. In the paper-pencil environment our data were constituted of the students' worksheets. In the computer environment the Camtasia Studio software was capturing (in a movie format) anything that was happening on the computer screen so as the researcher could have access to the intermediate phases of every student's problem solving process. The next day we called the students for audio-taped interview in order to use more direct questioning concerning the motivation of their working. The analysis of our data allowed us to study and highlight an additional dimension of problem solving. The geometry software (and in general the computer environment) besides the development of certain strategies for estimating the area of these shapes influenced the development of verification processes applied by the students. It is important to remind that verification plays an important role in primary school level since the issue of proof is completely absent in this level.

PRESENTATION AND ORDERING OF THE STRATEGIES

Observing the students' efforts to solve the problems we recorded a series of different problem-solving strategies as they are presented below:

Absence of any strategy. In this category the students did not apply any strategy. They worked using trial and error processes.

Strategy based on misconceptions. In this case the students were based on misconceptions connected to certain plane shapes. These misconceptions determined the students' reaction to a certain problem.

Strategy based on prototypes. This strategy has its origin to the prototypical examples of basic geometrical concepts which become the basis for prototypical judgment since the students use these examples as a model in their reasoning. Thus, these prototypes determine the first step of the reaction to a given task.

Visual and e-visual strategies. The students are based only on what they 'see'. The image, in front of them, is so strong that it can justify their choices. This strategy becomes more powerful in the computer environment since the students now can manipulate the image enhancing thus the legitimacy of this strategy. (Arcavi, 2003).

Reaction based strategy. This strategy is exclusively connected with the computer environment. The students had the possibility to make direct comparisons and to find relations between what they thought or what they expected as a result and what they saw on their computer screen. Thus visualizing the results of their activities on the screen, they could immediately react when there was a contradiction between them.

Incorporation of shapes in a recognizable frame through personal intervention. The problems included irregular shapes which were constituted by both segment lines and curved lines. Due to these curved lines the students could not identify the shapes as known ones and consequently they could not apply any known formula for the calculation of their area. So, some of them substituted the curved lines with segment lines creating thus shapes that were familiar and could be handled by them.

The strategy of creating a grid. Some students trying to solve a problem invented and created a grid. There was not any prompt in the problem's context (either explicit or implicit) towards the creation of the grid. On the contrary, the problem was oriented towards splitting the irregular shape to sub-shapes known to the students. However, some students instead of splitting the shape they preferred to draw this 'personal' grid in order to find the area of the shape.

"Cut-Paste" and "Decomposition to basic units". In the 'cut and paste' strategy the students divide a complex shape into pieces and rearrange these pieces so as to create a new shape of different form but of the same area in order to facilitate the calculation of the area of the shape. The only criterion for the transference of the pieces was 'which' one fits perfectly 'where' (Mamona-Downs & Papadopoulos, 2006). In the 'decomposition to basic units' strategy the students initially search for a basic unit that is iterated in the shape. After that they calculate the area in two steps. Firstly they count the whole basic measurement units in the interior of the shape. However since the shape is irregular it is not covered completely with whole units. So, in the second step, the students are dealing with the remaining partial units.

Definition or Properties based strategy. In this strategy we have solution procedures that show successful employment of mathematical knowledge concerning geometry. We consider this strategy as indicative of a higher level of 'mathematization' and this is why we put it in the top of the hierarchy of strategies. The students in order to justify their answers are based heavily on the definitions and the properties of the plane figures.

Trying to solve the tasks, the students initially applied a specific strategy (of the above mentioned ones) to estimate the area of the shape. This was a complex process since the shapes were irregular and there were not ready formulas or recipes for estimating their area. During this process the students had the chance to follow another of the already available strategies to verify now the correctness of their results. (In some cases they applied a completely new process for verifying the results). This is why we approach the ordering of these strategies in two ways: from the 'estimation' point of view and from the 'verification' point if view.

Table 1 presents the ordering of the strategies applied for estimating the area of the irregular shapes into six categories ('estimation' point of view). The ordering of categories 1, 2 and 3 is plausible. Next, since the obstacles from misconceptions and prototypes are overridden, we put the visualization and visually driven strategies (that are based but not completely on visualization) (category 4). Even though there is no 'real' reasoning in a certain mathematical level these strategies are characterized by an organized thinking. However, category 4 does not have the coherence of the category 5 strategies which presuppose executive control skills. Finally we put category 6 as the strategy with the greatest mathematical significance.

Categ – 1	Absence of any strategy
Categ – 2	Strategy based on misconceptions
Categ – 3	Strategy based on prototypes
Categ – 4	Visualization-Visually driven strategies (reaction, incorporation, grid)
Categ – 5	Cut and Paste - Decomposition
Categ – 6	Definition-Properties based strategy

Table 1: Ordering the problem solving strategies

From these strategies, visualization, reaction-based, and properties-based were applied for verification purposes also. The cut and paste was also applied in two different forms:

Verification through Erasing–and-Redrawing. Due to the fact that the students felt that they could not successfully accomplish the computational transferral of the pieces, after they identified the pieces that fit one to each other, they preferred to erase the first partial square unit and then to redraw it in its new place so as to fit with the second partial square unit. It is obvious that this strategy could not be applied in the paper-and-pencil environment and this is because the worksheets included the tasks printed and the students could not erase the already existed lines

Transformation-Based Verification. In this strategy the students tried to transform an unfamiliar shape to a familiar one of different known form but of the same area.

However there were also applied and some new verification approaches.

Outline and Auto-Measure Verification. The "Area" tool of Cabri would give the students an answer concerning the area of the whole irregular shape. Then and in order to verify this result the students decided to divide the shape to subshapes by drawing segments. However, these sub-shapes are not identifiable by the software. So they had to somehow overcome this difficulty. They thought then to re-draw these sub-shapes by using certain tools (e.g., the 'Polygon' tool). Working like that they made them recognizable to the program and now they were able to use the 'Area' tool to verify whether they correctly calculated the sub-areas

Formula based verification. This strategy was equally available to both environments and the students applied this strategy in a similar manner. After the initial estimation of the shape's area, the students split (when possible) the shape to subshapes. Using the appropriate tools, they measured bases and heights, they calculated the area of each sub-shape, they added the partial results and they compared the new final result with the initial one.

Copy-Paste" Verification. The students used this approach in the laboratory when they had to work in an electronic grid. The initial shape was constituted from complete, as well as partial, square units. These partial square units combined in two's formed a whole square unit. So the students, in order to verify their assumption, selected the partial square units (usually triangles) and by copy-and-paste transferred them to the remaining part of the grid on the computer screen. Then, they changed the orientation of these triangles so as to fit one to the other. Thus, they verified their initial thought.

All the verification strategies were ordered according to: a) whether they are completely based on empiricism and b) their mathematical 'significance' (Papadopoulos & Dagdilelis, in press). The analysis resulted into three main categories as can be seen in Table 2:

1. Empirical	2. Numerical	3. Idiosyncratic
a. Visual	a. Formula-based	a. Copy – Paste
b. Adaptation-based	b. Outline and Auto-Measure	b. Erasing -and- Redrawing
		c. Transformation-based
		d. Properties-based

Figure 2: Ordering Verification Strategies

Let see now how all this information could be summarized in a single table (Table 3).

Strategy	*Paper & Pencil environment For the estimation of area*	*Computer environment For the estimation of area*	*Paper & pencil environment For verification*	*Computer environment For verification*
Absence of strategy	X	X		
Misconceptions	X	X		
Prototypes	X	X		
Visualization	X	X	X	X
Reaction based		X		X
Incorporate into a recognizable frame	X			
Grid	X	X		
Cut-and-Paste & Decomposition	X	X		
Properties	X	X	X	X
Formula			X	X
Outline and auto-measure				X
Copy-and-paste				X
*Erasing-and-redrawing**				X
*Transformation**			X	X

* These two cases could be considered as special cases of Cut-and-Paste. However the intention of the user is different in each case. In the first one the solvers' intention is just to create complete units whereas in the second the solver tries to transform an irregular shape to a known one.

Table 3: Synthesis of the applied strategies

Studying Table 3 one could make some comments. First of all, we make comments about the number of strategies used for estimating the area of the shapes. We recorded nearly 14 different strategies concerning the specific problems posed to the students. Even though one could doubt about some of them, however, a large number still remains. The majority of these took place in the computer environment. Another interesting finding is that the verification processes are more or less equal in number to the ones applied for area estimation. The existence of the computer environment was in favour of the frequent usage of verification processes since in many cases verification was facilitated from the software itself. The kinds of verification processes in the computer environment were twice the number of the ones in the traditional environment. Moreover, some kinds of verification took place exclusively in the computer environment such as the verification through copy-and-paste or through transformation. On the contrary, there were not processes that took place exclusively in the traditional environment. Finally, there seems to exist a differentiation between the strategies employed for area estimation and the ones employed for verification. We agree with Margolinas (1993) that it is the intention of the solver that distinguishes an estimation process from a verification one. Thus, it is possible for the same strategy (for example cut-and-paste) to be used either as a way to estimate the area of an irregular shape or as a way to verify the already estimated area of the shape. The role that the solver attributes to the applied process is different to each one of them. Our expectation was that in both cases (estimation of area – verification of a result) the students would apply more or less the same processes. However, as can be seen at Table 3, there exists a clear distinction between the processes applied in these two cases. For instance, the usage of the grid seems to be used for the estimation of the area but not for verifying a result.

CONCLUSIONS

In relation to the primary school level, our research allowed us to record a series of certain processes for estimating the area of irregular plane figures as well as for verifying these estimations so much in the computer environment as in the paper and pencil one. We proposed a systematic ordering of these processes to certain categories. We summarized the total collection of the applied strategies in a single table. The findings allowed us to form the following statement: problem solving concerning area of irregular plane figures is in favour of the development of certain strategies especially when students work in the computer environment. Some of them are closer to empiricism while others are indicative of an exploitation of the acquired mathematical knowledge. Similarly, given that the students could not apply known processes or formulas due to the irregularity of the shapes, it seemed that the computational environment facilitated the development of a repertoire of verification processes. As can be seen from the relevant table, this development took place in a wider variety compared to the traditional environment. This in itself was an answer to our initial expectation about the equally distributed processes between the two environments. So, these processes are not equally distributed and this remains an open issue for future research. The relatively small number of participants (almost 100 students in total) could be regarded as a factor that restricts the generality of our findings. However, the fact that the duration of this study was approximately three years and the collected data were thoroughly examined reduced the impact of the factor of the sample size. In any case, future research could offer the chance for a more systematic study of our hypotheses as they are expressed in this paper.

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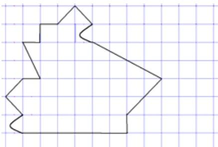
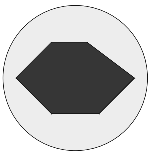
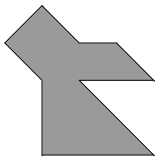
Appendix A







TOWARDS A TEACHING APPROACH FOR IMPROVING MATHEMATICS INDUCTIVE REASONING PROBLEM SOLVING

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The study aimed at proposing and assessing a training program that integrates both inductive reasoning problem solving and the development of mathematical concepts. This approach was developed on the basis of a general theory of inductive reasoning, which delineates six related classes of problems and the corresponding solution processes and it was implemented to sixth grade students. Data were collected through a written test consisted of mathematics problems of the six structures. Three repeated measurements were conducted with a break of 3-4 months between them. Findings revealed a significant improvement of mathematics inductive reasoning problem solving of the trained students while the training effect persisted for at least four months after the implementation of the program.

INTRODUCTION AND THEORETICAL BACKGROUND

Inductive reasoning is the highest-order cognitive skill that characterizes learning potential. It is considered to be a central component of critical thinking and one of the basic learning skills that contributes to problem solving (Haverty, Koedinger, Klahr, & Alibali, 2000). It is defined as the process of inferring a general rule by observation and analysis of specific instances (Haverty et al., 2000), and therefore it is a vital process for everyday life and for scientific investigation in particular.

In mathematics education inductive reasoning is enclosed among the most important goals of the curriculum (NCTM, 2000). It is closely related to the exploration and the generalization of different kinds of patterns that serve the basis of structural knowledge in mathematics learning (Johnassen, Beissner, & Yacci, 1993). As a generalization process, it is also fundamental to the development of many mathematical concepts, especially in algebraic concepts and in problem solving situations (Haverty et al., 2000; Orton & Orton, 1994; Warren, 2006). Consequently, mathematics teaching should focus on fostering basic skills in generalizing, and expressing and systematically justifying generalizations (Kaput & Blanton, 2001), as well as in developing strategies for solving various types of inductive reasoning mathematics problems. Nevertheless, in elementary education there is little emphasis on inductive reasoning as object of study; rather it is considered that could be developed as a by-product of the teaching of the content as defined in traditional curriculum (Hamers, De Koning, & Sijtsma, 1998). Classroom activities usually focus on mathematical products rather on mathematical processes and strategies. Inductive reasoning problems are likely to be marginalized by the press towards

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computational skills (Thompson, Philipp, Thompson, & Boyd, 1994) or appear in an abbreviated, arithmetic form (Blanton & Kaput, 2005). Consequently, many students have a lot of difficulties in solving problems.

Considering the valuable aspect of inductive reasoning in learning, research studies focused on designing teaching programs for improving inductive thinking in schooling (Klauer & Phye, 1994). Although these programs were oriented towards thinking processes and promoted inductive reasoning as a tool for problem solving, they were developed in a general content domain using content-free and daily life problem formats. Even in mathematics education, where inductive reasoning is an important process to investigate the gaining of a deeper understanding of mathematical cognition (Haverty, et al., 2000), there is a lack of the appropriate guidelines for designing comprehensive content-related approaches for improving inductive thinking within the content of the mathematics curriculum.

The present study, attempted to apply a teaching approach that integrate inductive reasoning problem solving procedures in a mathematics concept-development context. The proposed approach is based on a prescriptive theory of inductive reasoning (Klauer & Phye, 1994) while it is in line with the pedagogical principles and methods presented in the literature. The focus of the study was twofold: (a) to investigate whether the proposed approach improved students' mathematics conceptual knowledge and their ability to solve mathematics inductive reasoning problems of different structures, and (b) to specify the nature of change of students' mathematics inductive reasoning through the passing of time.

A prescriptive theory of inductive reasoning

Klauer (Klauer & Phye, 1994) suggests that inductive thinking could be improved through the teaching of the steps of an induction process that are necessary and sufficient to arrive at a generalization. These essential steps are resulted from an analytic definition of inductive reasoning, which delimits inductive reasoning problems from other types of problems (e.g. deductive) specifying their cognitive solution processes. This definition considers that inductive reasoning is the systematic and analytic comparison of objects aiming at detecting similarities and/or differences among them with respect to attributes or relations. It also presupposes that all types of inductive reasoning problems could be classified into two main subsets, the group-problems and the row-problems. The group-problems are dealing with attributes while the row-problems are dealing with scanning relations. Each subset comprises three different types of problems that could be discriminated in terms of the cognitive processes needed for their solution. That is, the group-problems set includes: (a) problems that require finding similarity of attributes among objects, (b) problems that are related with noting differences among objects with respect to attributes; and (c) problems that require a determination of both common and different attributes of objects. The row-problems set involves: (a) problems that require finding similarity among relationships; (b) problems that require detecting differences in relationships; and (c) problems that require finding either equivalence or dissimilarity of relationships. From a teaching perspective, the essential steps that are necessary for a successful inductive reasoning problem solving are the followings: First to train students to recognize the reasoning structure of an inductive task (group or row structure scheme), and then to apply the appropriate cognitive solution process. According to Klauer the mastery of these steps will improve students' ability to solve any type and complexity of inductive reasoning problems.

The proposed approach

The proposed teaching program was designed on the basis of a mathematics cognitive framework of inductive reasoning that delineates students' abilities in solving various mathematics problems that all require the use of inductive reasoning (Christou & Papageorgiou, 2007). These abilities correspond to the cognitive processes of similarity, dissimilarity, and integration, and are associated with the level of attributes and the level of relations, which specify the aspects that are compared in a mathematics inductive task. Thus, instruction aimed at helping students to distinguish whether a problem involves relations or attributes (reasoning structure) and then to identify whether there is a need to find similarity or difference or both similarity and difference (integration) in the attributes or in the relations involved in the problem (processing structure), in order for the problem to be solved.

In line with Klauer's training program, instruction proceeded through three hierarchical levels that correspond to three successive phases of knowledge development: (a) the conceptual-analytical level that corresponds to the development of the declarative knowledge; (b) the procedural level that corresponds to the development of the procedural knowledge; and (c) the strategic level, which is related to the development of the metacognitive knowledge. These three levels of instruction overlapped and proceeded developmentally throughout the time the program was carried out.

The conceptual-analytical phase aimed at the conceptual recognition of the different structures of inductive reasoning problems. Thus, instructional activities asked students to classify given problems into two main groups in terms of their reasoning structure, i.e. the kind of the objects needed to be compared (attributes or relations), and then to identify the different problem-formats included in each group. Teaching at this phase emphasized cooperation and discussion between students in order to facilitate discrimination of the various types of problems and to localize similarities and/or dissimilarities between different problem-formats. Furthermore, students encouraged to construct concept maps and diagrams to represent the relations existed among different problem-formats with respect to their solution comparison processes (processing structure) and then to relate unknown problems to the previous ones by analogy.

The procedural phase aimed at teaching students how to solve inductive reasoning problems of different structures. Problem solving at this phase involved also the attainment of new knowledge. Thus, instructional activities focused on constructing cognitive schemes of the new concepts that could be infused in an inductive processing structure. For facilitated learning, worked-examples were mainly used to relate problems involved new concepts to familiar ones by analogy. Furthermore, worked-examples and flowcharts were used to demonstrate the solution steps of each problem structure an to model problem solving processes in order for the students to localize similarities and/or differences between them.

Finally, the strategic phase aimed at accelerating the spontaneous application of the six reasoning processes in solving inductive reasoning mathematics problems. Therefore, the training activities encouraged students to solve problems of different representations and complexity, to describe the solution processes and to justify their thinking procedures in terms of the structures of the problems.

The problems used during the training were derived from 6^{th} grade mathematics curriculum and were related to attributes and relations of numbers and geometrical figures. Examples of some concepts intended to be developed through the proposed approach were: (a) the properties of numbers and numbers' operations, such as the multiplication of the odds and even numbers, (b) the attributes and features of the 2D and 3D geometrical figures, (c) numerical proportions, and (d) various types of number-sequences, like the sequence of the squared or the triangular numbers, arithmetic and geometric sequences, as well as patterns with geometrical figures.

METHOD

Participants and procedure

The sample of the study consisted of 137 sixth grade students (63 boys and 74 girls), from seven existing classes at elementary schools in an urban area of Cyprus. The study was based on an experimental-control design, thus sixty students comprised the experimental group while the rest comprised the control group. Students were assigned to the two groups according to their performance on mathematics inductive reasoning problem solving on the first measurement, which was carried out at the beginning of the study. The two groups were of the same performance.

Data were collected through a written test that measured students' performance on mathematics inductive reasoning problems of the six different structures delineated by the cognitive mathematics inductive reasoning framework. The same test was administered to students three times. The interval between successive administrations was 3-4 months.

Students of the experimental group were involved in the activities of the proposed program after the first measurement. The duration of the intervention was twelve 40-minutes teaching periods that were spread over nine weeks during the regular mathematics lessons. Students of the control group did not have any systematic instruction in inductive reasoning.

The assessment instrument

Twenty-one problems of six different structures comprised the test developed to measure mathematics inductive reasoning problem solving ability (see Table 1). Students had 60 minutes to complete the test. The scale reliability of the whole questionnaire was found to be very high. The Cronbach's alpha coefficients for the whole set of problems were 0.85, 0.88, and 0.88, at first, second and third measurement, respectively.

<table>
<tr><th></th><th>Attributes tasks</th><th>Relations tasks</th></tr>
<tr><td>Similarity</td><td>The numbers below have something in common. Write the common feature of the numbers:
4, 16, 8, 32, 20, 100, 40</td><td>Complete with the right number.
1 5 13 29 ……
(a) 33 (b) 37 (c) 45 (d) 61</td></tr>
<tr><td>Dissimilarity</td><td>Find the numbers that does not fit in with the others and put it in a circle. Explain.
9 21 11 15 12 6 23</td><td>One of the figures disturbs the sequence. Find it and define the right sequence.
△ Δ ○ ◯ △ Δ △ ○ ◯ △</td></tr>
<tr><td>Integration</td><td>Write the number 24 in the appropriate cell. Explain.
<table><tr><td>6 18
12</td><td>16 8
2</td></tr><tr><td>3 15
9</td><td>7 25
5</td></tr></table></td><td>Complete the empty cell with the appropriate number.
<table><tr><td>8</td><td>4</td><td>2</td></tr><tr><td>1</td><td>$\frac{1}{2}$</td><td>$\frac{1}{4}$</td></tr><tr><td>$\frac{1}{8}$</td><td>$\frac{1}{16}$</td><td>;</td></tr></table></td></tr>
</table>

Table 1: Examples of problem formats used in the test

Data analysis

A univariate analysis of covariance (ANCOVA) was implemented to the data in two parts. Initially, ANCOVA was used to examine whether the proposed approach improved students' ability to solve the whole set of problems included in the test. Thus, post-test attainments were used as depended variables, while the corresponding pre-test scores were used as covariates. Then, this kind of analysis was carried out to investigate the durability of the training effect; therefore it is preceded on comparing the scores of the two groups revealed from the third measurement to their pre-test attainments.

Finally, a multivariate analysis of covariance (MANCOVA) was used to explore students' improvement on solving each of the six different problem-formats as well as problems of the same reasoning (attributes and relations problems) or processing structure (similarity, dissimilarity and integration problems). Thus, in this case we set as depended variables the post-test scores related to each of the six kinds of problems or to each subset of problems, and as covariates we regarded the corresponding pre-test scores.

RESULTS

Regarding the whole set of mathematics inductive reasoning problems, results showed that students who received training outperformed the control group at the post-test ($F_{(1,\ 134)}$=32.779, $p<0.05$). This indicates that the training program has a positive impact on students' mathematics inductive reasoning problem solving after intervention. Furthermore, the training effect could be persistent at least four months after training, as revealed from the outcomes of the second part of the ANCOVA ($F_{(1,\ 134)}$=5.254, $p<0.05$). Table 2 presents the statistic indices resulted from the two-phase analysis of covariance.

Measurement	Group of Students	$\overline{X}$	Sum of Squares	F	p-value
Second	Experimental	0.871	0.641	32.779	<0.05
	Control	0.731			
Third	Experimental	0.768	0.149	5.254	<0.05
	Control	0.701			

Table 2: Students' attainments on the second and third measurements

With respect to each of the six kinds of problems, the results of the MANCOVA showed that the experimental group of students performed significantly better than the control group on the four kinds of problems after the implementation of the intervention (Pillai's $F_{(6,\ 129)}$=9.051, $p<0.05$). Specifically, the significant differences between the two-group performances were observed on attributes-similarity problems and on attributes-dissimilarity problems as well as on relations-similarity problems and relations-dissimilarity problems (see Table 3). Despite the improvement of the experimental group on solving both attributes-integration and relations-integration problems, this improvement was not significantly higher than the control group's ability. From a broader perspective, findings also revealed that the experimental group performed significantly higher than the control group on all the five sets of problems that formed on the basis of their reasoning structure (attributes and relations problems) (Pillai's $F_{(2,133)}$=14.506, $p<0.05$) or their processing structure (similarity, dissimilarity and integration problems) (Pillai's $F_{(3,132)}$=14.372, $p<0.05$). Given that the mathematics inductive reasoning of the two groups were equal at the beginning of the study ($t_{1,135}$=-1.948, $p>0.05$), these results indicate that the training effect have a positive impact on students' ability to solve various types of problems. Furthermore, the training effect contributed to the deeper understanding of the mathematical concepts involved.

Problem's Structure	Group of Students	Mean	SD	Sum of Squares	F	p-value

Attributes Problems	Similarity	Experimental	0.920	0.209	1.118	30.334	<0.05
		Control	0.782	0.290			
	Dissimilarity	Experimental	0.817	0.379	1.075	8.603	<0.05
		Control	0.708	0.424			
	Integration	Experimental	0.758	0.298	0.194	2.186	>0.05
		Control	0.714	0.329			
Relations problems	Similarity	Experimental	0.908	0.145	0.149	6.140	<0.05
		Control	0.867	0.197			
	Dissimilarity	Experimental	0.792	0.284	1.986	34.279	<0.05
		Control	0.584	0.321			
	Integration	Experimental	0.883	0.192	0.135	3.230	>0.05
		Control	0.847	0.247			

Table 3: Students' post-test attainments on the six types of problems

CONCLUSIONS

In this study we attempted to design a teaching model for developing inductive thinking within the content of the school mathematics in real-classroom situations. Findings revealed that this approach is effective since it improves students' ability to solve mathematics inductive reasoning problems of various structures. The use of the worked-examples in developing both the conceptual and the procedural knowledge of inductive reasoning mathematics problem solving seemed to guide students in their schemes construction and therefore in learning. This supports the idea that inductive reasoning problem solving and concept development could be integrated through appropriate training in a regular mathematics lesson. Though improvement was observed on all the six problem-formats included in instruction, the size of the training effect was different according to the problems' complexity. Integration problems seemed to need much more practice than the other types of problems for gaining spontaneous application of the solution procedures; they require the simultaneous application of two cognitive strategies and therefore they demand much more capacity of working memory to proceduralize the combination of the associated cognitive schemes, especially when they involve newly acquired concepts.

Nevertheless, the study is of great importance because incorporates problem solving and concept-development in real-classroom instruction. Thus, this approach could be used as a tool in teachers' instruction for developing inductive reasoning mathematics problem solving as well as the developing of specific mathematical concepts. Also, it could be used as a prototype for designing instructional programmes for improving thinking skills within the different subjects of the school curriculum.

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PRESCHOOLERS' SEMIOTIC ACTIVITY: ADDITIVE PROBLEM-SOLVING AND THE REPRESENTATION OF QUANTITY

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The present study aims to investigate preschoolers' semiotic activity during their attempt to solve an arithmetic problem concerning the addition of seven quantities. The analysis of 117 preschoolers' drawings in combination with their verbal descriptions resulted in the creation of 16 "notation types" distributed in four main categories: letters or words, pictograms, arbitrary non-conventional symbols and numerals. The results support a teaching perspective focusing on the integration of drawing in preschoolers' mathematics education.

THEORETICAL FRAMEWORK

Young children find difficulty in structuring a proper internal representation of an arithmetic problem. However, it has been suggested that the meaning of the problem could be grasped through its visualization, which is said to facilitate the solving process (DeWindt-King & Goldin, 2003). One of the strategies towards the visualization of the problem the researchers are studying is the integration of drawing activities (Edens & Potter, 2007; Saundry and Nicol, 2006; Smith, 2003). It is supported that when the children are encouraged to invent their own graphical representations of a mathematical problem, they are able to attribute a meaning and reflect on it (Dijk et al., 2004).

Another factor affecting the process for solving arithmetic problems is conventional symbols. Young children are not able to organize and manage data using them properly (Polland & van Oers, 2007). Although the children in the West every day come across numbers on several occasions, and not before long they master their ability to recognize them, this is not exactly the case when it comes to writing them and mainly to grasping their meaning (Munn, 1994). Relevant research on this field shows that the children graphically represent quantity in different ways (Hughes, 1986; Munn, 1994; Kato et al., 2002; Thomas et al. 2002; Carruthers & Worthington, 2005; Rogers, 2008). Even if they use arithmetic symbols, they do not always follow the conventional way as adults do. This fact shows that they have not fully understood the meaning and function of symbols (Munn, 1994; Thomas et al. 2002).

Nunes (1997) explains this difficulty by dividing symbolic representations into "*extended*" and "*compressing*" representations. The numeration system provides "*compressing*" representations while physical objects give "*extended*" representations (e.g., number 5 represents five objects). However, it is argued that if we encourage

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the children to invent and explore their own symbols through drawing activities ("*schematizing*" for Polland & van Oers, 2007), their representation ability and by extension their understanding of mathematical symbols is later facilitated. By using, recognizing and interpreting their self-invented symbols children are often able to talk about the problem and its solution (Carruthers & Worthington, 2005), to communicate their thoughts and ideas, to compare their own symbols with those of the others and to reconsider them. In this way, their symbolic capabilities increase (Polland & van Oers, 2007). In other words, they discover symbols functions regarding them as tools for thought and communication. This understanding is very likely to help them later, in formal education, realise the meaning and purpose of conventional symbols (Polland & van Oers, 2007; Carruthers & Worthington, 2005).

Although drawing activities seem to be a privileged teaching environment for mathematics education, research in this field has little developed particularly as regards young ages. Considering that drawing is a kind of semiotic activity – inventing symbols and attributing meanings– explored by the children already from a very early age (Van Oers, 1997) and that children's drawing ability is part of the development of their representation ability (Matthews, 1996; Kress, 1997), the integration of drawing in preschoolers' mathematics education constitutes a challenging teaching perspective. A number of issues arise. How do children represent quantitative data when they have to deal with arithmetic problems? What meanings do they attribute to their signs? Are they able to describe – talk about their "notations"? What kind of strategies do they develop and how are these strategies connected with their graphic representations? According to this perspective, the present study investigates the ways the children invent in order to graphically represent the quantitative data of an additive problem during its solving process.

METHOD

The research was carried out in eight nursery classes, including a total of 117 children, with 34 of them aged 4-5 years old (average 4,6 years, group A), and 83 of them aged 5-6 years old (average 5,7 years, group B), in normal class conditions. The children worked individually at the same time on the same additive problem derived from a fairy tale. The storytelling approach in mathematics education can both offer experiences meaningful to the students and promote their engagement in the mathematical task (Nicol & Crespo, 2005). The structure of the additive problem concerns the addition of seven quantities and has the form of 1+1+1+2+3+4+1=13 (referring to those who helped the pumpkin come out of the ground). It was presumed that this structure would allow us to record both the way the children would represent the quantities from one to four within a specific context and the way they would use these representations in order to manage the solution of the specific additive problem.

At first, the children listened to the fairy tale and then they only listened to the part including the problem, since they were given the following instruction: *"use your paper and pencil in any way you can in order to find out how many took the pumpkin* *out»*. The use of the words write and draw was avoided so that the children's production could remain unaffected. In addition, considering that talk helps us understand a child's internal representational capacity (Saundry & Nicol, 2006), personal interviews were later carried out during which every child was asked to describe his or her drawing, e.g., what have you done here and what's this, while reflective questions were also asked, e.g., how many of them finally took the pumpkin out, what did you do to find it, what have you written/ put down/ drawn, what did you do first (in case the student has used 2 or 3 notations at the same time).

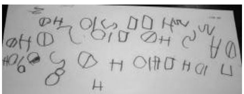
DATA ANALYSIS

At a first level, the different representation types detected in children's drawings were codified according to the categories resulting from previous research on written records of the quantity of objects (Hughes, 1986, Munn; 1994, Kato et al., 2002; Rogers, 2008) or numbers (Thomas et al, 2002). Four main categories emerged from this first analysis: (I) letters or words: any kind of writing except numerals, (II) pictograms: images representing features of the problem (III) non-conventional symbols: drawings of tally marks, dots, circles or squares and (IV) numerals: any kind of written numerals. At a second level, the detailed study of children's drawings in combination with their verbal explanations describing their drawings revealed a range of subcategories. While describing their drawings, the children provided additional information about their intentions and the way they used specific notations. It should be noted that 1 up to 4 different notations were recorded in each drawing.

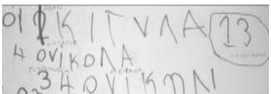
RESULTS

In this way, the four main categories gave a total of sixteen different "notation types" used for representing the problem data. Because a large number of children used more than one notation, the 117 drawings included 187 notations. In order to illustrate the "notation types", we below discuss representative examples for each main category separately.

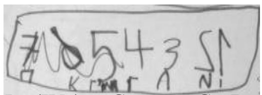
I. Letters or words (writing).



Ia Dimos, Gr. A



Ib, (IVe, IVb) Androniki, Gr.B



Ic, (IVc) Gregory, Gr. B

Figure 1: "Letters or words" (I)

Some children "write" the kind of data. Writing signs, usually combined with other notations, were detected in some of the drawings. The analysis of the drawings finally produced three different types of writing (Fig. 1).

Ia. *Pseudo-letters, pretending writing*. Dimos, just like other children of mainly group A, uses exclusively *pseudo-letters* and interprets his drawing saying "*I was* *writing the animals*". It is possible that the instruction *"use your paper and pencil in any way..."* may lead him to invent some symbols he considers a kind of "writing".

Ib. *Fortuitous letters* (trying to write the kind of data). Androniki, in combination with other notations (IVe, IVb), tries to "write" in fortuitous letters the persons and animals reported in the data, which she later "reads": "*Ivan, 4 cats, 3 pigs*, etc."

Ic. *Letters or the real words* representing the kind of data. Gregory writes correctly the first letter of each kind of data word (from right to left, he writes with Greek letters: Ι, Ν, Α, Γ, Γ, Κ, Π under the numerals). This notation helps him later recognize the kind of data.

II. Pictograms.

IIa Maria, Gr.A | IIb Katerina, Gr, A | Christina IIc, (IVb) Gr. B | IId,(IVb) Spyros, Gr.A

Figure 2: "Pictograms" (II)

Figure 2 shows four pictorial representations corresponding to the "notation types" of this category. The pictograms often include numerals (IVb), as shown in Christina's and Spyros' drawings.

IIa. *Related to the fairy tale but not necessarily to the problem*. Like Maria, who draws the pumpkin and Ivan's house. She possibly interprets the instruction according to the class habit of drawing after listening to a fairy tale.

IIb. *Global representation of quantity without accuracy*. Katerina makes a lot of figures and later says while describing her drawing: *"I made them all; there were a lot of them"*. She fails to recognize the different persons and data, but focuses on the fact there are lots of them.

IIc. *Representation of groups*. Christina draws one figure for each piece of data (e.g., one girl for Natalia but one pig for the 2 pigs, and one cat for the 3 cats, etc.). This does not help her later remember the arithmetic data; she describes the kind and at last she counts the seven figures one by one.

IId. *One-to-one correspondence*. Spyros draws one by one all the persons and animals described by the problem's data. He analyses every piece of quantitative data in the respective figures, e.g., he hears two pigs and draws two pigs, etc. This type of notation helps him later name all these figures one by one.

III. Arbitrary, non-conventional symbols (tally marks, circles, dots, etc.)

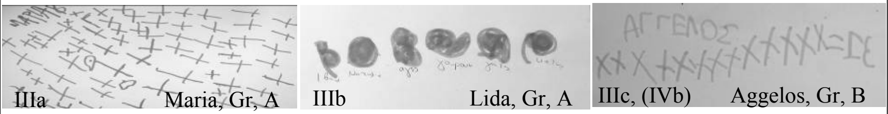


Figure 3: "Arbitrary, non-conventional symbols" (III)

Figure 3 shows three representative drawings of this category.

IIIa. *Global representation of quantity without accuracy*. Maria draws a lot of crosses and says: *"there are many, too many of them"*.

IIIb. *Representation of the groups*. Lida draws one dot for every piece of arithmetic data, meaning 6 dots for Ivan, Natalia, a cow, 2 pigs, 3 cats and 4 chickens. She focuses on the kind of data rather than on quantity.

IIIc. *One-to-one correspondence*. Aggelos uses the symbol X to represent one by one all those who took the pumpkin out. Like Spyros (Figure 2, IId), he interprets the "compressing" representations provided by the data (e.g., 3 cats) as "extended" representations (XXX) (Nunes, 1997).

IV. Numerals

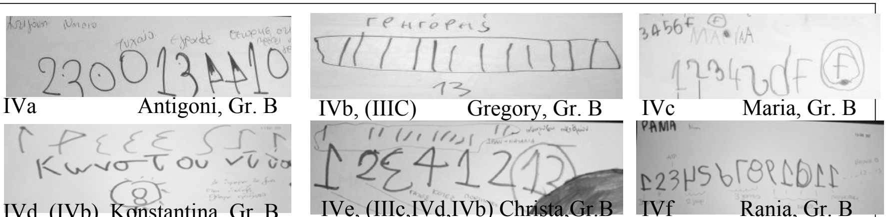


Figure 4: IV "Numerals"

The children used the arithmetic symbols in six different ways in order to represent the data and the solution of the problem (Fig. 4).

IVa. *Fortuitous numerals.* This type of number use seems to be impertinent to the arithmetic data of the problem, as shown in Antigoni's drawing (Fig. 4). They possibly write fortuitous numbers as a general and indefinite answer to a problem "including" and "asking for" numbers.

IVb. *Only one numeral, which represents the problem solution.* This type of notation was extensively and almost always used along with other notations (Fig. 2, 3, 4).

IVc. *Representation of groups* with successive numerals according to the order the data is read. Maria (Fig. 4) writes one numeral for each group. Although the children that use this notation use numerals, they seem to take into account only the qualitative rather than the quantitative character of the seven different pieces of data, as it happens with "notation types": IIc and IIIb.

IVd. *One-to-one correspondence with numerals*. The children who use this notation (Konstantina, Fig. 4) write a numeral for every person or animal, choosing for each of them the numeral that corresponds to the quantity of its particular group (e.g., from right to left, she writes for Ivan 1, for 2 pigs 22, for three cats 333, etc.)

IVe. *Numerals which represent every single piece of data*. In this case, children like Christa (Fig. 4) record the numerals that correspond to the data they hear.

IVf. *One-to-one correspondence with successive numerals starting from one.* The children that use this notation type do not use any other notations. They probably do not need them during problem solving. However, quite a lot of them find difficulty in writing double-digit numbers. This becomes more evident when they later describe their drawing and regard them as single-digit numbers (Rania, Fig.4).

	Ia	Ib	Ic	IIa	IIb	IIc	IId	IIIa	IIIb	IIIc	IVa	IVb	IVc	IVd	IVe	IVf
A	10.3	2.6	0	11.1	2.6	2.6	28.2	7.7	5.1	7.7	2.6	7.7	7.7	0	2.6	2.6
B	0.7	1.4	0.7	2.7	0	0	25.7	1.4	0.7	15.5	2	27.7	4.7	5.4	2.7	8.8

Table 1: Frequencies (%) of notation types for groups A and B

Table 1 shows the frequencies of the notation types analysed above for the two groups of children (A & B). Children of group A use mainly pictorial representations with those of type IId being in higher percentages (28.2%). There are some children that use arbitrary symbols and even numerals of different types. Some pretend to be writing (Ia: 10.3%), although it actually becomes evident from their explanations (Dimos, Fig. 1) that they pretend to be solving the arithmetic problem.

The notations mostly used by the children of group B are of three types. More specifically, they use pictorial representations with one-to-one correspondence (IId: 25.7%), a numeral which represents the problem's solution (IVb: 27.7%) and non-conventional symbols with one-to-one correspondence (IIIc: 15.5%). Some of them (8.8%) also use the notation type IVf, meaning one-to-one correspondence with successive numerals starting from one. Broadly speaking, group B uses non-conventional symbols and numerals more than group A does.

CONCLUSIONS

This study has produced two important findings, which are discussed below.

- Preschoolers invent a diversity of notations to represent the arithmetic data of an additive problem when they try to solve it through drawing activities.

Our results show that the children did not find any difficulty through drawing activities to invent their symbols and solve the problem. All notation types except one (IIa) were related to the arithmetic problem, although some of them were more pertinent (IId, IIIc, IVf) than others (Ia, Ib, IIb). Even the children that pretended to be writing (Ia) declare intentions relating to the problem. It is possible that the context of the arithmetic problem resulting from a fairy tale familiar to the children helps them realise the mathematical meaning of the problem (Nicol & Crespo, 2005). This finding supports what other researchers have said about the importance of drawing activities both for problem solving (Edens & Potter, 2007; Saundry & Nicol, 2006) and for the invention of symbols by the children (Dijk et al., 2004).

- Four main categories of graphical representations were found: (I) letters or words, (II) pictograms, (III) non-conventional symbols and (IV) numerals, each of them comprising several subcategories.

Similar categories are reported by different studies (Kato et al., 2002; Munn, 1994; Hughes, 1986; Rogers, 2008), which investigated through clinical-type experimental situations the way the children represent small quantities of objects (1-5). But our results revealed lots of different ways ("notation types") of using these representations, which are possibly related to the kind of task. In previous studies the children had to represent only one quantity at a time, while in our case they have to solve a complex additive problem. The children should initially virtually represent in their mind the arithmetic data and then represent it graphically before they use these representations in problem solving. Moreover, in previous research the children used mainly arbitrary symbols and numerals to represent the quantity of the objects presented (Hughes, 1986; Kato et al., 2002; Rogers, 2008). The present results showed that while they (mainly the children of group B) use these categories in high percentages, they also use pictograms in quite high percentages (both groups). In the former studies the quantities refer to objects rather than to persons and animals, as it happens in our study. It is very possible that this fact in combination with the fairy tale context may "lead" the children to the pictorial representation of the data.

However, there are still a number of issues to investigate. What is the meaning attributed to each notation type? What solving strategies do they develop and how appropriate is each of them? This seems to be a reasonably wide field for future research and development of propositions about the way such results could be exploited by teaching.

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THE GROWTH OF MATHEMATICAL PATTERNING STRATEGIES IN PRESCHOOL CHILDREN

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The development of patterning strategies during the year prior to formal schooling was studied in 53 children from two similar preschools. One preschool implemented a 6-month intervention focussing on Repeating and Spatial patterns. An interview-based Early Mathematical Patterning Assessment (EMPA) was developed and administered pre- and post-intervention, and again following the first year of formal schooling. The intervention group outperformed the comparison group across a wide range of patterning tasks at the post and follow-up assessments, most also being able to extend and explain Growing Patterns which they had not previously experienced.

Several initiatives in early childhood mathematics curriculum are promoting the development of mathematical patterning and algebraic reasoning (Carraher, Schliemann, Brizuela, & Earnest, 2006; Clements, 2004; Doig, 2005). For example, Australian state mathematics curricula, as well as national and state assessment programs, now incorporate a Patterns and Algebra strand (NSW Board of Studies, 2002; Ministerial Council on Education, Employment, Training and Youth Affairs 2008). However, despite a recent surge of research interest in early algebra (Kieran, 2004; Warren, 2003), the research base that might justify a greater emphasis on patterning is rather insubstantial.

In this paper, we describe a study designed to assess the effects of an early intervention, focused on patterning, on the development of children's mathematical thinking.

MATHEMATICAL PATTERNING

We understand the term pattern to mean any replicable regularity. In early childhood mathematics, children experience three main types of pattern (Papic, 2004):

- *Repeating patterns*, where a *unit of repeat* (Threlfall, 1999) is repeated indefinitely
- *Spatial structure patterns* such as triangles, blocks, arrays and grids
- *Growing patterns* consisting of a sequence of elements which change systematically

Recently, several mathematics education researchers have focused on the early acquisition of patterning and its role in mathematical development. A series of studies forming the Australian Pattern and Structure Mathematics Awareness Project (PASMAP) indicates that children's Awareness of Mathematical Pattern and

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 329-336. Thessaloniki, Greece: PME.

Structure (AMPS) in the first three years of schooling generalizes across a wide range of mathematical content domains to the extent that it can be regarded as a general cognitive characteristic (Mulligan & Mitchelmore, in press).

There are very few studies of patterning in early childcare settings. One recent observational study (Waters, 2004) found that Australian preschoolers initiate and talk about their own patterns, ranging from simple repetition to geometric forms. Waters's study also highlighted the limited knowledge of preschool teachers in relation to the types of mathematical patterns and their pedagogical potential.

The present study addressed the following research questions:

1. In what ways does a preschool intervention promoting mathematical patterning affect the complexity of children's patterning concepts and skills and the development of other mathematical processes such as multiplicative thinking?
2. Is the influence of such an intervention maintained after children's first year of formal schooling?

METHOD

An intervention preschool (IP) was selected in the south-western area of Sydney. A non-intervention preschool (NP), similar in size and structure to the IP, was identified within the same region. Both preschools had comparable enrolments (38 children in each), staffing levels, and resources and had identical approaches to the curriculum. The children in both preschools came from low to middle socioeconomic families, with a high percentage from non-English speaking backgrounds.

The participants comprised initially 53 preschoolers aged from 3 years 9 months to 5 years. There were 27 children (16 boys and 11 girls) in the IP group and 26 (16 boys and 10 girls) in the NP group. All children spent a minimum of six hours a day at preschool for at least two days a week. The teaching staff remained the same in both centres for the 6-month duration of the intervention.

Only 35 of the participating children were available for the assessment on completion of the preschool year. Thirty-two of these were traced to 27 different institutions on completion of the first year of formal schooling for the follow-up assessment. Despite the attrition, there was no indication that these samples were biased.

An Early Mathematical Patterning Assessment (EMPA) was developed prior to commencement of the intervention. Eleven task categories were devised to assess children's ability to reproduce, create, identify, extend, and copy from memory Repeating, Spatial and Growing Patterns, in a variety of modes (summarized in Table 1). The EMPA was administered as a semi-structured individual interview on three occasions: at the start and end of the intervention and again at the end of the first year of formal schooling. The first and second assessments included the same 29 Repeating and Spatial Patterns tasks, but the 22 tasks for the third assessment were increased in complexity or redesigned as Growing Patterns. All interviews were audio recorded and 20% were video recorded.

Category	Description
Repeating Patterns	
Tower	Copy, continue and represent simple (AB) and complex (ABC) repetitions, using blocks and by drawing.
Border	Complete border patterns using cut out tiles, identify whether a border pattern has a clear start or finish.
Hopscotch	Copy hopscotch patterns using square tiles, by drawing and draw from memory. Design own hopscotch pattern.
Number	Identify next numeral and colour in repeating patterns of two or three numerals that use two or three colors.
Spatial Structure Patterns	
Array	Copy array patterns using counters and by drawing.
Block	Copy rectangular block patterns using blocks and by drawing.
Grid	Copy rectangular grid patterns by drawing
Subitizing	Identify number of dots in regular and irregular patterns and within grids. Identify number of blocks in staircase patterns.
Triangular 1	Copy various triangular dot patterns using counters and by drawing.
Growing Patterns	
Triangular 2	Continue growing triangular dot patterns and justify response.
Square Tiles	Continue growing square patterns and justify response.

