RELEARNING MATHEMATICS - THE CASE OF DYNAMIC GEOMETRICAL PHENOMENA AND THEIR UNEXPECTED CARTESIAN REPRESENTATIONS

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Abstract This study describes the processes underwent by two mathematically sophisticated students during the investigation of geometrical phenomena with Dynamic Geometry. The investigation took place in an environment which included a computerized setting and ad hoc designed investigations to provide opportunities to deal with unexpected outcomes. In such an environment and in order to make sense of what they did and found, the students reviewed and enhanced their knowledge and thought processes, consolidating connections, establishing new ones, and developing new ways of work.

Introduction

In the last decades, there were considerable advances in the design of computerized environments to promote different and possibly more powerful ways of learning mathematics. Papert (1980) conceived the idea of "microworlds" in which students had the possibility to talk mathematics and to construct their ideas by designing. Pea (1986) defined "cognitive technologies" as a tool which may help transcend the limitations of the mind, supporting fundamental changes in the thinking processes. DiSessa (2000) described new "literacies" as rearranging "the entire intellectual terrain" (p. 19) making advanced knowledge more accessible.

In line with these and many others, our starting point rests on the broad assumption that computerized technologies enable students to approach, learn and understand mathematics in ways that were not possible before. There are many research studies reporting about learning in such settings (e.g. Kaput, 1992; Schwartz et al., 1993). Our focus is on "re-learning" (in the sense of Arcavi & Nachmias, 1989): we explored how students (and teachers) revisit and enrich their known mathematics in ad-hoc designed learning environments. In this paper, we briefly describe such an environment, and we present a case of two mathematically sophisticated students at the end of secondary school working with it. The mathematical topic of the exploration was familiar to the students, and they had all the needed pre-requisite knowledge to succeed, and they did. However, as in previous studies with students learning mathematical topics for the first time (Hadas & Hershkowitz, 1998, 1999, Hadas et al., in press), in this study the re-learning process of these students was led by their need to produce explanations of surprising findings which contradicted their expectations. We present data from their work and analyze the ways in which their prior knowledge and school habits (uses of procedures and formulae, repertoire of functions, expectations, courage to integrate and discuss intuitive ideas, etc) were engaged and changed during their work.

The environment

In our study, by learning environment we mean: (a) a Dynamic Geometry (DG) microworld with the possibility of juxtaposition and use of multiple representations (geometrical drawings, measurements, Cartesian graphs) and (b) specially designed geometrical investigations (involving unexpected outcomes) presented through a coherent sequence of open ended tasks (leading students to make and check conjectures and explain results not in line with their conjectures).

The following is one such sequence of tasks which includes two parts. The first deals with the area variation of an isosceles triangle (with equal sides of length 5 units) as a function of (i) the variable base (task 1), (ii) the variable angle between the two equal sides (task 2), and (iii) the variable altitude to its base (task 3). In the second part, the investigation turns to the area of a scalene triangle (sides of length 4 and 5), again, as a function of the variable third side (task 4) and as a function of the variable altitude to it (task 5).

Firstly, students are requested to examine the variation visually by dragging, measuring, and hypothesizing about the main graphical characteristics of the expected Cartesian graph of the relationship (domain, range, increase, decrease, maximum, symmetry, graph shape etc.) and to relate these to the geometrical features at stake (for a full description and analysis of the tasks, see Arcavi & Hadas, 2000).

After the students propose hypothesis/predictions, they are asked to make use of the software (Geometry Inventor, 1994) in order to draw the graph in real time, as they drag and change the triangle. Then they are requested to compare the graph on the screen with their predictions, and to explain the findings. Here, we concentrate on data from tasks 1 & 5, which in our experience are almost certain to produce surprises, contradictions and a subsequent motivation to search for an explanation.

The students

MI and US, 12 graders at the time of this study (age 17), were friends who studied together in the same advanced mathematics class, at the highest level of their last year in a prestigious (public) secondary school. They had had at least two years of calculus, geometry and algebra. They worked through the sequence of tasks described above in the presence of an interviewer who asked them questions to clarify what they said and requested for explicit verbalizations of their thought processes. At the end, they talked very openly about their views of the experiment in relation to their school experiences. The interview was videotaped and almost fully transcribed.

Framework for the analysis

Our analysis of the relearning focuses on three interrelated components:

- The role of prior knowledge - How pieces of the students' entrance knowledge determine what students see (and thus possibly hinder progress), and at the same time, how their knowledge is refined and enhanced by their experiences in this environment.

