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ABSTRACT

The first volume of this proceedings contains three plenary addresses: (1) "Visualization in 3-dimensional geometry: In search of a framework" (A. Gutierrez); (2) "The ongoing value of proof" (G. Hanna); and (3) "Modern times: The symbolic surfaces of language, mathematics and art" (D. Pimm). Plenary panels include: (1) "Contribution to the panel 'Putting mathematics into language and language into mathematics'" (C. Laborde); (2) "Language and the socialization of thinking" (T. Nunes); and (3) "Pupils' prompted production of a medieval mathematical sign system" (L. Puig). Research forums include: (1) "The role of representation systems in the learning of numerical structures" (L. Rico, E. Castro, and I. Romero); (2) "Mathematics teacher development: Connections to change in teachers' beliefs and practices" (J.R. Becker and B.J. Pence); and (3) "Teacher subject matter knowledge and pedagogical content knowledge: Research and development" (R. Even, D. Tirosh, and Z. Markovits). Short oral communications include: (1) "Understanding decimal numbers: From measurement towards the number line" (M. Basso, C. Bonotto, and P. Sorzio); (2) "The role of the graphic calculator as a mediating sign in the zones of proximal development of university students" (M. Berger); (3) "Professional development in performance assessment for Queensland years 1-10 mathematics teachers" (R. Bleicher, T.J. Cooper, S. Dole, S. Nisbet, and E. Warren); (4) "A student teacher's attempts to analyze a student's learning" (A. Boufi and S. Kafoussi); (5) "Teachers' and students' images about learning mathematics" (I. Branco and I. Oliveira); (6) "Prioritising mathematics teacher education choices at pre- and inservice levels" (C. Breen), (7) "Graphic calculators and precalculus: Effects on curriculum design" (C. Carulla and P. Gomez); (8) "The use of two different types of graphs in a statistical task (7th Grade)" (C. Carvalho); (9) "The influence of integrating an innovating project in pupils' ideas about mathematics" (M. Cesar); (10) "Using teaching cases: A portrait of teachers' thinking about reform" (M. Civil); (11) "Euclidean geometry: Cognitive gender differences" (L.S. Cronje); (12) "'The middle of what?': Students' images of mean, median, and mode" (D. Cudmore); (13) "Generalization in algebra problem solving and attitudes toward mathematics" (M.R.F. de Brito); (14) "Point configurations as representation system for the study of natural sequences" (M. de la Fuente and E. Castro); (15) "Learning and teaching percent problem solving" (S. Dole, A.R. Batturo, and T.J. Cooper); (16) "Cartesian graphs and

students conceptions: Looking for relationships between interpretation and construction" (M. Fabra and J. Deulofeu); (17) "Logo: Big step and small step" (E.K. Fainguelernt, F.C. Gottlieb, and J.B. Frant); (18) "Students' mathematical activity and cooperative work in the classroom" (E. Fernandes); (19) "Thinking processes used by two preservice teachers facing problem solving" (L. Fonseca); (20) "Personal strategies of generalization in linear generalizing problems" (J.A. Garcia-Cruz and A. Martinon); (21) "Conceptualizing non-unit fractions: An historical approach" (A. Goldblatt and C.L. Raymond); (22) "Transition between institutions: The case of algebra in the transition from vocational high schools to general high schools" (B. Grugeon and M. Artigue); (23) "The effects of a graphing-approach college algebra curriculum on students' understanding of function" (J.C. Hollar and K.S. Norwood); (24) "Assessment in geometry from two points of view: Levels of reasoning and SOLO levels" (M.P. Huerta); (25) "Researching computers and collaborative learning in pre-service mathematics teacher education" (A.J. Jones); (26) "An experiment on computer-assisted problem posing in undergraduate mathematics" (P. Kent); (27) "Student teachers' understanding of mathematics education" (S.H. Knudtzon); (28) "Knowledge of transformations of functions: Point-plotting as opposed to use of the graphics calculator" (P.E. Laridon); (29) "Learning from students' out-of-school mathematics practice" (J.O. Masingila); (30) "Assessing pre-service teachers' conceptual understanding of perimeter and area" (R. Menon); (31) "A framework for the quality of explanation in relation to the distributive law" (I.A. Mok); (32) "Division word problems: The construction of representation and procedures in young children" (L. Morgado and C. Abreu); (33) "The quality of students' reflection in problem solution" (M.D. Mosimege); (34) "An expert system to guide the geometry teacher" (L. Nasser, G.O.G. Zapata, and I.C.M. Bernardo); (35) "Going deeper into mathematical activity" (I. Plasencia, R.M. Guemes, and C. Espinel); (36) "Curriculum materials in mathematics education reform" (J.T. Remillard); (37) "Roots of teacher differences: Beliefs of preservice elementary vs secondary teachers" (B.F. Risacher); (38) "College students' knowledge and beliefs" (M. Risnes); (39) "The object and aim of teachers as regards pupils' learning in mathematics" (U. Runesson); (40) "Mathematics learning--Students' appropriation process of mathematical artifacts" (M. Santos); (41) "The initial growth of prospective mathematics teachers through participation in a teacher development project" (V.M.P. Santos); (42) "Mathematics, fractals and encoding images" (F. Sereno); (43) "Constructing knowledge together in the mathematics classroom: What do student teachers learn about teaching math from the practicum?" (R. Shane); (44) "The influence of 'supposed others' in the social process of making a mathematical definition" (Y. Shimizu); (45) "Developing new models of mathematics teaching: An imperative for research on mathematics teacher development" (M.A. Simon); (46) "Teacher beliefs and practices in primary mathematics" (S. Simon and M. Brown); (47) "Curriculum reform and teachers' conceptions of mathematics" (S.Z. Smith); (48) "Making connections: Representing understanding the number system" (N. Thomas); (49) "Deciding about studying mathematics at the senior secondary level of schooling in Australia" (R. Toomey, J. Deckers, B. Elliot, J. Malone, and R. O'Donovan); and (50) "Progress in mathematics education reform" (L.R. Van Zoest). Posters include: (1) "(divide) 2 litre" (J. Baena, M. Coriat, and P. Nieto); (2) "Evaluating an interactive CD-ROM designed for preservice teacher education" (L. Barron, J. Bowers, and K. McClain); (3) "The teacher as problem solver and his/her conception of problem solving in the mathematics classroom" (L.C. Contreras, J. Carrillo, and F. Guevara); (4) "The importance of relative code to young children's understanding of the division concept" (J. Correa and P.E. Bryant); (5) "Some adventures in Euclidian geometry" (M. de Villiers); (6) "Does language affect proportional reasoning?" (D. Desli); (7) "Making sense of science and mathematics through classroom dialogue" (B.A. Doig, S.C. Groves, and J.S. Williams); (8) "The assessment of algebraic partial knowledge: Model-eliciting activities for algebra" (F. Fernandez Garcia); (9) "Two case studies of future teachers' conceptions about mathematics, its teaching and learning" (P. Flores); (10) "Language and number notation: A study of deaf children" (M. Fuentes and L. Tolchinsky); (11) "An approach to analyzing the mathematics teacher's professional knowledge: The case of the concept of function" (M. Garcia, S. Llinares, and V. Sanchez); (12) "Mathematics teachers' beliefs concerning teaching and learning" (F. Gil and M.F. Moreno); (13) "The affective processes: A mathematical learning program in a socio-cultural context" (I.M. Gomez-Chacon); (14) "Algebraic thinking: Using magic squares with 5th grade students" (F.C. Gottlieb, J.B. Frant and R. Oliveira); (15) "Teachers responses to pupils' errors" (M.A. Inacio and G. Ramalho); (16) "A comparison of the geometric perceptions of Australian and American students" (C. Lawrie); (17) "A didactical engineering for a notion of limit of function" (A.L. Manrique and S.A. Almouloud); (18) "Supporting teachers of mathematics at a distance: Fifteen years of practice in the UK" (J. Mason, E. Love, C. Shiu, and D. Pimm); (19) "Arithmetic games with strategic components in the early primary mathematics classroom" (D. Mosel-Gobel, A. Peter, and P. Sorger); (20) "Number initiation--An African perspective" (R.K. Ngwa); (21) "Kites makes mathematics fly" (I. Oliveira, I. Branco, and J. Silva Pereira); (22) "Student decision making in a game of chance and misconceptions in probabilistic reasoning" (R. Peard); (23) "The use of cognitive maps for analyzing the understanding of transfinite numbers" (C. Penalva, C. Gaulin, and A. Gutierrez); (24) "Introduction of the concept of probability to teenagers (12/13 years old)" (C. Queiroz e Silva, I. Esteves, M. Henry, and T.M.M. Campos); (25) "Assessment of university students' mathematical generalization symbolization capacities" (A.M. Recio and J.D. Godino); (26) "Strategies used by mathematics university students in solving combinatorial problems" (R. Roa, C. Batanero, J.D. Godino, and M.J. Canizares); (27) "Additive operators in numerical tables" (F. Ruiz Lopez); (28) "Mathematics teachers' professional knowledge and their professional development" (M.J. Saraiva); (29) "Cognitive mapping: A study about the concepts of fractions in student teachers and elementary teachers (grades 1-4)" (V. Scomparim de Lima and M.R.F. de Brito); and (30) "Research of what-if-not strategy in problem posing" (H.S. Seo). (ASK)

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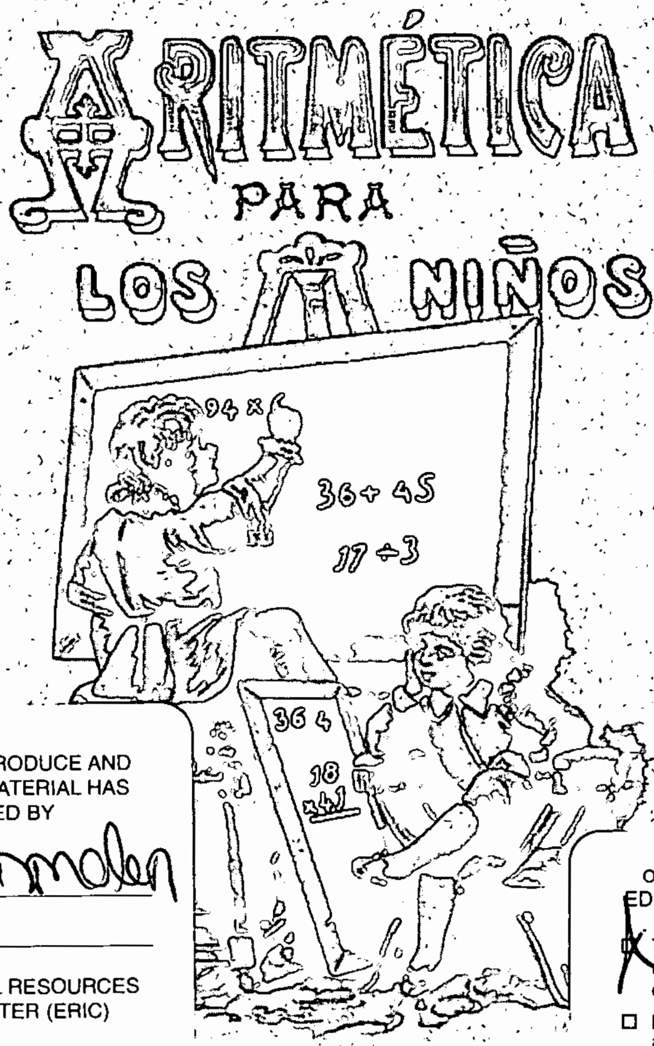

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Volume 1

International Group for the Psychology of Mathematics Education

PME 20

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Edited by

Luis Puig
Angel Gutiérrez

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Valencia (Spain), July 8 - 12, 1996

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Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education
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Preface

It was twenty years ago today, PME started to walk its way!

Piaget and Vygostky were born a hundred years ago, as it has been widely publicized and is well known by us all. The program of our conference includes an activity to celebrate this centennial.

Many of us know also that this year mathematicians and philosophers celebrate the birth of Descartes four hundred years ago.

Everybody who visits the cloister of the University of Valencia has to face the statue of Juan Luis Vives —whose 500th anniversary was only 4 years ago.

It's a pride and a challenge for me to be in the team organizing a PME conference with so many mentors, so many anniversaries.

Having been some days in preparation, a splendid time is guaranteed for all!

Luis Puig

Twenty Years of P.M.E.

Twenty years ago in Karlsruhe, during the Third Congress on Mathematical Education (Germany, 1976), a group of researchers and teachers decided to set up an international organization devoted to the study and promotion of the psychology of mathematical reasoning and mathematical education. The association, initially named IGPME (The International Group for the Psychology of Mathematics Education) later became known as the PME Group.

The creation of PME was preceded by two events. In 1969, during the First International Congress on Mathematical Education held in Lyon, France, Hans Freudenthal suggested the organization of a round table devoted to the psychological aspects of mathematics education. Though this meeting was not prepared in advance, it represented a remarkable success. As a consequence, the organizers of the Second International Congress on Mathematics Education (Exeter, Great Britain, 1972) decided to prepare and hold a workshop on the same topic. Speakers were invited to present papers. The workshop was a great success, both from the point of view of the quality of presentations and discussions and with regard to the number of participants. It became clear that the psychology of mathematics teaching and learning was not reducible to a theoretical, speculative discussion; rather, one had to admit that psychological concepts, investigations, interpretations, should be considered an integral part of every attempt to improve the quality and effects of mathematics education. It was during the Karlsruhe ICMI Congress, that the psychological workshop became an international, permanent research group.

A provisional committee and a president were elected and it was decided that the group would meet every year. The first PME meeting took place in Utrecht, Holland, in 1977, very much due to the organizational and financial support of Hans Freudenthal.

Since then we meet every year. The group grew in number and with regard to the quality of presentations. We started with about 100 members. We now have about 600 members. PME has become the most important and effective research group in the domain of mathematics education. Among its members, one may identify well-known personalities who have played and continue to play an effective role in shaping the modern theory and practice of mathematics education. The quality of investigations improved due to accumulated experience, international collaboration, and the imaginative creation of new concepts, new and more adequate research techniques, paradigms and theoretical interpretations. The "Proceedings" published every year represents a rich treasure of findings and ideas, a rich source of information and inspiration for everyone interested in mathematics education.

Before closing these lines, let me recall the names of some of those who were at the beginning of PME and are no more with us: Hans Freudenthal, Richard Skemp, Bob Karplus, Nicolas Herscovics. Although they are no longer with us, their fundamental contributions to the psychology of mathematics education continue to accompany and assist us.

Efraim Fischbein

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THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

History and Aims of PME

PME came to existence in 1976 at the third International Congress on Mathematics Education (ICME-3), held in Karlsruhe, Germany. Its past presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gérard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), and Carolyn Kieran (Canada).

The major goals of the Group are: To promote International contacts and the exchange of scientific information in the Psychology of Mathematics Education. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers. And to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

PME Membership

PME membership is open to people involved in active research consistent with the aims of the Group, or professionally interested in the results of such research. Membership is on annual basis and depends on payment of the membership fee for the current year (January to December). For participants on the PME 20 conference, the membership fee for 1996 is included in the registration fee. Others are requested to write either to their Regional Contact, or directly to the PME Executive Secretary.

Present Officers

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All members of the Local Organizing Committee belong to the Departament de Didàctica de la Matemàtica of the University of Valencia.

PME 20 Conference Chair

Angel Gutiérrez
Universitat de València
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The Review Process of PME 20

Six research forum proposals were received. Each proposal was reviewed by three PME members with well-known expertise in the topic of the research forum to which the proposal has been submitted. Three of the proposals were accepted on the grounds of these reviews.

The Program Committee received also 232 research report proposals encompassing a wide variety of topics and approaches. Each proposal was submitted to three PME members with expertise in the specific research domain. When there was not enough information on a paper from the reviewers' reports, one or more members of the Program Committee read the proposal. Based on these reviews, 160 research report proposals were accepted.

In addition, 67 short oral proposals and 35 poster proposals were received. Each proposal was read by one or more members of the Program Committee (two or more when a proposal was rejected). On the basis of these reviews, 50 short oral proposals and 30 poster proposals were accepted.

All written comments from the reviewers were forwarded to the authors along with the decision of the Program Committee.

List of the PME 20 Reviewers

The PME 20 Program Committee thanks the following people for their help during the review process:

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Universidad Autónoma de Barcelona
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Proceedings of Previous PME Conferences

The list below includes the proceedings of all PME International Conferences and all PME-NA Conferences having its own proceedings.

No.	Year	Place, Country	ERIC no.	Also available from
PME 1	1977	Utrecht, The Netherlands	Not available	
PME 2	1978	Osnabrueck, Germany	ED 226 945	
PME 3	1979	Warwick, U.K.	ED 226 956	
PME 4	1980	Berkeley, USA	ED 250 186	
PME 5	1981	Grenoble, France	ED 225 809	
PME 6	1982	Antwerpen, Belgium	ED 226 943	
PME 7	1983	Shoresh, Israel	ED 241 295	
PME 8	1984	Sidney, Australia	ED 306 127	
PME 9	1985	Noordwijkerhout, Holland	SE 057 131	
PME 10	1986	London, U.K.	ED 287 715	The Shell Centre
PME 11	1987	Montreal, Canada	SE 055 633	Carolyn Kieran
PME 12	1988	Veszprem, Hungary	SE 057 054	The Shell Centre
PME 13	1989	Paris, France	SE 056 724	
PME 14	1990	Oaxtepex, Mexico	SE 057 623	The Shell Centre
PME 15	1991	Assisi, Italy	SE 055 792	PME Executive Secr.
PME 16	1992	Durham, U.S.A.	SE 055 811	Karen Graham
PME 17	1993	Tsukuba, Japan	SE 055 789	PME Executive Secr.
PME 18	1994	Lisbon, Portugal	SE 055 807	PME Executive Secr.
PME 19	1995	Recife, Brazil	SE 057 135	PME Executive Secr.
PME-NA 3	1981	Minnesota	ED 223 449	
PME-NA 4	1982	Georgia	ED 226 957	
PME-NA 5	1983	Montreal, Canada	ED 289 688	
PME-NA 6	1984	Wisconsin	ED 253 432	
PME-NA 7	1985	Ohio	SE 056 279	
PME-NA 8	1986	Michigan	ED 301 443	
PME-NA 10	1988	Illinois	SE 056 278	
PME-NA 11	1989	New Yersey	SE 057 133	
PME-NA 13	1991	Virginia	ED 352 274	
PME-NA 15	1993	California	ED 372 917	
PME-NA 16	1994	Louisiana	SE 055 636	
PME-NA 17	1995	Ohio	SE 057 176	

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Visualization in 3-Dimensional Geometry: In Search of a Framework

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Friend, when I look back now and ask myself, what, properly speaking, have I done for the education of humanity? - I find the following: I have established the highest basic principle of education by acknowledging sensory-perceptual observation (visualization) to be the absolute basis of all cognition.

Pestalozzi, cited in Antonovskii (1990, p. 5)

Abstract. *The usefulness of visualization and graphical representations in the teaching of mathematics is being recognized by most mathematics educators and teachers of mathematics, but much research is still necessary. In the first part of the paper a framework aimed to organize the field of visualization in mathematics is presented. Visualization in 3-dimensional geometry seems to be a neglected area, since only a few reports of research can be found in the literature. The second part of the paper is devoted to raising some questions related to the analysis of primary and secondary school students' behaviour when solving tasks in 3-dimensional geometry by using dynamic software. The analysis will focus on students' ways of using screen images, the mental images they create, and the processes and abilities of visualization they use to solve the tasks.*

Introduction.

Although Pestalozzi exaggerated in giving visualization the role of *the absolute basis* of cognition, it is true that visualization is one of its main basis. When entering the area of visualization, several terms appear immediately: Visual reasoning, imagination, spatial thinking, imagery, mental images, visual images, spatial images, and others. When looking at the electronic databases for papers related to the terms "visualization", "spatial ability", or "mental image", one can find that most of the papers one encounters are published in journals of psychology, and only a few of them in journals of mathematics education. Many publications can be found

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concerning developmental stages of individuals (from early childhood to adults), the relationship of visualization to drawing, writing or speech, construction and handling of 3-dimensional objects, and other issues related to psychology, mathematics, or mathematics education. But one can also find titles concerning engineering, art, medicine, economy, chemistry, car driving, and some other surprising specialities.

Some conclusions arise: 1) Psychologists were aware of the importance of visualization long time ago, and they have developed detailed theories to frame their work, and tools to observe and test individuals. 2) Visualization is important for many more activities than we could initially suspect, although each speciality is only interested in certain specific abilities and environments, those narrowly related to their research problems. 3) Persons coming from different activities may have developed different meanings for the same words. 4) The field of visualization is so wide and diverse that it is not reasonable to try to encompass it all.

There is no general agreement about the terminology to be used in this field: It may happen that an author uses, for instance, the term "visualization" and another author uses "spatial thinking", but we find that they are sharing the same meaning for different terms. On the other hand, a single term, like "visual image", may have different meanings if we take it from different authors. Such an apparent mess is merely a reflection of the diversity of areas where visualization is considered relevant and the variety of specialists who are interested in it. The first section of this paper is devoted to identifying the field of activity I am interested in, and define the meaning those confusing terms will have here. I present a complete theoretical framework integrating partial results from researchers like Bishop, Hoffer, Presmeg, or Yakimanskaya, who have characterized different components of visualization.

Pestalozzi's statement was made more than 150 years ago, but only recently the relevant position of visualization has been broadly acknowledged by mathematics educators. Since a few years ago, mathematics educators have been underlining the need to increase the use of visual elements as a part of the ordinary teaching of mathematics in the different educational levels, particularly in secondary schools and universities. As far as research is concerned, we can find many papers reporting the results of experimental studies where the use of visual representations of the concepts to be taught have, in most cases, helped or, in some cases, hindered students in the formation of such concepts. As for curriculum development, more teachers and textbook writers are paying more attention to the use of drawings, diagrams, pictures, etc. in the math classes.

Geometry can be considered as the origin of visualization in mathematics but, if we examine the papers or books published in the last years dealing with visualization in mathematics education, we find many of them focusing on the teaching or learning of calculus (i.e. advanced mathematical thinking), quite a lot on (pre-)algebra and number systems, some on plane geometry, and only a few focusing on space geometry. In some way, this is reasonable since visualization has always been recognized as a necessary component for the teaching and learning of geometry (maybe the only exception is the period of 'modern mathematics'), and only recently has it gained the same recognition in other parts of mathematics. However, the technological revolution that has occurred in the last decade, with the popularization of computers and other multimedia tools, has provided teachers and researchers with new elements that may reshape the ways of teaching space geometry. These new possibilities have to be investigated and analyzed in depth, as a first step towards their implementation in the classrooms.

One of the new tools that can be used in the classrooms are computer programs giving a 3-dimensional representation of spatial objects and allowing users to transform those objects dynamically (transformations like rotations, translations, enlargements or sections by planes). In spite of the 3-dimensional aspect of the objects presented on the screen, they, like pictures, are plane representations of spatial objects, so some of the well-known difficulties students have when interpreting traditional plane representations of solids appear in computer environments too.

In the second part of the paper I raise some questions related to the analysis of primary and secondary school students' behaviour when using 3-dimensional dynamic software, their ways of analyzing screen images, and their mental images when they are working in such environment. Such questions are discussed under the theoretical framework organized in the first part of the paper, and they are exemplified by excerpts from students who were observed by Adela Jaime and myself as part of an on-going research project that has been carried out since 1989 at the Departamento de Didáctica de la Matemática of the University of Valencia.

Setting the Borders for Visualization and Other Related Concepts.

In cognitive psychology, one meaning of "mental image", supported by Denis, Kosslyn, Paivio, Shepard and others, is that of a quasi-picture created in the mind from memory, without any physical support. Kosslyn (1980) explains in detail his theory of mental images as having two major components: A *surface representation*, the quasi-pictorial entity present in the active memory, and a *deep representation*, the information stored in the long-term memory, from which the surface representation is derived. There are other cognitive psychologists, led by Pylyshyn, who argue against this concept of mental image owing to the many deficiencies they see in the picture-in-the-mind metaphor and the necessity they feel for a less vague definition. The work done by Minsky and Papert in the early 70s can be included in this paradigm. A third position is maintained by those who argue that the same representations are used in all kinds of cognitive processing, and mental images are just one case, so there is nothing special in mental images that makes them deserve a particular theory.

Researchers sharing the first or second positions are interested in the ways mental images are created and saved in a person's mind. For this reason, many tests designed to assess students' ability in the manipulation of mental images do not allow the use of paper and pencil or computers to answer the items (Zimmermann, Cunningham 1991, p. 3). A description of the main classical types of tests used by psychologists can be found in Denis (1989) and Clements (1981).

The meanings given by Kosslyn or Pylyshyn to the terms "visualization", "mental images", etc. are not shared by many educational psychologists nor by mathematicians, teachers of mathematics, or mathematics educators. These tend to give those terms a simpler more general meaning: A "mental image" is a mental representation of a mathematical concept or property containing information based on pictorial, graphical or diagrammatic elements. "Visualization", or visual thinking, is the kind of reasoning based on the use of mental images.

One of the main reasons for such disagreement is that, in mathematics, the use of drawings, figures, diagrams, or computer representations is part of everyday activity in the classrooms. In opposition to the approach of cognitive psychologists, mathematics educators consider that mental images and external (i.e. non-mental) representations have to interact to achieve a better understanding and to solve problems. Visualization is the context where this interaction takes place (Zimmermann, Cunningham 1991, p. 3). Furthermore, many mental images used in mathematics do not have a pictorial base, since they are based on diagrams, other visual ways of representation of concepts, or even textual or symbolic information.

Within mathematics education we can find several interesting pieces of theoretical work about visualization. Although they have elements in common, they have not been stated as parts of a single theoretical body. One objective of the research we are carrying out is to define a framework integrating those pieces, and to provide experimental support for the resulting general theoretical organization.

For Yakimanskaya (1991) a "spatial image" is created from the sensory cognition of spatial relationships, and it may be expressed in a variety of verbal or graphical forms including diagrams, pictures, drawings, outlines, etc., so she stresses the above mentioned interaction between spatial images and external representations. Furthermore, spatial images must be dynamic, flexible, and operational. She describes "spatial thinking" as *a form of mental activity which makes it possible to create spatial images and manipulate them in the course of solving various practical and theoretical problems* (p. 21), including verbal and conceptual operations, and several perceptual events necessary to form mental images. Yakimanskaya, then, considers that *images are the basic operative units of spatial thinking* (p. 26), and geometric objects are the basic material used to create and manipulate spatial images.

In the 60s and early 70s, when the Russian research reported by Yakimanskaya was carried out, geometry, geography, art, and other areas with strong geometrical support were the areas to observe spatial thinking and where most research experiments took place. Nowadays the role of geometry and geometric objects continues to be central as a support for visualization in mathematics, but there are also other useful elements coming from different areas of mathematics like calculus, algebra or statistics that are used very often.

Clements (1982, p. 36) considers that the concept of "image" as a picture in the mind is valid in mathematics education. Lean, Clements (1981, p. 267-68), following the dominant psychological theories of the time, define: "Mental imagery" as *the occurrence of mental activity corresponding to the perception of an object, but when the object is not presented to the sense organ.* "Visual imagery" as *mental imagery which occurs as a picture in the mind's eye.* "Spatial ability" as *the ability to formulate mental images and to manipulate these images in the mind.*

Presmeg (1986, p. 42) proposes to define a "visual image" as *a mental scheme depicting visual or spatial information,* with or without requiring the presence of an object or other external representation. As in Yakimanskaya's definition, Presmeg wants to include within the concept of visual image all those images having a graphical support different from a picture in the mind. She acknowledges that this definition is broader than most previous definitions, in particular the one proposed by Lean, Clements (1981) and Clements (1982).

Along the same lines, Dreyfus (1995, p. 3) defines "visual imagery" as the use of *mental images with a strong visual component.*

These broad definitions of mental images allow the possibility of having several kinds of images. Presmeg (1986) reports the results of a piece of research seeking to establish different kinds of visual images. Those observed in her students are classified as (ibid., pp. 43-44):

- Concrete, pictorial images: The kind of 'picture in the mind' images referred to by other authors.
- Pattern images: Images representing abstract mathematical relationships in a visual way.
- Images of formulae: Some students can 'see' in their minds a formula as it appeared written on the blackboard or the textbook.
- Kinaesthetic images: Those images that are created, transformed, or communicated with the help of physical movements.
- Dynamic images: Those images with movement in the mind.

Looking at the problem of visualization from another point of view, Bishop (1983, p. 177) recognizes two abilities in visualization: The "visual processing" of information (abbreviated to VP), including the *translation of abstract relationships and non-figural data into visual terms, the manipulation and extrapolation of visual imagery, and the transformation of one visual image into another.* The "interpretation of figural information" (abbreviated to IFI), involving *knowledge of the visual conventions and spatial 'vocabulary' used in geometric work, graphs, charts, and diagrams of all types ... and ... the 'reading' and interpreting of* visual images, either mental or physical, to get from them any relevant information that could help to solve a problem.

Bishop presents IFI and VP as abilities of persons, but, as defined, they fit better into the category of processes to be performed. The description of an ability should include information about the way it can be performed or the skills to be used. The description of a process should include information about the action to be done, but it is independent of the way of performing it in a specific case. For instance, the process of rotation of a mental image, which is a part of the IFI process, consists in converting the initial image into other one presenting the same object viewed while a rotation takes place or after it has been completed. The way such mental rotation is performed, i.e. the ability to be used, depends on the dimension of the rotation (in the plane or space), the position of the centre or axis of rotation relative to the figure (interior or exterior), the position of the axis of rotation relative to the subject (vertical, horizontal or orthogonal to the plane of vision), and the skills acquired by the subject. We can observe the presence of a variety of abilities by setting several problems to different students.

Yakimanskaya (1991, p. 101) describes two levels of activity in spatial thinking, the creation of mental images and their manipulation or use, as two closely interrelated processes (ibid., p. 102). The likeness of these processes to VP and IFI is evident.

Kosslyn (1980) identifies four processes applicable to visualization and mental images: Generating a mental image from some given information; Inspecting a mental image to observe its position or the presence of parts or elements; Transforming a mental image by rotating, translating, scaling, or decomposing it; Using a mental image to answer questions. The transformations investigated by Kosslyn are only part of those made with mental images in mathematics. For instance, it is quite usual to deform figures to solve problems of geometry. As an example, when students are learning relationships among quadrilaterals, they can imagine a rectangle shrinking to become a square and then again a rectangle.

Although Kosslyn's concept of mental image is different from that of the mathematics educators mentioned except Clements, the first process he defines is equivalent to the VP process, and the three others are parts of the more general IFI process. We can see, then, that Kosslyn, Yakimanskaya, and Bishop refer to the same processes of visualization, with the only difference being that Kosslyn's model is more detailed than the other two.

The list of abilities necessary to process mental images may be very long if we are interested in a general description of the field from the psychological point of view. For instance, McGee (1979), summarizing results from previous research on spatial abilities, describes ten different abilities, distributed into two classes:

Abilities of spatial visualization: 1) Ability to imagine the rotation of a depicted object, the (un)folding of a solid, and the relative changes of position of objects in space. 2) Ability to visualize a configuration in which there is movement among its parts. 3) Ability to comprehend imaginary movements in three dimensions, and to manipulate objects in the imagination. 4) Ability to manipulate or transform the image of a spatial pattern into other arrangement.

Abilities of spatial orientation: 1) Ability to determine relationships between different spatial objects. 2) Ability to recognize the identity of an object when it is seen from different angles, or when the object is moved. 3) Ability to consider spatial relations where the body orientation of the observer is essential. 4) Ability to perceive spatial patterns and to compare them with each other. 5) Ability to remain unconfused by the varying orientations in which a spatial object may be presented. 6) Ability to perceive spatial patterns or to maintain orientation with respect to objects in space.

Hoffer (1977) identifies several physio-psychological abilities relevant to the learning of mathematics: Eye-motor coordination; Figure-ground perception; Perceptual constancy; Perception of positions in space; Perception of spatial relationships; Visual discrimination; Visual memory (to remember mental images or objects no longer seen).

Some abilities in the previous lists overlap, part of them are general abilities useful in many ordinary life or professional activities, and others may be seen as specific to mathematized contexts. If we limit ourselves to the environment of geometry, only a part of the abilities are pertinent.

A complete framework characterizing the activity of visualization in mathematics can be defined by unifying the terminology used by several of the above mentioned authors, and integrating the concepts defined by them into a single network. In the following I will restrict myself to the context of mathematics, so the definitions given below do not try to go further or to be applicable outside the teaching and learning of mathematics. With respect to the vocabulary, the terms mental image, spatial image and visual image defined by Yakimanskaya, Dreyfus and Presmeg can be considered as basically equivalent, and the terms visualization, visual imagery, and spatial thinking can also be considered as equivalents.

I therefore consider "visualization" in mathematics as the kind of *reasoning activity based on the use of visual or spatial elements, either mental or physical,* performed to solve problems or prove properties. Visualization is integrated by four main elements: Mental images, external representations, processes of visualization, and abilities of visualization.

A "mental image" is *any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements.* Like Yakimanskaya (1991), I consider mental images as the basic element in visualization. The types of mental images identified by Presmeg (1986) can probably be completed if research is made to identify them in other specific areas of mathematics, like probability, functional analysis, or analytic geometry. Usually only a few types of mental images are necessary to solve a certain kind of task. For instance, only concrete, kinaesthetic, and dynamic images were used by our students to solve the tasks we proposed them.

An "external representation" pertinent to visualization is *any kind of verbal or graphical representation of concepts or properties including pictures, drawings,* *diagrams, etc. that helps to create or transform mental images and to do visual reasoning*. A research question can be raised at this point: Do external representations have an absolute or individual character? That is, does the property of being visual belong to an external representation or does it depend on the way a person uses the representation? This question has to do with the distinction between geometric (visualizers), analytic (non-visualizers), and harmonic (mixed) types of mathematical reasoning made by Krutetskii (1976).

A "process" of visualization is *a mental or physical action where mental images are involved*. There are two processes performed in visualization: "Visual interpretation of information" to create mental images, and "interpretation of mental images" to generate information. The first process corresponds to Bishop's VP. The second process corresponds to Bishop's IFI, and it is made up of three sub-processes, as identified by Kosslyn: Observation and analysis of mental images, transformation of mental images into other mental images, and transformation of mental images into other kinds of information.

Individuals should acquire and improve a set of "abilities" of visualization to perform the necessary processes with specific mental images for a given problem. Depending on the characteristics of the mathematics problem to be solved and the images created, students should be able to choose among several visual abilities. These abilities may have quite different foundations, the main ones being:

- "Figure-ground perception": The ability to identify a specific figure by isolating it out of a complex background.
- "Perceptual constancy": The ability to recognize that some properties of an object (real or in a mental image) are independent of size, colour, texture, or position, and to remain unconfused when an object or picture is perceived in different orientations.
- "Mental rotation": The ability to produce dynamic mental images and to visualize a configuration in movement.
- "Perception of spatial positions": The ability to relate an object, picture, or mental image to oneself.
- "Perception of spatial relationships": The ability to relate several objects, pictures, and/or mental images to each other, or simultaneously to oneself.
- "Visual discrimination": The ability to compare several objects, pictures, and/or mental images to identify similarities and differences among them.

To conclude, the diagram in Figure 1 summarizes the steps to be followed when using visualization to solve a task: The statement of the task is interpreted by the students as an external representation suitable to generate a mental image. This first image initiates a process of visual reasoning where, depending on the task and students' abilities, they use some of their visual abilities to perform different processes, and other mental images and/or external representations may be generated before the students arrive at the answer.

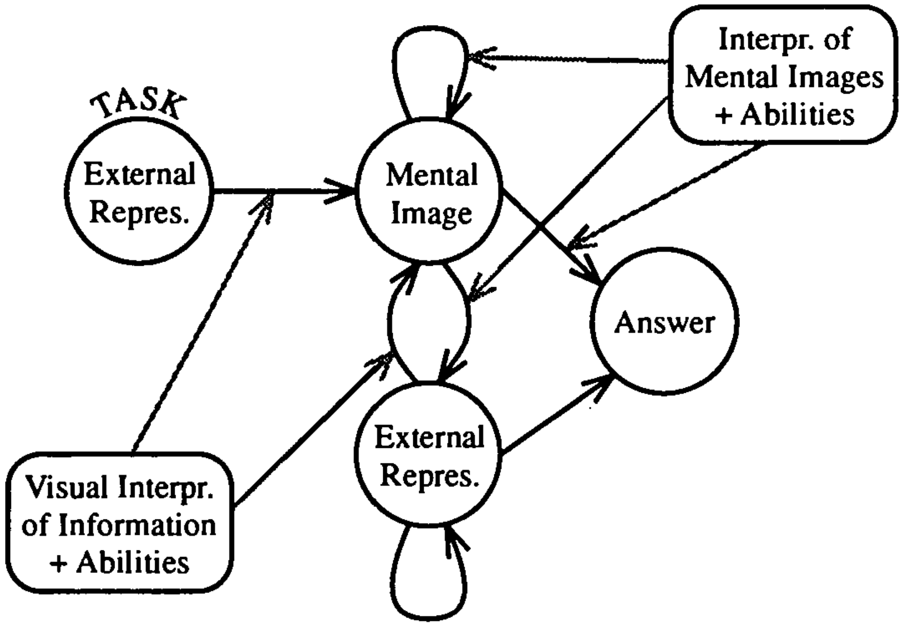

Figure 1. Main visualization elements integrating the solution of a mathematics task.

The Role of Visualization in 3-Dimensional Geometry. A Case.

I believe there is a general agreement that visualization is a basic component in learning and teaching 3-dimensional geometry. However, there is a very limited research activity in this specific area. On this research activity, most publications deal with students' difficulties when they have to move between 3-dimensional objects and some of their usual 2-dimensional representations, and only a few have approached the problems of the students using some specific kind of representation.

The only way textbooks have to present 3-dimensional geometry to students is by means of plane representations, usually perspective, parallel, or orthogonal projections. In many cases the teachers mitigate the limitation of textbooks by using wood or cardboard models. Until a few years ago, these were the only two possibilities available for most teachers all over the world, but now they can have access to a third way of representation: Computers with special software allowing students to see a solid represented in several possible ways on the screen and to transform it. Some important advantages students can gain from the use of this kind of software are:

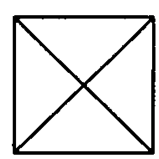

Figure 2.

Students will see polyhedra and other solids in many more different positions on the screen than in the textbooks. As a consequence, they will gain a rich experience that will allow them to form richer mental images than from textbooks. In particular, students will greatly improve their ability to create dynamic mental images. For instance, a student who has only seen the pictures in the textbooks will hardly recognize the drawing in Figure 2 as a representation of a pyramid or an octahedron. However, when rotating these solids on a computer, this special position appears as part of a continuum of images, and it gets meaning in this set of linked mental images.

As pointed out by Yakimanskaya (1991, p. 103), *the creation of images is possible because of the accumulation of representations that serve as the starting point and as necessary conditions for the realization of thought. The richer and more diverse the store of spatial representations, the more highly perfected the methods of creating representations and the easier it is to use images.* Computers can play a very relevant part in helping students to acquire and develop abilities of visualization in the context of space geometry.

When a person handles a real 3-dimensional solid and rotates it, the rotations made with the hands are so fast, unconscious, and accurate, even in the case of young primary school students, that one can hardly reflect on such actions; However, a software package limiting the directions of rotation forces the students to devise strategies of movement and to anticipate the result of a given turn.