Table 1: EMPA Task Categories

Another interview assessment, the 56-item *Schedule for Early Number Assessment 1* (SENA 1) (NSW Department of Education & Training, 2001), part of the state-wide numeracy program, was administered following the third EMPA assessment.

The first researcher worked closely with the IP teachers in developing, implementing and monitoring an intervention that provided explicit opportunities for children to develop patterning skills through problem-based tasks focused on the notion of unit of repeat. The intervention comprised two main components:

- *Structured individual and small group pattern-eliciting tasks* were based on the Tower, Subitizing and Hopscotch tasks (see Table 1). Children's responses to the Tower tasks were easily categorized into developmental levels of pattern recognition, which allowed the design of a framework to guide later instruction.
- *"Patternizing" the regular preschool program* emerged as the teachers realized how much children were learning from the initial patterning

activities. The first researcher worked collaboratively with teachers to scaffold rich patterning explorations within their regular curriculum.

RESULTS

Children's EMPA responses were first coded for accuracy, then their patterning strategies were described and classified. The first researcher's coding was verified by an independent coder, yielding an inter-coder reliability of 89%.

There is not sufficient space for us to describe the strategy classification. Instead, we shall present the quantitative results comparing children in the intervention preschool (IP) and the non-intervention preschool (NP) and simply quote supporting data from the strategy analysis. No statistical tests were applied because the two samples were not randomly selected.

Repeating Patterns

Figure 1 shows the change in performance in the Repeating Pattern tasks at Assessments 1 and 2 in the IP and NP groups. For each task category, the data show the average number of correct responses as a percentage of the total number of tasks.

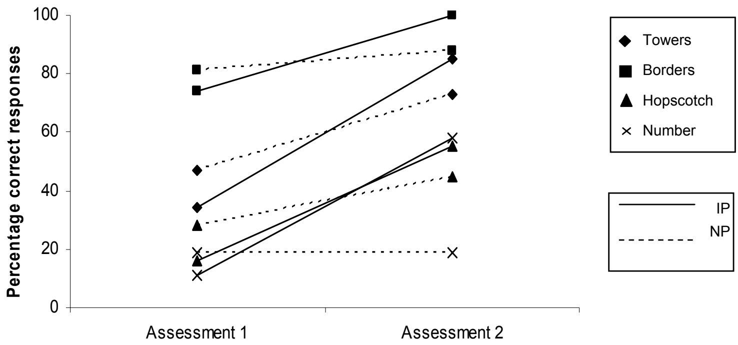

Figure 1. Repeating Patterns: Pre- and post-intervention assessment data.

At the initial assessment, the NP children were moderately more successful across all Repeating Patterns tasks than the IP children. However, at the end of the intervention period, the IP group consistently outperformed the NP group. The contrast was particularly evident for the Number tasks, where the IP children improved substantially but the NP children showed no improvement.

By the end of the intervention, there were very few of the apparently random responses seen initially in both groups. Many NP children continued to follow a direct comparison strategy, copying patterns one element at a time, but this strategy was no longer used by IP children. Most NP children used an alternation strategy (e.g., alternating colors) throughout, but most of the IP children had switched to a strategy based on recognition of the unit of repeat.

Spatial Structure Patterns

Figure 2 reports the results for the four categories of Spatial Structure Pattern. The two groups of children performed at a very similar level at Assessment 1. By Assessment 2, the IP children had made considerable gains in all task categories whereas the NP children had only improved on the Array tasks. The NP group found it particularly difficult to identify the number, shape, size, orientation, and spatial and numerical structure of the triangular dot patterns in the Triangular 1 task. The NP children actually regressed on Subitizing tasks, frequently reverting to counting individual items even for a simple three-dot pattern.

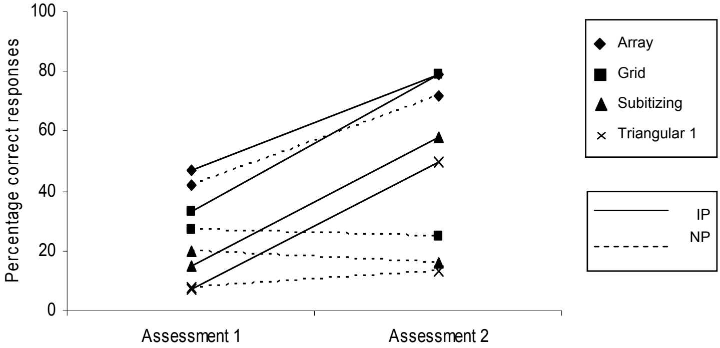

Figure 2. Spatial Structure tasks: Pre- and post-intervention assessment data.

Strategy analysis revealed four levels of structural representation similar to those reported by Mulligan and Mitchelmore (in press). By the end of the intervention, the percentage of highest-level responses among the IP children had increased substantially. By contrast, among the NP children the only level to show a substantial change was the lowest one: The percentage of random responses had decreased.

Follow-Up Assessment

Figure 3 shows a comparison between the IP and NP groups on the third administration of the EMPA. For the Repeating Pattern tasks (Towers, Borders, Hopscotch, Number), there were striking differences in favour of the IP group consistent with the pattern of responses shown at Assessment 2. The majority of the IP children continued to use the “unit of repeat” strategy they had demonstrated 12 months earlier, whereas most NP children still used the far less efficient direct comparison and alternation strategies.

Neither the IP nor the NP children had been exposed to Growing Patterns before Assessment 3. Although the IP children’s performance on these tasks was not as strong as for the Repeating Pattern tasks, many of them could identify, extend and justify both the Triangular 2 (1, 3, 6) and Square Tile (1, 4, 9) patterns. In comparison, none of the NP children gave a correct response. More than half of the

NP children treated the given pattern as the start of a repeating pattern and repeated the three given elements exactly.

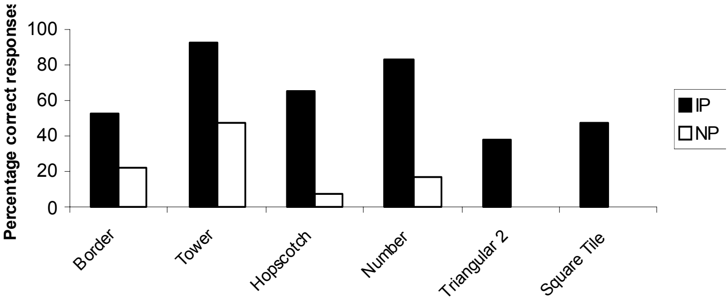

Figure 3. Follow-up assessment data: Repeating Patterns and Growing Patterns.

On the *Schedule for Early Number Assessment 1* (SENA 1), the IP children scored higher on average (82%) than the NP children (63%). They were superior in all assessment categories, including numeral identification, forward and backward counting, unitising, and simple arithmetical problem solving.

DISCUSSION AND CONCLUSION

This study has provided empirical evidence that children as young as 4 years can develop complex patterning concepts. The intervention resulted in gains in children's understanding of simple numerical and spatial patterns well beyond those made by the comparison group.

It seems that the intervention had acted to draw the IP children's attention to structure at a far deeper level than is achieved through regular pre-school activities. A likely source of this change is the teaching strategy, adopted throughout the intervention, whereby teachers repeatedly encouraged children to look for structural similarities and differences between the given pattern and their copy of it. According to Mason, Drury, and Bills (2007), "becoming aware of similarities and differences results in stressing or fore-grounding and consequently ignoring or back-grounding, which is the basis for both generalization and abstraction" (p. 55). We conjecture that the IP children had abstracted many concepts such as collinearity and equal spacing as well as primitive generalizations such as "many patterns have a unit of repeat".

It was the IP children's concept of unit of repeat that most clearly differentiated them from the NP children. The unit of repeat concept is particularly valuable because it leads naturally into the concept of multiplication—indeed, several IP children were observed skip counting. It may have been this increased level of understanding of number that was responsible for the IP children's advantage on the statewide SENA assessment at the end of their first year of formal schooling.

The most impressive result of the intervention was perhaps the success of the IP children on the Growing Patterns tasks one year after the end of the intervention. By contrast, the NP children seemed to be making random guesses. It is not clear why this difference occurred when none of the IP children had been exposed to Growing Patterns during the intervention (and probably at no time before the tasks were administered). It is possible that, as a result of the intervention, the IP children had not only become aware of many examples of numerical and spatial structure but had acquired a greater tendency to look for mathematical patterns. They would certainly have been better able to recognise the structure of the individual components (square and triangular patterns of dots). They may then have spontaneously generalised their idea of a pattern from one where the successive components are identical to one where there is a constant relationship between them.

If this explanation is correct, it would further support the argument put forward by Mulligan and Mitchelmore (in press) that Awareness of Mathematical Pattern and Structure (AMPS) is a general feature of young children's cognition that predicts their later mathematical achievement. The intervention included many activities that would have strengthened the preschool children's AMPS, and the result was a level of understanding that readily transferred to a more complex patterning task one year later.

Warren (2005) asserted that 9-year old children find growing patterns more difficult than repeating patterns because of an over-emphasis on repeating patterns in early mathematics curricula. Our findings suggest that it may rather be due to the way repeating patterns are treated. Responses from the NP group, as well as the state syllabus (NSW Board of Studies, 2002), suggest that teachers restrict their examples to alternating patterns—which would strictly reduce the likelihood of children gaining the powerful concept of unit of repeat. A simple curriculum change to the syllabus could have a significant impact on children's later learning.

Further research is needed to explore the impact of early patterning activities such as those used in our intervention on children's mathematical development overall, in subsequent years of schooling and in different populations of children. Possible gains may be found in several areas not fully explored in the present study—for example, the development of multiplication, functional thinking and symbolization. Research is also needed into how easily preschool teachers can learn concepts such as unit of repeat and incorporate them into their teaching. The results of this study are, to say the least, encouraging.

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DOES THE BUILDING AND TRANSFORMING ON LVAR MODES IMPACT STUDENTS WAY OF THINKING?

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In this paper we describe a twofold teaching experiment carried out in a secondary-level mathematics class in Greece which sought to investigate a) how the building and testing of LVAR (Linking Visual Active Representations) modes by the students supported by the Geometer's Sketchpad dynamic geometry software, impacts on students way of thinking with regard to the conjecturing and proving processes; and b) if transformations through different semi-predesigned LVAR modes lead students to structure mental transformations relative to the development of their van Hiele level.

LINKING VISUAL ACTIVE REPRESENTATIONS

The study presented here addresses one part of the fourth phase of the didactic experiment which was conducted in a secondary school Mathematics class in Athens, Greece. This process was linked to the developing of strategies for solving problems, or anticipating those strands of the solution relating to individual or collaborative thought processes by linking the steps in the constructional, transformational or explorative actions (or processes) in the proof via a sequence of actions using the different interaction techniques supported by the Geometer's Sketchpad v4 DGS environment (Jackiw, 1991). This mode of design in the software and the results of the testing with students led to the need to define the meaning of *Linking Visual Active Representations* (LVAR) (see Patsiomitou, 2008a; Patsiomitou & Koleza, 2008; Patsiomitou, 2008b). Some of the study's findings concerning semi-predesigned LVAR, the taking of decisions relating to the interaction techniques used, and the receiving of feedback on the students' proving processes were reported at an earlier PME meeting (Patsiomitou and Koleza, 2008), correlated with the developing of students van Hiele level. The current study sought to investigate how the building and transforming of different LVAR modes (Patsiomitou, 2008b), impacts on students thinking abilities during the proving process. The study focus was affected on concerns formulated by Dina van Hiele Hiele-Geldof (in Fuys et al., 1984) who had the objective "to investigate the improvement of learning performance by a change in the learning method". Crowley (1987) argues that "geometric thinking can be [made] accessible to everyone" by "refining the phases of learning, developing van Hiele based materials and implementing those materials and philosophies in the classroom setting." Many researchers who used the Geometer's Sketchpad have conducted studies, using the van Hiele model as descriptor for their analysis and concluded that students achieved significantly higher scores between the

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pre- and post-tests or significantly outperformed their peers who had received traditional instruction (see for example Almeqdadi, 2000). In Geometer's Sketchpad v4 DGS environment, LVAR are interpreted as "encoding the properties and relationships for a represented world consisting of mathematical structures or concepts" (Sedig & Sumner, 2006) in line with Goldin and Janvier (1998): a) "a physical situation, or situation in the physical environment" modelled mathematically embodying mathematical ideas; b) a combination of "*syntactic and structural characteristics*" enhanced by selected *basic* or *task –based* (Sedig & Sumner, 2006) different interaction techniques facilitated by the DG Sketchpad v4 environment where the problem is transferred or a geometrical theory is discussed. The semi pre-constructed LVAR have the following features: 1) "aid to make the final configuration appear less complex because all the inevitable auxiliary intermediate lines that must be drawn to achieve the final construction" (Schumann and Green, 1994), does not appear immediately but in linking dynamic illustration steps, keeping attention close to the aim of the overall construction and 2) enjoy an advantage over pre-constructed diagrams that can not only be manipulated and explored, since students can also *act* on them using the full range of program features (*which renders them **A**ctive*) and 3) provide the students with the guidance they require, and helps them replace their pre-existing knowledge by *assimilating* new knowledge or *accommodating* it as complementary to what they already know or by *confirming* / anticipating the pupil's thought processes. On the other hand the requirement for students to construct everything themselves even if the constructions are made in dynamic geometry environments may lead to "the actual construction process failing to correspond to the mental modular representation of the construction process" (Schumann and Green, ibid.). In the next section we will examine the correlation between the different LVAR modes and the van Hiele model.

LVAR modes and the van Hiele model

The link between visual and deductive way of thinking in the van Hiele model is the essence of the transition from the lower levels (Recognition and Analysis), to the upper ones (Formal deduction and Rigor). The original five-level classification is the following: Recognition (Level 1), Analysis (Level 2), Informal deduction (Level 3), Formal deduction (Level 4) and Rigor (Level 5). Another important aspect of this model is the five phases it specifies in the apprenticeship process, which are, in brief: information (inquiry), directed orientation, explicitation, free orientation and integration (Fuys et al., 1984). Instruction that takes this sequence into account promotes the acquisition of a higher level of thought. This model of teaching phases is used for the interpretation of the LVAR modes in this paper. Many researchers (for example Burger & Shaughnessy, 1986) support that sequencing instruction has positive effects on students' success. The different LVAR modes can be built using a combination of different transformational processes and interaction techniques supported by the Sketchpad environment. The LVAR modes corresponding to the apprenticeship phases reported above are described as follows (Patsiomitou, 2008b):

Mode A-the inquiry/information mode: In this phase of the problem, the students familiarize themselves with the field under investigation using the instantiated parts of the diagrams which lead them to discover a certain structure. **Mode B-the directed orientation mode**: In concrete terms, the sequential linked constructional steps of the solution to the problem emerge step-by step. **Mode C–the explicitation mode**: Transformations in increasingly complex linked dynamic representations of the same phase of the problem modify the on-screen configurations simultaneously. **Mode D–the free orientation mode**: Every phase in the solution can be displayed side by side on the same page of the software in an overview. **Mode E–the integration mode**: Successive configurations on different pages that are linked cognitively and not necessarily constructionally, compose the solution to the problem in global terms as a series of steps.

This categorization of the LVAR modes through experiment describes the way hypotheses are formed when the students face a problem in the dynamic geometry environment and how they manage the transition from Mode A (informational) and mode B (the directed orientation mode) to Mode E (integration), meaning the transition from the conjecturing to the proving phase (see for example Arzarello et al., 1998). In other words, the building of LVAR modes mirrors the constructs of inductive and deductive way of thinking with regard to the conjecturing and proving processes, carefully analysing every action and decision made as the students solved a problem using the LVAR modes in Geometer's Sketchpad. For Peirce (1960), conjecture is synonymous with *abduction* in which "we find some surprising fact which would be explained by supposing that it was a case of a certain rule, and thereupon adopt that supposition" (Fann, 1970). The theoretical framework includes the notions of instrumental genesis. During the instrumental genesis the user structures that Rabardel (1995) calls utilization schemes meaning the mental schemes that organize the activity though the tool/artefact (Trouche, 2004). The LVAR transformations which occurred due to techniques had a significant impact: during the instrumental approach, the student structured utilization schemes, of the tools, and consequently mental images of the operational processes, since any transformation of the pre-image figure (input) resulted in the transformation of the image (output).

RESEARCH METHODOLOGY

The didactic experiment was conducted in a class at a public high school in Athens during the second term of the academic year. Firstly the researchers examined the student's level of geometric thought using the test developed by Usiskin (1982) at the University of Chicago which is in accordance to the van Hiele model. During the 4-th phase of the research process, the pairs of the experimental group explored an open-ended problem (problem 1) within a dynamic geometry environment, *building* LVAR. Thereafter the problem was reformulated (problem 2) in an open–ended realistic problem taking into account the retroactions by the research group and was explored by the pairs of the experimental group, which interplayed with different LVAR modes to solve the problem aiming to proof every step. The methodology of the class experiment discussed in this paper includes the building and testing of semi-predesigned LVAR by a pair of students in the experimental group.The discussions were videotaped and examined simultaneously with the interviewers' fieldnotes during the inquiring process. The first part (problem 1) will describe how the students built the LVAR representations; the second (problem 2), how they manipulated and acted on semi-predesigned LVAR. We will focus on those key parts of a pair's dialogue, relating to the construction of meanings and deductive reasoning. M_1 is a male pupil (van Hiele level: 1 at the pre-test) and M_2 is a female pupil (van Hiele level: 2 at the pre-test) tested about 4 months ago. Van Hiele levels are used as descriptors in this analysis. The analysis of the results that follows is based on observations in class and of the video. Research questions: 1) How does the building of LVAR modes impact on students' transformation of verbal statements with regard to the construction of meanings, conjecturing and the proving process? 2) Does the building and transforming of LVAR modes lead students to structure mental transformations relative to the development of their van Hiele level?

Problem 1: Construct a triangle ABC with no angle greater than 120 degrees, and then construct equilateral triangles on the sides of the triangle ABC. Join the vertices of the equilateral triangles with the opposite sides of the triangle ABC. What do you observe?

Fieldnote1: With the definition and use of the "equilateral triangle" custom tool, the students developed a "*conversing*" (Sedig & Sumner, 2006) with the diagram on the screen using a *basic interaction*. This process can operate *in a complementary* manner to students pre-existing knowledge, or *as a confirmation* of the students mental approach. With a problem like this, the students are not confronted with auxiliary elements because dragging the vertices generates snapshots of several possible pair-congruent triangles that can often lead the students to obstacles relating to the equalities of the shapes. On the first page (Fig. 1), student M_1 is unable to find the pair of congruent triangles. He is not even able to focus on the segment as an element of the triangle to which it belongs; he therefore chooses to compare the triangle ZOB as a triangle whose side is ZC.

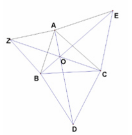

Fig.1

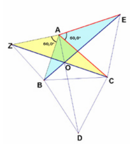

Fig.2

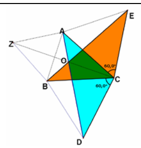

Fig.3

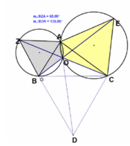

Fig.4

In the screenshots (Fig. 1-3) below, we can see the elements of the diagram that have been sequentially highlighted. The students explore the problem and start using *task–based interactions* like *highlighting, colouring, labelling* etc. (Sedig & Sumner, ibid.), to aid referencing and facilitate the visualization. This process can operate in a guiding / auxiliary manner whereby pre-existing knowledge is assimilated or accommodated. After the students have constructed the highlighted diagrams using the techniques provided step-by-step by the software, student M_1 was able to compare the triangles BEC and ACD (Fig. 3) despite their being differently orientated with regard to the ZAC, ABE triangles (Fig. 2). **Fieldnote 2**: The students try to prove that segment AD intersects the segments BE, ZC at a common point. They define the "circumscribed circle of equilateral" custom tool and activate it on the triangles AZB, AEC. The circle around the triangle BCD has not been constructed and the angles BOC, BDC have not been measured but they discuss:

M_2: Since these angles (BOC, BDC) are supplements, the quadrilateral..

She points to the shape and she tries to express the right geometric terminology but she doesn't know it, so the interviewer intervenes:

Interviewer …. can be **inscribed** within the circle.

M_1: ….**due to** the (opposite) angles.

The student verbalizes a hypothesis to confirm the conclusion she wants to reach. If the student knew that the angles had a sum equal to 180^o, it would be presented with a deductive way of thinking, meaning that if she knew Case A: the opposite angles have a sum equal to 180 degrees, she would apply the Rule B: a quadrilateral is inscribed within a circle if and only if the sum of two opposite angles equals 180 degrees and prove Result C: this quadrilateral is inscribed. In this case, however, she assumes/posits that she needs, in order to reach a conclusion. The students have therefore made abduction, because they select the right geometric property and 'what rule it is the case of' in Pierce language. Referring to the example by Peirce, we can say that M_4's reasoning is: Case A & Rule B, thus Result C. In this case, the student uses the property of the inscribed quadrilateral *inversely*, despite not having been taught beforehand. The students thus formulate the conjecture in a logical way, which reverses the stream of thought. By means of this process, the student has constructed the meaning of the inscribed circle and the suitable terminology *occurs* during experimentation with the software. They then apply the "circumscribed" tool to the third triangle. In this case, the tool functions *confirmatively*.

Problem 2: A power plant is to be built to serve the needs of the cities of A (Athens), P (Patras) and T (Thessaloniki).Where should the power plant be located in order to use the least amount of high-voltage cable that will feed electricity to the three cities? (see for example Patsiomitou, 2008b)

Mode A-the inquiry/information mode. Fieldnote 3: The problem can be modelled, representing the three cities by three points on the screen A, P, T and can be solved by finding a point K with minimum sum (KA+KP+KT) of distances to all three cities (Fig. 5). The researcher had constructed a table containing measurements of the angles AKP, AKT, PKT and calculations and the results on the table are linked to the effect of the movement of the mouse on the screen (Figure 5). Changing the position of point K by dragging, it is dynamically linked to the modifications in the resultant angles in the table and the upcoming modification in the sum of the segments. This process is an experimental process in which the students *act* on the diagram transforming the shape, by dragging point K and *making conjectures*, that the minimal sum is observed when the angles are close to 120 degrees. The students construct statements using *inductive reasoning*. That means the students allow inferring that A entails B from multiple instantiations of B: "A: the angles are close to 120 degrees", "B: the sum of the distances is minimum". So the inductive statements occur due to empirical evidence to be true. Their inductive inference from experimentation, confirms they can infer to hold the whole class of angle measurements from particular instances of angle measurements, meaning *they began with particulars and concluded with a general rule*.

Parts of Mode B and Mode E. Fieldnote 4: On screen, the students have diagram of figure 5, which did not remind them either of the problem they had already proved or of a possible solution to the problem in question. The researcher prompts the students to click on every hide /show action button that are connected with transformations on the shape or to act of their own volition. Here is an excerpt of student's discussion:

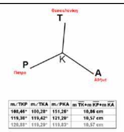

Fig.5

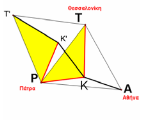

Fig.6

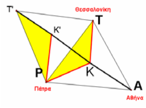

Fig.7

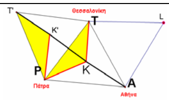

Fig.8

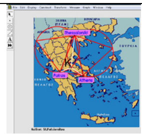

Fig.9

M_2:KK´P is an equilateral **because** angle P is equal to 60^o and K´P=PK (Fig. 6)

M_1:…**because** the triangles are equal (he points to the triangles TKP, T´K´P)

Interviewer: Has the sum KT+KP+KA been transformed to other segments?

M_2: KP is equal to KK´ and TK to T´K´ and the segment KA…..(Fig. 6)

M_1: has not been modified (it has stayed the same) (Fig. 6, 7) … the sum **becomes** minimum (the sum) **when** it (T´A) becomes a straight line (Fig.7)

Fieldnote 5: Student M_2 uses *deductive way* of thinking to prove that the triangle KK´P is an equilateral. The students interact with the LVAR and use a theorem to prove their reasoning, with students finishing each others' sentences. During the interaction with LVAR (enacting the rotation of the triangle TKP by 60^o) and through instrumental genesis student M_1 has constructed a utilization scheme which leads the student to conceptually grasp the meaning of the equality of the triangles. The students construct the segment T´T (Fig. 7). This is a crucial point in the research, when they perceive the relation between the diagram in the figure 7 and the diagram in the first problem: "It's like the equilateral we constructed before" they discuss.

M_2: we want the sum KT+KP+KA to equal the segment T´A, **which** means we have to construct a line equal to T´A. **That means** we will construct an equilateral on TA.

Fieldnote 6: In the dialogues, the phrases marked in bold are indicative of the students' levels. Student M_2 constructs the equilateral (Fig. 8) using the custom tool and then joins the points P and L. M_2 has *cognitively linked* the solution to the problem with the proof of the problem 1 they had worked through previously. The student use Problem 1 as a Proposition or a Rule in Pierce's logic which means they are employing *deductive reasoning*. They progressed from a (general) rule and presented results relating to the particular inferred case.After the students have acquired an overview of the Modes C and D the student M_1 in Mode E constructs the triangles directly on the map and notes "there is no need to construct the circles, only the equilaterals. Then we have to join the opposite points" (Fig. 9). This means that the students have developed thinking processes and *applied* skills, developing a mathematical model to interpret the realistic problem.

DISCUSSION

During the exploration of problem 1 through assimilation/accommodation generating by the building of the LVAR, the student M_2 "expresses her hypothesis not as a deductive sentence but as abduction namely a reverse deduction" (Arzarello et al., 1998). When students come to explore Problem 2 through different modes of LVAR, the reverse is true: they input the figure which was taken as output in Problem 1 through building LVAR. This leads the students to formulate conjectures, initially using an inductive way of thinking in Mode A. In Mode B, students actually implement logical connections in the form of articulated logical *concatenations* (Arzarello et al., ibid.) to produce meaningful arguments. In Mode E, M_1's logical sentence includes a semperasma. Although he was a level 1 in the pre-test, he condenses *actions* into an interpretation of the mathematical results, proposing/determining a simple solution process to demonstrate problem solving by applying interaction techniques. As the composition of the LVAR changes, there is a transformation in students' verbal formulations due to rules subjacent to the user's organized actions. Semperasma (Patsiomitou, 2008b): *the building and transforming of the semi-predesigned LVAR leads the students to pass from a visual way of thinking to a theoretical geometrical one, or to pupils' mental transformations. Students use verbal formulations to exchange their ideas meaning that they transform their mental objects into a language mapping, corresponding to LVAR transformations on pages in the software.*

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UNDERSTANDING AND REASONING IN A NON-STANDARD DIVISION TASK

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Abstract: *Here we focus on Finnish pre-service elementary teachers' (N = 269) and upper secondary students' (N = 1434) understanding of division. In the questionnaire, we used the following non-standard division problem: "We know, that 498 : 6 = 83. How could you conclude from this relationship (without using long division algorithm) what is 491 : 6?" The problem mainly measures adaptive reasoning. Based on the results we conclude that division seems not to be fully understood: only one fifth of participants produced a completely correct solution. The most central reason for mistakes was insufficient reasoning strategies.*

INTRODUCTION

Teacher education programmes face a major challenge in trying affect elementary teacher students' views of mathematics, that is, their beliefs, attitudes and knowledge. This paper draws on the work of the research project "Elementary teachers' mathematics" financed by the Academy of Finland (project #8201695), in which data were collected on 269 pre-service elementary teachers at three Finnish universities (Helsinki, Turku, Lapland). Two questionnaires were administered in autumn 2003 to assess the pre-service teachers' knowledge, attitudes and skills in mathematics at the beginning of their mathematics education course. The aim of the questionnaires was to measure their experiences of mathematics, their views of mathematics and their mathematical proficiency in certain topics. As part of the project we also collected comparison data on 1434 upper secondary students (grade 11, average age 17-18 years) from 34 Finnish schools selected at random. In the paper we concentrate on pre-service teachers' and upper secondary students' understanding of division and reasoning strategies used.

In the Finnish comprehensive school curriculum (NBE 2004) one of the principal goals as early as the second grade is that pupils should master and understand basic calculations. Therefore, many upper secondary school students may think that division is a "piece of cake". However, division is the most complex operation children have to learn in elementary school. Earlier studies show that also pre-service teachers and upper secondary students have clear weaknesses in understanding division (e.g., Simon 1993, Campbell 1996, Merenluoto & Pehkonen 2002). One of the main reasons for these weaknesses is that pre-service teachers have primitive models of division (e.g. Graeber & al. 1989; Simon 1993). Even after learners at school have had formal-algorithmic teaching, they continue to be influenced by primitive partitive and quotitive models (Fischbein & al. 1985).

THEORETICAL FRAMEWORK

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The division task used in this study measures several ot the strands of mathematical proficiency mentioned by Kilpatrick (2001), e.g. conceptual understanding, procedural fluency and adaptive reasoning. Yet, we view our task as measuring adaptive reasoning above all: to solve the task participants must reflect on and give justification of mathematical arguments, especially the relationships between operations.

Understanding division

Division is an important but complex arithmetical operation to consider in elementary teacher education. There are many reasons for its complexity: 1) division is taught as the inverse of multiplication, so understanding of division requires good understanding of multiplication; 2) division involving big numbers requires good estimation skills; 3) within the models of equal groups and equal measures two aspects of division can be differentiated: quotitive division (how many sevens there are in 21) and partitive division (21 divided by 7). (e.g. Anghileri & al., 2002)

People can use very different strategies in solving division problems. Some of them are useful and some are misleading. Prior research has identified the following useful strategies (e.g. Heirdsfield & al., 1999): 1) Several different counting strategies, (a) skip counting, (b) repeated addition and subtraction, (c) chunks; 2) using a basic fact; 3) holistic strategies.

In a study by Graeber & al (1989), 129 female pre-service teachers had high scores on all verbal problems involving the partitive model of division. They were less successful on the quotitive division problems and these primitive models influence pre-service teachers' choice of operations. Primitive models seem to reflect an understanding whereby a student separate things into equal size groups. The problem is whether pre-service teachers or upper secondary students can use this view to make sense of the abstract aspects of division. In Simon's (1993) study of pre-service elementary teachers the whole-number part of the quotient, the fractional part of the quotient, the remainder, and the products generated in long division did not seem to be connected with a concrete notion of what it means to divide a quantity.

Campbell (1996) studied 21 pre-service elementary teachers' understandings of division with remainder. He conducted clinical interviews with the students, who tried to solve four tasks with abstract contexts. The task we use here has some similarities in contrast to the following task used by Campbell (1996, 179): "Consider the number $6 \cdot 147 + 1$, which we will refer to as A. If you divide A by 6, what is the remainder? What is the quotient?" In Campbell's (1996, 182-183) study of the 19 participants who tried to solve this task, 15 calculated the dividend although it entailed additional trouble. Of those 15 respondents 9 calculated the dividend and relied upon long division in solving the task. Of those 4 who did not calculate the dividend, only 2 correctly identified the remainder and the quotient.

Zazkis & Campbell (1996) investigated 21 pre-service elementary school teachers' understanding of divisibility and the multiplicative structure of natural numbers in an abstract context. The following is an example of the tasks used: “Consider the numbers 12 358 and 12 368. Is there a number between these two numbers that is divisible by 7 or by 12?” Many pre-service teachers used long division as the procedural activity, but some degree of conceptual understanding was evident as well.

In a study by Silver & al. (1993), a total of 195 sixth, seventh and eighth graders from a large middle school solved three quotient division problems involving remainders with a real-world context (the number of the buses needed). The symbol forms of the word problems were a) 540:40; b) 532:40 and c) 554:40. Of the respondents, 91 % used appropriate procedures, and 73 % of them applied long division. Only 43 % of the participants understood that the result - the number of buses – was an integer.

Focus of the paper

In this paper we focus on the following research questions: What kind of reasoning strategies do pre-service elementary teachers and upper secondary students use in solving a certain non-standard division task? How do the reasoning strategies used by pre-service elementary teachers and upper secondary students differ from each other?

EMPIRICAL RESEARCH

Research participants and data

The study forms a part of the research project ”Elementary teachers’ mathematics” being carried out in three Finnish universities (Helsinki, Turku, Lapland). Of the 269 pre-service elementary teachers participating in the research, 35 % have completed advanced studies in school mathematics in upper secondary school. Two questionnaires were designed, the first measuring the pre-service teachers’ mathematical proficiency in certain topics, and the second their attitudes towards mathematics at the beginning of their university studies. The questionnaires were administered at the first lecture in mathematics education studies in all universities in autumn 2003. Students had 60 minutes time for the questionnaires and were not allowed to use calculators. Additional results of the project are described in Kaasila & al. (2008).

In conjunction with the project we also collected comparison data with the same questionnaires from upper secondary school. Altogether 50 schools were selected at random from all Finnish upper secondary schools. A letter was sent to the directors of the schools in the sample, in which they were asked to select from their school one group of students in the general and one in the advanced second-year mathematics course. We received responses from 34 schools representing a total of 65 student groups. Thus, in total obtained data on 1434 students.

The initial proficiency test contained a total of 12 mathematical tasks. The focal content areas were the rational numbers and related operations (in particular division), because previous research indicates that these are problem areas (e.g. Hannula & al. 2002). All in all, the initial proficiency test focused on content knowledge different from that tested in upper secondary courses and on the mathematics component of the matriculation examination.

The non-standard division task we used is the following:

> "We know that 498 : 6 = 83. How could you conclude from this relationship (without using the long division algorithm), what is 491: 6 = ?"

Data analysis

We did not find in the research literature a task similar to the one used in this study. As mentioned earlier, our task shares certain features with that used by Campbell (1996); however, it also differs in a number of respects: Firstly, in the task used by Campbell, the dividend is explicitly mentioned as the 'right hand side' of the division algorithm, whereby respondents have an opportunity to directly identify the quotient and the remainder. In our task, the starting equation is given in the form of division and does not involve a remainder. Secondly, unlike Campbell, we do not mention in the context of our task the concepts of remainder and quotient. Thirdly, the participants in our study did not have permission to use the long division algorithm or a calculator, which were central aids in Campbell's study.

In the first phase of this study (see Kaasila & al. 2005) we broke the 269 pre-service elementary teachers' solutions down into main categories and subcategories by applying analytic induction. This involves scanning the data for categories of phenomena and for relationships among such categories, developing typologies upon an examination of initial cases, and then modifying them on the basis of subsequent cases (cf. LeCompte, Preissle, & Tesch, 1993).

In the second phase of the study (see Hellinen & Pehkonen 2008), a deductive approach was used: the 1434 upper secondary students' solutions were categorized using essentially the same classification as used in the first phase when analysing pre-service elementary teachers' solutions. A number of categories were identified in addition to those formed in the first phase.

In the third phase we harmonised the categories we found in the phases one and two by reanalysing a part of the pre-service elementary teachers' solutions. At the end we compared the pre-service elementary teachers' reasoning (or solution) strategies with the upper secondary students' reasoning strategies. For more details see Kaasila & al. (2009).

RESULTS

The problem was solved totally correctly by one fifth of the pre-service teachers and the upper secondary students. The typical correct and erroneous categories of strategies used by the pre-service teachers and the upper secondary students are presented in Table 1 and Table 2, respectively. More details can be found in the paper Kaasila & al. (2009).

Table 1. Main categories of successful strategies used by the pre-service teachers and the upper secondary students (N1 = pre-service teachers, altogether 269; N2 = upper secondary students, altogether 1434).

Successful strategies	*N1*	*%*	*N2*	*%*
Using subtraction and division	43	16	196	17
Using multiplication and division	6	2	6	0.5
Other strategies	5	2	19	1.5
All	54	20	221	19

Table 2. Main categories of erroneous strategies used by the pre-service teachers and the upper secondary students (N1 = pre-service teachers, altogether 269; N2 = upper secondary students, altogether 1434).

Erroneous strategies	*N1*	*%*	*N2*	*%*
Almost correct strategy	60	22	231	20
Thinking limited to integers	59	22	165	14
Clear misconception	12	5	42	4
Other mistakes / irrelevant strategies	84	31	494	43
All	215	80	932	81

Successful strategies

According to Table 1 only 20 % of the pre-service teachers and 19 % of the upper secondary students used a strategy leading to the correct answer. These successful strategies we divided into three main categories, described below:

1) Using subtraction and division: 16 % of the pre-service teachers and 17 % of the upper secondary students used this strategy:

Example 1. The difference of 498 and 491 is 7. Hence, 6 is contained in 491 two times less, i.e. 81 times. As 5 units remain, making 5/6, the answer is 81 5/6 (2030[1])

2) The connection between multiplication and division: 2 % of the pre-service teachers and 0.5 % of the upper secondary students solved the task correctly using this method:

Example 2. $83 \cdot 6 = 498$, $81 \cdot 6 = 498 - 6 - 6 = 486$.

$491 - 486 = 5$, $491 : 6 = 81\ 5/6$ *(6089)*

[1] The four digit number in the bracket after the example refers to the test participant. When refering pre-service teachers, the first number is 1,2,3 or 4, and when refering secondary students the first number is 5 or 6.

3) Other strategies leading to the correct answer: 2 % of the pre-service teachers and 1.6 % of the upper secondary students solved the task correctly using other strategies. In the next example the pre-service teacher drew an auxiliary diagram, a circle divided into six equal sections:

Example 3. Seven units are to be subtracted from each quotient. This means one from each and makes 82. After that, one more unit remains to be subtracted from the total of six quotients, i.e. 1 : 6 = 1/6, or 0.17 as a decimal fraction, from each quotient. Thus the answer is 81.83. (4022)

Erroneous strategies

Misleading or otherwise erroneous strategies were used by 80 % of the pre-service teachers and by 81 % of the upper secondary students. We divided these strategies into four main categories (see Table 2):

1) Almost correct strategy: 22 % of the pre-service teachers and 20 % of the upper secondary students solved the task almost correctly. The solution of this group indicates a fairly high level of conceptual understanding but all the phases of the solution were not accurately reported, or students made some careless mistakes. In the next example the respondent did not justify in detail why the remainder was 5.

Example 4. 498 – 6 = 492. So 492 : 6 = 82; 491 : 6 = 81 remainder 5. (3093)

2) Thinking limited to integers: 22 % of the pre-service teachers and 14 % of the upper secondary students were not able to calculate the quotient. In the following example the respondent did not even seem to think that the answer might be something else than an integer:

Example 5. The number 491 is 7 units smaller than 498. Therefore 6 should go one time less into 491. I can't think of any explanation for the fact that 6 goes only 81 times into 491. (3016)

3) Clear misconception: 5 % of the pre-service teachers and 4 % of the upper secondary students had clear misconceptions in their answers. In the next example the respondent subtracted the difference of the dividends from the quotient:

Example 6. 498 – 491 = 7; 83 – 7 = 76. (3057)

4) Other mistakes / irrelevant strategies: 31 % of the pre-service teachers and 43 % of the upper secondary students obtained no answer at all or presented a solution that was not relevant to the research.

Example 7. I can't do it without a calculator (3079).

DISCUSSION The results indicate that the task was very challenging: only about one fifth of the participants were able to produce a totally correct solution. More than half of the participants either produced no result at all or used misguided strategies. Although division is known to be a difficult operation that has many interpretations, the result is still surprisingly poor. We were especially surprised that so many pre-service teachers and upper secondary students failed to provide justification with their responses, although it was specifically asked for in the instructions for the task.

When comparing the reasoning strategies used by the pre-service teachers with those used by the upper secondary students, we see that the results do not differ very much from each other. Altogether 20 % of the pre-service teachers and 19 % of the upper secondary students used a strategy in their solution that led to the correct answer. The reasoning strategies used in these groups were also quite similar. Almost all who obtained the correct result used both subtraction and division in their reasoning.

We identified three main reasons for mistakes or incomplete solutions: 1) Staying on the integer level: 10 % of the pre-service teachers and 4 % of the upper secondary student gave their answer as an integer, and it seems that in these cases they did not even think that the answer might be something else than an integer; 2) Inability to handle the remainder: Some of the respondents seemed to understand that the result was not an integer but a fraction, but they could not handle the remainder. For example, they expressed the remainder in the answer in tenths not in sixths (cf. Campbell 1996, 180). It seems that in school dealing with remainders has been a procedural matter, with too little attention focused on the idea that the fractional part of the quotient provides different (yet related) information from the remainder (Simon 1993); 3) Insufficient reasoning strategies: A little more than a fifth of the participants solved the task almost correctly. In these cases, all the phases of the solution were not accurately reported. The reason for insufficient reasoning strategies may be a lack of language skills, because the respondents had great difficulties in providing written explanations of their reasoning (see also Silver et al. 1993).

On the basis of this study we can suggest some guidelines for the content of mathematics courses in teacher education and in school: learners need a) a concrete, contextualised knowledge of division and b) the ability to examine division as an abstract mathematical object (cf. Simon 1993). Above all else learners need c) tasks and situations through which they can develop their adaptive reasoning skills. According to our study, a lack of reasoning skills may be the main factor causing students difficulties when solving non-standard division tasks.

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PROBLEM POSING: COMPARISON BETWEEN EXPERTS AND NOVICES

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In the present paper, we compare experts' and novices' problem posing. The analysis is made from process' point of view, looking for differences in trajectories that can be defined on a problem posing model. The model was defined on base of experiments done with high-school and university students, Olympiad participants and secondary / high-school teachers. The model is of a "generate and test" type consisting of five phases: setup, transformation, formulation, evaluation and assessment. The main finding is that novices' problem posing process consist of a, mostly, sequential follow of two particular phases; meanwhile in experts' case there is a cyclical parsing of the phases. The reason of this difference seems to reside on knowledge, especially in what we call strategic and control knowledge.

INTRODUCTION

Problem posing received increasing attention during the last decade from behalf of mathematics educators. One of the trends consists of using problem posing in teaching; Japan, for example, has a strong tradition in this line (for example, Imaoka, 2001; Kanno *et al.*, 2007). Another one, teaches problem posing; the book of Brown and Walter (1990) is good example of this. Both of them are practical approaches, however, from the point of view of the main interest of the conducted research the trends can be grouped into 1) relation between problem posing and problem solving; 2) problem posing abilities and the processes involved in the posing task; 3) classification of problem posing tasks and 4) problem posing and creativity. Although all these investigations contributed with valuable information about the problem posing process, there is no generally accepted problem posing model. Remains under debate whether there is a need for such model or, as a matter of fact, we are in a more general case of problem solving and, therefore, a problem solving model it would be enough to explain all the observed cases. The main challenge in problem posing research remains the definition of a framework, of a theoretical basis, that would allow to integrate the variety of results obtained on particular aspects.

In the paper we present a model of problem posing and we concentrate on comparing experts and novices by their trajectories defined by the phases of the model. Since the main interest is on comparison, the model will be described briefly such to create the context necessary for the purpose of the paper. We shall also discuss some differences between problem posing and solving from the point of view of the involved knowledge in the model.

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METHODOLOGY

For defining the model, the principle of analytic induction was applied to a series of experiments ran with high school and first year university students (considered as novice) along with Olympiad participants and secondary / high school teachers (experts). All the participants had to pose three sequence problems such to have one easy, one of average difficulty and a difficult one. After completing the task, they answered a questionnaire concerning the problem posing process. The participants were encouraged to freely comment the task. Complementary, few interviews were done with university students. Overall, in the experiments participated 44 high school students; 25 university students; 22 Olympiad participants; 41 middle school teachers and 22 high school teachers. The model represents a synthesis of several partial analysis (Pelczer & Gamboa, 2008a; Pelczer *et al.*, 2008b; Voica & Pelczer, 2009).

THE MODEL

The principle of analytic induction (Patton, 2002) was applied to the questionnaire and interview answers such to uncover common themes. In words of Taylor and Bogdan (1984, p. 124): “analytic induction, in contrast to grounded theory, begins with an analyst’s deduced propositions or theory-derived hypotheses and is a procedure for verifying theories and propositions based on qualitative data”. The questions of the questionnaire can be grouped as roughly corresponding to the: understanding, planning, implementing and looking back phases of the Pólya’s problem solving model. However, research posterior to Pólya on problem solving stressed the importance of metacognitive skills as ones that underlie the application of algorithms and heuristics.

As Flavell (1976) defined : “Metacognition refers to one’s knowledge concerning one’s cognitive processes and products or anything related to them….Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective”.

Distinguishing between cognitive and metacognitive behaviors is not easy. Roberts and Erdos (1993) noted: “definitional issues concerning metacognition are complicated and it is difficult to know how much of metacognition is meta and how much is cognition” (p. 259). In intent to systematize the study of metacognition, frameworks for metacognitive processes were proposed (Davidson, Dueser & Sternberg, 1994; Schoenfeld, 1985; Gieger & Galbrath, 1998). For example, Davidson *et al.* (1994) identified four metacognitive processes applicable in any domain: identify and define the problem; mentally represent the problem; plan how to proceed; evaluate what you know about your performance. These are very general categories of metacognitive behaviors, so our purpose is to refine them and identify the cognitive stages to which each belongs.

The common themes in the participant's problem posing were compared against Sharples' account of creative writing (Sharples, 1999). In his account, the process of creation is seen as cycle of cognitive engagement and reflection subject to a series of constraints. The model we define consist of five stages, not necessarily parsed in sequential order: setup, transformation, formulation, evaluation and final assessment. In the following table we describe these stages along with the sub-processes involved, however we shall not detail since our interest is to define trajectories based on the model.

Stage	*Sub processes*
Setup	Interpret task's restrictions; Define constraints and context (topic, domain); Preliminary definition of some evaluation criteria; Reflect on necessary knowledge, recall problems; Design main idea (strategy); Define start point (knowledge, theorem, problem, situation); Formulate an initial expression
Transformation	Analyze problem characteristics; Identify available techniques; Reflect on techniques from the point of view of their application; Perform transformation; Assess consequences; Reflect on the appropriateness of the technique.
Formulation	Identify possible questions; Reflect on the type of formulations in which these questions can have sense (formal, textual, everyday situation); Assess the value of each formulation and context; Select a formulation
Evaluation	Assess the proposed formulation (problem) with the initially set criteria; Assess problem's meta-characteristics, the technique and formulation; Decide if changes are needed (accept problem as finished or modify); If changes are needed, decide the aspect to be modified.
Final assessment	Reflect on the worth of the strategy, applied techniques and formulation; Identify the most important steps in the generation; Reflect on difficulty and interestingness; Reflect on the relation between technique, formulation and difficulty / interestingness; Reflect on one's confidence in handling the task, degree of satisfaction with the process and result.

