Of discs and cubes and magic squares

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Of discs and cubes and magic squares

a sort of algebra

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On some magic squares

6	7	2
1	5	9
8	3	4

Figure 1

A 3×3 magic square is an array as in Figure 1, in which the sum of the consecutive natural numbers in every row, column and (main) diagonal is the same, in this case 15. If we allow the numbers to be rational (or even real) and if we do not constrain them to be different, we can also have 'magic squares' like the following.

0	-0.5	-1
-1.5	-0.5	0.5
0	-0.5	-1

$2\sqrt{2}+1$	$-\sqrt{2}$	$2\sqrt{2}-1$
$\sqrt{2}-2$	$\sqrt{2}$	$\sqrt{2}+2$
1	$3\sqrt{2}$	-1

Figure 2

We have used the following problem with students (also as an activity to practice the arithmetic of negative numbers) and teachers who know very little about magic squares.

a) Complete the empty cells to obtain a 'magic square' with sum 9.

	3	
2		1

b) Complete the empty cells to obtain a 'magic square' with sum 6.

	2	
1		5

Depending on the class, we give one or two more simple magic squares, before the following, for which the requested sum is 8.

	4	
2		2

Because all previous examples worked out easily, this last one, which does not, is surprising. Usually a first move by students is to re-check the arithmetic or to try again a few times using different 'routes', until there is growing conviction that the mission is impossible. And then, of course the question, 'how come the others worked out and this one doesn't?'

In Arcavi (1995) we describe reactions to this problem in more detail, and we discuss student (or teacher) predisposition to call on algebra to solve the problem. We suggested there that, after raising conjectures and testing them, those who are proficient in algebraic symbol manipulation and its power to reveal the structure of certain numerical phenomena, will resort to the use of symbols to uncover (or prove) the relevant properties.

The point we were trying to make there is that proficiency with the syntactic rules of algebraic symbols does not necessarily imply 'symbol sense'. The aspect of symbol sense which we suggested may be missing for some students, is to know when to invoke symbols to generalise and justify, even when the problem does not explicitly suggest that we do so. Using symbols in this case means attempting to fill the square, for example, as shown in Figure 3 where a, b, and c are the given numbers, and S is the given sum.

[2] S–b–c		[1] S–a–b
[3] b+c–a	**b**	[4]
a		**c**

Figure 3

In order to complete cell 4, using the column sum we must write $a + b - c$. But, the middle row must add up to S. It follows easily that $S = 3b$ is both a necessary and sufficient condition to obtain a magic square.

	b	
a		c

Figure 4

This property is very simple. One can create (and check) problems of the form: 'fill in the blanks (in Figure 4) in order to obtain a magic square of sum S'. As long as we take care of $S = 3b$, a and c can be any two numbers. It is this simplicity and the fact that the magic works independently of the values we choose to place as a and c, that makes the property surprising and elegant.

On the need for symbols — part I

Although we did not say it explicitly, the discussion on symbol use and symbol sense in Arcavi (1994), seemed to imply that the only way to discover the property and to prove its universal validity is to resort to symbols. Somewhat later we were led to challenge that implication.

In other words we set out to *prove* without conventional algebraic symbolism, that any 3×3 arrangement (as above) is a magic square if and only if the number in the centre square is one third of the row, column, and diagonal sum.

The following describes such an alternative argument. Instead of symbols, we use cubes and a large (red) disc. The cubes represent the numbers in the cells, and the disc the sum. Even though the cubes may look identical, we can imagine that there are numbers written on them, but we are not interested in what specific numbers. First, in a blank 3×3 square, we place cubes to complete the constant sum in the rows, columns and diagonals which contain the centre square, one step at a time as shown in Figure 5, using the discs to keep record of the number of sums we complete.

Altogether we have completed four sums (as shown by the discs), and the sum of all the numbers (cubes) in the magic square is thus equal to four constant sums (discs). We now remove sequentially three rows and reduce the number of discs appropriately, so that the sum of the 'numbers' left in the magic square is equal, at all stages, to the sum of the discs remaining on the right. Figure 6 illustrates the process (at each stage we note what has been removed).

All we are left with is three times the number in the centre, and one (disc) sum. Hence the central number must be one third of the sum.

What we have proved is that a necessary condition for a 3×3 magic square is that the constant sum be three times the number in the centre square. It remains to prove sufficiency, namely that if we elect the sum to be three times the number in the centre cell, then we can complete the other cells so that all the columns, rows and diagonals add up to that same sum. In order to do that we introduce one new 'variable': the small (grey) disc to represent the number in the centre square. (We keep the consistent convention that for us discs represent given, known, numbers, and cubes unknown numbers.) Thus we know that three small (grey) discs are equal to one big (red) disc. If all we are given is that the number in the centre is one third of the proposed sum, then we can complete the magic square trivially by placing small discs in all positions (namely identical numbers). But this trivial magic square is hardly interesting. More reasonably, if, in addition, we are given any two numbers in, say, the two upper corners (as in Figure 7a), and the given sum as three times the centre element (namely, three small discs are equal to one big disc), then what we would like to show is that we can complete the magic square determined by the data.