Many advantages derived from the use of computers to teach geometry have been reported by research, and also several problems have already been highlighted in relation to the use of computers. In space geometry, students tend to base their arguments and conclusions on the appearance of the solid on the screen (Dreyfus, Hadas 1991, p. 87); For instance, they may accept a right angle as acute because it looks acute on the screen. Then, in the same way that students have to learn to interpret plane drawings correctly, they have to learn to interpret computer drawings correctly, and to use the tools provided by the software efficiently. On the other hand, when selecting a piece of software and a type of activity to be solved with that software, several variables should be taken into consideration (Gutiérrez, 1992 a; Gutiérrez, 1996): The type of representation of solids; The way the software allows a representation to be transformed, in other words, how user-friendly the software is; The range of students' abilities required by the software and the activities.

Working in that direction, we have made extensive experiments with students from a wide range of primary and secondary grades, aged from 7 to 17 years old. We have selected several computer programs that represent polyhedra in perspective and that allow the users to rotate them around the three standard coordinate axes (vertical, horizontal, and orthogonal to the screen), and we have asked the students to solve several types of activities. In particular, we asked them to rotate solids on the computer screen from an initial position to a target position drawn on paper (a hard copy of the computer screen). A more detailed description of this environment can be found in Gutiérrez, Jaime (1993). One of the aims of this line of research is to analyze the variables mentioned in the previous paragraph. Another aim, relevant to this paper, is to analyze the ways students solve the different activities, paying attention to the kinds of mental images and abilities of visualization they have used.

When trying to know the mental images created by students and the abilities put to work, one has to be aware that a mental image can be perceived only by means of some form of external support, verbal, graphical, gestural, etc. (Sutherland 1991, p.

71). From a methodological point of view, researchers should not ask the subjects to describe their mental images while they are solving a task, since the subjects may not be aware of their own images (as is the case of young primary school students) or, if they were aware of them, the dialogue could certainly distort the subjects' process of reasoning. Then, the best possibility for researchers seems to be to interpret the actions produced as a consequence of the subjects' activity with those mental images (ibid., p. 71), although, as Dreyfus (1991, p. 6) points, such an interpretation is influenced by the researcher's theory, and there may be some elements connecting the mental image and the researcher that influence and mediate the interpretation.

Figure 3 shows the three kinds of solid used in our research: A cube with pictures on the faces, several polyhedra with shaded faces, and the same polyhedra with transparent faces.

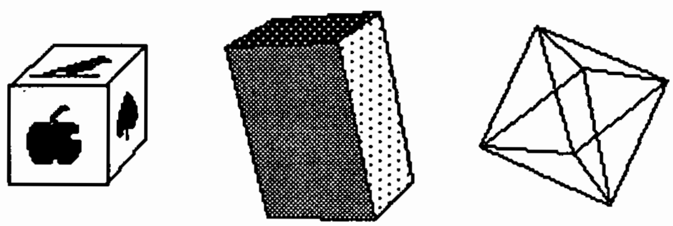

Figure 3.

The figurative cube was used in a HyperCard stack designed to help the younger primary school students to enter into the manipulation of 3-dimensional objects on the computer. This program only allows rotations of 90°, that are performed after clicking on one of a set of buttons shown on the screen (Figure 4).

One of the tasks presented to the students was to rotate a given figurative cube from its current position to match exactly a picture shown on a sheet of paper. The students were also asked to make the movement with the minimum number of rotations. Even this simple and easy task is rich enough to show different students' strategies, and the use of mental images and abilities of visualization with different grades of expertise.

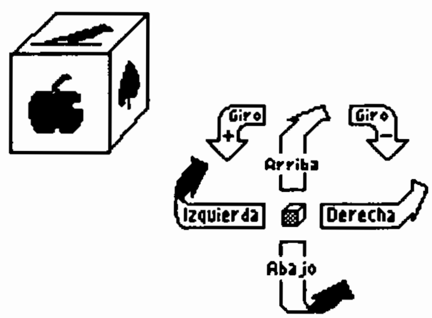

Figure 4.

After spending one hour solving several activities of this kind on the second day of the experiment, a second grader (7-8 years old) was asked to move the cube from its current position (schematized in Figure 5.1) to the target position (Figure 5.3). The first student's action was to move the cursor to the button ⇓.

1 Researcher: *Wait a moment before clicking. You are going to move it down. What is going to happen with the spade?*

2 Student: *It will go here* [pointing to the hidden bottom face of the cube on the screen].

- 1 - - 2 - - 3 -

Figure 5.

3 R.: *You will not see it, will you?*
4 S.: *No.*
5 R.: *What about the apple?*
6 S.: [It goes] *here* [pointing to the front face of the cube].
7 R.: *And, what about the candle?*
8 S.: ... [he hesitates, pointing to the left and right faces of the cube] *here* [pointing to the top face of the cube].
9 R.: *Let's see what happens.*
10 S.: [he clicks on ⇓ (Figure 5.2)] ... *No. It comes this way. And the bird.*
11 R.: [pointing to the cube on the screen] *The spade has come down, where you said, the apple has come here, but this one* [the candle] *has stayed on the same face but it has ...* [S interrupts R]
12 S.: *Now the bird is here* [pointing to the top face] ... [clicks on ⇓] ... *Now the turn minus* [clicks on ↴ (Figure 5.3)] ... *I got it.*

The student continued solving another task like the previous one. Now he has to rotate the cube from the position in Figure 6.1 to that in Figure 6.4.

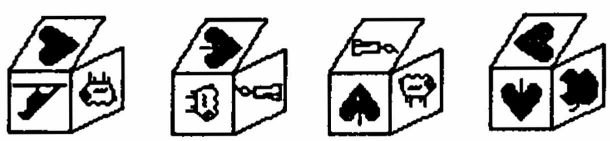

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Figure 6.

13 S. [talking to himself while working alone]: [clicks on ⇓, ⇐ (Figure 6.2)] *This one?* [clicks on ⇓] ... *Yes, but now I am going to make the turn plus* [clicks on ⇃ (Figure 6.3)] *I got it ... Oh, no. The apple isn't there.*
14 R.: *What has happened?*
15 S.: *The apple is there or there or there* [pointing to each of the hidden faces].
16 R.: *Where is the apple? Look at the* [target cube on the] *sheet of paper, where do you say the apple is?*
17 S.: *There on the back* [pointing to the hidden back face].
18 R.: *On the back? Why?*
19 S.: ... *No, here down here* [pointing to the bottom face] *because before ... No. It is here because I have made lots of turns and it has gone there.*
20 R.: *But, look, the apple is next to the spade, isn't it?*
21 S.: *Yes.*

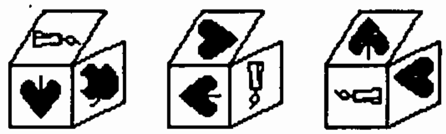
22 R.: *Then, may it be on the opposite face* [to the spade] *?*
23 S.: ... *Well ... This* [the spade] *has to be upside down* [clicks on ⤵] ... *Another one* [clicks on ⤵] ... *Yes, I got it.*
24 R.: *Where was the apple?*
25 S.: *Here* [pointing to the back face].
26 R.: *But you have rotated it* [the cube] *this way* [going ⤵ with the hand].
27 S.: *Then it was here* [pointing to the bottom face].

In the excerpt of the first task, we see that the student is able to create and use dynamic mental images, since he can anticipate the position of a figure after a turn (paragraphs 2, 6, and 8), although he cannot create a dynamic mental image of the whole cube, but only of one face, and he has difficulty in answering to the question in 7. When he is asked for the positions of the different faces after the turn ⇓, he creates a new mental image for each face. As a consequence, the student does not recognize his mistake when he thinks of the candle. many primary school students tend to pay attention only to the front face of the cube. This is the origin of the misconception of assuming that, after a rotation moving the front face's figure to another face, every figure is also moved to another face. From another viewpoint, this mistake reflects a lack of the abilities of perceptual constancy or perception of spatial relationships.

The last part of the excerpt (10 to 12) shows the student's resolution strategy for these problems: He looks for the figure on the front face of the target cube, making turns at random if it is hidden. When this figure appears on the screen, he moves it to the front face, and then he makes turns to move it to its target position. This student has difficulty in focusing on another figure to take decisions about the rotations to be made. This behaviour is typical of students in lower primary grades; Students in middle primary grades can improve this strategy and pay attention to any face with the help of the teacher, and students in the upper primary grades can improve it by themselves. However, to have this strategy does not mean to be able to apply it efficiently. I showed in Gutiérrez (1992 b) the case of a sixth primary school student who stated the strategy correctly while solving several tasks like the previous one. Although these tasks can be solved by sequences of four or fewer rotations, this student needed as many as 9, 12, or even 21 rotations to move the cubes to the target positions. The origin of her difficulties was the lack of certain abilities of visualization, like the ability of mental rotation and those of perception of spatial relationships or positions.

The excerpt of the second task confirms the conclusions drawn from the previous one. The student made the first series of turns (13) looking for the spade and trying to move it to its target position, although in this case the most efficient way of solving the problem would be to pay attention to the top face (the heart). The lack of the abilities of perceptual constancy or perception of spatial relationships is still more apparent in the second task (16 to 19). Although this child is able to rotate the mental image of a cube's face, he can only do rotations by a single turn. Now, as several rotations were made one after the other, the student cannot reproduce the movement of the faces (19).

Finally, I will show an eighth grade girl solving a similar task. This kind of activity is very easy for students in grade 8 or older, so now the researchers changed their approach and asked the students to predict the position of the cube after one or more rotations. The cube had to be moved from the position in Figure 7.1 to that in Figure 7.3.



-1- -2- -3-

Figure 7.

1 S.: *The candle has to be on the top the other way around.*
2 R.: *What should you do?*
3 S.: *A rotation like this* [making the rotation ⬐ with the hand], *and then the candle would be like this, wouldn't it?* [she shows the position with the hand] *Then to the right.*
4 R.: *Where are you going to move the candle first? How?*
5 S.: *Here* [pointing to the hidden left face].
6 R.: *On the hidden face?*
7 S.: *Yes, I think ... I can't see the heart ... I can't see the heart. Maybe it's here* [pointing to the hidden left face] *or down here* [pointing to the hidden bottom face].
8 R.: *Where do you think the heart is?*
9 S.: *Down here.*
10 R.: *Down here? How do you know?*
11 S.: *I think so, because if it is next to this one* [the spade], *it may be either here* [pointing to the left face] *or down here ...* [she rotates the sheet of paper with the target cube] *... It is here* [pointing to the left face] *... Then, if it is here, we will have to go ...* [she rotates the sheet] *... we will have to turn it like this. Then, if we rotate it this way* [going ⬎ with the hand], *it would go here, wouldn't it? The heart would come here* [pointing to the top face].
12 R.: *So, you want to put the candle here* [pointing to the right face] *to make the heart appear, don't you?*
13 S.: *Yes* [she clicks on ⬎ (Figure 7.2)] *... Now it* [the candle] *is like that* [pointing to the right face on the target cube on the sheet of paper and the screen].
14 R.: *What do you have to do now?*
15 S.: *Another turn* [going ⬎ with the hand] *to put the candle down here, and then another turn up* [going ⇑ with the hand], *don't you?* [she clicks on ⬎ and ⇑] *OK.*

The most noticeable fact from this excerpt is the extensive use the student makes of her hands to show the rotations, i.e. of kinaesthetic images. Another form of this type of mental image appears in paragraph 11 when she rotates the sheet of paper to see the position of some figures after a turn. In some moments (11 and 15) the student shows her ability to foresee the result of a series of rotations, thanks to the use of the ability of perception of spatial positions. In paragraphs 8 to 11 we can also observe how the student uses the ability of perception of spatial relationships by looking at the target cube, although with only partial mastery since, at first (9), she is able to determine that the heart is situated on a face next to the spade but she does not pay attention to their relative positions to determine the correct face of the heart.

To Conclude.

Five years ago T. Dreyfus' plenary address in PME-15 was a call for paying more attention to visualization and visual reasoning in the teaching of mathematics, since he *attempted to show that visual reasoning in mathematics is important in its own right and that therefore we need to develop and give full status to purely visual mathematics activities* (Dreyfus 1991, p. 46).

Since then, there have been several attempts in this direction, mainly designs of teaching units, but for the moment there is still a need for a theory (Dreyfus 1995, p. 16) emerging from mathematics education explaining how mental images of mathematical concepts are formed, how students can gain mastery in creating and using mental images, what role mental images play in the understanding of mathematical concepts and in problem solving, when visualization is more (or less) useful to students than analytical methods, how mental images can be transmitted, etc.

In this paper I have outlined a model characterizing the field of visualization in mathematics and defining its four main elements, mental images, external representations, processes, and abilities of visualization. This model is an attempt to integrate and complete several elements previously defined by Presmeg, Bishop, Clements, and others, that partially explained teachers' and students' activity when they use visualization as a component of their teaching, learning or reasoning in mathematics. On the other hand, the Van Hiele levels of reasoning give us the possibility to complete the model of visualization and to incorporate the assessment of activities of visualization into the context of assessment in mathematics (Gutiérrez 1992 b).

I have also shown the application of the model to analyze some cases of students solving tasks with the help of software allowing them to manipulate 3-dimensional objects. When completed this research project, we will probably give better answers to some of the many questions about the use of visualization and mental images in 3-dimensional geometry. Research and experimentation both in 3-dimensional geometry and other parts of mathematics is needed to complete the framework.

But there is still a long way to go.

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The Ongoing Value of Proof

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Introduction

Over the past thirty years or so proof has been relegated to a less prominent role in the secondary mathematics curriculum in North America. This has come about in part because many mathematics educators have been influenced by certain developments in mathematics and in mathematics education to believe that proof is no longer central to mathematical theory and practice and that its use in the classroom will not promote learning in any case. As a result many appear to have sought relief from the effort of teaching proof by avoiding it altogether.

In mathematics itself the use of computer-assisted proofs, the growing recognition accorded mathematical experimentation, and the invention of new types of proof that do not fit the standard mould have led some to argue that mathematicians will come to accept such forms of mathematical validation in place of deductive proof. The influence of these developments in mathematics has been strongly reinforced by the claims of some mathematics educators, inspired in part by the work of Lakatos, that deductive proof is not central to mathematical discovery, that mathematics is "fallible" in any case, and that proof is an authoritarian affront to modern social values and even hinders learning among certain cultural groups.

This state of affairs has caused great concern among other mathematics educators. Greeno (1994), for example, laid the blame squarely on misconceptions as to the nature of proof:

> Regarding educational practice, I am alarmed by what appears to be a trend toward making proofs disappear from precollege mathematics education, and I believe that this could be remedied by a more adequate theoretical account of the epistemological significance of proof in mathematics. (p. 270)

Schoenfeld (1994), in reply to the question "Do we need proof in mathematics education," also gives an unequivocal reply: "Absolutely. Need I say more? Absolutely."

This paper holds that none of the developments discussed really undermines the importance or value of proof, and that many of the assertions made in their wake are either simply wrong or based upon misunderstandings (primarily on the part of mathematics educators). It maintains that proof deserves a prominent place in the curriculum because it continues to be a central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding.

The Influence of Developments in Mathematics

A number of recent developments in mathematical practice, most of them reflecting in some way the growing use of computers, have caused some mathematicians and others to call into question the continuing importance of proof or indeed to announce its imminent death. John Horgan (1993), a staff writer of the magazine *Scientific American*, makes this prediction in article entitled "The death of proof" that appeared in its October 1993 issue.

Computer Proofs and a Potential Semi-Rigorous Culture

One of the developments that prompted Horgan's announcement is the use of computers to create or validate enormously long proofs, such as the recently published proofs of the four-colour theorem (Appel and Haken) or of the solution to the party problem (Radziszowski and McKay). These proofs require computations so long they could not possibly be performed or even verified by a human being. Because computers and computer programs are fallible, then, mathematicians will have to accept that assertions proved in this way can never be more than provisionally true.

This is a limitation in principle, but computing also has practical limitations, for all its ever-increasing power. There will always be tasks that take too long or are thought too expensive. Computer proofs are no exception, and so mathematicians have explored the implications that these limitations might have for mathematical practice. One prediction is that mathematicians, in the face of impractical times or prohibitive cost, will settle for "semi-rigour."

In an article published in 1993 in the *Notices of the American Mathematical Society* entitled "Theorems for a price: Tomorrow's semi-rigorous mathematical culture," the mathematician Doron Zeilberger predicts that with the advent of computer proofs a "new testament is going to be written." As "absolute proof becomes more and more expensive," he maintains, mathematicians will use proofs which are less complete, but cheaper. He points to the example of algorithmic proof theory for hypergeometric identities, where there is no lack of well-known algorithms. The problem is that some cases require computations which even on tomorrow's computers would take so long that they would exhaust the budget, if not the lifetime, of the researcher. He concludes that mathematicians will choose to limit the amount of computation allocated even to theorems which, in principle, are easily provable, opting for a less costly "almost certainty." Furthermore, he predicts that mathematicians as a whole will come to accept such "semi-rigour" as a legitimate form of mathematical validation.

A mathematical conjecture has always been considered no more than a conjecture until proven, so it is not surprising that Zeilberger's comments were quickly challenged by another mathematician. In an article published in the *Mathematical Intelligencer* (1994) with the dismissive title: "The death of proof? Semi-rigorous mathematics? You've got to be kidding," George Andrews maintains that Zeilberger's evidence is simply not convincing. That certain algorithms may prove too expensive to execute, he says, does not mean that mathematicians will now give up the idea of absolute proof with its "concomitantly great insight and, dare I say it, beauty" (p. 17).

And others have already pointed out that cheaper, non-rigorous proofs may prove costly in the long run. Saunders MacLane (1996) reported that in Italy during the years 1880-1920 several results in algebraic geometry were published without careful proving. The situation became so bad that "unverified rumour seems to have it that a real triumph for an Italian algebraic geometer consisted in proving a new theorem and simultaneously proposing a counter-example to the theorem" (p. 2). As a result Italian results in algebraic geometry were discredited until several mathematicians, including Emmy Noether, cleared up the difficult points by applying much more rigorous standards of proof.

New Types of Proof

Doubts about proof as a whole have also been raised by new types of proof that have little in common with its traditional forms. A particularly fascinating development is the recently introduced concept of zero-knowledge proof (Blum, 1986), originally defined by Goldwasser, Micali and Rackoff (1985). This is an interactive protocol involving two parties, a prover and a verifier. It enables the prover to provide to the verifier convincing evidence that a proof exists, without disclosing any information about the proof itself. As a result of such an interaction the verifier is convinced that the theorem in question is true and that the prover knows a proof, but the verifier has zero knowledge of the proof itself and thus is not in a position to convince others.

In principle a zero-knowledge proof may be carried out with or without a computer. In terms of our topic, however, the most significant feature of the zero-knowledge method is that it is entirely at odds with the traditional view of proof as a demonstration open to inspection. This clearly thwarts the exchange of opinion among mathematicians by which a proof has traditionally come to be accepted.

Another interesting innovation is that of holographic proof (Babai, 1994; Cipra, 1993). Like zero-knowledge proof, this concept was introduced by computer scientists in collaboration with mathematicians. It consists of transforming a proof into a so-called transparent form that is verified by spot checks, rather than by checking every line. The authors of this conccpt have shown that it is possible to rewrite a proof (in great detail, using a formal language) in such a way that if there is an error at any point in the original proof it will be spread more or less evenly throughout the rewritten proof (the transparent form). Thus to determine whether the proof is free of error one need only check randomly selected lines in the transparent form.

By using a computer to increase the number of spot checks, the probability that an erroneous proof will be accepted as correct can be made as small as desired (though of course not infinitely small). Thus a holographic proof can yield near-certainty, and in fact the degree of near-certainty can be precisely quantified. Nevertheless, a holographic proof, like a zero-knowledge proof, is entirely at odds with the traditional view of mathematical proof, because it does not meet the requirement that every single line of the proof be open to verification.

Experimental Mathematics

Zero-knowledge proofs, holographic proofs and the creation and verification of extremely long proofs such as that of the four-colour theorem are feasible only because of computers. Yet even these innovative types of proof are traditional, in the sense that they remain analytic proofs. More and more mathematicians appear to be doing all their work outside the bounds of deductive proof, however, confirming mathematical properties experimentally. A case in point is the Geometry Center at the University of Minnesota, where mathematicians use computer graphics to examine the properties of four-dimensional hypercubes and other figures, or to study transformations such as the twisting and smashing of spheres.

Even today one does not usually associate mathematics with empirical investigations, yet mathematicians have often carried out experiments to formulate and test conjectures (knowing full well that such testing did not constitute proof). Earlier mathematicians, limited to testing a small number of cases, would undoubtedly have done even more extensive experimentation if they had had the means. Thus today's experimental mathematics would not seem to differ *in principle* from what has been done all along. What does seem to be new is that more and more mathematicians spend their time almost exclusively on experimentation, and so naturally wish to assert a claim to its importance in its own right.

Horgan quotes several mathematicians who claim that experimental methods have acquired a new respectability. They have certainly received increased attention and funding following the growth of graphics-oriented fields such as chaos theory and non-linear dynamics. As a result, more mathematicians have come to appreciate the power of computers in communicating mathematical concepts.

Some are going well beyond communication, however. In a clear departure from previous practice, some now see it as legitimate to engage in experimental mathematics as a form of mathematical justification. Horgan maintains that:

> .... some mathematicians are challenging the notion that formal proofs should be the supreme standard of proof. Although no one advocates doing away with proofs altogether, some practitioners think the validity of certain propositions may be better established by comparing them with experiments run on computers or with real-world phenomena. (p. 94)

The implication of such a view is that experimentation has become not only a prestigious mathematical activity, but also an alternative to proof, an equally valid form of mathematical confirmation. This would seem to redefine "experimental mathematics" as a new discipline which is self-governing, so to speak, no longer subject to the criteria by which mathematical truth has traditionally been judged.

The founding of *Experimental Mathematics* in 1991 might be seen as a portent of such a new and independent discipline. This new quarterly does differ markedly from traditional journals, in that it publishes the results of computer explorations rather than theorems and proofs. But does this mean that its editors think proof is dead? This would not seem to be the case. In their paper "Experimentation and proof in mathematics" the editors of *Experimental Mathematics*, Epstein and Levy, first point out the enhanced potential of experimentation in the age of the computer: "the use of computers gives mathematicians another view of reality and another tool for investigating the correctness of a piece of mathematics through investigating examples" (1995, p. 674). They then go on, however, to make very clear how they believe experimentation fits into the mathematical scheme of things:

> Note that we do value proofs: experimentally inspired results that can be proved are more desirable than conjectural ones. ... The objective of *Experimental Mathematics* is to play a role in the discovery of formal proofs, not to displace them. (p. 671)
> We believe that, far from undermining rigor, the use of computers in mathematics research will enhance it in several ways. (p. 674)

A New Division of Labour within Mathematics?

Many mathematicians are nevertheless very concerned that the recognition of experimentation as a valid full-time mathematical activity may obscure the fact that its results cannot satisfy the criteria of proof. They do not agree on what, if anything, should be done about this. Some propose separation: that heuristic results be isolated as a clearly separate category.

Jaffe and Quinn (1993), for example, in their paper "Theoretical mathematics: Toward a cultural synthesis of mathematics and theoretical physics," stress how important it is to distinguish unequivocally between results based on rigorous proof and those based on heuristic arguments. They even suggest labels for the two kinds of activity, proposing the former be called "rigorous mathematics" and the latter "theoretical mathematics," by which they mean heuristic or speculative.

Jaffe and Quinn are motivated by a concern for standards of rigour, which they propose to preserve by isolating rigorous from non-rigorous mathematics (or "theoretical mathematics," as they dub it) through a new division of labour. They suggest that non-rigorous mathematics be considered a valid branch of mathematics in its own right, and that mathematicians be evaluated by the standards of the branch to which they choose to belong.

The suggestion that mathematicians be divided into two camps brought swift and varied reactions, sixteen of them in the *Bulletin of the American Mathematical Society* (1994). William Thurston, for example, responded in an eighteen-page essay entitled "On proof and progress in mathematics," in which he opposes the division suggested by Jaffe and Quinn. In his view the important question is not "how do mathematicians prove theorems?" or "how do mathematicians make progress in mathematics?" but how they "advance human understanding of mathematics," and accordingly he believes it wrong to split mathematics on the basis of standards of rigour. Though he does not question the role of proof in validation, he sees its main value in its ability to communicate ideas and generate understanding. Accordingly he proposes to mathematicians, who have traditionally gained recognition among their peers primarily by proving theorems, that they all undertake to recognize and value the entire range of activities that advance understanding in their common discipline.

Fifteen other prominent mathematicians gave shorter responses. Most rejected the proposal put forward by Jaffe and Quinn to recognize two separate branches of mathematical activity (Atiyah *et al.*, 1994). James Glimm wrote that if mathematics is to cope with the "serious expansion in the amount of speculation" it will need to adhere to the "absolute standard of logically correct reasoning [which] was developed and tested in the crucible of history" (p. 184).

Though driven, as Jaffe and Quinn were, by the growth of experimental mathematics and by a concern for rigour, it is apparent that Glimm has come to precisely the opposite conclusion. While Jaffe and Quinn seem to believe that identifying and welcoming heuristic mathematics as a separate (though perhaps lesser) discipline would prevent it from establishing itself as a method of mathematical confirmation equal in value to rigorous proof, Glimm appears to fear that such isolation would have the opposite effect of allowing heuristics to stake this parallel claim.

But the responses also revealed differing views on the role of rigorous proof. Saunders MacLane stated that "... mathematics does not need to copy the style of experimental physics. Mathematics rests on proof—and proof is eternal" (p. 193), while Atiyah conceded that "Perhaps we now have high standards of proof to aim at but, in the early stages of new developments, we must be prepared to act in more buccaneering style" (p. 178). And, not surprisingly, Mandelbrot asserted that rigour is "besides [sic] the point and usually distracting, even when possible."

It should be added that Mandelbrot also takes exception in his response to the customary practice of awarding credit only to those who prove conjectures, slighting those who came up with them in the first place. Indeed, one cannot ignore that the recent controversies over the place of experimentation and other heuristic approaches may be motivated as much by a concern for professional recognition as by disagreement over the nature of mathematical truth.

Certainly in these controversies the issue of the importance and prestige of heuristics has become intertwined, often confusingly, with the issue of the role of proof as the arbiter of mathematical truth. In the recent discussion triggered by Jaffe and Quinn, however, there is a perhaps surprising degree of agreement. All the participants would seem to agree with Albert Schwartz that heuristic mathematics is an important and legitimate part of their discipline (p. 198). But none suggested that mathematicians carry out their work without a view to the ultimate test of proof. Those who agreed, as most did, that mathematicians should accord more recognition to those who come up with interesting and productive heuristic results, were nevertheless of the opinion that such results remain conjectures until validated by proof.

The Influence of Lakatos

The thinking of Imre Lakatos, first published as a dissertation in 1961 and finally as *Proofs and refutations* in 1976, provoked much discussion among philosophers, and in particular among philosophers of mathematics (Agassi 1981; Feyerabend 1975; Lehman 1980; Hacking 1979; Steiner 1983). Whatever their assessment of his claims as a whole, they tended to accept Lakatos' principal insight that the critique of mathematical results by others has been the motive force in the growth of mathematical knowledge. Practicing mathematicians were impressed by his work as well, in particular by his detailed study of how the proof of Euler's theorem evolved over time. This study shed light upon many previously unappreciated aspects of mathematical activity, and for many mathematicians Lakatos' account of the dynamics of mathematical discovery rang true.

Lakatos' ideas were brought to the attention of mathematics educators primarily by Davis and Hersh (1981) in their book *The Mathematical Experience*. Their enthusiastic exposition of Lakatos' approach gained for it broad acceptance among mathematics educators, who assumed this approach to be more widely applicable in mathematics itself than in fact it is.

It is not surprising that such a fascinating new way of looking at mathematical discovery diverted attention from its weaknesses. The method of proof analysis is admittedly engaging, but the case for it as a general method rests upon a single sample, the study of polyhedra, an area in which it is relatively easy to suggest the counterexamples required. This method does not even begin to explain some important cases of mathematical discovery, however. It has nothing to say about set-theory research and the acceptance of the Zermelo-Fraenkel axioms, or about the emergence of non-standard analysis, or in fact about the many mathematical discoveries that did not start with a primitive conjecture.

It is not difficult, in fact, to cite cases in which a proof was found or a mathematical discovery made in a way radically different from the process of heuristic refutation described in *Proofs and refutations*. Even in the proof of Euler's theorem cited by Lakatos, for example, refutation is redundant; as soon as adequate definitions have been formulated the theorem can be proved for all possible cases without further discussion. Whenever mathematicians work with adequate definitions (or an adequate "conceptual setting," to use Bourbaki's term), in fact, the process of proof is not one of heuristic refutation. Indeed, in "A renaissance of empiricism in the recent philosophy of mathematics" (1978, p. 36), Lakatos himself says:

> Not all formal mathematical theories are in equal danger of heuristic refutations. For instance, *elementary group theory* is scarcely in any danger; in his case the original informal theory has been so radically replaced by the axiomatic that heuristic refutations seem to be inconceivable.

In *Proofs and Refutations* Lakatos defines proof as a "thought experiment ... a decomposition of the original conjecture into subconjectures or lemmas" (p. 9). In his interpretation of the history of Euler's theorem for a polyhedron, $V - E + F = 2$ (where V is the number of vertices, E the number of edges, and F the number of faces), for example, Lakatos describes a thought experiment in which one imagines stretching a rubber polyhedron and observing the results of its manipulation. He goes on, however, to describe a broader process which allows proofs and refutations to interact, generates counter-examples and "informal falsifiers," gives rise to happy guesses, and ends with a well-formulated result.

This approach can be viewed as an attempt to examine mathematics from Popper's point of view, to erect a critique of deductivism in mathematics parallel to Popper's critique of inductivism in the physical sciences. Taking "induction" to mean the verification of general laws on the basis of observational data, Popper hoped to show that "empirical science does not really rely upon a principle of induction" (Putnam, 1987). Similarly, Lakatos hoped to show that verification in mathematics does not rely on "Euclidean deductivism." In describing the heuristic process, Lakatos constantly attacks what he calls the "Euclidean programme," which in his opinion aims at making mathematics "certain and infallible."

But the truth is, first of all, that when mathematicians have undertaken the heuristic method which Lakatos describes, or one similar to it, it has almost always been for the purpose of arriving at certainty. In the case of Euclid's theorem, for example, the long heuristic process did lead, in fact, to a proof which satisfies the accepted criteria of mathematical certainty. As Ian Hacking (1979) put it: "Critical discussion can enable a conjecture to evolve into logical truth. In the beginning Euler's theorem was false; in the end it is true ... The theorem has been 'analytified'."

Secondly, the concept of fallibilism would seem to be a red herring. Mark Steiner has shown that in the eyes of present-day topologists Euler's theorem is "not about a polyhedron so much as about the underlying space the polyhedron divides" (p. 514). (He also shows that the modern proof is more explanatory than the one from the 19th century which Lakatos studied.) Steiner comes to the conclusion that the history of Euler's theorem in the 20th century not only provides a case in which Lakatos' model does not work, but, more importantly, demonstrates that we "can have progress without fallibilism" (p. 521). He also states that "despite the title of his book, Lakatos' mathematical realism can be profitably disengaged, not only from his fallibilism, but from the method of proofs and refutations itself!" (p. 510).

John Conway has remarked recently that Lakatos' Proofs and refutations "is a very interesting book, but I fear is definitely misleading as regards mathematics in general" (Sept. 1995, request for advice, www.forum.swarthmore.edu). And in words which seem to sum up the present discussion, Conway adds:

> It is misleading to take this example (Euler's) as typical of the development of mathematics. Most mathematical theorems do get proved, and stay proved; the original proof may not be quite satisfactory according to later standards of proof, but that is a fairly trivial matter. In many cases there has been a significant omission or error in the first attempt at a proof, which later had to be corrected; but there have been very few cases like Euler's theorem, in which the discussion continued for several centuries.

Let us now turn to the difficulties that may arise in applying Lakatos' ideas to the classroom. While Lakatos may have chosen, perhaps with good reason, to state some of his ideas over-dramatically, some mathematics educators have taken many of them literally and sought to translate them directly into classroom practice. He dismissed certainty and infallibility with the rather dramatic statement "we never know, we only guess," for example, and this has led some educators to present all mathematical knowledge as provisional. (One cannot but wonder if they would be prepared to fund a research project with the goal of finding the largest prime number or a counter-example to the Pythagorean theorem.) As well, the concepts of "informal falsifiers" and the "fallibility" of mathematics seem to have led many mathematics educators to believe that we should eliminate any reference to "formal" mathematics in the curriculum and in particular that we should downplay formal proof (Dossey, 1992; Ernest, 1991).

This attitude is surely misguided. In the first place, formal proof arose as a response to a persistent concern for justification, a concern reaching back to Aristotle and Euclid, through Frege and Leibniz. There has always been a need to justify new results (and often previous results as well), not always in the limited sense of establishing their truth, but rather in the broader sense of providing adequate grounds for their plausibility. Formal mathematical proof has been and remains one quite useful answer to this concern for justification.

Secondly, it is a mistake to think that the curriculum would be more reflective of mathematical practice if it were to limit itself to the use of informal counter-examples. The history of mathematics clearly shows that it is not the case, as Lakatos seems to have implied, that only heuristics and other "informal" mathematics are capable of providing counterexamples. Indeed, formal proofs themselves have often provided counterexamples to previously accepted theories or definitions. For instance, as Mark Steiner (1983) points out, Peano provided a counterexample to the definition of a curve as "the path of a continuously moving point" by showing *formally* that a moving point could fill a two-dimensional area.

Gödel's famous incompleteness proofs are another example, with an interesting and ironic twist. In this case *formal* proofs were employed to demonstrate that the axiomatic method itself has inherent limitations. Gödel could not have produced these proofs without using a comprehensive system of notation for the statements of pure arithmetic and a systematic codification of formal logic, both developed in the *Principia* for the purpose of arguing the Frege-Russell thesis that mathematics can be reduced to logic. His proofs could certainly not have been produced in informal mathematics or reduced to direct inspection.

Nor does it seem reasonable to assume that Gödel's conclusions could have been arrived at through a discovery of counterexamples ("monster-barring") followed by a denial ("monster-adjusting"), or by finding unexplained exceptions ("exception-barring") or unstated assumptions ("hidden-lemmas"). Curiously enough, however, when some educators make a case that formal proof and rigour should be downplayed in the curriculum they rest their case on Gödel's most formal proof.

The Influence of Social Values

In the minds of many mathematics educators the status of proof has also been called into question by the claim put forward, primarily by other educators, that it is a key element in an authoritarian view of mathematics (Confrey, 1994; Ernest, 1991; Nickson, 1994). This claim owes much to Lakatos (1976), who not only challenged the "Euclidean programme" for an "authoritative, infallible, irrefutable mathematics," as noted, but also wrote of the dangers of elitism in mathematics.

What supporters of this claim would add is that the "Euclidian" view is in conflict with the present values of society, which dictate that one not defer to authority and not regard any knowledge as infallible or irrefutable. They appear to see proof in general, and rigorous proof in particular, as a mechanism of control wielded by an authoritarian establishment to help impose upon students a body of knowledge that it regards as infallible and irrefutable.

Now, it may be true that mathematics has sometimes been presented as infallible and taught in an authoritarian way, but one could hardly maintain that there has been a recent consensus among educators that it should be. Whatever the case, one can only find it strange that proof should have become the main target of what in the end may be no more than a misguided desire to impose a sort of political correctness on mathematics education.

It is not easy to understand, in the first place, what it means to say that mathematics or a mathematical proof is "authoritative." Certainly a proof offered by a very reputable mathematician would initially be given the benefit of the doubt, and in that sense the fact that this mathematician is considered an "authority" by other mathematicians would play some role in the eventual acceptance of the proof. But the claim seems to be that the very use of proof is authoritarian, and this claim is hard to fathom.

In fact the opposite is true. A proof is a transparent argument, in which all the information used and all the rules of reasoning are clearly displayed and open to criticism. It is in the very nature of proof that the validity of the conclusion flows from the proof itself, not from any external authority. Proof conveys to students the message that they can reason for themselves, that they do *not* need to defer authority. Thus the use of proof in the classroom is actually *anti*-authoritarian.

Of course one could claim that the use of proof requires that the students accept certain "authoritative" rules of deduction, and so move the argument to a new, meta-mathematical plane. But one would hope that those who challenge the role of proof are not also challenging the very idea of rules of reasoning. It would be disturbing to see mathematics teachers ranging themselves on the side of a revolt against rationality itself.

In the second place, it is hard to understand how the use of proof strengthens the idea that mathematics is infallible. Looking at the issue first from the point of view of theory, it is clear that any mathematical truth arrived at through a proof or series of proofs is contingent truth, rather than absolute truth, in the sense that its validity hinges upon other assumed mathematical truths and rules of reasoning. Nor would infallibility seem to be an issue from the point of view of mathematical practice. Mathematicians are as prone to making errors as almost anyone else, in proof and elsewhere. The history of mathematics can supply many examples of erroneous results which had to be subsequently corrected. Thus it is difficult to see just how proof strengthens "infallibility," and the concept would seem to be irrelevant to the teaching of mathematics in general and the teaching of proof in particular.

The use of proof in the classrom has also been called into question on the grounds that it would encourage the idea that mathematics is an *a priori* science. The supporters of this claim see a conflict between this idea and their own view that mathematics is "socially constructed" (Ernest, 1991). Though their use of the term *a priori* is not entirely clear, it would seem that what they reject is not that mathematics is *a priori* in the sense of being analytic, non-empirical, but rather that it is *a priori* in the sense of given, pre-existing, waiting to be discovered. Of course this is a view of mathematics which they might well see as standing in opposition to "socially constructed."

On this point, however, Kitcher (1984) is surely right when he says that the pursuit of proof and rigour in mathematics does not carry with it a commitment to looking at mathematics as a body of *a priori* knowledge. Nor need it do so in mathematics education. As Kitcher put it: "To demand rigor in mathematics is to ask for a set of reasonings which stands in a particular relation to the set of reasonings which are currently accepted" (p. 213). Whether the set of reasonings currently accepted is regarded as given *a priori* or as socially constructed has no bearing on the value of proof in the classroom.

Those who challenge the use of proof in general would challenge even more strongly the use of rigorous proof in particular. Yet in mathematical practice the level of rigour is often a pragmatic choice. Kitcher states that it is quite rational to accept unrigorous reasoning when it proves its worth in solving problems, as it has in physics. Mathematicians worry about defects in rigour, he adds, only when they "come to appreciate that their current understanding ... is so inadequate that it prevents them from tackling the urgent research problems that they face" (p. 217). When is it rational to replace unrigorous with rigorous reasoning? Kitcher's answer is: "when the benefits it [rigorization] brings in terms of enhancing understanding outweigh the costs involved in sacrificing problem-solving ability." Mathematics educators, whose goal is surely to enhance understanding, would be well advised to adopt this guideline.

Rigour is a question of degree in any case. In the classroom one need provide not absolute rigour, but enough rigour to achieve understanding and to convince. An argument presented with sufficient rigour will enlighten and convincc more students, who in turn may convince their peers. It is the teacher who must judge when it is worthwhile insisting on more careful proving to promote the elusive but most important classroom goal of understanding.

Coda: Proof in the Classroom

With today's stress on teaching "meaningful" mathematics, teachers are being encouraged to focus on the explanation of mathematical concepts and students are being asked to justify their findings and assertions. This would seem to be the right climate to make the most of proof as an explanatory tool, as well as to exercise it in its role as the ultimate form of mathematical justification. But for this to succeed, students must be made familiar with the standards of mathematical argumentation; in other words, they must be taught proof.

Teaching students to both recognize and produce valid mathematical arguments is certainly a challenge. We know all too well that many students have difficulty following any sort of logical argument, much less a mathematical proof. We cannot avoid this challenge, however. We need to find ways, through research and classroom experience, to help students master the skills and gain the understanding they need. Our failure to do so will deny us a valuable teaching tool and deny our students access to a crucial element of mathematics.