Table 1: Stages and sub processes in problem posing.

Based on the model, we identified the stages novices and experts pass (the trajectory) and compared them. In the following we present the result of the comparison.

COMPARISON BETWEEN NOVICE AND EXPERTS

High school students (considered as novices, since was the first time they were taking an introductory analysis course) had a very simple way to generate the problems. In most cases, the setup phase ended with the final problem. They do not perform subsequent transformations in order to get the final expression. Instead, they try to solve the problem and, in case of no exit, abandon and propose a new expression. In this case, problem posing oscillates between a first proposal (setup), formulation and evaluation (see figure 1). The evaluation is done by an attempt to solve it, but when the student fails to solve he doesn't try to modify the expression such to become solvable but rather (randomly) define another expression.

In this particular case, it does not seem to occur any final assessment. Very few students tried to reach a new problem and their search is more random then goal-oriented. They neither seem to dispose of strategic knowledge (that is, how to outline the process) since they propose problems that are weak modifications of recalled examples. Knowledge of techniques (that is, modifications that can be performed) is also weak; even when they are able to identify the type of problem (and they can solve it) they can't invent ways of proposing a new problem of that type.

We give two examples. In the first one, the student tries out several expressions, don't know how to continue and finally opt for a very simple problem expression. This behavior corresponds to the loop *Engagement –Evaluation –Engagement*, since there is really no transformation, formulation but instead a random invention of an expression about which it is hoped to be easy to solve (figure 1). The other type of observed trajectory involves transformation instead of a new beginning on the task. Once evaluation is performed (like trying to solve) there is a return to the transformation step. Several transformations can be done at once, so we have a successive *Transformation-Evaluation* phase (figure 2).

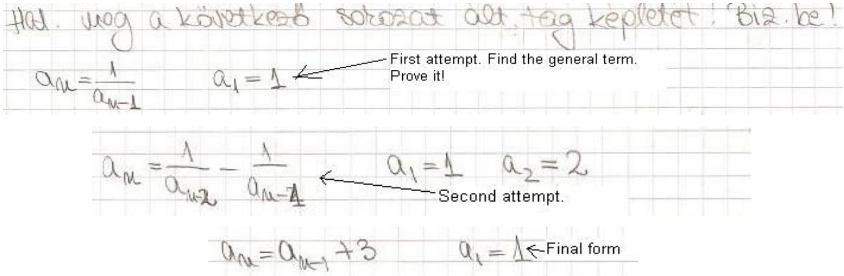

Figure 1. Example for the *Setup – Evaluation - Setup* loop; Two abandons before the proposal of final form

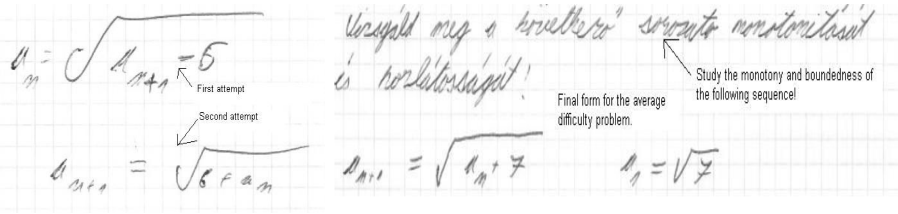

Figure 2. Example for the *Setup –Evaluation –Transformation - Evaluation* loop;

Since there it is no indication on how he got to this expression, we suppose that he wrote it down without any further verification. These expressions seem to be taken directly from memory (rather then "reconstructed" on base of some stored features of the problem). The student recalls that he saw something of this form and tries to write it down. He modifies the expression (it is not really clear why) and, in the end, gives a new one. Although at this moment we have a transformation and a kind of evaluation of the proposed expression, the problem posing process remains at the level of a complete novice. At this level it seems that memory and recall has a strong accent, meanwhile there is almost no trace of strategic, domain–specific or task specific knowledge.

The most complex problem posing process that we could identify from the work of the high school students shows the existence of a simple strategy: using a domain-specific rule in order to get new problems (figure 3, two different examples).

1. $\begin{cases} a_n = \sin n \quad \textit{bounded} \\ \lim \dfrac{n}{n^3+2} = 0 \\ A = \lim \dfrac{n}{n^2+2} \sin n \end{cases}$	"I want a sequence that goes to a constant. For this, I need a constant (4) and a sequence that tends to zero: $\begin{array}{l} y_n = 4 + \textit{something} \\ \qquad\quad \downarrow \\ \textit{example } 1/n,\ 1/n+1,\ldots \end{array}$. So, consider $y_n = 4 + \dfrac{2}{n+1}$."

Figure 3. The use of domain specific rule for posing a problem

In these examples we can see that the students realized that some of the domain-specific rules can be employed also for new constructions not only for solving problems. It remains an open issue whether they have tools and knowledge to construct an arbitrary sequence that fits into the restriction (like in above cases, constructing a sequence with given property – tending to zero) or they just rely again on memory retrieval. However, the student's process can be described in terms of *Setup-Formulation-Evaluation* stages and no cycles are present.

We state a preliminary conclusion: high school student's problem posing process is mostly a linear process that relies heavily on the retrieval of problems seen (and solved) before and, at most, uses a very limited domain specific knowledge (in form of domain-specific rules).

Olympiad participants' and teachers' problem posing process is more complex: they try to formulate more explicit criteria for assessment, make transformations and even try to build "interesting" problems. In some cases from an initial formulation new problems are proposed. Also, the last step, final assessment is present in some cases. The most interesting fact is that here we have an influence between the different behaviors: for example, evaluation can impose changes (and going back) to previous phases and cause a change in formulation or in transformations, and in cases, even the start over of the process. Final assessment has the role of overall assessment of the process and can influence future executions of similar tasks. We shall give some examples in order to illustrate some of the backward transition between different stages.

One of the teachers commented at the end of the first problem: "Meanwhile I was posing this problem I thought it is too easy and a recurrence relation would be more proper…" As second problem then he proposes the same one, but rewrites the generic terms as a recurrence relation. This is a clear example on how a process from *Final assessment* can affect the whole process in subsequent tasks (we have an "*Final assessment-Setup*" feedback). The influence can come in form of a reformulation of difficulty level or as the necessity to use more complex transformations.

Another teacher wrote the following comment at the end of the proposed problem: "Initially I thought of a sequence like 566555666655555…and wanted to give three questions about it such to have an easy, an average and a difficult one. It was difficult for me to find a formulation for difficult problem and I abandoned the idea". In this case, the *Formulation* phase affects the whole process and the lack of a satisfying question leads to start over ("*Formulation-Setup*" feedback).

A third teacher reports that he thought of combining three theorems in the problems (in particular, all three in the difficult problem – see figure below). In his words "For the difficult problem it was necessary to think more and I tried out different expressions until getting the final formulation" ("*Evaluation-Transformation*").

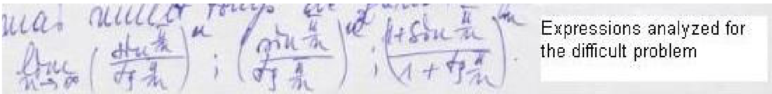

Figure 4. The expressions analyzed for the difficult problem (he selected the last one)

In figure 4 we see how several transformations were effectuated before the final formulation was given (*cyclic process on Transformation*). The same phenomena can be observed at some University students and Olympiad participants: often they start from a known results and re-describe the constituting elements (*repeated Transformation*). In some occasions, this get concretized in formulating several questions related to a problem.

It is important to underline that these participants also thought of the whole process and frequently expressed their feelings about it. These feelings, related to the task solving experience, are likely to influence future approaches to similar task.

In conclusion, novices' and experts' trajectories are different. We synthesize our results in figure 5. The *transformation* stage can miss at novices.

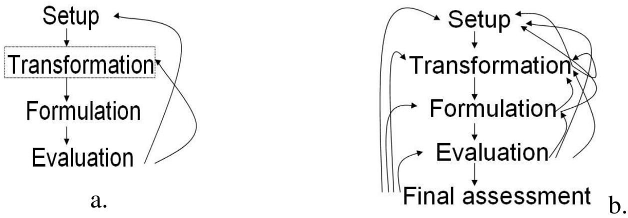

Figure 5. Trajectories of novices (a.) and experts (b.)

The distinct trajectories illustrate the differences between novices and experts and can serve as basis for teaching problem posing.

CONCLUSIONS AND DISCUSSION

In this article we discussed: 1) a model for classroom problem posing and 2) differences between novices and experts as it can be identified on the proposed model. The model consist of five stages: setup, transformation, formulation, evaluation and final assessment. Each stages contains several sub-processes. At a closer look the proposed model resembles to some extent the problem solving model proposed by Yimer and Ellerton (2006). Naturally, the question that arises is: do we need a model for problem posing or, in fact, problem posing is "covered" by a problem solving model. We argue that there is a need for a separate model, since the sub-processes involved in the stages are different and the knowledge has to be applied in a distinct way. In particular, when we look at textbook problems, most of them are well-formulated so one can build a completely and clearly defined problem space with well-identified start and end points. More than that, most often that space is quite common to most of students (like the one associated to the most common solution). However, during problem posing one has to invent the problem space even if there is a clear start and final state. The meaning of these two states can present huge variations when they appear in different problem spaces. Therefore, we argue that differences between problem posing and solving models occur especially at meta-cognitive level.

On other hand, the analysis we carried out shows that novices' and experts' problem posing defines different trajectories. By this way, the model helps also to follow the novice's advance in problem posing. When analyzing the relation of different stages with knowledge types involved in problem posing we get hints about how to teach problem posing and how to address to personalize instruction. However, these lines remain open for future investigation.

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THE CASE OF KARLA IN THE EXPERIMENTAL TEACHING OF FRACTIONS

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The case study we present is a part of a doctoral research, which was carried out in a fourth grade group (children aged 9) of a public elementary school. This research dealt with the notion of fraction and some of its meanings (part-whole relationship, intuitive quotient, measure, and rudiments of multiplicative operator) in the development of an experimental teaching. This study was composed of: initial questionnaire, teaching program, final questionnaire, and interviews. Karla was chosen as one of our cases because, in solving the questionnaires, she made use of various pictorial representations and she also showed an outstanding performance during the teaching sessions.

THEORETICAL FRAMEWORK

Freudenthal (1983) and Goffree (2000) propose that the education which is taught through the development of concepts accentuates the formal attitude of definitions. These authors suggest that teaching should be founded upon the experience of the student so that concepts do not remain isolated in his/her mind and they could be applied in solving everyday problems.

Freudenthal (1983) and Streefland (1991, 1993) establish the relationship that exists between the didactic approach and the mathematical reasoning regarding the teaching of fractions in elementary education. The lines of work of these researchers have been adopted in the designing of our methodological instruments as well as for their application in the development of the present doctoral study, which is partially described here.

Thomas Kieren has conducted studies on the construction of fractional numbers; his objective is to find out which is the genesis of such numbers. This author recognizes various intuitive constructs (measure, quotient, multiplicative operator and ratio) which serve as the basis for the constitution of the concepts which are relative to the fractions. Moreover, he identifies a fifth intuitive construct: the part-whole relationship which acts as a support for the construction of the four constructs previously mentioned (Kieren, 1983).

In the same way, Kieren (1980) considers the part-whole relationship as a whole (continuous or discrete) subdivided into equal parts, pinpointing as fundamental the relationship that exists between the whole and a number designed in parts. The fraction as a measure is recognized by him as the assignation of a number to a region or a magnitude (of one, two or three dimensions), the final product of the equitable

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partition of one unit. In relation to the fraction as a quotient, he regards it as the result of the division of one or several objects into a determined number of people or parts (Kieren, 1980, 1983, 1988, 1992). As for the fraction as operator, he identifies it as a multiplicative transformator of a set towards another equivalent set, this transformation may be thought too as the amplification or reduction of a geometric shape into another shape associated with the use of fractions. The fraction as a ratio is distinguished as the numerical comparison between two magnitudes (Kieren, 1980). In our teaching program, we treat the meanings of the fraction linked to the **part-whole relationship**, **measure**, **intuitive quotient**, and **multiplicative operator** proposed by this researcher.

Bergeron y Herscovics (1987) indicate that the quantification of the part-whole relationship leads us to differentiate three levels of the notion of **measure**, the **iterative measure** which implies the reiterative use of one unit of measure, when the quantity measured is an exact multiple of the unit of measure, the **fractional measure**, understood as the resulting of the equidivision of the whole or one of its parts and the **sub-unitary measure**, which refers to the fraction considered as a new unit to take more accurate measures.

On the other hand, Kieren (1993) presents a resourceful model for the comprehension of mathematics. This model comprises eight incrusted levels of knowledge or efficient actions, which are: *primitive making, make an image, have an image, notice property, formalize, observe, structure and invent.* For this study we took into account the first three levels that correspond to the most intuitive thought of the individual, that is to say, partition as a "primitive action", "make an image" as the problems of distribution that anticipate in the use of different partitions and fractions to represent the same quantity, and "have an image" as equivalent fractions generated by means of a given fraction.

Furthermore, we have taken into account the ideas produced by Kieren (1983, 1992, 1993) with reference to the construction of the fractional number on behalf of the student, and also those of Solé and Coll (1999) which refer to the way how the child learns the content which is intended to be taught. What has been mentioned above provided us a point of view about the cognitive aspects of the subjects in our study that we consider to carry out the experimental teaching and the case study.

RESEARCH PROBLEM

Various researchers (Figueras, 1988; Valdemoros, 1993, 1997, 2001; Pitkethly and Hunting, 1996; Perera, and Valdemoros, 2002) immersed in the field of educational studies in mathematics, indicate that fractions are one of the mathematical contents that causes difficulties both in their teaching and learning, in elementary levels. Together with these problematic issues, researchers such as Steencken y Maher (2003), Bulgar (2003), Nabor (2003) among others, have devoted their time to conduct teaching experiments on the subject of the knowledge of fractions with students of elementary education.

Taking into account the complexity that the construction of fractional numbers represents for children, we came out with a research question: *What teaching situations should be employed by the teacher in order to facilitate the learning of fractional numbers in children incorporated to the fourth grade of elementary education?*

Consequently, the problem of this research is directed to know ***how does a realistic and constructivist mathematical teaching, influence the child in the acquisition of the notions relative to the fraction.***

METHOD

The study of the case reported here is a part of a doctoral research, which was carried out in a fourth grade group (children aged 9) of a public elementary school in Mexico City, where we develop an experimental teaching with a constructivist approach that contains activities referred to different scenarios of the children's daily life. The methodological instruments for this experiment were: initial questionnaire, teaching program, final questionnaire and interviews. With the case study we concluded the field work, carrying out in its middle part a deep assessment of the teaching.

Methodological Instruments

The initial exploratory questionnaire consisted of 13 tasks related to the meanings of fraction: **part-whole relationship**, **quotient** (in relation to the sharing tasks), **measure** and the rudiments of the **multiplicative operator.**

The teaching program was composed of activities that revolve around various "scenarios" associated to the children's real life, in which there were several situations where fractions could be applied.

The final questionnaire comprised 13 analogous tasks to those items posed in the initial questionnaire and to those carried out in the teaching sessions. The purpose of this instrument was to assess the advances reached by students in the teaching program.

The interview was the fundamental methodological instrument of the case study. Its purpose was to delve into the relevant learning processes of the students interviewed. The interview was semi-structured (according to Cohen and Manion, 1990). This instrument consisted of six tasks, each one was directed to the application of one meaning of the fraction, the first and the third items were directed to the part-whole relationship, the second one to measure, the fourth and the fifth tasks to the quotient (sharing problems) and the sixth one to the multiplicative operator. Three children were interviewed for the case study; they were chosen according to the results obtained in the questionnaires and in the teaching sessions.

The Case of Karla

The case of Karla is the only one to be considered in this report. The mentioned girl was chosen for her evident ability to express her ideas through the use of pictograms[1], while solving problems.

In the following paragraphs we present the most relevant aspects that we identify in the qualitative analysis carried out to the methodological instruments applied to the case of Karla.

Initial Questionnaire

We observed that Karla possessed a scarce knowledge about fractions, since she only obtained 4 correct answers out of the 13 tasks comprised in this questionnaire.

Karla had trouble subdividing a continuous or discrete whole into a determined number of equal parts and she gave as an answer a natural number. The girl had difficulties identifying the whole and its resulting partition, in sharing tasks. Upon increasing to double or decreasing to half the sides of a given shape, Karla expanded and reduced the figures, ignoring the corresponding multiplicative operators; for this girl, expanding to double only meant "making it bigger", while decreasing to half, only meant "making it smaller".

The Teaching Program

In the sessions of the teaching program, Karla identified and wrote the fractions represented in a whole. In the same way, she was able to produce equivalent fractions when a fraction was given. In the sharing problems, she conveyed symbolic-arithmetic expressions of a fraction in order to name the part of the whole that was distributed. These tasks brought about in her the anticipation to the addition of fractions with equal denominator. The applications of multiplicative operator favored Karla's intuitive recognition of the multiplicative operator 1/2.

We noticed that Karla mentally reconstructed her reality, in the way she solved the tasks that composed the didactic "scenarios", emerging in her answers the connection and the use of various meanings of fraction (according to the constructs of Kieren, 1993). Also, the activities in the instructional "scenarios" brought about interaction, exchange of ideas, discussion of viewpoints among her and her classmates, but above all, the advance in her knowledge since the reflection upon the posed problems was caused.

On the other hand, we should emphasize that the representations given by Karla in her answers proved an accurate application of some of the semantic contents of fraction, such as partition, equivalence, the identification of a unit, the reconstruction of the whole, and the part-whole relationship.

Final Questionnaire

[1] Pictograms: They are drawings in which the qualitative and quantitative aspects are linked. Furthermore, it is possible to operate with pictograms (Valdemoros, 1993).

Karla showed noticeable progress in her knowledge regarding fractions, after having participated in the experimental teaching. This is proved by the 12 tasks that she solved correctly out of the 13 items comprised in the final questionnaire.

Karla overcame the difficulty she showed in the initial questionnaire, when subdividing a continuous or discrete whole into a determined number of equal parts. Similarly, we observed that she manifested progress in her knowledge when making equitable and exhaustive sharing, since in the initial questionnaire she presented distributions without concluding or carrying them out. She used symbolic-arithmetic expressions of the fraction to name the parts she obtained as the result of her strategies, knowledge that she did not exhibit in the initial questionnaire. Moreover, she worked intuitively with operators 1/2 and 1/3 to decrease the sides of a given shape, a process she did not carry out in the initial questionnaire.

RESULTS OF THE INTERVIEW OF KARLA

In the following paragraphs we present the analysis of the tasks involved in the interview, which revolved around the most important passages in the elaborations of Karla, both in the initial and final questionnaires.

In the task that involves the reconstruction of the whole out of one part, Karla recognized the complement of the fraction without difficulty. She reiterated the part (1/6), drawing it successively until she obtained the whole. In order to quantify the whole, the girl made use of a pictorial representation, as the basis to obtain symbolic-arithmetic expressions, which she added up; this strategy is shown in Figure 1.

On children's day celebration, 1/6 of the cake the teacher brought was left. Draw the missing part of the cake to complete it.

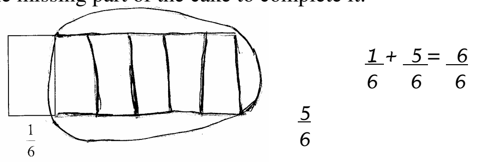


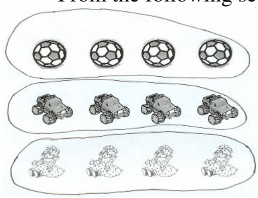
Figure 1. Pictorial representation of Karla in her reconstruction of the whole.

Regarding the task referred to the part-whole relationship, Karla developed the solution to the problem without difficulties, she subdivided the whole into three equal parts, recognizing the parts as thirds; she wrote 1/3 to answer the question of the task. Karla manifested that 1/3 is equals to 2/6 and, at the same time, it is equals to 4/12; it is noticeable that she manages the equivalence of fractions by multiplying the numerator and the denominator by 2, Figure 2 illustrates this situation.

For the sharing task, Karla succeeded in the way she faced it; she carried out partitions and equitable and exhaustive distributions, this led her to identify what was bound to be given to each child in the distribution. In order to solve this task, Karla made use of a pictogram in which she represented her strategy and expressed her answer through the development of an addition (see Figure 3).

In the task of measure (which involves covering a rectangular shape with a unit of measure of a quadrangular figure), we observed that Karla had difficulties covering the given surface with a preestablished unit of measure, this brought about that she obtained different fractions from those required in the task. That is to say, the limitation was that Karla did not show ability to cover the given surface with the unit of measure that was indicated.

From the following set of toys:

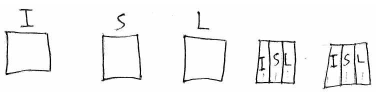


What fraction do balls represent? $\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$

balls and dolls $\frac{2}{3}$ *all toys* $\frac{3}{3}$

Figure 2. Answer of Karla to the part-whole task.

Ivan, Sergio and Luis distributed five sandwiches into equal parts among themselves.



How many sandwiches were given to each child? $1 + \frac{2}{3}$

Figure 3. The strategy Karla used when solving a sharing problem.

Perla drew a vase in exactly the same shape of the one below, but of a different size; she reduced to half each of the sides. Draw it yourself!

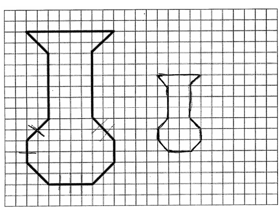


Figure 4. Karla's drawing relating to the multiplicative operator.

We noticed in Karla the ability she had to solve tasks related to the multiplicative operator. Upon reducing to half the sides of a shape, she used the algorithm of division, she counted the little squares of each side and she divided the result in two. In this activity, Karla intuitively admitted the existence of other fractional operators (1/3 and1/4), by expressing that we can divide each one of the magnitudes of the sides of a shape into 3 or 4 if we want to reduce it to 1/3 or 1/4 of its original size. Figure 4 illustrates this result.

CONCLUSIONS

We emphasize that Karla privileged pictorial referents in the development of the strategies she used to solve the given tasks. The representations of the problematic situations through the use of pictograms made it easy for her to obtain correct results in the activities.

Pictorial representations are appropriate resources that Karla used to clearly express her thoughts (according to what was proposed by Valdemoros, 1993, 1997).

In the same way, the pictorial representations used by Karla in order to contextualize the problematic situations in the "scenarios", meant a useful resource for her to strengthen the links between one and the other meaning of fractions. Furthermore, upon solving tasks, the connections she established between the different meanings of fraction made it possible for her the construction of mental images.

Globally, the experimental teaching and the didactic "scenarios" included in it enriched the knowledge of Karla about fractions, multiplying her intuitive resources, meanings and strategies.

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LESSON STUDY WITH A TWIST: RESEARCHING LESSON DESIGN BY STUDYING CLASSROOM IMPLEMENTATION

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In this paper we propose that 'lesson study' may be adapted from its primary use as a professional development strategy for use as a valuable research strategy, especially to identify principles of good lesson design. We report on a project undertaken in two Australian schools where lesson study research was used to investigate the design of a lesson which aimed to access some of the pedagogical affordances of new technology (TI-Nspire). An example of principles for use of multiple representations is given. Using lesson study as a research strategy allowed the collection of rich data suitable for the thematic analysis of lesson design and also secondary analysis for other purposes, from all stake holders and under varying conditions. Professional development was a valuable outcome for participating teachers.

LESSON STUDY

Changing classroom teaching to take advantage of potential opportunities offered by some new approach or technology requires the identification of principles of lesson design that capitalise on its affordances while minimising constraints. This is a challenge that deserves more than a trial and error approach. As will be discussed below, the careful analysis of the presentation of a lesson through *lesson study* has been established as an effective method for professional development of teachers. This paper reports on the potential of harnessing *lesson study* as a fresh strategy for research as well as for professional development.

The focus for this study was the design of lessons that take advantage of technology that provides graphing, symbolic algebra and dynamic geometry capabilities, in this case Texas Instruments' Nspire. After two decades of technological development, the latest generation provides both hand-held and computer software for individual student work and classroom presentation, along with a mathematically able document system, that provides students with instructions, dynamic diagrams etc. We therefore wanted to consider how to gain pedagogical value from such new technologies. We see this as an important research question, given the extent of investment of many school systems in this equipment, its obvious potential, contrasted with the many reports of difficulties that school systems experience in realising this potential.

In the sections that follow we briefly discuss *lesson study* and our adaption of it as a research strategy; the details of one study using this approach; brief results of the study in terms of the insights gained by using a *lesson study* approach; and finally some conclusions about using lesson study as a research strategy.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 369-376. Thessaloniki, Greece: PME.

Lesson study has its origins in Japan where it has been used with the primary goal of professional development for teachers. For comprehensive details of Japanese lesson study see for example Isoda, Stephens, Ohara & Miyakawa (2007). In traditional lesson study a group of teachers plan a lesson; the lesson is taught by one of this group of teachers with the others from the school observing, perhaps with some visitors. Then, at a debriefing, the lesson design and teaching practice is analysed in detail so that the lesson can be revised and the teaching methods can be refined. This process is repeated over an extended period of time with the goal of producing quality lesson plans and improvement in teachers' understanding of student learning.

In recent years lesson study has been used to support professional development in many countries. See for example APEC – Tsukuba (no date). Lesson study as a professional development process has been researched in settings outside of Japan, for example: Indonesia (Marsigit, 2007) Australia and Malaysia (White, Lim & Chiew, 2005; White & Southwell, 2003), and the Chicago Lesson Study Group in USA. In each case the researchers have observed the need for some cultural adaption to make lesson study work. Marsigit reports on teachers initially observing a discussing a video of a lesson taught elsewhere rather than their own while White, Lim and Chiew commented on the importance of participants in Australia being volunteers and the importance of active support of the school leadership in Malaysia.

Our adaption of lesson study also focuses on the discussion and development of a single lesson but in our case, in addition to the goal of professional development, we wished to use lesson study as a strategy for researching design principles for lessons using TI-Nspire (and similar technologies). In-school lesson study clearly fits into an action research paradigm with local goals, but we wished to use it as a setting for traditional research to develop principles of lesson design robust beyond the school setting and with traditional standards of research evidence. We hypothesised that the lesson study research approach would allow input (direct or indirect) to researching the design process from all stake holders in a classroom setting. At the same time it would offer teachers the opportunity to reflect on their own practice and so gain professionally from participation in this research.

THIS STUDY

In the study reported here, we chose to focus on exploring the principles of design for a lesson that used the dynamic simulation of a real situation, linked many representations, and used the document facility. Pierce and Stacey (submitted) have developed a framework that lists 10 types of pedagogical opportunities afforded by mathematical analysis software. This study addresses the opportunities of simulation and multiple representations, and is part of a research program to systematically study the design of lessons exploiting the other pedagogical opportunities.

As part of the cultural adaption, it was decided that the research team would write the lesson. This was for two reasons: the project teachers were new to the use of TI-Nspire and further, the critique of the lesson in the debriefing would be strongly focussed on the lesson and would not be seen as criticism of the teachers themselves. At the request of our project schools we planned a lesson to be taught at the end of a unit of work on quadratic functions for year 10 (16 year old) students. The introduction to the lesson is shown in Figure 1. The lesson plans in their final version are available from http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/.

Marina owns a fish shop, and wants to create a new sign above the shop. She likes geometric ideas, and thinks a square with a triangle looks like a fish. Marina draws a square with a horizontal diagonal, starting from the left wall of her shop. This makes the body. Then she extends two sides of the square as far as the right wall of the shop. This makes the tail. The shop is 10 metres wide. Marina soon realises that there is more than one possible configuration (see above) and wonders, "What is the best possible sign?" She uses mathematics to investigate.

Figure 1. Introduction to Marina's fish sign lesson

In the first lesson version, students used both pen-and-paper measurements, calculations and algebra, and TI-Nspire assisted mathematics to explore the relationship between the length of the body of the fish and the total area. The students used multiple representations in order to understand the various mathematical aspects of the problem (Figure 2). Next students were asked to solve an optimisation problem: *Marina wishes to illuminate the total area of the fish but wishes to minimise her use of power* (i.e. minimise area). Finally students found the relationship between body length and total area for fish signs of this given shape with arbitrary total width.

THE LESSON STUDY PROCESS AND DATA COLLECTION

Two schools volunteered for the lesson study. The schools were principally interested in the professional development opportunities it offered. Each phase of the lesson study process was taken as an opportunity for research data collection. First the researchers drafted the lesson and sent it to the two potential Cycle 1 teachers at School 1 for feedback. Researchers met with these teachers to discuss their feedback on the lesson and this discussion was audio-recorded. The lesson was revised and sent to the teacher who volunteered teach the lesson with observers present.

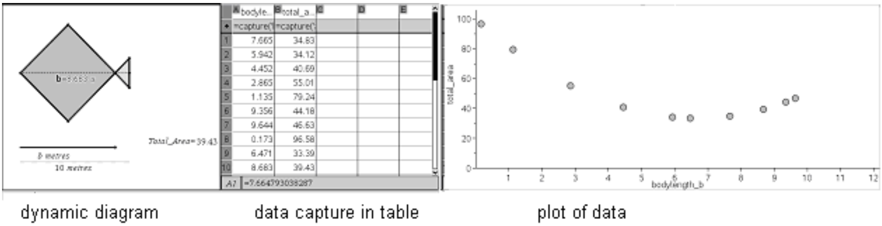

Figure 2. Dynamically linked multiple representations

Lesson Cycle 1 started with the presenting teacher preparing to teach the lesson. This led to some clarifying phone and email conversations which were recorded. At the appointed time the presenting teacher taught the lesson, as set out in the lesson plan, in front of the other 4 year 10 teachers, the research team and an international visitor. This lesson was video and audio-recorded, digital photos were taken of students' work including calculator screens and of the teacher's board work. Following the class all participants (including students) were asked to answer the following questions. Most space was allocated for answering the last question.

- What did you think was the key point of this lesson?
- What new mathematical ideas do you think students learnt?
- What new use of TI-Nspire do you think students learnt during this lesson?
- What do you think was the best feature of the lesson?
- How do you think this lesson could be improved?

For the teachers and researchers these survey items set the direction for the focus group conducted after the lesson, over a simple lunch. This discussion was audio-recorded and later transcribed. Initially only 2 teachers at this school had volunteered to teach this lesson in front of us and colleagues, but following this first discussion of the lesson, all of the other year 10 teachers volunteered. Each time the lesson was taught at least two observers were present (including other teachers when their timetables allowed), and an audio-recording, digital photos and observation notes were collected. Following each lesson, the presenting teacher was interviewed. Finally, we held another whole group meeting to consolidate these reflections.

The *redesign phase* involved thematic analysis of the data through reading, re-reading, studying of digital images and further discussions among the research team as to how to solve the problems that emerged in the original design. The revised lesson was then sent to the presenting teacher at School 2 for Cycle 2, who then met with the research team to discuss the revised lesson. This was also audio-recorded.

Lesson Cycle 2 followed the same pattern as Cycle 1 and, in addition to teachers from the Cycle 2 school and researchers, two teachers from School 1 also chose to attend the first and main teaching of the revised lesson. Data were collected from the same sources as Cycle 1 with the addition of a 12 Likert scale item survey for students to try to identify the parts of the lesson and technology use which they found hard or easy. In total the lesson (in initial and revised form) was observed 9 times.

The lesson study format provided rich data collected from multiple sources which showed the lesson from the perspective of students, teachers and researchers (lesson writers). Focus group discussions and presenting teachers' interviews gave sufficient information for analysis, reworking of the lesson and evaluation of the revised lesson.

FINDINGS ON PROFESSIONAL DEVELOPMENT

Since the major goal of the collaborating schools is staff professional development, we report briefly on this. As noted in other studies, some cultural adaptation was required. Having the research team write the lesson and strongly focussing the follow up discussions on improving the lesson design rather than the teaching made the teachers feel comfortable to participate first in critiquing and then in teaching the lesson. This was clearly evidenced when, following the first discussion of the lesson at School 1 and the first teaching of the lesson at School 2, more teachers volunteered to have their teaching of the lesson observed and discussed. We began with 3 volunteers, and ended with 9. This was a major step in the context of Australian secondary schools, where typically teaching has occurred behind closed doors.

Effective professional development did occur. In particular teachers learnt about teaching with technology by observing each other and sharing ideas. This was evident in the quality of technology use observed in the lessons of teachers who had initially not thought they had the technology expertise required. Further evidence of professional development success is that the project schools have committed to a second year of lesson study with the research team. They have gone ahead, although they now know from experience that participation places high demands on the participating teachers and in particular the school coordinator.

FINDINGS ON LESSON STUDY AS A RESEARCH STRATEGY

We have noted above that lesson study as a research strategy enabled the collection of very rich data, from the perspectives of all categories of participants, and enabled triangulation of results. Opinions often varied (there is after all no one right way to design a lesson), but there was substantial agreement on many issues. Although we observed only one 'lesson' being taught, the circumstances varied more than might have been expected given that we only used two schools: some classes were girls only, others boys only; some high ability and others mainstream classes. The conditions for teaching with technology also varied considerably due to teacher preference and equipment actually present in different classrooms. Nspire was used with and without computer data projector, with and without teachers' calculator view-screen; with and without interactive whiteboard. In one case, there was only the handheld with view-screen projected on a shiny whiteboard. In another, the teacher used an interactive white board, computer display and calculator viewscreen. These varying conditions have permitted deep analysis of the lesson and highlighted issues that are common across the different teaching contexts. This should lead to the design of lesson plans, student worksheets and TI-Nspire files that are robust for use with minimal adaption in a wide range of contexts.

The lesson study strategy allowed us to study particular aspects that were deliberately built into the lesson design, for example in this case, the use of multiple representations. While the data were suitable for thematic analysis of the lesson design, our guest observers also noted it is suitable for re-analysis for a range of diverse purposes. Given the cost of collecting data, the diversity and potential for secondary analysis of data we were able to collect during lesson study makes 'lesson study' a valuable research strategy. To illustrate the type of results obtained, we briefly describe below three findings that inform the design of lessons taking advantage of easy access to multiple representations.

EXAMPLE RESEARCH RESULTS FROM THE STUDY LESSON

The lesson was designed to allow students to explore many different mathematical representations of the problem. It was envisaged that the students would move quickly through initial activities using practical measuring, dynamic diagrams and data collection, and that working with these simpler representations would support students' understanding of the abstract symbolic work that followed. However this lesson feature needed to be handled much more carefully than we did in Cycle 1. Some students found difficulties with early activities and this presented the teacher with the dilemma of either dealing with these problems or moving on. Some of the teachers we observed saw great value in particular representations (for example measuring a pen-and-paper scale diagram or analysing a graph of *total area* versus *body length)*. They spent extra time on these representations and had little or no time left for the symbolic algebra, which we saw as an essential part of the lesson. In addition working with multiple representations required working with different 'applications' in the software each requiring some special technical as well as mathematical knowledge.

The number of representations needed to be restricted and the lesson plan needed to give clearer guidance regarding the priorities for student learning in the lesson.

Nearly all participants agreed that using multiple representations also caused students to lose motivation in the Cycle 1 version. We intended that students should explore the problem from a number of perspectives in order to gain a rich understanding of the mathematics involved, but when students sensed they had 'solved' the problem (e.g. by dragging a dynamic diagram, or putting a regression curve through collected data) they were reluctant to go further. An experienced researcher/observer asked:

> …What's the motivation for the algebra? I mean, if they've done this regression and got this answer, there's no motivation for the algebra.

In the Cycle 2 version, each representation was given a different purpose, and this solved the motivation problem. For example, students used the dynamic fish diagram without any measurements only to make a guess from visual estimation of whether (and then how) the *total area* of the fish sign varied with the *body length*. In this case lesson study highlighted the value of asking open questions.

The Cycle 1 lesson observations also highlighted issues of cognitive load that arise when using multiple representations. Dealing with different screens for different representations and having the 'clutter' of several features and variable names on a single screen created extraneous cognitive load. As mathematical novices students' attention was not immediately attracted to what their teacher saw as important but rather they were distracted by other features. A presenting teacher reflected:

> The information's on the screen. Right? To you and I, the answer's there, it's just there. … it staggers me that it's so hard for a lot of kids to see that. …

The Cycle 2 files which 'hid' all information not of immediate concern to the student were much more successful. Seufert, Jänsen & Brünken, (2007) showed that working with multiple representations may create extra cognitive load and so impede learning and Sweller, van Merrienboer and Paas (1998) pointed out that the cognitive load involved in an activity will not be the same for an expert as a novice. These warnings have not been prominent in the algebra education literature. Lesson study enabled us to identify the sources of extraneous cognitive load and so improve the lesson design.

CONCLUSIONS

The aim of this paper has been to discuss the use of lesson study as a research tool. As others have found, some cultural adaptation was required to undertake lesson study within the schools. Teachers' increasing participation indicated that the adaptation was successful. The schools' desire for professional development was met, which has led to their requests for involvement in future lesson study research. Lesson study research proved to provide rich data of a range of types, under surprisingly varied conditions (given only two schools were involved) and from all perspectives, including students. The study reported above indicates that the intensive collection of data provides sufficient information to progress the improvement of a lesson through a design cycle process. More importantly, the study of the one lesson was a test-bed for lesson design principles of wider applicability. The present paper has given, as an example, findings about principles for using multiple representations. This has great potential, extensively discussed in the research literature, but we identified that potential loss of focus, loss of motivation and cognitive overload needed to be addressed. The second cycle of lesson study tested designs to overcome these issues. We commend lesson study research as a fresh approach to research with significant benefits to researchers and teachers.

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MATHEMATICAL CREATIVITY AND COGNITIVE STYLES

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In our increasingly technological world, it is more and more important to encourage students to develop their abilities to reason and think creatively, especially in mathematics. The aim of this study is to investigate whether individuals' cognitive styles are related to their mathematical creativity. Mathematical creativity is measured in terms of fluency, flexibility and originality whereas individuals' cognitive styles are measured in terms of object imagery, spatial imagery and verbal. The study was conducted with 96 prospective elementary school teachers. The results suggest that spatial imagery cognitive style is related to mathematical fluency, flexibility and originality. On the other hand, object imagery cognitive style is negatively related to mathematical originality and verbal cognitive style is negatively related to mathematical flexibility.

INTRODUCTION

The need for individuals who can provide innovative solutions to our problems becomes particularly great in today's world. Creativity is undoubtedly the result of a complex process, one not easily broken down for more manageable examination. Yet, to break it down we have to, and must, understand its internal structure.

Research in the field of creativity has revolved around three orientations: study of the products of creativity, the creative personality and the cognitive processes involved in the creative arts. The orientation of this paper attempts to examine not the products of creativity or the creative individual, but the cognitive variables necessary for the creative process. It is the purpose of this paper to demonstrate that the creative process might be related to various cognitive styles such as spatial imagery, object imagery and verbal (Blazhenkova & Kozhevnikov, in press).

THEORETICAL FRAMEWORK

Creativity

Creativity is a complex construct and as such, it has been defined in several ways. A widely accepted definition of creativity which is used in this study is the one provided by Torrance (1994). Torrance (1974) defines fluency, flexibility and originality as the main characteristics of creative individuals. Fluency is the ability of producing many ideas, while flexibility refers to the number, the degree and the focus of approaches that is observed in a solution. Originality refers to the possibility of holding extraordinary, new and unique ideas (Gil, Ben-Zvi & Apel, 2008).

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Creativity in mathematics is often looked at as the exclusive domain of professional mathematicians (Shriraman, 2008). Few studies have examined creativity in classrooms, particularly in its manifestation in individuals with different cognitive characteristics. It is only in the last ten years that there has been a renewed interest in the phenomenon of creativity. In this paper the notion of mathematical creativity is developed from literature in psychology. In particular the dimensions of fluency, flexibility, and originality are used to construct a framework for studying and assessing mathematical creativity of prospective elementary school teachers.

Cognitive styles

The construct of cognitive style has been widely researched in psychology (for a review, see Rayner & Riding, 1997). It can be defined as "an individual's characteristic and consistent approach to organising and processing information" (Tennant, cited in Riding, 1997). Although there appear to be various conceptualisations of cognitive styles (for a classification, see Sternberg & Grigorenko, 1997), most of the researchers agree that cognitive style is a construct which is relatively stable over domain and time.

In the field of mathematics education, the verbaliser/imager distinction was the one that attracted most attention. However, it needs to be noted that this distinction was not referred to as "cognitive style" but as preferred type/mode of thinking, or type of students (Kruteskii, 1976; Lean & Clements, 1981; Presmeg, 1986a, 1986b; Breen, 1997; Pitta & Gray, 1999). The broad idea documented by a number of researchers was that visual-spatial processes are distinct from verbal processes and that mathematics involves not only verbal processes but also visual reasoning (Presmeg, 1986; Sfard, 1991).

Recently, Blazhenkova and Kozhevnikov (in press) suggested that there exist two distinct imagery subsystems that help individuals process visual information in different ways. Specifically, they suggest that there is an object imagery system and a spatial imagery system. The object imagery system processes the "visual appearance of objects and scenes in terms of their shape, color information and texture", while the spatial imagery system processes "object location, movement, spatial relationships and transformations and other spatial attributes of processing" (p. 1475). Thus, recent research identified two distinct types of visualizers. Object visualizers who use imagery to construct images of objects and process visual information globally and holistically as whole perceptual objects and spatial visualizers who use imagery to represent spatial relations, make complex spatial transformations and process visual images analytically and sequentially, part-by-part (Kozhevnikov, Kosslyn, & Shephard, 2005). Rosenberg (1987) and Kozhevnikov et al. (2005) also found that object imagery can be beneficial for visual art tasks, whereas Kozhevnikov, Hegarty and Mayer (2002) found that spatial imagery can be beneficial for physics, mechanical engineering tasks, technical drawing and mathematics.

Purpose of the study

The realm of creative thinking offers an opportunity for examining the performance and competence aspects of different cognitive styles that "have potentially profound implications for learning and the structuring of knowledge" (Messick, 1994, p. 129). Thus, the purpose of the present study was to investigate whether mathematical creativity is related to specific cognitive style, namely to verbal, spatial imagery, and object imagery.

METHOD

Participants and procedure

Ninety six prospective teachers participated in the study. A mathematical creativity test and a self-report cognitive style questionnaire were administered to participants at the same day. The mathematical creativity test was used to measure participants' mathematical creative abilities in area, shape, pattern, problem solving and number. The questionnaire, which was a translation of the computerised Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) (Blazhenkova & Kozhevnikov, in press), measured participants' cognitive styles.

The mathematical creativity test

The mathematical creativity test included seven tasks: two on the area of polygons, one on shapes (triangles), two on shape patterns, one on problem posing and one on reasoning with numbers. This variety of tasks was purposeful in order to provide a balanced test towards the three cognitive styles under investigation. It was hypothesised that spatial visualisers might perform better in spatial tasks such as those on polygon areas and shapes. Object visualisers might be more successful with pattern tasks, whereas verbalisers might find problem posing and reasoning with numbers easier. In the area tasks prospective teachers were provided with a square and an L-shape and requested to divide them in a number of shapes having the same area. The shape task presented a triangle with many embedded triangles inside it and participants were asked to state the number of triangles that they could see (Spatial imagery task). One of the pattern tasks asked participants to use two different types of triangles and construct as many different kinds of patterns as they could, whereas, the other pattern task required students to use colours to create a pattern in a diagram with overlapping shapes (Object imagery task). There were also two tasks that called upon participants' verbal abilities. One of them asked prospective teachers to write three different word problems which could be solved with the operation $51 \div 4$, but would have three different correct responses: (a) $12\frac{3}{4}$, (b) 13 and (c) 12. In the other verbal task the numbers 2, 3, 4, 5, 7, 9, 10, 15, 21, 25, 28, 49 were presented. Participants were asked to construct groups of four numbers with a common characteristic and name these groups (Verbal tasks). It needs to be stressed that all of the above tasks required participants to find multiple solutions, a characteristic of creative mathematical activity (Leikin & Lev, 2007).

To measure mathematical creativity three dimensions were evaluated: fluency (number of correct responses), flexibility (number of different types of responses or categories of responses) and originality (extraordinary, new and unique responses) (Torrance, 1994). Every response of the mathematical creativity test was given a score from 0 to 4 for each one of these three dimensions (fluency, flexibility and originality).

The cognitive style questionnaire

The self-report cognitive style questionnaire (OSIVQ) (Blazhenkova & Kozhevnikov, in press) was used to assess the individual differences in spatial imagery, object imagery and verbal cognitive style. Prospective teachers were asked to read 45 statements and rate each item on a 5-point Likert scale with 1 indicating total disagreement and 5 total agreement. Ratings 2 to 4 indicated intermediate degrees of agreement/disagreement. Fifteen of the items measured object imagery preference and experiences, fifteen items measured spatial imagery preference and experiences and fifteen items measured verbal preference and experiences. These items addressed qualitative characteristics of the images (*My images are colourful and bright*), preferences to specific types of visual images or verbal thinking (*When remembering a scene, I use verbal descriptions rather than mental pictures*), habitual and learning preferences (*I usually do not try to visualize or sketch diagrams when reading a textbook*), professional preferences (*If I were asked to choose among engineering professions or visual arts I would choose visual arts*) and individuals' estimations of their abilities in using spatial or object imagery or verbal processing *(My verbal skills are excellent*).

For each participant, the fifteen item ratings for each factor were averaged to create object imagery, spatial imagery and verbal scale scores.

RESULTS

The main purpose of the study was to learn more about the relations between cognitive styles and creativity in mathematics. In this study, we used the object-spatial-verbal cognitive style dimension as predictor variable for students' creativity. Specifically, through multiple regression analyses with criterion (dependent) variables the mathematical creativity dimensions of fluency, flexibility and originality, and predictors (independent) the spatial imagery, object imagery and verbal cognitive styles.

The correlations among students' cognitive styles and their performance in fluency, flexibility, and originality are presented in Table 1. As can be seen from Table 1, the spatial imagery cognitive style significantly correlated with students' creative abilities in fluency, flexibility and originality, while object imagery cognitive style did not correlate with any of the components of creativity. Verbal cognitive ability negatively correlated with flexibility (r=-0.292), i.e., the ability of students to find a number of different types of responses or categories of responses. We furthered examined the nature of these correlations and non correlations between cognitive styles and the components of creativity by analysing the data with multiple regressions.