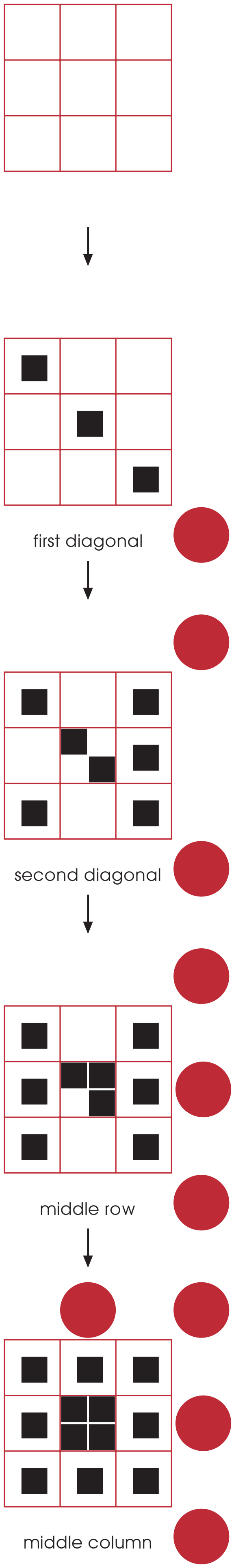

Figure 5

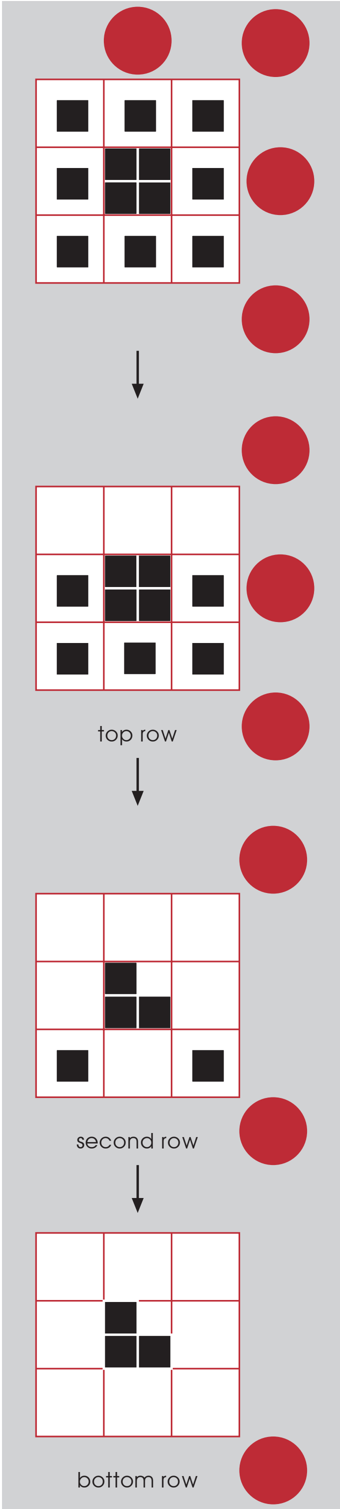

Figure 6

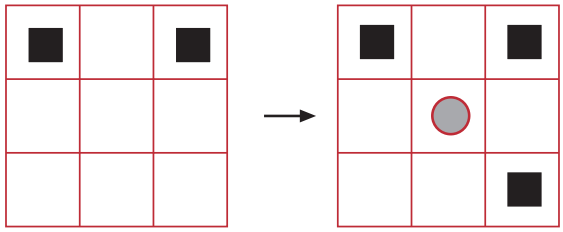

Figure 7a Figure 7b

Our starting position is then as Figure 7a. We now complete one diagonal so that the sum of its numbers is the given sum (i.e., three small discs). What do we know about the two cubes in this diagonal? Since they are unknowns, we know nothing about them individually, but we do know that their sum must be two small (grey) discs. In order not to forget this we place two small discs outside the magic square as in Figure 8a, and then continue to complete the other diagonal (Figure 8b).