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MODERN TIMES: THE SYMBOLIC SURFACES OF LANGUAGE, MATHEMATICS AND ART

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'Mathematics is the languageless activity of the human mind.'
(Brouwer)

'Whereof I cannot speak, thereof must I remain silent.'
(Wittgenstein)

In his recent book *River out of Eden*, biologist Richard Dawkins (1995) uses the metaphor of a river of DNA, a fluent stream of digitised information (base four, naturally) being carried through geological time rather than geographical space. The focus of his work generally is on the nature of self-replicating systems, and although 'naked replicators' (genes) are his chosen unit, he draws attention to a 'co-operative principle' selected for increased survival that leads to cells and larger biological organisms. He writes movingly of the beauty he finds inherent in complex biological organisms, of the linking of mechanisms and apparent purposes, of inheriting successful genes (our ancestors, in order to become ancestors, needed to have survived), of our own striving drives to become ancestors.

Yet there is an oxymoronic opposition in this watery image – the discrete nature of digital information temporally 'flowing' as a most continuous phenomenon, a river. Or is continuity merely a problem of discrimination, an epiphenomenon of scale? It does raise again the question of appropriate units of analysis (for biology, for language, for mathematics education – units where we can see continuities and ones where we can see discontinuities), which reminded me of Plato's observation about those discrete-minded arithmeticians who, whenever someone divided up the one, they respond by changing the unit in order to preserve an integral system. Certainly Dawkins himself offered a surprising attribution of intentionality at a particular biological level via the catachresis in the title of his celebrated earlier book *The Selfish Gene*. When pushed, he prefers to say this was merely a metaphor and 'the world becomes mechanically filled with those organisms that behave as if they had a purpose because they are good at surviving' (Dawkins, 1996).

A river can be a shared resource: offering, among other things, a source of *aqua vita*; a means of transportation and connection; a place for purifying (poet T.S. Eliot talked of a need 'to purify the dialect of the tribe'). We can think of a river of language ('River out of Babel?') connecting humanities together (as well as bitterly dividing them) at both numerous times and places, in a space-time continuity linking us to our ancestors, as well as to our contemporaries, and on into the future. A language too can be seen as a self-replicating system: what are its units? The utterance? But this comprises structured forms (e.g. ordered strings of words perhaps, spoken or written), functions (purposes and intentions) and meanings. Philosopher Paul Grice (1989) has written persuasively about a 'co-operative principle' underlying human communication in any natural language, which itself can be seen as akin to the result of speciation. The question of ancestors and antecedents is always of relevance in a meeting such as this, the annual ritual gathering of the PME tribe, for in part we constitute what has been termed 'a community of memory'.

Too often, I think, the forward-looking, the what-is-to-come, the avant-garde, is seen as the only place to be. I think there is also something to be said for the position of *après*-garde, dwelling with what has already been. Or is this merely a sign of increasing age and conservatism? In terms of being 'up-to-the-minute', I might have been expected to talk under the heading of 'post-modern times'. However, I prefer to revisit Chaplin's haunt to examine aspects of a *post-hoc* awareness raised into a doctrine: one which constituted Modernism.

Northrop Frye (1967, p. 59) wrote:

> I have chosen a slightly different approach from the question assigned – What knowledge is most worth having? – because, like everyone else, I want to quarrel with the assumptions in that question. In the first place, the knowledge of most worth, whatever that may be, is not something one has: it is something one is, and the correct response to such a question, if a student were to ask it, would be another question – with what body of knowledge do you wish to identify yourself?

Frye's use of the word 'body' is quite resonant for me, as what we gain from our ancestors are our genes which help significantly to shape our bodies, our physical forms. Unlike with our parents, we have more possibility of choice with our intellectual ancestors, with what material we deploy to help form our mental selves.

There is always the complex question of influences. Philip Jackson (1992) has lovingly written a book, *Untaught Lessons*, about the residues of teaching, the effects of teaching on all of us. It also concerns how absorbent we humans are. His book is a personal exploration of both where and how one might look to gain insight into the question of the (perhaps unseen or unacknowledged) influence of teachers. (In one chapter, 'B(e)aring the traces: reflections on a sense of being indebted to a former teacher', he explores the residues of an eighth-grade algebra teacher, Mrs Henzi.) He comments (p. 5): "In other domains of my life, I have often had the experience of belatedly coming to realise that someone or something has left its mark on me without my knowing it."

We are all in large measure a product of our times (which includes the flood of past times brought to us through the tributaries of language in books and other storage containers) – as well as being partial constituters of these times. The existing literature can be a crucial source for multiplying experience. These are changing times (truthfully, when were they not?) which include variation in prevalent attitudes and beliefs about 'the' nature of mathematics as an activity: mathematics seen as a complex interlocking system of propositional knowledge, concepts, problems, techniques, methods, working practices, communities, traditions and beliefs. There are a similar set of systemic elements concerned with mathematics teaching / learning (not to mention a widespread assumption that the two sets are causally connected).

There are also comparable sets of attitudes and beliefs about language, its teaching and learning/acquisition (the presence of a split descriptor such as this is always worthy of interest – why do we not talk much about acquiring mathematics?). The linguist Michael Halliday (1973) has characterised a language as comprising three interlocking systems: the forms, the functions and the meanings. Key words for me here are 'interlocking' and 'system'. I shall look at elements of all of these in subsequent sections. Language provides an example of a social practice, *par excellence* within a speech community. But it is also worth bearing in mind that there is the linguistic notion of *idiolect*, the individual speaker's unique speech style which nonetheless draws on and partakes of many of the communal, conventional, historical forms, functions and meanings.

Why might these two areas – mathematics and language – have anything of mutual interest if viewed in parallel, not least in the light of the Dutch philosopher-mathematician Brouwer's provocative assertion?

Let me start by explaining why I am not going to do (in print, at least) what I might have been expected to do here. I am not going to produce a general survey of the area of 'language and mathematics', identifying and highlighting future trends and past themes (though in my actual PME talk, I will do some of this). This is in large part because I have made my attempt at that particular task quite recently for the International Encyclopaedia of Education (second edition) entry on 'language and mathematics' (Pimm, 1994a), with an updated version to appear in the International Encyclopaedia of Developmental and Instructional Psychology this year – and pretty much to repeat that did not seem an appropriate use of this public text space.

I also looked forward in Pimm (1994b) to suggest some themes in this area that might come into greater focus in time to come. My other reason is that it might have proven too disparate and fragmented a piece: people seldom read encyclopaedia entries at a single sitting. The challenge I have set myself is to talk around one theme only, albeit in different settings. Your challenge, should you choose to accept it, is to try and figure out by the end what that one thing was.

There is so much I could talk about. For instance, here are just a few possibilities:

> Developments in the use of discourse analysis, exploring large-scale 'naturally occurring connected spoken or written discourse' above the sentence level (Stubbs, 1983a, p. 1) Is language in a mathematics classroom setting 'naturally occurring'? What are some of its characteristic features that make it cohere, and what distinctive stylistic features does it display?
>
> Stubbs claims (1980, p. 115): "A general principle in teaching any kind of communicative competence, spoken or written, is that the speaking, listening, writing or reading should have some genuine communicative purpose". Yet this is at odds with my view of the mathematics classroom being an avowedly, un-natural, artificial setting, one in which the structure and organisation of the discourse by the teacher has some quite unusual features.
>
> Even within the linguistic study of mathematical word-problems, an area which received considerable attention in the 1970s, recent work in genre analysis as well as sophisticated attention to time coding among the problem elements has contributed to new characterising distinctions and insights (see e.g. Gerofsky, 1996).
>
> Issues regarding text and mathematics education are explored, for instance, in Morgan (1995) and Love and Pimm (1996). Morgan draws on Halliday's meta-functional categories among other linguistic resources to analyse student investigative coursework write-ups and teacher valuations of them. High-level functional differences exist between speech and writing: How does pupil meta-knowledge about them in relation to mathematical language develop? What can teachers usefully attend to in their students and their own discourse?
>
> Classrooms have both spoken and written forms present: How are the transitions between them signalled and handled, are the differences in functionality available through them apparent, and are all differences a matter of degree or are there certain aspects that only the one can manage? What roles do written symbols play in expressing mathematical ideas and in transforming expressions?
>
> Study of sophisticated pragmatic judgements which pupils make encoding their beliefs about and their degree of commitment to propositions or generalisations they express. Working with teachers on their awareness of the linguistic information which is available. Where is their attention and how is this reflected in their spoken contributions? How attuned are teachers and pupils to the linguistic potentialities of modality and hedging. See Rowland (1995) on both themes in a context of vague mathematics talk.

But before launching into more specific discussion about language and mathematics, there are some remarks I want to make about mathematics education in general. I am interested in the large-scale organisation and development of newly-emergent disciplines such as mathematics education. We would do well to remember that mathematics education only really started in earnest in the late 1950s along with what I shall identify as the Modernist thrust of 'Modern' Mathematics in schools. Even a brief look at *Educational Studies in Mathematics* confirms how much of the concerns of those times were caught up with mathematical content. So I might also have entitled this talk '(Almost) Forty Years On'.

David Wheeler (1989, pp. 282-3) has written:

> Dewey said somewhere that subject matter is a prime source of pedagogical insights. Almost no educators really believe this, I think, except in the trivial sense of hoping that teachers, textbook writers, and curriculum designers "know their mathematics". Even many mathematicians, who ought to know better, have no interest in looking below the instrumental or formal surface of mathematics in order to get clues about how to present it more effectively.

I was also recently struck by Thurston's (1994, pp. 162-3) somewhat unexpected claim of mathematicians as being centrally involved 'in finding ways for *people* to understand and think about mathematics', an aim I believe should be fundamental to the emerging discipline of mathematics education. The most important related discipline for my work is that of mathematics not psychology (and the second is linguistics to which I shall turn shortly).

Mathematics education is an applied discipline, not dissimilar to medicine or law. It draws upon a number of more established cognate disciplines, including psychology, mathematics, sociology, linguistics and philosophy, each of which has its own phenomena of interest and parallel structures, modes of enquiry, conceptual tools and methodological norms, structures of knowledge and means of validation. These elements constitute some of the requirements for an area of study to form a discipline. To the extent that someone's work is situated closer to certain discipline boundaries than others, it is more susceptible to their intellectual rationales. The historical 'P' in the title of this group alludes to one particular cognate discipline as being seen as dominant.

I am interested in the emergence of a specific discipline of *mathematics* education, seen as an area of study with its own methods, concepts and practices in its own right. The word that is often used in other areas (e.g. linguistics, art criticism, as you will shortly see) is 'autonomy'. Mathematics education has had a transitional period *en route* to becoming disciplinised, though not necessarily resulting in the same discipline (see e.g. Balacheff *et al.*, 1992). One earlier focus of English mathematics education work was on pupil errors and misconceptions, trying to characterise what could and could not be done, creating baselines and generating expectations: the unit of study was the individual child in isolation. While there was some attention to spoken or written interactions between researcher and subject, it was not until classroom learning (and the teacher's central role in this) moved to the fore and became the unit of study, that certain fundamentally linguistic phenomena were attended to.

I am particularly concerned with the practices (both actual and potential) of mathematics teachers in classroom, specifically language practices, as I take seriously Stubbs' (1983b, my emphasis) observation that: "There is a sense in which, in our culture, teaching *is* talking". One of my starting presuppositions is that there are important particularities about mathematics which strongly mark attempts to engage with it in classroom settings. Another is in the value of close-focused, specific, detailed investigative work, carried out in the belief that the particular is much richer than any general theory.

Linguistics, language teaching and mathematics education[1]

In this section, I wish to use a broad brush to delineate some of the links between linguistic theories and second language teaching methods, while all the time listening for both echoes and silences from mathematics education. I shall not insist on the metaphor 'mathematics is a language', though I still believe it has useful work it can do for us in helping to see mathematics and mathematical activity more clearly. Or perhaps, better said, I shall undertake the converse activity required in order to see 'mathematics is a language' as *only* a metaphor, namely by looking at and stressing differences as well as consonances. Accounts often distinguish between natural, first-language (speech) acquisition and taught, second-language learning, so might an account of mathematics vary depending on whether it were seen as more akin to a first or second language.

Once again, a similar set of differences serve to structure an academic area: a theorising approach, formalised frequently by mathematical ideas and symbolisms, including a moving away from actual transient or generated phenomena to 'idealised' data (e.g. the unit of analysis being a perfectly-formed spoken 'sentence' – a written artefact – in which very few humans, one being the strikingly erudite speaker Chomsky himself, ever expressed themselves). Yet many modern-day writers about language teaching seem to want to deny that children do attend to the form. Chomskian linguistics privileges the regularities of structural form.

Ironically nevertheless, despite claims to be working on speech, this work on theoretical language *competence* continued a bias towards the written that traditional language study (under the heading of rhetoric) had promoted, either privileging the written over the spoken (seeing the latter as an lesser, flawed copy) or ignoring the differences between them. Often in contrastive reaction to this approach, there are others whose focus remains with actual rather than idealised phenomena themselves: eschewing any other than carefully gathered 'authentic' speech data.

The contentious issue within linguistics about 'naturally-occurring' language data is deep and old: many linguists relied on their own 'native-speaker competence' to judge the acceptability of hypothetical examples used as data for theorising. The philosopher of language John Searle, writing in the early 1970s about 'Chomsky's revolution in linguistics', observed:

> Throughout the history of the study of man there has been a fundamental opposition between those who believe that progress is to be made by a rigorous observation of man's actual behavior and those who believe that such observations are interesting only in so far as they reveal to us hidden and possibly fairly mysterious underlying laws that only partially and in distorted form reveal themselves to us in behavior. Freud, for example, is in the latter class, most of American social science in the former.
>
> Noam Chomsky is unashamedly with the searchers after hidden laws. Actual speech behavior, speech *performance*, for him is only the top of a large iceberg of linguistic *competence* distorted in its shape by many factors irrelevant to linguistics. Indeed he once remarked that the very expression "behavioral sciences" suggests a fundamental confusion between evidence and subject matter. Psychology, for example, he claims is the science of mind; to call psychology a behavioral science is like calling physics a science of meter readings.. One uses human behavior as evidence for the laws of the operation of the mind, but to suppose that the laws must be laws of behavior is to suppose that the evidence must be the subject matter. (1974, p. 2)

Mathematical knowledge can be seen as the epitome of conscious, articulated, communicable knowledge, while even the presumed nature of linguistic knowledge is less clear. The marking of the distinction between learning and acquisition, due to Steven Krashen, is between consciously learning about part of the target language and being able to use it *without* thinking about it. There is a comparable distinction in working mathematically, whether to do with mastering algebraic symbolism or set theory (mathematician Paul Halmos' injunction in his book *Naive Set Theory* was to study it then to forget it'). Experience with the grammar–translation method (described below) suggests that language learning does not necessarily lead to acquisition. Experience with attempts to teach algebra through explicit attention to the transformation rules mirrors this.

Sometimes pedagogical practices seem to cluster around beliefs about the particular nature of an area of knowledge (and some arose specifically from theoretical writing). It is, however, still an open question for me as to the nature of the interaction between pedagogic practices on the one hand and beliefs about and more explicit theorisations of the subject matter discipline itself on the other. Can shifts in mathematical pedagogy likewise be seen in relation to shifts in perceptions of the nature of mathematical knowledge? Or is this too simplistic a view? It is perhaps worth pointing out that only some linguists are interested in the problems of second language acquisition, and it is sometimes seen as a poor relation to theoretical descriptive linguistics.

Foreign language teaching in Britain over the past fifty years has moved away from a stable and explicit teaching method almost exclusively concerned with grammar and translation. The presumption was if pupils knew the grammar and vocabulary, then they would know the language (both spoken and written) in consequence of a predominant focus on the written forms, gaining access to the target language through their understanding of their own language.

The next development in teaching practice came in the 1960s with audiotape-slide presentations and a massive exposure to *spoken* utterances in the target language (albeit no naturalistically recorded ones). It was connected to a behaviourist account of first language learning, namely children learn to speak simply through response to pure language input. One switch was from almost complete attention to written language to almost complete attention to spoken language. However, little attention was paid to understanding, and little explanation was provided; exposure was considered enough.

Interestingly, despite the increasing domination of English-language linguistics by a heavily mathematised formal programme based on the work of Noam Chomsky from the late 1950s onwards, there is no obvious teaching method that can be linked to his work. He endeavoured to offer a systematic description and explanation of the structures of human language. One pedagogic consequence that is there, though, is a strong injunction against teaching grammatical rules explicitly, the language student is presumed able to work these out for herself.

This belief stemmed from Chomsky's account of 'universal grammar', components of an innate 'hard-wired' language acquisition device, consisted originally of the intersection of the rules of each language, then (partly in response to the breadth of counter-examples available from a range of languages) it became a set of rules available to be drawn on by any particular language. These were expressed in an increasingly abstract manner that it became unclear precisely what they were and the programme eventually fell foul of Chomsky's own clearly-articulated criteria of theoretical adequacy in linguistics cogently expressed in his 1965 book *Aspects of a Theory of Syntax.*

The issue of explicit teaching of rules or not and whether they should be taught by:

- moving from the (given) general to the particular;
- generalising from particular instances to general rules which then get 'tested' by producing other particulars, which are then tried out in some way;

- no use of meta-language formulations, and only key patterns are exemplified by means of paradigm exemplar utterances;

remains under discussion.

The next major shift came in about 1970 with the development of 'communicative methods', a direct and explicit link of Speech Act theory (see Searle 1969), one of the earliest representatives of the developing linguistic area of pragmatics, concerned with language in use. Its features included a continued emphasis on the spoken at the expense of the written, a view of foreign language competence as being able to produce sentences appropriate to particular situations, and a focus on a widely understood but poorly executed belief in various functions of language in different circumstances. It also seemed to involve an analysis of what the learner need to be able to say, as well as presupposing studies of real, actually-occurring linguistic data.

The emphasis on the particularity of language context and function at the expense of the generality of grammatical rules has remained, although the teaching of rules, the learning of structures and the insertion of vocabulary into them never really ceased. Another key move was from simple spoken immersion as if one were learning a second language as a languageless infant to an emphasis on communicative competence (where the functions and intentions of speakers were emphasised as well as contextual understanding of the speech situation). One of the hangovers from behaviourism linked to pure language input is a belief in teaching in the target language only, and not through or in the pupils' first language, and another is strongly discouraging the learner from translating. It may come as little surprise to hear that there has been little feedback from language teaching into theoretical linguistics.

There are some outstanding issues in need of work. I am struck by a number of parallels, which I will draw out in my talk. I have framed them here as a series of catechismic questions which have both a second-language teaching and a mathematics teaching interpretation.

Q: Is a spoken or written emphasis to be preferred?:

Q: The area can be characterised by certain rules in certain circumstances: should these be offered to pupils, and if so how?

Q: What place and function is there for particular contexts in teaching?

Q: From where does meaning derive?

Q: What do they not need to be taught as they are increasingly masters of their native tongue?

A 'Modern' perspective

The rough history of British and particularly North American linguistics since the second world war can serve as a guide for changing interests and methods. There was a long tradition of linguistics as a sort of classificatory natural history of languages ('verbal botany' according to John Searle (1974, p. 3)), followed by intensive use of field methods. Much earlier, in a more traditionally scientific context, Goethe, writing in the preface to his theory of colours, commented:

> It is sometimes unreasonably required by persons who do not even themselves attend to such a condition, that experimental information should be submitted without any theory to the reader or scholar, who is himself to form his conclusions as he may list. Surely the mere inspection of a subject can profit us but little. Every act of seeing leads to consideration, consideration to reflection, reflection to combination, and thus it may be said that in every attentive look on nature we already theorise. But in order to guard against the possible abuse of this abstract view, in order that the practical deductions we look to should be really useful, we should theorise without forgetting that we are so doing, we should theorise with mental self-possession, and, to use a bold word, with irony. (1810; 1967, pp. xx-xxi)

The linguist John Lyons, in his account of Chomsky's work, talks of one of modern linguistics' distinguishing features as: 'its autonomy, or independence of other disciplines. [...] When the linguist claims 'autonomy' for his subject he is asking to be allowed to take a fresh and objective look at language without prior commitment to traditional ideas and without necessarily adopting the same point of view as philosophers, psychologists, literary critics or representatives of other disciplines.' (1970, p. 17).

I want to add a second area to this discussion, namely that of visual art. The metaphor of 'art as a language' is at least as prevalent as the corresponding formulation for mathematics. It too has seen the incursion of linguists and semioticians into the preserves of art historians and art critics, as well as attempts by the latter groups to import methods and notions from these other disciplines. By comparing and contrasting the uses of linguistic ideas and procedures in the two disciplines, I hope to approach a question that interests me, namely what is particular about mathematics in the discipline of mathematics education.

A similar question about art, and about painting in particular exercised the American Modernist art critic Clement Greenberg. In a talk first broadcast on the American propaganda radio station *Voice of America* (itself an interesting choice of forum) at the height of the cold war in 1961, he asked what is the essence of each artistic discipline.

> Modernism includes more than art and literature. [...] It happens, however, to be very much of a historical novelty. Western civilization is not the first civilization to turn around and question Its own foundations, but it is the one that has gone furthest in doing so. I identify Modernism with the intensification, almost the exacerbation, of this self-critical tendency that began with the philosopher, Kant. [...]
>
> The essence of Modernism lies, as I see it, in the use of the characteristic methods of a discipline itself, not in order to subvert it but in order to entrench it more firmly in its area of competence. [...] Each art had to determine through its own operations and works, the effects exclusive to itself. By doing so it would, to be sure, narrow its area of competence, but at the same time it would make its possession of that area all the more certain. [...]
>
> Realistic, naturalistic art had dissembled the medium, using art to conceal art; Modernism used art to call attention to art. [...] Manet's [pictures] became the first Modernist pictures by virtue of the frankness with which they declared the flat surfaces on which they were painted. [...]
>
> The Old Masters had sensed that it was necessary to preserve what is called the integrity of the picture plane: that is, to signify the enduring presence of flatness underneath and above the most vivid illusion of three-dimensional space. The apparent contradiction involved was essential to the success of their art, as it is indeed to the success of all pictorial art. The Modernists have neither avoided nor resolved this contradiction; rather, they have reversed its terms. One is made aware of what the flatness contains. Whereas one tends to see what is in an Old Master before one sees the picture itself, one sees a Modernist picture as a picture first. (pp. 309-310)

This is a lot to take in, even after more than one reading. But first I also note (from a post-modern perspective) a number of features of this piece as a whole. The first is its isolationist essentialism, offering a single criterion (flatness) for guaranteeing the 'autonomy' of pictorial art. The second is the language of the whole piece is steeped in the cold war rhetoric of territoriality ('entrench', 'possession of that area'), of "purity and danger" (Mary Douglas' ever-useful categorisation of anthropological fears underlying and underlining some intentions behind many systems of categorisation). Greenberg talks about a wish for 'autonomy' and 'purity' and the values of separateness. Post-modernism identifies this polarity as a characteristic marking of difference through distinction as well as the two poles not being equally valued (one being marginalised and sometimes vilified as 'the Other').

However, whatever his reasons for doing so, what he is drawing our attention to is significant, I believe. What I also take from this is not 'self-criticism' as much as greater self-awareness of and directing of attention to the form, the surface, the means of representation as well as what is represented. Greenberg would argue that the point came with abstract expressionism (the work of certain American artists such as Jackson Pollock in the 1940s and 1950s whose work he wanted to see as the pinnacle of high art – for his own personal and political ends, among others) where the medium was the message. This shifting of attention is always possible because of the human symbolic practice of offering and working with substitutes. A substitute so functioning always lives a double life: it is a thing and it is standing in for something else. It is a particularly-coloured and shaped piece of wood, and it is a Cuisenaire rod. It is a particularly-shaped mark on a piece of paper and it is a 'four'. My experience with mathematics and with both spoken and written language tells me it is hard to attend to both at the same time. Yet, for me, one of the essential features that allows the possibility of mathematics is precisely this shifting.

(a) Modernist themes functioning in mathematics

One 'flatness' of modern mathematics in the 1950s and 1960s was its apparent eschewing of any concern with the subject of connection with the three-dimensional material world, preferring itself to engage with its own autonomy and existence. 'Art for art's sake' became first a rallying cry and now is a much-derided politicised aberrant abdication to be rooted out and denied within art criticism. 'Mathematics for mathematics sake' is not an expression I have ever come across, partly because of the still-respected twentieth-century notion and mythologising of the pure mathematician.

But another which I wish to think about publicly here is that of symbol as object and symbol as signifier. My parallel with Greenbergian flatness is about seeing the representation as a representation rather than the thing represented, something a number of paintings by Belgian artist René Magritte also alluded to. In Pimm (1995), I discussed at length the work of Robert Schmidt (1986), a historian of mathematics, who makes a useful distinction between the mathematical language *functions* of symbols serving as *signs* and serving as *counterparts*. A sign names or points to something else, but bears no necessary relation to the thing named. A counterpart stands for something else, but does not name or point to it (an indicative function): however, there is an actual relation, a resemblance or connection, between the object and its counterpart. These two functions can coalesce on the same symbol, but there may be confusion when this occurs.

Schmidt uses the example of lines drawn on a nautical chart to illustrate the notion of a counterpart: a nautical chart in no way names what it stands for, but it allows computations and actions to be made upon it which can be directly transferred to actions on the actual object represented. Technical drawings, as opposed to geometric diagrams, are counterparts, though when teachers invite pupils to use rulers and protractors on geometric drawings they are shifting the drawing's status to that of a counterpart. Counterparts are to be acted on, and then the results interpreted via the connection. Counterpart forms can also provide substitute images. Schmidt claims:

> It is also the nature of counterparts to draw attention to themselves, while it is in the nature of signs to lead our attention away from themselves and towards the thing signified.[2] [...] Furthermore, it is in the nature of counterparts to turn their object into themselves, while it is the nature of signs to disclose their objects. (p. 1)

I see the signification and counterpart functions of symbols as complementary; neither one suffices for mathematics, yet they seem to conflict with one another, pulling in opposite directions. In mathematics, two important contrastive foci occur, interact and recur. One is the nature of the objects about which generalisations are made and the other is the nature of the language used for their generation and 'manipulation' (as with geometric images, algebraic forms need to be both conjured and controlled). More than with arithmetic or geometry, and despite its abstract air, 'doing' seems to be central to algebra. In the process, attention is moved away from what, if anything, is being 'manipulated'. With algebra, 'manipulation' comes into its own, with symbols as counterparts very much to the fore; the 'true' nature of the algebraic object becomes ever more confused.

Fluent symbol users report two awarenesses when working with algebraic expressions: being able to see them as structured strings of symbols (and hence symbolic objects in their own right) *and* seeing them as descriptions connected with some 'reality' or situation they are concerned with. Maintaining this dual perspective, of substituting counterpart and indicating sign, is of central concern when working on mathematical symbols at whatever level, and places a heavy burden on novices. As Schmidt (1986) points out, algebra offers both a calculus and a language.

(b) Modernist themes in mathematics classrooms

In Pimm (1994c), I wrote about the phenomenon of meta-commenting which teachers sometimes engage in, and talked of it as an instance of the language asymmetry between teacher and pupils. The choice is always there, at any moment, because of the permanent double function available to any symbol. As I mentioned earlier, this is an issue of particular relevance to mathematics, because becoming a fluent functioner with *mathematical* symbols requires a double fluency. I can attend to the form, and function metonymically, moving along the chains of signifiers, noting similarities, creating associations. I can attend to the referential 'meaning', functioning metaphorically, returning to the domain of discourse and interpreting each 'utterance'.

In the above, I alluded to how belief about the nature and forms of knowledge can interact with pedagogic practices. Adler (1996) has been working in multilingual mathematics classrooms in South Africa, and has identified a fundamental pedagogic tension between implicit and explicit practices with respect to language issues in such classrooms. These issues are present in all classrooms but are present in particularly heightened form there. Some of her teachers attended to pupil language expressions as a *shared public resource* for class teaching, as well as providing a verbal display tacitly taken as indicative of an individual's grasp and understanding of the content. The latter attention acknowledges the fact that all I (as teacher) have access to is the forms: if they can say it 'right', do they know it? If they don't say it right', can I let it go? There is a world of difference between 'what they are saying is wrong' and 'I can't get at what they are trying to say to me'. What significance are teachers attaching to the use of a 'wrong' word? How does what people say inform what they do? By 'shared public resource', Adler is pointing to the fact that a characteristic of classrooms that is not shared by many other speech settings, is that the language itself can become the explicit focus of attention. So it is no longer the medium of expression as the 'thing' to which pupils are invited to attend.

In the talk, I will give an analysis of an example from a secondary algebra classroom. Underlying these particular concerns is a more general issue about direct teaching in general. Hewitt (1996) has drawn attention to a powerful practice of teaching through task subordination, so what is being desired to be learnt is not the direct focus of attention.

The external world we all live in is a world of forms. Teachers, in order to teach, need to develop or acquire linguistic strategies. This is in order, among other things, to direct pupil attention to salient aspects of the discourse – or indeed to the nature rather than the content of that discourse itself – while still remaining in 'normal' communication with the pupil. The teacher must be able to stand outside the discourse as a commenter on it in order to teach, yet still be seen as a participant within it. One artifice of the teacher is in rendering these excursions sufficiently unexceptionable that the conversations that occur, eddies in the river of language, can flow smoothly and effectively.

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[1] I am very grateful to the linguist Joanna Channell for discussions on this topic.

[2] John Mason (1980), in his article 'When is a symbol symbolic?', talks of 'seeing through' a symbol.

PLENARY PANEL

Panel discussion

Language in Mathematics Education

Panellists:

Colette Laborde
Luis Puig
Terezinha Nunes

About the Panel

The aim is to promote a discussion of language in mathematics education from different perspectives. The panellists have chosen a research protocol, which is transcribed next, and have each prepared a statement to pose issues concerning language and mathematics education.

At the beginning of the plenary session, each panellist will expand on their initial position. These initial statements will be followed by a discussion drawing on the protocol as a common ground where issues raised by the panellists can be discussed and other ideas about language in mathematics education can be raised.

All PME participants are asked to read the protocol and the contributions to this session before the session so that they share a frame of reference from which the discussion might start. Ideas presented by the panellists, research about the topic of the panel, and gaps in the positions put forth can be starting points.

PUTTING MATHS INTO LANGUAGE AND LANGUAGE INTO MATHS

The task

The pupils were asked to solve the following problem: "Two year five classes had 75 books to share between the two classes. However, one class had more pupils than the other and the head teacher decided that it would get 13 books more than the other. How many books will each class receive?" After the pupils had solved the problem, they received the following instructions:

"Your friends will now solve the same problem you just solved and you want to help them. You will write a message with instruction for them on how to solve the problem. But there is something about the message you are going to write for them. Their problem will have different numbers from those in your problem. That means that your message cannot use your numbers in the explanation. You have to explain to them how to solve the problem without using any numbers in the message."

Solution

The pupils wrote all the information on one page and then decided to divide 75 by 2, obtaining 37 and a half; they checked the calculation over and again had 37 and a half as the result. (The names used here are fictitious).

1. Jane: *Thirty seven and a half?* (Both laugh). *Thirty seven and one book left.*
2. Ann: *Take away 13.*
3. Jane: *No, that won't work. That's 14 books left.*
4. Ann: *Because we did the take away?*
5. Jane: *Yes.*
6. Ann: *35 and 35 is 70.*
7. (They re-check the division sum).
8. Jane: *Thirty seven and a half, right? When you divide 75 by 2* (looking at the interviewer).
9. Interviewer (Int.): *Did the classes have the same number of books?*
10. Jane: *No, we have to take away some and give to the other class.*
11. Ann: *From where?*
12. Jane: *I know, we'll take the 13 books away first* (crumples up page). *Seventy five take away thirteen.*
13. Ann: (does)

$$\begin{array}{r} 75 \\ -13 \\ \hline 62 \end{array}$$

14. Jane: *Then one class has half that. Thirty one.*

15. Ann: (writes 31 under the 62 without indicating the computation and says) *31.*

16. Jane: *One class gets 31. And the other gets 13 more.*

17. Ann: (does) *31 + 13 = 44*

$$\begin{array}{r} 31 \\ +\ 13 \\ \hline 44 \end{array}$$

18. Jane: *44 and 31 is the answer.*

19. Int.: *Are you sure?*

20. Jane (to Ann): *Do they add up to 75?*

21. Ann: (does)

$$\begin{array}{r} 44 \\ +\ 31 \\ \hline 75 \end{array}$$

22. Both: *Yes* (emphatically and laughing).

Putting the solution into language

The pupils approached the task by looking at the computations and writing a description of what they had done as the instructions. They continuously checked where they were with respect to the computations. (Underlined portions were spoken and written; pupils took turns at writing). The page with computations looked as below.

$$\begin{array}{r} 75 \\ -\ 13 \\ \hline 62 \end{array}$$

31

$$\begin{array}{r} 31 \\ +\ 13 \\ \hline 44 \end{array} \qquad \begin{array}{r} 44 \\ +\ 31 \\ \hline 75 \end{array}$$

23. Jane: *Get the number of books and ... take away ...*

24. Ann: *Take away what?*

25. Jane: *Take away.*

26. Ann: *Take away what?*

27. Jane: *Take away the amount.*

28. Ann: *Take away the number that one class had more.*

29. Jane: *Yes.* (Reads what was written) *Yes. Take away the amount that is bigger.*

30. Ann (seems to interrupt): *Take away* (crosses out number) *the amount of how many more books one class had.*

31. Jane: *Add.*

32. Ann (shows computation on paper): *Split into half.*

33. Jane: *Oh, yes, split the number, the amount into half and add the number of how many more.*

34. Ann: *Of how many more ... you want to put here?*

35. Jane: *Of how many more books - or how many books more? We've got that far* (pointing at computations used in solving the problem; adds books to the text) *one class had.*

36. Ann: *We're here* (pointing at previous solution).

37. Jane: *Then see if you are right, add up the two answers to see if it comes to the amount in the beginning.*

38. Ann: *The amount you started with.*

39. Jane: *Of books you started with.*

When the message was finished, the interviewer gives the new set of instructions:

40. Int.: *That is brilliant, that is very good indeed. Now, you wrote your message in English and I want you to try to write it in mathematics. Using signs like in mathematics.*

41. (The pupils ask how and the interviewer continues).

42. Int.: *Now, because you don't know the numbers they will have in their problem, you can use a letter in its place, like a code. You can start like this, instead of saying get the number of books* (reading from the pupils' message) *you tell them that the total number of books will be called A in your message.*

43. Pupils: *What? The number of books will be called A?*

44. Int.: *Yes, you see, instead of saying the number of books, you say A, like a code, and you explain to them what the code means. You can call the total number of books A.*

45. Jane: *Oh, I get it. Call the number of books they have A.* (To interviewer) *Like this? And what?*

46. Int.: *You can tell them to call the difference - do you know what difference is?*

47. Both: *Yes.*

48. Jane: *Call the difference between the classes?*

49. Int.: *Yes* (encouragingly).

50. Jane: *What? B?*

51. Int.: *Yes, good. Then you can tell them what sums to do. Have you done this in your classroom?*

52. Ann: *No.*

53. Int.: *Do you do formulas in science?*

54. Ann: *No.*

55. Int.: *Or in maths?*

56. Ann: *No.*

57. Jane: *I know, call the difference between the two classes B.*

58. Int.: *Now you can tell them what mathematics to do.*

59. (Jane wants to change the old message but is asked to do a new one).

60. Ann: *Get A and take away B.*

61. Jane: *Split B.*

62. Ann: *Split A.*

63. Jane: *A?*

64. Ann: *A? Take away B and split.*

65. Jane: *Oh yes, get A and take away B* (reading from message), *and call A take away B = C.*

66. Ann: *Split in half.*

67. Jane: *Yes. Split C in half. And then get the number of how many more books* (reading from old message).

68. Ann (pointing to computations for solution): *What's that?*

69. Jane: *That's A.*

70. Ann: *That's B. I'm up to there.*

71. Jane (following with Ann and also looking at earlier message): *Split C in half* (from present message) *and then add the amount of how many more books one class had* (looks at old message) *and add B.*

72. Ann: *To what?*

73. Jane: *To C.*

74. Ann: *No, C is that one* (points at earlier solution).

75. Jane: *To D.*

76. Ann: *What's D?*

77. Jane: *Get the second answer. Call the second answer. Half of C.*

78. Ann: *What is the second answer? Half of C?*

79. (The pupils check everything up to then, reading the message and comparing it to the computation).

80. Jane: *C : 2 = D.*

81. Ann: *Call the answer.*

82. Jane: *Call the answer to B and D = E.*

83. Ann: *Call the answer to B and D = E*.

84. Jane: *Then add E and D and see if you can get A*.

85. Int.: *How will they know which ones are their answers?*

86. Jane: *What do you mean?*

87. Int.: *How many books each class gets.*

88. Jane: *These* (points to paper).

89. Int.: *Which?*

90. Jane: *D.*

91. Ann: *E.*

92. Int.: *Do you want to tell them that then?*

93. Ann: *One class has D and the other has E*.

The interviewer finally asks them which message they think will be the best for their friends to use and they agree immediately that the second one is the best. The message was written on three different pieces of paper and the pupils arranged them on the table for their friends to use later. Total time: approximately 21 minutes.

Solving the problem with the message

The task for the second pair of pupils was to use the message to solve the problem. The interviewer read the problem to them and they wrote down 75 and 13 underneath. The first pair had put the three pages on the table and the message read from left to right like this:

Left page:	Call the answer to B and D = E.

Centre page:	Call the total number of books A.
	Call the difference between the two classes B.
	Call A take away B equals C.
	C : 2 = D

Right page:	Get A and take away B.
	Split C in half
	and add B to D.
	Add E and D to get A.
	One class has B and one has E.

The pupils read the message in this order, skipping "C : 2 = D". They re-read the centre and the page on the right (not shown on video to save time).

94. Lynne: (Inaudible).
95. Carey: *The difference between the two classes? A. Get A and take away B.*
96. Lynne: *Sixty two.*
97. Carey: *Sixty two.*
98. Lynne: *Sixty two. There.*
99. Carey: (Writes 62 and B next to it).
100. Lynne: *The difference between the two classes.*
101. Int.: *I don't think I will be able to hear.*
102. Carey: *OK.*
103. Carey: *Sixty two. B.*
104. Int.: *So, do you know what they say you should call A?*
105. Carey: *Yes, that* (points to the number; then writes and checks over) *A. Call the difference between the two classes B. No, that's not B* (showing 62 and crossing out the B next to it), *that's B* (showing 13, writes B next to it).
106. Lynne: *Take away, they said here.*
107. Carey: *Take away B. Call A. Call A take away B equals C. That's sixty two, isn't it?*
108. Lynne: *Yes.*
109. Carey: *So, sixty two is C* (writes 62 C).
110. Lynne: *Yes.*
111. (Both read together: Get A take away B equals C).
112. Carey: *Split C in half, 31, then add B to D.* (Pause) *Call the answer to B and D E. What?*
113. Lynne (to interviewer): *We don't know what's D.*
114. Carey: *What's D?*

115. Lynne (to interviewer): *Call the answer to B and D E.*

116. Carey: *Then add D and E - What's E? - to get A. One class has D and one class has E* (reading from message).

117. Lynne (to interviewer): *We don't know what E is.*

118. Int.: *I see.*

119. Carey: *What's E?*

120. Int.: *Did you read both sides? Perhaps they wrote on both sides.*

121. Carey: *Call the answer ... Both sides?* (Turns page over) *No. Because.*

122. Int.: *Oh, there it is. Call the answer to B and D equal E. B and D. Is what?*

123. Carey: *B and D. Is forty four.*

124. Lynne: *Yes.*

125. Carey: *Forty four.*

126. Int.: (Unintelligible).

127. Carey: *Forty four. E. Forty four is E. Oh! Then. Add B and D to get A.*

128. Lynne: *Seventy five.*

129. Carey: *One has D and one has E.* (Speaks and writes) *Forty four one.*

130. Int.: *One class, right?*

131. Carey: *And the other thirty one.*

132. Int.: *Do you think it was hard?*

133. Both: *No.*

134. Carey: *Once we knew what B and D were.*

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Contribution to the panel
PUTTING MATHEMATICS INTO LANGUAGE AND LANGUAGE INTO MATHEMATICS

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We were given a protocol without indication of the aims of the investigation the protocol came from. We were not told about the previous knowledge and experience of the observed pupils in algebra. This offered to me an excellent opportunity of focusing on the meaning of the task for the pupils because in absence of information I had to reconstruct myself this meaning. This is indeed exactly the work of a researcher. A researcher cannot take for granted what is told about the experiment and must construct an interpretation of the situation within his/her theoretical framework.