	Fluency	Flexibility	Originality
Spatial Imagery Cognitive Style	.199*	.321*	.208*
Object Imagery Cognitive Style	-.053	-.113	-.130
Verbal Cognitive Style	-.078	-.292*	-.135

* Correlation is significant at the 0.05 level (2-tailed).

Table 1: Correlations among creativity and spatial imagery, object imagery and verbal cognitive styles.

Table 2 presents the results of the multiple regressions. Two distinct characteristics arise from the analysis of data. First, it seems that object imagery and verbal cognitive styles are negatively related to creativity, since all standardized regression coefficients are negative numbers. This trend appears systematically in all creativity dimensions (fluency, flexibility, and originality) as shown by the betas in Table 2. However, not all the negative relations are statistically significant. Object imagery cognitive style can somewhat predict the originality of students in a statistically significant manner and it can explain more than 15% of the variance in originality. However, the negative sign of beta (b=-0.293, p=.016) means that as the object imagery cognitive style of prospective teachers' increases, their performance in originality decreases. In the same way we can interpret the negative relation between the verbal dimension and flexibility (b=-0.220, p=.031). As prospective teachers' verbal cognitive style increases, their flexibility in solving mathematical tasks decreases in a statistically significant way.

The second important observation of the data analysis refers to the relation between spatial imagery cognitive style and mathematical creativity. Spatial imagery cognitive style is a statistically significant predictor of prospective teachers' abilities in fluency, flexibility and originality (see betas in Table 2), and explains a large proportion of variance (more than 30%) in creativity. This means that as far as prospective teachers tend to prefer the spatial visualization processing, their performance in fluency, flexibility and originality is much higher than those who seem to prefer the verbal and object visualization processing of information. This result confirms the importance of distinguishing between spatial and object imagery as it was noted by Blazhenkova and Kozhevnikov (in press).

	Fluency		Flexibility		Originality	
	b	*p*	*b*	*p*	*b*	*p*
Spatial Imagery Cognitive Style	.195	.050*	.226	.027*	.297	.004*

Object Imagery Cognitive Style	-.186	.070	-.140	.157	-.293	.016*
Verbal Cognitive Style	-.104	.317	-.220	.031*	-.082	.412

* Statistical significance p<0.05

Table 2: Multiple regression analyses with dependent variables fluency, flexibility and originality, and independent variables spatial imagery, object imagery and verbal cognitive styles.

DISCUSSION

A number of studies have been carried out to understand the role of visual and verbal thinking in mathematics (Kruteskii, 1976; Lean & Clements, 1981; Presmeg, 1986a, 1986b, 2006; Breen, 1997). However, most studies investigated the effects of visual imagery as a unitary, undifferentiated construct. This study provides evidence that different types of visual imagery, namely object and spatial imagery, may have significantly different effects on mathematical creativity. This result may be an explanation why previous studies found no relation between the use of visual representations and problem solving (Lean & Clements, 1981; Presmeg, 1986a, 2006).

The results of this study showed that spatial imagery cognitive style is a significant predictor of students' mathematical creativity and its dimensions: mathematical fluency, flexibility and originality. This is in accord with previous research by Kozhevnikov et al. (2002) who found that spatial imagery is beneficial for mathematics. On the other hand, object imagery cognitive style appeared to have a negative relation to originality and verbal cognitive style a negative relation to flexibility.

It is likely that spatial visualizers were able to manipulate abstract, schematic, spatial images in a more analytic and step-by-step way, whereas object visualizers were not able to suppress concrete, pictorial information which were irrelevant to the solution of the mathematical problems. Object visualizers may have tended to see the problems in a more holistic, iconic manner which was inhibiting their mathematical creativity and especially the creation of original and novel mathematical solutions. This finding is consistent to previous studies (for example, Gray & Pitta, 1999) who found that concrete, vivid images may be persistent during mathematical operations and tend to inhibit mathematical solutions. Similarly, there may have been cases where verbal processing interfered, resulting to the obstruction of mathematical flexibility.

From the results obtained in the present study, it cannot be generalised that object imagery or verbal cognitive styles are obstacles to all mathematical domains and tasks. It is possible that object imagery or verbal cognitive style may be facilitating certain mathematical domains and tasks. For instance, it is possible that object imagery or verbal cognitive style may enhance the learning of mnemonic rules and mathematical formulas. However, these hypotheses need to be tested.

Overall, the results of the current study suggest that it is helpful to know students' cognitive styles since it can be useful to develop appropriate teaching materials and methods in the mathematics classroom. The question remains however, whether those who are creative in mathematics have a spatial imagery preference as a result of experience or whether they reflect inborn abilities. Therefore, an interesting direction for future research is to investigate whether students can be trained to use their spatial visualization. It might be that training may enhance spatial imagery and subsequently facilitate mathematical creativity.

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LEVELS OF SOPHISTICATION IN REPRESENTING 3D SHAPES

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This paper focuses on the identification of students' levels of sophistication in representing 3D shapes. Unstructured interviews were conducted in order to obtain data from forty 5th to 9th grade students in Cyprus. The results of the study showed that students have difficulties in representing 3D shapes, in interpreting 2D representations of 3D shapes, in building 3D objects based on plane representations and in mentally manipulating 3D shapes. In addition, the analysis of the data revealed that students' representational processes and thinking are developed through four levels of sophistication.

INTRODUCTION

The representation of three dimensional (3D) shapes provides opportunities for learners not only to develop spatial awareness, geometrical intuition and the ability to visualise, but also to develop knowledge and understanding of, and ability to use, geometrical properties and theorems (Jones & Mooney, 2003). The Principles and Standards of the National Council of Teachers of Mathematics (2000) recommends that two dimensional (2D) and 3D spatial visualization and representation are core skills that all students should develop to become experienced in using a variety of representations for three-dimensional shapes. Gutierrez (1996) asserts that it is essential for children to acquire and develop abilities allowing them to handle different 2D representations of 3D objects. A single type of representation can hardly represent a complete real-world object. Thus, switching between 2D and 3D representations is usually very important (Ho & Eastman, 2005). In an attempt to develop a coherent framework of students' representation processes and thinking in 3D geometry, the present describes 5th to 9th grade students' levels of sophistication in manipulating different representational modes of 3D objects.

THEORETICAL CONSIDERATIONS

Students' ability to manipulate different representation modes of 3D objects is one of the most substantial ingredients to be successful in 3D geometry tasks. A 3D object can be represented in a 2D drawing or in a 3D model. A model is considered as a close representation of a 3D object because it resembles the geometric object. Parzysz (1988) asserts that a drawing is a distant representation because there is a loss of information when moving from 3D space to plane representations. This loss of information can have various causes because under this projection each point of the plane corresponds to infinitely many points of the space (Bako, 2003). Thus, it often becomes difficult to guess, from the drawing, some properties of the 3D object. Some

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properties of the representation can be understood if the reader tries to visualise non displayed parts of the object (Parzysz, 1988). However, plane representations are the most frequent type of representation modes used to represent 3D geometrical objects in school textbooks (Gutierrez, 1992). Even though students in grades 6 through 8 are familiar with a variety of types of representation modes, they have great difficulty in successfully communicating spatial information (Ben-Chaim, Lappan, & Houang, 1989). Bishop (1983) and Parzysz (1988) point out that the representation of a 3D object by means of a 2D figure demands considerable conventionalizing which is not trivial and not learned in traditional school curriculum. Students are asked to use conventionalisation rules without any attempt to directly teach conventions. Students' ability to represent 3D objects is directly connected with the difficulties of the various representation modes and the mental representations involved (Parzysz, 1988). Researchers (Clements & Battista, 1992; Parzysz, 1988; Presmeg, 2006) studied these difficulties and concluded that there is a need to explicitly interpret and utilise conventions in drawing 3D objects. Otherwise, students may misread a drawing and do not understand whether it represents a 2D or a 3D object.

Gutierrez (1996) describes four types of plane representations that are most frequently used in 3D geometry teaching: perspective, layers, orthogonal or side views, coded orthogonal and isometric. Perspective type of representation is the kind of drawing naturally made by children. In this type of representation, lines which are parallel in real life are drawn to intersect at the vanishing point. A layer representation is made of several horizontal sections of the solid at some particular heights, to present the variations from bottom to top. An orthogonal type of representation consists of three views which are displayed separately: the front view, the side view and the top view. Coded orthogonal representation is an enriched type of orthogonal view with information about some characteristics of the solid. Finally, isometric view is a type of parallel projection in which the three Cartesian axes form angles of 120 degrees. There are important differences in the difficulty between building solids and drawing their plane representations based on the type of representation. For example, Gutierrez's study (1996) showed that drawing side views is easier than building from side views, but drawing isometric projections is more difficult than building from an isometric representation.

Students and adults have great difficulties in drawing 3D objects, such as representing parallel and perpendicular lines in space (Mitchelmore, 1980). For example, in the perspective and isometric views right angles are drawn differently. In addition, students have to combine into a unified mental image the different perspectives of an object. Initially, a student draws a rough sketch based on the front view of the object and then modifies the sketch based on other perspective of the object.

THE PRESENT STUDY

The present study tries to add to the research literature on students' ability. Through observations and insights into students' 3D geometry thinking, this study attempts to provide a starting point for better understanding students' levels of sophistication in representing 3D shapes. In this study, by the term "representing a 3D shape", we refer to the interpretation of a 2D representation of a 3D shape, the building of a 3D object based on a plane representation and the drawing of a 2D representation of a 3D shape.

Subjects

Forty students from fifth to ninth grade at two primary and two middle schools in urban districts in Cyprus formed the population. The school population is representative of a broad spectrum socioeconomic background. Eight children, from each grade, were randomly selected and served as case studies in this research.

The Tasks - Data Collection and Procedures

Students were involved in a number of tasks that referred to the building of a 3D object based on a 2D type of representation, to the drawing of a 2D representation of a cube, to the interpretation of various forms of 2D representations of 3D objects and to the translation of one type of representation to another one. Due to space limitations, in the present study we present students' work in two tasks.

Task 1: The orthogonal view of a 3D object was presented to students (see Figure 1) and they were asked to build a 3D object by using multilink cubes. Before working with the cubes, we asked them to visualise the object and describe it. While working with the cubes, students had to explain their actions. Finally, after the completion of the construction, students had to decide whether they could remove one cube from the construction without altering the orthogonal view of the object.

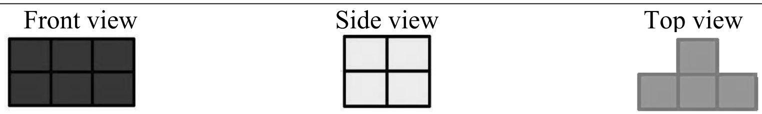

Figure 1: The orthogonal view of the 3D object

Task 2: A plastic cube was placed in front of students and they were asked to observe it and draw it. Based on students' drawings, appropriate questions were raised. For example, in the case that a student drew the cube in a transparent view, we asked him/her whether he/she could match the edges that appeared in the drawing with the edges of the concrete cube. In addition, students had to identify parallel and perpendicular edges in the drawing and explain why some edges that intersect perpendicularly appear in a different way in the drawing.

We videotaped students' interviews to capture their actions as they discussed their work. The setting was informal with students being able to analyse and build constructions that they thought would help them without time limits. Unstructured interviews were used in order to keep ourselves open to noticing hidden structures and processes in students' thinking. Each interview lasted around 30 minutes.

RESULTS

A qualitative interpretive (Miles & Huberman, 1994) framework was used in the analysis of the data. After multiple readings and examinations of the data generated during the study, four levels of sophistication were identified in representing 3D shapes. The four levels were warranted based on the emerging patterns in students' processes and thinking in interpreting 2D representations of a 3D object, building a 3D object based on plane representations and drawing a plane representation of a cube. Table 1 makes easy the conceptualisation of the developmental patterns in students' processes and thinking in the two tasks across the four levels of sophistication. Although the comparison of different grades' students thinking in representing 3D shapes goes beyond the purpose of the study, it is important to report that each level consisted of students from all the grades

Level	Students' processes and thinking
1	Construction of three different objects instead of one based on the three sides of the orthogonal view
	Drawing isolated squares to represent a cube
	Inability to understand the 3D nature of objects in 2D representations
2	Coordination two of the three sides of the orthogonal view in the building of a 3D object
	Drawing a cube using a procedural method without conceptualising the conventions applied, fragile thinking
	Absence of mental manipulation of 2D representation of 3D objects
3	Description and construction of a 3D object based on its orthogonal view
	Drawing a cube in transparent and non transparent format
	Not able to mentally manipulate 3D objects
4	Description and construction of a 3D object based on its orthogonal view and mental manipulation of the object
	Drawing a cube by utilising, conceptualising and reflecting on the necessary conventions

Table 1: Characteristics of the four levels of sophistication

First Level of Sophistication

Students of the first level of sophistication had significant difficulties in representing 3D objects. Their answers exhibited that they could not conceptualise the three dimensions of objects in 2D representations. For example, students at this level did not realise in the first task that the three faces of the orthogonal view represent the same object. A fifth grade girl asked which of the three objects had to be constructed. Thus, students of this level decided that the front, side and top view of the orthogonal drawing were not linked and built three different objects by using multilink cubes that corresponded to the three faces (see figures 2(a), 2(b) and 2(c)). When the researcher reminded that he asked them to build one object, they just joined the three solids they had built up (see figure 2(d)). Students did not understand the spatial relations between the cubes presented in the side and top view of the object. Then, students were prompted to examine whether their construction fitted the given orthogonal view. Students spent much time working with the multilink cubes and built several objects. Most of them rotated their constructions and tried to figure out whether they corresponded to the three given views. However, none of them succeeded because they insisted on putting one column of two cubes on the right edge of the front view to fulfil the requirements of the side view, as presented in Figure 2(e). Moreover, the majority of the students put a column of three cubes behind the side view to form the shape of the top view (see Figure 2(f)). They did not realise that the back column of the inversed "T" shape of the top view represents also the right column of the side view.

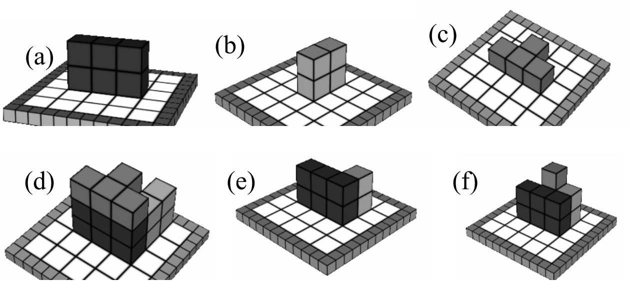

Figure 2: First level students' constructions

Students of the first level did not succeed in drawing a cube in a proper way. Students neither accepted nor used the necessary conventions in representing a solid cube in a 2D drawing. The majority of the students drew isolated squares. A small number of students drew two squares (the one covering the other), as it is presented in Figure 3(a), and then they drew segments to join the vertices (see Figure 3(b)). However, when students were asked to explain their procedure, they could not do it. They answered that they had memorised this procedure. They could not explain which edges of their drawing were viewable in the concrete cube and which ones were viewable only in the drawing due to the transparent format. When they were asked whether the marked angle (see Figure 3(b)) is a right one, they answered that it is not. Then, we asked them to show this angle in the concrete cube. This question confused them and responded that if we measured the angle with another protector the angle could be a right one.

Figure 3: First level students' cube drawing

Second Level of Sophistication

Students of the second level of sophistication failed in building the 3D object. However, based on their justifications, we concluded that their thinking had qualitative differences compared to previous level students' thinking. Students of the second level realised that the three faces of the orthogonal view represent the same object. They coordinated only the front and side view of the object in the construction of the object (see Figure 2(e)). Students thought that the edges of the front and side views have to be joined and as a result they did not take into consideration the top view. A sixth grade student supported that there was a mistake in the orthogonal view because the top view could not be combined with the other two views. When students were prompted to re-examine the correctness of their construction, the majority started over and built-up a new object taking into consideration the front and top views. However, they did not realise that their construction was correct (see Figure 4(a) and extended their construction by adding one column in the right part of the object to match the side view (see Figure 4(b)).

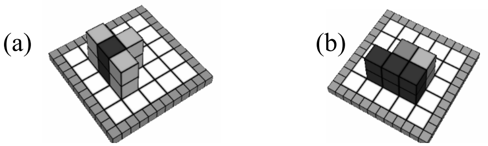

Figure 4: Second level students' constructions

Students' inability to utilise the necessary conventions in drawing 3D objects was evident in Task 2. The majority of the students did not observe the concrete cube at all, but they implemented blindly the typical procedure of drawing two squares and then joining their vertices. Students did not conceptualise their actions and they failed in drawing the cube in a non transparent view. None of the students interpreted the necessary convention for the drawing of the right angles in the projection view. Instead, they answered that the marked angle (see Figure 3(b)) did not look a right one because they had made a rough sketch. Their thinking seemed to be very fragile.

Third Level of Sophistication

Students of the third level implemented a variety of processes in representing 3D objects and used in a proper way the necessary conventions in drawing 3D objects. Students in the first task described their construction based on its orthogonal view. Their first attempt to build the object resulted in the construction presented in Figure 4(b). However, they realised on their own that their construction was wrong; they rotated it and observed at the same time the object and the orthogonal view and made easily the necessary changes. When students corrected their construction they were asked whether they could remove a cube from the construction without changing the orthogonal view. Students failed to identify the two possible answers mentally (appear in darker colour in Figure 4(a)), but managed to do it after several trials.

Students of the third level seemed to utilise the necessary conventions in drawing 3D objects. The majority of the students drew a perspective view of a cube and explained which of the edges in their drawing were visible due to the transparent view of their sketch. They had no difficulties in making another drawing of the cube in a non transparent format. They displayed pairs of parallel edges and edges that intersect perpendicularly. However, the majority of students did not explain why the marked angle was not ninety degrees in the drawing. They just mentioned that "it is a right angle, but makes a visual illusion that it is smaller than ninety degrees".

Fourth Level of Sophistication

Students of the fourth level described accurately the object before constructing it since the orthogonal view of the object helped them to create a mental image of the object. In addition, the majority of the students drew the perspective view of the object. They constructed very easily the object correctly, after observing very carefully the three sides of the orthogonal view. It seemed that they coordinated the three sides mentally. Students were very confident for their actions and justified all their actions. Students' internal mental image of the object helped them to grasp the spatial relations. Thus, they had no difficulty in identifying mentally the two possible cubes that could be removed without changing the orthogonal view of the object.

Students were successful in the drawing of the perspective view of a cube. An illustrative difference from the remaining students was the fact that they understood and implemented appropriately the conventions used in the 2D representation of 3D objects. As a result, they explained that although all the edges of a cube intersect perpendicularly, in the drawing some angles appear differently. They justified this procedure by claiming that the marked angle represents a right angle that looks smaller due to the perspective view. Our questions did not confuse them but prompted them to think more convincing arguments instead. Most of them explained that the marked angle would have been drawn ninety degrees if the angle view of the observer was different.

CONCLUDING REMARKS

The results of the study showed that the majority of 5th to 9th grade students have great difficulties in representing 3D objects, as supported by previous researchers (Ben-Chaim, Lappan, & Houang, 1989). A number of students do not even understand the 3D nature of objects in their 2D drawings and cannot conceptualise the necessary conventions in drawing and interpreting 2D representations of 3D objects. Students exhibit these problems in interpreting plane representations, building objects based on 2D representations and in drawing 3D shapes. This result might suggest that students lack the necessary spatial visualisation skills (Gutierrez, 1996). Four levels of sophistication were identified in representing 3D shapes based on students' representation processes and thinking. The differences between the four levels of sophistication students' thinking suggest that the development of students' spatial visualisation skills may induce their 3D representation processes. From the perspective of teachers, the present results may be used in order to include in their instruction appropriate activities aiming at improving students' ability to read, interpret and draw plane representations of 3D shapes. Students should be involved in a variety of activities that will help them to investigate in a constructivist learning environment the conventions applied in the plane representation of 3D shapes.

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LANGUAGE SWITCHING WITH A GROUP OF BILINGUAL STUDENTS IN A MATHEMATICS CLASSROOM

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This paper presents a report on how a group of immigrant bilingual students use their languages in the learning of mathematics. We have developed our research with immigrant bilinguals in Catalonia, Spain, that arrived at a young age from South-American countries. We propose a critical sociolinguistic approach, which draws on social theory in the analysis of how language is involved in the construction of teaching and learning opportunities. Our data points to the differences that the Spanish dominant bilingual students have in the use of Catalan and Spanish during their engagement in mathematical activity. They tend to use the two languages for different purposes, depending on the complexity of the mathematical practices, and in relation to different social settings that coexist within the classroom.

INTRODUCTION

A majority of students in immigrant bilingual mathematics classrooms in Catalonia, have Spanish as a first language, however, they learn mathematics in Catalan, the official language of learning and teaching mathematics. How do these Spanish dominant bilingual students use languages during mathematics teaching and learning? Do they switch languages during mathematical activity? If so, what are some of the factors that seem to promote their language switching within the context of specific lessons? In this paper we explore the above questions by drawing on a wider study involving immigrant bilingual children that were either born in Catalonia or went there at a young age from South-American countries and attended a Catalan school. We discuss two of the most recurrent themes in the data: 1) the acquisition of specific vocabulary in the second language; and 2) the development of mathematical argumentations in the first language.

By studying bilingual learners, we hope to come to a better understanding of how the use of the languages is mediated by the interpretation of the different contexts of mathematical school practices. Although many researchers recognize that the choice of language in bilinguals may vary depending on where and how the language is used (Daller, Van Hout & Treffers-Daller, 2003), we still have to further develop research on this topic in the case of contexts of mathematical practices.

A SOCIOLINGUISTIC APPROACH TO BILINGUISM

In this study we propose a critical sociolinguistic approach, which draws on social theory in the analysis of how language is involved in the construction of teaching and learning opportunities. We consider the construction and use of the language

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essentially as a social process embedded in social interaction that takes place in social participation structures where issues of identity and power are structural. This process of construction and use of language is about how individuals, groups and the contexts mutually constitute one another and work to maintain certain power relationships and change others. Our approach is influenced by the perspectives of several researchers on mathematical learning and multilinguism such as Barwell (2005), Clarkson (2007), Moschkovich (2007), and Setati (2005). These authors assume the integrity of the language and culture of minority groups, and emphasize the need for developing a sense of "language awareness" in the domain of mathematics education research. They have in common: a) the attention to the on-going struggles over power, and resulting inequalities, in multilingual contexts of mathematical practice; b) the analysis of both social structure and agency; c) the need for sociolinguistic work to draw on social theory in our understanding of school mathematics; and d) the interest in how difference and dominance may be created in face-to-face classroom interactions.

The phenomena of language contact and linguistic diversity are particularly represented by bilinguism, in relation to the knowledge of and the ability to use two languages. In our work, a bilingual is someone who has learned to understand and speak the world by means of two languages, although the understanding may not be the same depending on the language that is being used. We focus on how bilinguals integrate the diversity of understandings, and how they put together their knowledge of two languages to use in communication. The integration of languages and the construction of joint knowledge are always problematic as language contact involves one kind or another of social imbalance that reflect tensions among groups. The selection of one language as well as the maintenance or the eventual shift to the other, are types of imbalance that are related to the differences in the knowledge of the languages, and to the social contexts where the languages are used. Caldas and Caron-Caldas (2002) argue that a bilingual's preference for either of her/his two languages is context sensitive: the shift to the language with a higher status may be favoured by the students' perception of conditions of gaining access to social goods, while the maintenance of the language with a lower status may be associated to the perception of conditions of segregation and marginalization.

CONTEXT AND METHOD

Catalan is a Romanic language that shares many linguistic structural properties with Spanish. These structural similarities distinguish this research from studies on immigrant bilinguals in Europe that are faced with the problem of comparing the use of language pairs with large structural differences, such as Norwegian and Turkish or Arabic. While Catalan and Spanish are both common street languages in Catalonia, Catalan is the official language of teaching and learning. This means that teachers are required to produce written texts in Catalan and to use Catalan in their oral talk.

The main data for this study came from regular lessons in a secondary school bilingual class in Barcelona, Spain, with twenty-four students about twelve years old and an experienced bilingual Catalan native speaker teacher. Data was collected over five consecutive lessons of fifty minutes each. The lessons were planned for the students to spend time working in small groups. The goals in the class included giving the students the experience of "thinking like mathematicians" and "learning basic facts about the mathematics". Students had been informed that they were expected to "develop some ability to think critically about mathematics in open-ended situations". For the five lessons that were video-recorded, the contents were related to geometrical transformations. The unit, "Our dynamic planet", included a variety of mathematical activities that were thought as a way to allow students to pose questions and solve problems in real contexts. In the third lesson, the students were asked to mathematically represent a tornado.

In the class, there were eight students from South-America who were Spanish dominant bilingual, whereas the other sixteen students from Catalonia, mostly from Barcelona, were Catalan dominant bilingual, except for one of them who was a second generation immigrant and came from a Colombian family. All the students had a different bilingual proficiency profile due to the differences in their biographies. Although they were not "balanced bilinguals", most of them could be seen as almost native-like competent in their second language. They all had similar working class backgrounds; most of their parents had not completed high school, were limited Catalan proficient and immigrated to Catalonia for work reasons. Our research was focused on the nine students who spoke Spanish at home. The data for this report comes from one of the regular small groups (WG1), whose members were Máximo (M) –a second generation Colombian boy–, Luna (L) –a girl born in Peru– and Nicolás (N) and Eliseo (E) –two boys born in Colombia. The teacher described the four students in this group as having an average mathematical competency.

For the five class periods, the teacher and one of the students in each group wore a wireless microphone. There was also a static camera placed in one corner to capture the general picture of the entire classroom environment. For the analysis, different portions of the students' interactions within the small groups and with the teacher were first isolated and then transcribed. After having examined that language switching occurred, by quantifying the shifts from Catalan to Spanish and from Spanish to Catalan, we drew on ethnomethodology and interactional sociolinguistics to describe the contents of the talk that were observable and interpretable when reading the interactions. The use of a constant comparative method led to the development of interrelated themes that seemed to be promoting language switching. We now discuss two of the more recurrent themes.

RESULTS

Our data points to differences in the use of the languages during the Spanish dominant bilingual students' engagement in mathematical activity. When the students are getting familiar with the task and the new mathematical vocabulary, they tend to use Catalan, both with their small group peers, who are Spanish dominant, and with the teacher. However, when they start reflecting on the resolution of the task, they tend to use Spanish as if it was easier for them to complete and communicate their mathematical processes in this language. The experience of searching for mathematical explanations seems to be a factor that initiates students' switching between languages. We know about this type of findings in relation to English as a second language and with pairs of languages that have many structural linguistic differences such as Vietnamese and English or Iranian and English. But there is not literature regarding language switching by Catalan and Spanish bilingual students in Catalan mathematics classrooms. The language context given by the socio-political situation in our country, where Spanish has a low social standing, makes it relevant to pay attention to the particularities of this group of bilingual students and their efforts towards the public use of Catalan, the language with a higher status.

The acquisition of specific vocabulary in the second language

In the five lessons, during the first minutes the teacher gives priority to the introduction of mathematical vocabulary concerning geometry. She asks the students if they know the meaning of a certain word that has been written on the board or orally introduced, and urges them to use it in the context of the task. She does not translate the word into Spanish neither do the students ask for a translation. She begins by only explaining vaguely the mathematical meaning of the new word and leaves the students to explore in small groups the underlying concepts in the context of the task. In the interactions with the teacher, the immigrant students tend to use Catalan, their less proficient language, when they are prompted to introduce new terms. The following two excerpts from the third lesson are entirely in Catalan. The first excerpt shows part of the moment when the words "helicoidal", "helicoid" and "helix" are presented. Particularly interesting here is the way in which the speakers co-learn individual vocabulary terms by repeating one another's talk, completing the other's turns and providing supportive feedback. The teacher (T) models language behavior by only using the Catalan and suggesting the idea of "word family":

T: Sabeu què és un moviment helicoïdal? / Do you know what a helicoidal motion is?

M: Bé, sabem el que és un tornado. / Well, we know what a tornado is.

N: I sabem que un tornado es mou amb facilitat i rapidesa. / And we know that a tornado moves easily and quickly.

T: Un tornado va recte endavant i també gira. És un moviment helicoïdal. / A tornado goes straight forward and it also turns around. This is a helicoidal motion.

M: Un tornado va recte i cap avall. Com es diu? Helicoïdal? / A tornado goes straight forward and down. How do you say it? Helicoidal?

T: Es diu igual. Un moviment helicoïdal. Una helicoïde. Una hèlix. És el mateix. / You say it the same. A helicoidal motion. A helicoid. A helix. It is the same.

N: I un tornado és un moviment helicoïdal. / Then a tornado is a helicoidal motion.

M: És un moviment helicoïdal que va recta avall i gira. / It is a helicoidal motion that goes down straight forward and turns around.

In the interactions with the Spanish dominant students in the small group initial discussions, the immigrant bilinguals also tend to use Catalan. They go on with the use of Catalan in the context of getting familiar with the new words when talking to their peers, even when the teacher is not standing next to them. In the excerpt below, they do not accept incorporating the word "spiral", which does not fit into the helicoid's word family but could be seen, however, as part of a "concept family". Both the helicoid and the spiral have in common the idea of representing a curve in motion and some of the shared geometrical meanings associated to these words are helpful in the representation of a tornado. Eliseo points to the idea that the understanding of the concept "tornado" is more important than the words we use for it, but then rejects talking about spirals. One can sense in this excerpt possible tensions between the focus on the language and the focus on the mathematics, specifically between the idea of practicing the new vocabulary ("We are talking about helixes, not spirals") and the idea of exploring geometrically similar mathematical concepts ("It is a bit of a spiral"). All the utterances were in Catalan except for the last one, below we only reproduce the English translation in order to reduce the length.

L: We need to make the spiral.

N: Not a spiral, a helicoid, a helix.

E: What we really need to make is a tornado. And we need to name it a helicoid.

L: Do we need to make the arrows like yesterday?

E: We need to understand what a tornado is and then we find a name for it.

N: But now they are arrows of a helicoidal motion. It is a bit of a spiral.

E: We are talking about helixes, not spirals.

L: We are talking about helicoidal arrows.

M: We need to decide the arrows that we draw and that's all.

E: First we think about the arrows, then we draw them and then we talk about it.

M: [Spanish] This idea of the arrows is not easy. We have to imagine the different movements that exist within the tornado.

Máximo uses Catalan when reproducing the new terms and changes to Spanish when starting to develop more sophisticated arguments based on the coexistence of different simple motions within a helicoid. The students' initial interactions around the notion of helicoid are centered on how this notion is represented in the context of the task and in relation to the teacher's language priorities. They give priority to the use of the new words (a word family) instead of making distinctions or stating similarities between a spiral and a helicoid (a concept family). In the next section, we show that, in general, the students' switching in the middle of certain conversations seems to be sensitive to the type of practices. They tend to switch from Catalan –when "getting familiar with the new vocabulary"– to Spanish –when "solving the task"– and maintain the switch to Spanish during the time devoted to argumentation in the small group.

The development of mathematical argumentations in the first language

The excerpts above show how well particular mathematical vocabulary is used in the context of the Catalan language but do not give information concerning which of the mathematical meanings for the new words ("helicoidal", "helicoid" and "helix") are known, neither do they inform about the process of further exploring some of the geometrical concepts that are being represented by these words. The Spanish dominant students go back to their first language in the small group when they start experiencing some difficulties in the process of resolution and when they try to complete their explanations and, more generally, the mathematical task. This is the case with WG1 in the third lesson. Below, we reproduce the English translation of a conversation that happens entirely in Spanish:

N: It can be a diagonal arrow.

L: But the tornado does not follow a diagonal direction, it goes down and turns around at the same time.

M: The helicoid is like a broken arrow [he makes the drawing on the left in Figure 1].

E: A tornado is much more complicated. I will do it like this [the drawing on the right].

L: I don't think that a tornado may be represented with arrows. When you look at it, it doesn't go by staggering, now this direction and then the other.

N: None of your drawings are real. A tornado moves like a circle and you have only made rectilinear lines.

E: Now it does make sense to talk about the spiral.

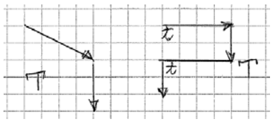

Figure 1. Some of the students' drawings representing a tornado.

The following excerpt starts with Luna asking for help in Spanish (S). Nicolás goes back to Catalan (C) and points to some key terms for the understanding of the task. As soon as Luna shows to have understood the meaning of the key words in the context of the task, Nicolás goes on with the mathematical explanations of the arrows in Figure 1 and uses again Spanish. Eliseo insists on introducing the notion of spiral, which is now accepted, probably because now it is not seen as an obstacle in the learning of new vocabulary. The references to the curves of a spiral will help to complete the linear representations in Figure 1 with curves (see Figure 2). It is interesting to note that while the approach to the task seems to be centered on the learning of a word family, the notion of spiral is not accepted and the idea of curvilinear lines is not considered. Later, when the approach seems to be centered on the learning of the mathematics, the consideration of the notion of spiral is allowed and a more accurate representation of a tornado is achieved.

L: [S] The question asks to represent a tornado, doesn't it?

N: [C] Yes, it says that we need to mathematically represent a tornado.

L: [C] It's not to talk about a tornado, it is to mathematically represent it.

E: [S] The drawing of a tornado can be useful before its representation.

N: [S] It is clear that only one arrow is not enough, a tornado is more than a translation.

E: [S] We need to think about the drawing of a spiral. We would draw curves.

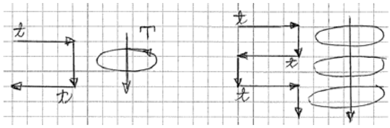

Figure 3. Eliseo's final drawings representing a tornado.

These four immigrant bilinguals use their two languages for different purposes. They use Catalan when getting familiar with new vocabulary, when situating the use of this vocabulary in the context of the given task, and when beginning to organize approaches to the resolution of the task. However, they use Spanish, their dominant language and the language that they share with their small group peers, when arguing at various degrees of specificity and developing more complex comprehension processes that are not centered on the repetition of some of the teacher's words and sentences. Our findings, concerning the use of the first language when elaborating on an argumentation, fit with Moschkovich's data (2007) where Latino students use Spanish to justify an answer or elaborate on an explanation and return to English when being asked by the teacher to give priority to the acquisition of new vocabulary.

The data from the whole group interactions in the five lessons shows that the group of Máximo tends not to speak when the teacher asks the groups to present their reasoning. Their engagement with the mathematics in Spanish does not lead to an increased participation in Catalan outside the context of the small group, although they are allowed to use the Spanish language. On the few occasions that the teacher asks these students to interact, they make short interactions in Catalan. Conversely, the local bilingual students tend to volunteer information unprompted, even interrupting the teacher to do so.

FINAL REMARKS

We have illustrated data concerning the use of the two languages by a group of bilingual students. These students tend to use each of the two languages in different domains of mathematical practices (acquiring vocabulary vs. explaining and arguing), and in relation to different social settings within the classroom (small group vs. whole group). First, when the Spanish dominant bilinguals in our study are prompted by the teacher to get familiar with the task within their small group and learn new mathematical vocabulary, they change to Catalan, which is the language in which this vocabulary is introduced. Second, when these students go more deeply into the resolution of the task within their small group, they change to Spanish although eventually they may go back to Catalan for certain clarifications. Third, when the time for the whole group discussion starts, they only intervene if they are directly asked by the teacher to do so and, when this happens, they use Catalan. For a further interpretation of this sort of language switching, we need to frame it in terms of the students' expectations about what they might achieve –or lose– by speaking in one of the two languages, given their different levels of language and mathematical proficiency and the role of each language within that classroom.

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"…BECAUSE 'OF' IS ALWAYS MINUS…" - STUDENTS EXPLAINING THEIR CHOICE OF OPERATIONS IN MULTIPLICATIVE WORD PROBLEMS WITH FRACTIONS

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The correct choice of operations is well known to be an obstacle for students when solving word problems. The presented study contributes to the discussion on possible explanations by investigating explicitly given reasons for choices in a written test with 269 German grammar school students. It shows that no uni-dimensional account can be given for the multi-faceted phenomenon of choice of operation.

Various empirical studies have documented difficulties in students' performance with word problems. Different obstacles were specified for a successful mathematization of word problems, some of them concerning the external appearance of word problems like length and readability of texts, others concerning their internal structure, like familiarity of contexts, necessary choice of operations, number type, didactic contracts for questions of validation etc (cf. Verschaffel et al., 2000, for an overview). Among all these important aspects, the choice of operations and their background gained a special attention for word problems with non-natural numbers. Many researchers studied the choice of operations for one-step multiplicative word problems with two decimal numbers (e.g. Fischbein et al., 1985; Bell et al, 1981; Bell et al., 1989; Harel et al., 1994). This study builds upon them, but extends them by

1. a focus on fractions rather than on decimals (demanded by Harel et al., 1994),
2. an enriched test design, including various models of multiplication and items for other layers of competence, and
3. a deeper analysis of reasons for choices given by the students themselves. By this, we attempt to enlarge usual quantitative research designs.

EXISTING RESULTS AND THEORETICAL EXPLANATIONS FOR DIFFICULTIES WITH THE CHOICE OF OPERATIONS

Solving word problems is only one aspect in a multi-faceted landscape of competences that are to be developed for fractions. Following Fischbein et al. (1985), this landscape can be structured in a *multi-level model for competence with fractions* (elaborated especially on the intuitive level in Prediger, 2008):

- *Formal Level,* including the definitions of concepts and of operations, structures, and theorems relevant to a specific content domain; formally represented by axioms, definitions, theorems and their proofs,
- *Algorithmic Level,* comprising procedural skills - here of multiplying - and the capability to explain the successive steps of the standard procedures,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 401-408. Thessaloniki, Greece: PME.

- *Intuitive Level,* characterized as the type of mostly implicit knowledge that is often accepted directly and confidently as being obvious, including different layers:
  - *mathematizing competence*, i.e. ability to translate word problems into terms,
  - *intuitive rules*, i.e. individual conceptions about existing laws and coherences,
  - *individual interpretations of operations*, and
  - *individual interpretations of numbers* (decimals, fractions etc.).

Individual interpretations of operations and numbers have been conceptualized as (mental) models (Fischbein et al., 1985; Greer, 1994; Usiskin, 2008) or 'Grundvorstellungen' (GVs, see vom Hofe et al., 2006; Prediger, 2008). They constitute the meanings of mathematical concepts based on familiar contexts.

In order to give more precise accounts for students' deficits in their mathematizing competences for multiplicative word problems, various researchers investigated into students' choice of operations. A robust finding is that number types involved in the problem statement strongly affect the difficulty of the mathematization process, the so-called multiplier effect: For multiplicative problems with an integer multiplier, the correct choice of operation is easier than for decimal multipliers > 1, and those are easier than for problems with multiplier < 1, from which one knows that the result must be smaller than the factors (e.g. Bell et al., 1981; Bell et al., 1989).

Basically, two different theoretical accounts have been given for the multiplier effect: First, Bell et al. (1981) emphasized the importance of the intuitive rule "multiplication makes bigger" (here shortly called the 'order property') and its generalization from natural to fractional numbers as the main obstacle for choosing multiplication for word problems with multiplier < 1 (cf. Bell et al., 1981; vom Hofe et al., 2006). Fischbein et al. (1985) gave empirical evidence for an explanation situating the difficulty one layer underneath: They emphasized that the pertinacity of the *intuitive rule* "multiplication makes bigger" is often connected with the continuing maintenance of the *interpretation* of multiplication in the repeated addition model (which does not work for decimal or fractional multipliers). Both accounts can be integrated, as shown in Prediger (2008), since the intuitive rule is often based upon uncompleted conceptual changes on the layer of interpretations of multiplication (similarly Greer, 1994). That means, that those students who have widened their repertoire of interpretations for multiplication (and hence mastered the discontinuity of the repeated addition model) can also change their intuitive rules concerning the order property of multiplication.

Although later studies started to widen the structure of situations in view (e.g. Bell et al., 1989, considered not only repeated addition models, but also prices, speed and currency-conversion), the great variety of other individual models for the multiplication of fractions and naturals are still to be explored more systematically.

CENTRAL METHODOLOGICAL IDEA AND RESEARCH QUESTIONS OF THE PRESENT STUDY

In order to specify aspects in students' thinking and in word problems that influence students' choice of operations, three main research strategies have been adopted so far:

1. Studying effects of factors by comparing difficulties when systematically varying operation-choice test items (e.g. Bell et al., 1989; de Corte / Verschaffel, 1996).
2. Searching for statistical coherences in a written test, covering different layers of competence (e.g. vom Hofe et al., 2006; Bell et al., 1981; Prediger, 2008).
3. Qualitative in-depth analysis by clinical interviews (for example Bell et al., 1981 in their first phase, Wartha, 2007).

Research strategies 1 and 2 can only give statistical coherences (by comparing, in contingency tables or with correlations), but no account for causal connections. That is why some quantitative studies have been complemented by qualitative in-depth studies in clinical interviews, but they only allow small numbers of participants.

4. Qualitative deeper analysis of written answers
   This study tries to combine advantages of qualitative and quantitative strategies by applying an intermediate strategy: We conducted a written test with open items, coded answers in an explorative procedure and quantified frequencies of constructed codes afterwards. This generated insights beyond statistical coherences.

RESEARCH DESIGN: CORE ITEMS, PARTICIPANTS, DATA ANALYSIS

The empirical material presented here was part of a study conducted by a written test with twelve test items (see Prediger / Matull, 2008). This paper focuses on two core items (the other ten items that are shortly characterized in Table 1):

Item 7 a.) One kilogram tangerine costs 1.50 €. Kate wants to buy 3/4 kg. How can she calculate the price?
□ 1,5 – 3/4 □ 1,5 : 3/4 □ 3/41,5 □ none of these, but this:
b.) Give reasons for your answer given in a)

Item 9 a.) How can you calculate 2/3 of 36? □ 36-2/3 □ 36 : 2/3 □ 2/3 · 36 □ none of these, but this:
b.) Give reasons for your answer given in a.)

Item 7 and 9 follow the choice of operation methodology (cf. Fischbein et al., 1985, Bell et al., 1989), in which students are asked to give or choose a term without calculating the answer. Crucial for research strategy 4 is the added part b.), asking for *reasons of choices* in an open item format.

Item 7 asks for a mathematization in a situation acting across quantities, which is (according to Usiskin, 2008), especially difficult for students. Item 9 has the same structure, but refers to a situation in which the multiplication is used for taking a part of a whole number, one of the most important models for the multiplication of fractions.

The paper and pencil test was written by 269 students in five Grade 7 classes (age about 12 years) and five Grade 9 classes (age about 14 years) in German grammar schools which comprise the (assumingly) higher achieving 40 % of students.

The students' answers were evaluated quantitatively in a points rationing scheme. The values were used for statistical investigations on correlative coherences and contingencies of performances for different items.

For a deeper exploration, the self-constructed word problems to Item 5 and 6 and the reasons given for operation choice in Item 7 and 9 were analysed qualitatively by coding the manifested individual conceptions and strategies. Whereas a coding scheme for Item 5 and 6 pre-existed (from Prediger, 2008 with an interrater agreement of Cohen's kappa 0.93), the coding scheme for the reasons in Items 7 and 9 first had to be constructed from the data. In an explorative coding procedure, categories were built by comparing answers due to their similarity. Some could be anticipated by the existing literature (like the pertinacity of the order property "multiplication makes bigger and division makes smaller", see Bell et al., 1981), but other interesting, unforeseen codes (see Table 2, e.g. restructure strategy) had to be constructed in the process. Once finally established, the coding scheme of Item 7 and 9 achieved an interrater agreement of Cohen's kappa 0.83.

RESULTS

1. Statistical results

Table 1 gives an overview on the scores and frequencies of complete solutions for all test items. Item 7 and 9 are among the most difficult, with only 0.7 % complete solutions for Item 7 and 4 % for Item 9 and average scores of 14 % and 12 %, resp. The distributions of performances in Item 7a and 9a are compared to other operation choice items in Figure 1.

Item	Content	Frequency of complete solutions	Average of reached scores	
1	Multiply fractions (technically)	62 %	3,15 of 4	79 %
12	Confirm commutativity of multiplication	54 %	1,55 of 2	77 %
5	Find a word problem for a given equation with addition	54 %	1,39 of 2	70 %
4	Explain the meaning of a given fraction	49 %	1,37 of 2	68 %
3	Identify a multiplication in rectangle picture	67 %	1,33 of 2	67 %
8	Mathematize situation allowing repeated addition (natural multiplier)	20 %	1,21 of 2	60 %
2	Order property (multiplication makes bigger?)	34 %	0,77 of 2	38 %
10	Specify part of a fraction and mathematize	1 %	1,24 of 5	25 %
11	Mathematize a situation of scaling down	1 %	0,30 of 2	15 %
9	**Mathematize situation with part of whole number (2/3 of 36)**	**4 %**	**0,27 of 2**	**14 %**
7	**Mathematize situation acting across quantities (kg x €)**	**0,7 %**	**0,24 of 2**	**12 %**
6	Find a word problem for a given equation with multiplication	3 %	0,11 of 2	6 %

Table 1: Scores of items, ordered due to average scores

The far best results were reached for Item 8a, in which 91% of the students activated the well-known model of repeated addition and chose the multiplicative terms 2/10 · 15 or 15 · 2/10 (More precisely, 38 % chose one, 53 % both.). In contrast, in all other items, no more than 93 of 269 students chose the multiplicative term (< 35 %, the guess probability was 33% for Item 8 and 9).

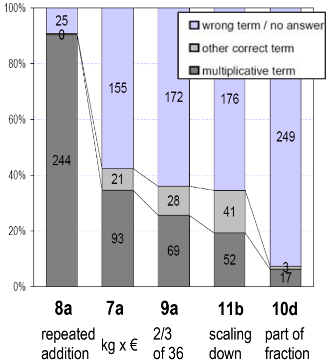

Figure 1: Comparison of chosen terms for Items 7-11

Reasons for these operation choices can first be studied by statistical methods. Chi-squared tests for independence in the contingency tables of Item 7 and 2 (or Item 9 and 2, resp.) gave evidence for an association between unsuccessful operation choice and wrong intuitive rules about multiplication making bigger: The null hypothesis of independence of outcomes in Item 2 and 7 could be rejected with a chi-square of 9.65, being highly significant ($p < 0.008$). For Item 2 and 9, a chi-square of 42.34 allowed to reject the null hypotheses of independences with $p < 0.001$. In contrast, the contingency tables between Item 7 and 3, and Item 9 and 3 showed no significant dependence of item outcomes. By these results, we could confirm classical findings on the importance of the intuitive rule "multiplication makes bigger", shortly said the order property of multiplication.