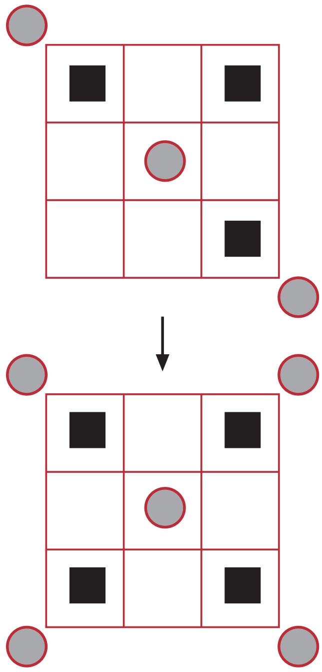

Figure 8a–b

We now complete the top and bottom rows with cubes so that their sum is also 3 small (grey) discs or one large (red) disc, and record the fact by placing large discs outside the square, on the right of these rows (Figure 9).

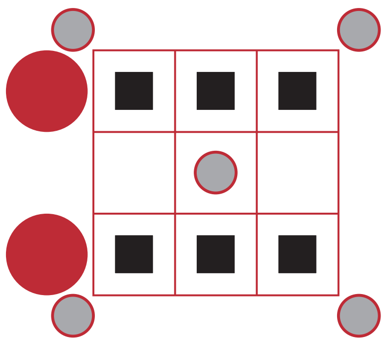

Figure 9

But now there is a problem. By completing the top and bottom rows, we have also completed the middle column, and how do we know that the sum of the two (unknown) cubes and the (known) small disc in this column is also a large disc?

As in the previous proof, we can complete the argument by removing appropriate cubes from the figure. What we have to do here is to remove the cubes from the four corner squares; we know their sum, because the sum of each pair of diagonally placed cubes is two small discs. So we remove a total of four small discs from the top and bottom rows; i.e., from two large discs (= six small discs). The result is clearly two small discs (see Figure 10).

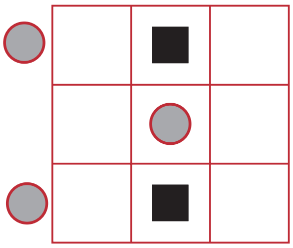

Figure 10

Similarly we can complete the two remaining empty squares in Figure 9, by completing the first and third column, and the middle row sum will be correct.

On the need for symbols — part II

From the strictly mathematical point of view one may argue that the cubes and discs are nothing but place holders for variables. And thus our avoidance of symbols is more apparent than real.

However, using algebraic symbols as we did in the first section of this paper, required more symbols to represent three unknowns explicitly, since, even if we are not interested in their possible values, we cannot represent possibly different numbers by the same letter. Cubes can have numbers on them, and two cubes do not have to show the same number. So our ‘physical’ proof is not only not ‘isomorphic’ to, but it is also much simpler than the conventional algebraic proof (at least the ‘necessary’ part).

Furthermore, one can handle cubes, move the ‘unknowns’ about by placing and removing cubes and discs, which helps to keep meaning in front of your eyes. With symbols, one cannot easily ‘pick up’ an x. Thus cubes have the advantage of physical manipulation as opposed to syntactic manipulation of symbols, which may be more error prone and less ‘transparent’.

The immediate temptation is to believe that this manipulative proof is appropriate for pre-algebra students, young and old. Some of the anecdotal evidence we collected would seem to support this.

When we showed this to teachers, at in-service workshops, their enthusiasm was evident, One teacher, for instance, had been playing magic squares with her 7 year-old, only to be dismayed to find that she was unable to respond to the child’s request for an explanation why the ‘three-times’ condition worked. She was delighted that she could now go home and explain. On a number of occasions — once even with the cutlery at dinner, with knives and forks as cubes and spoons as discs — we have explained the proof to mature adults whose mathematics was confined to elementary arithmetic.

Again an ‘average’ fifth grader playing with her mother (a math curriculum developer in our department) had some difficulties with the arithmetic of the magic squares, and was helped by her mother to discover the property of the sum. When her mother asked whether she was interested in knowing why this is the case, she seemed to get interested. Her mother reported that she not only understood the proof (the necessary condition) but was delighted and decided to play teacher and show it to her classmates. Apparently, she succeeded.

Finally, in another case, a group of pre-service teachers who had been shown the proof with the discs and cubes, spontaneously applied the idea successfully to discover properties of 4×4 magic squares.

The attraction of the manipulative proof seems to lie in its playfulness and its immediacy. Cubes and discs are concrete and familiar and would seem not to require the abstraction needed to fully understand and manipulate a symbolic representation. Bypassing symbols seems to make the essence of the proof and its meaning more accessible. When a proof is mediated by symbols, these may divert mental energy and become the focus of attention, and thus the proof is viewed as symbolic wizardry, rather than as an explanation of why.

Nevertheless, we would not like to claim more than that this idea is just another one in the assorted repertoire we should bring to the classroom, like those beautiful occasional flashes of proofs without words, which can enrich and entertain.

Reference

Arcavi, A. 1994 ‘Symbol sense: informal sense-making in formal mathematics’, *For the Learning of Mathematics*, *14*(3), 24–35.

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