I - Methodology

We as researchers are faced with the problem of interpreting what the pupils did and why they did so. In this interpreting work, we consider that a meaning of a behaviour does not exist *per se* but as one among all possible ones. Searching for the reasons of the actual pupils strategies becomes identifying why such behaviour occurred and not other ones.

Our way to construct an interpretation is to contrast the observed behaviours of the pupils and the text they produced with other possible behaviours or other possible writing strategies. Attempting to systematically find all possible ways of solution developed by pupils allows a decentration of the researcher and avoids interpretation directly and too much inspired by the beliefs and ideas of the researcher. This search for all possible ways of solutions is done in taking into account the situation and its meaning for the pupils.
This paper deals only with the two tasks of writing a message.

II - Analysis of the phase of writing a description

II.1 - What is the meaning of the task for the pupils ?

The task is not to solve the problem; the pupils already did it. They must explain to peers how to solve the problem with different numbers that they ignore. The statement of the task even stresses the fact that they do not have to use numbers. It implies that our analysis does not directly deal with the mathematical solving process of the problem but it deals with the linguistic ways used by pupils to formulate how to solve the problem. In particular, our aim is to make explicit possible linguistic strategies of expressing the two given numbers of the task, which play the role of parameters, i.e. the total number of books and the difference between the numbers of books of the two classes.

But even if the task is not to solve the problem, the explanation written by the pupils depends on the way they solved the problem. Two main ways can be distinguished:
- an arithmetical way of solution consisting of starting from the known numbers given in the statement and producing numbers as intermediate results of operations until finally obtaining the required unknown numbers (this is, as noted by several researchers, the reverse way of the algebraic way).
- an algebraic way of solution, consisting of denoting the unknown numbers of books in the two classes by letters and representing the relations between the unknowns by a system of equations which is solved in a second step.
A third way can be evoked here, a geometrical way in which the unknown numbers of books are represented by segments. In this case, the solution does not depend on the known numbers and the message would probably present the schema with the segments with a legend in natural language. This kind of solution is unlikely to appear unless it has been taught. So our discussion will focuse on the arithmetic and algebraic cases.

II.2 - In case of an arithmetic solution
There are several possible ways of solving arithmetically the problem, which have already been described in research papers (Bednardz & Dufour-Janvier 1994). Arithmetic solutions differ from algebraic ones in that they generally start from the known numbers. The task for the pupils is to describe the sequence of operations they do on the given numbers and on the results of these operations. It means that the task is of linguistic nature. The pupils must develop linguistic means for describing the known numbers, the operations and then the results.

It is possible to propose several linguistic strategies, according to the degree of use of symbolism in this description. Predicting the possible strategies must be done in taking into account that the situation is a communication situation which is successful from the point of view of the pupils if the receivers are able to solve the problem by means of the message. What is important is not so much the mathematical correctness of the expression of the numbers and results but the fact that receivers identify the right numbers by means of the description, i.e. that they recognise the right numbers among all numbers involved in the problem.

Three systems of signifiers can be used by the pupils for describing numbers and operations: natural language, algebraic symbolism, geometrical diagrams (segments representing numbers). We consider here the two first ones which give rise to two kinds of strategies:
Arithmetics and natural language (AR. NL)
Arithmetics and parameters (AR.P)

AR. NL (arithmetic and natural language)
The given numbers and results of operations are described in natural language. It is possible to consider variations in the use of natural language. Pupils may have recourse to various ways of describing the numbers in order to allow the receivers

to identify which number it is about (Laborde 1990): either by referring to the context or by referring to the text itself. There are here at least two kinds of context: the story on which the problem is based, the context of problem solving. This leads to identify four possible linguistic strategies of description of the numbers and results in natural language:

i - numbers are described with reference to the story and results with reference to operations by which they are produced

ii - use of a reference to a time scale in order to distinguish between various results of operations

iii - use of anaphoric language means like pronouns

iv - use of the position of numbers or results in the statement or the solution.

Below are presented in more details each of these strategies.

i - expressions referring to the context of the story of the problem for describing the given numbers

"the amount of books", "the number of books of the class which has less books", ...

expressions referring to the operations done on the numbers for describing the results

"the difference between the amount and the amount of how many more books one class had"

ii - use of a reference to time

"the number we have just found" as opposed to "the number we found earlier" or "the number we found at the beginning".

These two last strategies include a contextual meaning, referring either to the story or to the context of performing actions for solving the problem. In this latter case, the pupils themselves may be involved in the description as *actants*. The two further strategies offer a decentration, in that they are based on the *text* and not on the *context*.

iii - use of anaphoric language means such as pronouns "this one" "that one"

We can hypothesise that these means are mainly used for expressing the results "this result" as opposed to "that result" which has been found earlier in the solution.

iv - expressions referring to the position of the number in the problem statement and of the result in the sequence of operations

"the first given number", "the second given number", "the first starting number", "the first result", "the second result"

This is close to a strategy of denoting by letters and it has been observed at pupils as an intermediate step between natural language and the use of letters.

Of course, the described categories result from a theoretical definition and in fact pupils can have recourse to mixed strategies.

AR.P. (arithmetic and parameters)
The known numbers are denoted by letters (for example T for 75 and D for 13). This leads again to a gradation in the use of symbolism. We distinguish complex symbolic designations like a+b from simple designations like a, b, ... The latter ones have only a designation function whereas the former ones fulfil a double function, designation and expression as a function of other objects denoted by letters.
Two strategies can be used.
- using complex designations for the results of the operations:
"T-D", "1/2 (T-D)", "1/2 (T-D)" and "1/2 (T-D) + D"
- using only the designations of the parameters T and D as simple designations and possibly other simple designations for the results. Operations are described in natural language "subtract D from T, you obtain A, divide A by 2..."

The choice of the relations involved in the problem is crucial for the possibility of using only natural language in the description. If the operations to perform are too complex and too numerous, using natural language becomes difficult and it may compel pupils to search for another way of description and therefore to have recourse to letters for designating the parameters, and even in some cases to use complex symbolic designations (this belongs to a higher conceptual step).

In the case of the present problem, the relations are very simple and it allows a description based on natural language. In case of use of letters for the parameters, it is also possible to use them only as simple designations. And even if complex designations are used, the letters keep a mere status of designation and do not receive an operative function since it is not necessary to perform algebraic reduction except may be for 1/2 (T-D) + D but the description can be made and efficient without saying that 1/2 (T-D) + D = 1/2(T +D). Briefly speaking, this communication situation is not a situation favouring the use of algebra. It may be a situation favouring the use of letters as simple designations (so to speak as a way of memorising results) but it is not really necessary.

II.3 - In case of an algebraic solution

An algebraic solution is based on the following system of equations with variables (denoted for example by n1 and n2),

$$n1 + n2 = 75$$

and

$$n1 - n2 = 13$$

We distinguish two possibilities.

AL.P (algebraic and parameters)
The pupils writing the message say to the receivers that they denote the parameters by letters (N and D for example); and they write down the system of equations and perform an algebraic solution using N and D, i.e. operating with N and D.
The message consists of two parts:

a first part in natural language in which the mapping between the parameters and the letters is expressed. The parameters are described in natural language (cf. above the possible ways of describing the parameters)
a second part which is purely algebraic.

The probability of such a strategy is very weak, since it consists of denoting both unknowns and parameters. We know from the history of symbolism that the use of the letters for parameters appeared later than the use of letters for the unknowns and constitutes a conceptual step (it was achieved by Viète.[1]). So it is very likely in case of an algebraic solution that the pupils will not have recourse to letters for the parameters but will develop the following strategy.

AL. EX (algebraic and explanation)
Because the parameters are not denoted by letters, the pupils do not write down the system to be solved but explain to the receivers that they have to write a system with two unknowns n1 and n2 , the first equation expressing that the sum n1+n2 equals the first parameter, the second equation n1-n2 equals the second parameter. The parameters can be described in various ways like in the arithmetic case (AR.NL).
The solution may be given or to be done by the receivers.
If the solution is given, the formulas expressing n1 and n2 in function of the parameters are formulated in natural language:
n1 equals half of the sum of the two given numbers, n2 equals half of the difference of the two given numbers. It is to be noted that the solutions can be easily described in natural language because of the choice of the relations between n1 and n2.

The second strategy AL.EX is made possible by two kinds of choices in the problem: i) the fact that the given numbers are exactly the second member of the equation ii) the simplicity of the relations between the unknowns.

Note : We did not consider a strategy in which the pupils keep the numbers of their own problem statement, and explain to the receivers that they have to do a similar solution but with their own numbers replacing the numbers of the solution, because they are explicitly told that they are not allowed to use numbers.

In conclusion of this prior analysis, it appears that the choice of the problem favours more strategies based on the use of natural language and that this problem is not the most appropriate for allowing an evolution of the pupils towards the use of letters.
From the protocol of Jane and Ann, we learn that they never used letters in the mathematics class. The algebraic solution will certainly not occur.

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[1]"Quod opus, ut arte aliqua juvetur, symbolo constanti et perpetuo ac bene conspicuo date magnitudines ab incertis quaesititiis distinguantur, ut pote magnitudines quaesititias elemento A aliave litera vocali, E, I, O, V, Y datas elementis B, G, D aliisve consonis designando" (in Opera mathematica, opus of Viète gathered and edited by F. Schooten, 1646, Leyden)

III - Analysis of the protocol of Jane and Ann
III.1 First task: writing a message without numbers

Jane and Ann performed an arithmetical solution:

on one page	on the other page
75 for 2cl	75-
I get 13 more	<u>13</u>
75 - 13 = 62	
31 + 13 = 44,	62
	31
	<u>44</u>
	75

In their message, they adopted a strategy AR.NL without hesitation and did not discuss the possibility of using letters. It was done straightaway with only some discussions about some elements of the formulations. We can notice how the written solution of the problem affects the writing of the message. The message is only the expression in natural language of the arithmetical operations written in their own solution, except for the division by 2 which is added in their message.

Numbers and operations were expressed in natural language. Here are listed their formulations:
- for the total amount of books
"the number of books", "the amount of the beginning"
- for the difference between the numbers of books of each class
"the number one class had more" (Ann orally)
"the amount that is bigger" (Jane orally)
"the amount of how many more books one class had" (twice in the message)
- for the results of operations
"the amount", "the two answers"

For the given numbers they referred
- to the context of the story:
"the amount of how many more books one class had"
- or to a time scale of the solving process:
The total amount of books is expressed as "the number of books" when they did the subtraction but is called "the amount in the beginning" (Jane) and then "the amount you started with" (Ann) in the final checking process. We recognise here a means of solving the ambiguity between several numbers: referring to time allows to identify the right number. Note that Ann introduced a reference to the *actant* solving the problem which was not present in the formulation of Jane.
The formulation "answers" refer to the context of problem solving in which the two girls and the receivers were placed.

In conclusion, Jane and Ann did not experience any difficulty in writing their message and did not feel the need of other means of formulation for improving their message.

III.2 Second task: writing a message with letters

III.2.1 What was the task for the pupils ?

The statement of the second part of the task by the interviewer introduced a strong *contract* : English is not mathematics and mathematics is using signs, i.e. letters like a code. It means that in this second task, the use of letters by the pupils was not conceived as a way of improving their message but as a way of satisfying both the interviewer and the image of mathematics he was giving. The interviewer triggered in addition the expression by the pupils of the mapping between the code and the letters ("and you explain to them what the code means"). He did more than triggering, he proposed formulations for this mapping: "You can call the total number of books A" and later "You can tell them to call the difference ...".
The interviewer modified the problem for the Jane and Ann through a *Topaze like effect* (Brousseau 1986). The task was then for the pupils not a communication task but mainly a translation task of their first message, consisting of changing the formulations for numbers into appropriate letters.

III.2.2 Analysis of the protocol

The expected strategy of translating the previous message and recognising the numbers from their formulations in natural language appears clearly. At first Jane wanted to change the old message but was asked to do a new one. At several places, Jane and Ann come back to their previous message, reading or pointing numbers and saying "that's A" or "that's B". Another sign is that the interviewer had to trigger them to indicate at the end of their message how many books each class received. The fact that they had to communicate the solution of a problem was forgotten by Jane and Ann.

Jane and Ann perfectly understood the rule of the game since they decided on their own to denote each result with a new letter:
C for A-B, D for C/2, E for D + B. They use the strategy of simple designations and because they systematise this, they could not be aware of the operative power of letters.
D did not appear as 1/2(A-B) nor E as 1/2(A-B) + B.

But the systematic use of letters as simple designations lead them to be aware of the missing step of their old message. When they tried to express the addition 1/2(A-B) + B, they recognised that B must be added to something, because they recognise that it could not C, it provoked the designation of C/2 by D (Jane).

The fact that operations were expressed by verbs in natural language also supports our interpretation of the meaning of the task for the pupils. It also reinforced the function of pure designation of letters.

IV - Conclusion

As expected from our prior analysis, the first task did not promote the use of letters by pupils nor the use of similar strategy like using ordinals, "first number", "second number"... The choice of the problem played a crucial role and changing the problem into a more complex would probably not lead to the same pupils' behaviours. The second task was strongly affected by injunctions of the experimenter. The use of letters was achieved by the pupils for satisfying the new explicit contract. From the protocol we could learn that the systematic use of letters as simple designations of each result of an operation may hinder the operative use of letters hiding the possibility of simplifying symbolic expressions.
A too systematic introduction of letters could be a *didactical obstacle* to algebra. This is a point for the discussion with the participants in this PME meeting.

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LANGUAGE AND THE SOCIALIZATION OF THINKING

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I want to approach this discussion as a psychologist and reflect on mathematics as a way of thinking rather than a scientific discipline. In this sense, mathematics involves a set of practices used by different communities (such as schools and mathematicians but also in other spheres of life). These practices involve a variety of systems of signs, which are public, external forms of representation, such as numeration and measurement systems, representations of arithmetic relations (in signs such as +, -), literal representations of unknowns, representations of points in a system of coordinates and of relationships in space, by means of names for shapes, measurements, and constant values, to name just a few. These external systems of signs can be used by subjects for their own purposes, for example in making records, computation etc, and in communication with others. Mathematical systems of signs represent the signified in particular, conventional ways, which may or may not match the way learners organize the world when not using mathematical representations. In order to use mathematical representations, learners must provide the meaning for the signifiers, regardless of whether they are using the system for their own purposes or for communication, and, in the latter case, regardless of whether they are at the producer or at the receiver end of the communication. That means that one of the major tasks of learners must be to discover how their meanings are represented by the signifiers in these systems of signs. For this reason, a psychological account of mathematics learning involves finding out how learners develop meanings that can be related to mathematical signs and how they come to establish a relationship between signs and meanings. Once a system of signs is acquired, it has consequences for learners: Old meanings might be thought about in new ways and learned systems can become the meanings for new systems of signs. The overall process of adjusting meanings to conventional systems of signs and then using these as meanings for learning new systems I call socialization of thinking. This socialization is what I want to focus on in this presentation. In a previous PME presentation and in different publications I have discussed the characteristics of oral mathematics, where language is the tool used to think with, but this will not be my focus here.

My aim in this discussion is to consider the role that natural language plays in the process of learners constructing meanings and relationships between meanings and mathematical practices. As a method, I will raise hypotheses and illustrate them with examples. The examples do not prove the points, they are starters in a discussion about language, which is the goal of the panel.

1. Beginning with an analogy to written language.

What is represented in the signs of written language? For a literate adult, there is little difficulty in understanding that written English, for example, represents oral English: script is a conventional system of signs that represents another system of signs, oral language. In this sense what is represented by the signs is quite straightforward (although the meaning of the text is another matter). But even in this simple situation there is still quite a lot for learners to figure out to become readers. They need to discover how written English is connected to oral English. In English, the sounds (phonemes) and basic elements of meaning (morphemes) are represented. We say /kist/ and write "kissed": the first few letters represent the sounds, the end, "ed", is a representation of the morpheme for the past of regular verbs. In Chinese, sound and meaning are also represented, not by letters but rather by stroke patterns: a character corresponds to a syllable, not a phoneme, and the majority of characters contains a semantic radical, which gives a clue to meaning, and a phonological component, which gives a clue to sound. This comparison simply illustrates that there are very different ways of representing oral language through written language.

Children don't necessarily know from the beginning how script represents language and often make wrong hypotheses about how language is represented by script before they understand how alphabetic representation works. Emilia Ferreiro and Ana Teberosky have shown that young Spanish speaking children who have discovered that written Spanish has something to do with sounds might still hold a wrong hypothesis about how script and oral language are related: they might think that each letter represents a syllable. A four-syllable word, like "mariposa" (butterfly), will be written with four letters by these children.

Thus learners have to provide the meaning for the graphic signs. Because alphabetic scripts represent sounds, learners must think about the sounds of their language in order to connect script and oral language. This requires thinking about language - that is, developing metalinguistic knowledge. Children's progress in reading and spelling can be predicted by their metalinguistic knowledge.

In the process of developing metalinguistic knowledge, natural language plays an important role. First, teacher and learner can talk about "what does this say?" when they look at a word or a phrase together. Second, teachers can point out aspects of the written signs and their meanings, "this is an ef, it makes the sound /f/, f-f-fun". We can now turn to the question: How are mathematical meanings represented by mathematical signs? It is clearly not the case that mathematical signs represent oral language. What are then the meanings of mathematical signs? Piaget suggested (and I am perfectly happy to go along with this idea) that the basic meanings of mathematical concepts stem from children's schemas of actions: children can compare, put things in order, join and separate, count in several ways in order to solve problems, make correspondences etc. When they carry out these actions, either on objects or on representations, they can make deductions in the absence of perceptual information, and thereby show the logico-mathematical nature of their schemas. For example, if children of about 6 years know that A is larger than B and B is larger than C, they conclude that A is larger than C even without comparing them directly.

Schemas of action can be applied to objects but also to external symbols, such as fingers or marks on paper. When children solve a problem about sweets using their fingers to represent the sweets, they can use their action schemas. This is already an example of modelling, even if elementary, because children assume that whatever result is obtained with fingers also applies to sweets. Children can operate on signs as long as their schemas can be used to structure the relations in the problem situation.

When children learn arithmetical representations in school, however, there is no simple match between their schemas of action and the mathematical signs which they learn: schemas of action cannot be used on numbers because numbers offer a compressed representation of objects (8 is one representation for eight separate objects). Further, the job done by several schemas of action needs to be done by two arithmetic operations in the first years of school, addition and subtraction. Several studies on addition and subtraction have already demonstrated this point: Children may be able to solve a problem using action schemas without knowing which arithmetic operation is adequate to calculate the result formally.

This bring us to the end of the analogy: whereas in literacy learning the match between script and oral language is simple, when learning mathematical systems of signs, language will have to play a different role in instruction. I suggest that language can be used in at least four different ways in mathematical instruction:

- to allow children to participate in linguistically created realities;
- to instigate the use and coordination of action schemas;
- to invert the relationship between figure and ground and promote metacognitive mathematical thinking;
- to promote the process of representational redescription necessary for establishing relationships between everyday meanings and mathematical signs.

2. Language as a tool in mathematics learning.

Participation in linguistically created realities. Mathematics learning in the classroom is not meant to be the same as everyday life events. The problem: "In the morning Mary had some sweets. Her father gave her 5 sweets in the afternoon, and now she has 8. How many sweets did she have in the morning?" might sound like an everyday situation: but when would Mary really want to figure out how many sweets she had in the morning and why? The aim of story problems in the classroom is to involve children in linguistically created realities, which give us more possibilities than everyday life to explore children's logico-mathematical meanings, but still allow them to use their action schemas for interpreting the problem. But in the school we want learners to use their schemas under constraints that do not apply in everyday life in order to extend their schemas (for example, through inversion).

For Vygotsky and Luria, participating in linguistically created realities was equivalent to developing a theoretical attitude, which allows the learner to work on possibilities and their implications rather than events and their consequences. And thinking about implications is a fundamental way of thinking in mathematics.

Instigating the use and coordination of action schemas. Language is clearly connected to many action schemas: for example, when you ask children "How many...?", they count. Teachers can thus instigate the use of particular schemas through language (see poster by Desli for an example) and provoke the coordination of previously disconnected action schemas. Peter Bryant and I carried out a study with children in the age range 5 to 7 years which had the goal of provoking the coordination of two action schemas, correspondence and adding/subtracting, in order to help the children understand comparison problems, normally very difficult for children. Young children succeed in solving problems that evoke their correspondence schemas, such as: "There are 6 children and 4 balloons: How many children won't get balloons?" However, they cannot indicate which arithmetic operation might be used to solve this problem. They can also answer the question: "How many more balloons do we need so that all the children have a balloon?" We hypothesized that if we led the children to coordinate the action schemas used in solving each of these problems, they would see their way through solving comparison problems. We attempted to build this connection by asking the children to solve a few problems in a teaching session, where the children started from equal sets which they put in correspondence ("We both had 4 sweets one day"); the equality was then destroyed either by addition or by subtraction (e.g.: "and then you were a really good girl and you got 3 extra sweets; how many more do you have now? ... I have 4 sweets and you have 7, how many more do you have?"). This group of children showed more progress in their ability to solve comparison problems than a control group, who had a session where they solved the same number of comparison problems but did not have the linguistic input evoking the two schemas together in the same situation.

Inverting the relationship between figure and ground: developing metacognition. Vygotsky, Luria, and their colleagues stressed the role of language in leading children to focus attention on particular aspects of situations. Through language it is possible to make children focus on something which normally constitutes the background rather than the figure in a stimulus. For example, young children (4 years) easily learn a discrimination task involving a figure but have great difficulty in learning a discrimination attached to the background (the colour of the paper on which figures are drawn); through language, it is possible to make them respond to the background. This is the basic idea in metacognition: making what was the background into the figure, the tool into an object of reflection (to use Regine Douady's language). When learners solve problems, they focus on solutions rather than the methods. A lesson that takes full advantage of problem solving does not end when the problem is solved: having succeeded in solving a problem, learners can then focus on the process of solution, compare different methods, re-represent their schemas of action or their intellectual moves. Language is here both the means of expression ("let me show you why my solution works") and the bridge between different types of solution. Language can be used to cast the solution in general terms which are already part of the learner's vocabulary ("take the number of extra books away from the total number of books") rather than the specific conditions in a problem (Do 75 minus 13).

Language and representational redescription. Annette Karmiloff-Smith has proposed the idea that much of development depends on children's representations becoming progressively more manipulable and flexible, for the emergence of conscious access to knowledge. This process involves representational redescription. I will use the term "redescription" here in a slightly different way, to stress the fact that meanings derived from action schemas are reshaped - that is, redescribed - when learners establish a connection between their old meanings and the new systems of signs learned in the classroom. There are several aspects in this redescription.

First, different action schemas may be compressed into a simplified system of representation with fewer options. For example, all the different action schemas related to additive structures can be represented in the expression a+b=c and the implied forms c-a=b and c-b=a. It can signify a transformation (+b), a static difference (c-b=a), the union of sets (a+b) etc. For the addition and subtraction signs to become related to these different meanings, a representational redescription is necessary. Language, as argued earlier on, can play a special role in promoting the coordination of different schemas so that this redescription becomes possible.

Second, redescription may involve establishing connections with schemas previously excluded from the concept. For example, it is easy (and current practice) to teach children about multiplication by connecting it to addition but the price may be that the meaning that young learners connect to multiplication will need much redescription for them to progress beyond initial stages. Peter Bryant and I have hypothesized that the basic meaning for children's understanding of multiplication should be sought in an action schema quite different from those used in addition: the schema of one-to-many correspondence. When we treat multiplication as a special case of addition, the ideas of correspondence and ordered pairs are overlooked and may not be represented in any way by the learner. In order to recover these ideas, it may be necessary to question some of the current conceptions. Language is usually the way in which such conflicts can be created in the classroom.

Finally, redescription may involve creating entities out of implicit relations. This is another case of linguistically created realities, where language nominalizes relations (and, as Gerard Vergnaud points out, makes them into objects): by this manoeuvre, learners are presented with entities which are not encountered in everyday life. A: "In the time of dinosaurs..." B: "What's a dinosaur?" is a perfectly possible dialogue. Dinosaurs, and the need to know what they are, are created in the discourse. Similarly, in the dialogue A: "This function..." B: "What's a function?", an entity is created out of a relation between two variables. Construction foremen, who learn much about multiplication in everyday life and clearly use their understanding of one-to-many correspondence as a basic reference in multiplication situations, still do not easily draw on functional solutions to solve problem and focus mostly on scalar transformations. It is possible (but I have no evidence for this conjecture) that in this sort of situation language can play a very special role: by denoting the unperceived relation as an object it may create the need for knowing it.

3. Conclusion.

The role of language in mathematics learning discussed here relates to the process of socialization of thinking that results from participation in any cultural practice. Language can be used as a means of promoting the redescriptions needed in going from action schemas to mathematical systems of signs and of promoting the development of new concepts. In some cases, language may itself offer intermediary redescriptions; in others, it may not. Although language is not the only means to promote the process of redescription, it is a privileged way of communicating, and communication is likely to be the essential element in the socialization of thinking.

PUPILS' PROMPTED PRODUCTION OF A MEDIEVAL MATHEMATICAL SIGN SYSTEM

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Introduction.

I have faced the task of making sense of the protocol that Terezinha provided us in a way I daresay that makes use of Foucault's way of dealing with historical documents. Having meager information on the aims and context of the research where the protocol was produced, I have stuck to the text I had, trying to turn this document into a monument. That means, in Foucault's words, "not the interpretation of the document, nor the attempt to decide whether it is telling the truth or what is its expressive value, but to work on it from within and to develop it" (Foucault, 1969, p. 14). Moreover, I have confronted this text in a dialogue with a text from the 13th century —*De Numeris Datis*, by Jordanus de Nemore—, using the history of mathematical ideas like Filloy does (see, for instance, Filloy, 1990), i. e., like a way to shed light on pupils' productions. I don't claim then to have found the meaning of the protocol, instead I'd rather say that my work on this text has produced new senses that I would like to be fortunate.

By making this statement I'm also merely introducing the semiotic idiom in which I feel more at ease. In it, actually, each new reading of a text takes it as a textual space whose transformation through the act of reading produces a new text together with new senses. A new sense for a sign or a text becomes a new meaning if this sense is fortunate, that is, introduces a new use for the sign or text that get to be shared by a community, entering therefore the encyclopedia[1].

A problem and several tasks.

Two pairs of pupils were given a problem to solve and several tasks to perform. The first pair had to solve the problem and to give the solution to the interviewer. Once they gave the solution to her, they were asked to write a message explaining their solution to some friends that had to solve a similar problem. They were requested not to use numbers in this written description. Once they produced such a description, as it happened to have been written in plain English, the interviewer gave them another task: to write it anew "in

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[1] See Talens & Company (1984) for a discussion of the terms 'text' and 'textual space', 'meaning' and 'sense' that is in the origin of the use of them I'm making here. See Voloshinov (1992) for the stress on the social character of the process of signification – the original Russian edition dates from 1929 and it seems to have been written at least partially by M. Bakhtin.

mathematics. Using signs like in mathematics": She prompted them also by saying "you can call the total number of books *A*".

The second pair of pupils were given the task to solve the problem using the message sent by the first one. I am not going to analyze this part of the protocol, but only the tasks accomplished by the first pair.

A problem or a task, considered as a text / textual space, conveys meanings as a result of all uses of this problem or task in the history of a community. The movement between meaning and sense that I have sketched accounts for the process of semiosis in general, and the way in which semiotic practices produce an *episteme*. However, when a problem or a task is given to some pupils by a teacher, we have to deal with this process in a teaching-learning situation within a school system. As a consequence it becomes necessary to consider the meanings the problem or task conveys to these pupils as a result of their personal history, that is, the uses they have done of the set of notions involved and the social practices, both discursive and non-discursive, in which they have encountered this kind of problem or task. Moreover, it becomes necessary to consider also the competence model (Filloy, 1990) as the good fortune of the senses produced by pupils heavily hangs on their adequacy to the competence model. In the case of the situation described in the protocol, we can assume that pupils are supposed to become competent in the use of the mathematical sign system of school algebra, specially in its use when solving arithmetic problems of several combined operations.

Mathematical sign systems in the school system.

It's not unusual that a description of the language in which mathematical texts are written distinguishes between two subsets of signs. One subset is seen as containing the signs that are considered really mathematical —often qualified as "artificial"—, and to the other belong the "natural" signs from some vernacular tongue. Even if this distinction can be done, it leads usually to focus the study of mathematical sing systems on the study of mathematical signs, and their teaching on the teaching of the use of these signs. It should not be then a surprise if pupils end up identifying mathematics with the use of letters.

I'd rather like to consider instead that what is qualified as mathematical in the expression 'mathematical sign systems' are the systems and not the signs, i. e., that mathematical sign systems means mathematical systems of signs and not systems of mathematical signs. Thus, for instance, Freudenthal shown in the chapter *The Algebraic Language* of his *Didactical Phenomenology* how the usual expression in German arithmetic classrooms 'sieben minus vier' is not "normal German" and that there are similar expressions in Dutch that are not "normal Dutch" either (Freudenthal, 1983, pp. 486-487). This is the case also in Spanish, and is more conspicuous in not Indo-European languages, of which we have one in Spain - the Basque language. A mathematical sign system for arithmetic can have segments with the same matter of expression that a vernacular tongue, but even when the very words of this tongue are used, grammar, semantics and pragmatics are not exactly the same as those of it.

Mathematical sign systems do have of course segments whose signs don't belong to any vernacular tongue, but mainly or only to mathematical sign systems; what I want to stress is that we should not identify mathematical language with these signs, since we are not interested in analyzing mathematical texts that only exist in Bourbaki's heaven —they not even exist in Bourbaki's books—, but actual mathematical texts. Furthermore, we are specially interested in analyzing mathematical texts that are produced by pupils in the school system while they are taught mathematics, and, as Filloy pointed out, we need a notion of mathematical sign system wider enough to account for this kind of texts.

From this point of view, I have discuss in Puig (1994a) some characteristics of mathematical sign systems that are worth considering here, briefly stated:

1) Mathematical texts are produced by means of stratified mathematical sign systems whose matter of the expression is heterogeneous.

2) The heterogeneity of the matter of the expression is shown by the existence in mathematical texts of segments that, when seen isolated, seem to have been produced by different languages. Nevertheless, these segments are not ruled separately by the rules of those languages, but by new rules that result from a specific combination of them.

3) There are pointers that refer mutually signs from segments of different matter of expression.

4) Both in the history of mathematics and in the history of individuals, mathematical sign systems are the product of a process of progressive abstraction. As a consequence, those that are actually used are stratified. The strata come from different moments of the process of abstraction, and are related among them by the correspondences established by this process.

5) The autonomy of the transformations of the expression from the content plays an important role in the process of abstraction that leads to the production of a new mathematical sign system.

A medieval mathematical sign system.

The earlier known mathematical texts that use letters to stand for quantities are those written by Jordanus de Nemore in the 13th century, namely *De Numeris Datis* and *De Elementis Arithmetice Artis*. The critical edition of *De Numeris Datis* has been published by Hughes (1981), who gives 1225 as the more likely date of publication.

De Numeris Datis is written in Latin and is straightforwardly organized: three definitions at the beginning and 115 propositions distributed in four books, without any explanation of the aims of the book, neither an introduction nor transitions between books.

The propositions are presented always in the same three parts:

1) A statement asserting that if some numbers (or ratios) have been given, along with some relations between them, then some other numbers (or ratios) have also been given.

2) A series of transformations of the numbers (or ratios) and the relations that either show that the numbers have indeed been given or convert them into the numbers and relations of the hypothesis of some previous proposition.

3) The calculation of an example with concrete numbers.

Like in Diofanto's *Arithmetic*, the statement does not involve concrete numbers. Unlike in Diofanto's *Arithmetic*, the argument does not involve either concrete numbers, moreover the quantities mentioned are often represented with letters. This use of letters in the arguments of the propositions is the main reason why *De Numeris Datis* has been considered as the first medieval advanced algebra. However, neither Jordanus de Nemore uses letters always in *De Numeris Datis* —but only in two thirds of the arguments—, nor he uses them only in his so called advanced algebra treatise —but also in his elementary arithmetic, *De Elementis Arismetice Artis*, whose Latin text has been recently edited by Busard, with a paraphrase in English (Busard, 1991).

I have reported an analysis of the mathematical sign system of *De Numeris Datis* in Puig (1994b). I will present here two propositions from the beginning of the book and a summary of the part of my account of the characteristics of its mathematical sign system that is relevant to the discussion of the protocol we are analyzing here.

The statement of I-1 is "Si numerus datus in duo dividatur quorum differentia data, erit utrumque eorum datum", that can be translated[2] as follows: "If a number that has been given is divided in two parts whose difference has been given, then each of the parts has been given".

The argument is: "Since the lesser part and the difference equal the larger, the lesser with another equal to itself together with the difference make the given number. Subtracting therefore the difference from the total, what remains is twice the lesser. Halving this yields the smaller and, consequently, the greater part".

The statement of I-3 is "If a number that has been given is divided in two parts whose product has been given, then each of the parts has been given".

[2] The translation of Hughes is more liberal than mine, but even if the literality gives a coarse English, his liberality makes impossible to see some of the characteristic of the original text that are essential to its description.

The argument to prove it can be schematized, preserving Jordanus de Nemore use of letters, as follows:

"Let *abc* be the number that has been given, divided in *ab* and *c*[3].

ab by *c* makes *d*, given.

abc by himself makes *e*.

Let the quadruple of *d* be *f*.

Taking *f* from *e* remains *g*.

g is the square of the difference between *ab* and *c*.

The square root of *g* is *b*.

And *b* is the difference between *ab* and *c*.

Since *b* has been given, *c* and *ab* have been given".

What follows is the relevant part of my account, that is grounded on the analysis reported in Puig (1994b).

The quantities that appear in the arguments are named sometimes, like in proposition I.1, using its meaning by reference to an initial number divided in two parts (the lesser part, the larger part, etc.) and sometimes, like in proposition I.3, with a letter (or some letters put together).

Whenever letters are used, all quantities, both known and unknown, are represented by letters; the letters are marks to denote the quantities that are built in the course of the argument and appear in alphabetical order, without any distinction between known and unknown quantities.

Besides, a quantity can be denoted by more than one letter. Each letter does not represent then a number, but the instance of appearance of a number in the course of the argument.

There is a lack of syntactic operativity, excepting for juxtaposition to mean addition; thus, when a new quantity is built using quantities already denoted by letters with an operation different from addition, the only way to denote the new quantity is to introduce a new letter to do it, there is no way of using the letters denoting the quantities involved (i. e., to denote the product of *a* by itself the *a* is useless: it is necessary to introduce a *b*).

Moreover, the relations between quantities can not be produced on the expression level of the sign system, they have to be produced on the content level instead.

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[3] Hughes translates "Let the given number *a* be separated into *x* and *y*", making impossible to see in its translation that Jordanus de Nemore does not distinguish between known and unknown quantities.

The problem and pupils' solution.

The problem posed to the pupils corresponds to the first problem in Diofanto's *Arithmetic* and to proposition I.1 of *De Numeris Datis.* They are not however the same. The problem posed to the pupils tells a story about classes and books to share and the quantities involved are concrete quantities, namely "numbers of books". Diofanto's first problem tells a story about pure numbers and arithmetic operations on them. Proposition I.1 of *De Numeris Datis* is actually a theorem – but a theorem on the possibility of solution of a problem (or a class of problems) and contains then its solution. Moreover, one of the quantities involved is expressed in Diofanto and Jordanus de Nemore as a "difference", while in the problem posed to the pupils is expressed by means of the comparative "more than".

Nevertheless, the page with the computations shows that the pupils gave the same solution that Jordanus de Nemore includes in the argument "Subtracting therefore the difference from the total, what remains is twice the lesser. Halving this yields the smaller and, consequently, the greater part".

The task of writing a message without numbers.

To write the solution without numbers they have to name the quantities involved in the operations that lead from the data to the unknowns, to express the operations and to construct sentences that link the appropriate quantities and operations in due order. In a word problem of several combined operations, the data and the unknowns have names in the statement of the problem, but the auxiliary quantities can be mentioned in the statement or not.

Jane begins by writing the name of the quantity that correspond to the first number they have used in the solution ("Get the number of books and...") and expresses the subtraction by writing "take away". Then Ann and Jane look for an expression for the quantity taked away, ("Take away what?"). They try "the number that one class had more" (Ann) and "the amount that is bigger" (Jane) till Ann writes the complex expression "the amount of how many more books one class had". Both quantities are data named in the statement of the problem, and the names that they have given them are only slight modifications of those that appear in the statement.

The next quantity they have to name is an auxiliary one that is not mentioned in the statement. Jane simply calls it "the amount" writing "split the amount into half and add the number of how many more", completing the sentence after an interchange with Ann: "books one class had".

They write also a final sentence that corresponds to the checking of the correctness of the answers.

Jane and Ann do not seem to have got into much trouble in writing their solution without numbers – nor letters. Jordanus de Nemore did not use letters either, although he did know to use them.

The task of writing a message with letters.

The nature of this task for the pupils is very different from the previous one. Pupils are requested to write a text with a mathematical sign system that is new for them. They have to produce a new mathematical sign system at the same time that they are translating the text they have written to the mathematical sign system they are producing.

The interviewer introduces the substitution of the names of the quantities (that already stand for the numbers in the first text of the solution written by the pupils) by letters, saying that it is "like a code" (i. e., each different number or quantity, a different letter).

After some questioning, Jane seems to understand ("Oh, I get it") and writes the exact words said by the interviewer ("Call the number of books they have *A*"), but she does not know what to do next.

The interviewer goes on introducing the next quantity, but she changes its name ("the difference", a pure arithmetic name, instead of "the amount of how many more books one class had"). Jane complete the name of the quantity ("the difference between the classes" referring to its meaning in the context of the story) and ask if she has to call it *B* ("What? *B*?"). The interviewer agrees opening up the way to other letters in alphabetical order. Jane writes "call the difference between the two classes *B*", and Ann writes the first sentence to express an operation ("Get *A* and take away *B*").

The crucial step is given by Jane. She begins saying "Split *B*", making the next operation to the last mentioned letter in a kind of sequence. But as Ann wants to split *A* instead, and explain to her questioning "*A*? Take away *B and* split" (my stress), she interprets 'and' as 'and then', and produces an unexpected mathematical sign system: "call *A* take away *B* = *C*".

Ann has not yet understood the rules of the new game and says "Split in half". Jane corrects her and states clearly "Split *C* in half".

The new game is definitively established when some items later Jane writes "and add *B*", Ann asks "To what", and as Ann does not agree with the answer of Jane ("To *C*"), Jane solves the problem by saying the next letter ("To *D*") and looking afterwards for its meaning ("Get the second answer. Call the second answer. Half of *C*." "*C*:2=*D*").