Nevertheless, contingency tables and chi-squared tests cannot account for *causal connections* between layers of competence. That is why the analysis was deepened by coding the reasons given by the students.

2. Deeper analysis of reasons

The answers in Item 7b and 9b were coded according to the reasons given for the choices of operations. As we were especially interested in those answers that gave access to the choosing strategy behind the given reason, we filtered all answers that did not allow the interpreters any access to the strategy, as for example "Because you have to take this." (Kim).

Table 2 gives a quantitative overview on the answers that were filtered or coded according to the reconstructable choosing strategies behind the given reasons. The explorative coding process ended with a categorization of codes into three main categories: order strategies, restructuring strategies and keyword strategies, by which most of the reconstructable answers could be captured (79 %). The categories shall be explained by examples in the following.

In Item 9, 69 of 269 students chose correctly a *multiplicative term* for mathematizing 2/3 of 36. Only in 5 of the reasons given, the researchers could identify points that allowed any access to their thinking. One of the students activated an order strategy, i.e. she successfully made use of her intuitive rule: "For fractions and multiplication,

	Item 7				Item 9				
	Mult	Div	Sub	Oth	Mult	Div	Sub	Oth	
Chosen Term	93	81	36	59	69	120	19	61	**Overall part of reconstructable strategies**
Answers allowing no access to strategy for interpreter	90	72	29	55	63	73	11	51	
Reconstructable strategies behind the formulated reasons:									
Order Strategy: multiplication makes …	1	7	1	0	1	2	0	0	12 / 95 = 13 %
Keyword Strategy: of-tasks are… tasks	0	0	0	0	4	25	3	0	34 / 95 = 35 %
Restructure Strategy Use other parts	0	3	6	0	0	17	5	0	28 / 95 = 29 %
Others	2	0	0	3	0	5	0	10	20 / 95 = 21%
Guessing Strategy	0	0	0	0	0	0	0	0	0 / 0 = 0 %

Table 2: Operation choice and frequencies of reconstructable strategies of choice

you always get less." (Liza) (Although Liza's rule is not of sufficient generality, it worked here).

Five of the students referred to the keyword "of". For them, this seems to be a rule guiding their operation choice, for example, "because of is the same as times" (Eddie) or "of-tasks are times-tasks" (Sam). We subsumed these answered under the so-called keyword-strategy.

Whereas Liza, Sam and Eddie drew upon their strategies successfully, order strategy and keyword strategy were more often used for choosing the wrong operation *division*: Among all 120 wrongly chosen divisions, 49 were justified in a way that allows access to the underlying strategies and conceptions. 25 of these 49 belonged to misleading order strategies: "When you multiply, it becomes more, when you subtract, it also becomes less, but just wrong." That is why Karen chose division.

17 other participants restructured the situation in an idiosyncratic way, by mixing wholes and parts like Paul, writing "because you need a part". Anna's answer was also categorized as restructure strategy, as she tried to solve the following task: "You must calculate, how often 2/3 fit into the 36 for getting the new fraction."

The rate of restructure strategies was even higher for *subtraction*: 19 students wrongly chose subtraction for Item 9, and among the 8 reasons that gave access to their thinking, there were 5 with restructure strategies. Most of them explained their constructed term 36 – 1/3 while referring to false referent wholes, like Ali "When you want to have e.g. 2/3, then 1/3 is left! They are subtracted (in this case) from 36."

The strategy that was most often reconstructable for the operation choice in Item 9 was the keyword strategy: 4 choices of multiplication were (correctly) explained by this (see above), but even more (25) choices of division, like in Terry's answer: "Well I want to know how much is 2/3 <u>of</u> 36, not 36-2/3 or 2/3 ·36, though, division." With the same argument, 3 students explained their choice of subtraction by the keyword strategy, like Eve: "2/3 <u>of</u> 36, thus minus", another one already cited in the title.

Two of the three main strategies reconstructed for Item 9 also reappeared in Item 7: Whereas the keyword strategy was not that important here, restructure and order strategies also appeared in the reasons given for Item 7 (see Table 2).

In conclusion, we resume that order strategies were not as important as suggested by Bell et al. (1981) for our sample. Only 12 of 96 interpretable answers showed order strategies (13 %). In contrast, keyword strategies (35 %) and restructure strategies (2 9 %) were found in significantly more cases. This relativizes other findings.

Guessing Strategies have not been formulated by any student of this sample (of higher streamed students). This is very different in another sample of 561 lower streamed students, in which 14% of the answers to the same two items were "I have guessed" (see Prediger / Matull, 2008). Apparently, higher streamed students know that this answer is not accepted in mathematics classrooms for higher achieving students, so even if they guessed, they did not write it.

DISCUSSION

The findings of this study affirm the thesis (formulated in Prediger, 2008) that difficulties on one layer of competence (here the mathematizing competence, operationalized as choice of operations) cannot be explained uni-dimensionally because they might be located on different layers of competence. The reconstructed categories of choosing strategies have their roots on different layers:

- The *order strategy* comprises all reasons given with reference to a sustainable or non sustainable intuitive rule on the order property of multiplication (making bigger or smaller). But although being privileged in existing studies, like Bell et al., 1981, it could only be reconstructed in 13 % of the accessible cases in this study.
- In contrast, the restructure strategy (that was reconstructable in 29 % of all accessible cases) does refer to misconceptions on the layer of interpretations of fractions, a layer that has not yet had sufficient attention in empirical studies.
- The guessing strategy (not appearing in the grammar school sample) and the keyword strategy (reconstructable in 35 % of all accessible cases) are interpretable as (sometimes misleading) schemes on the layer of mathematizing competence itself. The use of a keyword strategy alone cannot be taken as empirical evidence for deficits on layers underneath, but it shows that the students did not activate a deeper layer of interpretations in this situation. But additional cross-references in the raw data table showed that none of the 25 students who activated a keyword strategy for choosing division in Item 9 had been able to find a correct word problem for a given multiplication in Item 6. This indicates that wrong keyword strategies might often be connected to missing interpretations for the multiplication of fractions.

To sum up, the study could show some connections between layers of competence that can account for wrong operation-choices better than pure correlative results. This gives hints for new research strategies in this field.

Obviously, the short paper must leave many questions unanswered, especially more precise connections between the reconstructed choosing strategies and the answers to other items. In further research, we plan to investigate a larger sample of students of all achievement levels.

Remark. The study was conducted within the research project "Stratification of student conceptions – The case of multiplication of fractions", financed by the research fund DFG - Deutsche Forschungsgemeinschaft (Prediger / Matull, 2008). I thank Ina Matull and all coders for their work.

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“THREE EIGHTHS OF WHICH WHOLE?” - DEALING WITH CHANGING REFERENT WHOLES AS A KEY TO THE PART-OF-PART-MODEL FOR THE MULTIPLICATION OF FRACTIONS

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One important meaning of the multiplication of fractions is the part-of-part-model, by which 4/5 x 2/3 is interpreted as 4/5 of 2/3. Students’ understanding of this model is often constrained by the difficulty of changing referent wholes. The paper presents first investigations of a learning arrangement that was designed in order to deal with this obstacle and to increase students’ awareness about changing referent wholes by associating different representations. The qualitative analysis of prospective teachers’ products and processes gives insights into individual constructions of meanings and terms for part-of-part-situations.

THEORETICAL BACKGROUND AND EXISTING FINDINGS

Theoretical background: Grundvorstellungen in different representations

The (individual and normative) *meaning of operations* can be conceptualized in different ways. This paper draws upon the notion of *Grundvorstellungen*, shortly *GV* (vom Hofe et al., 2006), being the cognitive building blocks for interpreting and mathematizing in processes of modelling (see Figure 1). We take this notion nearly synonymously to mental models in Fischbein’s sense as a “meaningful interpretation of a phenomenon or concept” (Fischbein, 1989, p. 129). While mathematizing, GVs are activated to find models of a situation; while interpreting, GVs provide models for the formal mathematical expression.

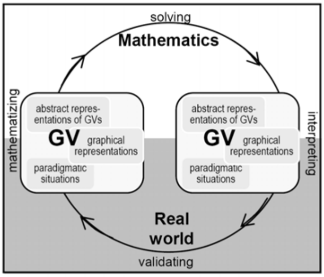
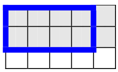


Fig. 1: GVs as translation - tools in modelling processes

GVs are not only represented by their abstract form, like by saying “multiplication of fractions can be interpreted by the part-of-part-interpretation”, but also by paradigmatic situations or graphical representations. For example, the meaning of 4/5 x 2/3 as 4/5 of 2/3 is more accessible in a picture or a context: Jim has 2/3 of a pizza left from lunch. For dinner, he eats 4/5 of the rest of the pizza. So, he eats 8/15 *of the original pizza*.



Our research illustrates how individual processes of constructing GVs can be enhanced by intermodal transfer, i.e. by associating different representations for GVs. By this we follow Gerster & Schultz (2004) and others who conceptualize understanding of operations as a successful interplay between different modes of representations.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 409-416. Thessaloniki, Greece: PME.

Empirical starting point: Changing referent wholes as obstacle to understand the part-of-part-model for multiplication

Although the already mentioned part-of-part-interpretation is one of the most important mental models for the multiplication of fractions, empirical studies show limited success in students' acquisition of this GV (vom Hofe et al., 2006; Prediger, 2008).

Previous findings point at one important obstacle for students to construct the part-of-part-model: the change of referent wholes (e.g. Mack, 2000). In Schink (2008), this is exemplified by a student who approached parts of parts through paper-folding (original paper complete and zoomed in Figure 2). Having successfully folded 1/8 of 1/5, he correctly obtained 40 rectangles in his paper. The obstacle is manifest in his notation "1/8" for 8 rectangles (in the left 1/5-stripe of Figure 2). Realizing that the partner wrote 1/40, he corrected one of them into 1/40.

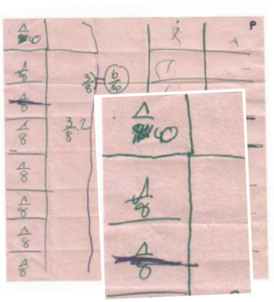


Fig. 2: $\frac{1}{8}$ of $\frac{1}{5}$ is $\frac{1}{8}$ or $\frac{1}{40}$?

The problem with changing referent wholes appears for interpretations in all representations: Whereas the 2/3 in the pizza-situation refers to one original whole pizza, the second factor, 4/5, refers only to the rest of the pizza, thus it has another referent whole. But the result 8/15 again refers to the original pizza, i.e. the whole one. Hence, constructing meaning for the part-of-part-model necessitates a clear orientation on the question "What is the whole?" (cf. Mack, 2000).

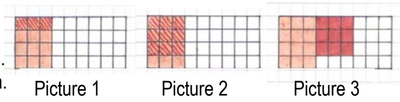
As mentioned by many researchers, problems with changing referent wholes or units also appear in non-multiplicative contexts, and not only for fractions (e.g. in Harel/Confrey, 1994), Well known is that when students formulate word problems for given additions, one of the most typical mistakes is to join parts of two different referent wholes (Prediger, 2008).

Design of a learning arrangement built upon associating representations

Starting from these empirical findings on typical difficulties, a learning arrangement was designed that allows students to develop or enlarge their individual GVs of addition and multiplication of fractions and to gain awareness for different referent wholes and the question "What is the whole for this fraction?".

The so-called Excursion-Problem shown in Figure 3 is one part of this learning arrangement that demands the construction of terms for single-step and two-step additive and multiplicative situations. It builds upon the interplay of different representations by giving three texts of (paradigmatic) situations and three pictures which have to be associated. The case study presented here investigates how this arrangement supports the construction of adequate GVs and terms.

Excursion-Problem
Here, you see different situations and different pictures.
Which belong together? And why?
Give answers to the questions and terms for the situations.
One picture will remain unused, construct a fitting situation.
Two situations belong to one picture.



1 The class 6d has 36 students. Each child is in one club. 1/3 of them are in a music club, 1/4 of them are in a sports club. Which part of the class is in a sports or music club?

2 The class 6a also has 36 students. 1/3 of the class are boys. 1/4 of these boys are in a football club. Which part of the class are male football players?

3 In the class 6c, 1/3 of the students want to go to the ocean for their excursion. 3/8 of the rest of the students prefer the mountains. Which part of the class wants to go to the ocean or the mountains?

Fig. 3: "Excursion-Problem", designed for enhancing students' GVs

RESEARCH QUESTIONS AND DESIGN

As a first step in an ongoing design research project, this study investigates the didactical potential of the above problem in a preservice teacher education course for prospective (mostly middle-school) teachers in their 2nd or 3rd year. Their working processes in groups of 2-3 were observed, partly video-taped and transcribed. 66 written answers were collected that document the products. The data analysis of products and processes followed four research questions:

(1) How do the participants relate situations, pictures and terms? Which terms do they construct especially for the more complex Situation 3?

(2) How do the participants deal with changing referent wholes, and in how far do they gain an increasing awareness during the process?

(3) How does the requested activity of associating representations (situation, pictures, terms) influence the process of constructing or choosing models and terms?

(4) Which obstacles hinder participants in their construction processes?

Research question (1) and (2) were addressed for all participants by analyzing and categorizing the 66 written documents (see quantitative overview in the next section). Research question (3) and (4) were in the core of a more detailed case study on Laura and Paul, two prospective middle-school teachers. The transcript and the video of their process were analysed qualitatively turn by turn, then coded and categorized in a procedure of open coding. Selected dimensions and results of the analysis are presented here.

ANALYSIS AND DISCUSSION OF PROCESSES AND PRODUCTS

Variety of constructed terms - Quantitative overview on products

Nearly all participants could assign Situation 1 and 2 to suitable pictures. The describing additive (1/3 + 1/4) and multiplicative terms (1/3 x 1/4 or 1/4 x 1/3) were successfully constructed in 45 of 66 (and 42, resp.) documents. Hence, the big majority of participants could activate a part-of-part-model in this setting (although only 23 % of them chose a part-of-part-situation in an earlier problem when asked to invent a situation for a given multiplication of fractions).

As anticipated, Situation 3 was more challenging for the participants, since it demands a complex combination of GVs to construct the term 1/3 + 3/8 x 2/3. In sum, 23 different terms (not including multi-step calculations) were constructed, with varying appropriateness for describing the situation (see Figure 4). Only 15+4 documents gave a complete term description for the group of ocean- and mountain-fans, in fractions or absolute numbers.

The categorization also took revised terms into account. As far as they appeared in the written documents (e.g. scratched out or later corrected), they allow interesting insights into the processes of conjecture and refutation. The fact that 12 participants notated wrong terms but revised them afterwards, gives a first evidence that the designed Excursion-Problem offers the desired potential to affiliate the intended processes of developing GVs. Only 5 documents ended with terms in which wrong operations were chosen or fractions of different referent wholes were combined. However, these 5 and 11 more documents without terms emphasize the importance of the issue.

Most documents (7+11+9+3+1=31) contain partly adequate solutions which refer only to a subgroup (e.g. the subgroup of mountain-fans) or describe the situation in a

Categories: appropriateness of term	**Subcategories**: What and how does term describe? (abs. num. = absolute number)	**Examples** for written terms	**Frequency** of occurrences of this type of term as final results (in brackets: frequency of revised terms)
Adequate terms	group as part	“1/3+2/3x3/8” or “1/3+3/8x(1-1/3)”	15 (1)
Nearly adequate terms	group in abs. num.	“1/3x36+3/8x24” or “1/3x36 + (36-1/3x36)x3/8“	4 (0)
Partly adequate terms or calculations	subgroup (as parts) in more steps or one single term	e.g. “2/3x3/8” or “(36-36/3)x3/8 [(36-36/3)x3/8]/36=1/4”	7 (1)
	subgroup (in abs. num.) in more steps or one single term	e.g. “36x2/3x3/8” or “36:3 36-24 24x3/8”	11 (0)
	group (as part) in more steps	e.g. “1-1/3 2/3x3/8 1/3+9/36”	9 (0)
	group (in abs. num.) in more steps	e.g. “1/3x36=12 36-12=24 3/8x24=9 21”	3 (0)
	others	“1/3+1/4” [fraction taken from picture]	1 (2)
Wrong or no terms	no term or fractions	no written term or all scratched out	6
	only fractions or verbal descriptions	“21/36”; “1/3 of the whole (=12 children)+3/8 of the rest (=9 children)”	5 (1)
	wrong term	e.g. “1/3x3/8”, “1/3+3/8” or “1/4x1/3”	5 (12)

Fig. 4: Overview on terms, constructed for Situation 3 in 66 documents

multi-step calculation instead of a single term. Hence, most participants were apparently able to give meaning to the situation itself and to realize the importance of changing referent wholes, but were nevertheless not able to give a complete term description. The underlying reasons cannot be reconstructed by an analysis of products alone; a case study gives more insights into the process and its obstacles.

A long search for a term – The case of Laura's and Paul's process

Laura and Paul, two prospective middle-school teachers, intensively worked on the Excursion-Problem for 25 minutes. A detailed analysis of their interesting case provided insights into patterns and obstacles of an unfinished process (cf. research question (3) & (4)). Due to place restrictions, only the key results can be sketched which were drawn from the qualitative coding procedure (shortly documented in Figure 5).

Within 6 minutes, Laura and Paul successfully assign Picture 1 and 3 to Situation 2 and 1, resp., and find the terms 1/3 x 1/4 and 1/3 + 1/4. Unlike many colleagues, Laura immediately activates the part-of-part-model for multiplication.

Their work on Situation 3 starts by drawing individual pictures (based upon the assumption that 3/8 refers to the whole class, not to the non-ocean-group). When they hear that they should only use existing pictures, they restart by re-reading the text:

104	L	…oh, 3/8 of the **rest of the students** prefer…

Once having realized that the remaining group of non-ocean-fans was meant (abbreviated by "nog" in Figure 5), Laura immediately associates Picture 3. So, 8 minutes after having started with Situation 3, the right picture is found, and they recognize the difference between the class and the non-ocean-group as referent wholes for the first time. The remaining 11 minutes are dedicated to the search of a term.

110	L	*[writes down* "1/3 x 3/8, because you take 1/3 of the class and then 3/8 of the rest". *She types on the calculator, apparently receives 3/24, later she scratches out the equation]*
112	L	*[looks on her picture]* No, these are 9/36. Thus 1/4.
114	L	So 1/4 times 1/3 *[writes* "1/4 x 1/3 = 1/12", *types on the calculator, scratches it out]*

Although she is aware that 1/3 and 3/8 have different referent wholes (evidenced by her verbal formulation in line 110), she combines the fractions unconventionally, apparently because "3/8 of the rest" signals multiplication, even if this rest is not described by 1/3. The calculator serves her (here and later) as important tool for falsifying constructed terms.

As the fractions extracted from the text do not work, Laura controls their meaning in the picture. Keeping the multiplication, she exchanges 3/8 by 1/4 in her next term, because she extracts 9/36 by counting from the picture and by syntactically reducing 9/36 = 1/4. Although all fractions in this second term refer to the class, the calculator does again not produce the desired results.

Paul studies the text intensively and recognizes the abstract structure "join of parts" that is to be mathematized by an addition (119). Adopting this insight (125), Laura

Trans-cript Line and Actor	**Intermediate terms** (w=written, s= only spoken, p=only pointed at already written terms, c= apparently entered into calculator)		**Referent whole for each number** (nog=remaining non-ocean-group, abs.n.= absolute number)	**Activated GV for chosen operation**	**Associated Representation** (most often: number taken from… t = text, p = picture, r= already obtained result, tpr = t or p or r)	**Prompt for dropping the term** (f calc = falsified by calculator, p=control of result in the picture, T/P= communication with teacher /Paul)
110 L	w	"1/3 x 3/8"	class x nog	x part of ?	t x t	f calc, p
114 L	ws	"1/4 x 1/3"	class x class	x part of part	p x tpr	f calc
119 P	s	take together	?	+ join parts	t	(not dropped)
125-128 L	ws	"1/3 + 3/8"	class x nog	+ join parts	tpr + t	f calc, p
152 L	s	"3/8 of 24"	rest of abs.n.	(no term)	t pr	does not help
152 L	ws	"4/12 + 3/12"	class + class	+ join parts	pr + pr	no fit to text
154 L	w	"36 – 1/3"	abs. n. – class	- not reconstructable	r – tpr	f calc
159 L	p	"4/12 + 3/12"	class + class	+ join parts	pr + pr	no fit to text, T
169-181 L	ws	"1/3 + 3/8"	class + nog	+ join parts	tpr + t	L: f calc P: diff ref wholes,T
184 L	c	"1/3 x 3/8" (?)	class x nog	x part of part	tpr x t	f calc
194 L	s	"3/8 ref. to 36"		(no term)	tp r	does not help, T
198 L	wsc	"24 x 3/8"	abs. n. x nog	x part of whole	pr x t	(not dropped), T
202 L	s	"36 ÷ 9"	abs. n. : abs. n.	1/9 as part	r	(not dropped) ,T
205 L	ws	"1/3 + 1/4"	class + class	+ join parts	r + tpr	no ideal fit to text, used as one step
208 L	s	"1/3 ref. to 36"		(no term)	r	P
211 L	s	"24 x 3/8"	abs. n. x nog	x part of whole	r	P
211 L	s	"1/3 + 1/4"	class + class	+ join parts	r	(not dropped) P
212 P	w	"3/8 x 24"	nog x abs.n.	x part of whole	r	(not dropped)
213 L	w	"1/3 + 1/4 = 7/12= 21/36 Z 3/8 of 24, referred to the whole"	class + class abs.n.	+ join parts x part of whole	r	final expression

Fig. 5: Chronology of Laura's & Paul's search for a term for Situation 3

varies the term into 1/3 + 3/8. In this variation, she keeps the numbers from the text and changes the operation (128). Again, she falsifies the result 17/24 given by the calculator by counting in the picture.

128 L *[writes* "1/3 + 3/8", *enters it into the calculator, writes* "=8/24 + 9/24 = 17/24",
130 *counts in the picture] [laughs]* I still cannot calculate

Paul extracts 3/12 from the picture as a fraction that describes the part of the mountain-group when referred to the class. His attempt to associate this fraction also to the text of Situation 3 raises a new question: How to receive the 3/12 by a term involving 3/8? Paul knows in principle what to do, but cannot mathematize it:

151 P Well, we should first calculate what these 3/8 *[3 sec. break]* are of the whole
152 L ...3/8 of 24 are 3/12 because there are 36 – err, 24 – no 9/24 *[scratches out the "1/3 = 4/12" and "3/8=9/24"]* I cannot!

Anyway, the result is *[writes at the same time]* 1 err 4/12 + 3/12 = 7/12, isn't it?

Although already having distinguished the class and the non-ocean-group as different referent wholes for 1/3 and 3/8, the relation between both is still not completely clear. Line 152 shows the conceptual difficulty in a nutshell: within one utterance, Laura refers the mountain-group (described by 3/8) to the non-ocean-group (described by the absolute number 24), then to the whole class (described by the absolute number 36), and then (with the 9/24) to the non-ocean-group (this time conceptualized as 1). In contrast, they never describe the non-ocean-group as 2/3 or as 24/36, which would allow to conceptualize the non-ocean-group as part of the class (as 1), so that the referent wholes could be nested in each other. This constraint hinders them to relate 3/8 to 1/4 by mathematizing 3/8 x 24/36 = 1/4.

Although the term 1/3 + 1/4 at least gives the right result 21/36, they consider it not to be sufficient, because they cannot relate it to the text:

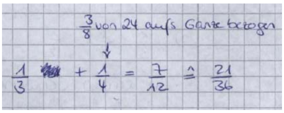
162 P Well, from the picture, we can justify it *[he means the term* "3/12 + 4/12"*]*, but we do not find a term that associates it more directly

163 L Because this is not direct, it is only read off *[meant is from the picture]*

As they do not succeed, Laura comes back to 1/3 + 3/8, although the calculator falsified it, and Paul explicitly states that it is not appropriate:

181 P But you cannot add 1/3 and 3/8, because the 3/8 do not refer to the whole picture here

After trying other terms and discussing with the teacher (see Figure 5 for a complete chronology), the teacher signals that the time is over. Laura is finally satisfied with a verbal description of the connection as printed here:



Translation: "3/8 of 24, referred to the whole"

Discussion: First answers to the research questions in the case of Laura & Paul

(1) Like most of the participants, Laura and Paul quickly relate situations and pictures. Unlike many of their colleagues, both immediately mathematize Situation 2 by a multiplicative term. We conclude that they are familiar with the part-of-part-model for multiplication and also the join-model for addition. But these building blocks alone do not enable them to combine them in one complete term for the more complex Situation 3; so they end with an intermediate result that associates 3/8 of 24 to 1/4 only verbally.

(2) Whereas Paul is quickly aware of different referent wholes involved in Situation 3, Laura first confuses them. Associating representations helps her to gain awareness for the difference between the whole class and the non-ocean-group, but she continues to mix parts and absolute numbers. Finally, she gains awareness for the referents.

(3) Although Laura seems to know abstract representations of elementary GVs ("of is multiplication"), she cannot activate them for mathematizing the more complex situation. In this challenging constellation, the interplay of a *greater variety* of

representations for GVs in pictures and paradigmatic situations is a big help for Laura and Paul. By linking representations, they construct the meaning of the context and gain awareness of different referents. Each interpretation or construction of a symbolic element can always be validated in another representation. However, representations offer benefits *and* difficulties: The picture also suggests a misleading absolute view instead of focussing on parts; the calculator is an important tool to falsify terms, but also distracts Laura from controlling the meaning of her terms.

(4) Two further obstacles hinder Laura and Paul to find the term: Their unquestioned implicit premise that a term always consists of only two numbers, and the missing attempt to make use of *nested referent wholes* instead of varying referent wholes: They refer to 24 or to 36, but never to 24/36 or 2/3 as the part of the non-ocean-group *of* the whole class. We hypothesize that Laura and Paul could have found the term if the teacher would have given them an impulse to overcome these constraints.

CHANGING REFERENT WHOLES – A MANAGEABLE CHALLENGE

Although the Excursion-Problem was originally designed for students of age 11, it proved to be productive for enhancing learning processes even for prospective teachers. It might be received as disappointing that the prospective teachers could only find a complete term in 19 of 66 documents. On the other hand, the first insights into their processes give hope that well designed learning arrangements can help them to manage the challenge (this is in line with Greer 1992 who gives kindred recommendations for designs with emphasis on construction of models and meanings).

Our empirical findings on the complex interplay of representations might be of major importance far beyond the concrete subject "part-of-part-model for fractions". Learners can fruitfully rely on different representations, but we are far from understanding in detail what happens in these processes. Future research should focus on this point more intensively.

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RESOURCES USED AND "ACTIVATED" BY TEACHERS WHEN MAKING SENSE OF MATHEMATICAL SITUATIONS

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In previous work, we have outlined points that appeared significant for the mathematical *education of mathematics teachers. Issues of school mathematics, in opposition to academic mathematics, and of engaging teachers in a practice of mathematizing (Bauersfeld, 1998), in opposition to being exposed to standardized knowledge, were argued as fundamental dimension of an approach better articulated with mathematics teaching practices. A two-year project was developed along that perspective with a group of secondary mathematics teachers. To illustrate the potential of the approach, we analyse the resources used and activated by teachers when making sense of a mathematical situation proposed by the teacher educators.*

As Ball and Bass (2003) made clear, the manner in which mathematics teachers engage with a mathematical situation in their teaching is quite different from the way mathematicians would. This obvious, albeit insightful, assertion leads us as a community to profoundly rethink the mathematical education that is provided to mathematics teachers (at the pre- and in-service levels). One issue that has received recent attention is the gap between the *mathematical experiences* encountered in university courses/in-service education and in the practice of teaching mathematics in schools. Critiques are abundant, mainly in relation to the *nature* of the mathematical content explored in academic courses (focused on formalism, compacted ideas and abstract forms [see Ball & Bass, 2003; Moreira & David, 2008]) and to the *manner* in which this content is approached (through lecturing and exposition modes [see Bauersfeld, 1998; Burton, 2004]). This divide raises questions about the current orientations of mathematics teacher education programs and suggests a rethinking of the mathematical experiences and learning opportunities teachers are exposed to through their professional formation.

Our research interests are located within that perspective: conceptualizing and studying an approach to in-service education that attempts to offer teachers experiences better aligned with their mathematics teaching practices. That is, as we have presented elsewhere (Proulx & Bednarz, 2008), an initiative focused on (1) the development and enrichment of teachers' knowledge of the mathematics they teach (i.e., school mathematics) and on (2) their immersion in an engaging practice as mathematical "doers" – taking into consideration both aspects of *nature* and *manner* highlighted above. One main research objective is to characterize the mathematical meaning and practices teachers (as mathematics doers) develop throughout this initiative, and in return to study the potential of such an approach. In this paper we present preliminary findings of this ongoing project in which we focus on the

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 417-424. Thessaloniki, Greece: PME.

manners in which teachers made sense of, and the knowledge they enacted about, a mathematical situation related to "school mathematics."

THEORETICAL ORIENTATIONS UNDERLYING THE APPROACH

The in-service approach developed with teachers is articulated on the exploration of what we call "school mathematics" and on an engagement in a culture of mathematizing; two interrelated components we define below.

Defining school mathematics

Our understanding of "school mathematics" is oriented by Moreira and David's (2005) theoretical distinction of academic mathematics and school mathematics as different fields of knowledge. They use the term *academic mathematics* to refer "to the scientific body of knowledge produced by the community of professional mathematicians," whereas *school mathematics* is defined as "the set of validated knowledge, specifically associated with the development of school education in mathematics […] includ[ing] knowledge produced by mathematics teachers in their school practices […] as well as knowledge produced by research on teaching and learning of mathematical concepts and processes at school" (pp. 1-2). Thus, school mathematics represents not only concepts present in curricular documents outlining what teachers have to teach, but also the mathematical elements that surround them and emerge in its learning and teaching. For example, when teaching mathematical concepts, various related *mathematical* issues unfold: key reasoning, specific approaches/ways of making sense, range of specific procedures and representations, different conceptions or difficulties experienced, etc. For teacher education practices, theorizing school mathematics in this way "move[s us] away from the idea of school mathematics as a discipline taught at school to re-conceptualize it as a body of knowledge specifically associated with mathematics teaching at school" (p. 2).

Manners of doing/mathematical practices: Immersing teachers in a culture

The manner in which content is approached also appears fundamental. The intention is to engage teachers in a practice of doing mathematics, in what Bauersfeld (1998, p. 215) refers to a "culture of mathematization as a practice." As Burton (2004) suggests, issues of mathematical practices requires a shift, from mathematical knowledge to mathematical knowing; from mathematics as an object-oriented discipline for someone to know, to mathematics as something that one does. Participants in a culture of mathematics are seen as producers of mathematical knowledge and meanings. In the construction of such a culture, where concepts are explored and worked on, participants are encouraged to generate ideas, questions and problems, to make explicit and share understandings, to develop explanations and argumentations, to negotiate meanings and explore different ways of understanding problems, concepts, symbolisms and strategies and to validate other's understandings and explanations (see, e.g., Seeger, Voigt & Waschescio, 1998). Teachers' immersion in mathematical practices requires that these aspects be encouraged.

METHODOLOGICAL CONSIDERATIONS

A group of 6 voluntary secondary mathematics teachers (7^{th} to 12^{th} grade) participate in the 2-year project. The initiative is structured around day-long monthly workshop during the school year. All sessions are videotaped to keep a record of the sessions' unfoldings, and a researcher journal is kept about occurring events and reflections these provoke. Our role in this research is two-dimensional as we take the position of both teacher educators and researchers (see Proulx, 2007). This posture is aligned with recent innovative participative research approaches that combine both research and educative concerns (Cochran-Smyth & Lytle, 2004; Tabach, 2006). As teacher educators, we design and conduct sessions where we participate actively in the development of the mathematical practices and understandings occurring in them. This position, as Wong (1995) explains, offers us, as researchers, a privileged access to the meanings built in action and enriches the possibilities for making sense of and understanding their intricacies by being intertwined in them.

The sessions activities revolve around "school mathematics" tasks for teachers to engage with, on mathematical topics teachers want to explore (e.g., algebra, fractions, measurement). Tasks are designed on the basis of conceptual content analyses, many inspired from the mathematics education literature. In regard to mathematical culture, tasks are also designed on the basis of enacting significant aspects of mathematical practices (e.g., validation, argumentation and negotiation of meanings, using symbolism, posing and solving problems).

SOME PRELIMINARY FINDINGS: TEACHERS' SENSE MAKING

The analysis we offer here is centred on teachers' exploration of a "school mathematics" task, in which a non-traditional solution for dividing fractions (coming from a student) is presented to the teachers for discussion. The analysis focuses on the ways teachers engaged in the situation, documenting the ways in which they made sense of it through the exploration. (Because of space constraints, aspects of mathematical culture are not elaborated on here.) The analysis sheds light, as we will see, on the intricate nature of the various dimensions (mathematical, didactical and pedagogical) teachers bring forth as they engage in and explore this task:

> A colleague reported this procedure, used by an 11-year-old to divide fractions:
>
> $$\frac{26}{20} \div \frac{2}{5} = \frac{26 \div 2}{20 \div 5} = \frac{13}{4}$$
>
> Is this procedure adequate/correct? Does it always work? How?

Moment 1: Multiple entries to the appropriation of the situation

The first reaction came from Ana who right away mentioned that "It does not work" and offered what she called a counter-example ($\frac{14}{5} \div \frac{3}{4} = \frac{14/3}{5/4}$), adding: "You only need a counter-example to show that something is false. I have found a counter example, thus it is false." (Note that Ana appears to collapse issues of obtaining a simplified answer and of mathematical validity of a procedure. This issue comes back later.) Whereas Ana's reaction to the task was mathematical, Joe and Marco's reply to the situation was more at a didactical level, preoccupied with what students would do with Ana's example. They mentioned that the problem students would face is not of mathematical inappropriateness, but of trying to avoid obtaining another division of fractions with, this time, $\frac{14/3}{5/4}$ as an answer; having them going in circles.

As the discussion continued, Neli and Marco changed the focus and began questioning the "question" posed in the problem, because for them it was not clear enough for students to know what to do. In particular, Neli asked for clarification of the question: if "give a simplified fraction as an answer" was present in the question then Ana's answer would not be acceptable. Neli and Marco's questioning of the question mainly focused on pedagogical issues: their intentions were not explicitly about the nature or the sort of knowledge that could be provoked by the question's formulation, but about making sure to eliminate possible confusion in students, in order to make everything clear. Thus, already in this first moment, the analysis show that teachers engaged in different ways in the task and its discussion, using mathematical (Ana), didactical (Joe & Marco) and pedagogical matters (Marco & Neli) as points of entry.

Moment 2: New questioning, orienting teachers toward "why does it work?"

Oriented by our intention of engaging teachers in a mathematical culture and since teachers appeared to agree that both answer and procedure were correct, we probed teachers to question the meaning behind the procedure (Why did it work?).

Many of the teachers answered that they checked it by using the "multiply by the inverse" algorithm. As they explained this, Pia realized that she did not know why the "multiply by the inverse" algorithm worked – something others agreed to. She added that when students ask why it works, they tell them "this is how it is," but in fact they themselves do not really know. Thus, as teachers attempted to justify why the first algorithm worked, they realized that their own means of dividing fractions needed justification (as a mirror effect, where the validation of the algorithm appeared to lead teachers to question and validate the very argument they were putting up.)

After teachers slightly drifted away and began sharing various mathematical ways of doing the multiplication by the inverse algorithm (i.e., $\frac{26}{20} \div \frac{5}{2}$), we questioned them anew to know if they would accept $\frac{14}{5} \div \frac{3}{4} = \frac{14/3}{5/4}$ as an answer to a division of fractions problem. As some said yes and others no, they all appeared to agree that "something" was missing: some questioned again the question; some said that the student would not get much far and would stay with a "similar" question; Ana mentioned that this answer could be simplified using a calculator and then become valid ($3\frac{11}{15}$). This led back to the issue of difference between a mathematically valid and an efficient answer. As Joe insisted, $\frac{14/3}{5/4}$ is mathematically valid, as it is not false, but does not lead one far and thus is not much efficient to solve the problem. This led teachers to become sensitized to the difference between a valid procedure, being true or non-false, and an efficient one – a significant meta-mathematical reflection.

In the midst of questions like "why does the algorithm work?" and "would you accept this answer?" Marco expressed positively that it was mathematically true since he proved that it worked "all the time." And before offering his proof, Marco explained what he would do in front of a student who would give that sort of answer. Since he, like the other teachers asserted, had never seen this algorithm, he would first try it for himself, at a mathematical level, to make sure it worked and gave a correct answer. And if so, he would give the points to the student. But then, at a more didactical level, he would ask the student to explain what he did, and why, to make sure the student understood what he/she did and had not calculate this by "chance." Asked how he would make sure that it worked, Marco offered his proof.

(1) $\frac{a}{b} \div \frac{c}{d} \quad = a \div b \div (c \div d)$

(2) $\quad = a \div b \div c \times d$

(3) $\quad = a \div c \div b \times d$

(4) $\quad = a \div c \div (b \div d)$ (this step was added afterward in the explanation)

(5) $\quad = \frac{a \div c}{b \div d}$

This proof is based on symbolic manipulations. Marco's strategy was to transform in line (2) a division of a division (÷(c÷d) into a multiplication (÷c×d), and the reverse in line (4) – something that is reminiscent, albeit mathematically wrong even if it leads to a right answer, to the issue that "– followed by a – becomes a +", forgetting the distinction between a sign and an operation, which inhibits the reasoning on divisions to be valid. We find two aspects intertwined in Marco's explanations of what he would do with this student answer: there is a mathematical component where he wants to check the answer for himself and positions himself as a mathematical doer, and there is a didactical component where he positions himself as a mathematics teacher and wants to verify the student understanding. Within both components, there is the teacher's intention to show that the procedure is correct for him and, in a mirroring sort of effect, to see the reasoning underlying the use of such a procedure by the student, illustrating well how both dimensions are intertwined.

Thus, in this second moment, validation aspects are central, in relation to mathematical, didactical as well as meta-mathematical aspects. What this shows is that the mathematical non-familiarity of the division procedure offered and the fact that its correctness is not obvious, triggered interesting teachers' reactions. We see this in Marco's need to prove for oneself that the procedure worked (a mathematical action triggered by didactical intention to be able to position oneself on students' work) and at the same time check for students' understanding (a didactical intention). We see also this in the group's meta-discussion of the meaning and difference between a mathematically valid procedure and an efficient one – drawing out the unclear distinction that implicitly lied here for teachers.

Moment 3: And how would students justify this answer?

Marco's comments that he would ask the student for a justification led us as teacher educators to ask in return for the sort of justification they thought students could offer. Again, some teachers mentioned that students would do the "multiplying by the inverse" algorithm. Another solution brought forth was to do the same thing Marco did, but with numbers, transforming the operation $\frac{26}{20} \div \frac{2}{5}$ in (26÷20)÷(2÷5). (But Joe and Neli mentioned that students would not be able to do this.) Another solution offered by Neli would be to see division as the inverse of multiplication and use that to validate, in that if $\frac{26}{20}$ divided by $\frac{2}{5}$ gave $\frac{13}{4}$, then $\frac{13}{4}$ multiplied by $\frac{2}{5}$ should give $\frac{26}{20}$. Another way of validating, offered by Ana earlier, was to enter numbers in the calculator and check the answer – obtaining $3\frac{11}{15}$ as an answer, which represented a simplified and acceptable answer for Ana. Through teachers' answers of potential students' responses, we see again intertwined aspects at a didactical level when they assert that "students could do… students could not do…" and aspects at a mathematical level where they make sense mathematically and probe for potential ways of explaining.

Moment 4: Reflections on the relevance of the procedure

At this point, we thought that it could be interesting to present what other teachers (to whom we have presented this task on other occasions) have found. One teacher had suggested that both fractions be placed under a common denominator, something that would always render a division by 1 at the denominator (and since teachers in the current group had previously suggested that what is needed is for both numerators and both denominators to have common factors, this made a lot of sense). [e.g., $\frac{14}{5} \div \frac{3}{4} = \frac{14/3}{5/4}$ would give $\frac{14}{5} \div \frac{3}{4} = \frac{56}{20} \div \frac{15}{20} = \frac{56/15}{20/20} = \frac{56/15}{1} = \frac{56}{15}$.] This solution pleased teachers, especially Neli who saw the link with the addition and subtraction algorithm, opening to a "general way" of operating with fractions that would simplify what students are taught, that is, to place them under a common denominator. (This idea led some teachers to flag the fact that the division, in the case of $\frac{26}{20} \div \frac{2}{5}$ becomes much more evident as it becomes $\frac{26}{20}$ divided by $\frac{8}{20}$ which could then be reduced/simplified to the operation 26 divided by 8. But, as Joe and Neli explained, this is a very difficult step to understand for students and they probably would not be able to reason these steps.) Albeit mathematical in essence, both reasoning were also influenced by a didactical issue of offering a more accessible approach to solve divisions of fractions.

Commenting on this procedure, Marco raised the fact that it is not the most efficient algorithm since calculations can be long and it seems more relevant to multiply by the inverse. Implicitly, here, Marco was raising a pedagogical issue related to "time constraints." Another discussion began about issues of efficiency, not in regard to mathematics itself, but to students since Joe mentioned the importance of letting students develop their own ways of solving through developing personal algorithms, where this was an illustration of. We see here reflections concerning the advantage of this algorithm in regard to other algorithms, to the limits of one algorithm or its limits in regard to its accessibility to students and concerning issues of time, to the possible justification students could use, and so on. Again, there is an important integration of mathematical, didactical and pedagogical dimensions through the discussion and teachers' ways of engaging with/in the task.

DISCUSSION OF OUTCOMES

To push further our analysis of meanings teachers developed through their interactions during the exploration of this "school mathematics" task, we use Lave's (1988) concept of *structuring resources*. Lave has elaborated this concept to underline, from a situated cognition perspective, the importance of context and of the actions of a person in an ongoing activity for the "structuration" of a specific social practice. For Lave, an "activity" (in our case, exploring "an answer proposed by a student to divide fractions") gains a certain structure from its specific context (e.g., the proposed activity is one among many possible solutions to divide fractions and thus orients reactions), and in return provides structuring resources to the activity itself as well as for other subsequent activities (e.g., for making sense of a student's solution in class, for approaching division of fractions in teaching). This iterative view of being structured and of structuring is opposed to assumptions that see activities and settings as isolated and unrelated, as well as to the notion of universal and generalizable forms of knowledge that can be inserted and transferred into any situation and to anybody. This concept of structuring resources sheds a refreshing light on the diversity of resources that teachers draw from in order to make sense of and develop an understanding of a situation they are confronted with, and which in return structures the manner in which they appropriate this same situation (and other situations related to their mathematical teaching practices).

As the data illustrates, teachers "activated" *mathematical* resources of different order to make sense of the situation (e.g., offering a counter-example, proving the adequacy of the procedure, offering validation processes, discussing meta-mathematical issues of efficiency and validity). They also enacted *didactical* resources to appropriate and give meaning to the situation (e.g., on students' difficulties, conceptual limits and possible solutions, ways of assessing their understanding, possible interventions), as well as *pedagogical* resources (e.g., issues of clarifying the question to avoid confusion, issues of time as a constraint, issues of efficiency and of rapid solving). These various resources are structuring the activity of the teachers and the way they see the situation, and are iteratively being structured in return as the interactions and explorations unfold. The "interaction" between the situation, the ongoing activity and the resources developed by the teachers seems to be characteristic of the sessions' unfolding, as well as descriptive of teachers' activity and engagements in the tasks.

These resources appear also strongly intertwined, as teachers enacted as much mathematical, didactical and pedagogical issues: some teachers appropriated the "same" situation through different angles, some engaged themselves at different times through different angles, and some entered in ways that had implicitly a double nature (e.g., mathematical and didactical). All those points of entry appear to play a role in their work. They do not play out in isolation but act in connection and are nested in one another, and influence each other. When confronted with mathematical situations, teachers appear to not only use their mathematical resources, but also build on their didactical and pedagogical ones, forming as a whole a very specific, teacher-oriented, professional way of engaging in a "mathematical" situation; one which we are only beginning as a community to develop an appreciation of.

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REIFYING ALGEBRAIC-LIKE EQUATIONS IN THE CONTEXT OF CONSTRUCTING AND CONTROLLING ANIMATED MODELS

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This paper reports on a design experiment conducted to explore 17-year-old students' constructions of meanings, emerging from their interpretations and uses of algebraic-like equations in the context of constructing and controlling animated models. We particularly focused on the students' engagement in reification processes, e.g. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie their animated models' behaviour.

THEORETICAL BACKGROUND

In this paper we report on a classroom research [1] aiming to explore 17-year-old students' construction of meanings emerging from the use of algebraic-like equations employed as means to create and animate concrete entities in the form of Newtonian models. The students worked collaboratively in groups of two or three using a constructionist computational environment called "MoPiX" [2], developed at the London Knowledge Lab (http://www.lkl.ac.uk/mopix/) (Winters et al., 2006). MoPiX allows students to construct virtual models consisting of objects whose properties and behaviours are defined and controlled by the equations assigned to them. We primarily focused on how students interpreted and used the available equations while they engaged in *reification* processes (Sfard, 1991), e.g. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of their models.

Recognising the meaning of symbols in equations, the ways in which they are related to generalisations integrated within specific equations and the ways in which a particular arrangement of symbols in an equation expresses a particular meaning, are all fundamental elements to the mathematical and scientific thinking. Research has been showing rather conclusively that the use of symbolic formalisms constitutes an obstacle for many students beginning to study more advanced mathematics (Dubinsky, 2000). Traditional approaches to teaching equations as part of the mathematics of motion or mechanics seem to fail to challenge the students' intuitions since they usually encompass static representations such as tables and graphs which are subsequently converted into equations. Lacking any chance of interacting with the respective representations, students fail to identify meaningful links between the components and relationships in such systems and the extensive use of mathematical expressions (diSessa, 1993).