- The role of the views of the subject matter and on "doing" mathematics.

- The role of real time experimentation - How the objects of their observations and the features of the software support the change in their knowledge and in turn how the growth of their knowledge support new uses of the software?

Data and Analysis

At the beginning, MI and US correctly worked out the values of the domain of the variable base (0 -10). When they turned to the range, without any prompt, they said:

US: *between 0 and…, a right angled triangle will be the largest.*
MI: *When it is 90, its five…* [inaudible, maybe "squared"] *divide by 2.*
US: *its 25/2.*

Later, US explained this conclusion:

US: *I simply saw it is 12.5, the only variable* [between the two equal sides] *is* α *and when the angle is 90º the sine is 1, and if it is not 90º then the sine value is less than 1 and that decreases the area.*

The calculation they used ("*5 squared divide by two*") corresponds to the formula side-times-side-times-the sine of the enclosed angle which was the most common for them from their trigonometry course. Then they were asked to predict the graph of the area as a function of the triangle's base.

US: *It is growing like* ... [traces with his hands a parabolic shape with a minimum], ... *you have to use a theorem…*
MI: *Why not linear?*
I: *A nice question.*
MI: *If it is the base times the altitude over 2 it must be linear… but the altitude decreases and the base increases ... Anyway, that means that is not linear, but it is not parabolic either, I think. ... for a parabola there must be here a quadratic function, and I don't see it.*
US: *He is right, shall I explain?*
I: *Yes.*
US: *I am not sure, just a second... Suppose I define the length of BC as x, so how do I find the area? I divide it by two and draw the altitude, I can do x times the altitude...*

US and MI brought many knowledge resources to this problem: they identified the maximum area correctly and efficiently. Their repertoire of functions was beyond the prototypical (*"It is not linear, but it is not parabolic either"*). The connections between the graph and the formula were properly and spontaneously invoked ("*for a parabola there must be here a quadratic function, and I don't see it"*). They could use another area formula ("*Suppose I define the length of BC as x…*").

We propose that in this part of the session, their actions and discussions were clearly determined by their looking of how to apply what they already knew (which they did and quite successfully). They switched from one formula to another according to the context, and spontaneously discussed why the function is neither linear nor quadratic. However, they made almost no use of the environment and the findings it may suggest as a source of information or insight. One obvious reason for that, is their lack of previous experiences with a dynamic software. But, we claim that another, more subtle reason for not relying on the environment was their implicit views on doing mathematics. Their actions and discussions revealed a “formalistic approach”, (e.g. finding quickly what is easy to find -the maximum- via a known calculation, attempts to invoke 'ready-to-go' procedures – “*you have to use a theorem…*” - and proceeding to a “secure” building of a symbolic model, as the only basis to envision the characteristics of the graph rather than from the features of the situation observed on the screen.

Their initial reliance on displaying and using their knowledge and their initial reluctance to use the dynamic environment, was confirmed by MI and US themselves, when they were debriefed after the interview. They mentioned that at school they “simply follow the formulae”, sometimes even “without understanding”.

In the following exchanges, they gradually began to rely more on the information provided by their experimentation with the software in order to produce qualitative arguments.

US: *Symmetry? I don't think that there will be symmetry, here there will be higher values* [points to the zone in the graph up to the maximum area value*] and here* ["after" the maximum*] it will come down slowly.*

I: *Why do you say so?*

US: *I imagine the rate of change*

I: *You imagine it from seeing it?*

US: *From sort of physical vision, not formula or anything.*

I: *MI, do you agree?*

MI: *It looks like it first grows a lot and then it decreases a lot.*

I: *So you agree with non-symmetry here?*

For the first time, in order to predict the shape of the graph, they abandoned the formulae. Instead, they relied on their visual perception of the area variation in order to predict how the rate of change should be reflected on how the graph should look like. They seemed to agree on the qualitative difference between rates of change on both sides of the maximum. However, MI made a surprising comment which reflected the interference of his symbolic knowledge.