A medieval mathematical sign system has been produced by Jane. We know from the analysis of *De Numeris Datis* its differences with the mathematical sign system of school algebra. I wonder whether these differences are only a step to it or whether its lack of operativity can be a serious obstacle.

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RESEARCH FORA

The role of representation systems in the learning of numerical structures

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1. Introduction

The learning of number concepts within the school system and the role that the representation notion has to analyze and interpret the understanding of number concepts in schoolchildren are important topics in numerical thinking research. Our research team is interested in the difficulties young people find on managing numerical structures when they face advanced mathematical questions. The work presented here will show the general aims and some results of a piece of research done by our team in this field.

We have chosen the representation concept to point out some curricular lacks and to observe students' work on learning numerical concepts, and consequently to interpret their numerical thinking construction. Such an idea has been continuously considered as it is interesting and useful for the mathematics education researchers (Janvier, 1978, 1987; Kaput, 1987, 1992; Goldin, 1993; Duval 1993, 1995). Though most of the considerations we make here are suitable for other mathematical subjects, our actual contribution is limited to numerical concepts and structures.

Since there is not an univocal meaning for this term, it is important to specify the sense in which we are going to use the representation concept. We will carry out such task discussing three different approaches to this concept. Once we have done this, we will present the results of our research.

2. About the representation notion

2.1. General features

The history of both philosophy and science show the richness of the different interpretations that this concept has (Ferrater, 1981). Some of them are interesting for current lines of research in mathematics education.

A first point of interest for us is to underline that the representation idea implies something to be represented. It is generally assumed that *any concept of representation must involve two related but functionally separated entities*. One of these entities is called the *representing world* or *representation* and the other is the *represented world*, which implicitly presupposes some kind of connection between the objects of the representing world and the objects of the represented world.

Thus "*any particular specification of a representation should describe the following five entities:*

1º the represented world

2º the representing world

3º what aspects of the represented world are being represented,

4º what aspects of the representing world are doing the representing

5º the correspondence between the two worlds

In many of the interesting cases one or both of the worlds can be hypothetical entities or even abstractions" (Kaput, 1987).

Therefore we consider necessary to distinguish the representation systems from the numerical concepts and structures for which they stand. When we identify natural numbers with the numerals that we get by the writing rules of numbers of the decimal system, we forget that the decimal system is only a way of writing numbers, statements and proofs by lineal combination of successive powers of 10. Using the Arithmetic's Fundamental Theorem, it is possible to write each number as a product of prime factors, and this shows its multiplicative structure; this is another representation system for natural numbers.

Though it is not usual, we will consider which different features and properties of natural numbers are highlighted by each kind of symbolization. Each of the natural numbers representations, together with its own rules, proposes a different description of the natural number concept. It is a simplification to identify numbers with any of its notations and what is worse it is inadequate for mathematics education research. So, we will differentiate between numbers and its kinds of representation.

A second important idea is the contemporary philosophical use of the representation term to refer to *anything that can be semantically evaluated* (Dancing & Sosa, 1993). It can be said that representations are true, that they refer to, that they are true with regard to something, that they are about something, that they are accurate and so on. *Contents* is the technical term used for naming *what makes a representation semantically evaluable*; thus of a statement it is said that sometimes it has a proposition or truth condition as its content; of a term it is said that it has a concept as its content; of a graphic, that it expresses a proper relationship between its elements. A representation's content is just whatever it is that underwrites its semantic evaluation. From this point of view, symbol expressions, statements, diagrams, graphics, tables and other common notations are mathematical representations.

2.2 Numerical structures and representation systems

Current number conceptualization is based on the system notion; talking accurately we are not just referring to number concepts but to number systems or structures. A numerical structure is a set of abstract entities expressed symbolically, provided with operations or ways of composing numbers and with relationships to make the comparison among its entities possible. What characterizes a numerical structure is the consideration as a whole of its entities, their operations and their relationships (Feferman, 1989). For number systems a rather small collection of big and powerful ideas determines the structure of each system (Fey, 1990).

Mathematicians work with meaningful symbols and representations (Kaput, 1987) whose nature and use have been of great interest for mathematics thinkers and researchers along the history of this discipline.

The set of signs, symbols and rules to express or represent a numerical structure must satisfy its systemic nature. That is why we can hear about *sign mathematical systems* (Kieran & Filloy, 1989), *notation systems* (Kaput, 1992) or *semiotic systems* (Duval, 1993). We prefer to use the term *representation systems* when talking about the several modes of expressing and symbolizing numerical structures by means of some specific signs, rules and statements. The decimal numeration system is a paradigmatic example of a well-known representation system for natural numbers.

The structural consideration of numbers and our choice of distinguishing between numbers and their representations lead us to the formalist foundation of mathematics. Numerical fields are established as operative fields by the formal approach of Peano and Hilbert (Badiou, 1990). The formalist foundation of mathematics stresses a technical consideration of numbers, as some kind of tools to carry out some processes, following some rules and with the possibility of establishing a variety of relations among numbers. In the formalist school, signs and symbols play a central role, together with the syntactic rules by means of which they combine to cause more complex expressions and formulas, which are necessarily complemented by finitist procedures to prove statements and formulas of each numerical system (Von Neumann, 1964).

On the other hand, in our position about its epistemological base, mathematical concepts do not refer to objects or physical phenomena but to the relations among objects, phenomena or concepts, and consider mathematical concepts as abstract entities that need to be expressed by some symbolic system; that is to say, mathematical concepts are given by means of one or several specific representations. We consider two different levels of representation: facts or particular concepts (i.e., the unity) represented by specific symbols (i.e., 1), and the relationships between concepts (i.e., one plus one makes two) represented by symbolic statements (i.e., $1 + 1 = 2$) (Körner, 1977). We assume a phenomenological base for the numerical concepts and relations.

Besides, there is not a symbolic system completely suitable to express the complexity contained in each mathematical concept; this is the reason why each concept has more than one representational system which at the same time emphasizes and sets out some important properties but also blurs or makes other properties more difficult to understand. We accept as mathematical representation systems: natural language, drawings and graphics, different symbolical writings, tables and the algorithmic notations that describe an operating rule.

2.3 Representation and cognition

In mathematics education mathematical concepts should be linked with the mental activity of human beings. Following Wittgenstein when he analyzes several mathematical language games and among them the number concept (Wittgenstein, 1988; §§ 65-68), we claim that every mathematical concept is supported by its different uses and meanings and so by its representations. All this in the sense that the use of each concept is what establishes its semantic field by extension and that each other meaningful mode of understanding a concept needs its own symbolization system or representation to be recognized. This leads us to the well-known distinction between external and internal representations. Internal representations or thinking objects, which are supposed to be placed within individual human minds, are different from external representations whose semiotic character is given by signs, symbols or graphics.

The wide use of the representation notion to characterize human mental conditions and activities is an outstanding feature in the current development of Cognitive Psychology (Guttenplan; 1994). We assume that cognitive processes are those that deal with representations. What establishes the difference between cognitive processes and those that are not is exactly that the former but not the latter can be epistemically evaluated. Since only something with contents can be epistemically evaluated, only processes can be considered as being cognitive as they involve representations. A proper internal domain of external representations is essential in the development of numerical thinking processes; this is a basic tenet for the understanding of number concepts in human beings.

We consider understanding as a representation, which is structurally or conceptually directed, of the relationships between the pieces of information that should be learnt, and between that information and those ideas and our knowledge and experience basis (Wittrock, 1990). We admit that different subjects present different understanding about the same concept or mathematical structure because their representations have different contents. The links between external and internal representation are clues to study understanding phenomena.

2.4 Balance

The representation concept in mathematics education must consider its duality. "*To think about and to communicate mathematical ideas we need to represent them in some way. Communication requires that the representations be external, taking the form of spoken language, written symbols, pictures or physical objects. (...) To think about mathematical ideas we need to represent them internally; in a way that allows the mind to operate on them*" (Hiebert & Carpenter, 1992).

Mathematical knowledge is only reachable by external representations, which are the facts for this knowledge. Representation is also involved in the actual working of our thought, and it has a central position in the learning of mathematics.

This duality of the concept converts it in a suitable tool to study understanding phenomena; for the researchers' aims it is useful when deciding to inquiry on the different ways by which human beings process numerical structure.

We have decided to use the term *representation systems* though we are aware of the problems that have been pointed out by Kaput (1992). He considers this term leads to the distinction between the representation system (representing) and the numerical concept (represented), and so it is necessary a self definition for the second one. Nevertheless we consider that we have the same problem if we talk about signs and symbols instead of representations, because symbols must express or denote a concept whose characterization has to be done outside these notations, at least from a non-nominalistic point of view. This is why we have discussed some of the previous ideas.

From the analyzed complexity we have been able to emphasize phenomenological and cognitive dimensions of numerical thinking. We have also been able to move away from the platonic foundation that claims for the reality of mathematical concepts out of space and temporary conditions and also out of human beings mental activity (Kitcher, 1984).

3. Scope of this work

3.1 Background

At the beginning of 80's there are two conceptual fields whose study is based on the notion of representation.

One of these fields is related to the concept of function; the studies that have been carried out emphasize the different systems for the representation of functions and detect some difficulties for the understanding of this concept due to translation problems among these systems. Among the most famous are Janvier's works, which ended in his thesis in 1978, and which later were used for the materials created by the Shell Center in Nottigham University. These materials undertake a kind of diagnostic teaching on this field, based on graphic representations.

The second of these fields deals with the concept of rational number, considering and analyzing different representation systems for this number field. Behr, Lesh, Post and Silver's works (1983) are among the pioneer ones in the study of this number set, which is still offering useful results.

In 1984 a symposium is held in the University of Quebec (Montreal) organized by CIRADE, in order to present and discuss the last stages of a research project on representation. The result of this symposium is the book "*Problems of Representation in the Teaching and Learning of Mathematics*" (1987), where it is discussed the usefulness of the concept of representation in mathematics education.

The interest in the topic is specially shown by the existence in the *International Group for the Psychology of Mathematics Education* till 1995 of the *Working Group on Representations*. Goldin, who was the coordinator for this working group, expresses the general concern on this topic: "*Representations are a key theoretical* *construction in the psychology of mathematics education. The meaning of this term is quite broad and it includes:*

a) external physical embodiments (including computer environments): an external structured physical situation or set of situations that can be described mathematically or seen as embodying a mathematical idea;

b) linguistic embodiments: verbal, syntactic and related semantic aspects of the language in which problems are posed and mathematics discussed;

c) formal mathematics constructs: a different meaning of representation, still with emphases on a problem environment external to the individual, is that of a formal structural or mathematical analysis of a situation or sets of situations;

d) internal cognitive representations: very important emphases include students' internal, individual representation(s) for mathematical ideas, such as "area", "functions", etc., as well as systems of cognitive representation in a broader sense that can describe the processes of human learning and problem solving in mathematics" (1993).

From a semiotic approach Duval, from the University of Strasbourg, is working since the beginning of the 80's on the representation notion and on the understanding of mathematical objects; his work *Semiosis and Noesis* (1993) is a valuable contribution in this sense.

Nevertheless, we have not found any previous work on the representation systems for natural numbers neither on the understanding of the general term of a sequence that have been based on these several representation systems.

3.2. Aims and assumptions

The main aim of this work is to make clear the plurality of representation systems by which number structures are expressed. We maintain that each number system, as a complex set of entities, relationships and operations, cannot be expressed as a whole by only one representation system. Conventional number structures need the coordinated action of several representation systems in order to underline essential features of such structures. Particularly, graphic representations play an important role in understanding number structures. These are some of the conclusions in our work "*Exploring number patterns by means of point configurations*" (Castro, 1994). Here we study the integration of three representation systems for natural numbers in order to deepen on the concepts and procedures used by 12-14 years old students in relation to the notion of general term of a sequence of natural numbers.

4. Representation Systems for Natural Numbers

4.1 Decimal numeration system

Decimal numeration system is a powerful mathematical tool, the result of a long historical evolution, inspired by economical principles, not only semiotic but also operational, by which men have developed and expressed their counting, classifying, measuring and ordering skills. In our current society the domain of this system is a basic cultural fact; its knowledge establishes one of the criteria to determine that a human being has acquired the basic skills that allow him/her to hold a deserving intellectual position in society. That is why educational systems give such a value to transmitting and learning decimal numeration and basic arithmetic operations, using the decimal numeration system as the only one.

This is the way we come to identify each number with its decimal notation and the set of natural numbers with their Arabic notations. Such an identification, although culturally useful and economical, is still a limitation for the learning of natural numbers.

4.2 Arithmetical analysis

The dynamic character of the natural numerical system gets blocked by the inertia of the common decimal representation; its dynamism requires that numbers be determined by their intertwined relationships. So, knowing, for instance, what 15 means is not just reading it as 1 ten and 5 units, but also interpreting it as 3 times 5, 5 times 3, next to 14, preceding 16, the sum of two consecutive numbers: 7+8, the sum of three consecutive numbers: 4+5+6, the sum of five consecutive numbers: 1+2+3+4+5, coming before a square 4^2 -1, the sum of two numbers multiplied by their difference:$(4+1)\cdot(4-1)$, half of 30 and so on. From this point of view, each number is a knot in which several relationships intertwine, it is an element of a complex net closely connected whose wider or smaller domain will determine the real understanding reached by each subject in the natural numbers system (Rico, 1995).

The former considerations show that, on the basis of decimal notation, there are other representation systems for natural numbers. The *arithmetical analysis* of numbers is one of them; this analysis consists of considering each number as a sum or as a product of simpler numbers. The former examples are expressions of number 15 by means of its arithmetical analysis.

4.3 Graphical systems

Nevertheless, we still have not taken into account ways of graphic representation. The number line is the standard graphic representation. We choose two points arbitrarily on a line and we give them the values 0 and 1; by agreement, the point that matches 0 is on the left of the point that matches 1:

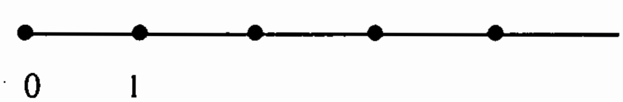

Fig. 1

From 1 to the right we write down points that keep the same distance between them than that of the two initial points and mark on them consecutive natural numbers. This representation is an useful tool to understand numbers, carefully studied for the domain of the natural numbers system (Resnick, 1983).

4.4. Point configurations

History sets us in touch with another powerful system of representation for natural numbers that has been ignored by the current school mathematical curriculum. We are talking about point configurations used to represent figurative numbers and whose origin and development was the Pythagorean number concept.

For those who followed the Pythagorean doctrine a number was not just a label for a collection, the symbol for a quantity or an intellectual construction, but something that was consistent by itself; numbers were like atoms that, in their varied compositions and relationships, gave the essence itself of the plural real world.

This number notion had its best expression in the representation that we know as *point configurations* or *figurative numbers*, completely different from the usual numeration systems.

The basic idea of this representation system is considering each number as an aggregate of points or units distributed on a rectangular or isometric scheme and according to a flat or space geometric figure. This way triangle, square, rectangle, pentagon, pyramid and cubic numbers come across, as many as different geometric figures are considered; they allow us to think in each number as a whole arranged with regard to a fixed geometrical structure

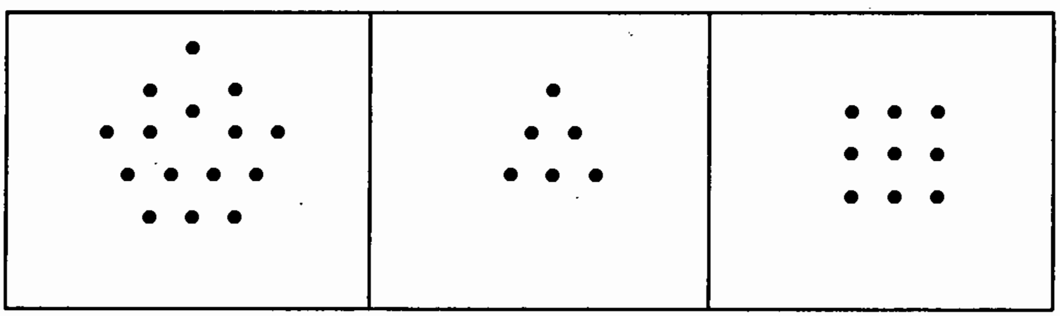

Fig. 2

To summarize:

Point configurations are a number representation system based on:

*a single symbol: the point;

*a structured space of representation, commonly the square or isometric scheme when working on the plane;

* a way of arranging the quantity of points that satisfies some agreed criteria of symmetry or regularity and that can be explicited easily.

These three conditions establish a new representation system for numbers (Beiler, 1966), whose advantage lays in providing graphic models that help to visualize and analyze the arithmetic structures of each number.

Two important pieces of information emerge when arranging, geometrically, the units that embody a number. On the one hand, we see an arithmetical analysis of the number: a triangular number is the sum of consecutive numbers starting from 1, a square number is the product of a number by itself, a rectangular number is the product of two consecutive numbers. This visual analysis allows us to know several properties of each number and relate it with many others. Besides, the same number could be considered as belonging to several kinds of figurative numbers.

On the other hand, different numbers share the structure that represents each kind of point configurations. The shared configuration shows several arithmetical analysis and each one becomes a common property for all these numbers; this property can be generalized. So the representation system of point configurations becomes a useful tool to establish general properties of numbers and find new relationships among them.

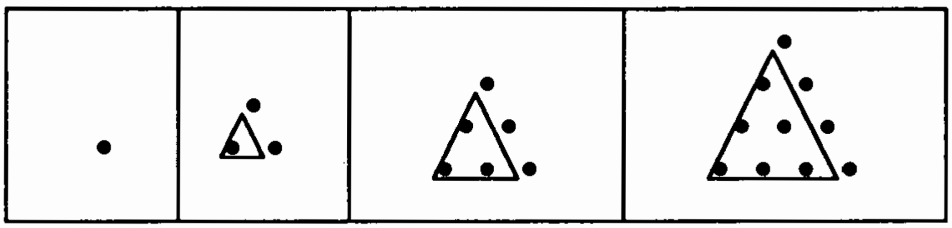

Fig. 3

Historically this representation system allowed mathematicians to establish general number properties and algebraic identities without having the current symbolic signs of algebra. We can find the use of figurative numbers along the Theory of Numbers history.

5. A curricular problem

5.1 Starting with sequences

The concept of sequence of natural numbers is a complex one; it is based on two notions: the one of a totally ordered set and the one of infinite process, by which every term of the sequence has a following one. When we have several terms of the sequence and we are challenged to go on with it, the proposal is to find new numbers related with the ones just known trying to use the same relations they have among them. There are several ways to relate a limited number of terms; to the question: "1, 2, 4, ..., which is the next term?" there is a multiplicity of feasible answers (Sloane, 1973). The possibilities of finding new relations among the terms of a sequence are decreasing when its number is increasing, and the options to continue the sequence could be reduced to a single one. Recognizing the relations among the given terms of a sequence can let the students find new ones, that is to say, continue the sequence. Nevertheless, the characterization of a sequence is given by its general term.

5.2 The general term of a sequence

To find and express the general term of a sequence offer some understanding difficulties, and there are many students who are not able to find a proper meaning for this idea because the high abstraction level implied on it. What does the general term of a sequence mean? The general term of a sequence is the algebraic expression of the rule that is followed by all its terms taking into account its corresponding ordinal place. The general term of a sequence expresses the common structure shared by all its terms when they are considered as members of an ordered set. The usual way to write the general term of a sequence is by means of algebraic notation. So, the formula $a_n = (n^2 + 2n)/2$ expresses that all the terms of the considered sequence can be found taking the square of its ordinal, adding up its double and dividing the result by 2. Nevertheless, this idea of a common structure or *the shared structure of all the members of the sequence* cannot be captured by the analysis of the relations among two or three consecutive terms.

To have several numbers written in the decimal system at your disposal does not allow us to observe the common structure they have; in order to know this structure is necessary to have the numbers written by a shared arithmetical analysis or, better than this, to have them expressed by means of point configurations following a single pattern. Triangular numbers visualization shows that the numbers 1, 3, 6, 10, 15, ..., share a common pattern (fig. 2). The arithmetic version of the pattern:

$$1, 1 + 2; \; 1 + 2 + 3; 1 + 2 + 3 + 4; 1 + 2 + 3 + 4 + 5; \ldots$$

advances the shared structure by means of the first numbers of the sequence: each one is the result of summing up consecutive numbers from 1 on till the corresponding number to its ordinal position in the sequence. But it is still necessary to consider many understanding phenomena to establish that the general rule of this sequence is, precisely, $a_n = (n^2 + 2n)/2$.

5.3 Curricular context

The work "*Exploring number patterns by means of point configurations*" (Castro, 1994) poses and studies the viability of a representation system for natural numbers, as an adequate tool for visualizing and analyzing sequences, similar to the graphic representation of functions, in the mathematical curriculum of the Compulsory Secondary Education. We study the strength of point configurations to express numerical relations and properties; we also study how students discover and use the numerical properties by means of such representations.

Our study is summarized in the following considerations:

* the coord'nated use of three representation systems for natural numbers: point configurations, decimal numeration system and arithmetical analysis or development of numbers;

* the work and reflection on the pattern by which linear and quadratic sequences are defined on terms of point configurations and arithmetical development;

* the performance of the following tasks: to continue a sequence, to extrapolate terms; to generalize; to find out the general term and use it to obtain specific ones.

6. Findings and discussion

6.1 Sequences and representation systems

Point configurations allow us to represent sequences of first and second degree taking integer values by means of a graphic display. So, arithmetic progressions allow simple point configurations, generally with rectangular shape and with constant base or height. They are called linear sequences because the pattern representation of their terms can be analyzed decomposing them by lines, and the difference between two consecutive terms can be described as the aggregation of a line.

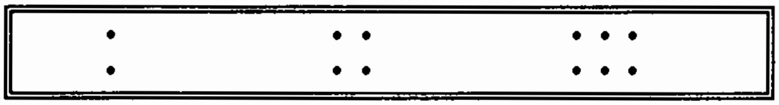

Fig. 4

The sequences with constant second differences are those whose general term is expressed by a second degree polynomial function. The simpler cases are the sequence of square numbers: $C_n = n^2$, and the sequence of rectangular numbers: $R_n = n(n + 1)$.

It is possible to make a graphical representation of these sequences having in mind that their two dimensions vary; the change from one term to the following is defined by its growth in both dimensions. The structure of these numbers is called quadratic and the change from one term to the following is not constant but variable with a linear variation.

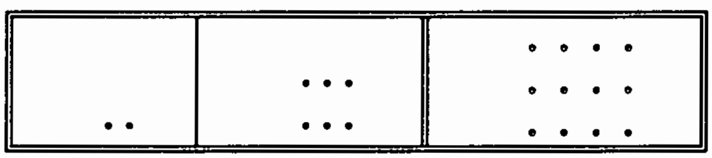

Fig. 5

In general, if the rule of a sequence is $a_n = an^2 + bn + c$, such a sequence has a constant second difference. If a_n takes natural values for every n, then each of its terms can be represented by a polygonal plane configuration, regular o irregular, and all its terms have the same pattern of representation.

6.2 The diversity of analysis

We have introduced 12-14 year old secondary education students to the symbolic representation system of point configurations. We have used these representations as an alternative symbolic system for the purpose to carry out the following tasks: to visualize the representation pattern shared by the terms of the sequence, to continue the sequence and represent some advanced terms with the pattern. Thus, in the example of figure 5, students recognize the geometrical shape shared by the three represented terms, they are able to add the two or three following terms and also to represent the 11th or 15th terms.

Likewise, point representation provides a structural analysis of the terms of the sequence and allows us to express new terms by means of the arithmetic analysis obtained. For the figure 5 example, there are several correct arithmetic analysis found by the students:

a) $2,\ 3+3,\ 4+4+4,\ 5+5+5+5, \dots$
b) $1+1,\ 2+2+2,\ 3+3+3+3,\ 4+4+4+4+4, \dots$
c) $2 \times 1,\ 3 \times 2,\ 4 \times 3,\ 5 \times 4, \dots$
d) $1^2+1,\ 2^2+2,\ 3^2+3,\ 4^2+4, \dots$
e) $2^2-2,\ 3^2-3,\ 4^2-4,\ 5^2-5, \dots$

We can observe that there is a variety of different analysis of the point configuration pattern; and each one provides a possible arithmetic development (sometimes additive and sometimes mutiplicative) which is shared by all the terms of the sequence. In this way it is possible to obtain several expressions for the terms of the considered sequence, with the representation system we have called arithmetic analysis. When the same sequence is displayed in the decimal numeration system: 2, 6, 12, 20, ..., students have not enough information to find a common arithmetic development for all these terms.

6.3 Findings

We have studied how the students understand and generalize the common structure that the terms of a sequence have using the established connections among the terms of the point configuration sequence and the terms of its arithmetic development. That is to say, we have tried to explicit the general term of a sequence notion that 12-14 year old students have by means of the question "How can we write the n-th term?" The answers to this question are different according to the representation system considered.

So, in the decimal number system, the most common expression given for the general term is **n**, which is the immediate symbolic translation for the expressions: "a number in general", "any number of the sequence", "any term of the sequence", or the like.

When the geometrical pattern is used, as it is necessary to leave some wide spaces between the points to indicate the generalization to **n**, this leads some students to change the model by a continuous shape for the general term. That indicates the difficulty of this representation system for expressing the general term.

In representations by means of the arithmetic development system we find that it is not difficult to move to the general term in a successful way. Nevertheless, when pupils have several arithmetic expressions about the terms of the same sequence it is not easy to accept as equivalents the general term expressions obtained.

A strong obstacle for finding the expression of the general term of a sequence has been detected in this study. Both, the point representations and the arithmetic developments, express in some way the structure shared by several numbers. The decimal notation of the same numbers does not allow us to capture the common structure.

When we ask for obtaining the general term of a sequence we really ask for the general expression of the common structure of all its terms, by means of an algebraic symbolism. Because in the number decimal system each term is shown by a single symbol and the common structure is not considered, the former question (how can we write the n-th term?) cannot be answered in this system. This fact explains that the most common given answer is "**n**", which is a single symbol and expresses "a general term". With the point configurations representation system it is possible to appreciate the common structure, but the concrete manner of such representations makes difficult to find the general term. Only with the arithmetic analysis system it is possible to generalize the expressions of the terms of a sequence.

The question: "Which is the general term of this sequence?" is a more abstract version of the question "How can we write the n-th term?" and it has its answer on the arithmetic development representation system but not so in the decimal system. In our study, there are very few students capable of understanding the question, because the most of times it is posed in the former system (decimal) and it must be replied necessarily in the latter one (arithmetic analysis).

Only with the integration of the representation systems, as different expressions of the same idea, it is possible to talk about the understanding of the concepts of a sequence and its general term.

7 Conclusion

We have introduced 12-14 year old secondary school students to the point configurations representation system. For that purpose we have used these representations as an alternative symbolic system to carry out a structural analysis of numbers sharing the same visual representation pattern; in this way we have obtained the arithmetic development shared by the terms of the same sequence.

Our work was focused on the study of linear and quadratic natural numbers sequences by using the three symbolic systems just mentioned: figurative, decimal number and operational or arithmetic development. Thus, it is possible to stress the development patterns of point sequences as well as number sequences. We have considered point configurations as models that express development patterns of number sequences, supplying the lack of visualization of these contents in the decimal system writing.

The results obtained in our study have highlighted that students accept, without difficulty, the point configurations system for numbers and they use it properly working with different geometrical models; students find a great variety of relations for triangular and squared numbers and they establish arguments to connect the geometrical pattern with its respective arithmetic translations by means of point configurations.

The data provided by the students, from the proposed tasks, have shown that the most intuitive of the three representation systems is the point configurations one, due to its graphic character, which favors a visual analysis and allows the processing of the quantity structure. However the maximum strength of this system is reached when it is conjointly used with the arithmetic developments and the current decimal number system. A point configuration is meaningful when it is used as the visualization of a singular arithmetic development for a specific number (or a family of numbers). The variety of developments suggested for the same point configuration shows the intuitive character of this system.

To the arithmetic level, the new symbolic system provides an operative character to the natural numbers, which is seen while carrying out the assigned tasks; so that a variety of arithmetic developments are performed for every number. The idea that there are numbers with the same *arithmetic structure* is also strengthened; this structure is visualized by a *geometric pattern* and it is expressed by an arithmetical development. This notion is a first step to the generalization of an arithmetic base.

A third aspect related to the students' knowledge clearly shown in the data analyzed is the richness of relations performed with numbers that share the same pattern.

We have checked that there is a weak integration among the three symbolic systems and which is specially clear in the low performance obtained with the tasks designed to express the notion of the general term of a sequence. There are very few students able to identify the general term of a sequence with the operative structure shared by the specific terms given for this sequence; this structure is understood more easily when it is expressed by its arithmetic development.

Though students are able to perform a variety of tasks with the new representation systems, we can say that the 12-14 year old students' understanding on the general term notion is virtually nonexistent, because we have not appreciated for this topic strong connections among the three representation systems neither any kind of structuration among the mental representations corresponding to the different representation systems used. The arguments given by students show that decimal system and arithmetic system are not clearly seen as two views of the same facts, and there are scarcely some translation rules between them. Only a few students, who coordinate more or less the three systems and incorporate them, show a kind of control for the general term notion, though several understanding levels are appreciated.

There is enough evidence to maintain our main hypothesis: the richness of the numerical structures and their complexity need several complementary systems to be understood; the contribution of graphics representations is essential to understand certain structural notions and to develop numerical thinking.

The integration of several representation systems has been necessary to show the difficulties of some concepts as it is the case of the general term of a sequence, and also to establish ways to overcome these difficulties through the understanding of the underlying structures.

Numerical thinking does not end with the study of the different representation systems, which are useful for the development of a concept, though this analysis is an unavoidable step as well as the notion of representation system. The understanding of numerical structures has a complexity that is still unknown and it needs to be carefully explored. This has been the aim of this study.

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MATHEMATICS TEACHER DEVELOPMENT: CONNECTIONS TO CHANGE IN TEACHERS' BELIEFS AND PRACTICES*

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Abstract

This paper synthesizes the work of the research team over many years as we have sought to identify critical aspects of staff development which translate into change in teachers' beliefs and classroom practices consistent with U.S. reform efforts. The research has been done with high school teachers involved in two staff development projects, "Building Bridges to Mathematics for All" and the "San José Mathematics Leadership Project." Using a variety of quantitative and qualitative research approaches, we have found several aspects of the programs positively related to teacher change: a support network as teachers tried to implement change; the opportunity for teachers to engage in extended conversation about teaching and learning mathematics; and the length of time in staff development.

Resumen

Este artículo sintetiza el trabajo de investigación de nuestro equipo durante muchos años en que hemos intentado indentificar aspectos críticos del desarrollo del personal docente que se traducen en cambios de actitud en los maestros y en los métodos usados en los salones para que éstos sean consistentes con los esfuerzos de reforma en los EEUU. Esta investigación se ha hecho con maestros de escuelas preparatorias involucrados en dos proyectos de desarrollo de personal, "construccion de puentes para la matemática para todos" y "proyecto de liderazgo para la matemática en San José". Utilizando diversos metodos de investigación cualitativos y evaluativos hemos encontrado aspectos de los programas positivavente relacionados con el cambio de los maestros: una red de apoyo al traves de la cual los maestros tratan de implementar el cambio, oportunidades para que los maestros discutan acerca de la enseñanza y del aprendizaje de la matemática y el tiempo que hay que invertir en qel desarrollo del personal.

Introduction

The inclusion of mathematics teacher development as a category for a research forum for PME 20 is an indication of how far we have come in research in this area in the last decade. In the last edition of the Handbook of Research on Teaching, the chapter on mathematics education (Romberg & Carpenter, 1986),

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hardly mentions research on teacher education. Grouws & Schultz (1996) also point out the lack of substantial research on mathematics teacher development and contribute the dearth of such studies to the length of time it takes to conduct them as well as the lack of funding support.

One of the critical issues being studied relative to staff development for teachers is how one creates a milieu which fosters change in teacher behavior in the classroom. A number of researchers have highlighted the importance of teacher beliefs to change in teacher behavior (Cooney & Jones, 1988; Ernest, 1991). As late as 1988 Grouws pointed out that there was little information available about the overall design features of inservice education programs which produce changes in teacher beliefs and classroom practices. He called for careful studies which focus on the impact of various features of inservice education on classroom practice. Since then there has been some work done, much of it descriptive but some theoretical as well. Our efforts in this regard will be described here.

Theoretical Background

Ernest (1991) has postulated a model relating one's views of mathematics, role as a teacher, and intended classroom outcomes. His three views of mathematics include: the instrumentalist view, that mathematics is an accumulation of facts, rules and skills; the Platonist view, that mathematics is a static but unified body of certain knowledge; and the problem solving view, that mathematics is a dynamic, changing field of human creation. Ernest has related these three views to the teachers' roles and intended outcomes in the classroom, as shown in the table below.

View of Mathematics	Teacher's Role	Intended Outcome
instrumentalist	instructor	skills mastery
Platonist	explainer	conceptual understanding
problem solving	facilitator	problem posing and solving

This model has informed our work as we have tried to relate teachers' beliefs, and changes in them, not only about mathematics but also about mathematics teaching and learning, to classroom practice.

Thompson (1992) also makes the point that one's view of what mathematics is affects how one teaches mathematics. An alternative conception of views of mathematics is presented in Lerman (1983). He identified two theoretical perspectives on the teaching of mathematics that are helpful in examining mathematics teacher development programs: absolutist and fallibilist. From an absolutist perspective, mathematics is based on universal truths and so is certain, value-free and abstract. From a fallibilist perspective, mathematics develops through human-made conjectures, proofs and refutations, and uncertainty is a part of the discipline as with any science. Lerman has connected these two views of mathematics to teaching, finding that teachers who had these two perspectives viewed teaching in very different ways. Absolutist teachers would tend to be directive, while fallibilist teachers would act more as facilitators in the classroom.

Obviously connections can be made between these two alternative ways of categorizing views of mathematics. Lerman's absolutist and fallibilist perspectives parallel Ernest's Platonist and problem solving perspectives. We have found that, in different contexts, it is helpful to have these slightly different lenses through which to view teachers' conceptions of mathematics and classroom behaviors.

However, while connections between teachers' beliefs about mathematics and their classroom behavior can be made, beliefs may be influenced by other factors in the context of the school and the classroom. As Cooney (1993) has pointed out, there are several metaphorical ways of examining teacher beliefs. Considered in different ways, teacher beliefs might seem to contradict classroom practice. For example (Cooney, 1993), a teacher might believe that technology should be used to teach mathematics and thus that students should be allowed to use calculators. This would be a primary kind of belief. However, the psychological strength of this belief may not be very strong, so that when faced with school or classroom impediments to implementation, the commitment to use of calculators diminishes. This view of beliefs has also informed our work as we have tried to ascertain not only what works to stimulate teacher change, but also what impedes it in the school setting.

Description of "Building Bridges to Mathematics for All"

This staff development program for high school mathematics teachers has been in operation since Fall, 1992 and includes several facets. In addition to a 13-day intensive summer institute, the project included five days of academic year full-day workshops, classroom coaching, and purchase of manipulative materials, software and classroom sets of graphing calculators. These aspects of the program were designed to help teachers as their districts eliminated tracking and placed all ninth grade students in algebra 1. During the summer institute, the teachers also visited summer bridge classes which were designed to help eighth grade students bridge more easily to algebra in ninth grade; these classes were taught by exemplary teachers who modeled thc reforms in teaching mathematics being promulgated in the institute. Figure 1 below shows how the pieces of the program fit together.

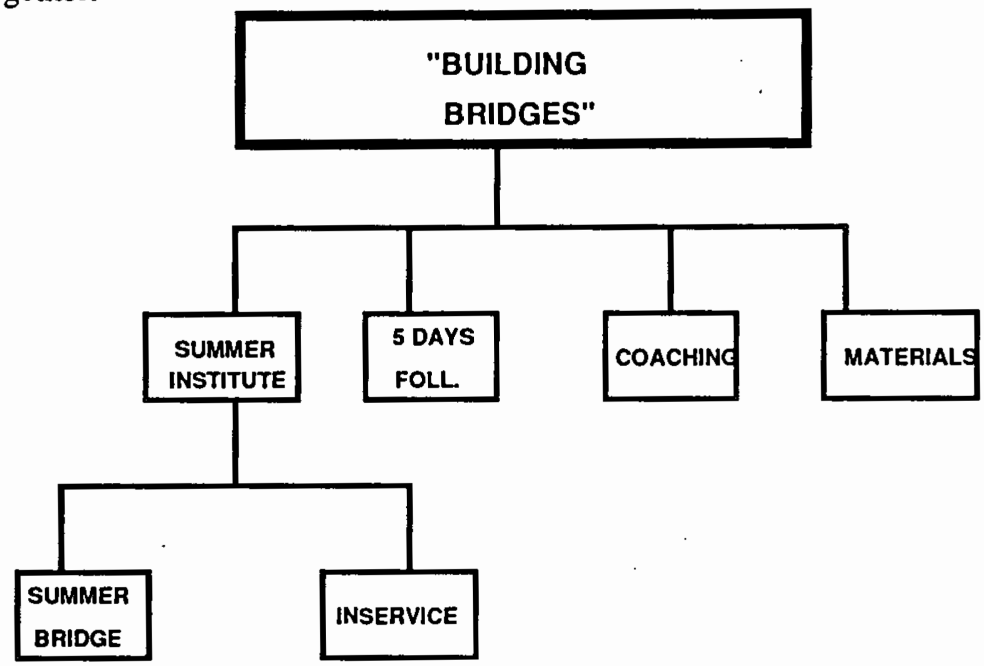

FIGURE 1

The teachers in this project participated for one year; the main foci of the workshops related to teaching algebra for all students, including use of innovative curriculum materials and technology such as graphing calculators and computers. Equity issues were also discussed. The main goals of the staff development were that:

- teachers will learn new ideas about how students learn mathematics, changing their instruction to a student-centered one using a variety of instructional modes to serve a diverse population;
- teachers will learn how their expectations of student achievement can make a critical difference on student achievement and what their role is in achieving educational equity for all students;
- teachers will learn new content and learn to extend content across topics and strands as needed for implementation of integrated courses.

The classroom coaching was our first attempt in a staff development program to connect what teachers were experiencing in the inservice education with what was happening in their classrooms. In the coaching, an attempt was made to ascertain those aspects of classroom behavior teachers were trying to improve; an informal "contract" was made with the participant teacher to collect data relative to that issue during classroom visits. Then the visitor (one of the staff) would have a post-observation visit with the teacher in which they discussed what was observed and collectively tried to determine ways to improve.

The San José Mathematics Leadership Project

Noting that considerable time is needed for teachers' conceptions of mathematics teaching and learning to change (Thompson, 1992), this project planned for a lengthier involvement of participants. In the Leadership Project teachers committed to either a three-year or two-year program of staff development (See Figure 2 below.)

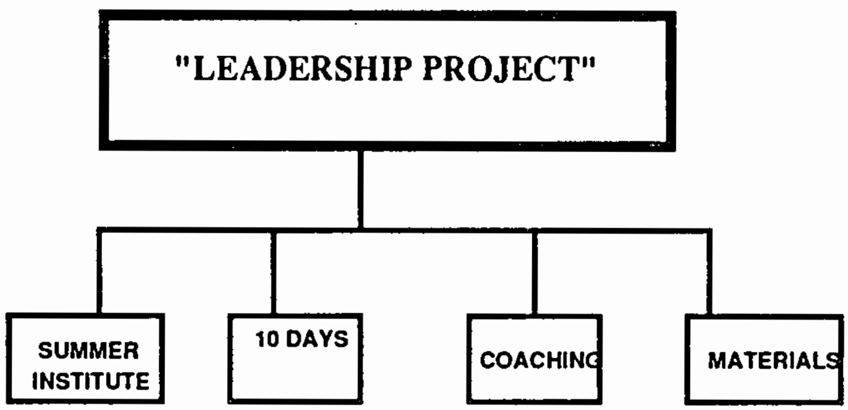

FIGURE 2

This project included a summer institute (16 days) and 10 days of academic year followup workshops, three of which were flexible so that teachers could use the time for other staff development of their choice or to visit their peers in their school or other schools. The program also involved classroom coaching and the purchase of materials such as graphing calculators or software.