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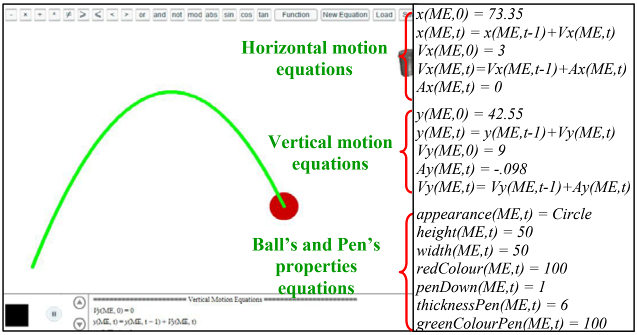
In the relevant research in the mathematics education field, a central question concerns the nature of equations and the ways in which they can be understood by students. Most of the respective studies are based on the distinction between the two major stances that students adopt towards equations: the process stance and the object stance (Kieran, 1992; Sfard, 1991). The process stance is mainly related with a surface "reading" of an equation concentrated into the performance of computational actions, following a specific sequence of operations (i.e. computing values). The object stance, however, is related to the actual entity -the equation itself- and the outcome of the computational actions performed (i.e. the computational product).

Elaborating further on the distinction between the above stances and the ways by which students understand algebraic expressions (and thus equations), mathematics educators brought into play the idea of students moving from process–oriented views to object–oriented ones via a process of abstraction which has been termed *reification* (Sfard, 1991) and has been considered to underlie the learning of algebra in general. Sfard's theory of reification (1991) describes three levels of mathematical conceptual development which eventually lead to the construction of a new concept. In the first level -the stage of *interiorisation*- the learner gets acquainted with processes involving operations performed on lower-level mathematical objects. At the second level -the stage of *condensation*- the learner is able to condense processes, viewing them as a whole. At the third level, corresponding to *reification*, the learner is able to view mathematical concepts as objects in their own right and use them as inputs in higher-order processes which might be precursors to new constructs.

Adopting a constructionist framework (Harel and Papert, 1991) in the present study we used a computational environment that is designed to enhance the link between formalism and concrete models, allowing us to study the ways in which the use of formalism, when put in the role of an expression of an action or a construct (a model), can operate as a mathematical representation for constructionist meaning-making. Our central research aim was to study students' construction of meanings, emerging from their uses of the available mathematical formalism, when engaged in reification processes. Particularly, we were interested to shed light upon the relationship between the evolution of students' understandings with their emerging engagement in different aspects of the abstraction processes (i.e. interiorisation, condensation and reification) concerning the conception of equations as objects.

THE COMPUTATIONAL ENVIRONMENT

MoPiX (constitutes a programmable environment that provides the user the opportunity to construct and animate models representing phenomena such as collisions and motions. In order to assign behaviours and properties to the objects taking part in the animations, the user attributes equations that may already exist in the computational environment's "Library" or equations that she constructs by herself. The MoPiX equations incorporate both formal notation symbols (i.e. Vx, x, t) as well as programming – natural language utterances (i.e. Circle, appearance).



However, their main characteristic is that they constitute functions of time. The environment constantly computes the attributes given to the objects in the form of equations and updates the display, generating on the screen the visual effect of an animation.

MoPiX offers a strong visual image of equations as containers into which numbers, variables and relations can be placed, allowing students to make easily connections between the structure of an equation and the quantities represented in it. It also allows the user to have deep structure access (diSessa, 2000) to the models animated as the equations attributed to the objects do not constitute "black boxes", unavailable for inspection or modifications. The manipulations performed to a model's symbolic facet (i.e. the equations) generally produce visual results on the Stage, from which students can get meaningful feedback.

TASKS

For the first phase of the activities we developed the "One Red Ball" microworld which consisted of a single red ball performing a combined motion (Figure 1). The students were asked to execute the model, observe the animation and discuss with their peers the behaviours generated. In order to stimulate students to start using the equations themselves, we asked them to try to reproduce the red ball's motion. In this process, we encouraged them to interpret and use equations from the "Library" and link the equations they used to the behaviours they had previously identified. As we deliberately made the original red ball move rather slowly we expected students to start expressing their personal ideas about their own object's motion (e.g. make it move faster) and thus start editing equations so as to ascribe it new behaviours.

For the second phase of the activities we designed a half–baked microworld (Kynigos, 2007), i.e. a microworld that incorporates an interesting idea but it is incomplete by design so as to invite students to deconstruct it, build on its parts and change it. The "Juggler" half-baked microworld (Kynigos, 2007) consisted of three interrelated objects: a red ball and two rackets. The ball's behaviour was partially the same as the "One Red Ball's", however, when it hit the rackets, it bounced, moving away in specific ways. We asked the students to execute the Juggler's model, observe the animation and identify each object's behaviour. The students were encouraged to discuss with their peers on how they would change the "Juggler" microworld and embed in it their own ideas. In this process, students were expected to deconstruct the microworld so as to link the behaviours animated to the equations and reconstruct it, employing strategies that would depict their ideas about their model's behaviours.

METHOD

The experiment took place in a Secondary Vocational Education school in Athens for 25 school hours with one class of eight 12th grade students studying mechanical engineering and two researchers. The adopted methodological approach was based on participant observation of human activities, taking place in real time. A screen capture software was used so as to record the students' voices and their interactions with the MoPiX environment. The data corpus also included the students' MoPiX models and the researchers' field notes. We verbatim transcribed the audio recordings of two groups of students and also several significant learning incidents from other workgroups. The unit of analysis was the episode, defined as an extract of actions and interactions performed in a continuous period of time around a particular issue. The episodes presented were selected (a) to involve interactions with the available tool during which the MoPiX equations were used to construct mathematical meaning and (b) to represent clearly aspects of a reification process.

ANALYSIS AND INTERPRETATIONS

Conceiving the MoPiX equations operationally– The phase of Interiorisation

Attempting to make their objects move exactly like the "One Red Ball", the students interpreted and used during the previous phase of their experimentations several motion equations that they found categorised in the environment's "Library". However, as they gained familiarity with the MoPiX formalism, they didn't seem willing to confine themselves in merely reproducing a given motion. Expressing their own ideas about the way their objects should move, the students started modifying the pre-defined library equations.

The students of Group B, for instance, looking in the library for equations that would make their object move vertically, came across the "Vy(ME,0)=0". This equation prescribes the velocity of an object in y axis at the zero time instance (left part of the equation) to be 0 (right part of the equation).

S2	[*To S1 who attributed the "Vy(ME,0)=0"*] Press "Play". You didn't do anything. You just made the velocity 0 at the zero time instance. Its initial velocity is 0. You did nothing to it. It didn't change, to move downwards.
S1	Yes, yes.
S2	That's what I'm saying. Change it. Give it some initial…, we should give it an initial velocity. Isn't it better?
R2	Whatever you like.
S2	Give "3" as an initial velocity. The equation you used before, with the difference that after the equal sign, we will place a "3". There, move it up.

S1	No. That was stupid. Let's change the velocity. Increase it.

After attributing the "Vy(ME,0)=0" equation to their object and started the animation, the students realised that the equation they used didn't make their object move downwards as they had expected. This observation triggered the implementation of a series of changes on the right part of the equation, beginning with converting "0" into "3". The students went on replacing the arithmetic value on the right side of the equation with other ones, attributing each time the new equation to their object so as to verify its effect. However, this procedure seemed to be rather mechanical. All the new equations were in the form of "Vy(ME,0)=___" which means that they merely determined the object's velocity at the *zero time instance* and thus had no apparent effect to the object's motion. Nevertheless, the students continued replacing the arithmetic value on the right side of these equations with new ones, a process that implies an operational conception of the notion of equation. The continuous replacements indicate that students viewed the expression "Vx(ME,0)" on the left side as an unknown quantity (an "x") which had be equal to a specific arithmetic value ("x=____"). They seemed to disregard the structure of the "Vx(ME,0)" expression and the meaning its comprising symbols conveyed and considered the right part of the equation as a placeholder into which numbers should inserted in order to provide an arithmetic value for the unknown quantity.

Conceiving the MoPiX equations operationally – The phase of Condensation

Having determined the meaning of the symbols in the "Vy(ME,0)=0" equation, the same students, continuing their experimentations, sought for ways to make the velocity of their object to constantly be "3".

S2	How can we insert the 2nd, 3rd time instance… in there? [*the equation*]
S1	In the 0 time instance, it's 3…
S2	Do we need a symbol for this?
R2	Do we need a symbol? It's a good question. How you plan to express it?
S2	With symbols we usually express something that we can't describe accurately.
S1	Plus… t! [*He types Vy(ME,t)=3 and points at "t"*] So, when I look at this symbol
S2	I'll know it represents the infinity.

In the above extract, the students seem to have relocated their focus from just replacing arithmetic values to determine an unknown quantity into forming functional relations. Although they could have followed the previous strategy and start replacing the "0" on the left side of the equation with other numerical values (i.e. 2 , 3, …) so as to form several equations that would define the velocity at different "time instances", the students took a moment to think of ways to incorporate all the numerical values in one equation that would describe the velocity at all "time instances". This approach led them to introduce a symbol which they would "look at and know that it represents the infinity". The students seem to have dethatched their mathematical activity from simply attributing specific arithmetic values to an unknown quantity and to attempt to form a functional relationship among two varying quantities: the velocity and the time. Symbolising all the upcoming time instances with "t" –an in-built MoPiX symbol– the students seem to have taken a more decontextualised stance towards the notion of equation and at the same time to have condensed the up to that point number attributing processes into a whole.

Conceiving the MoPiX equations structurally – The phase of Reification

During the two previous phases of their experimentations, the students edited Library equations and manipulated in-built MoPiX symbols, such as the "t". The changes performed restricted to the content of the existing equations (e.g. substituting a numerical value for another one or for a variable) while structure was left intact. The next episode describes how the Group A students, in the course of changing the "Juggler" microworld, didn't just edit existing equations but constructed two new ones from scratch. The idea these students wanted to bring into effect was to "make a ball change its colour according to an ellipse's position".

S1	When it [*the ball*] is situated in a Y below the Y of this one [*the ellipse*] for example…
R1	I'm thinking… Will the ball know when it is below or above the ellipse?
S2	That's what we will define. We will define the Ys.
S1	This. The: "I am below now". How will we write this?
S2	Using the Ys. Using the Ys. The Ys. That is: when its Y is 401, it is red. When the Y is something less than 400, it's green!

Having conceptualized the effect they would like their new equation to have, the students decided about two distinct elements regarding the equation under construction: its content (i.e. the symbols to be used) and its structure (i.e. the signs between the symbols). However, as there was no in-built MoPiX symbol to express the idea of an object becoming green under certain conditions, they had to invent a new symbol: the "gineprasino" (i.e. "become green" in Greek). To represent the ball's position they used its Y coordinate in the form of a quantity varying over time (i.e. "y(ME,t)"), whereas to represent the ellipse's position, its Y coordinate in terms of the constant arithmetic value (i.e. "274") corresponding to the object's position at the time. Adding a "less than" sign in between, the equation eventually developed was the "gineprasino(ME,t)=y(ME,t)≤274".

Since this equation described the event to which the ball would respond (being below the ellipse) and not the ball's exact behaviour after the event would have occurred (change its colour), the students decided to construct another equation. A Library equation which explains what happens to a ball's velocity when it hits on one of the Stage's sides, led students to duplicate this equation's structure, eliminate any content and use it as a template to designate what will happen to the ball's colour when it goes below the ellipse. The second equation encompassed in-built MoPiX symbols, the "gineprasino" variable in two different forms and numerical values (0 and 100) to express the percentage of the green colour the ball would contain in each case (i.e. being above or below the ellipse). The second equation developed was the: “greenColour(ME,t) = not(gineprasino(ME,t))×0 + gineprasino(ME,t)×100”.

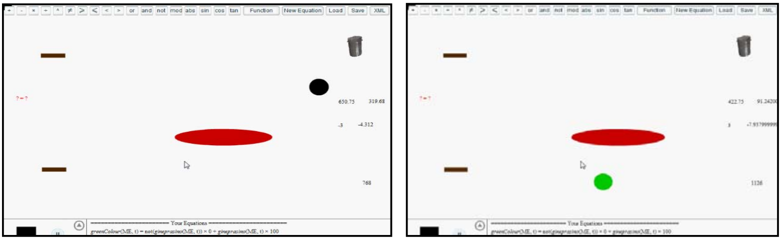


Figure 2: The ball’s different percentage of green colour according to its Y position

The above episode implies a qualitative transformation of the students’ mathematical activity form process-oriented into object-oriented. The operational conceptions regarding the MoPiX equations seem to have now given their place to more structural ones. In the process of constructing their first equation, the students invented a new – meaningful to them– symbol to which they attributed the properties of a varying over time quantity, used and manipulated in-built MoPiX symbols and inserted an inequality operator to specify the relation between their symbols. Translating their own ideas into algebraic equations, defining both their content (i.e. the symbols) and structure (i.e. the relation among the symbols) indicates that students have conceptualised the MoPiX equations as a “fully-fledged mathematical objects” (Sfard, 1991 pp.12).

A structural conception of the MoPiX equations is also advocated by the students’ series of actions in the process of constructing the second equation. Striving to transform their idea into a MoPiX equation, the students indentified a mapping between an existing Library equation and the one they attempted to create. Subtracting the Library equation’s structure and eliminating its content, the students formed a template whose fields they completed using terms relevant to the behaviour they wished to attribute to their object. This is a clear indication that the students were able to recognise the existence of structures external to the symbols themselves and to use them as landmarks to navigate their equations’ construction process.

It is noticeable, however, that in the process of constructing the second equation, the students’ conceptualisation of the first equation partially shifted to become operational again. Viewing the “y(ME,t)≤274” as a iterate comparing process between two numbers and the “gineprasino(ME,t)” as the outcome of this comparison, the students integrated the “gineprasino(ME,t)” varying quantity into their second equation treating it as an algebraic object. This aspect suggests the co-existence of both a structural and an operational conception of the MoPiX equations.

CONCLUSION

Our purpose in this paper was to illustrate the students’ development of a structural conception of the notion of equations in the context of constructing and controlling animated models. Editing ready-made algebraic-like equations and constructing new ones so as to assign behaviours to their objects, the students reached different *degrees of structuralization* (Sfard, 1991) (i.e. interiorisation, condensation, reification) shifting gradually their view of equations from process-oriented into object-oriented, without, however, those two approaches being mutually exclusive.

Concluding, under the constructionist theoretical perspective, in the present study reifying an equation was not a one-way process of understanding hierarchically-structured mathematical concepts but a dynamic process of meaning-making, webbed by the available representational infrastructure and the ways by which students drew upon and reconstructed it to make mathematical sense.

NOTES

1. The research took place in the frame of the project "ReMath" (Representing Mathematics with Digital Media), European Community, 6th Framework Programme, Information Society Technologies, IST-4-26751-STP, 2005-2008 (http://remath.cti.gr).
2. "MoPiX" was developed at London Knowledge Lab (LKL) by K. Kahn, N. Winters, D. Nikolic, C. Morgan and J. Alshwaikh.

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APPLYING A MATHEMATICAL LITERACY FRAMEWORK TO THE IRANIAN GRADE 9 MATHEMATICS TEXTBOOK

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This study presents a content analysis of the new Iranian grade 9 mathematics textbook, examining the extent to which it includes aspects of mathematical literacy. Two chapters were analysed for the extent of use of real world contexts, and the extent of emphasis on the mathematical modelling processes of formulation and interpretation. The new textbook used real world contexts in about a third of items. About 5% of items required complex formulation and/or interpretation of the mathematical result in real world terms. Most formulation was strongly supported by the text. This textbook has moved towards mathematical literacy, but more development is required before students will see mathematics as a human activity.

INTRODUCTION

Iran is a country with a very strong record of achievement in the International Mathematics Olympiad, so it was both unexpected and of great concern when it was found that Iranian students' performance in TIMSS in 1995,1999, 2003, 2007 was well below the international average. This leads to the hypothesis that the education system, with its emphasis on abstract mathematics, does well for the best students but is not meeting the needs of most students for their schooling or for their future lives. Many countries have participated in the international survey PISA because they value its assessment of 'mathematical literacy', which is concerned with how learning mathematics prepares students for their future. We consider mathematics literacy as defined by the OECD and cited by de Lange (2003):

> "Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments, and to engage in mathematics in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen." (p. 76)

Iran has not participated in PISA, so there is no assessment of Iranian students' mathematical literacy. This paper begins to address this question, by looking for evidence of whether Iranian textbooks build mathematical literacy.

The education system in the Islamic Republic of Iran is centralized and mathematics education goals are set at the national level. The Ministry of Education develops the syllabi and textbooks (Kiamanesh, 2005). There is a general expectation that teachers will follow textbooks and official documents closely. An investigation into textbooks, as in this article, is therefore more significant in a centralized educational system, such as Iran, than in some other countries.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 433-440. Thessaloniki, Greece: PME.

In Iran, all students use the same textbook for mathematics at Grade 9. After more than 15 years of using one textbook, a new version of the grade 9 book was introduced in the school year 2008/2009. The new version was based on the previous version and has similar chapters, although their order is changed. Importantly, some methods of introducing topics are different. In this new version, there are some new features of mathematics education such as recognising the relation between mathematics and the real world. There is some evidence from the preface of the new textbook that mathematical literacy is becoming a goal of math education in Iran.

In this article we report an initial content analysis of the new grade 9 Iranian mathematics textbook in terms of a theoretical framework based on the PISA concept of mathematical literacy. This study is part of a larger study that is investigating all aspects of mathematical problem solving in Iran and comparing it with some other countries, with a view to establishing a new curriculum framework for mathematical problem solving in Iran. The aim of this article is to establish to what extent the new Iranian grade 9 textbook presents mathematical literacy to students.

LITERATURE REVIEW

Several studies address research questions by investigating textbooks. One early study is by Howson (1996). He investigated grade 8 textbooks in 8 countries as part of the TIMSS study. He found that some students could respond correctly to some probability TIMSS items by using their common sense, whereas using school-taught procedures was error-prone. Gooya and Rafiepour (2004) observed this phenomenon in Iranian students' mathematics performance in TIMSS 1995. Some students in grade 7 had a better mathematical performance on some items than students in grade 8. Having learned mathematics without relation to real world considerations appeared to cause them to apply mathematics procedures without common sense.

Several studies show that there are links between the general nature of school curricula (in some countries represented well by textbooks) and results on international tests. For example, Kendal and Stacey (2004) gave examples of how different national emphases can be reflected in different performance in international comparative studies. One example they cited analysed different performances on 2 items in the algebra section of TIMMS-R (1999). Russian and Australian students had similar performance overall, but very different performances in these two items, one of which focussed on expressing generality and pattern and the other on understanding symbolic notation. These differences aligned with national curriculum orientation. Wu (2007) aimed to quantify such effects by measuring what she called “content advantage” to show how countries’ national performance on international tests is affected by the alignment of its curriculum with the assessment. These studies reinforce the view that if mathematical literacy is regarded as a valuable outcome of schooling, then it should be well represented in school curriculum and textbooks.

METHODOLOGY

This study is a content analysis of the new grade 9 textbook. We selected grade 9 because these students are about 15 years old, the age of the students tested in PISA. Additionally grade 9 is the final grade that all Iranian students have the same math curriculum and textbook. The first author and a colleague choose 2 of the 9 chapters of this textbook by considering three factors. First, each should be a good subject for supporting mathematical literacy. As de Lange (2003) noted, some mathematics topics are more suitable than others for supporting mathematical literacy. The second factor is that the topics should be introduced for the first time in Grade 9, so that they are treated fully. Vincent & Stacey (2008) noted that topics introduced in earlier years are often given abbreviated treatment in subsequent textbooks. The third factor is that the topics would also be suitable for later international comparisons at Grade 9 level. One chapter was on linear and first degree equations, and it contained subsections on equations and equation solving, equations of straight lines including gradients, perpendicular lines and distance between two points in the plane. The other chapter was on trigonometry relations, containing sections on trigonometric ratios and relationships between them, gradients, angles of lines etc.

Every chapter in the Iranian Grade 9 mathematics consists of 5 labelled sections: explanation, worked examples, activities, exercises in the classroom, and problems. The explanation section introduces new information and reasons. Worked examples give model problems and solutions for students to follow. “Exercises in the class” gives questions for students to practise the methods introduced in the worked examples. Sometimes this section also provides some opportunity for reflection and for linking different mathematics topics. The activities section is an innovation which did not appear in the previous textbook. In this section, we find some further explanation, guided reflection, and activities to build relations and connections within and outside mathematics. Activities in this new textbook are a rich source for mathematics education and strength of the new book. They give opportunities to students for doing mathematics, comparing different results and conjecturing. The final part is a problems section which reviews all work in the chapter or sub-chapter and includes routine exercises and real world problems. A few illustrations of the material from these sections are given in Figure 1.

In this study we consider all sections except the explanations, which are more complex to investigate. The units of analysis (referred to below as ‘items’) are either individual problems (e.g. Items 1 and 2 in Fig. 1) or sequences of instructions and problems grouped as one in the textbook (e.g. Items 3 and 4 in Fig 1). In total there were 78 items in the 2 chapters. Each item is rated separately on 6 criteria to make a mathematical literacy framework, as described below. It was intended that the rating would be done independently by the first author and a Farsi-speaking colleague, so that the rating reliability could be reported. This has not yet been possible, so only one set of ratings is reported. The other ratings will be completed before the conference paper is presented.

Item 1 (Problem): A lizard lays 30 to 50 eggs in spring. Eggs take 90 days to hatch. The length of a new born lizard is about 30 cm, and increases by 22.5 cm per year on average. At what age does a lizard reach 80 cm in length?

Item 2 (Problem): A person 1.70 metres high wants to raise a 3 metre bar to an angle of inclination of 60°. At first she is standing with one end of the bar on the floor and against a wall, holding the other end exactly up to her height. Then she walked toward the wall raising the bar until the angle of inclination became 60°. How many metres did she walk to the wall? (A diagram was supplied with this problem)

Item 3 (Exercise in the classroom): Solve the equation 3(2x-7) =81 and check the correctness of each operation giving the reasons. Solve the above equation in another way and explain the correctness of your work in every step.

Item 4 (Activity): Sara and Maryam are two sisters. When Maryam was born, Sara was 4 years old. When Sara is 7 years old, how old will Maryam be? When Maryam is 20 years old, how old will Sara be? Let the age of Maryam be x and the age of Sara be y. Write an equation that shows this relation. If Maryam becomes 5 years older what will happen to Sara's age? If Maryam becomes 8 years older what will happen to Sara's age? Complete the table (table given with blanks to complete) comparing Sara's age and Maryam's age. Putting Maryam's age on the x-axis and Sara's age on the y-axis, draw a graph by using the table.

Figure 1. Four illustrations of items from textbook (translated and abbreviated)

Theoretical framework

"Mathematics literacy" is the central concept of PISA mathematics (de Lange, 2003), and the processes of mathematical modelling (called mathematisation by de Lange, 2006) are all key components of mathematical literacy. These processes relate to *formulating* real world problems in mathematical terms (called vertical mathematisation by Freudenthal, 1991), so that they can be *solved* as mathematical problems, and then the mathematical solution can be *interpreted* to provide an answer to the real world problem. In the formulation stage, the problem solver faces a problem situated in a real context, and then gradually trims away aspects of reality, recognizing underlying mathematical relations, and describes the stripped down problem in mathematical terms. In the interpretation stage, the problem solver considers the mathematical result(s), and uncovers their meaning in terms of the real context.

In this paper we report on 6 criteria for mathematical literacy, which focus on the formulation and interpretation processes. The first component records whether there is any real world context for the problem, the next three components focus on aspects of formulation and the final two relate to interpretation. Table 1 gives examples of the classifications for the 4 items in Fig. 1.

Criterion 1. Does this item contain any reference to a real world context? (If there is no real context, the formulation/ interpretation criteria below are not applicable).

Criterion 2. Is the required formulation complex? Factors making formulation complex include having extra data present (e.g. Item 1), a shortage of data so that gaps have to be filled by the problem solver, multiple steps of formulation (e.g. Item 2), etc.

Criteria 3 and 4. Is formulation supported by the textbook or left to the student alone? In some cases formulation is completed by the textbook or supported (e.g. by specifying variable names and axes in Item 4). These criteria apply whether the formulation is simple or complex.

Criterion 5. Is the problem question open or closed? An open question will have more than one acceptable solution. None of the illustrations are open questions. A problem that has two routine methods for solution (e.g. Item 3) is not an open question.

Criterion 6. Does the solution require interpretation? After solving the mathematical problem, does the meaning of this strictly mathematical solution in the real world setting require consideration? Item 2 is classified as requiring interpretation because two mathematical results have to be linked together.

	Set in real world context	Complex formulation required	Textbook - supported formulation	Formulation by students alone	Open question	Solution requires interpretation
Item 1	yes	yes	no	yes	no	yes
Item 2	yes	yes	no	yes	no	yes
Item 3	no	no	no	no	no	no
Item 4	yes	no	yes	no	no	no

Table 1. Illustrating the classification system with items from Fig. 1

RESULTS

In the 2 chapters, there were a total of 78 items to be analysed according to the 6 criteria. Table 2 shows the percentage of items in each textbook section meeting each criterion.

Textbook section	Set in real world context	Complex formulation required	Textbook - supported formulation	Formulation by students alone	Open question	Solution requires interpretation
Worked Examples (N=16)	18.8	0.0	12.5	0.0	0.0	0.0

Exercises in the classroom (N=13)	38.5	0.0	23.1	15.4	0.0	0.0
Activities (N=19)	36.8	0.0	31.6	0.0	0.0	5.3
Problems(N=30)	40.0	16.7	16.7	16.7	0.0	13.3
Total (N=78)	34.6	6.4	20.5	8.9	0.0	6.4

Table 2: Percentage of all items meeting framework criteria by textbook section.

Table 2 shows that about one third of the items in these two chapters use a real world context. Further analysis (not reported in the table) showed that the real world contexts were varied, and it was not the case that a few standard contexts (e.g. the ladder against a wall in Item 2) were used repeatedly. Given that these chapters were selected because they had strong real world applications, it is likely that other chapters would have fewer items involving real world contexts. However, this already appears to be a major departure from the previous textbook, where topics were mostly treated abstractly.

Only a few items (5 problems, including Items 1 and 2) required complex formulation. Many more items would have required complex mathematical procedures for their solution, but here the complexity refers only to the formulation (vertical mathematisation) stage.

There were 23 items which required formulation to translate the problem into mathematical terms. Columns 4 and 5 of Table 2 show that in most items, the textbook provided strong support for this step, leaving students to carry it out alone in only 7 items (9%). In keeping with their instructional purpose, the worked example and activity sections always provided support for formulation.

With the interpretation criteria, none of the questions were classified as open questions. In 5% of items (1 activity and 3 problems), the mathematical results needed some interpretation. However, this interpretation is generally very shallow. For example, Item 2 was rated as requiring interpretation of results because the answer is found by comparing the distance from the wall at two times, both distances having to be calculated using trigonometry.

DISCUSSION & CONCLUSION

As noted earlier, the grade 9 Iranian textbook has now been revised after more that 15 years. In general this new book has made a substantial effort towards reform. The results above show that the application of mathematics has some presence in this textbook whereas the previous version introduced subjects mostly in the pure mathematics form. However the results of this study show a gap between it and the concept of mathematics literacy introduced by PISA. In particular, the results in

Table 2 show that very few items require interpretation. The interpretations required are very simple and straightforward, whereas items that properly represent mathematical literacy need significant interpretation of mathematical results in terms of the real problem situation. Some authors (for example, Stillman, 2008) make a distinction between applications of mathematics (where students learn how to apply mathematical procedures routinely to real situations) and mathematical modelling. The new textbook has, in these terms, moved towards applications of mathematics, but needs to include more mathematical modelling. In almost all units, the textbook presents the relevant mathematics component to students, without leaving space for students work on this phase of modeling. The role of interpretation and checking and looking back in these units is very colorless.

The new activities section of the textbook provides some rich materials for enhancing mathematics education for Iranian pupils. However, there is room to improve these activities. There could be more items requiring interpretation and checking a strictly mathematical answer against a real world situation. Almost all activities are introduced in a simplified situation and the relevant mathematics is identified by the textbook while really real world problems are often messy with too much or too little data. Therefore the activity sections could be improved by encouraging students to discover the mathematical concepts that exist in the real world situation by themselves and giving them some opportunity for interpretation, and to deal with the type of multi-dimensional ill-defined problems (such as choosing a route by public transport) that occur in real life.

The previous textbooks appear to have been based on a widespread myth that a person has learned mathematics well will be able to apply it in a real world situation. However, we agree with de Lange (2003) that although pure mathematics is essential for doing mathematics in the real world, it is not enough.

There are many different points of view from which the new textbook should be studied, and we recognise that this is only a partial analysis from the point of view of mathematical literacy. However, even this preliminary analysis of the new textbook shows that the effort at improvement is good, but not sufficient. There are too few opportunities for students for interpretation and to refine again and again a mathematical model. We have a long journey to reach the desirable point and at this time we stand at the beginning of this path. We need further textbook development, to be written in a new paradigm. The new paradigm should take "basic mathematics literacy" for all students as a fundamental principle. In this type of textbook, students may find mathematics as a human activity.

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HIGH SCHOOL MATHEMATICS TEACHERS' DIDACTICAL BELIEFS ABOUT ERRORS IN CLASSROOM

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Math educators frequently point to the teacher's central role in establishing the mathematical quality of the classroom environment. As people act in the light of their beliefs, this quality depends on the teacher's own didactical beliefs and, in particular, their beliefs about ways to address errors in their classroom. In this study we examine this issue while distinguishing between teachers' error-related beliefs in optimal class-situations, and their related, declared, decisions in realistic settings.

THEORETICAL SETUP

It widely documented that individuals', and particularly mathematics teachers' behaviours are strongly affected by their beliefs (e.g. Clark & Peterson, 1986; Thompson, 1992). In our study we adopt Green's commonly held definition that beliefs are psychological propositions held by an individual to be true (Green, 1971), as expressed, for instance in the statement: "students must be given much homework".

When investigating mathematics teachers' beliefs, some researchers refer to their beliefs in general (e.g. Hoffmann, 2003), while others focus on specific issues such as teachers' interpretations and beliefs of educational reform recommendations (Nathan & Knuth, 2003), or teachers' beliefs about technology adoption (Suger et al., 2004). Since errors are an inevitable and potentially stimulating part of teaching-learning situations, we find it important to *investigate teachers' beliefs regarding the use of errors in mathematics classrooms.*

Teachers' beliefs to the use of errors in mathematics classrooms

In this section we report on research into the beliefs of teachers regarding the use of errors in mathematical classrooms. For example, Gasatsis and Kyriakides (2000) examined elementary school teachers' attitudes towards their pupils' mathematical errors. The interpretations of 254 school teachers of their pupils' errors, showed that the teachers saw the *pupil* factor (pupils' abilities, attitudes and psychological situation) as the most important source of errors, and the *knowledge* factor (special characteristic of concepts involved in the task) as the next most important source of errors. However, this research did *not* refer to teachers' attitudes to possible ways of *using* their pupils' errors in class.

Tsamir and Tirosh (2003) found that out of 14 *elementary* school teachers, a small majority thought that error-triggering tasks should be presented; as to whether teachers should initiate the presentation of error-based activities, opinions were

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 441-448. Thessaloniki, Greece: PME.

evenly divided. We found it important to investigate *high school* teachers' related didactical beliefs.

Furthermore, teachers in Tsamir and Tirosh (2003) mentioned the relevance of the students' ability-level concerning whether or not students' errors should be discussed in the classroom, and this issue also arises in Groose and Renkel (2004). In our experience, mathematics teachers often revealed in casual conversation that the issue of student ability-levels is a weighty factor in their attitudes regarding the use of errors in the classroom. *Therefore, in our research we address the issue of varying student mathematical-(matriculation exam)-levels*.

In this paper we investigate high school teachers' beliefs and didactical decisions about the use of errors in their mathematics classroom. We have posed the following questions: (a) what are teachers' beliefs regarding the use of errors in their mathematics classroom in *optimal* circumstances, and do they differ for students of different math-levels? (b) Is there a difference between the teachers' beliefs in optimal conditions and their related didactic decisions in their classroom?

METHOD

Eighty-five high school mathematics teachers answered in writing a two-part questionnaire regarding: (1) their beliefs about the use of errors in the mathematics classroom in *optimal* circumstances; and (2) their related, declared didactical-decisions in their *actual* mathematics classroom .

First, the teachers were asked to imagine a situation in which they had no constraints preventing them from teaching optimally, and to express their justified views on the following four teaching approaches to mathematics exercises: (1) Presenting *only the teachers' correct solution*; (2) Presenting *students' correct solutions*; (3) Presenting *students' correct and incorrect solutions*; and (4) Presenting *incorrect solutions that didn't arise in the class*. The teachers could express their attitude to each approach (they were not limited to choosing just one) by marking in what frequency they would use this approach: always, frequently, rarely and never. In order to simplify the results, we put together the percentage of *"always"* and *"frequently"* under the heading "often", and the percentage of *"rarely"* and *"never"* under the heading "rarely".

In the second part, the teachers were presented again with the four approaches mentioned above, but this time asking in which ways they *actually* act in their class, and why. In both parts, the teachers were asked to differentiate between classes of three math-exam-levels. We would like to note that only about 20 teachers provided the requested explanations for their answers.

DATA ANALYSIS

In our analysis of the data, we used Cramer's V to measure the relationship between teachers' support-level of a certain approach ("often" or "rarely") and students' exam-level ("low", "medium" or "high"), as these are two categorical variables, and a 2-by-3 contingency.

We present the results as answers to the two research questions.

How would the teachers treat errors – optimally?

Table 1: percentage[#] of teachers declaring with what frequency (often or rarely) they would optimally use each approach, in the three exam-level classes

Approach	1 presenting *teacher's correct* ideas only		2 discussing *students' correct* ideas		3 Discussing *students' correct* and *incorrect* ideas		4 discussing *incorrect* solutions that didn't arise in class	
	Often	Rarely	Often	Rarely	Often	Rarely	Often	Rarely
Exam level								
Low	80	17	45	50	46	47	23	64
Medium	72	23	77	19	76	16	50	40
High	59	34	81	12	81	11	60	29
Cramer's V	rc= 0.184*		rc= 0.385**		rc= 0.386**		rc= 0.345**	

[#] The total does not add up to 100%, some teachers did not answer each question.

*p < .05 **p < .01

Table 1 Indicates that 80% of the teachers expressed support of approach 1 (presenting *only the teacher's* correct solution) for students in low exam-level classes. This support decreases as the students' level rises. As for approach 2 (discussing *students' correct* ideas) the support is inverted: teachers declare that it should optimally be used more frequently in the better classes and less frequently in the weak classes. Approach 3 (presenting *students' correct and incorrect responses*), similarly to approach 2, was poorly supported (45%) for lower-exam students, and highly supported (81%) for the higher exam-level students. Only a minority of the teachers stated that they would optimally use approach 4 (presenting *errors that had not necessarily arisen in class*). The percentage of support increased (to a maximum of 60%) as the *exam- level* increased. This increase in support, for approaches 2, 3 and 4, as the students' exam-level in Mathematics goes up, was found significant by Cramer's V.

Explanations to the above expressed varied opinions. Teachers *supporting* approach 1, (presenting *teacher's correct* ideas only), explained that "it is important to show the students the steps and stages we go through, so that they, later, can solve the problems independently". As *opposed* to that, others explained that it is not healthy to give the impression that the teacher's method is the only one: "There isn't just one correct and effective way …" Furthermore, students should be given opportunities to deal with mathematical problems on their own, rather than being "spoon-fed". One teacher noted: "You don't learn much from just watching – you learn from doing." Some explained their *conditional support* for this approach; they said it is appropriate only for low level exam students, whose mathematical creativity is under-developed.

Teachers *supporting* approach 2, (discussing *students' correct* ideas), explained that one must take into account students' ideas "So as to understand the student's ways of thinking and to be able to progress accordingly." Another was so as to "open up new ways of thinking." A third was affective – "it motivates them…" Some of the teachers who *opposed* to approach 2 to low-exam level students, explained: "For the students with difficulties, this approach can make them frustrated at not having a solution to the problem". Other teachers expressed the feeling that this approach didn't go far enough: "I agree with the idea of asking for students' solutions, but I don't agree with only relating to the correct answers – part of the teaching process lies in relating to mistaken answers, so that students can analyze their mistakes and develop their critical faculty". Of the teachers who gave *conditional support* for approach 2, some teachers said that this approach should optimally be used more often with higher exam-level students, and less with lower-exam level students, because "Wrong solutions sometimes become embedded in the minds of weaker students, and then it is hard to correct them."

Teachers *supporting* approach 3, (using *students' correct and incorrect responses* in class), explained that "This way I will get most of the students involved, and we shall learn from the mistakes", that this approach is "stimulating". One of the teachers was more pragmatic: "So that students know where they were mistaken and how they can avoid it for the next time." Teachers *opposing* approach 3 wrote: "When we teach new material, I think it's difficult to begin with confusing the students (strong ones too) with complicated ideas. Weak pupils can't always absorb the information, and remember what is correct and what is incorrect; in the long term, I'm worried my pupils will remember precisely the mistakes". Explanations of *conditional support* of this approach were, for example, "I will refer to everything, but with lower level students I would relate to the errors personally; with stronger students, it would be interesting to discuss everything." There are also teachers who fear that the openness demanded by this approach will lead to unforeseen directions in the lesson, and the weaker students will "loose track, there will be noise and loss of discipline."

The few teachers *supporting* approach 4 (discussing *incorrect solutions that didn't arise in class*), explained that "Through the error you can understand the logic of the material." Teachers who *opposed* to this approach were concerned lest their students acquire mistaken concepts. "In principle I don't like the approach of analyzing erroneous solutions; it shifts the thought from the right to the wrong, and may firmly fix mistakes and confusion." A large number of teachers expressing *conditional* *support* for this approach supported it for good students only: "Among weak pupils, I'm fearful of the fact that *they* didn't err may confuse them, and they might think this is the correct answer. This fear exists less with students taking the higher level exam, they have a higher sense of control". Other, affective reasons for this opinion were: "The good students would like this, it would encourage them and give them self confidence, but the medium and weak students may get bored and not like it."

While teachers may agree that a certain method of teaching is desirable, this does not mean that they carry it out.

What are the differences between optimal and actual use of errors in class?

In Table 2, we present a comparison of teachers' responses to all four approaches. In this table we show the percentage of teachers who declared they *would* (optimally) or *do* (actually) use each approach *often*. We also bring the mean of all answers to each approach for the optimal and actual situations, and the T value of the t-test we used to compare them.

Table 2: Percentages of teachers who declare they would, optimally, or do, actually use each approach *often*.

Approach	1 *teachers' correct* ideas only		2 *students' correct* ideas		3 *students' correct* and *incorrect* ideas		4 *incorrect* solutions - didn't arise in class	
	Optimal	Actual	Optimal	Actual	Optimal	Actual	Optimal	Actual
Exam level								
Low	80	80	45	42	46	34	23	7
Medium	72	75	77	64	76	60	50	31
High	59	54	81	67	81	71	60	39
Mean[#]	3.00	3.07	2.92	2.76	2.97	2.69	2.44	2.08
T	-.806		2.224*		4.132**		4.477**	

*Sig<0.05, **Sig<0.01

[#]1: Never; 2: Rarely; 3: Frequently; 4: Always.

<u>Approach 1</u> (presenting *teacher's correct* ideas only):

The respondents reported similar results for their optimal and actual behaviour in class. In fact, far from being constrained to use this approach *less* than they would like, it seems that for the medium level students, teachers actually use this method slightly *more* than they believe it should optimally be used. This is contrary to the other three approaches, which the teachers used less in practice than they believed they should ideally.

Approach 2 (discussing *students' correct* ideas):

There is a significant difference between optimal and actual use of this approach in class. Teachers believe they should ask the students for ideas as to how to solve exercises and take up the correct suggestions, significantly more often than they actually do.

Approach 3 (discussing *students' correct* and *incorrect* ideas):

In this approach too, there is a highly significant difference between teachers' ideas of what *should* be done in class in the optimal situation, and what they *actually* do. Again, they thought that this approach should be used much more than they actually use it.

Approach 4 (discussing *incorrect* solutions that *didn't arise* in class):

In this case the difference between what teachers would do in *optimal* situations, and what they *actually* do in class, is the largest. While 23% of the teachers declared they would give students with difficulties *'Find the mistake'* tasks, only 7% say that they actually do so. This difference is found regarding all levels of students, and it is statistically highly significant.

DISCUSSION

Various beneficial error-based experiences in the mathematics classroom are mentioned in the literature (e.g., Bainbridge, 1981; Borasi, 1996; Groose & Renkel, 2004; Melis, 2005); and many advantages are listed for using students' errors in practice, including: promoting students' motivation, promoting students' knowledge of concepts, promoting students' understanding of the nature of mathematics, and promoting students' reflection and inquiry performances (e.g., Borasi, 1996; Melis 2005). However, in order that teachers implement error-based class discussions, they should believe that this is "the right thing to do", and that "doing it" is feasible. Research examining the issues of mathematics teachers' error-related beliefs is scarce and publications address elementary school teachers, with no differentiation between "optimal" and "practical" circumstances. The contribution of our study is in examining mathematics teachers' error-related beliefs, in addressing *secondary school mathematics* teachers, and in refining our research tools to differentiate between the examination of teachers' beliefs about "what error-related approaches should be used in their class in optimal conditions", and "what realistic conditions allow them to implement".

Our findings indicate the following teachers' beliefs about error-related issues in optimal classroom conditions: (1) presenting the teacher's correct solution is beneficial for all students; (2) error-based activities are beneficial for high level students (e.g., Groose and Renkel, 2004); (3) usually, errors should be discussed when they were made by students in class; and (4) weak students should only be presented with correct solutions, because errors may confuse them and cause them frustration (e.g., Tsamir & Tirosh, 2003).

As mentioned before, our questionnaire allowed the teachers to distinguish between their beliefs in optimal settings, as opposed to the actions they actually take, giving them the opportunity to explain the discrepancy and to express their frustration at not being able to achieve the aims they had set themselves. Indeed, the gap found between "optimal" and "actual" error-based activities, in all items of the questionnaire, showed that there was a significant statistical difference between teachers' beliefs in optimal conditions and their declared didactic decisions taken in the practice. This outcome is concordant with studies that show discrepancies between teacher beliefs and teacher behaviours and decisions (e.g. Thompson, 1992). The innovation in this study is that the participants themselves were aware of this gap, and gave their own reasons for it.

Most teachers declared that they actually teach, most of the time, by traditional methods. They present correct solutions to mathematical problems, without taking into account students' various correct solutions and students' incorrect ideas. This is especially true for weak students. As the level of the students rises, teachers stated that they tend to engage their students in posing and discussing solutions, and of addressing students' correct and incorrect ideas during class discussions. As for initiating activities based on errors that did not arise in the class being taught, as recommended by some mathematics educators (e.g. Avital, 1981; Smobol & Applebaum, 2003), a small number of teachers declared that they actually use this method, and, in accordance with their attitudes, this percentage is smallest when teaching the low level math groups.

One may wonder, what *is* the best way to address students' errors in the Mathematics classroom? Is some teachers' belief, that presenting mistakes may confuse the weak students, proven in research? It has been shown that working with incorrect solutions can lead to enhanced learning outcomes for students with a *strong* mathematical background (Groose and Renkel, 2004), and "junior school children did not appear to suffer from being systematically exposed to their own mistakes" (Bainbridge, 1981, p.12). So, if it can contribute to stronger students, and doesn't harm the others, should this method be used in class? Is it worth precious teaching time it consumes? We see this as a fertile ground for further research about teachers' beliefs, their didactical decisions and classroom experiences and about students' knowledge and feelings.

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REVERSIBLE REASONING IN RATIO SITUATIONS: PROBLEM CONCEPTUALIZATION, STRATEGIES, AND CONSTRAINTS

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The primary aim of this study is to identify the strategies and constraints that middle-school students encounter in reversing their thought process in ratio situations. Using Vergnaud's (1988) construct of theorems-in-action and Thompson's (1994) notion of quantitative reasoning as analytical lenses, we analyse the concepts and operations that the six participants in grades 6, 7 and 8 deployed when asked to solve a set of specifically chosen multiplicative comparison problems, algebraically equivalent to $(a+b)x = q_1 + q_2$ *and* $(a-b)x = q_1 - q_2$. *Our case study illustrates how the same problem can be conceptualized differently by different students, cueing different cognitive resources and solution paths. Difficulties in conceptualizing the problem quantitatively led to observable constraints on students' reversible reasoning.*

BACKGROUND, CONTEXT AND OBJECTIVES

The concept of ratio is a key constituent of school mathematics and aims at consolidating and extending students' multiplicative reasoning (Hoffer & Hoffer, 1992). While it has a more numerical aspect at the elementary level, at the middle grades, students are required to analyse the functional aspect (in terms of tables, graphs, verbal rules, and equations) of a ratio to understand how two quantities vary in relation to each other. A ratio can be interpreted in different ways. From a mathematical perspective, Freudenthal (1983) considered a ratio as "an equivalence relation in the set of ordered pairs of numbers (or magnitude values)" (p. 180). A ratio ($a:b$) can also be understood from the perspective of a fraction (*a/b*) and as a quotient (*a* divided by *b*). In terms of units-coordination, a ratio can be interpreted as requiring the coordination of two number sequences. From the point of view of quantities, a ratio can be regarded as a multiplicative comparison between two extensive quantities (Schwartz, 1988) in the same measure space. From a cognitive perspective, Thompson (1994) considers a ratio as "the result of comparing two quantities multiplicatively" (p. 190) and extends the concept of ratio to that of rate which is defined as a 'reflectively abstracted ratio'. Kaput & West (1994) used the notion of quantities to analyse students' building-up pattern of reasoning with ratios. Clark, Berenson, & Cavey (2003) distinguish between two conceptions of ratio: descriptive ratios and functional ratios. Examples like comparing the number of boys and girls in a class constitute a descriptive ratio as they are concerned with static counts of objects in two sets. On the other hand, functional ratios deal with consistent linear relationships between two variables.