MI: *No, I think it will be symmetry because again if we consider the formula side times side times the sine of the angle between them, then, kind of, if sinus from 0° to 180° is symmetrical. From looking at it as it seems...*

I: *So it looks like that but it is not like that. Is that what you say?*

MI: *Yes. Also usually functions like these are symmetrical. I think on the basis of what...*

Two strong reasons made MI disregard what he had just “saw” and to reject their agreed qualitative conclusion about the asymmetry of the graph. The first is the knowledge of the formula for the area which focus on the angle as the independent variable (in which case the graph is indeed symmetrical) and not the base. From what they said later, we learn that their school experiences did not include calculus problems in which students model the same dependent variable as a function of different independent variables. Thus, in this case, MI could not disentangle what he imagined as symmetrical (a function of the angle) from the situation he is investigating (a function of the base). The second is related to MI's views of their past practices of mathematics, rather than to a dominant piece of prior knowledge: the area functions they have met were symmetrical. Prior knowledge and past school practices "join forces" here in order to reject a qualitative correct observation.

After drawing the graph with the software (see figure 1), the interviewer asked them whether they got what they have expected, and they both laughed:

MI: *No, absolutely not.*
I: *Does it make sense?*
US: *I said the opposite... I thought it was non-symmetrical from an absolutely non-mathematical view. I imagined the side varying and I imagined the increase and saw that at first the area increases with a fast rate and when it decreases, it decreases with kind of a slower rate, therefore I thought it is not symmetrical..*
MI: *It does not, it does not make sense... Why it is not symmetrical?*

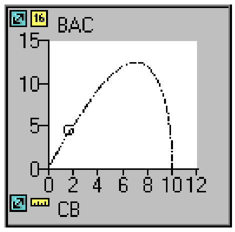


Figure 1

US focuses on why the asymmetry he mentioned before turned out to be the opposite of what they obtained. Whereas, MI was very puzzled by the asymmetry itself. Later on, when discussed the algebraic representation, he returned to his dilemma:

MI: *I know that in order to get a symmetrical graph, the maximum value ought to be when the angle between the equal sides is 60°, and it is at 90°, but from the point of view of the formula it still does not make sense to me.*

When they obtained the graph for task 2 (Figure 2a below), MI realized that his expectation of symmetry was right, but it referred to the area as a function of the angle and not as a function of the base. MI realized that a dependent variable can be a function of two different independent variables, and thus settled his dilemma. Figures 2b and 2c show the graphs they obtained for tasks 3 and 4. Their expectations were correct and made them feel comfortable with both the environment and their work.

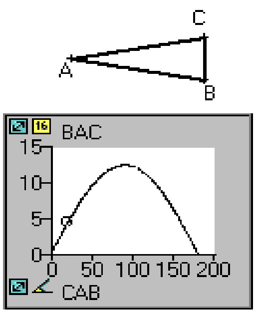


Figure 2a

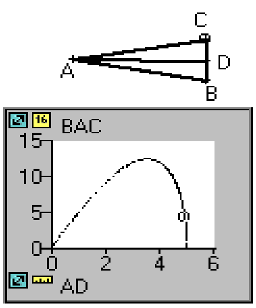


Figure 2b

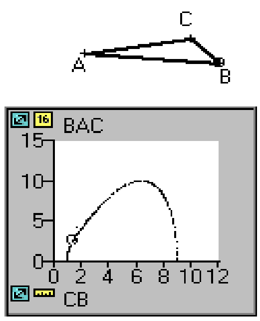


Figure 2c

In task 5 (investigating the area of a scalene triangle of sides 4,5 and x, as a function of the altitude to the variable side), MI and US found another point of confrontation with their previous knowledge and habits of learning presenting another opportunity for the creation of new connections. Firstly, they predicted that the graph would be similar to the three graphs they already saw in the previous tasks.

MI: *why should it be different than before?*
US: *I don't think that it makes any difference if the triangle is isosceles or not.*

But, at this point and in contrast to their work on the first task, they felt the need to experiment before making a final prediction. After discussing the domain of variability of the altitude and the range of the area, US suggested to use the DG tool to measure the altitude AD while changing the triangle. As US dragged the vertices on the screen and drew his predicted graph on the worksheet MI suddenly took the mouse from US, dragged the figure on the screen and looked unsatisfied.

I: *What disturbs you, MI?*
MI: *It is like we have a double domain. You have a certain length* [of the altitude AD] *here and also…*
US: *Something is wrong, it is returning back, the altitude values are growing from 0 till 4 and then, if you continue, it is going back.*
I: *I want to understand, does it change your prediction about the graphs' shape?*
US: *It is not the graphs' shape, I am trying to solve MI's question. It is a function. It is a graph that satisfy some sort of function so it can't be that there are two images for the same pre-image, two different values of the area fitting the same value of the altitude. It simple can't be.*

In order to strengthen the argument, MI measured:

MI: *For 3.12 you have the maximum* [10 square units] *and also this area* [2.18]. [He produced by dragging the two situations on the screen one after the other, as seen in figure 3.]