The main goal of the Leadership Project was to develop teacher leaders who could go back to their schools and school districts and lead the continuing reform in mathematics education curriculum and instruction. In the first year of work with each group of teachers, we concentrated on careful examination of innovative materials for algebra 1/integrated first year of high school mathematics (hereafter referred to as Course 1). Information on programs such as Core-Plus, Computer Intensive Algebra, or the Interactive Mathematics Project can be found in the November 1995 issue of the Mathematics Teacher (NCTM, 1995). Technology was a heavy component as these new materials were examined, and teachers were encouraged to pilot materials and report back to the group their experiences.

Other major foci of the Leadership Project were: equity as the schools in our area continue to work for mathematics for all; assessment, as the teachers endeavored to incorporate a variety of alternative forms such as projects, journals and portfolios; and content as the teachers experienced some newer areas of mathematics, leading to their own professional growth as mathematicians.

In the second year of the project, geometry (integrated Course 2) formed the core of the curriculum reviews. Here the technology moved from use of graphing calculators, such as the TI 82 to use of dynamic geometry programs such as the Geometer's Sketchpad and the Cabri-Geometri on the TI 92 calculator.

In the third year, the first group of teachers had settled on a new curriculum to pilot and considerable time was spent sharing experiences with new curriculum and the accompanying pedagogy. Teachers formed focus groups to investigate issues of keen interest for them as individuals, and action research was begun in some classes.

Project Schools

The schools (≈ 20) involved in both of these projects are in the San José, CA metropolitan area, which has a high representation of Latino/Latina and Asian students. In fact, in many of the participating schools, "minority" students represent the majority of the student body, at least 65%. Many of the students are English-language learners; that is, their first language at home is not English and they may come to school knowing little or no English. Equity has been a major concern in all of California because minority students, especially African-American and Hispanic students, are underrepresented in the study of college-preparatory mathematics, in attendance and graduation from college, and especially in the pursuit of mathematics-related careers.

Methodologies

The research we have conducted with project participants has included a variety of quantitative and qualitative approaches to ascertain the impact of the programs on teachers' beliefs and classroom practices. One of the first items we developed was the Instructional Practices Scale (Becker & Pence, 1990), a Likert-type scale built on concepts from the National Council of Teachers of Mathematics' Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) (see Appendix A). We have used this scale as teachers entered the programs, at intermediate points, and at the end of the programs, and compared their responses across time. In particular, we asked teachers to project, at the end of a summer institute, how much they **thought** they would use each instructional practice, and then at the end of the academic year, we asked them how much they had **actually used** that practice so we could compare intent with actualization.

Other methods used were qualitative in nature, and included in-depth interviews with individual teachers, a focus group discussion with a group of teachers, individual interviews with the principals of the schools in which participant teachers worked, and daily teacher reflections collected at the end of each workshop. Results were triangulated where possible with data from the Instructional Practices Scale and observations during classroom coaching.

Questions

The following are questions which were primarily directed to teachers, although the answers might be informed by principal responses as well.

1. Which aspects of staff development are most important to participant teachers?
2. Was the staff development successful in changing teachers' beliefs about the learning and teaching of mathematics?
3. Did changes in teachers' beliefs accompany changes in classroom practice? Is it possible to determine which, if any, causes which? That is, if teachers change classroom practice and see success from doing so, does that help change their beliefs? Or must change in beliefs come first? Are changes in beliefs sufficient for change in classroom practice to occur?
4. What are impediments to change in teachers' beliefs and practices?

These are questions directed to principals.

General questions:

1. What changes have been noted in the way the math faculty has worked together over the past five years?
2. How have project participants had the opportunity to experiment with new curriculum and pedagogy?
3. Where did support for school experimentation come form?
4. What constituted any barriers and/or obstacles?
5. In what ways has the teaching of mathematics changed over the last five years?
6. In what ways has available technology been integrated into the math classroom?

Content questions:

1. What mathematics courses are available to incoming (ninth grade) students?
2. What support systems are available for students who have difficulty?
3. How is the school moving toward an integrated sequence of mathematics courses?
4. Have you noted any increase in the number of students taking college preparatory mathematics beyond the first year?
5. What provisions are made for special education and limited English proficient students?
6. What do you see as the next step to be taken in terms of math reform at this school?

Pedagogy questions:

1. If we were to walk into a math class right now, what kinds of things would we be likely to see?
2. How aware are your teachers of recent findings of how students learn? How is this knowledge influencing their use of new pedagogical techniques such as cooperative learning, problem solving, use of manipulatives, and the infusion of technology?
3. How are students assessed in their math classrooms?

Attitude questions:

1. How would you rate your teachers' understanding of equity goals?
2. How would you rate their commitment to those goals?
3. Which of those goals, if any, do your teachers feel most strongly about?
4. What changes over the last five years have you seen in the teachers' expectations for their students?
5. What do you consider the teacher's role to be in achieving and sustainig equity?
6. What role do you feel this project has played in bringing about the changes in content, methodology and attitudes that you have described to me?
7. What are the needs now, and how could such a project as this help to satisfy them?

Results

In this section we briefly discuss some of the major results from this ongoing research program. The results are centered around the four questions relative to the teachers; the principals' responses to questions posed to them are integrated as appropriate. Some of these results have been discussed in more detail in Becker & Pence (1991), Becker, Pence & Pors (1995), Peluso, Becker & Pence (in preparation), and Peluso, Pence & Becker (1994).

1. Which aspects of staff development are most important to teachers?

From the perspective of the teachers, the opportunity to network with other teachers was the most important part of their involvement in staff development. This network provided support, a "community of risk takers" as we have called it, as teachers went back to their school sites and began to change curriculum and pedagogy. Such networking is rarely available in the regular school day in US schools. An additional aspect of the projects of critical importance to teachers was the instruction relative to technology. It was obvious that technology was highly valued by teachers, because much of the materials requests were for technology such as graphing calculators, CBL units, and software.

From the perspective of the research team, the length and depth of involvement in staff development were clearly most important. In one of our studies, we found a clear relationship between <u>length</u> of time in staff development, which affects <u>depth</u> of involvement, and both the amount of change in teachers' views of themselves as teachers and changes in their instruction.

From the perspective of the principals, the projects provided the teachers with a common focus and increased commitment to ongoing conversation about how to improve the mathematics education of all students. Alternative means of assessment were being implemented, and the schools currently are experimenting with a variety of support systems to help students succeed.

2. Was the staff development successful in changing teachers' beliefs about the learning and teaching of mathematics?

From the principals' perspectives, their mathematics teachers had gleaned a good understanding of the goals of the "Building Bridges" project, that is eliminating tracking and increasing the numbers of African-American and Hispanic students succeeding in college preparatory mathematics to prepare them better for college. The principals interviewed felt that 100% of their staffs had such understanding. Commitment to the goals, however, varied from a low of 70% to a high of 90%.

Change in the beliefs about teaching and learning mathematics on the part of some participants is exemplified by the following quotes from individual interviews with two teachers in the Leadership Project, who had also previously attended "Building Bridges" for one year. The first teacher is a one-year participant in the Leadership Project; the second quote is from a three-year participant.

> *Karen: When I went to college the ideas as far as the changes in [the] teacher's role were taught to me and I started teaching with that idea. But I think I got a better idea by going to the summer institutes . . . I still do lecture, but I do have portions of it where they're working and I'm trying to help them. It's really changed my view in so many things - - is there really a vital thing that I have to be teaching any more?*

> *Belinda: It [staff development] has reoriented me to a new way of teaching . . . And it involves the kids more because they are more active in the learning. . . I just think the more involved they are the more they'll learn. Obviously it has affected me a lot because I changed from a lecture teacher to a facilitator.*

In contrast, Brandon is a teacher who only participated in the one year "Building Bridges" project. While his quote indicates his understanding of the new ideas the project was promulgating, there is little indication that he has fully endorsed those ideas.

> *Brandon: One thing I'm finding more of is that they want me, when I say they I guess the powers to be have decided how it ought to be done, more of a facilitator than someone just delivering information. I don't have a problem with that as long as we understand that I think some days need to be different.*

These quotes indicate that changes in teachers' beliefs proceed slowly. Belinda and Karen, while struggling to make changes in their teaching, were experimenting and trying various modifications in both curriculum and classroom organization. Even more important, they were reflecting on these experiments and using these reflections in their own professional growth.

3. Did changes in teachers' beliefs accompany changes in classroom practice?

Quantitative data relative to this question were collected on the Instructional Practices Scale (see Appendix A). This scale has been administered at various points in both projects, including before the summer institute, at the end of the summer institute, and at the end of the academic year. A sample of results is provided here (see Pence & Becker, 1994 for more details). On this scale we found four items on which teachers made substantial change: they used less memorization of facts in their instruction; they made more use of cooperative learning; their instruction became less textbook-driven; and they provided more instruction that interrelated various areas of mathematics. Perhaps more interesting, when asked at the end of the summer institute, teachers expressed their intention to use more instructional strategies consistent with the National Council of Teachers of Mathematics' Curriculum and Evaluation Standards for School Mathematics (1989) than they did in fact use the next academic year when we asked them to recall their actual practice.

Additional data relative to this question were collected through interviews and classroom coaching visits. Teachers were asked to describe a typical one of their classes, and a class that was most rewarding for them. We compared teachers who had been in our staff development for one, two, or three years. Brandon described a fairly traditional class as his typical day.

Brandon: . . . in a typical day, say it's inequalities and equations, is to give them a wide range of examples . . . I would do so many, because I engage them from the beginning, I would do so many and they would do so many primarily on the board . . . and then have at least 15 minutes for them to work on homework in class.

Another teacher who had been with the Leadership project one year is using a new curriculum material and much more use of cooperative learning.

Aretha: To begin my class I have a warmup question, one or two . . . take 10 minutes for the warmup . . . That's just a problem from yesterday, you know typical problems, a little review before I go on. And how I organize my classroom we work in groups all the time, all the time, except for the days we have tests.

Finally, Belinda was using two different innovative curriculum materials and two different instructional modes in her algebra and geometry classes.

Belinda: In the algebra 1 I do some lecturing as well as going around helping the kids on the computers. In the geometry, I circulate around the room and I listen and I make sure that everybody in the group has the same question before I'll answer the question.

Rewarding lessons revolved around lessons that had students more actively involved in instruction. For example, one teacher described a bouncing ball investigation, in which she gave students general instructions and allowed the students to structure the activity themselves. She found that even students who were not normally involved in class got interested and engaged by this activity.

4. What are impediments to change in teachers' beliefs and practices?

The Instructional Practices Scale was used to identify teachers who had increased scores from first to second administration, but then decreased considerably when asked at the end of the year if they actually made the changes they intended. These teachers (n=5) were involved in a focus discussion group in which this question and others were explored.

During this discussion, it was clear that had it not been for the "Building Bridges" project, the teachers would not have taken the risks necessary for change.

The project had developed a community of risk takers. However, implementation of recommended changes in curriculum and instruction was much harder than teachers thought it would be, and the level of staff development offered was not sufficient for full implementation to take place. The teachers felt that as innovators they were acting as salespeople with students, parents and administrators as well as other colleagues. They did not feel they had the skills in group dynamics to communicate effectively with all of these constituencies or counter the opposition that arose.

More planning and staff development time were highlighted as critical needs. Teachers need more opportunity for networking and sharing as they try innovations in curriculum, technology, and pedagogy. By sharing successes, the teachers felt better able to change the attitudes of reluctant colleagues who were unwilling to consider new ways of teaching.

The interviews with the principals identified three additional impediments to change. First, although the projects have supplied teachers with materials such as classroom sets of graphing calculators, many schools still are quite deficient in technology, especially computers. This is very frustrating to teachers who experience the power of software such as the Sketchpad or Cabri but are unable to use it because of lack of hardware availability. Second, principals felt the lack of substantial evidence that the new materials translated into increased student achievement. And third, it was apparent that some principals have minimal understanding of the mathematics reform movement, so that they are unable to provide critical leadership for fundamental change in curriculum and instruction

Summary

Our research program has substantiated many of the findings of Clarke (1994). For example, we found that involving groups of teachers from the same school rather than lone individuals helped provide the support system for change back at the school site. Changes in teachers' beliefs come slowly with in-depth involvement over a long period of time, but do accompany changes in classroom practice. And it is critical that teachers have time for reflection, planning and support to implement change in a school site that might not be committed to the same goals as they. As Clarke says, change is a gradual, difficult and often painful process; a support network is essential for ongoing professional development to occur.

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APPENDIX A

INSTRUCTIONAL PRACTICES SCALE

Please rate each instructional practice on the scale of 1-5 to indicate how frequently you used each practice in your classroom this past year. This non-evaluative information will remain strictly confidential.

	very freq.	often	sometimes	seldom	never
1. Teacher presentation of new content.	5	4	3	2	1
2. Active involvement of students.	5	4	3	2	1
3. Memorization of facts and procedures.	5	4	3	2	1
4. Extensive questioning of students.	5	4	3	2	1
5. Use of small group, cooperative learning.	5	4	3	2	1
6. Use of calculators and computers as instructional tools.	5	4	3	2	1
7. Problem solving as a means as well as a goal of instruction.	5	4	3	2	1
8. Student communication of mathematical ideas orally.	5	4	3	2	1
9. Student communication of mathematical ideas in writing.	5	4	3	2	1
10. Text as main source of knowledge.	5	4	3	2	1
11. Paper and pencil skill work.	5	4	3	2	1
12. Testing mainly to assign grades.	5	4	3	2	1
13. Interrelating two or more mathematical strands.	5	4	3	2	1
14. Use of projects that take more than two class periods to complete.	5	4	3	2	1
15. Assessment as integral part of instruction.	5	4	3	2	1
16. Use of hands-on, manipulative materials.	5	4	3	2	1
17. Class discussions.	5	4	3	2	1
18. Problem solving to introduce mathematical concepts.	5	4	3	2	1
19. Activities which focus on the whys and hows of mathematics.	5	4	3	2	1
20. Development of mathematics through an integrated approach.	5	4	3	2	1
21. Activities which stress the connections between mathematical topics.	5	4	3	2	1

Name________________________

TEACHER SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE: RESEARCH AND DEVELOPMENT

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This paper describes a long-term research and development project. Its major overall aims are: (1) to explore sources of a main component of pedagogical content knowledge: teacher presentations of subject matter; (2) to develop and study research-based teacher education programs aimed at promoting teacher subject-matter knowledge (SMK) and pedagogical content knowledge (PCK); and (3) to prepare research-based materials for use in teacher education. This presentation elaborates on each of these components of our project. Special attention will be paid to the impact of research on teacher education programs and to the participating teachers' learning processes, including the changes in their SMK and PCK.

In recent years, there is a growing awareness of the critical role of the teacher in changing the traditional ways in which mathematics has been taught and learned in schools. According to the current reform movement in mathematics education, teachers are expected to set mathematical goals and create classroom environments in which these goals are pursued. These environments must enable students to encounter, develop, and use mathematical ideas and skills in the context of genuine problems and situations; where the teacher chooses appropriate ways to represent the subject matter, asks questions, suggests activities and conducts discussions (NCTM, 1989, 1991).

This growing awareness of the teacher's role, along with increased attention to teacher knowledge and its development, led to changes in the design of professional development activities for mathematics teachers. Previous work with mathematics teachers tended to focus on implementing curricula developed by "experts". In recent years, however, there is a growing trend based on different guiding principles, a trend which both aims to enhance the professionalism of teachers and to empower the teacher as a decision maker. This trend is evident in the growing number of publications related to teacher education in professional journals as well as in the numbers of PME research reports and working groups that focus on teachers and teaching.

Our long-term research and development project, which began nearly a decade ago, is part of this trend. Its major overall aims are: (1) to explore sources of a main component of pedagogical content knowledge: teacher presentations of the subject matter (e.g., their planned explanations, their responses to students' questions, remarks and ideas); (2) to develop and study research-based teacher education programs aimed at promoting teacher subject-matter knowledge (SMK) and pedagogical content knowledge (PCK); and (3) to prepare research-based materials for use in teacher education.

We use qualitative analysis with the following research instruments: open-ended questionnaires that include non-standard mathematics problems and examples of students' work to be appraised and commented upon; interviews in which teachers are asked to respond to hypothetical students' questions or ideas, and to react to other teachers' responses to the same situations; in-class observations accompanied by lesson plans and post-teaching interviews; observations and video-documentation of teacher education meetings; assignments (including interviewing students) and portfolios prepared by the participants.

In this paper we elaborate on each of the three main components of our project. Special attention will be paid to the impact of research on teacher education programs and to the participating teachers' learning processes, including changes in their SMK and PCK.

Sources of Teacher Presentations of the Subject Matter

We study two main sources of this component of PCK: knowledge about the subject matter and knowledge about students.

Knowledge about the Subject Matter

The initial focus of our studies, as is typical of mathematics education research about teachers, was teacher SMK. We studied teachers' conceptions of various mathematical concepts and topics, such as infinity, rational numbers, functions, estimation and number sense. In these studies, we found the distinction between "knowing that" and "knowing why" useful. These two kinds of knowledge are frequently discussed in the mathematics education literature and there is consensus that the understanding of subject matter requires both types (e.g., Hiebert, 1986; Nesher, 1986). Nonetheless, whereas "knowing that" and "knowing why" are rather acutely defined in the literature, our research (Even & Tirosh, 1995) suggests that it neither straightforward nor easy to define "knowing". When it comes to a specific topic, its scope becomes vague, and teachers' SMK seems to depend heavily on the context.

The distinction between "knowing that" and "knowing why" contributes to our analysis of teacher SMK. Yet, it does not indicate necessary SMK characteristics for teachers in order to teach a specific topic. In response to this deficiency, an analytic framework of necessary SMK for teaching a specific mathematics topic was constructed and applied to the function concept (Even, 1990). The framework consists of seven aspects: (i) essential features of the concept; (ii) different representations; (iii) alternative approaches; (iv) strength of the concept; (v) basic repertoire; (vi) different kinds of knowledge and understanding; and (vii) epistemological knowledge about the nature of mathematics. This framework is both useful for research on teacher SMK and for the design of mathematics courses aimed to deepen and integrate teacher SMK of specific mathematical topics.

Our studies indicate that, in many cases, teachers' knowledge of mathematics influences their pedagogical content-specific decisions (e.g., Even, 1993; Even & Markovits, 1993; Even & Tirosh, 1995; Markovits & Even, 1994; Tirosh, Even & Robinson, in preparation).

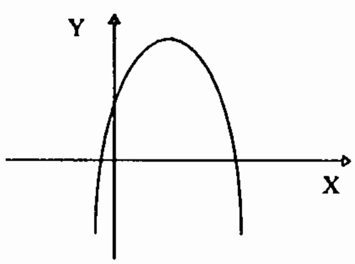
For example, in a study that investigated prospective secondary teachers' conceptions of functions (Even, 1989) they were presented with the following question.

****

This is the graph of the function $f(x)=ax^2+bx+c$.

State whether a, b and c are positive, negative or zero.

Explain your decision.



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Almost all the prospective teachers correctly stated that when the graph of a quadratic function looks like $\cap$, "a" (in the equation) will be negative. A vast majority of the subjects stated a rule as an explanation: "a < 0 since the graph opens downward." When asked to explain <u>why</u> the rule works, most did not explain the "why" but rather admitted that they just memorized it.

Not knowing why the rule works influenced the prospective teachers' pedagogical content-specific choices. The subjects were presented with a situation where a student asked them <u>why</u> "a" in the quadratic equation had to be negative if the parabola looked like $\cap$. Those who treated the relationship between the graph and "a" as a rule to memorize, suggested a nice exploration of quadratic functions with positive and negative "a"s so that the student could recognize the pattern. The problem with this approach is that it does not help the student understand <u>why</u> the rule works:

> I think that the best way to teach this is when you're having students graph them. Give them a whole set of these, interchanging negatives and positives... And then have them see if they can find the pattern. ...Have them make several different graphs. And then cut them out and try to put them in groups. ...Some

would, probably, group all the ones that are down together and all the ones that are up... And that way, kids can say, "What does this have in common?" And then, as a class, I think that they could come up with it. And then they would remember it. By reading it in a book I don't think they will.

By asking students to try several examples and detect a pattern, these prospective teachers ignored the fact that the student already found a rule, and they did not relate to the essence of the student's question: Why does this rule hold?

In contrast with the above approach, the following excerpt illustrates an approach based on better subject matter understanding--understanding why the rule holds. Both the above approach and the following one begin with the sketching of graphs. However, instead of suggesting the sketching of several graphs in order to find a pattern, the following approach suggests sketching the graph of graph of $y=x^2$ and then follow the change in the graph when the y-values are multiplied by a negative number:

> Begin with graphs of several parabolas, using $y = x^2$. And showing how that changes (going back to the translations) and having them realize they're essentially multiplying the y values by a negative so they are rotating the whole thing about the x-axis. And that what forces it to point down, the opening down.

The above example illustrates how teacher knowledge of the subject matter influences their ability to focus on the essence of students' questions. Our research also indicates the importance of this knowledge since lack of such knowledge impedes teachers' ability to determine the correctness of a student's answer, leads them to respond to students in ways which are mathematically inadequate, and to provide ad-hoc responses that do not take their long-term impact on students into consideration.

<u>Knowledge about Students</u>

In light of a constructivist approach to learning, the existence of an extensive body of knowledge on student conceptions and thinking in mathematics raises several issues concerning teachers and teaching. This component of our research focuses on three main questions: What do teachers know about students' ways of thinking related to specific mathematics topics? To what do they attribute them? Do they take this knowledge into account when designing and teaching specific topics? Substantial work on elementary teachers' knowledge of students' ways of thinking has been done within the Cognitive Guided Instruction framework (e.g., Carpenter, Fennema, Peterson, & Carey, 1988). Our studies focus on teacher knowledge of students' conceptions related to arithmetic, rational numbers, algebra and functions .

We have suggested (Even & Tirosh, 1995) that the distinction between "knowing that" and "knowing why" is useful when dealing with teachers' knowledge about students' ways of thinking. "Knowing that" in this context refers to research-based and experience-based knowledge about students' common conceptions and ways of thinking about the subject matter. "Knowing why" refers to general knowledge about possible sources of these conceptions, and also to the understanding of the sources of a student's reaction in a specific case. We use this distinction to investigate teacher knowledge about students' conceptions and ways of thinking in several mathematical domains. The teachers' responses and reactions are analyzed according to the following dimensions: (i) awareness of the student's conceptions, (ii) types of teacher responses (i.e.; concentration on the student's misconception, ritual versus meaning orientation, teacher versus student centered, short- versus long-term implications, richness of responses), and (iii) content knowledge (e.g., Even, & Markovits, 1993; Markovits & Even, 1994; Tirosh, 1993; Tirosh, et al., in preparation).

Our studies show a wide range of teachers' awareness of students' ways of thinking and common difficulties. Many simply ignore the former, and evaluate students' work as either right or wrong. Moreover, even when teachers do understand the student's difficulty, many respond in ways that do not support powerful constructions of mathematical knowledge. In doing so, they focus on mechanical thinking and achievement of short-term objectives--mastery of skills--, without considering the long-term implications of their subject-matter presentations. Even more profound is the tendency to try to "transfer knowledge"--teach by telling, where the student is expected to remain relatively passive. There is also a wide range of teachers' attitudes towards students of different age groups. Whereas most elementary school teachers phrase their responses in ways to make the children feel good even if their answers were incorrect, the junior-high teachers relate to the correctness of the children's solutions alone, not taking their feelings into consideration.

Teacher Education Programs

Our research findings, as well as those of others, have become an integral part of several teacher education programs that we are currently developing. These programs are accompanied by extensive research which affects both the actual operation of the programs and our on-going study of teacher SMK and PCK. Accordingly, special attention is paid to the impact of these programs on the development and changes in the participants' SMK and PCK. Following are brief descriptions of three of these programs, followed by a more detailed description of a fourth program.

Mathematics Classroom Situations

One major component of a one semester course aimed at improving PCK of elementary school teachers (Markovits & Even, 1994) is devoted to the analysis of "mathematics classroom situations", both actual and hypothetical, in which the teacher responds to a student's question or idea. The situations were selected to highlight students' ways of thinking and misconceptions, as known from research and personal experience. Another main component of the course concentrates on exposure to research on understanding how students learn. Several studies and articles on learning and thinking in mathematics are presented and discussed with the teachers. Special attention is directed toward learning as construction of knowledge, as opposed to transfer of knowledge from the teacher to the student. Different learning styles and teaching methods are also discussed, emphasizing their potential contribution to student reasoning and justification. During the course, the teachers explore students' ways of thinking about mathematical situations and teachers' explanations by interviewing students.

The research, which is in its third year, investigates the participants' PCK and its development. It focuses on examining their individual responses to mathematics classroom situations, as well as their reactions to other teachers' responses to the same situations. We also analyze their ways of discovering how students think, as expressed in the interviews conducted by the participating teachers.

Improving Mathematics Teaching: A Teacher Development Approach

This project, currently in its fifth year, aims to improve teaching mathematics in elementary schools in a development town by fostering teacher development (Markovits & Niv, 1995). The teachers from all eight schools in the town participate in the project, that focused on: a) enhancing SMK and PCK of the participating teachers, b) developing local leaders, c) guiding local leaders in their work with teachers. The project activities include: workshops for the entire group, for each school team, and for the selected local leaders; observations and discussions of lessons; individual assistance for the leaders in their work with teachers and preparation of materials. The research focuses on changes in the teaching of mathematics as conceived by the teachers themselves, by the supervisor and the school principals, and by junior-high teachers. We also trace the formation of the local leading team.

Teacher-Leader Preparation

There is a need to help successful experienced teachers to advance beyond their excellence as teachers to become teacher-leaders (e.g., teacher-educators, teacher-mentors, school mathematics coordinators, staff of projects for improving mathematics teaching). A three-year project for teacher-leader preparation, material development and research was therefore designed (Even, 1996). Now in its final year, the program includes: a) providing academic background in mathematics education (at graduate level); b) methods of integration of new technologies; c) enhancing SMK and PCK; d) developing leadership skills. It operates in four interconnected mutually supporting modes: (1) theoretical presentations and workshops for the whole group, (2) workshops to prepare teacher development activities and to support initiation of change in school mathematics teaching and learning, conducted in teams, (3) practical work with teachers, and (4) preparation of individual portfolios. The research focuses on several dimensions of the participants' growth, including development of participants' awareness of students' ways of thinking and changes in characteristics of teacher development activities conducted by the participants.

SMK and PCK of Rational Numbers

This one-year course for preservice elementary teachers is aimed primary to promote their algorithmic, formal, intuitive and psycho-didactic understanding of rational numbers (Tirosh, Fischbein, Graeber, & Wilson, 1993). The main concepts related to rational numbers and to operations with numbers are taught from various perspectives, including mathematical, historical, psychological, and didactic. Common systematic difficulties are analyzed, their sources are discussed, and various suggestions for dealing with them are made by the preservice teachers and instructors. Several articles on children's conceptions of rational numbers are used as sources for information about alternative conceptions, and as a basis for suggestions for ways of helping students overcome their misconceptions. The first version of this course was developed in 1990, and has since been team-taught yearly by two college instructors. The research focuses on changes in the participants' mathematical and psycho-didactic knowledge of rational numbers. Revisions in the course are made in accordance with the results of research on its effects, and the reported findings of studies on children's and preservice teachers' conceptions of rational numbers. Typical sessions begin with an opening activity in which preservice teachers solve mathematical and/or pedagogical content problems individually or in groups. These problems, which are aimed to expose the teachers' own ways of thinking, constitute the basis for meaningful discussions about the mathematical, pedagogical and didactic aspects of the domain of rational numbers.

Here, we present one activity about addition of fractions and discuss our impressions about using this activity with classes of preservice elementary teachers. In developing this activity, we used research findings on children's ways of thinking about addition of fractions as a means of enhancing prospective and/or in-service teachers' SMK and PCK of rational numbers.

There are several possible reasons for students' tendency to add fractions by "adding the tops and the bottoms", including (1) Viewing fractions as representing quantities but seeing them as four separate whole numbers to be combined in one way or another. Each fraction is viewed as two numbers separated by a line, and it seems reasonable to add the numerators to obtain the numerator of the sum and to add the denominators similarly (Carpenter, Coburn, Reys, & Wilson, 1976); (2) Confusing the rule of adding fractions with that of multiplying fractions (Herskowitz, Vinner, & Bruckheimer, 1978); (3) Believing that the adequate way to perform addition is to add "like items", that is, numerator to numerator, and denominator to denominator (Herskowitz, Vinner, & Bruckheimer, 1978); and (4) There are some life situations in which such a way of operating is appropriate (e.g., Mochon, 1993).

This activity is called " Can fractions be added in an easier way?" It includes four tasks, a description of two follows.

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Task 1

Multiplication of fractions is simple. Addition of fractions is much more complicated.

Why wasn't it decided to add fractions by "adding the tops and adding the bottoms?" Try to give more than one reason.

Task 2

Ran, Dan, Dorit and Alon solve the problem $\frac{5}{8}+\frac{7}{12}$ in the following way:

$$\frac{5}{8}+\frac{7}{12}=\frac{5+7}{8+12}=\frac{12}{20}$$

Ran explains that "It is reasonable to add the numerators and the denominators, as in this way we do the same thing in both addition and multiplication."

Dan explains that "When we add, we always add things of the same kind, for instance, ones to ones, tens to tens. In this case, we shall do the same; add numerators to numerators, and denominators to denominators."

Dorit explains that: "It is easy and simple to add this way."

Alon gives an example: "In a basket-ball game, Miki Berkovitz [a well-known Israeli basket-ball player] hit, during the first part of the game, 5 out of 8 attempts, in the second part, he succeeded in 7 out of 12. So, during the entire game he hit 12 out of 20 trials, that is

$$\frac{5}{8}+\frac{7}{12}=\frac{5+7}{8+12}=\frac{12}{20}$$

Assume that you are the teacher in this class. How would you react to each of these responses?

****

Prospective teachers who participated in this activity were asked to think about each of these tasks. Then, for Task 2, they were asked to work in pairs, to simulate the four situations, each teacher acting as the student in two of the four cases, and as the teacher in the other two cases. Afterwards, they presented their simulations to the class.

This activity opened the stage to a thorough discussion on algorithmic, formal and intuitive aspects of addition of fractions and their interconnections. A large part of the discussion was devoted to various issues concerning the nature of mathematical operations and definitions, children's conceptions and ways of thinking and the learning and teaching of mathematics. We briefly describe here some of the issues (and emotions) evolving in the class.

The prospective teachers argued, in response to the first task, that it is impossible to add fractions in the suggested way (i.e., adding the tops and the bottoms). They were surprised to encounter such a question, and commented that they never thought about the sources of the definitions of operations, or about the possibility of an alternative definition totally different from the one currently used. Aviv, for instance, explained that "Such questions should not be asked. Addition is defined in a certain way. There are some rules in mathematics, and we operate accordingly. These rules often do not seem reasonable, but still we have to operate according to them... We have to accept these rules as they are, and not wonder about them."

Other prospective teachers used two types of arguments to justify their conclusion that addition could not be defined in the suggested way: mathematically-based arguments and practical arguments. Prospective teachers whose reactions were classified as mathematically-based explained that the suggested definition does not satisfy certain mathematical principles (e.g., "$\frac{1}{2}+\frac{1}{2}$ should be greater than $\frac{1}{2}$, since you added a positive number to a given number. But, if we use the suggested way to add these numbers, we get $\frac{2}{4}$, and $\frac{2}{4}$ is not greater than $\frac{1}{2}$"). Practical arguments showed that the suggested addition does not match real-life situations in which addition of fractions is an appropriate operation (e.g., "If we consider the recipe for a cake, in which $\frac{1}{2}$ kilogram wheat is added to $\frac{1}{4}$ kilogram sugar, altogether we'll have $\frac{3}{4}$ kilogram, and not $\frac{2}{6}$ kilogram").

The second task was very demanding for the preservice teachers. The request to react to students' conceptions and ways of thinking encouraged them to reflect on their own conceptions of addition of fractions and to appreciate the complexities of understanding and explaining this operation. While working on the simulation, they raised various issues and discussed them among themselves. These issues related to <u>addition of fractions</u> (e.g., Sometimes the correct solution to $\frac{5}{8}+\frac{7}{12}$ is $\frac{29}{24}$, at other times it is $\frac{12}{20}$. However, mathematical operations should result in only one number. What should be done in such situations?); <u>definitions of operations</u> (e.g., How do mathematicians make decisions about definitions of operations? Should definitions be proved? What is the relationship between mathematics and real-life situations?); <u>children's ways of thinking and instruction</u> (e.g., What are the sources of students' intuitive solutions to addition of fractions? Should we allow students to listen to incorrect, apparently reasonable suggestions made by other students in the class? What type of justifications should be used in elementary classes?)

Similar activities were developed and tested in each of the four teacher education programs described above. Our impression is that such activities have several potential uses in teacher education. First, they can serve as a means of exploring teachers' conceptions and understanding of mathematical concepts, and their beliefs about mathematics and mathematics instruction. Our findings indicate that when asked to respond to specific suggestions made by students, teachers are pushed to articulate their own understanding. Thus, in turn, they provide teacher educators with an opportunity to study adult learners' cognitive processes and conceptions.

Secondly, such activities can serve as a means to raise teachers' awareness of their own inadequate conjectures, and to encourage modification. Our findings indicate that teachers' exposure to students' suggested definitions encouraged many of the former to clarify their knowledge, to reflect on their responses and to reconsider and re-evaluate their judgments.

Thirdly, these activities serve as a springboard for discussing a number of central issues related to the nature of mathematics and mathematics instruction (e.g., the nature of the arguments they would like to use in their future classes and the relationship between stages of development, maturity, and types of justifications).

Finally, activities which are based on students' known difficulties, serve as a means of increasing teachers' awareness of students' understanding and misunderstanding, and thereby improving their PCK. The teachers were curious to learn about other students' conceptions related to specific topics they intended to teach.

While working with pre-service and in-service teachers in each of the four above described programs, attention was also paid to the social interaction and the social climate in the class, and, in particular, to the teacher's role in attempting to encourage students to describe their arguments in a way comprehensible to the entire class. The teachers acknowledged that the open, accepting and non-evaluative atmosphere created a feeling that it was legitimate to make mistakes. Moreover, they felt that these mistakes often enhanced the entire class's mathematical understanding as well as their familiarity with children's ways of thinking. It was our impression that the teachers realized that knowing mathematics, much like other types of human knowledge, is not a clear-cut matter; there are different, legitimate ways of exploring mathematical situations. Perhaps most important: the teachers saw that they themselves can enjoy learning mathematics.

Materials for Teacher Education

Two of the teacher education programs described above are accompanied by development of research-based resources. These materials are for preservice teachers, in-service teachers, and teacher-leaders. We briefly describe one (Rational Numbers for Elementary Teachers) and further elaborate on the second (Teacher-Leaders' Resource Files).

Rational Numbers for Elementary Teachers

One product of this course for preservice teachers is the development of a book for use in preservice and in-service elementary school teacher education (Fischbein, Tirosh, Barash, & Klein, in preparation). The book presents various general issues related to the nature of mathematics and mathematics education (e.g., mathematics and reality, mental models and mathematics education, consistency and mathematics education, the concept of schema and its relevance to mathematics education). It also discusses the main concepts related to rational numbers and to operations with numbers from various perspectives (mathematical, historical, psychological, and didactic).

Teacher-Leaders' Resource Files

As part of the Teacher-Leaders' Project, resource files for main curriculum and didactic topics (e.g., algebra, analysis, heterogeneous classes) are being developed by the project staff in cooperation with course participants. These materials focus on central topics and domains of mathematics teaching and learning. They are field-tested throughout the course. Making the course participants part of the development of the materials contributes to their motivation to use the materials, provides a "test-bed" for the materials, and ensures that the materials are relevant to the needs of the people in the "field".

For illustration we describe the "Algebra Resource File" (Hirshfeld, Robinson, Radai, & Even, 1996) recently completed. The file focuses on a number of ideas and issues that seem central to the contemporary discourse on school algebra and the attempts to reform school mathematics. The main components of the file are:

- Historical view on the development of algebra.

- Various (and sometime confusing) meanings of letters in algebra.

- Students' conceptions of algebraic concepts, emphasizing an operational approach and a structural approach.

- Characteristics of a "good problem" in school algebra and ways to design such problems and activities.
- Teacher role in algebra classes.

The "Algebra Resource File" contains 11 detailed suggestions for teacher development meetings that touch on mathematics, cognitive and didactic aspects of teaching and learning algebra. Each suggestion contains brief, "user-fricndly" theoretical background on the suggested meeting topic, includes an instructional guide and teaching aids (e.g., worksheets, transparencies, video clips for illustration and discussions). The suggested meetings in the "Algebra Resource File" exemplify four models for such meetings:

Model 1: Raising a research question, presentation of a relevant study, discussion of the results, and actual replication of the study.

Model 2: Presentation of a pedagogical-content question, working on a related task, crystallizing components for a framework for the question, closing the circle--re-discussion of the opening question.

Model 3: Presentation of written and/or video documented teaching/learning events, analysis of the events, conclusion.

Model 4: Working on a task, reflective discussion on the task, connecting to learning processes.

All of the suggested teacher development meetings included in the "Algebra Resource File" were tested in several in-service courses. Following is a description of teacher development meetings based on the first model as conducted in the "teacher-leader" program.

Several of the course meetings were based on the first three components of Model 1. In these meetings, research questions focused on learners' (students and teachers) ways of thinking when working in algebra were raised, and relevant studies were presented and discussed. Later in the year, the fourth component came into play. The participants were asked to replicate a study presented in the course and to submit a written report.

Most of the participants emphasized that replicating a study helped them to understand students' ways of thinking when learning specific topics. Alma and her colleague, for example, chose to focus on generalizations and justifications in algebra. They based their mini-study on Lee and Wheeler's (1986) study, which examines the types of justification students apply to their judgments of the truth or falsity of propositions in an algebraic situation. In the introduction to their paper, Alma and her colleague explained that this topic was presented during the course from different perspectives, generating curiosity about how their own students would behave in problem situations in which one has to generalize and justify. They especially wanted to learn to what extent algebraic tools are used by students for these purposes since common textbooks do not do enough to develop inductive reasoning and the need to justify generalizations. In her interview, Alma explained that the results of heir study (which was conducted with 7th and 8th grade students and teachers) were so interesting that she decided, in addition to the course requirements, to "compare students from 2nd grade up to grades 7 and 8." This comparison helped her "understand what is going on in elementary school."

In their reports, some of the participants emphasized their surprise about the results of their study. Sarah, for example, declared in her interview, that she found out that the students exceeded her expectations:

> Even though I have taught for 30 years, I was surprised by some of the things that we found about the group of students we studied. The students reached much higher levels of thinking than what I would have expected. So it was very interesting.

Dalia, on the other hand, pointed out how she wanted to "prove" that her students would do better than the ones in the original study, because she (and the other teachers in her school) finished teaching them the material just several months before her study was carried on. However, she was surprised:

> We did a replication of the lecture. Simply, I was amazed by the results. I said, well, this is a topic that we mention at school... It was several months after we had taught the material. And I said, OK, no problem. Our students, for sure, would know better than those students at the university. And we were shocked that actually with us it was the same as there. That was the interesting part. We also shared with the senior-high teachers on this topic. This was an interesting part.