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Reversibility of thought

Piaget differentiated between two forms of reversibility of thought: (i) negation which refers to the idea that every direct operation has an inverse which cancels or negates it (e.g., multiplication is cancelled by division) and (ii) reciprocity which refers to the idea that a relation can be interpreted bi-directionally (e.g., combining two quantities to form a new quantity and decomposing a given quantity in terms of its constituents). As can be inferred from Inhelder & Piaget (1958), the second form of reversibility, i.e., reciprocity gets its essence whenever a relation is defined. Ratios are constitutively statements of relations, more specifically multiplicative comparison relations. However, despite being a central component of mathematical reasoning, reversibility of thought has not been given much attention in mathematics education research, partly because of its implicit nature. As pointed out by Lamon (2007): "researchers know very little about reversibility or about multiplicative operations and inverses, and these could be subjects for a valuable microanalysis research agenda" (p. 661). Further, our review of literature leads us to conclude that not much attention has been given to the analysis of ratio situations from a reversibility perspective. In the present study, we analyse the strategies and constraints that students encounter as they were required to solve a set of reversibility situations in the domain of ratio. We also illustrate how the same problems were conceptualized differently by the participants, cueing different resources and leading to different solution paths.

THEORETICAL FRAMEWORK

In this study, we use Thompson's (1994) notion of quantities and quantitative relationship and Vergnaud's (1988) concept of theorems-in-action to understand how reversibility operates in ratio situations. Thompson (1994) asserts that characterizing students' reasoning in terms of quantities allows us to capture important structural characteristics of their reasoning when they are asked to deal with complex situations. Thompson (1990) defines a quantitative relationship as "the conception of three quantities, two of which determine the third by a quantitative operation" (p.13) and a quantitative operation is regarded as "the conception of two quantities being taken to produce a new quantity" (p.11). For instance, a multiplicative comparison like "Paul is twice as old as his brother" is an example of a quantitative relation. It involves reasoning with two quantities (Paul's age and his brother's age) without using their numerical values but rather using the relationship between the quantities. In other words, one can reason about two (or more) related quantities without knowing the actual numerical values. The quantitative relation may assume different forms like additive and multiplicative comparison, additive and multiplicative combination, as well as composition of quantities and generalization of relations (Thompson, 1990). In this study, we are particularly interested in investigating quantities that are in multiplicative relationship and where some of the values of the quantities in the relationship are unknown. Reversibility involves working with known and unknown quantities and consequently requires reasoning about relationships among quantities.

One of the aims of the present study is to identify the ways in which students use quantitative reasoning as a basis for reversing their thought process, in contrast to using numerical reasoning.

Theorems-in-action

Vergnaud (1988) defines *Theorems-in-action* as the "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (p. 144). They are "held to be true propositions" (Vergnaud, 1997, p. 14) for a certain range of situations and may even be flawed. As clearly defined by Hiebert & Behr (1988), Theorems-in-action may also be viewed as an action (physical or mental) on the part of the cognizing subject "that provides behavioral evidence of implicit knowledge of a more formal property or method or 'theorem' of mathematics" (p. 11). In addition, these relationships may not be expressed verbally by the students. In other words, Theorems-in-action are the implicit mathematical operations that students use in solving mathematical problems. In one sense, they can be regarded as the mental counterpart of the symbolic operations, relations or transformations.

METHOD

Data for the present report come from a larger study about the strategies and constraints that students encounter in reversing their thought process in multiplicative situations. We used task-based interviews (Goldin, 2000) to collect data from three pairs of above-average students in grades 6, 7 and 8 in an urban middle school. The six participants in the study are: Ted and Cole (the pair of sixth-graders), Aileen and Brian (the pair of seventh-graders), and Jeff and Eric[1] (the pair of eighth-graders). The first author interviewed the students on a range of tasks involving multiplicative comparison situations for a period of two weeks in May 2008. We used two cameras to record the interviews so as to produce a restored view (Hall, 2000). The first camera focused on students' movements, gestures and facial expressions, besides recording their spoken words and interjections, while the second camera focused on their writings, drawings and actions (e.g. pointing movements with their fingers) with external materials. We also kept a record of what students wrote or drew on the worksheets that were provided to them.

DATA AND ANALYSIS

The two situations under study are algebraically equivalent to $(a+b)x = q_1 + q_2$ (Type I) and $(a-b)x = q_1 - q_2$ (Type II). Due to page constraints, we present the findings for only one representative problem in each category.

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[1] All names are pseudonyms.

PROBLEM 1. Joe had some marbles. Then his friend gave him 5 times as many marbles as he had initially. Now Joe has 42 marbles. How many marbles did Joe have initially? (algebraically equivalent to $(1+5)x = 42$)

Ted and Cole: The pair of six-graders

Their very first move was to divide 42 by 5 rather than six. The students did not take into account that Joe also had a share. In this problem, Joe's marbles are both the referent and the compared quantity and this may be one reason why they divided by 5 instead of 6. Here, the referent quantity (Joe's initial amount) has an implicit nature. It was the mixed number quotient ($8\frac{2}{5}$) resulting from their division of 42 by 5 that cued the students to realize that their solution could not be correct (Joe could not have $\frac{2}{5}$ of a marble). They did not conceptualize the problem context as a ratio situation (1:5) between two unknown quantities (initial and added marbles) and this hindered them from coordinating the known and unknown quantities in both the additive and multiplicative relations that constituted the quantitative structure.

Aileen and Brian: The pair of seventh-graders

Brian: He has 7. Because if you have, uh, five times as much, so you still have to have that one that you have originally. So you would have 6 of how many sets whatever. So 42 divided by 6 that's 7 because 6 times 7 equals 42.

Aileen: Because I started plugging numbers like, I knew it couldn't be 1 through like 3 because that would only give you 15 and if you added that to the original then you wouldn't have 42. So I went up higher, and started up at 5 and 5 times 5 is 25 plus 5 is 30 and I was getting close to it. I went to 6 and 6 times 5 is 30 and plus 6 is 36. Then I went up to 7. Seven times 5 is 35 plus 7 is 42.

Brian could conceptualize the problem explicitly from a ratio perspective. In addition, he could interpret the end result 42 as being constituted as 6 'sets' of unknown numerosity when he said "6 of how many sets whatever" to decompose 42 as a partitive division situation, which indicates reasoning reversibly. In contrast, rather than decomposing the end quantity to find the two constituting quantities, Aileen's strategy was to work forward by systematically plugging-in numbers until she reached the end result of 42 – a building-up strategy.

Jeff and Eric: The pair of eighth-graders

Both Jeff and Eric started by dividing 42 by 5 to get 8.4. The decimal part (0.4) led them to observe that their answer was not correct and prompted them to look at the problem from another angle. Their responses are highlighted below:

Jeff: Uh, I got that initially he would have 7 marbles because it says that he had some marbles then his friend gave him 5 times as many. So that would be the same as, uh, well what I came out [with] was $x + 5x = 42$. So I just added the x's together and divided 42 by 6 and got 7.

Eric: Uh, not really. I just started out trying, you know. I just divided 5 into 42 and I kind of knew that I get a decimal. So, and then I was getting restarted you know, figuring out another way to get it because I know 8.4 probably is not the correct solution because you can't have a fourth of a marble.

Like Ted and Cole, the eighth graders started with the division of 42 by 5 and it was the decimal result that cued them to choose an alternative solution path. By defining the unknown quantity as x (the number of marbles that Joe had initially), Jeff could articulate the additive and multiplicative quantitative relations within the quantitative structure of the problem. Constructing the quantitative structure explicitly as an algebraic expression lowers the cognitive load required in thinking about an unknown quantity in a quantitative relationship and allows one to solve the problem directly rather than reasoning reversibly. On the other hand, Eric tried to plug-in numbers after observing that 42 divided by 5 did not result in a whole number solution. Note also his confusion between decimal and fractional notation (.4 as one fourth).

PROBLEM 2. A sum of money was divided between Alan and Bill. For every \$5 that Alan received, Bill received \$3. Given that Alan received (i) \$10 more and (ii) \$7 more than Bill, calculate how much Bill received? (algebraically equivalent to $(5-3)x=10$ and $(5-3)x=7$). For the sake of brevity, we report only the result for case (ii) for the sixth graders and for case (i) for the other participants.

Ted and Cole: The pair of six-graders

They generated members of the equivalence class 5:3 by making a table of values with rows: 5, 3; 10, 6; 15, 9 and so forth thus working in a forward direction, starting from the given ratio. Ted deduced that the answer should be between the pair (15, 9) and (20, 12) since 7 lies in between the differences of 15 and 9 (i.e. 6), and 20 and 12 (i.e. 8). This is one of the instances where they were directly confronted with a reversibility situation in that knowing the difference (i.e., 7) of the components of a ratio of two quantities, they had to find a ratio that satisfies it. In their first approximation, they added half to 15 and 9 respectively to obtain $15\frac{1}{2}$ and $9\frac{1}{2}$. Then Cole observed that there is a difference of 3 between 9 and 12 and half of this interval would be $1\frac{1}{2}$ which led him to obtain \$$10\frac{1}{2}$ for Bill. However, he added the same amount ($1\frac{1}{2}$) to 15 to obtain $16\frac{1}{2}$. After observing that his solution ($16\frac{1}{2}$ and $10\frac{1}{2}$) did not produce the required difference of 7, he re-evaluated his calculation and figured that the middle value of the interval 15 to 20 is $17\frac{1}{2}$. In other words, Cole used linear interpolation as his theorem-in-action to find the components of the 5:3 ratio that corresponds to a difference of 7 units. While Cole's approach is very creative, it avoided reasoning reversibly. Our intention in using an odd number for the total difference in this problem was to make it difficult to use a forward process of generating successive values for the ratio until the desired difference was obtained, thus possibly encouraging an approach that could involve reversible reasoning. We were unsuccessful in provoking reversible reasoning but the problem did provoke a creative solution on Cole's part involving linear interpolation.

Aileen and Brian: The pair of seventh-graders

Aileen's strategy for solving problem 2(i) was similar to Ted's and Cole's. She scaled up the ratio in increments of one unit until she reached a difference of 10. On the other hand, by relating the difference in the components of the ratio (5 – 3 = 2) to the difference in amount ($10), Brian could observe that the multiplier for calculating the total difference is 5 as evidenced by the following interview quotes:

Brian: Fifteen [his answer for how much Bill received]. Because, if there is $10 more, then 5 is two more than 3, and 2 times 5 is 10. Five times 3 is 15.

Aileen: I knew that it had to get to $10 more. So I did 5 times 2 and 3 times 2 and that gave me 10 over 6. And I did 5 times 3 and 3 times 3 and that gives me 15 over 9. And then I did 5 times 4 and 3 times 4 and that was 20 and 12. And then I got to 5, and 5 times 5 and 3 times 5 is 25 to 15.

Compared to Aileen's numerical scaling-up method, Brian's reasoning can be considered to be more quantitative in nature and involves reversibility. He posited that the money was shared out in increments of $5 and $3, thus each time Alan received $2 more than Bill. All he had to do was to find how many shares they each received, and to do this he divided the final difference, $10, by $2 to get 5 shares.

Jeff and Eric: The pair of eighth-graders

Like Brian, Jeff compared the difference between the components of the ratio (2) and the total difference of $10 to deduce that each share should be multiplied by 5, and from this result he could compute Bill's money (3 times 5). On the other hand, Eric's strategy was to plug in numbers like Aileen, constructing the ratios 15:9, 20:12, and 25:15, i.e., by incrementing the scale factor by one until the required difference between the two quantities was $10.

CONCLUSION

From a conceptual perspective, decomposing a given quantity (say $q = q_1 + q_2$) in terms of the components of a given ratio $a : b$ requires reversible reasoning as one has to interpret the sum of the components ($a + b$) as one entity and determine the number of times that this entity is contained in the initial quantity q (a statement of division algebraically equivalent to $(a+b)x = q_1 + q_2$) – Type I problems. Simplistic as it appears from a numerical/algorithmic perspective, such a conceptualization essentially requires quantitative reasoning as one has to coordinate the relations among the quantities rather than the numerical values of the quantities. Additionally, one has to posit an unknown scale factor between $a + b$ and the given q. Difficulty in conceptualizing such a quantitative structure led 4 of the 6 students to work in a forward direction, by incrementing the ratio $a:b$ using the building-up strategy or other guess-and-check procedures until the required sum was obtained.

Type II problems ($(a-b)x = q_1 - q_2$) proved to be even more demanding as they require the coordination of a multiplicative comparison and a difference in a network of relations (Thompson, 1990) but no particular values of the quantities are stated. One needs to coordinate these two quantitative relations to figure out the specific quantities. Further, this problem involves positing an unknown as a quantity in a multiplicative relationship. The propensity of the more primitive building-up strategy among the participants shows that most of them could not readily conceptualize Type II situations from a quantitative perspective. For instance, in problem 2(ii), ($(5-3)x = 7$), where the final difference between the quantities was not even, the building-up strategy based on integer increments proved to be insufficient and hence the students were constrained to seek alternative approaches. The prevalence of the building-up strategy shows that students worked in a forward direction reasoning numerically using a guess-and-check strategy (with the exceptions of Brian and Jeff). Such numerical reasoning as a fallback strategy is consistent with the results of Kaput & West (1994) and can be explained by students' difficulty in conceptualizing the problems in reverse by analysing these situations from the perspective of the quantitative relations. *Thus we conjecture that constructing the network of relations within a quantitative structure is an important resource for reasoning reversibly in ratio contexts.* This is the main conclusion of this particular study.

The different ways in which the participants solved the same problem led us to focus our attention on the important issue of problem conceptualization. The way that a mathematical situation is conceptualized depends both on problem features as well as the cognitive resources available to the problem solver. The problem solver may possess specific resources but may not be able to deploy them at specific instances. In our attempt to identify the factors that influence the cueing of cognitive resources during problem solving situations, our data lead us to posit the following elements: semantic and syntactic structure of the problem, numerical characteristics of the data, students' failure and success at intermediate stages of the problem solving process and problem-solver conceptualization – factors that are currently under study.

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MATHEMATICS TEACHER DEVELOPERS' ANALYSIS OF A MATHEMATICS CLASS[1]

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Mathematics teacher developers are an important and diverse group of professionals who prepare teachers, yet there is limited research on their professional development. In this study we use learning theory constructs to examine mathematics teacher developers' experiences during a summer institute that focused on mathematical knowledge for teaching. We found that participants' analysis of their observations of a mathematics class varied according to the extent that their observations conflicted with their existing understanding of teaching.

Mathematics teachers participate in a variety of professional development experiences with the intent of improving their practice, but such participation will not ensure their learning; a more fundamental change is required. "Professional development experiences must challenge teachers' current assumptions about what mathematics is, who can do mathematics, and what it means to be successful in mathematics classrooms" (Smith, 2001, p. 44). The underlying assumption is that teachers may learn when they experience disequilibrium or a dynamic state of cognitive imbalance, and, consequently, change their teaching practices. Though a change in practice is a possible response to disequilibrium, it is also important to recognize that disequilibrium may cause teachers to reject ideas as well. Studying teachers' disequilibrium during professional development experiences provides insight into how they might make sense of their experience (Ledford, 2006) and ultimately contributes to understanding more about professional development.

The majority of the literature regarding professional development specifically addresses the needs and challenges of teaching K-12 mathematics, yet those who teach teachers, who we refer to as *mathematics teacher developers* (MTD), also have a vital role in mathematics education. The limited research focused on MTDs, in combination with the significance of disequilibrium during professional development, provides a strong rationale for this study. The focus of this paper is to

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examine the disequilibrium that MTDs experience while observing, analyzing, and discussing a mathematics content class for preservice teachers.

THEORETICAL FRAMEWORK

Mathematics teacher developers who teach courses or organize workshops are a diverse group who range in job titles. For example, mathematics teacher developers include university mathematicians who teach content courses for prospective teachers as well as school district leaders who offer workshops for teachers. The common thread among members in this professional group is that they all work with mathematics teachers. Consequently, *mathematical knowledge for teaching* (MKT) (Ball & Bass, 2003) is a significant area of work that permeates all mathematics teacher developers' work (Sztajn, Ball, McMahon, 2006). While MKT is a common area of MTDs' work, it is also important to note that some MTDs may not have considered how learning MKT is different than learning mathematics.

Mewborn (2003) stated, "the design of effective professional development opportunities should be grounded in sound theories about learning" (p. 49). We have extended this idea by using a learning theory-based framework to *study* professional development. Teachers are learners; therefore models of learning provide a lens for examining MTDs' experiences while observing, analyzing, and discussing a mathematics class. The constructivist ideas of assimilation, perturbation, and accommodation (Piaget, 1970) can be used to discuss professional development experiences that create disequilibrium for teachers, providing a framework for studying professional development that is based on learning theories (Ledford, 2006).

Much of the literature on assimilation, perturbations, and accommodations examines students' mathematics learning (e.g., Steffe & Olive, 2002), but Ledford (2006) studied teachers' learning in the context of a professional development experience. In particular, she studied teachers' experiences in a mathematics course that indirectly addressed issues of student learning and pedagogy. She found constructivist ideas useful in analyzing and characterizing teachers' learning experiences.

When learners face a new challenge, they experience disequilibrium and may need to make adjustments to their current understanding. Assimilation is a process in which a learner incorporates new learning into his existing knowledge through small adjustments made to his or her current understanding (Goldsmith & Schifter, 1997).

When it is not sufficient to incorporate new information into existing knowledge, a perturbation arises. A perturbation is a mental agitation that occurs when the learner's assimilation response to disequilibrium fails; new knowledge cannot be assimilated into existing representations (Piaget, 1970). In this instance, "new experiences or challenges are so different from individuals' current knowledge and skill that they have few resources for operating on these experiences in a cognitively productive manner" (Goldsmith & Schifter, 1997, p. 41). Learners have a natural inclination to maintain equilibrium between their cognitive structure and their environment, and consequently they will seek to eliminate or resolve the perturbation. Accommodation occurs when the learner reorganizes her thinking to reconcile a perturbation by making a modification to her existing cognitive structure.

METHODOLOGY

During the summer of 2004, the Center for Proficiency in Teaching Mathematics hosted an eight-day residential institute for MTDs entitled "Developing Teachers' Mathematical Knowledge for Teaching." The central feature of the summer institute was a university-credit mathematics content course taught by Deborah Ball at the University of Michigan entitled *Mathematical Content and Applications for the Teaching of Elementary School Mathematics.* Sixteen prospective elementary teachers attended daily sessions of the course, referred to as the *laboratory class* by institute organizers and attendees. According to its syllabus, the focus was "meaning and representations of fractions" with special attention paid to the mathematical practices of explanation, representation, recording, and language. As "a shared specimen for observation and manipulation" (Sztajn, Ball, and McMahon, 2006, p. 156), this class enabled the attendees to develop hypotheses and look for confirming or disconfirming evidence. Prior to each laboratory class session, the attendees met with institute organizers to review the plan for that day's lesson, explore the tasks to be used, and provide critical feedback. At the conclusion of each class session, the attendees convened again, first in smaller groups and then as a whole group, to discuss and analyze what they observed.

The 65 institute attendees were MTDs selected from a pool of 140 applicants. The goal of assembling a diverse group of MTDs with respect to current work and prior experiences guided the selection process (Sztajn, Ball, & McMahon, 2006). Institute attendees included graduate students, privately practicing professional developers, school district mathematics curriculum administrators, and community college and university faculty from both departments of mathematics and mathematics education. In this study, we analyzed 16 of these 65 MTDs. Institute organizers selected these 16 participants, who mirrored the diversity of the institute attendees, for additional data collection.

Primary data consisted of participant notebooks, ranging from 40 to 187 pages, and field notes detailing participant discussions during sessions. At the onset of the institute, all attendees received a notebook in which they could record and reflect upon their institute experiences. Notebook entries referring to the laboratory class and all sets of small group field notes were coded. The notebooks gave insights into attendees' thoughts and reactions to the laboratory class, but they did not necessarily capture everything the attendees were thinking or noticing. Furthermore, attendees used the notebooks in various ways. Some attendees gave detailed descriptions of their observations while others analyzed or reflected on their observations.

Secondary data included field notes from discussions of the entire large group of institute attendees, participants' pre-surveys, video recordings of the lab class, and focus group interviews with a subset of attendees two and a half years after the institute. The secondary data was not coded and was only used to confirm or disconfirm findings from the primary sources.

The first level of analysis involved coding the data. The data were coded according to their function and content. The function codes were based on the four professional noticing skills identified by Jacobs et al. (2007), (a) *identifying* noteworthy aspects, (b) objectively *describing* them, (c) *interpreting* them by providing links to relevant knowledge, and (d) *responding* to them with respect to the goals for observation. Content codes aligned with the corners of the instructional triangle (Kilpatrick, Swafford, & Findell, 2001): the mathematics, the teacher, and the students (the preservice teachers). We further classified the content codes (e.g., student understanding of tasks, student mathematical claims, teacher content knowledge) as needed. The coding scheme was the product of an interpretative process and evolved over the course of this level of the analysis; as more data was coded, new codes were added and the coding scheme was refined. After the notebooks were coded individually, each pair of researchers compared codes, discussed inconsistencies, and produced one final coded notebook file for each participant.

The second layer of analysis included input from the four members of our research team to determine recurring themes across participants. Once themes were identified, we re-examined each participant's primary and secondary data to find confirming and disconfirming examples of those themes. Using these examples, we then composed an overall summary for each participant and a detailed explanation of the major themes as they related to that participant.

This paper examines one such major theme: the state of disequilibrium that surfaced among some of the participants. Because this phenomenon was inferred from what participants said in discussions and wrote in their notebooks, our conclusions may be limited. However, triangulation of data provided multiple sources of evidence to make claims about participants' experiences.

RESULTS

Our findings will be illustrated through a discussion of participants' responses to the Cookie Jar Problem (CJP). Though selected for the productive discussions it generated throughout the week, instances of cognitive imbalance were not limited to the CJP.

The Cookie Jar Problem

The CJP was presented to attendees prior to their observation of its introduction to students during the second day of the laboratory class. The problem was stated as follows:

> There was a jar of cookies on the table. Kira was hungry because she hadn't had breakfast, so she ate half the cookies. Then Steve came along and noticed the cookies. He thought they looked good, so he ate a third of what was left in the jar. Niki came by and

decided to take a fourth of the remaining cookies with her to her next class. Then Kayla came dashing up and took a cookie to munch on. When Pam looked at the cookie jar, she saw that there were two cookies left. "How many cookies were in the jar to begin with?" she asked Kira. (Ball, lab class lesson plans)

Though four student solutions were presented in the laboratory class during the two-day development of this problem, we focus on one algebraic solution that was salient for many attendees, providing the context for discussion, reflection, and, for some participants, perturbations. For some participants, algebra had an elevated status, and yet, others considered algebra too sophisticated, inhibiting meaningful conceptual understanding of unit fractions. Participants took note of students' discussion of an algebraic solution, and experienced disequilibrium when some of the laboratory class activities conflicted with their beliefs about presentation of content and the mathematics that should be valued. This was often evident in their notebooks or field notes when participants questioned or were critical of some aspects of their observations (coded as *question/wonder*), drew comparisons between observations and their own teaching practice (*connect to practice*), or suggested alternatives to their observations (*respond*). We classified the participants' responses by the degree of imbalance experienced and how it was accounted for: awareness or noticing, experiencing disequilibrium and resolution by assimilation, and experiencing perturbations. In the following sections, we present three participants who illustrated these reactions.

Kay, Val, and Jon (Pseudonyms)

Kay exemplified the participants who were aware of issues surrounding the algebraic representation and wrote detailed notes about the student solutions, yet did not experience disequilibrium. Kay's notebook included several student quotes concerning the algebraic solution; she noted one student's lack of confidence, "we can't figure out the algebra to prove what we're thinking" (p. 25). She also commented in the laboratory class session that the students may have viewed the algebraic solution as the "grown up way" to solve the problem and expressed concern that students didn't want to share their geometric solutions because of the perceived lack of sophistication. Based on this evidence, Kay noticed students' struggles with the algebraic solution, but noticing alone does not suggest a state of disequilibrium. Kay's notebook and discussion contributions showed no evidence of her experiencing disequilibrium or perturbations related to the algebraic solution during the institute.

The students themselves gave a high priority to the algebraic solution, viewing it as a means to verify the other solution methods. Like the students, several participants attributed more value to the algebraic solution, considering it a means of justification. We use Val as an example of a participant holding this perspective and as a participant who experienced disequilibrium and assimilated the new information into existing cognitive structures. To Val, a solid foundation in the correct use of algebraic properties and rules was essential to study higher-level mathematics. While difficulties with algebra encountered by students in the laboratory class were reminiscent of her students, the teacher's moves to address student misconceptions were dissimilar to her practice. Val experienced a state of disequilibrium; her critical questioning stance was clear in her notebook. She wrote:

> Will it be possible to think and communicate if the support structure is weak? Three such cases from Spring 2004 stand clearly in my mind. Conversations with them... showed problems with algebra and knowing what rules were inappropriate. [The] importance of algebraic rules was absolutely unclear to them.... [and] not communicated to them as being important since it was not thought of as important. Why?" (p. 65).

Val viewed exploration not as a way to develop a given rule or procedure, but as a way to justify a given rule. Though this view contrasted sharply with the development of these ideas in the laboratory class, it did not lead to a restructuring of her thinking. Instead, her institute experiences confirmed her beliefs about the importance of algebra instruction, contributing to her existing ways of thinking. In concluding remarks to organizers she stated, "connections between <u>all</u> representations was not <u>clearly</u> discussed. One of them being the algebraic solution… and the arithmetic solution" (Val's notebook, p. 88).

Val's consistent references to this problem throughout the week provide evidence that her professional understanding of teaching was challenged, but not so strongly that she needed to significantly change her thinking to account for it. The disequilibrium she experienced about the algebraic representation and the teacher's management of instructional time during the CJP was resolved by assimilation. These new experiences were incorporated into her existing thinking, providing her additional evidence for the need to focus on algebraic rules and properties.

Whereas Val assimilated institute experiences, perturbations arose for Jon that resulted in substantial changes to his ways of thinking. Unlike Val, Jon feared that the focus on algebraic representations hindered students' development of a conceptual understanding of the problem. He expressed this idea saying, "By going into the algebra so deeply, it hindered the mathematical thinking of the problem. It distracted them from a deeper understanding of fractions" (field notes, June 7, 2004). This issue remained with him throughout the week, reaching aspects of the institute not directly connected to the CJP. Though initially doubtful of the mathematical rigor offered by the CJP and the laboratory class in general, his attention to the algebra allowed him to analyze the complexity of its mathematics. He wrestled with the mathematics of the CJP and how those ideas were reflected in other problems. This concern about the mathematical rigor led him to consider pedagogical decisions about when and how to introduce fractional unit, explicitly, or through student discovery. In field notes, institute researchers described Jon's struggle to understand how tasks were chosen and implemented, "He really struggled ...to articulate and understand his own thinking about the difficulty of tasks and how to draw out the nuances of a problem" (field notes, June 8, 2004).

In discussing the problem that follows the CJP, Jon questioned, “What are the characteristics of a problem that prompt such discussion?” (field notes, June 8, 2004). Jon becomes intensely interested in the interplay between the teaching and the mathematics, wondering if some problems are inherently good, or if any problem can be good because of the teacher’s pedagogical decisions regarding the development of the mathematics. He returned to the CJP at the week’s end by offering an extension, “turn the cookie problem around and take the fractional parts away in the reverse order from the original problem. The result is exactly the same as before. Why?” (field notes, June 11, 2004). At the institute’s conclusion, he intended to encourage his fellow mathematics faculty colleagues to “think about how deep mathematics is disguised as mathematics for elementary teachers” (field notes, June 11, 2004). His inability to immediately resolve his struggles indicates a perturbation and his deeper understanding of rigor in elementary mathematics suggest he was in the process of making an accommodation. Yet, we have no evidence that this transition was complete at the end of the institute.

CONCLUSION

Of the 16 participants, eight did not show evidence of experiencing disequilibrium during the institute. This may be due to limitations of our data sources. Though some of these participants used the notebooks to only transcribe events rather than document their analytical thoughts, others wrote very little. Three of these eight participants attended every session, followed instructions from institute organizers, and actively participated in discussions. They viewed the laboratory class as a model class taking detailed notes with the intent of emulating observed teaching strategies. We believe that viewing the laboratory class as a model class made it difficult for these participants to experience disequilibrium.

Eight participants experienced disequilibrium and perturbations to varying degrees on possibly multiple issues, with seven participants’ concerns related to MKT and one’s related to classroom social dynamics. The prevalent issues that participants were conflicted about were: 1) acceptable mathematical explanations, 2) rigor in content for elementary preservice teachers, 3) the role of definitions in teaching, and 4) reasons for differences in student participation. There were participants who left the institute without achieving resolution to their perturbations and we hypothesize that they continued to consider these issues in relation to their work with teachers (e.g., Jon).

There are two potential catalysts for achieving disequilibrium in professional development settings. First, the content of the professional development experience should be accessible to all participants and relevant to their work. At the institute, MKT allowed the MTDs opportunities to explore multiple issues related to their own teaching. Second, diverse views of participants provided opportunities for them to experience disequilibrium. The differences in MTDs ideas about mathematics and teaching mathematics at the outset of the institute surfaced throughout the week and initiated important, even intense, discussions. Professional development leaders need to develop strategies that encourage diverse views of participants to surface as well as be carefully considered. One participant reflected on his institute experience:

> During my personal time [at the institute]—I got the most out of tense conversations where there's some kind of opposition or polemic going on… Those are the things that I remember and they cause me to think past the conversations. (focus group, January, 2007)

Analysing MTDs' disequilibrium while observing and discussing a laboratory class provided insights into the complexity of learning about teaching. In our study, the participants who had perturbations showed evidence that they were being analytical in their thoughts and were struggling to merge their own teaching experiences with their observations of the laboratory class. We are not suggesting that participants who didn't experience perturbations didn't have a meaningful experience. However, participants who experienced disequilibrium during the institute examined their beliefs about teaching, students, and mathematics.

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FROM REASONS TO REASONABLE: PATTERNS OF RATIONALITY IN PRIMARY SCHOOL MATHEMATICS CLASSROOMS

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The paper characterizes some of the patterns of rationality –Cartesian, rhetorical, operating, by habituation and by motives- that arise in primary school mathematics classes and which objective is to convince participants of the mathematical results and facts. The patterns described were identified in the didactic practices of a grade 6 teacher, who was observed during a year-long longitudinal study. A change of heading takes place in the research, going from the search for reasons to investigating what is 'reasonable' (Cfr. Bourdieu, 1977).

Context and presentation of objectives

For the community of mathematicians, certainty and *convincingness* appear as an unavoidable process in constructing the discipline, particularly in the processes of proofing and certifying results. René Thom, for instance, sustains that: "Any demonstration capable of causing a state of mind in a sufficiently instructed and prepared reader that leads him/her to show agreement is rigorous" (1980, p. 122); Thomas Tymoczko feels that what characterizes a mathematical proof is that "… it be convincing, able to be examined (manually or step by step) and able to be formalized" (1986, p. 247) and Hersh asserts that "a proof is just a convincing argument, as judged by competent judges" (1993, p. 389).

Convincingness is thus an indispensable component in the practice of experts. Does a similar phenomenon occur in the teaching and learning of mathematics within the school classroom context?

Hersh sustains that:

> The role of proof in the classroom is different from its role in research. In research its role is to *convince*. In the classroom, convincing is no problem. Students are all too easily convinced. … In a first course in abstract algebra, proof of the fundamental theorem of algebra is often omitted…. Nevertheless, the students believe the unproved theorems. (Ibid., p. 396)

Whereas De Villiers concludes from an experiment undertaken with secondary school students "that pupils displayed a need for an explanation (deeper understanding) for a result which was independent of their need for conviction" (1991, p. 26).

The general objective of the research –which includes the partial results reported in this paper- consists of examining several phenomena related to the roles played by convincingness in the educational processes that take place in primary school

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classrooms. The emphasis has been put on the teaching process because the study focuses on the figure of the teacher. This is an issue that is practically unexplored and with reference to which there is scarce agreement among the experts as can be deduced from the foregoing quotes.

In order to examine the above-mentioned phenomenon, it is necessary to identify, describe and characterize, inter alia, some of the arguments, proofs or justifications presented by members of the class for the purpose of convincing participants of mathematical results and facts. And this is the objective of this paper, which specifically consists of proposing a language and classification for some of the adherence mechanisms promoted by primary school teachers –with the participation of their students- in their mathematics classes.

The analysis will be carried out following the steps of Klein "whose objective is not to investigate what rational or correct argumentation consists of, rather to examine just how men argue…" (cit. in Habermas, 2001, p. 50), hence attempting to avoid reducing a research paper on justification in primary school mathematics classes to a treaty on proofs as was suggested by Krummheuer (Cfr, 1995).

The majority of experts who undertake research on mathematical justifications in the classroom setting or on proofs in general intend to explore the *rationalis* of the subjects presenting the argumentation. Unlike those works, the intention of the research partially presented here is not limited to the purely rational processes of justification; although mathematical reasons and intentional justification processes are of course taken into account, the work is oriented toward an exploration of what, at a given point in time, seems 'reasonable' to a classroom community. As such the research takes on a change of direction that shifts from a search for reasons to investigating the practical reasons that make sense for a given group (Cfr. Bourdieu y Passeron, 1977).

Method and collection of empirical data

The empirical evidence that substantiates the research is derived from a longitudinal case study which received follow-up for a period of one year. The study focuses on a grade 6 primary school teacher –Ms Ale- who has taught class in a public school in Mexico City for 25 years. She has distinguished herself due to the high grades usually obtained by her students in official evaluations. Since the research aimed to understand the phenomena of teaching and learning mathematics in the manner in which they take place in an ordinary school context, no experimentation of didactic engineering was undertaken and observation was favored as the ethnographic and ethnological method of collecting empirical data. Of the classes observed, those that involved proportional reasoning (nine lessons lasting approximately one hour each) were video-taped and transcribed.

Justification in primary level arithmetic classes

For purposes of this paper, references to proofs, reasonings, arguments and argumentations, application of formulae and algorithms or analysis of particular cases will be made by way of the term 'justification', taking the latter in its generic sense. The idea of justification is confined to classroom practices that –generally- have a dual purpose: an epistemological end that consists of implicitly or explicitly sustaining the truth of an assertion with mathematical content; and a psychological end that consists of obtaining –intentionally or unintentionally- the understanding of the listener and a degree of adherence to that truth.

The justifications usually offered in primary school mathematics classrooms are not explicit. They also have vague or blurry limits since they do not generally have clearly delimited beginnings and ends, and since they are part of verbal communication they are reiterative and cumulative. They are similar to those that Duval calls 'argumentations', those that "…are added to one another, complement each other and sometimes overlap" (1999, p. 215), circumstances from which emerges their linear structure.

The justifications examined herein lay within the context of solving the mathematics exercises (non routine for the students) raised in class. This is due to the fact that Ms Ale organizes her didactic activities based on the official textbook, which didactic and pedagogical proposals are articulated based on the solution of this type of exercises.

Justification patterns

In a classroom the likes of Ms Ale's, justification and convincingness practices are structured in line with different patterns of rationality. In this portion of the paper, we classify the patterns used in the classes observed to explain and sustain the truths bearing mathematical content, as well as to promote convincingness, conviction and persuasion. Some of these patterns are illustrated with an extract of the class observed.

Cartesian Rationality. The justifications that falls under the Cartesian pattern of rationality are underpinned by sufficient mathematical reasons from which the necessary truths are derived. They bring to mind *more geometric,* Euclidean demonstrations, yet they also include all types of proofs that meet the strict cannons of logical and mathematical thoroughness. In primary level classrooms, one almost never finds argumentative pieces that meet the above cited formalism. Nonetheless, (implicitly) deductive reasonings arise fairly frequently, from which the necessary results emerge and, albeit tacitly, present traits and tendencies of this type of rationality pattern. In the class observed, we detected a large number of justifications that respond to that Cartesian pattern, especially with respect to instantiations of general formulae or to verification of hypothesis (Cfr. Reid, 2002). We have not illustrated this with examples since they are so frequently reported on in the literature.

Rhetorical Rationality. The pattern of rhetorical rationality includes justifications that are based on mathematical reasons, but from which one can only extract plausible truths. Qualitative-type argumentations fall into this pattern, as do the proofs that Balacheff (in 2000) classifies within naive empiricism and the crucial experiment. Also included in this pattern are reasons-based arguments incompletely presented in the classroom, but unlike Cartesian arguments the group is not able to re-construct or complete them due to their referential framework. This pattern evokes the rhetoric vindicated by Aristotle (in his *Rhetoric*) and later by Perelman (among others) (Cfr. Perelman and Olbrechts- Tyteca, 1989), for whom the competence of reason is not limited to the field of logic-mathematics. Concrete, practical, situated reasons, with their categories of things verisimilar, plausible or reasonable are for them also the object of understanding.

Operating Rationality. The justifications that meet the pattern of operating rationality are supported by the confidence of the person applying them in the algorithm or procedure used. In this paper, we sustain that usage of any given algorithm –and especially systematic usage- is always accompanied by a justification or explanation from the person applying it. Said justification may be sufficient and abide by mathematical reasons, but it may also be tacit or even unconscious and respond to a 'faith' or conviction of the validity of general mathematical formulae, in which case the justification is consolidated by extra-mathematical reasons that are not directly related to the *logos* of the argument.

Rationality by Habituation. When an idea or an interpretation of an event is systematically reiterated within a given cultural or social context, the partisans involved usually end up believing in it. Under such circumstances, custom most likely supports the credibility or adherence, with which as of the *habitus* (Cfr. Bourdieu and Passeron, Ibid.) and processes of practical familiarization that do not go through the conscious mind, they become 'familiar' or 'normal' to the community. Believing in or sustaining the veracity of a mathematical fact or the validity of a formula based on habituation is very common in mathematics classes. Frequently after mechanizing an algorithm, for example, and after persistently encountering it in class or in the textbook or hearing it often from the teacher, the students end up believing in it and taking it as valid simply because it has become natural and known to them. If they are additionally made to repeat the algorithm out loud as a group, the confidence in and certainty of the results of applying it will be boosted because repetition in unison stimulates socialization and the feeling of belonging to a group (Cfr. López Eire, 2001), while also aiding the memorization process.

In traditional schools it was quite common to find teachers who considered habituation to be a mnemonic technique for learning and, perhaps unconsciously, for validation activities as well. The case study selected in this research is interesting because, among many other reasons, it shows that the practices for constructing the convictions and securities of mathematical results, based on verbal reiteration, have not yet been unearthed from today's schools. And this is not just the case of schools located in Arab countries, as stated by Kilpatrick (2007), but also in the schools of Mexico, as seen in Ms Ale's classroom.

Rationality by motives. In mathematics class there is undoubtedly a rationalist *éthos* that orients students and teachers in their justification practices of mathematical formulations. There are however also diverse 'motives' that are derived from supra-rational sources and that have a bearing on said practices, as is confirmed by the empirical data obtained in the study. 'Motive' is understood to be:

> … all that which induces a person to act in a certain manner in order to achieve an end. …Motives include conscious purposes (the 'practical reasons'), as well as more general and deeper motivations (irrational forces, desires and drives) that urge the obtainment of satisfactory states." (Villoro, 2002, p. 103)

For instance, when Ms Ale resorts to inflections of speech, her silences, body language and illocutory acts to express her personal experiences of convincingness or conviction, she is conveying (inducing or imposing) those epistemic states (very possibly in an involuntary manner), supported by her authority, that of the textbook or of mathematics itself. In this case, she is justifying based on a logic of motives, which transcends the sphere of things rational and mathematical contents.

Extract of class: Illustration of rationality patterns

The rationality patterns alluded to above were discovered and classified as a result of coming from and going to the theoretical orientations of the paper and an analysis of the results that arose from the longitudinal case study undertaken in Ms Ale's classroom. Indeed, they are called 'patterns of rationality' in the study because they correspond to the justification and adherence mechanisms that were systematically and regularly found throughout the classes observed.

Below is an extract of said classes (Lesson 80, episode 3), which was not only chosen because it exemplifies the rationality patterns cited, but also given that it provides evidence that mechanisms complying with different rationality patterns are brought into play in non routine mathematics exercise solution activities undertaken in class.

239. G: [In unison reading from the textbook]: The following table contains the cumulative times it takes Dario to swim five sections of his swim.

250 m.	500 m.	750 m.	1 000 m.	1 250 m.	1 500 m.
6 min.	12 min.	19 min.	26 min.	32 min.	40 min.

Table from the textbook

Which section did he swim the fastest?

257. T: Let's see (suggesting) … we can do it like Diego was saying [divide the distance by time]. … For example, in the first section of two hundred and fifty meters, how many meters did Dario swim in one minute?

258. S: Forty one point sixty-six

261. T: Meters, right?

262-324 [Under the teacher's direction and with her help the group applies the d/t quotient to calculate the speed per minute swum by the swimmer in each of the sections that appear in the textbook table].

325.	T:	… now we have these quantities [showing satisfaction]. First question, reading out loud!
326.	G:	Which section did he swim the fastest? [reading from the textbook]
329.	G:	[Several answers are given] In the second,… In the fifteen hundred section
330.	T:	In the first section how many meters did he swim per minute? In one minute he swam?
331.	G:	Forty-one...
332.	T:	Forty-one point sixty-six. And in the second section?
333-341		[The group repeats the results calculated in 258-324]
342.	T:	Which section did he swim the fastest? (directive act, awaiting the correct answer).
343.	G:	[Different answers are provided]
344.	T:	What do you mean?
345.	S:	In the fifteen hundred section
348.	T:	In the fifteen hundred section, which one is the …? Ah! He did thirty-seven meters (confirming, without emphasis)
349.	S:	Ms Ale, it was in the two hundred and fifty meter section.
350.	T:	That's right (without much confidence)

G: group; T: teacher; S: student.

The teacher's insistence on the use of the (d/t) quotient (in this exercise as well as in another) generates the impression that she possesses a vectorial idea of velocity and speed. Yet in her interventions one can see that she (and her students) have difficulties signifying mixed magnitudes and promoting that vectorial idea (amongst other things, because for all practical purposes they ignore those magnitudes throughout the entire solution process). It is perhaps due to such conceptual problems that the teacher leaves one of the most significant and complex parts of the solution to the students –which consists of comparing the mixed magnitudes, the interpretation of the results of that comparison in terms of the speed, and the conversion processes needed into order to go from the physical arithmetic domain to the table register.

The solution falls within rhetorical rationality because although it complies with a deductive structure of specification, neither the students nor their teacher possess the conceptual elements needed to provide a well-founded explanation of the truth, from the mathematical point of view, of the results derived from the formula. It is rhetoric; not because the justification is incomplete, but because it fails to include sufficient or conclusive reasons in keeping with the rationality of the group.

By the same token, the justification also complies with an operating rationality. In the solution presented, one can see how the results that arise in class are shored up by the confidence and 'quasi faith' that Ms Ale has in the symbolic formulae of mathematics, a confidence and faith that may very possibly be shared by her students too. It is there that one can see the credibility that the teacher attaches from the onset to the formula; she trusts the velocity quotient without even seeking to give it some form of plausibility with the conceptual elements that the group has at hand.

The solution is further supported by a rationality by habituation. Although it is very possible that the teacher does not intentionally or consciously do so, her interest clearly lays in having the children memorize and become familiar with the formula and the results it provides. She very patiently applies the quotient twice (from 257 to 325 and from 330 to 350), with the participation in unison of the students, proposing each of the answers.

The solution is likewise underpinned by a logic that complies with the teacher's motives and interests. At 257, for instance, she subtly imposes application of the d/t formula in order to calculate the speed, supported by her authority and that of mathematics proper. At 325 and 342, she conveys (or attempts to impose) confidence in the results obtained for the purpose of having the group propose the correct answer.

Consequently Ms Ale's students may at the end of the day have begun to memorize the velocity quotient formula and identify the resulting magnitudes. Although they may not have experienced a convincingness based on diaphanous reasons, they may have acquired a feeling of security based on the confidence that the teacher and group usually have in mathematical formulae. That confidence would have been shored up by the familiarity they acquired in using the formula and mixed magnitudes, resulting from repeated use, and also backed by the authority proper of the teacher who is directing the solution process with all of the security and confidence that she generally conveys in her teaching practices.

Final Remarks

The results presented here represent the first stage of the findings encountered in a broader research study, in which the objective is to analyze the mechanisms that enable a teacher (or student) to convince the members of his/her mathematics class of a mathematical result. The backbone of the research is the concept of a 'culture of class rationality', which includes the idea of an 'epistemic state' –convincingness, conviction and persuasion- because they provide a systemic view of complex aspects of the phenomenon being studied[1]. As of this comprehensive and holistic perspective, the results reported here were discovered. In particular we were able to identify processes of rational validation in the class observed, as well as other 'types of discursive persuasion' (Habermas, Ibid., p. 45) that are both local and specific to that class, and very possibly also present in other classrooms. Foremost among those results are the manners of convincing that are neither explicit nor specific, that even go well beyond the sphere of things voluntary and conscious, based on very subtle adherence techniques that are rationed out throughout the course/term, as well as

[1] We are not including those concepts in this presentation in view of space limitations, but this paper will serve as a reference point for presentation of the terms, as well as the complement of the work in future contributions. In fact, preliminary data regarding the study appears in the Memoirs of PME 32, in Rigo, Rojano & Pluvinage.

those supported by operating rationality and rationality by habituation. The foregoing shows that in the mathematics classroom, as in other everyday settings, the natural law of the land depends on what the community deems to be reasonable. It is not a matter of being irrational, rather simply that decisions and interactions fall into logic that does not always coincide with a formal or purely rational logic. (Cfr. Bourdieu y Passeron, Ibid.).

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VISUAL TEMPLATES IN PATTERN GENERALIZATION*

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Eleven 7th- and 8th-graders obtained pattern generalizations on three unfamiliar tasks months after a teaching experiment on linear patterning. The study explores additive, multiplicative, and pragmatic visual templates in patterning activity.

INTRODUCTION AND RESEARCH QUESTIONS

Well-defined patterns such as the ones shown in Figures 1 and 2 are structured sequences of objects. The primary task for learners is to coordinate their perceptual and symbolic inferences so that they are able to establish and justify a structure that could be conveyed in the form of a direct formula. The term *pattern generalization* is used throughout the report and refers to both mathematical practices of construction and justification of direct formulas for a given pattern. Further, we assume that pattern generalization is a subjective process as a result of the interpretive nature of structure discernment and construction in patterning activity.