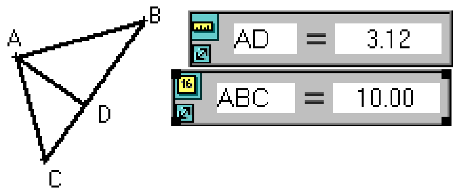
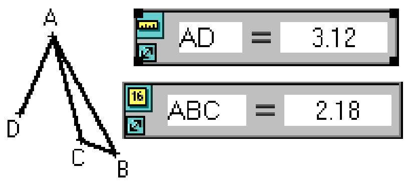


Figure 3



We note that at this point, MI and US used freely the DG environment to challenge their own initial prediction ("*why should it be different from what happens with an isosceles triangle*"). However, what they found seemed unnatural and different from their school experiences with "nice" graphs, let alone those who do not represent functions. (Reluctance from relationships which are not functions is described in detail in Even, 1993.) Thus when the interviewer asked them again if they want to correct their predicted graph, they requested to draw the graph with the software, at once, instead of risking a prediction. Although they already knew that "it will return back" (see figure 4) they were surprised and laughed:

MI: *How come that we didn't guess it.*
US: *It is very surprising.*

MI's comment can be interpreted in at least two different ways. It may mean that given their analysis, they could have guessed the shape of the graph. However, judging from the somehow ironic tone of his voice, he could have meant that this graph is something impossible to anticipate.

Figure 4

Then they tried to reflect on their prediction and explain the result.

US: *It is as MI said, for the same altitude there are two area values.*
MI: *And this fits what we see here.*
US: *In one case the altitude is inside the triangle and in the other outside.*

But they were still concerned.

MI: *It is impossible to write it as a function.*
US: *I never saw a graph that does not satisfy the definition of function…*
MI: *After what we have said … it is not so surprising.*

The interviewer suggested using the algebraic representation. As they were used to do so, they had no problems in writing two symbolic expressions for the two situations [$0.5x[\sqrt{(25-x^2)} + \sqrt{(16-x^2)}]$ and $0.5x[\sqrt{(25-x^2)} - [\sqrt{(16-x^2)}]$. When they proceeded to make sense of how these expressions depict the situation, their comments seemed to add another layer of meaning in connection to the geometrical phenomenon.

MI: *Looking at the graph, the functions meet when the altitude length is 0 or 4. 4 is not the maximum of the function, it is the maximum of the altitude. At 4 it will be the same in the two formulas, the value of $\sqrt{(16-x^2)}$ is 0.*

Discussion

We take the data above as an illustration of what it may mean to re-learn a known mathematical topic within a new dynamic environment which creates the need to settle results contradicting expectations. MI and US have not seen these tasks before. However, from their school experience, they were used to explore several types of functions and solve max-min problems by modeling and using calculus tools (e.g. derivatives). Thus the mathematical topic of their exploration was very familiar to them, and the following are some of the characteristics of their re-learning process:

- The unfulfilled expectation of symmetry (in task 1) and the non-function (in task 5) created engaging dilemmas. In task 1, MI's correct envisioning of the variation as a function of the angle, affected what he saw when the variation was a function of the basis. In task 5, their notion (and repertoire) of functions did not allow a non-function to depict a situation. In the first case, their knowledge helped them realize that the same dependent variable can vary according to different independent variables (something they were not used to but were able to understand and experience). In the second case, their knowledge and ability to produce a symbolic model and make sense of it, coupled with their initial qualitative analysis led them to accept the surprising graph as the right representation. In both cases, they enlarged their repertoire of functions.

Therefore, we can say that the students previous knowledge which seemed to hinder their learning was also the basis of their re-learning. Their knowledge was precisely the basis upon which meaningful dilemmas were created. They were able to integrate the DG tools in order to: (a) produce qualitative arguments, (b) make connections between different representations and (c) resolve their dilemmas and apparent "contradictions".

- The students approach to the first task indicated habits of work and views of mathematics clearly established by their school practices. As their work progressed, they started to use the tool to allow themselves to operate differently from what they were used to. Later, in task 5, they incorporated the dragging and graphing options to support more qualitative reasoning. They began to think about the predicted graph not only in terms of what kind of formula it is tied to, but also on how the graph has to reflect the features of the situation using the dragging and measuring tools offered by the software.

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