Most of the participants felt that they learned a great deal from doing the task. However, there were also some who did not think it was a worthwhile task. Betty was the only one who really resented the task. She thought there are no benefits replicate a study. In her interview she stated,

> I don't understand what I should get from a replication of a study. If the study was already done, it has findings. It was done by much more reliable people than you or me. So why should I work on this?

Nonetheless, most of the participants became very involved in conducting the mini-research. Almost all of them used the original study only as a base for developing a study which would help them answer questions that were important to them. Sarah, for example, explains why it was important for her to replicate a study:

> In a mini-research, in contrast to an article which is completely theoretical, you have question marks about the findings. Could it be like this? Is it only a coincidence that this happened? Will it happen to students I know? My students? It is very interesting to see what really happens. To duplicate the study and see, to support the original findings or refute them...
>
> When you read a research article, it is one level of depth. When you have to re-do it and implement it again, it is another level. I mean, what I know now about the study, about its hypothesis, its findings and the theoretical material, I

certainly wouldn't know after reading it once or twice or even if I had summarized it--it is much more. It became mine.

In sum, most of the course participants said that they enjoyed the meetings in mathematics education research and learned a lot from them about student thinking. As we can see, actually replicating a study that was presented at the course helped the participants better interpret the findings, learn about their "flesh and blood" students, and, as one of the participant put it, "It became mine."

So far we have described one model of the suggested teacher development meeting that can be used by teacher leaders/educators. The participating teachers commented that the other three models where useful as well.

In general, it should be clarified that there are almost no structured resources in Israel to be used in mathematics teacher education programs. In our project we continue to develop different types of such resources.

Conclusion

Research about learning and learners, and research on teaching and teachers have been following separate tracks for a long time. Recently these two tracks have began to intersect in several ways, three of which are demonstrated in this paper. The first is an extension of cognitive studies on students' conceptions and ways of thinking about specific mathematics topics. We investigate teachers' own understanding and their conceptions of students' ways of thinking of several mathematics topics, and describe sources of teacher presentations of the subject matter. The second is the development of research-based teacher education programs. We trace the impacts of the participation in various models of teacher education programs (all focused on students' ways of thinking in mathematics) on teacher knowledge growth and professional development. The third is the development of the prototype materials for teacher education programs that extensively use cognitive research on students' conceptions .

So far, most of our research was done outside of the classroom setting. At this stage we feel that the time is ripe to follow the participants of our teacher education programs into their own classes, and to study the long-term effects of their education on their actual teaching. The next stage of our research will deal with this central issue.

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WORKING GROUPS

Working Group

Teachers as Researchers

Chris Breen (University of Cape Town), Vicki Zack (St George's School and McGill University), Judy Mousley (Deakin University).

The General aims of the *Teachers as Researchers* working group of PME are to engage participants in discussions about the work of teacher-researchers, to review issues surrounding this work and its contexts, and to facilitate and promote collaborative work in this area.

The group explores the dialectical relationship between teaching and classroom research in the belief that teachers can and should carry out research in their classrooms and that mathematics educators should research their own teaching and its effects in broader fields. Discussion and other activities generally relate to teaching as a reflective practice and continuous learning process, the nature of the theory/practice interface, the critique and dissemination of research findings from various contexts, the types of research problems being generated in classrooms and methods of finding solutions within the context in which questions arise.

The 1995 working sessions focused on the challenges facing teachers in different world contexts and contributions from South American countries were particularly welcomed as they highlighted some of the problems facing teacher-researchers in those countries. At the end of the programme, the group started a process of framing contributions which would be used to form the basis of a publication which would reflect the work of this group over the years at PME.

In 1996 the group will continue to discuss items falling into the general brief of the group but will also specifically focus on the research problems facing teachers in challenging contexts demanding change. Where possible this will be done by means of task-driven activities rather than through talks so that we can try to take maximum benefit of the combined wisdom and experience of participants. We will also continue planning on the book and attempt to broaden the base of participation. New participants are welcome to attend and participate in the group sessions.

WORKING GROUP ON

CULTURAL ASPECTS IN THE LEARNING OF MATHEMATICS

Foundations

Although the process of learning mathematics takes place in the school environment, this educational process cannot be isolated from the child's cultural context.
In other words, school mathematical knowledge is a product of schooling based on explicit patterns (day-to-day customs, practices), filtered through the implicit patterns (cognitive activity) of the culture(s) of the students. The cultural factors can interact at a collective or individual level.
The group is concerned with all studies that take into account the effect of cultural factors on cognitive processes in the learning of mathematics.

Objectives of the group

With respect to cultural aspects in the learning of mathematics, our aims are:

- To exchange information about relevant research
- To encourage interdisciplinary studies (across for example psychology, sociology and mathematics education).
- To contribute to the formulation of a methodological and theoretical framework useful to the PME group.
- To identify issues for further research.
- To produce a joint publication on relevant research in this area.

Current work

For PME XIX, a number of the members of the Group contributed draft papers to a Booklet which formed the working document for the sessions. These papers dealt with previous research, research in progress or were literature reviews. As a result of discussions during PME 19 and subsequently it became clear that many of the contributions addressed the following question:
Considered in cultural context, what is *real mathematics* ?
This theme will be pursued during PME 20. The interrelationships between academic mathematics and the mathematics evident in various cultural practices and in everyday usage will be explored. Improved drafts of the PME 19 papers together with solicited additional contributions will be collected in a second edition of the Booklet.
These papers will be grouped into sub-themes. These sub-themes will be dealt with in the sessions of PME 20 through the Group responding to, developing a critique of and thus further refining the contents of the Booklet.

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RESEARCH ON THE PSYCHOLOGY OF MATHEMATICS TEACHER DEVELOPMENT

The Working Group *Research on the Psychology of Mathematics Teacher Development* was first convened as a Discussion Group at PME 10 in London in 1986, and continued in this format until the Working Group was formed in 1990. This year, at PME 20, we hope to build on the foundation of shared understandings which have developed over recent years.

Aims of the Working Group

The Working Groups aims to:

- develop, communicate and examine paradigms and frameworks for research in the psychology of mathematics teacher development;
- collect, develop, discuss and critique tools and methodologies for conducting naturalistic and intervention research concerning the development of mathematics teachers' knowledge, beliefs, actions and reflections;
- implement collaborative research projects;
- foster and develop communication between participants;
- produce a joint publication on research frameworks and methodological issues.

Current Working Group Activities

A strong feature of the Working Group for Research on the Psychology of Mathematics Teacher Development in the last three years has been its cohesiveness, and its wide representation across many countries.

Group members have been discussing different approaches to research on mathematics teacher development, and sharing results from their research. In 1995, the Group focused its discussion on papers which will form the basis of a book whose working title is "Research on Mathematics Teacher Development: An International Perspective."

The book provides a unique opportunity to bring together the research expertise of mathematics educators currently working in the area of mathematics teacher development in a wide range of different cultural contexts, and to disseminate both their approaches to research and their findings more widely.

The Working Group Sessions in 1996 will concentrate on finalising submissions for the book, on coordinating the Group's presentation for ICME-8, and on building plans for the future work and directions of the Group.

Nerida F. Ellerton, Convenor

Geometry Working Group

The last meeting of the Geometry Working Group (Recife July 1995) was devoted to introduce the discussion of a new theme: "Different external representations in the geometrical field: their dialectic relationship with geometrical knowledge."

The activity of the Group was centred on two presentations; they were devoted to introducing the participants to two different approaches to geometry through the use of different systems of representation. One of them consisted in "linkages" (a workshop was held by M. Bartolini Bussi) and the other consisted in a specific didactic software (Paula Moreira Baltar briefly but effectively presented the microworld CABRI).

The two presentations succeeded in stimulating the discussion focused on the comparison of the two different approaches as far as the different theoretical frameworks are concerned. The interest shared by the participants, the complexity of the questions arisen suggested to continue our discussion on this theme.

Although it seems nearly impossible to conceive geometry without 'figures', the ambiguity of the term *figure*, as often pointed out, focuses the deep link between the two aspects, the mathematical object and to its 'concrete' representation and, and witnesses the interrelation between images and geometrical ideas.

Historic analysis shows the basic contribution to geometrical theorization given by experiences and theories about graphical representations provided by outstanding artists and artisans.

The availability of purposeful software, which provide images differently linked to geometrical topics raise the problem of images anew and open new perspectives on geometrical education. But, using new technologies does not exhaust the complexity of the problem. At the same time, although their presence adds new elements to the analysis, focusing to computers risks to hide the rich contribute coming from other sources. The analysis of the interaction between external representations and geometrical knowledge rises a main issue

> how to use the dialectic relationship between external representations and geometrical knowledge in the educational field.

It would be interesting to hear about the current research on this topic, thus those who are directly involved in this field of research are invited to participate and contribute.

Whoever is interest in contributing to the discussion, please contact us at the following addresses:

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Advanced Mathematical Thinking

The AMT working group is concerned with all kinds of mathematical thinking, developing and extending theories of the psychology of Mathematics Education to cover the full range of ages. This interest includes gathering information on current research, discussions of both the mathematical and psychological aspects of advanced mathematical thinking, and research into thinking in specific subject areas within mathematics. This will be the eleventh meeting of the working group.

Our first session will begin with short reports of work arising from the topics of last year's discussions: informal mathematical thinking, and the relationship between social contexts and mathematical thinking. This will be followed by discussion of the working group's current project, the production of a book intended for mathematicians teaching at the post-secondary level. This book will include material on: teaching practice; a vocabulary for discussing learning and teaching; research which gives voice to students' beliefs; concerns and conceptions about mathematics; and the character of successful and unsuccessful students at the post-secondary level.

This book is intended to complement the group's last publication (Advanced Mathematical Thinking, D. O. Tall, Ed.) by making current research accessible to the general population of teachers of post-secondary mathematics, in a form which can be easily applied to their practice.

Our second and third sessions will include presentations and critiques of proposed contributions to the book. The third session will conclude with planning of work for the coming year and at PME-21.

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SOCIAL ASPECTS OF MATHEMATICS EDUCATION

PME XX VALENCIA, SPAIN, 1996

Coordinator: Leo Rogers, Roehampton Institute, LONDON SW 15 5 PH

The Social Aspects of Mathematics Education considers the range of economic, political, historical origins and motivations which combine to produce a unique mathematics curriculum in each country. Here we look principally at the "external" influences on the curriculum. We need to construct a modern curriculum in such a way that it pays attention to the individual processes of learning, including the affective aspects, and empowers students through relevant mathematics. Rapid changes in working practices in the adult world can be reflected in the mathematics done in the school, but we also need to address the problems of creating suitable mathematics curricula for mature students.

It is intended to provide participants with copies of the lead papers to be discussed in each of the first three sessions below.

Session 1: Noel Geoghean.
Emotional Issues in Mathematics Education.

We need to become aware of the complexities that make up each individual learner's capacity to accommodate to change, and thereby to learn. Emotional states and mathematical experiences reflect conjoint dimensions in learning and teaching. In this paper it is proposed that emotional factors such as motivation, confidence and self - esteem have a dynamic effect on the lives of young learners, and can offer added dimensions to our conception of Mathematics Education.

Session 2: Vic Parsons.
Gender and Computing in Further Education; a "life story" approach.

This project uses biographical / life story methods to explore the under - representation of women on computing courses. The account concentrates on an exposition and analysis of the methodology of the study. In this way, by refining and extending the methods described here it is hoped to explore further the idea of the "gendering" of activities and interest in the field of computing

Session 3: Leo Rogers.
Trends in Research into Social Aspects of Mathematics Education.

This presentation makes a survey of the growth of interest in the social aspects of mathematics education; it notes the developments in the philosophy of mathematics which have supported the development of the social - constructivist approach to teaching and learning. The implications for the changes in research methodology are considered and compared with current controversies in the social sciences.

Working Group
Algebraic Processes and Structure

Organizers:
Teresa Rojano, Matemática Educativa, CINVESTAV, México.
Luis Radford, Université Laurentienne, Canada

This Working Group has been interested in studying the algebraic thinking from different perspectives (e.g. epistemology, history, semiotics, problem-solving, generalization, modelization) and its links to arithmetical thinking.

One of the achievements of the group is the development of different frameworks and experimental research that aim to provide a better understanding of the complexity of the constructing processes of students' algebraic ideas. Many of the different approaches discussed in our previous PME Working Group meetings crystallized in the chapters of a book that the Group is preparing to publish.

Various debates arose in our last meeting in Recife, Brazil. Among them, we discussed the status that may be attributed to algebra in the school. Even though there may be a consensus in seeing algebra as a teaching subject that needs to evolve from a non-formal achieved discipline to an abstract-structured one, it is not clear enough how to didactically control the abstracting process under question.

The members of the group recognized that one of the questions that deserves to be discussed in depth is the role that problems may play in such a process. Particularly, what would be the role of "real" problems?
This question includes different sub-questions:

- What is a "real" problem? May a "real" problem be characterized from its *context*? from its "structure"?
- What is it that makes a problem difficult to solve by algebra?
- Should algebra be introduced through "real" or "non-real" problems?
- What is the role of "non-real" problems? When to use them?

We want to tackle these questions during the Valencia PME Conference. In order to do so, some participants will present a short talk that will be followed by a discussion. A booklet containing the short talks will be available to the participants.

PME XX WORKING GROUP

CLASSROOM RESEARCH

Organizers: Anne Teppo, teppo@math.montana.edu

Carolyn Maher, cmaher@math.rutgers.edu

The purpose of this group is to examine issues and techniques related to research involving the learner in the classroom.

The topic of discussion will focus on the role of a theoretical framework in classroom research. Issues addressed will include the ways in which the choice of such a framework influences the research design and data analysis, and how research frameworks may evolve during ongoing research.

Short presentations by Erna Yackel (USA) and Susan Pirie (Canada) will provide examples of the ways in which the results of ongoing research lead to a deeper articulation and evolution of the underlying theoretical framework of each project. Participants are invited to share their own experiences with the process of understanding and articulating the theoretical frameworks under which research is conducted.

DISCUSSION GROUPS

Discussion Group: **"Vygotskian Research and Mathematics Teaching and Learning"**

Convenors: Kathryn Crawford and Steve Lerman

The aim of the discussion group is to continue an examination the contribution of Vygotsky and some of his compatriots and the implications of their theory of learning in a socio-cultural context for mathematics education. The assumptions underlying Vygotsky's position differ in several ways from the tacit philosophical and psychological position of the mathematics education community. In particular Vygotsky challenges the centrality of the individual in meaning-making and insists on a social ontogeny of consciousness. By identifying meaning as the unit of analysis for psychology, Vygotsky offered an alternative programme to mentalism, one that focuses on the socio-cultural settings in which the child grows up, on the tools, both physical and psychological, which mediate experience, and on internalisation as the process by which the internal plane of consciousness is formed (Leont'ev). In addition his position transcends traditional Cartesian dualities such as self/other, mind/body, feeling/thinking and subject/object. His historical-cultural method of research differs significantly from predominant methodologies which typically focus on a part of the learning situation.

At PME 18 in Lisbon, the first session developed from some key aspects of Vygotskian approaches chosen and offered by the convenors, and in the second session some research issues/implications were proposed and discussed. At PME 19 in Recife the interrogation of the body of work was continued looking for its relevance for our research, for its complementarity or contradiction to other approaches, and to its deficiences and its strengths.

In PME 20 we will aim to focus the discussion on a critical examination of some of the current research in the psychology of mathematics education which draws on the Vygotskian perspective and to attempt to identify the implications of these research perspectives for future directions in research and the teaching and learning of mathematics. A particular focus of these discussions will be the implications of a Vygotkian perspective for research designed to examine the cognitive processes and meaning making that occurs as students increasingly use shared computer-based environments for mathematical activity.

PSYCHOLOGY OF EXPERIMENTAL MATHEMATICS

GARY DAVIS & KEITH JONES, UNIVERSITY OF SOUTHAMPTON, U.K.

Computers provide an environment in which mathematicians can experiment. The aim of this discussion group is to provide some examples of that, to discuss the need for proof and the conjectural nature of experimental mathematics, and to discuss the implications that would come from classroom sessions in experimental mathematics. The relevance for the study of the psychology of mathematics education is the possibility to record and analyse the creative and social aspects of students and teachers in their engagement with unsolved mathematical problems.

FIRST 60 MINUTE SESSION

- Demonstration of the ways in which computer programs allow students to generate data relevant to unsolved, but elementary, mathematical problems.

- Discussion by the participants of these examples, and others suggested by them.

SECOND 60 MINUTE SESSION:

- Discussion of issues relating to classroom experiences with experimental mathematics: positive and negative aspects of experimental mathematics; proof and explanation in experimental mathematics; how would experimental mathematics sessions run? as science prac labs, or in some other way? How could the mathematics education community provide a resource for teachers who want to run experimental mathematics sessions? (For example, like the Web Access Excellence site for biological science).

- Discussion of the cognitive and social aspects of experimental mathematics: effect on teachers and students of the authenticity of gathering and organising data on unsolved mathematical problems; development of students' language to record and describe their data and experiences; development of small group explaining.

Both sessions will be video taped using Super VHS, and the video tapes will be made available to the PME committee for distribution to the mathematics education community. The prepared examples will all be made available, prior to the PME meeting, on the University of Southampton's School of Education Web page (http://www.soton.ac.uk/~gary/crime.html), in MicrosoftWord, TeX, and Postscript formats. There will be some hard copies to hand out at PME.

Embodied/Enactive/Ecological Cognition and the Psychology of Mathematics Education

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An important goal of the psychology of mathematics education is to understand the thinking involved in doing and learning mathematics. The field of cognitive science constitutes a resource for addressing this goal. Unfortunately, the term "cognitive science" is generally understood to refer to a particular theoretical approach focused on individual reasoning, often explained in computational terms. Traditional approaches endorse the idea that in explaining human cognition it is necessary to refer to mental representations, symbol manipulation, and information-processing. These concepts are rooted in an objectivist tradition, in the sense that it is assumed that knowledge exists independently of the knower: the objects and world being represented and manipulated pre-exist the knower's mind. These approaches have had enormous difficulties to explain everyday cognitive realities such as common sense, sense of humor, natural language understanding, interactions in the classroom, apprenticeship learning, and so on. The result of this interpretation has been that many mathematics educators, especially those concerned with social and cultural factors, have overlooked the potential contribution of cognitive science as the scientific study of knowledge.

We claim that the scientific study of knowledge and learning must not be constrained to this narrow view of cognitive science, and that the field of mathematics education can benefit from alternative views. In this discussion group, we invite participants to consider one new paradigm in cognitive science, a view called "enactive/embodied cognition". This view sees cognition as a biological, embodied, and ecological phenomenon which is realized via a process of co-determination between organisms in their communities and the medium in which they exist. We explore the relevance of this view to the psychology of mathematics education. To do so, we propose to:

1) analyze some entailments and advantages of non-objectivist paradigms.
2) discuss what is meant in this view by "enaction", "embodiment", "ecology", "perception-action", and "experience".
3) discuss one kind of work done in this field. In particular the work by G. Lakoff and M. Johnson on Systems of Conceptual Metaphors. According to this view much of our thinking is based on metaphors and metonymies. We intend here to focus on the technical meaning of metaphor, that is, in the mapping between the source and target domains that make understanding possible. This mapping is consensual, cultural, largely unconscious, and is not arbitrary nor random. It is grounded in our bodily experiences.
4) explore how this alternative paradigm can provide powerful tools for both research and practice in the psychology of mathematics education; and,
5) invite participants to bring their own research and practice issues into the discussion and to examine them from the perspective of the framework provided by this non-objectivist view of cognitive science.

Because of the nature of the problem described above, this Discussion Group is theoretical in its orientation. It has been conceived in order to provide a forum where participants, on the one hand, may try to actively make sense of different experiences of the mathematics education world in the perspective of new theories of cognition, and on the other hand, may be exposed to more precise, technical and rigorous concepts relative to scientific aspects of cognition, perception, action, and language. We believe that research and practice in mathematics education can only be enriched by drawing from new perspectives provided by non-objectivist approaches to cognitive science, and the overall purpose of this discussion group is to begin to build a bridge from mathematics education to these new paradigms.

References: Will be presented during the sessions.

USING OPEN-ENDED PROBLEMS IN MATHEMATICS

Erkki Pehkonen, Dept Teacher Education, University of Helsinki (Finland)

The method of using open-ended problems in classroom for promoting mathematical discussion, the so called "open-approach" method, was developed in Japan in the 1970's (Shimada 1977). For example in the paper of Nohda (1991), one may find a nice description of the paradigm for the open-ended approach. This discussion group began three years ago in the PME–Japan, and had a continuation in the PME–Portugal and PME-Brazil. In these sessions, the topic of discussions was the concept "open-ended problem" and its classroom usage, with examples from different countries (e.g. Australia, Finland, Germany, Japan, Taiwan, UK, USA).

In Japan (1993), we concluded that open-ended problems pertain to a larger class of open problems (i.e. problems with openness in the initial or goal situation). Furthermore, open problems contain e.g. problem posing, project work, and most real life problems (Pehkonen 1995). The presentations there (Nohda, Silver, Stacey) which are published in the ZDM journal (2/1995) give a good view into the variety of problems. In the following PME-meetings, Portugal (1994) and Brazil (1995), we continued the basic discussion, and concentrated on different open problems used in different countries.

In the last discussion group, we decided to focus on the following questions: "What is a 'good' open-ended problem and how to realize that in classroom?", "How can we combine open-ended problems and text books?", and "What kind of concepts can we develop with open-ended problems?" There will be 3–4 brief presentations (about 10–15 min) from different countries, containing the presenter's view point to some of the mentioned questions. Thus presentations will give us some starting points for discussion.

References

Nohda, N. 1991. Paradigm of the "open-approach" method in mathematics teaching: Focus on mathematical problem solving. International Reviews on Mathematical Education 23 (2), 32–37.

Pehkonen, E. 1995. Introduction: Use of Open-Ended Problems. International Reviews on Mathematical Education (= ZDM) 27 (2), 55–57.

Shimada, S. (ed.) 1977. Open-end approach in arithmetic and mathematics – A new proposal toward teaching improvement. Tokyo: Mizuumishobo. [in Japanise]

Proposal for a Discussion Group at PME 20, Valencia

The Learning and Teaching of Probability and Statistics

John M. Truran (University of Adelaide)

Kathleen M. Truran (University of South Australia)

Formal stochastics teaching is a relatively recent activity, especially in schools, and effective pedagogical skills are not yet widely spread. Much recent research has focused on establishing children's understanding of parts of the topic. More general theories are slowly emerging, but are still not widely accepted. This discussion group will aim to provide a forum to assist the increasing number of researchers in this field to see the broad picture within which to set their own work.

Shaughnessy (1992) presented a wish list for future research in stochastics that included both conceptions and misconceptions held by both students and teachers, assessment issues, cross-cultural studies and the effect of metacognition in decision-making under uncertainty.

Some of these issues were addressed at the Fourth International Conference on the Teaching of Statistics in Marrakech in 1994 and in a Discussion Group at PME in Lisbon in 1994 chaired by Kath Hart. These discussions, and also the papers presented to PME in Lisbon and at Recife in 1995 have shown that current work often addresses some of these isues but tends to concentrate on either probability or statistics, and much less often on the links between the two.

It is proposed, therefore, that the two meetings of the discussion group will look especially at forging links between probability and statistics. The first meeting will address the "pure" issue of conceptions and misconceptions in both topics. The second will address the "applied" issue of the learning in service courses of stochastic concepts by secondary and tertiary students who want to use statistics as a tool in other disciplines. Issues of assessment and metacognition and cultural background will be of special relevance here.

Reference

Shaughnessy, J. Michael (1992) "Research in Probability and Statistics: Reflections and Directions" in Grouws, D.A. (ed.) (1992) *Handbook of Research on Mathematics Teaching and Learning* (pp. 465- 494) New York: Macmillan

Understanding of Multiplicative Concepts

Convener: Tad Watanabe, Towson State University, U.S.A.

Mathematical concepts that are tied to what Vergnaud (1988) called "multiplicative conceptual fields" include multiplication, division, fraction, ratio, proportion and linear functions. These concepts play a central role in school mathematics, and, as a result, they have been studied by a large number of mathematics education researchers all around the world. It appears that there is a growing awareness that these concepts do not develop in isolation (Harel & Confrey, 1994). Rather, their developments are closely connected. However, traditionally, mathematics education research has investigated these concepts more or less independently.

The purpose of this discussion group is to bring together researchers who have studied children's understanding of these various mathematical concepts and to share their current understandings. We will discuss how our understandings can inform each other's future investigations. It is hoped that such a discussion will lead to a coordinated framework to approach future research on children's understanding of concepts in the multiplicative conceptual fields.

References

Harel, G. & Confrey, J. (1994). *The development of multiplicative reasoning in the learning of mathematics.* Albany, New York: SUNY Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, Virginia: National Council of Teachers of Mathematics.

SHORT ORAL COMMUNICATIONS

UNDERSTANDING DECIMAL NUMBERS: FROM MEASUREMENT TOWARDS THE NUMBER LINE

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In recent research the interplay between thought and symbolic meaning has been elucidated (Confrey 1995). Children develop spatial representations that provide the conceptual grounds for the acquisition of the meaning of symbolic systems (Bialystok 1992, Resnick 1983). Arithmetical concepts can be mentally represented in terms of a number line; counting and arithmetical inferences are understood by children as transformations of points represented on the number line.

The understanding of decimal numbers implies a more encompassing representation of the mental number line. We think that the ruler can be a useful tool to enable children to focus on the salient properties of the line. Measurement highlights the interrelationships among decimal number and facilitates comprehension of the meaning of some arithmetic operations on them. However, children must overcome the metric representation on the ruler in order to understand a representation with no unit of measure and move towards the conceptualization of pure numbers. Furthermore, a ruler can be a cultural artifact for bridging the mathematics that children practice in their ordinary life with the school objective of thinking by mathematical models (Saxe, 1991).

This exploratory study is part of a project involving 21 third grade children (ages 9-10) and dedicated to the use of measurement for introducing the concept of decimal numbers in the normal classroom curriculum. We are interested in gaining an insight into the opportunities and constraints implicit in the children's use of the ruler for understanding decimal numbers. Crucial points in learning about decimal numbers through measurement are: reasoning processes in the representation of decimal numbers; the matching of representations on the ruler with representations in written symbols; abstracting from a given unit of measure to represent decimal numbers. Data sources consist of researchers' observations, children's protocols, and tape recordings.

References

Bialystok, E. (ed): 1992, 'The emergence of symbolic thought', *Cognitive Development*, 7 (3), 269-381.

Confrey, J.: 1995, 'A theory of intellectual development. Part II', *For the learning of Mathematics*, 15 (1), 38-48.

Resnick, L.: 1983, 'A developmental theory of number understanding', in H. P. Ginsburg (ed) *The development of mathematical thinking*, New York Academic Press.

Saxe, B. G.: 1991, *Culture and Cognitive Development. Studies in Mathematical Understanding*, Hillsdale, NJ: Lawrence Erlbaum.

THE ROLE OF THE GRAPHIC CALCULATOR AS A MEDIATING SIGN IN THE ZONES OF PROXIMAL DEVELOPMENT OF UNIVERSITY STUDENTS

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This analysis forms part of a qualitative study exploring and describing the ways in which the use of the graphic calculator effects the mathematical conceptions, manipulations and performance of first-year mathematics students at Witwatersrand University. The analysis is situated primarily within a Vygotskian paradigm.

20 students, out of a class of 400 students, were each loaned a graphic calculator for the duration of the academic year and encouraged to use these during support tutorials. Since the class as a whole did not have such tools, graphic calculators were not allowed in examinations. At the end of the year, 7 students (4 with the graphic calculator, 3 without) were asked to solve a mathematical problem speaking aloud in an interview situation.

Each interview was explored using two complementary procedures: an analysis in terms of the tool, sign or agent used for mediation and an interpretation of this analysis with reference to the relationship (if any) between the mediating tool or sign and the apparent mathematical conceptions, manipulative skills, flexibility and performance of the student. A comparison was made of these analyses and their interpretations, between the students. It is suggested that the calculator functioned primarily as a tool for cognitive amplification (Pea, 1985) in the zones of proximal development of the students, increasing efficiency and speed of the completion of the task, rather than as a semiotic system which had been internalised. This role of the graphic calculator is understood within the social context of this study in which the use of the graphic calculator was limited to non-examination work. Furthermore, and consistent with a Vygotskian framework, it seems that learner use of the graphic calculator is not uniform but depends on the zone of proximal development of the particular student in relation to the specific task within the interview context.

Vygotsky, L.S. (1978). Mind in Society. Cambridge, MA: MIT Press.

Pea, R.D. (1985) Beyond Amplification: Using the computer to reorganise mental functioning. Educational Psychologist, 20, 167-182.

PROFESSIONAL DEVELOPMENT IN PERFORMANCE ASSESSMENT FOR QUEENSLAND YEARS 1-10 MATHEMATICS TEACHERS

R. Bleicher, T.J. Cooper, and S. Dole (Queensland University of Technology), S. Nisbet (Griffith University), and E. Warren (Australian Catholic University)

This oral communication is a summary report of research on a professional development project on performance assessment for Years 1 - 10 mathematics teachers throughout Queensland, Australia. The two-year project provided inservice courses aimed at introducing performance assessment classroom practices in gathering and recording student information, analysing student behaviour, and making achievement level decisions for reporting to students and parents. In this, the project complemented the inservice training provided by the Queensland Department of Education.

The research program included both quantitative and qualitative data collection. The quantitative data was gathered through surveys: of a random sample of Queensland mathematics teachers before the inservice activities began and one year later in the project; and of participants of the inservice activities at the activity and approximately 6 months after. Analysis of the data from the random sample revealed both gender and school level (primary/secondary) differences. A report of the first survey can be accessed in Bleicher, Cooper, Nisbet and Warrenn (1996). The second survey revealed two pertinent points: (1) the response patterns across survey questions remained markedly similar to the first survey; and (2) a new section in the second survey about teachers attitudes to the performance assessment initiative revealed a widespread reluctance for teachers to embrace the reform. It also indicated an apparent poor dissemination of information about performance assessment through the efforts of both the project and the Department of Education. Analysis of the data collected from participants also indicated a lack of knowledge about and confidence in the application of performance assessment in the mathematics classroom.

The qualitiative aspect of the research was gathered through interviews and classroom observations. A constant comparative methodology combined with anecdotal support was used to identify categories of teacher activities which represented exemplary classroom practice in performance asssessment techniques as they emerged from the data. Attitudes of inservice leaders and participants to both the project's inservice model and the content of the inservice activities were also collected and categorised. These data showed a consistent reluctance throughout the project for mathematics teachers to embrace performance assessment.

Reference. Bleicher, B., Cooper, T.J., Nisbet, S., & Warren, E. (1996). Assessment and reporting in mathematics: The effect of teacher gender and teaching level. In B. Atweh, & S. Flavel (Eds.), Galtha (Proceedings of the 18th Annual Conference of MERGA). (pp. 102-108) Darwin, NT: MERGA.

A student teacher's attempts to analyze a student's learning.

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Current recommendations for a change in mathematics education necessitate the reform of teacher preparation programs. New ideas concerning students' learning of mathematics have their impact on the design of these programs. Student teachers' training should prepare them for a practice that does not ignore the active and interactive nature of students' mathematics learning (Bauersfeld, 1995). This report is part of a broader study that aims to enhance the effectiveness of our university program for preparing teachers to teach elementary school mathematics. The organization of our program consists of several aspects. In parallel to the university class lessons, student teachers visit school classrooms and are involved in microethnographical investigations of their life. Also, they are assigned to teach individual students or small groups of students, and later on in whole classrooms. Our purpose in this paper is to present some preliminary results related to a student teacher's learning in the context of her interactions with a second grade student.

The readings and the activities in which our student teacher was involved at the university class provided her with several opportunities to reflect on general ideas related to the teaching and learning of whole number arithmetic. So, in the course of her assignment she could integrate these ideas to her practice. In this assignment, she had to meet with her student once a week for 2 months and she was obliged to keep a diary for recording her decisions and observations from teaching the student. Her notes along with our close observation of her teaching allowed us to study the process by which she developed her ideas and practices in relation to the teaching of multiplication and division.

From the beginning of her assignment, she experienced surprises and conflicts that allowed her to reorganize her beliefs and practices. First, her student's solutions of the tasks she gave him in the initial interviews, her observations in his classroom, as well as her analysis of the textbook's approach helped her to interpret his difficulties as a consequence of the traditional way of teaching. Second, through the activities she designed and the way she interacted with him, she came to appreciate his thinking and trust his abilities. Furthermore, she was in a better position to anticipate possible learning paths and use activities that might support him through these paths. From our own perspective, her practice included some elements of a realistic approach to mathematics instruction (Gravemeijer, 1992). In our presentation, several examples for illustrating the process of our student teacher's learning will be given.

References:

Bauersfeld, H. (1995). Remarks on the education of elementary teachers, preservice and in-service. In P. Cobb & H. Bauersfeld (Eds.), Emergence of mathematical meaning: Interaction in classroom cultures. Hillsdale, NJ: Lawrence Erlbaum Associates.

Gravemeijer, K. (1992). Micro didactics in mathematics education. Paper presented at the AERA Conference, San Francisco.

TEACHERS AND STUDENTS IMAGES ABOUT LEARNING MATHEMATICS

Isabel Branco - Escola Secundária António Arroio, Portugal

Isolina Oliveira - Escola 2,3 Damião de Góis, Portugal

The present study is oriented by questioning the role of the past mathematics experience in students´learning and their relationship with mathematical tasks in classroom - how images about learning mathematics and the subject matter influence teachers and students atitudes towards mathematics in the teaching/learning process?

In this study, our focus was the experience with math that was recorded and crystallized in the memory of students and high school teachers and is presented as images. Experience here is a started point to understand how practical knowledge is formed. We looked for the meaning that students and teachers give to their experiences with math following the conceptual framework of images considered by Elbaz (1983: p. 254) as "a brief descriptive and sometimes metaphoric statement". And also, how this images have expression in action during classroom interactions with mathematics.

The images allows us, to study: a) the socio-affective aspects that are related with learning; b) the practical knowledge that teachers and students have c) how they project their images in action.

Symbolic interactionism and biographical approaches supported the procedures (written and oral narratives, interviews and observations) used to collect and analyze the data of this study.Two high school teachers and twelve of their students gave meaning to their past experiences and present actions with mathematics in the context of school and out side of it. The study gives some evidence how different images and beliefs about mathematics and its learning, that these teachers and students bring to the classroom, affects their relationship with mathematics in classroom situations.

References

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Clandinin, D. J. (1986). *Classroom Practice: Teacher images in action.* Philadelphia: The Falmer Press.

Elbaz, F. (1983). *Teacher thinking: A study of Practical Knowledge.* New York: NIchols Publishing Company.

PRIORITISING MATHEMATICS TEACHER EDUCATION CHOICES AT PRE- AND INSERVICE LEVELS.

Chris Breen
University of Cape Town

The tasks of pre- and inservice mathematics teacher education have tended to take place in isolation of each other. Professional contact with mathematics teachers in the School of Education at the University of Cape Town has traditionally been restricted to the preservice Mathematics Methodology courses. However the formation of the Mathematics Education Project (MEP) in 1989 brought the priorities and problems of inservice education strongly onto the agenda. As a result a research project with three strands was established and funded by the Centre for Science Development to explore the possibilities for transforming the teaching of mathematics in South African schools. The different sections of the project looked at: the experiences of teachers who had completed the preservice course at UCT (Breen and Millroy 1994); the potential for running a support programme for first year teachers (Coombe 1995); and an ethnographic examination of the realities of teaching mathematics at an urban Black high school (Walters 1996).

This short oral communication will highlight some of the findings of the project as a whole and attempt to draw some conclusions as to the tensions and choices that need to be made in prioritising activities to optimise the potential of mathematics teachers.

References:

BREEN, C. & MILLROY, W. (1994) "An investigation into the longer term effects of a preservice mathematics method course". Paper presented at the 19th Conference on the Psychology of Mathematics Education in Lisbon, July 1994.

COOMBE, J. (1995) "From Student to Teacher" in Z.Davis (ed.) *Exploring Mathematics Teaching and Teacher Education,* Cape Town: Mathematics Education Project, University of Cape Town.

WALTERS, B. (1996) "An ethnographic study of a Black South African high school with special reference to its mathematics classrooms". Unpublished M.Phil. thesis, University of Cape Town.

Graphic calculators and precalculus. Effects on curriculum design

Cristina Carulla and **Pedro Gómez**

"una empresa docente", Universidad de los Andes, Bogotá, Colombia

This paper reports on a project that studied the effects of graphic calculators use on the curriculum design of a first-year precalculus course in a private university in Bogotá, Colombia.

Three curriculum levels were considered. The *macro* level in which social, political, economic and cultural factors intervene and define the visions, values and traditions about mathematics, its teaching and learning. The *meso* level in which the educational institution expresses its visions about the teacher, the student and mathematics as a cultural and teaching–learning knowledge. And the *micro* level in which teacher and students interact in the construction of the mathematical knowledge through the implementation of a curriculum design.

The curriculum design existing before the introduction of graphic calculators was determined on the basis of course documents, the textbook and some workshops and assessment tests. The curriculum design of the course once the graphic calculators were introduced was deduced from working documents of the research program this project was part of, the meeting minutes that were produced during the three semesters during which the project was done, and from almost one hundred workshops that were designed and used in the course.

Differences between the two curriculum designs were appreciated at the *meso* level. These differences affected several elements of the *micro* level (Gómez y Rico, 1995). Before the technology was introduced, the institution had a vision of mathematics as a structured body of conceptual knowledge and expected the student to be able to solve repetitive exercises, construct his knowledge in an individual manner and develop a vision of mathematics as a true *a priori* knowledge. After the calculators were introduced, the institutional vision was different. Mathematical knowledge is now seen as a socially constructed knowledge, in permanent evolution, open to experimentation and to conjecture posing and testing. The student is expected to socially construct a mathematical knowledge that takes into account its practical applications and includes a vision of a globality of mathematical objects that can be seen and manipulated from multiple dimensions (conceptual and procedural) and representations. The student is expected to be able to solve the type of complex problems associated with higher–order mathematical thinking (Resnick,1987).

References

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Resnick, L.B. (1987). *Education and learning to think*. Washington, DC: National Academy.

THE USE OF TWO DIFFERENT TYPES OF GRAPHS IN A STATISTICAL TASK (7TH GRADE)

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During the last twenty years, the history of Psychology of Mathematics Education has been studying how the mathematical knowledge is constructed, which is the role of daily life experience in the acquisition of mathematical knowledge, which is the role of social representations refering to mathematical notions in the construction of the different symbolic systems. This new epistemological paradigm originates new interpretations of the learning process going further than the simple evaluation of performances (seen as a final product) and the acquisition of skills. It rises new questions about the roles of context, tasks, instructions, situations and relations among the different actors of the didactic process.

The research that we are presenting in this paper studies the role of the task in the performance and learning process of the pupils. It was asked to 50 pupils from a secondary school in Lisbon, attending two classes from the 7th grade, to solve two "usual" tasks (described as typical by teachers) related to their curriculum (Statistics). These tasks envolved the construction of graphs of two different types: bar and pie graphs.

In a first analysis of their written productions there are no main differences in terms of level of performance reached by pupils. Anyhow, a deeper analysis of the data shows that the construction of pie graphs leads to a more complex approach once they are obliged to use a conversion to another system, the grades. This represents a greater difficulty to the subjects which is stressed by a slower answering process and with a lot of corrections within itself. So, even having similar final results, the answering processes are quite different.

These results make us believe that the study of the tasks presented to pupils is essencial to the comprehension of the way they acquire their knowledges and skills.