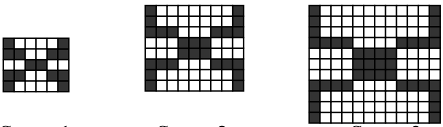


Fig. 1 Square Frog Pattern

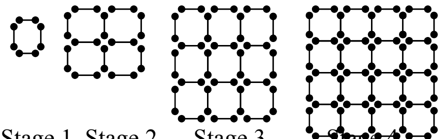


Fig. 2 Square Array Pattern

We address the following research questions: (1) How do seventh- and eighth-grade students (ages 12 and 13) develop a structure and perform pattern generalization for figural patterns such as the ones shown in Figs. 1, 2, and 3? In particular, what strategies do they have at their disposal and within their developmental capacities for perceptual and symbolic inference? (2) Considering the fact that this age-level group does not have extensive experiences in structure formation and development, how do they impose a structure and pattern generalize on well-defined patterns such as the ones shown in Figs. 1 and 2?

CONCEPTUAL FRAMEWORK

We use Giaquinto's (2007) notion of *visual templates* in developing provisional answers to the above research questions. But our appropriation of the term aligns with the characteristics identified by Neisser (1976) who explored the

On the right are the first two stages in a growing pattern of squares.
1. Continue the pattern until stage 5.
2. Find a direct formula in two different ways. Justify each formula.
3. If none of your formulas above involve taking into account overlaps, find a

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 473-480. Thessaloniki, Greece: PME.

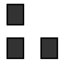
direct formula that takes into account overlaps. Justify your formula. 4. How do you know for sure that your pattern will continue that way and not some other way? 5. Find a different way of continuing the pattern and obtain a direct formula for this pattern.	 Stage 2

Fig. 3 An Ambiguous Patterning Task

idea of *template matching* in the context of pattern recognition of everday objects. Giaquinto's (2007) appropriation is derived from Resnik (1997) who describes a template as a "concrete device for representing how things are shaped, structured, or designed" (Resnik, 1997, p. 227). When we manipulate a template much like how we use blueprints, somehow the concrete experience should provide either a structurally isomorphic or structurally contained context that enable us to make sense of the corresponding abstract patterns and their properties. For example, when second-grade children begin to group concrete objects in a particular way, then the concrete understanding that comes with the grouping experience should somehow foreshadow what eventually would be known as the general structure of place value notation or counting systems. There is, thus, an expectation of a percept-to-concept projection process that is involved with the visual component assisting in constructing or contextualizing the abstract elements involved. In Neisser's (1976) case, children use prototypes or canonical forms as a standard or basic model that help them either learn a characteristic of a new object or compare the new with an existing object. Thus, visual templates provide a skeletal structuring tool that learners use to systematically capture and organize an intended content. Learners use them to compare and/or to assist in generalizing tasks. They also act as a heuristic, that is, they work in some situations or may fail in others.

In this report, we discuss visual templates in relation to the work of a group of 7th and 8th grade students (ages 12 and 13) in an urban school who obtained pattern generalizations for the patterns in Figs. 1 through 3. We focus on three template-types, as follows: additive, multiplicative, and pragmatic. Students who use an *additive template* express grouping relationships in additive form, while those who use a *multiplicative template* express those relationships multiplicatively. The initial conceptualization of grouping is rooted in students' acquired understanding of the concept of multiplication of two integers, that is, $a \text{ x } b = b + b + b + \ldots + b$ (a times) or a groups of b objects. Students who use a *pragmatic template* implement a combined numerical-visual strategy, which could also be interpreted as a task-induced coping mechanism in order to make sense of an emerging pattern generalization. These characterizations are explained in greater detail in the Findings section.

METHOD

Participants. Eight 8th Graders from Cohort 1 and three 7th Graders (4 males and 7 females) were members in an Algebra 1 class that participated in a month-long teaching experiment on linear pattern generalization in December 2007. Then, they each participated in a 55-minute clinical interview that took place in May 2008. No patterning activity was conducted between January and April. Two methodological issues were dealt with in relation to establishing a case for the existence of visual templates. First was the necessity of a prolonged period of time in which no patterning activity was pursued on purpose. In our study, while the clinical interviews immediately after the teaching experiment gave some indications that such templates existed, however, we were interested in establishing their durability and stability over time. Second, we saw to it that none of the tasks were familiar to the students since we wanted them to articulate their visual templates when they were confronted with unfamiliar patterns.

Task and Task Protocol. The first author conducted all the clinical interviews with the 11 students. Each participant worked on the patterns shown in Figs. 1 through 3 one at a time in the following order: 3, 1, and 2. The complete task protocol for Figs. 1 and 2 are shown in Figs. 4 and 5. Each student was asked to think aloud. Square blocks were provided in the case of the Fig. 3 task, while a centimeter graphing paper was available in the case of the Fig. 2 pattern.

1. What does Stage 4 look like? Either describe it or draw it on a graphing paper.
2. Find a direct formula for the total number of gray square tiles at any stage. Explain your formula.
3. How many gray square tiles are there in stage 11? How do you know?
4. Which stage number contains a total of 56 gray square tiles? Explain.

Fig. 4. Task Associated with the Fig. 1 Pattern

A. Find a direct formula for the total number of sticks at any stage in the pattern. Justify your formula.
B. Find a direct formula for the total number of points at any stage in the pattern. Justify your formula.

Fig. 5. Task Associated with the Fig. 2 Pattern

Data Analysis Process. The data analysis process took shape following the steps described in Healy and Hoyles (1999). Individual case studies that consist of transcripts, written work, and analysis of relevant interview segments were initially developed, followed by individual cognitive maps with the aim of schematically capturing the trajectory of their generalizing processes. The summative evidence was then compared, analyzed, and categorized using grounded theory that led to the three labels corresponding to the three template types. Several iterated reading-and-analysis processes of the interview data were done to ensure a correct categorization of student work and also to obtain a sufficient characterization of the template-types.

FINDINGS

Table 1 is a summary of the students' direct formulas on the three pattern tasks and categorized by template type. We use Diana's visual process as an exemplar of a student who pattern generalizes *using a multiplicative template*. Her template had her seeking out for groups of parts that had the same count and then connecting the count with the appropriate stage number. In the case of the pattern in Fig. 1, Diana initially obtained the formula $n = x(x + 1) + 4(2x + 1)$ which she later simplified to $n = x^2 + 9x + 4$. She explained her pattern generalization in the following transcript below.

Task	Additive Template	Multiplicative Template	Pragmatic Template
Fig. 3	**Dung**: s = n + n–1 **Emma**: s = n+1; s = n + (n – 1) **Karen**: x = n–1 + n–1 + 1	**Dexter**: B = 2s – 1 **Diana**: n = 2x – 1; $n = x^2 – (x – 1)^2$ **Dung**: s = 2n – 1 **Earl**: 2(s –1) + 1 = n; 2s – 1 = n **Karen**: x = 2n–1	**Anna**: n = 2s – 1, n = 2s – 2 + 1 **Chloe**: s = 2n – 1; x = 8n – 4 **Frank**: S = 2x – 1; S = 2(x–1) + 1 **Shaina**: s = nx2 – 1 **Tamara**: x = 2p – 1
Fig. 1	**Emma**: S = n(n+1) + 4(2n+1)	**Anna**: g = 8s + 4 + s(s+1) **Chloe**: $x = (n^2+n) + 4(2n+1)$ **Dexter**: B = 4(2s+1) + s(s+1) **Diana**: x(x+1) + 4(2x+1) = n **Dung**: $g = n^2 + n + 4n+4n+4$ **Earl:** s(s+1) + 4(2s+1) = n **Frank**: n(n+1) + 4(2x + 1) = S **Karen**: x = n(n+1)+[(n+1)+n]4 **Tamara**: $x = n^2 +(nx8) + (4+n)$	**Shaina**: s = g + 8 (incorrect; counted just the four legs)
Fig. 2		**Anna**: n = s(s+1) + s(s+1) **Chloe**: x = n(n+1)2 **Diana**: 4x + 2(x–1)x = s **Dung**: s = n[4n–(n-1)] – (n-1)n **Earl**: 4s + [s(s-1)]2 = n **Frank**: s = x(x+1)2 **Tamara**: (p+1)p + (p+1)p = x	**Dexter**: P = 4s (incorrect; just counted the perimeter sides) **Emma**: s = n(2n + 2) **Karen**: x =4n+n(2n-2) **Shaina**: s = n x 8 – 4

Table 1. Summary of Students' Direct Formulas Categorized by Template Type

Diana: Well, basically you always, like, to this number here, to this part her [*referring to stage 2*], you added 1 and on this side you add 1 to make it longer [*referring to the growing legs on every corner*.] You always add 1 to everything to make the legs longer. Instead of like 2 x 2, you make it 3 x 3. And for this one, too [*the middle rectangle*], instead of 1 by 2, you make it 2 x 3. [*She then finds a direct formula and obtains* $x(x + 1) + (2x + 1) = n$].

FDR: Okay, so tell me what's happening there? Where did this come from, $x(x + 1)$?

Diana: This, the little square, x times $x + 1$.

FDR: So where's the x times $x + 1$ here [*referring to stage 3*]?

Diana: Like 3 x 3, or 3 x 4.

FDR: So where's the $2x + 1$ coming from?

Diana: This. I mean I can look at it like 2 times (x + 1) minus 1 but I just made it, like, 3, 3, and 1, so $2x + 1$. [*She initially saw that each leg had two overlapping sides that shared a common square.*]

FDR: But this $2x + 1$ is just for this side [*referring to one leg*], right?

Diana: For all of the legs, oh, [*then adds a coefficient of 4 to her formula*: $x(x + 1) + 4(2x + 1) = n$].

FDR: Okay, so are you happy with your formula?

Diana: I think I could simplify it. I'd like to see what happens if I simplify it. [*She then simplifies her formula to* $x^2 + 9x + 4 = n$.]. 4 would be these [*the corner middle squares*] I'm pretty sure. 9x would be, 8, oh, yes, I see it. I see how it works. There's an x squared here [*referring to the rectangle which she saw as the union of an n by n square and a column side of length n*] if you see one square here and the 9x would be these legs [*referring to the (n + 1)th column of the rectangle of length n and the eight row and column legs minus the corner middle squares*]. Plus 4 would be the center of each leg.

The consistent manner in which Diana used a multiplicative template became especially useful when she established and justified a direct formula for the Fig. 2 pattern. Initially, she obtained a formula for the number of sticks on the perimeter of each square, 4x. Next, she counted the interior sticks as follows: (1) "[*in stage 2*] 2 minus 1 would be 1 so there would be 1 row going down and another row so that would be rows of 2 sticks;" (2) "in stage 3, there's 2 rows of 3 sticks;" (3) "[*in stage 4 there's*] three, okay, it's two sets of three rows of 4." She then wrote $4x + 2(x - 1)x = s$ which she simplified to $2x + 2x^2 = s$.

Karen's work on the Fig. 3 task exemplifies the *use of an additive template*. Her pattern of five stages is shown in Fig. 6. In justifying her direct formula, $s = n - 1 + n - 1 + 1$, she visualized it in groups, as follows: "You group these [*the row and column squares excluding the common square*] and add 1 [*the corner square*]."

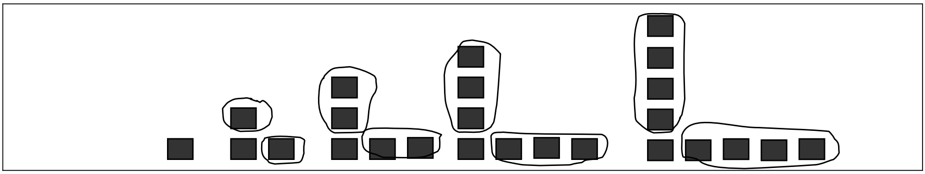


Fig. 6. Karen's Constructed Pattern in Relation to the Fig. 3 Task

Emma used an additive template in dealing with the Fig. 3 task. She produced the same pattern in Fig. 6, constructed $S = n + (n - 1)$, and reasoned as follows:

FDR: What helped you in transitioning from these visual squares to a direct formula?

Emma: Grouping it, I guess. This is stage 1 [*referring to the one square*]. This is stage 2 [*the column of two squares*]. This is stage 3 [*the column of three squares*] and this is stage 4 [*the column of four squares*]. And then so ahm when I figured that, I try to see what's left. So if it's 1 [*the remaining square on the row of stage 2*], if you subtract the stage number from 1, you get 1. If you subtract 1 from the stage number [*stage 3*] you get 2 [*the two remaining squares on the row of stage 3*]. If you subtract 1 from this stage number [*stage 4*], you get 3 [*the three remaining squares on the row of stage 4*].

Emma's thinking in relation to the Fig. 2 pattern exemplifies the use of a *pragmatic template*. When Emma was presented with the Fig. 5 task, she initially inspected a part or parts in each cue that reflected a particular stage number. She then tried to use an additive template but failed. She then counted once again and observed that stage 1 had 4 sticks, that is, "four of stage 1." In stage 2, she counted the total number of sticks (*12*) but then saw it as "[*stage 2*] 6." She then verified it in stages 3 and 4. She noticed that since stage 3 had 24 sticks, it was equal to "[*stage 3*] 8." In stage 4, she claimed that 40 was "[*stage 4*] 10." Next, she concluded that her direct formula for the total number of sticks was $s = n \quad (2n + 2)$. In explaining her formula, she said that n referred to the stage number and that the numbers 4, 6, 8, and 10 were "even numbers and that to get to 10, I multiply it by 2 and add 2, so $2n + 2$." But when she was asked to explain how the formula might make sense in the given stages, she said, "I don't know, I don't see it in the picture."

Karen's work on the Fig. 2 pattern also demonstrates the use of a pragmatic template. Initially, she saw that the total number of sticks on the perimeter of each square was $4n$ ("there are four sides and four groups of the stage number"). Then, she noticed that the interior sticks had the following relationship which she organized in a table: stage 1 had 0 sets of horizontal and vertical sticks; stage 2 had 2 sets of 2 sticks; stage 3 had 4 sets of 3 sticks; stage 4 had 6 sets of 4 sticks; …; stage n had (2n – 2) sticks. In obtaining the expression $2n - 2$, she employed the same numerical strategy that Emma used above. Thus, a pragmatic template was used in cases when students tried to construct a direct formula first using a numerical method that they would then try to justify visually.

DISCUSSION

Where the fundamental difference lies between multiplicative and pragmatic templates and, thus, a weakness in the latter, is at the stage of justification. Those who used a multiplicative template saw formula construction and justification to be two complementary processes, which illustrates the powerful view in which "representat-ion is explanation" (Leyton, 2002). In this view, representation emerges as an observer tries to describe a target stimulus "as a state (or sets of states) in a history that causally explains the stimulus" to him or her (p. 157). For example, Chloe saw that each cue in the Fig. 2 pattern consisted of several rows and columns of sticks. She then counted as follows: "[in stage 2,] there's 2, 2, 2, 2, 2, and 2; [in stage 3,] there's 3, 3, 3, 3, 3, 3,3, and 3; so there's two groups of 3 twice [and] there's two 4 groups of 3 [and] two five groups of 4. So $x = 2$ times $n + 1$ times n." In Dung's case, he first specialized using stage 4. He initially saw *4* disjoint rows of squares and counted the total number of sticks per row. In counting the number of sticks per row, he saw *4* disjoint squares for a total of *16* sticks and then subtracted the overlapping vertical sticks, *4* – 1 = 3. Since there were *4* rows, the total number of sticks was 4 x [(4 x 4) – (4 – 1)]. But he was aware that when the four disjoint rows of squares were combined to form stage 4, the interior horizontal sticks have been counted twice. In counting the total number of such non-overlapping interior horizontal sticks, he saw that there were (*4* – 1) groups of 4 = 12 overlapping horizontal sticks. Hence, his formula was $s = [4n - (n - 1)] \times n - (n - 1) \times n$. Those who employed a pragmatic template, on the other hand, established correct direct formulas but either were unable to justify them or produced inconsistent explanations. There was, it seems to be the case, an apparent disconnect between formula construction and justification. For example, while Emma was able to establish her formula, $s = n(2n + 2)$, pragmatically for the Fig. 2 pattern, she, however, could not justify it.

CONCLUSION

In this report, we provide evidence of visual-template use in the development of structure and pattern generalization. Both additive and multiplicative templates are endowed with conceptual content; they provide powerful building-block models that are oftentimes accompanied by *imagistic reasoning* in which stages in a pattern are seen as conveying some kind of mathematical relationship among their parts within an interpreted structure. In patterning activity, they enable a mapping between familiar (i.e., something already known) and unfamiliar patterns within an interpreted relational structure that lead to an algebraic generalization. The relation-mapping nature of such templates explains why, among the students, patterns with ambiguous stages such as Fig. 3 become well-defined and well-defined patterns such as Figs. 1 and 2 are generalized accordingly with relative ease.

Healy and Hoyles (1999) have offered a construction approach framework in which they categorized the content of children's generalizations on linear patterns to be falling under two approaches. A primary dilemma we had with their categorization was how to justify the conceptual divide between symbolic and iconic approaches. As data in this article show, the students' visual templates reflect a synergistic relationship between the symbolic and the iconic that early research studies have oftentimes assumed to be two opposing approaches. On the basis of our findings, we propose a shift away from such binary practices in favor of visual templates that model a dynamic relationship between "phenomenological modes of production" and a system that is "mobilized for producing [the relevant] representations" (Duval, 2006, p. 105). For example, the phenomenological use of visual templates in patterning activity has mobilized the use of the concept of multiplication and relevant analogical processes as reflected in the students' imagistic reasoning across pattern tasks.

In this report, we may have given the impression that the generalizing route to figural patterns is bound to take shape inevitably when a visual template is tapped. Unfortunately, we do not have full access to all types of figural patterns. This means to say that, while a visual template can assist in generalizing, it may be limited to patterns of a particular type.

Another unresolved issue is that we have no empirical knowledge of factors (say, visual attention) that contribute to choosing one template over another. Also, due to the small number of students who employed an additive template in this study, we could not establish the possibility of continued template preference, especially with more learning. For example, if Emma were exposed to more activities that explicitly encouraged her to use a multiplicative template, would her preference for an additive template diminish over time? There is also the question of asymmetrical (versus mutual) translatability in template use, that is, while most students with a multiplicative template could easily transition to an additive template, we observe the considerable difficulty of, say, Emma, in transitioning from an additive to a multiplicative template.

Also, research is needed in assessing the existence, nature, and content of visual templates among children and adult learners (especially teachers) who deal with patterns in their mathematical experiences. If both groups use them in obtaining a generalization for a given pattern, it is interesting to compare such templates within and across grade levels or age groups by way of, say, constraints in both developmental and non-developmental senses. Also, it is possible that other types of visual templates exist on the basis of personal theories and/or other learning experiences or contexts that influence the content of their relevant pattern generalization strategies. In this report, for example, the cognitive content of the students' visual templates could be traced to their classroom experiences on patterning. Younger children and older adults may derive the sources of their visual templates in some other way.

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STUDENTS' CONSTRUCTIONS FOR A NON-ROUTINE QUESTION

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This paper reports preliminary findings of a research study which aims to explore how students' preparation for the examinations conducted by different examination systems, in Pakistan, influences their construction for non-routine questions. The study uses the students' construction in problem solving as a window on the enculturation processes of teaching and learning through the curriculum as written and tested. This paper presents analyses of the parts of the curricula in relation to the one of the non-routine questions, and responses of the students to that question.

INTRODUCTION

This paper reports some findings from a study which entailed clinical interviews with students who had completed their secondary school education in one of the three different examination systems, in Pakistan. The study is an attempt to understand students' preparedness for higher secondary education where students from all systems come together to study the same course.The study aims to find out about students' reasoning* in non-routine questions; identify similarities and differences in reasoning within and across the students from different examination backgrounds; and infer the influence of secondary school curricula on students' reasoning. Ten pairs of students who were recently qualified from each secondary examination system and had just started higher secondary course in a college participated in the study. In the same setting, each pair comprised of students from the same examination system was asked to solve four non-routine questions designed for the study. This paper will present analyses of the parts of the exam curricula which relate to the one of these non-routine questions and responses of students from to that question**.** This paper is concerned with only two of the three examination systems which portray distinctly contrasting images of mathematics.

THEORETICAL FRAMEWORK

The basic assumption of this study is that the shared activity of teaching and learning in a school is mediated through tools or artefacts available in the cultural and historical context within or outside the school (Vygotsky, 1978). In a classroom, the objects of both activities, i.e. learning and teaching, are the concepts of mathematics. Both the actors (teacher and student) mediate mathematical concepts through curriculum material. Here 'curriculum' refers to any written text that teacher and students use during the course of teaching and learning respectively. In real classroom situation, the two activities are not disjointed. In a successful situation, teaching shapes learning and learning informs teaching. Within sociocultural

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 481-488. Thessaloniki, Greece: PME.

perspective, teaching and learning are conceptualized as socially shared activities in the construction of knowledge (John- Steiner & Mahn, 1996, Voigt, 1996).

Like many countries where teachers derive their teaching approaches mainly from textbooks (Fan & Kaeley, 2000), in Pakistan, too and to a greater extent, teaching and learning revolve around the written curriculum, mostly textbooks and past exam papers, from which teachers and students derive sign, symbols, and language of representations (Riaz, 2008, Fakir, 2004). Bernstein's (1977) notion of "classification" and "frame" across three message systems i.e. pedagogy, curriculum, and evaluation, in realization of formal educational knowledge, can be used in conceptualising how construction of concepts of mathematics occurs. In Pakistani classrooms, strong "classification" and "frame" (p. 206) reduce the power of students and teacher over what to learn and what to teach respectively. In this situation, it is not unrealistic to think that the students' formal mathematical knowledge is the product of interactions between teacher and student mediated through curriculum material in order to meet the external demand of the examinations.

In this study participating students came from the schools where they experienced mathematics learning in different cultural and historical contexts due to the different dominant images and approaches in curricula. Therefore, the assumption was they had different ways of learning mathematics. This study will explore how their way of learning mathematics helped them construct, extend, or modify their mathematical concepts in order to fulfil the demand of non-routine questions they encountered in setting of this study.

NON-ROUTINE QUESTION

The term non-routine refers in this study to questions which students cannot solve by rehearsed procedures; rather they need to construct novel methods to find solutions. Detailed curriculum analysis and pilot testing helped me consider the questions as non-routine from students' perspective. One of the non-routine questions used in the study is given below.

"Suppose there is a big circle on your school ground.
You need to locate its centre. What will do?"

After analysing curricula, textbooks and examination questions of the past ten years, I designed the question for this study with the assumption that the prerequisite knowledge for the chosen question, i.e. the concept of the circle and the related terms, postulates, theorems, and geometrical constructions, was available to all the students. The proof of the theorem about the perpendicular bisector of chords, along with the other theorems on angle properties of a circle, is presumably underlying the mathematical concepts of the chosen question. These properties have been predominant in the syllabi, textbooks and examination papers of all the systems.

Despite having familiarity with the underlying mathematical concepts, most of the students would not find any ready-made solution for the question, rather they would synthesise their idiosyncratic and novel solution strategy by trying out different ideas they had worked out through studying mathematics in schools.

Though the chosen question would not limit students to using any particular method, (it could be solved using variety of informal and formal reasoning), the question was intentionally set to investigate the influence of students' curricular experiences on their reasoning or change of reasoning during working on the question. That is why I will describe briefly the analysis of each curriculum in relation to the chosen question to hypothesise how curriculum experience could supposedly influence students' reasoning for the question. Then this hypothesis will be checked in the light of empirical data from the students' working on the problem.

Analysis of curricula

The curriculum of one of the examinations, International Cambridge Examination (ICE) lays emphasis on the use of mathematical content, i.e. angle properties of a circle in a specific activity such as calculating angles in a variety of figures drawn on paper. In doing this activity, the mathematical content serves as a set of tools.

In the standard textbook, though they present formal proof to justify the properties, their focus is on solved examples and exercises giving students practice in calculating angles using the properties. The exercise questions vary in their complexity. There are questions which can be solved by producing one argument linking directly the given attribute to the required attribute, and also questions which require a chain or chains of arguments linking given attribute to the several intermediate stages and then finally to the required attribute. 'Calculating angle' questions have appeared in exam papers for last ten years but there are no questions on proofs and proving. This shows that ICE curriculum has a clear inclination towards viewing mathematics as a tool for calculating unknown attributes of a figure and this follows uniformly through the curriculum , textbooks and finally to exam questions.

There is a separate chapter on locus and constructions. They introduce major categories of loci by providing instructions to carry out practical activities. Proving is one of those activities. Further there are some worked examples followed by exercises on how to locate the position of a point by tracing intersecting loci. These types of questions have also quite frequently appeared in the exam papers.

I conclude the analysis of the specific content of the textbooks used in preparation of ICE by noting that their overall approach differs from the deductive structure of mathematics. It does not even fit Van Hieles's (1986) levels of understanding, which suggest that the student might apply certain new rules unconsciously until at a certain moment he becomes conscious about them. Rather the text intends to make students conscious about the rule from the very beginning.

Contrary to its counterpart, the emphasis of the curriculum of Public Examination System (PES) is on the mathematical content *per se,* i.e. theorems on circle, and it suggests formal proofs and proving as means to explore the content. The curriculum seems to portray the absolutist perspective that mathematics is a set of truths, discovered by purely deductive reasoning proceeding from accepted axioms to established truths. The breadth of the curriculum is almost the same to that of ICE, as it covers theorems on the same angle properties which are covered in ICE curriculum.

The standard textbook for PES presents definitions of the related key terms with some discussion about them. Then it presents two-column statement and reason proofs of each theorem mentioned in the curriculum. Finally, it presents some exercise questions on the related concepts.

In the exercise, there are mainly factual questions where students need to recall exact definitions or draw geometrical objects by recalling their definitions or images. There are two questions on proving and construction, which expect students to use the given information in establishing new facts. These questions match the definition of non-routine question used in this study. These questions are as follows.

"Prove that a rectangle is a cyclic quadrilateral." and "Take any three non-collinear points. Draw a circle passing through the points." (Q6 & Q13. p.240)

As these questions come in the text just after introducing the standard method and layout (two column-statement and reason format) of proving the theorems, it provides students with a context in which they can use their experience of learning through proving theorems. Nevertheless, the exam papers do not have this type of questions, rather they only ask students to prove traditional theorems using standard format, so the two textbooks' non-routine questions seem to be merely a cosmetic phenomenon.

Another chapter on practical geometry provides stepwise geometrical construction of the circumcircle, incircle, and escribed circle of a triangle. At the end, there are some exercises for providing practice in drawing these geometrical objects with different measurements and with slight variations in the conditions.

The two chapters appear in the textbook entirely isolated from each other. There is no attempt to mention the connections between the ideas presented, through proofs and proving the theorems in one chapter and the instructions for drawing geometrical objects in another chapter. For example, Chapter 11 (p 235) provides a sequence of arguments to prove that *if a diameter of a circle is perpendicular to a chord, it bisects the chord* and *if a diameter of circle bisects a chord, it will be perpendicular to the chord*, whereas Chapter 12 (p. 249) provides stepwise instructions for the construction of circumcircle of a triangle, where they demonstrate that right bisectors of the sides of a triangle meet at one point and that point will serve as a centre for the required circle. I reiterate that for many teachers the textbooks are the only resources and their "lessons are confined solely to the textbooks" (Riaz, 2004, p. 146). So, it is likely that the teachers do not encourage students make connections between the ideas present in these two chapters, rather they engage students in, as Swan (2006) mentioned, "practising skills and notation without having any substantial understanding of underlying concepts" (p. 16). I conclude on the basis of curriculum analysis and the understanding we have about the general trend of teaching and learning in Pakistani schools that the chosen question for this study will indeed be non-routine for many students. Though one cannot disregard the role of individual teachers, the expectations of the curriculum and examination suggest that students would have had minimal experience in going beyond the periphery of the curriculum. However they would be familiar with several mathematical concepts which they could integrate and employ in the question. I decided that a clinical interview, where they would work with an 'expert" and a peer, would provide them with an opportunity to put their mathematical ideas into perspective and demonstrate quality of their learning. Through this activity the research aims to generate evidence about their "zone of development".

ANALYSIS OF STUDENTS' WORK

The following broad categories of students' ways of approaching to the question were identified through juxtaposing the data collected through the interviews, written work and field notes. The numbers in the brackets with each response show the frequency of responses among students of particular examination systems.

Using visual perception

This is the category of responses where students did not use mathematical knowledge explicitly. For example, one student from PES argued that "the centre of circle is the point where vertical and horizontal straight lines [drawn across the centre] meet"(Fig 1). Another example from the same system is shown in Fig 2. They drew a "square" around a circle, joined the opposite "corners" of the "square" by drawing lines [diagonals] and described that the centre would be the point where these lines met.

Fig 1: (PES: 3)

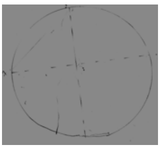
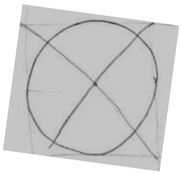


Fig 2 (PES: 2)



Informal use of factual knowledge

Here students used strategies based on trial and error methods using mathematical knowledge explicitly. For example, the way they used to find the longest chord of the circle shown in Fig 3. Some students suggested drawing an equilateral triangle, by trial and error method, on the circle. They mentioned that the centre of the triangle would be the centre of circle, though they were not sure how to find that centre.

Fig 3: (PES: 6, ICE: 4)

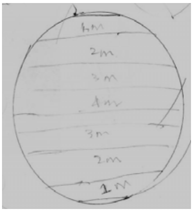
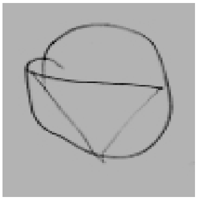


Fig 4: (PES: 1, ICE: 2).



Using hands-on activity (PES: 5, ICE: 3)

The students suggested making a tool with three sticks joining to make an angle of 120 degree with each other) and placing it uniformly on the circle. They thought that the point where all three sticks met would be the centre of circle. Some students used a string to put it around a circle to find circumference of a circle.

By using formula (PES: 5, ICE: 2)

After finding the circumference by hands-on activity, they suggested using the formula $c= 2 \pi r$ to find the radius of the circle.

Formal use factual knowledge (ICE: 7).

They used methods based on angle properties of a circle. The example below shows one of the responses which falls in this category

"Draw a chord and then bisect it [perpendicularly]. Let the bisector touch the two points on the circle. Find the mid point of the bisector. That will be the centre of the circle." (A response of a student from ICE)

DISCUSSION

Analysis of the responses of students from PES shows that they mostly used methods based on visual perception. The most common and immediate response was their effort to divide the circle into four equal parts by drawing two lines across it. Some of them used trial- &-error by applying particular mathematical facts explicitly like 'the largest chord of a circle will pass through its centre', or they marked some points as centres at random and then measured the distances between each of those points and the points on the circle to check which marked point fulfilled the condition of being centre by having equal distances from different points on the circle. Their use of mathematical terms like "straight line", "vertical line", or "line touching to the centre" was also not mathematically precise. They preferred to use freehand drawing although geometrical instruments were available to them. There was no evidence that the students' experiences of learning the theorems on circle or geometrical construction could help them design strategies for this question. At the end of the session, on request, they successfully recalled the theorems they had learnt to prove or the geometrical objects they had learnt to draw in their last year of schooling, but even retrospectively they could not identify the links between the question and their experience of learning geometry in the past year. The main issue was that the students faced difficulties in perceiving the type of the problem. The following quote from one of the students exemplifies their perception about the nature of the question.

"In a theorem question, there are always some things given and some things we have to prove but in this question no information are given and nothing is there to prove...I think it is like a practical question". Another student from the same system argued, "Usually we are given radius and we are asked to draw a circle, but in the question the circle is given and its radius is not."

Many students from the ICE background also started with trial-&-error but they soon realized they need to have a formal concise method. After that, most of them did successful and unsuccessful attempts to recall and apply angle properties of a circle like "we should draw chords because their [perpendicular, as it appears from their drawings] bisectors will pass through the centre of circle." However their use of those properties indicates that they used them as factual knowledge and produced only empirical evidence rather than reasoning, when asked for justification of the facts. This phenomenon seems to be a representation of their experience of learning the rules of the discourse during their previous year of schooling.

CONCLUSION

From this analysis of one question and the relevant parts of curricula, it looks as if the systems indeed do prepare students differently. As evidenced by their distinctly different responses to the same prompts, it seems that the students who learned the proofs in a "context free" environment could not use them in representing a new situation. And the students who learnt to use the proofs to solve problems seem to have difficulty in representing their understanding in the form of deductive arguments. While difference is not in itself a surprise, the nature of the difference is of concern when students from different backgrounds are compared to each other.

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*Note:

Here students' work is referred as responses, construction and reasoning used in the paper. Responses are the actual utterances, drawing or actions which manifest students' mathematical construction. Every mathematical construction is implicitly or explicitly based on certain type mathematical reasoning e.g. looking for patterns; reasoning by analogy; exploring specific cases; generalizing; inductive reasoning (based on observation); deductive reasoning (based on pre-established premises).

A TALE OF TWO LESSONS DURING LESSON STUDY PROCESS

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Improving quality of mathematics lessons is a well known purpose of teacher education. Lesson Study is considered as a process that leads teachers to a better understanding of student thinking in order to develop lessons that advance student learning of mathematics. In this paper we will focus on two lessons which were planned, conducted and reflected on during one Lesson Study cycle with one team of elementary mathematics teachers. We consider these lessons as personal stories of one teacher's development. For each of the two lessons we analyse two levels: macro-level analysis for the structure of the lessons and micro-level analysis that zooms in on students' learning.

BACKGROUND

The problem of how to improve teaching and learning of mathematics has been talked about by policymakers, researchers, teacher educators, teachers, parents and the public all over the world for some decades. Stigler and Hiebert (1999) wrote:

> Lesson study is a process of improvement that is expected to produce small, incremental improvements in teaching over long periods of time (p. 121).

During Lesson Study (LS), groups of teachers meet regularly to work on design, implementation and improvement of one or several lessons. Members of the team plan the lesson, and one member of the team teaches the lesson while fellow teachers observe and collect data on students' learning. After the lesson, the team discusses the lesson. Data from the discussion session are used to refine the lesson for a repeated teaching of the lesson in a second class. Then the teaching, observation and discussion session repeat. At the end of this cycle (see Figure 1) new ideas about teaching and learning based upon a better understanding of student thinking, is constructed within the team of teachers (Wang-Iverson & Yoshida, 2005).

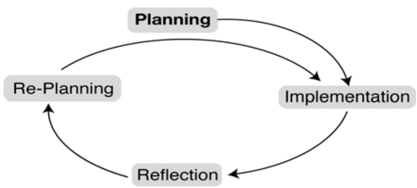

Figure 1: The LS Cycle

The LS process has some unique characteristics. LS keeps students' thinking and learning as the focus of the teachers' attention. LS provides teachers with opportunity to carefully examine student learning and understanding, and to reflect on and for a

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 489-496. Thessaloniki, Greece: PME.

lesson, and thus encourages teachers' developing understanding as a basis for their life-long learning (Hiebert, Morris, & Glass, 2003; Jaworski & Gellert, 2003). During LS teachers reflect on a particular lesson in which all the team is involved, observe and collect data on students' learning. And, the collaboration of teachers during LS, can support teachers' life-long learning, and have a long term influence on students' learning of mathematics (Lewis & Tsuchida, 1998; Wang-Iverson & Yoshida, 2005).

In this paper we will focus on two lessons which were planned, conducted and reflected on during one LS cycle with one team of teachers. We analyse the effect of this process on the quality of the mathematics lesson as reflected in the differences between the first and the second lessons in the cycle.

THE SETTING AND DATA COLLECTION

The data presented in this paper is a part of a wider study with two teams of elementary school teachers, teaching mathematics in grades 3-6. While the larger study explores for each team three learning cycles with two mathematics lessons, in this paper we analyse data collected during one (second) cycle of LS with one team of five teachers. We use the following data: (1) lesson planning team session; (2) lesson taught by one teacher and observed by the other members of the team; (3) reflective discussion of the lesson aimed at planning an improved version of the lesson; (4) teaching an improved lesson by the same teacher and observation of the lesson by the same team members; and (5) final reflective discussion.

All the data was video-recorded and the artefacts (e.g., lesson plans, lesson observations recorded by the teachers and written protocols) were collected. Videotaped meetings, lessons and interviews were transcribed.

DATA ANALYSIS

Similarly to Leikin & Rota (2006) we perform a comparison between the lessons based on two types of analysis. *Macro analysis* focuses on the lesson structure as consisting of: (1) opening, (2) students' mathematical activity independent of the teacher, and (3) summarising discussion. *Micro analysis* zooms in on: (a) the nature of mathematics tasks, their suitability to a particular classroom, and (b) teacher's attentiveness to student responses, and managing a lesson according to these responses.

Since personal stories have been acknowledged as means of presenting meaningful processes of teaching and learning to teach (Krainer, 2001; Zaslavsky & Leikin, 2004), we choose to present here the story of the lessons as the story of teacher's development through LS. Our story is about the teacher (Leah) who taught the two lessons. The topic of the lesson chosen by the teachers for grade five was "perimeters and areas". Leah volunteered to teach the lessons.

THE FIRST LESSON

Planning Lesson 1

Before the lesson, during the planning session the team discussed the purpose of the lesson. Ronnie, one of the fellow teachers, proposed to present the students with a “complex” (in her words) figure, for which they would find the perimeter and the area. She claimed that this is the best way of raising and coping with the mistake of confusing the ways of finding the area of the “complex” figure (addition of the areas of the sub figures) with finding the perimeter the same (addition of the perimeters of the sub figures).

Ronnie: I think they need to know how to find a perimeter and areas of a “complex” figure, for instance, figure of “house” with rectangle, triangle…

Tali: But it will be difficult for them to find the perimeter…

Ronnie: This is the point. I know it will be difficult but I want them to face this difficulty. This is the best way of raising and coping with the mistake of confusing the ways of finding the area of the “complex” figure by addition of the areas of the sub figures, with finding the perimeter by addition of the perimeters of the sub figures.

Leah: O.K. So the purpose of the lesson is that each student should find the area and the perimeter of a figure which composed of a triangle, a square and a rectangle.

Performing Lesson 1

During the opening of the lesson (that lasted twenty minutes) Leah repeated important concepts for the lesson, such as: What is the perimeter of a figure? What is the area of a figure? How to find the area or the perimeter of a square? Of a rectangle? Of a triangle?

Leah: What is a perimeter of a figure?

Student: The whole thing that is outside the figure.

Leah: Do you mean that this is the perimeter? [draws a rectangle on the board and colours the area outside it].

Student: No. I mean the sides of the figure.

Leah: O.K. Now, I want to ask you how we can find the area of a triangle.

When the introduction part was finished Leah asked students to find the perimeter and area of the figure (Figure 2). Students worked on the task independently of the teacher, sitting in six heterogeneous groups.

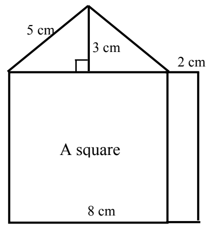

Figure 2: The figure presented to students during the first lesson

While the students worked on the task, Leah realised that for some students the task was too difficult and for the others it was not challenging enough. Some of them finished the work fast, whereas some others could not cope with it and needed the teacher's help. After fifteen minutes of work in small groups, Leah stopped this part of the lesson.

The summarising discussion that Leah managed in the classroom was first based on her observation of the students' work in small groups. Leah saw that students finished with two different results for the perimeter of the figure. She recognized that many students made the mistake of finding the perimeter of the figure by adding the perimeters of the sub-figures. Thus she asked:

Leah:	I want to hear now how each group found the perimeter of the figure?
Gal:	Each side of the square is of 8 cm, and the rectangle has two sides of 8 cm and two (sides) of 2 cm, and the triangle has two sides of 5 cm and one 8 cm, so we added it all and got 70 cm.
Leah:	Did anybody get a different result?
Yuval:	We got 38 cm. We didn't count the interior sides of the figure.
Leah:	Why?
Yuval:	It is a perimeter of the figure. We need to add the sides that surround the figure and not the sides inside the figure.

During the summarising discussion, the teacher asked all the students two questions: Can you draw another figure with a perimeter of 38 cm? Can you draw another figure with the area of 92 cm^2 (as the given figure)? This time, misunderstanding of the connection between perimeter and area of the same figure came up. Leah handled it in a very adequate way, by giving a counter example.

Adi:	I think the perimeter always less than the area.
Leah:	Why? Do you think that the perimeter of every figure needs to be smaller than the area?
Adi:	Yes
Leah:	Who wants to comment on this statement?

No one responds during 2 minutes

Leah: What is the perimeter and the area of this square? [draws a square 3x3 on the board].

Ron: The area is 9 cm^2.

Dan: The perimeter is 12 cm.

Adi: I understand now.

Finally Leah summarized the main points of the lesson (e.g. how to find perimeter and area of a ”complex” figure? What is important in computing the perimeter – not to add the inside sides).

THE SECOND LESSON

Planning Lesson 2

Before the second lesson the team of teachers participated in the meeting devoted to the reflection on the first lesson with the purpose to refine the lesson plan and "improve the lesson". The teachers discussed the changes that needed to be done in the structure of the lesson in order to adapt it to a heterogeneous classroom:

Leah: I felt during the students’ work that some of them couldn’t approach the task. The figure was too complex for them. …

Nili: I think we have to divide the class into homogeneous groups and let each group work on a different figure according to the students’ ability.

Ronnie: I agree. When students are able to cope with the task they can take an active part in the lesson.

Leah: This will let us challenge the better students. It is a good idea.

As the results of the discussion the ***task was transformed***. The teachers prepared three different tasks (see Figure 3) that varied in level of complexity for students of different levels.

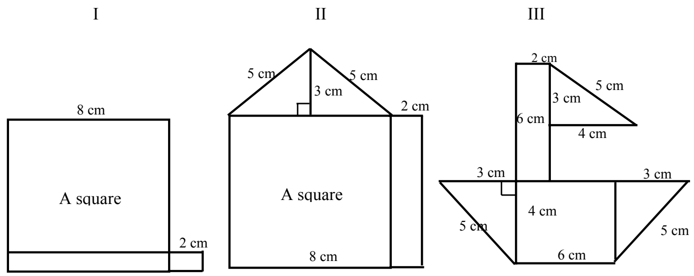

Figure 3: The three figures presented to students during the second lesson

Performing Lesson 2 (in a different class)

When opening the lesson Leah presented the concepts of perimeter and area of a figure (e.g. square, rectangle, triangle, and parallelogram). As planned at the reflection session, she prepared a power-point presentation to enable an efficient presentation of the main concepts and ways of finding areas and perimeters of some geometric figures. Students answered Leah's questions and it was obvious from their reactions that they enjoyed this part of the lesson very much.

The opening part lasted seven minutes. Then Leah divided the students into six homogeneous small groups and gave each group one of the figures (depicted in Figure 3) so that each figure was given to two small groups. The students in all the groups were asked to find perimeter and area of the figure. After computing the area and the perimeter of the figure, each group was asked to find another figure with the same perimeter (as their figure) or another figure with the same area.

During the small group work Leah saw that this time, the lesson developed “smoothly”: All students were involved in the activity. The teacher let students work collaboratively for twenty-five minutes as she felt they were succeeding with the tasks.

Leah was surprised that the students repeated the mistake from the first lesson. However, based on her previous experience, she asked the students about the difference between finding the area and the perimeter of a “complex” figures, and led students to the conclusion that to find the area of a figure one need to sum areas of (disjoint) parts, whereas one cannot use sum of the perimeters of to find the perimeter of the whole figure. During the following episode Leah asked students to focus on the “boat” task (Figure 3III)

Tomer: I see a trapezoid.

Bar: I see two triangles and a rectangle instead of a trapezoid.

Leah: Is it the same area if you find the area of the trapezoid or the area of the two triangles and rectangle?

Students: Yes, yes it is.

Leah: What about the perimeter?

Daniel: If we want to find the perimeter we need to add only the sides of the trapezoid. We need to remember not to add the inside sides.

In the summary discussion, each group sent a representative to draw on the blackboard a figure that had an area and perimeter equal to the figure they had investigated. There were some interesting suggestions:

Leah: I want suggestions of figures with the same area as the “house” (Figure 3II).

Ori: A rectangle of 10x6 and another rectangle of 4x8 connected to it.

SUMMARY (OR CONCLUDING REMARKS)

In this paper we introduced two lessons that were part of a LS process with the mathematics teachers in the elementary school.

Table 1 demonstrates distribution of the time among three main parts of the lesson.

Table 1: Time distribution (in minutes)

Lesson / Part of the lesson	1	2
1. Opening	20	7
2. Small group activity	15	25
3. Summarizing discussion	20	13

From table 1 we see differences in lesson structure. In comparison with Lesson 1, Lesson 2 had a clearer and shorter opening and the students were provided with more time for the whole group activity. Shortening of the summarising discussion was due to the inclusion of the mathematical task of designing a new figure of a given area or perimeter into the group work. Along with previous studies (Leikin & Rota, 2006; Stigler & Hiebert, 1999) these changes in the structure of the lesson are considered as one of the indicators of teachers' proficiency in teaching.

There was a clear distinction between the mathematical tasks presented to the students. The teachers made these changes based on their observation of students' difficulties that occurred during Lesson 1. Based on these observations the teachers were aware that there was a need for task adaptation to the different levels of students in the classroom. Additionally the students were asked "to design a new figure" in small groups instead of doing this during the discussion. This allowed students to provide richer examples during the discussion.

It must be noted that Leah conducted her lesson respecting students' replies during the two lessons. However during the second lesson students' mistakes were less surprising for her and she was "better equipped" to scaffold the learning process when needed.

Our analysis demonstrated that there were two main components that supported the changes in the lesson quality and structure: (1) Noticing and awareness of students' ways of learning, their difficulties and success based on teachers' lesson observations became resources for changing (Leikin & Zazkis, 2007; Mason, 2002), (2) Collaborative reflections – on and for – lessons, that teachers performed during the LS cycle served as springboards for the changes performed (Artzt, 1999; Jaworski, 1998; Mason, 2002).

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Author Index
VOLUME 4

O

P

R

S

T

V

W

Y

Z