THE INFLUENCE OF INTEGRATING AN INNOVATING PROJECT IN PUPILS' IDEAS ABOUT MATHEMATICS

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The importance of pupils' social representations of the subjects they learn at school has been stressed by many reseachers. The way pupils perceive Mathematics deeply influences how they deal with the mathematical knowledge, how they acquire it and the expectations they have toward this subject. A previous study (César, 1995) showed that pupils from the 7th grade, attending a school in Lisbon, had quite traditional ideas about Mathematics. Is was seen has important for their future life but they associated it mainly with computation and "memorizing things".

During the last years an educational reform has been implemented in Portugal, namely for the compulsive grades of education (till the 9th grade). We decided to study the influence of this reform in pupils' ideas about Mathemetics as well as the influence of participating in a project which promotes peer interactions in the classroom. Our main goal is to understand the influence of teachers' practices upon the construction of pupils' representations of Mathematics.

All pupils from the 7th grade and the same school of the previous study were asked about their ideas about Mathematics. These pupils ideas about Mathematics will be compared to the ones of the previous study to stress the effects of the new educational reform. Simultaneously, we asked teachers which strategies they used in their classes so that we could see if the new reform was really changing their practices.

Another sample was formed by the pupils of three classes from the 7th grade, attending a school in a rural region, but integrating an innovating project which promotes peer interaction in the classroom. These pupils ideas about Mathematics were followed along the school year so that we could see the effect of participating in such a project in the ideas they had about Mathematics.

The comparison of the data from these three samples (previous study and the two new ones) may help all those who are concerned with the construction of social representations to understand better the influence of teachers practices and other social factors in pupils' ideas about Mathematics.

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USING TEACHING CASES: A PORTRAIT OF TEACHERS' THINKING ABOUT REFORM [1]

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This paper focuses on two components of a model to encourage and support teacher reflection in a professional development project for teachers working on changing their teaching of mathematics. The model rests on the use of cases: teaching cases around specific mathematics content and cases by or about teachers immersed in reforming their mathematics teaching. In the first component, teachers read and discuss cases about the teaching of fractions (Barnett, Goldenstein, & Jackson, 1994). The cases were selected on the basis of their potential for discussions around mathematical content (e.g., understanding multiplication of fractions) and pedagogical content knowledge (e.g., when and how to use manipulative materials). In the second component, teachers write a case based on their reflections as they try to change their teaching of mathematics. These cases provide yet another source of data (in addition to observations of participants as learners and teachers of mathematics, interviews, and journals) to assist us in understanding participants' interpretation and adaptation of the reform ideas.

The social interactive characteristic of case discussion allowed to bring into the open those preconceived ideas that teachers hold about each other, largely based on the variety among schools. (The 30 participants come from seven school districts that have both differing philosophies of education and student population.) Although the school culture plays a large role in teachers' implementation of reform ideas, these case discussions brought to the surface a key factor in this process, namely teachers' beliefs about teaching and learning mathematics. Ernest's (1991) work on the teaching approaches stemming out of different philosophies of mathematics (and of mathematics education), as well as current research on teachers' approaches to reform (Lambdin & Preston, 1995; Peter, 1995) served as the theoretical framework to analyze the case discussions and written cases. This analysis shows some recurrent topics, such as mathematical authority in the classroom: the difficulty of letting go; use of hands-on activities: are they driving the teaching to the expense of the mathematics?; and affective aspects: isolation among teachers trying to implement reform. These topics will be used as the basis for a critical discussion of some of the interpretations of the reform movement in mathematics education.

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EUCLIDEAN GEOMETRY: COGNITIVE GENDER DIFFERENCES

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Cognitive gender differences in mathematics, which were noted in past research have latterly been found to be declining, but are still observed at secondary school level in the field of spatial skills and geometry. However, due to divergent geometry curricula and the multidimensional nature of spatial skills, results from gender studies have led to broad conclusions being drawn from investigations which were affected by random elements. Research results have been influenced by imbalanced representation of the two gender groups in samples and also by differential gender exposure to previous mathematical experience. These limitations in research have produced outcomes reflecting negatively on females. Mostly only mean performances are reported.

The research in question addressed some of the limitations:

1. Only Euclidean geometry was considered, with emphasis on spatial skills, as well as deductive reasoning, proof writing and hypothesis testing.
2. Due to a standardised system in most South African secondary schools and a uniform core curriculum for mathematics, a sample representative of males (410) and females (406) could be selected. Theoretically, both gender groups were equally exposed to the teaching of Euclidean geometry.
3. Multiple choice questions and open-ended questions were used.
4. Achievement at a continuum of different ability levels was investigated by means of the Item Response Theory.

No statistically significant gender difference in mean performance on Euclidean geometry was found. Further analysis at item level was undertaken for the multiple choice component of the test instrument by means of the one dimensional three parameter logistic model in Item Response Theory. This model is considered to be very sensitive in identifying differences at different levels of ability - a shortcoming in existing research. Four out of a total of twenty Euclidean geometry items were identified as producing differential gender performance. Similar gender tendencies were observed for all four items. The performance was in favour of most males for each of these items. Females of lower ability (as measured by the item) did not perform as well as males of a similar ability level. However, in all four cases females of higher ability surpassed males of a corresponding ability level.

"THE MIDDLE OF WHAT... ?": STUDENTS' IMAGES OF MEAN, MEDIAN AND MODE

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Gus: {Reading question} 'Compare the mean, median and mode.' Median?
Jeffrey: What is that? The middle?
Gus: I think so.

This presentation is concerned with students' understanding of data handling, with particular emphasis on the mental images that students have for mean, median and mode.

It is based on an analysis of video recordings of upper school students, aged 13-15 years, engaged in classroom activities in which they are involved in the creation of a data set and then proceed to pose and solve questions concerning the data. This is part of a much larger study investigating students' posing and critiquing of mathematics problems.

In the data handling portion of the study, evidence was collected from three classrooms. Each classroom was set according to ability (low, middle, and high, respectively); two pairs of students were filmed in each class. All ability groupings used essentially the same data set.

Evidence of students' mathematical thinking was analysed according to the Pirie-Kieren model for Growth of Mathematical Understanding. Particular attention was paid to incidents in which students were observed to articulate, apply or modify their images for measures of central tendency.

This presentation will consider these images and their connection to the difficulties that students were observed to encounter when solving problems involving mean, median and mode.

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GENERALIZATION IN ALGEBRA PROBLEM SOLVING AND ATTITUDES TOWARD MATHEMATICS

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In addition to previous studies (Neumann and Brito, 1995; Brito, 1995) the main objective of this work is to explore the process of generalization, perception and mathematical memory related to attitudes toward Math in "good" mathematical students. According to Krutetskii (1976), the psychological abilities of a student can be investigated using problems of various types.

The subjects in the present study were 4 students of a private school (13 years old), classified as "Good"or "Excelent"in Mathematics. Those pupils were asked to solve a set of problems which involves a gradual transformation from concrete to abstract. In another study developed by the research group (Lopes and Spalleta, 1995) five types of problems (Serie VII proposed by Krutetskii) were tested with brazilian students and here the same set of problems are used.They were also interviewed, answered a questionnaire and a instrument to measure attitudes toward Mathematics (Aiken, 1970; Brito, 1995).

The students were asked to answer five series of problems and each problem had four to five variants (1E, 1D, 1C, 1B and 1A or 4E, 4D, 4B, etc..). The *E* variant has only abstract terms and it is the most difficult problem for the student; the variants *D*, *C* and *B* are intermediate and composed with abstract and concrete terms representing a transition from concrete to abstract. The variant *A* is only concrete . First of all, the student is asked to solve the variant 1- *E* , and when the problem is solved or after five minutes, the examinee ask the student to solve variant 1- *A*, and after solving variant 1- *A* the subject returns to variant 1- *E* and the examinee asks him about the previous solution and they can think about a new solution or mantain the first one. The order of problems is E, A, E, B, E, C, E, D, E; and the sets 1, 2, 3, 4 e 5 (easiest to more difficult). The subject is asked to think aloud and the session is taped.

The analysis of the data (protocols, questionnaire, transcripts and interview) showed that they can work better with the intermediate problems. They also informed that this kind of problem is similar to that used by the teacher in the classroom. The set 1, 2 and 3 were considered easy for all the students, and they solved this part quickly, but they found difficulties on set 4 and 5 and two of them informed that when the problem is in a general level and completely abstract (first presentation of variant *E* in all setst) hey can not deal with. During the presentation of intermediate problems one can perceive in the record when the generalization occurs. Comparing the students' performance while they solve the problem, the solution in the protocols, their attitudes toward Math and their performance in Math, one can say that those students present abilities to deal with this content.The ability to generalize, the flexibility of thinking and positives attitudes toward Mathematics are components of schoolchildren's mathematical abilities and the analysis of data revealed that it is present in those students.

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POINT CONFIGURATIONS AS REPRESENTATION SYSTEM FOR THE STUDY OF NATURAL SEQUENCES

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Representations, and in particular external representations, are one of the centers of interest in mathematical education research , as stated in the Working Group on Representations within the International Group PME.

Kaput (1987) highlights the role of representation systems, the common tendency to underestimate as well as their systemic nature.

Janvier's works (1978, 1987) stress the importance of consideration of several external representations for a same notion (the function notion)

Hiebert, J. & Carpenter, Th.(1992) or Duval (1993) have emphasized the relationships between representation and understanding.

Our interest is centered in the study of the number field and in showing the potentialities of the several representations in the curricular development of numbers.

We have included, in a systematic way, "point configurations" and "figurate numbers" like representation system for natural numbers and natural sequences.

Some first outputs have been stated by Castro, E. (1994) working with 12-14 year-old students.

In this paper we are presenting a first phase (development and outputs) of the continuation of our study; a case study carried out with 15-16 year-old students, in the metodology frame of Action-Research.

In this experience we have proposed our students a systematic work with several representation systems for natural numbers and natural sequences: point configuration, point structured configuration, usual decimal notation, arithmetic development of the number and symbolic notation regarding the form or structuring of configuration. We have centered our interest in this phase of research basically on the semiotic aspects of these representation systems.

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LEARNING AND TEACHING PERCENT PROBLEM SOLVING

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As have highlighted, percent has a multiplicity of meanings (i.e., number, common fraction, decimal fraction, ratio, proportion and multiplier) which have to be derived from their context (e.g., is a change of sales tax from 5% to 6% a 1% or 20% increase?). Percent knowledge has also to be applied in three different problem types (Ashlock, Johnson, Wilson & Jones, 1983): *Type A problems* - finding a percent of a number (e.g., 25% of 20 is __); *Type B problems* - finding what percent one number is of another (e.g., __% of 20 is 5); and *Type C problems* - finding a number when a percent of that number is known (e.g., 25% of __ is 5).

Experience with classrooms and inservice indicates that percent is poorly understood and applied to problems. For example, pertinent results of the fourth National Assessment of Educational Performance (Kouba, Brown, Carpenter, Lindquist, Silver & Swafford, 1988) show the following performances at Years 7 and ll: Type A - 32% & 62%; Type B - 20% & 34%; Type C - 22% & 43%; 2 step Type A - 9% & 37%; and 2 step Type C - 2% & 5%. There are also a variety of approaches to teaching (Parker & Leinhardt, 1993) that appear to be used with mixed success: using cases, formulae and equations, proportion and unitary.

This oral communication reports on research in progress (Dole, 1996) to develop effective methods to teach percent problem solving. The research has found that Queensland students' performance is also poor, particularly for Type B and C problems, as the following results for Year 8 show: Type A - 58%; Type B - 15%; Type C - 7%; 2 step Type A - 5%; and 2 step Type C - 0%. Classroom studies are being undertaken to trial a pictorial representation of percent and an approach to problem solution that encompasses all three types. These studies are based on an approach to teaching which provides a technique to enable students to unlearn prior erroneous knowledge and have control over their own relearning. The communication will focus on describing the approach to teaching and its theoretical rationale.

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SHORT ORAL COMUNICATION (PME XX - VALENCIA 1996, SPAIN)

CARTESIAN GRAPHS AND STUDENTS CONCEPTIONS: LOOKING FOR RELATIONSHIPS BETWEEN INTERPRETATION AND CONSTRUCTION

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Our research deals with the conceptions that secondary students have about functions and graphs, in the framework of Advanced Mathematical Thinking. In particular (Fabra 1995) we analyse the students (high secondary 16-18 years) conceptions about maximum and minimum of a graph function.

The main task presented to students was adapted from PAAU (Entrance examination to University, Barcelona 1992). They were asked to draw a functional graph given by verbal conditions (intersection with axes, maximum and minimum, asymptotes). We design two parallel tasks: to draw the graph and to interpret the conditions of a given graph. One of the research goals was to look for the coherence between pupil's answers to both tasks, and to stablish students conceptions (150 pupils of 3rd grade of BUP, 16 years old, 100 pupils of COU, 17 years old).

Concerning the interpretation task, we find four categories to express the meaning of a maximum / minimum. P: the maximum as the great value of x. G: the maximum as the highest point. F: the maximum as the point were the function change from increasing to decreasing. FF: the maximum related to the variation of the sign of the tangent's gradient (the same for the minimum).

In relation with the construction task, we find four main models for the drawing of the graph of the function. A: plotting points and drawing a prototype (segment, parabola). B: plotting points, drawing the relative maximum and "inventing" a minimum to draw a continuous function. C: plotting points, drawing the relative maximum and the discontinuities. H: drawing the graph as in C but also introducing an implicit minimum to draw the horizontal asymptote.

The research has pointed out that the analysis of interpretation and construction process in parallel allowed us to stablish levels of coherence between what students say when they interpret a graph and what they do when they draw the graph. The following main conceptions were stablished: Primitive (16%): students interpretation is P and (more than a half of the students) draw the graph like A; Geometric (44%): students interpretation is G and (more than 2/3) draw like B; Functional (29%): students interpretation is F and (more than a half) draw like C; Double Functional (11%), students interpretation is FF. and (more than 2/3 of the students) draw a graph like H.

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Title: **Logo: Big Step and Small Step**

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The purpose of the study was to better understand the impact of teacher´s speech on students´understanding.

The study took place in a private school in Rio de Janeiro, Brazil. The subjects were six third and five fourth grade students age 8 to 11 and the Computer Laboratory teacher. Students and teacher were videotaped and the analysis of those tapes revealed the path that the words took from teacher mouth to student´s misconceptions.

Almost all the students were being introduced to the Logo graphic commands (turtle commands). When introducing the four basic commands FRONTWARD, BACKWARD, RIGHT and LEFT the teacher told the students that there were big steps and small steps. The transcription of her words is below.

> *The turtle can walk using two different steps, a big step and a small step. When it walks frontward or backward it uses the big step, when it walks to the right or to the left it uses the small step.*

It is worth stressing that the teacher did not transfer the concept of angle to this context and used to step and to turn as synonymous. The impact of this misconception on the students could be observed in many students´ talking. For example, student A said:

> *I asked the turtle to RT 20 and she didn´t walk.*

They expected that using the command RIGHT 20 the turtle would indeed walk 20 steps to the right.

It was observed that the students were confused in their way constructing the angle concept by the words of the teacher. In a Logo environment it can be almost immediately diagnosed and overcome while in a regular math classroom sometimes this misconceptions can be carried out for a longer time without being perceived nor by the student nor by the teacher.

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Students Mathematical Activity and Cooperative Work in the Classroom*

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The idea that the way people learn is interwoven with the nature of the knowledge that we claim to learn, in opposition to the idea that we learn by a single way, and similar in all areas of knowledge. On the other hand we can think on learning as "situated in practice, as an integral part of generative social practice in the lived-in world" (Lave, 1991, p.35). Lave still argues that all activity imply learning. There is no learning without activity. The way we see the activity, will influence the way we interact with others. In this sense, knowledge is not independent, it is part of activity, context and culture in which it takes place. School activity tend to be hybrid -implicitly framed by one culture while explicitly being attributed to another. "When, for pedagogic purposes authentic domain activities are transferred to the classroom, their context is usually transmuted; they become classroom tasks and part of school culture" (Brown et al, 1989, p.9). It is not the task that will make emerging a different context. There are the expectations that the students have, that will make him or her act in one manner and not in another. Cooperative work can help to create an adequate environment.

In this presentation, it will be presented and discussed the theoretical framework of a study which aim is to contribute to the knowledge of the characteristics of school mathematics activity, in the mathematics classroom, trying to answer questions as: (a) How can we characterise students mathematics activity on classroom? How do students act? How do they dialogue? (b) Who and why has initiatives in mathematics classroom?
(c) What is the role of discussion in mathematics classroom? (d) What is students perception about mathematics activity?

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THINKING PROCESSES USED BY TWO PRESERVICE TEACHERS FACING PROBLEM SOLVING[1]

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There are many studies related with problem solving and we have some knowledge about conceptions and beliefs of preservice teachers, but there are also few information about the ways they use to solve problems. This research studied thinking processes used by preservice elementary school teachers (Grades 1 - 6) when they solve problems. The following were the main questions of the study: (1) What thinking processes teachers use when they solve problems?; and (2) How can we explain the dificulties revealed?;

In order to answer those questions two qualitative case studies have been conducted. Data has been collected through interviews and observations. A male and a female participated in the study and they didn´t enjoy the discipline in the same way: the female don´t like maths and the male like it. They agreed that problem solving can motivate students to learn mathematics. In this study we understand thinking processes like all the actions that occur when we solve problems. They can be cognitive, metacognitive or affective processes.

Principal results about thinking processes that underlie mathematical thinking reveal that these preservice teachers had dificulty in generalizing and convincing. They agreed that problem solving tasks can be used whith elementary school students as a means to motivate them to learn mathematics. Preservice teachers needs to solve diferent types of problems, to gain knowledge about problem-solving and to develop metacognition and reflection capacities.

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Personal Strategies of Generalization in Linear Generalizing Problems

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Over the last years some research studies have been conducted on the students' ability to perceive and generalize patterns. In an early work Stacey (1989) reports a classification of responses and methods of generalization used by students (aged between 9 and 13) when confronted with linear generalizing problems, questions which require students to observe and use a linear pattern of the form $f(n)=an+b$, $b\neq0$. Here we report briefly on the responses of linear type given by students and how they are affected by the drawing accompanying the questions. We guess the answers given by students should be viewed from two different levels. The first level is related to the particular way the students try to solve the question, how she or he uses the data and develop an answer, that is what we should call a strategy. Thus, a strategy is a procedural response. In the second level it is the concept underlying the answer given by students, the mathematical object, and to which the students are not always aware, that is what we call a model. Thus, the model concerns the mathematical object underlying the response given by students. Accordingly we have classified the students' responses in models and strategies as follows:
(*C*) Direct Counting: Counting from a drawing or generating a sequence by successive addition. (**W**) Direct Proportion: Difference (W1), whole-object (W2), both have been described in the works quoted above, and finally *rule-of-three* (W3).(L) Linear: Using a pattern that recognizes both multiplication and addition are involved and that the order of operation matters. Under the last model we have analyzed the students' responses and detected a paticular strategy, thus responses given by students are in direct reference to the drawing accompanying the problem, showing links between features of this drawing and the calculations. Another characteristic is that students develop their solution by their own, and that makes the difference with other strategies which are dependent on the instruction received. The starting point is a mental action introduced by students with reference to the concrete object showed in the drawing , these are basic transformations or groupings of certain elements that lead them to construct a particular calculation from a concrete case. Doing these basic transformations or groupings of elements, allows children to detach from a particular case some invariant and variant elements associated to the size of the object. These invariant and variant elements are related to *a*, *b* and *n* that are characteristics of the underlying mathematical object an+b.

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CONCEPTUALIZING NON-UNIT FRACTIONS: AN HISTORICAL APPROACH

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An analysis of the historical development of fractions can yield useful insight into the difficulties the subject poses for modern math students. The study of Egyptian fractions provides an excellent example of how an individual's steps in mastering a skill mimic those of the predecessors who first attempted the mathematical manipulation. It also serves as an example of how an historical analysis of a concept's development can reveal a more effective pedagogical approach to a subject.

Although the ancient Egyptians possessed a sophisticated technology, they never developed a fractional notation that advanced beyond the unit fraction. Instead, all fractions were written as the sum of other unit fractions. For example, 5/6 was written as the sum of 1/2 and 1/3; 2/43 as the sum of 1/42, 1/86, 1/29, and 1/301.

Egyptian fractions were designated by placing the symbol 𓂋 over the symbol for an integer. Since the symbol for 4 was IIII, the symbol for 1/4 was 𓂋IIII.[1] However, the symbol 𓂋 was also the hieroglyph for "mouth,"[2] and it, therefore, is reasonable to interpret 𓂋 as the instruction "bite into four equal parts." To convey the concept of 3/4, an additional instruction is needed, either to add or to multiply. It is immediately clear that the leap from unit to non-unit fractions requires the mathematics student -- in any millennium--to perform two operations consecutively, operations of fundamentally different natures whose relatedness may not be immediately obvious.

In our study of 58 middle school students, we used a mathematics diagnostic inventory to examine students' skill with rational numbers. Preliminary findings indicate that difficulty in moving from unit to non-unit fractions is a common problem among students who have difficulty in working with rational numbers. These students consistently exhibited the same difficulties, even when working in the notation-free environment of manipulatives. Our results suggest that the initial introduction to fractions should be restricted to unit fractions and that the transition to non-unit fractions be afforded a greater degree of emphasis. A broader implication of this study is that Ernst Haekel's maxim:" Ontogeny recapitulates phylogeny," applies to pedagogy as well as it does to biology.

[1] D. E. Smith, History of Mathematics vol. 2 (New York: Dover Publications, 1953) 210.

[2] A. Gardiner, Egyptian Grammar (Oxford, England: Oxford University Press, 1969) 197.

TRANSITION BETWEEN INSTITUTIONS : THE CASE OF ALGEBRA IN THE TRANSITION FROM VOCATIONAL HIGH SCHOOLS TO GENERAL HIGH SCHOOLS

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The research we present deals with transition problems in educational systems. Our aim is to understand the problems encountered in such transitions and to identify conditions necessary to a positive evolution. A specific case is analysed : transition from vocational to general high schools, for tertiary programs, and a crucial mathematical domain is chosen : elementary algebra.

This research articulates two theorical backgrounds : an anthropological one inspired by Chevallard (Chevallard, 1992) and a cognitive one. It is based on the following principles :

- mathematical knowledge is hisghly dependent from the institutions where it has to live, develop or be taught,
- knowledge with respect to a mathematical concept or a mathematical domain is necessarily a multidimensional object which, in each of its dimension, is only partially ordered,
- in order to understand the nature of such a multidimensional structure, it is necessary to look for the more or less local coherencies which structure it.

According to these principles, from a methodological point of view, our research is based on the definition of a multidimensional structure for algebraic knowledge.

• Firstly, this structure has been used in order to analyse the institutional relationships with algebra at play in the transition process. Analysis shows that beyond official texts for curricula which look very similar, the algebraic culture and effective practices of each institution present evident discrepancies which are not seriously taken into account in usual mathematics teaching of transition classes. Moreover, the analysis of students' copy-books and exercise-books, allows to identify strong regularities which can be connected with some characteristics of previous teaching and could explain some important difficulties encountered by students, in the transition process.

• Secondly, the structure has been used in order to elaborate a set of 19 diagnostic tasks as well as to define criteria of analysis for students' productions. Research also allowed to identify some positive coherencies in terms of "students' profiles" which, in our opinion, could be used as starting points for an educational action in this area.

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THE EFFECTS OF A GRAPHING-APPROACH COLLEGE ALGEBRA CURRICULUM ON STUDENTS' UNDERSTANDING OF FUNCTION

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The function concept is of fundamental importance in college mathematics, but, it is also one for which students seldom develop a satisfactory understanding . One possible explanation of the difficulties that students have with the function concept is that the students may develop only naive conceptions of functions. Researchers hypothesize that students acquire the function concept in two stages: *procedural* in which the function is viewed as a process such a assigning values, and then *structural* in which the function is an object on which operations can be performed (Sfard, 1991). The transition from procedural to operational is termed reification and is very difficult for students. Some researchers suggest that technology has the potential to assist students in develop structural conceptions of some algebraic concepts.

This quasi-experimental semester- long study examined the effects of a graphing approach curriculum along with the TI-82 graphing calculator on students understanding of the function concept, traditional algebraic skill, and mathematics attitude. The function posttest had four subtests: modeling, interpreting, translating, and reifying. The research questions were based on the process/object (reification) theoretical framework. Four classes at a large state university served as the population with the balanced design: two teachers, each teaching one experimental and one control class. The classes studied the same topics, but the traditional algebra curriculum focused more on by hand calculations.

Findings indicate that students in the graphing-approch classes had a better understanding of functions as a group than did the traditional algebra group. They had significantly higher scores on the function test and on all four subtests. No significant differences were found on the departmental final exam which measures traditional algebraic skill without a graphing calculator, or on the attitude measure.

Current mathematics theory suggests that technology has the potential to help facilitate the development of structural conceptions of functions. This study supports that theory and suggests that the graphing calculator has some effect in bridging the gap between operational and structural conceptions.

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ASSESSMENT IN GEOMETRY FROM TWO POINTS OF VIEW: LEVELS OF REASONING AND SOLO LEVELS.

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This paper reports part of a longer research on assessment in geometry. Manly, it tries to show how we can use more than one learning theory to evaluate students' answers: in this case, van Hiele theory and SOLO Taxonomy. To reach this objective, we have to solve some problems related to the research methodology.

The first problem we have to solve is to construct a set of items letting students demonstrate both their reasoning abilities and the quality of their answers. We have used the Collis, Romberg & Jurdak (1986) superitem structure to construct items to evaluate SOLO levels: "the term superitem describes a set of questions that are asked about a particular problem situation. The problem situation is typically described in the item stem, which consist of a paragraph describing the problem, and the items consist of a series of questions that can be answered by reference to the information in the stem" (p. 211). If in questions one involves reasoning processes like identify, read and use definitions, classify and demonstrate (Jaime & Gutiérrez, 1994), we will have the instrument that assesses students' answer from both points of view.

The second problem we have to solve is how can we assign levels to the students. On the one side, we have assigned reasoning levels using Gutiérrez, Jaime & Fortuny (1991) technique. This will let us classify students in relation with their grade of acquisition of the van Hiele level (Gutiérrez, Jaime & Fortuny, 1991). On the other side, we will assign SOLO levels to the students following Collis, Romberg & Jurdak (1986) and Collis & Watson (1991). Both classifications probably let interpret us which is the place in which a student is in his/her learning process of a geometrical topic.

In congress we will present a superitem and students' answers to this superitem and its assessment from both points of view: levels of reasoning and SOLO levels.

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RESEARCHING COMPUTERS AND COLLABORATIVE LEARNING IN PRE-SERVICE MATHEMATICS TEACHER EDUCATION

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In 1996 it is probable that every school in Australia has some computers, but not every mathematics teacher is able to use a computer. In some mathematics classes computers are used to provide students with opportunities for exploring patterns and relationships. Some rural teachers and students participate in senior secondary mathematics classes using computers for communication. However in the majority of mathematics classes, at both primary and secondary levels, use of computers by students is not an integral part of the mathematics curriculum.

The content of school mathematics has changed little in the past decade. Recent developments, for example NCTM (1989) in the US and Board of Studies (1995) in Australia, have arguably focused more on methods of teaching and learning mathematics than on revising mathematical content. In particular, these and other reforms have not successfully come to terms with computer use as an integral part of mathematics.

This report focuses on a research project with the aim of developing a model, for pre-service mathematics teacher education, that provides a theoretical underpinning as well as practical experiences in collaborative problem solving using computers as a computational medium. Participants worked in small groups and were given tasks to solve using a designated piece of computer software. As well as recording some group work at computers, a reflective discussion at the end of each problem solving session was video-taped and analysed.

As reported in previous studies with school children, there were agreements and disagreements among the mathematics education students as they attempted to solve problems. The role of disagreements during the process of collaborative learning in a technologically rich environment will be a major focus of this paper.

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An experiment on computer-assisted problem posing in undergraduate mathematics

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The idea of *problem posing* (Brown 1984; Brown & Walter 1990, 1993; Silver 1994) strikes one as being both intellectually and morally exciting. Few would not agree, in principle, with a pedagogy for problem solving that takes account of the fact that a crucial part of the activity of "problem solving" as it is practiced by real mathematicians, scientists, *etc.* is not the "solving" itself but the finding (or generating), assessing, choosing and refining of problems to be solved.

Creating problem posing situations for students is a challenge; as Paul Goldenberg (1993) has written: "What I need is a kind of guidance for my students that neither leads them by the nose nor leaves them wandering aimlessly: a way of helping students have an adventure without my knowing in advance exactly what adventure they are likely to have."

I will report on a modest experiment concerned with devising some computer-based materials, using the software *Mathematica*, that try out various styles of problem posing activity with undergraduate students (in particular, non-mathematical specialists). I will attempt also to tease out the meta-problem posing issues: how I resolved *my* problems in designing problem posing activities.

Having a powerful computer system like *Mathematica* to hand, which can handle the "routine" algebraic and graphical manipulations, removes certain barriers that keep students from freely exploring in problem posing situations. One of the aims of the experiment is to see how other, more fundamental kinds of barriers (cognitive, psychological, social) can be dealt with in the design of situations.

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STUDENT TEACHERS' UNDERSTANDING OF MATHEMATICS EDUCATION

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This project comprises the major part of a PhD program under the Department of Psychology at The University of Bergen, Institute of Practical Pedagogy (IPP).

The theoretical background for my analysis is constructivism (Jaworski, Lerman, von Glaserfeld) and social constructivism (Ernest). I wish to explore whether "Activity Theory" (Vygotsky, Mellin-Olsen) can help to shed light on whether students perceive their pupils as individuals who may have their own goals in the mathematics classroom.

My focus at the moment is students' conceptions of the word 'understanding'. Some students seem to conceive 'understanding' as the ability to mechanically cary out mathematical operations - 'how' to multiply, divide etc; while other students see 'understanding' as a matter of having genuine insight in the underlying principles of - 'why' we carry out these operations.

I am interested in students' ideas about how children learn mathematics. Do they envision their teaching to be dominated by mathematical rules introduced by themselves. Or do they wish to create a situation in which the children themselves get a chance to explore and discuss? When the students teach, what do they do, and what do they think they do? Is the focus on learning?

I have interviewed 28 students in the beginning of their first mathematics course. Half a year later I observed 22 of the students in their teaching practice. After the observation I interviewed them about the lesson.
I plan to meet some of the students for a third time in 1996/97.

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KNOWLEDGE OF TRANSFORMATIONS OF FUNCTIONS: POINT-PLOTTING AS OPPOSED TO USE OF THE GRAPHICS CALCULATOR

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An aspect of the research entailed in a broader project related to the effectiveness of the graphing calculator in the learning and teaching of function transformation concepts at grades 11 level in the South African context, is reported on.

Materials based on the work of Confrey et al. (1992), and Wenzelburger (1992), were developed. These materials focused on concepts relating to transformations of a range of functions through the use of appropriate investigations. Two groups of learners were established either in the same schools or in similar neighbouring schools. One of the groups used ordinary scientific calculators to do immediate pencil and paper plots of the graphs of the functions but otherwise the same materials and approach as the group using graphics calculators. Data gathered by pre-post tests, and a delayed post-test were used in this analysis.

The teaching-learning process was designed to enable the learner to draw out generalisations and thus get a global view of a number of instances of common behaviour. The test items were therefore designed to force students to call on the generalisations made rather than deal with the particularities of specific functions. The generally low scores obtained in the post- and delayed post-test indicate that students had difficulties in achieving this extent of generalisation. The point-plotting group means for the post- and delayed post-test were significantly higher than those of the group that used the graphics calculator. The delayed post-test means indicated that the retention of the point-plotting group was higher. A finer analysis which provides scores on specific conceptual elements within the test did not detect any qualitative differences across the two groups.

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LEARNING FROM STUDENTS' OUT-OF-SCHOOL MATHEMATICS PRACTICE

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Research on cognition and learning has pointed out the need for closing the gap between learning and doing mathematics in and out of school. Furthermore, recent proposals for teaching and learning mathematics in school have encouraged educators to connect mathematics with other subjects and out-of-school mathematics practice. However, in order for teachers to help students make these connections we need to know how persons use and perceive how they use mathematics in out-of-school settings. But as Pea (1991) stated: "Even though that field [mathematics education] calls for relevance of mathematics learned to everyday settings, there has been remarkably little ethnographic investigation of mathematical activities by children in settings outside classrooms."

As part of an ongoing study, I examined the mathematics practice of six middle school students and used Saxe's (1991) research framework: (a) to gain insight into the goals that emerged during students' out-of-school activities, and (b) to explore the cognitive forms and functions students constructed to accomplish these goals. Through activity sampling with electronic pagers, logs, interviews, and observations, I investigated (a) the activities in which students participate, (b) the mathematics they perceived using in these activities, and (c) the mathematics she perceived the students using. In the paper, I identify the principal functions for mathematics concepts and processes in the respondents' mathematics practice, and describes the implications these functions have for the respondents' construction of mathematical goals (cf. Saxe, 1994). I also identify a range of strategic forms that the respondents used to realize these functions. These analysis methods are similar to the ones used by Saxe, Guberman and Gearhart (1987) in their work examining the enculturating processes that support children's number development.

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ASSESSING PRESERVICE TEACHERS' CONCEPTUAL UNDERSTANDING OF PERIMETER AND AREA

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Inadequate mathematical competency of teachers and learners have resulted in research that, among others, suggests the need to identify deficiencies in teachers' own knowledge (Eisenhart, Borko, Underhill, Brown, Jones & Agard, 1993), that students revert to rote-learned procedural knowledge when under pressure to complete a task (Tall, 1995), and that pencil-and-paper tests do not effectively measure mathematical understanding (Clements & Ellerton, 1995). To assess preservice teachers' conceptual understanding of perimeter and area, eighty four postgraduate preservice teachers (elementary school)--with mathematics knowledge ranging from high school to university level mathematics--were given three problems and asked to decide whether there was sufficient information given to work out a solution. If sufficient, they were to work out a solution, and if not, they were to explain what information was missing. The results show that although all the questions had sufficient information, an alarming number of the preservice teachers thought otherwise and some used questionable reasoning or pedestrian computational procedures. This is cause for concern because these teachers had done well in standardized public pencil-and-paper mathematics examinations, yet seem to have inadequate conceptual knowledge to teach these elementary school topics effectively. If such poor conceptual understanding is widespread, then preservice mathematics education courses have to address and redress this.

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A Framework for the Quality of Explanation in relation to the Distributive Law

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The SOLO taxonomy (Biggs and Collis, 1982) uses learning quality as a point of departure and measures students' performance at a particular time. The hierarchical nature of the SOLO levels can be used a reference for comparing students' proficiency and development at a particular discipline. The non-domain specific nature of the taxonomy gives the advantage of applicability in a range of school subjects. Nonetheless, in terms of assessment, it seems necessary to consider the interaction between domain-specific knowledge and general cognitive skills (Webb, 1992). Drawing upon the above ideas, this presentation describes a framework based on both the SOLO taxonomy and domain-specfic components.

This research used a set of interview tasks developed in an earlier study (Mok, 1994). In order to describe the quality of students' explanations in relation to the distributive law, a framework was developed from the responses of secondary students. The framework extends the SOLO taxonomy to incorporate domain specific components of mal-rules and misconceptions, strategies and perception of a problem which are directly related to students' algebraic knowledge. The components of the framework are described below.

- *Mal-rules and misconceptions*: Mal-rules refer to ad hoc rules used by students which may be wrong or partly correct. For example, the distribution works for both multiplication and division. Here, a shorter list under this component implies a better explanation
- *Strategies*: Strategies like applying rules, substitution, fraction transformation, exhaustion and proofs were found to be used by students. The proficiency in these methods gives another measure of the quality of explanation.
- *Perception of a problem*: This component considers whether students can anticipate the variations of a statement when the operation or variable factors of the statement vary.
- *SOLO*: This component is described in terms of the SOLO descriptors.

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DIVISION WORD PROBLEMS
THE CONSTRUCTION OF REPRESENTATION AND PROCEDURES IN YOUNG CHILDREN

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Introduction

Developmental psychology, mathematics education an learning have always shared a common interest in problem solving. As a matter of fact this domain of research is important to understand the relationship between the construction of general knowledge and the individual procedures leading to the solution of a problem. The letter can be different from one subject to another according not only to its structural level and social and cultural background but also to the specific task proposed. Researchers from several countries (De Corte, 1987; Morgado; 1991; Vergnaud, 1983) have studied these procedures in verbal arithmetic problems, trying to relate them to the construction of their general representation(*). The aim of this investigation is to analyse, in division word problems (multiple groups, proportion and keyword problem, according to Vergnaud's classification) the relationship between the construction of a correct representation of the data and a correct solution to the word problem, regardless of the kind of procedure employed by the child to solve it.

Methodology

In our study we chose four word division problems and a subtraction word problem, to avoid the general idea that all of them could be solved by the same arithmetic operation.

The sample was composed of 75 subjects (aged from 7; 0 to 9; 8.). 25 attended the 2nd grade (Group A); 25 the 3 rd grade (Group B) and 25 the 4 th grade (Group C). Children from Group A didn't have any specific knowledge of division word problems. 1. All problems were presented randomly and individually to each child. 2. The child began to read the word problem and he had to explain it verbally to the experimenter. 3. The experimenter gave him realia (manipulative material) adapted to each problem and asked him to represent the word problem with it. 4. Finally the experimenter asked the child to give the result and to explain verbally how he found it.

Results

1. The number of children who solved the division word problems correctly increased with age and grade. 2. A strong relationship between a correct representation and a correct answer in each problem was found. 3. Several procedures were employed by the children to solve each problem (addition, subtraction, partitive division, multiplication). 4 Finally it seems impossible to establish a relationship between age/grade and kind of procedure, except in the keyword problem where more than 90% of the children employed a multiplicative one.

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(*) Representation: The child's organisation and understanding of the semantic struture and meaning of problem's data.

THE QUALITY OF STUDENTS' REFLECTION IN PROBLEM SOLUTION

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AIM AND CONTEXT OF STUDY

George Polya (1957) has suggested four phases of problem solving and they are :

(i) Understanding the problem
(ii) Making a plan
(iii) Carrying out the plan, and
(iv) Looking back.

The fourth stage of looking back or reflection on an attempt that one has made is crucial in the problem solving process because it assists one to determine whether the solution obtained is the correct one, and whether any improvement or change in the attempt made is necessary. Inability to reflect on the solution deprives the problem solver of a chance to 'consolidate knowledge gained and development of ability to solve other problems' (Polya, 1957). When this is not done, students fail to appreciate the broad range of beneficial consequences to be derived from looking back (Taback, 1988, 429). It also helps the student to think about his/her thinking and anticipating the results of the potential action (Wheatley, 1992, 537).

This paper reports on the findings of the study which was carried out with Standard 9 and 10 students at a scho·.l nearby the University of The North, Northern Province, South Africa. The purpose of the study was to investigate the students' thought processes as they attempted the problems given to them, with some more focus on the extent to which students review the solution attempts. Some of the specific questions of the study were:

(a) Do students reflect at all on the solution attempts to the problems they are given or try on their own?
(b) If and when they do reflect, does this help in any way in identifying the problematic areas in the solution attempts?

METHODOLOGY

Ten students - five standard 9 and five standard 10 - were selected from a high school near the University of The North. The number of students was kept as small as possible to enable them to spend more time solving the problem without the limitation of time. Three mathematical problems (exercises) were given to the students to solve. These were based on the work that they had covered in the syllabus at the time of the study.

FINDINGS

The results of the study indicate the following:

- very little reflection (in some instances none), takes place after arriving at some kind of a solution when students engage in mathematical problem solving (exercises);
- students seem to spend very little time in trying to understand the problem before they start with their solution attempts;

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AN EXPERT SYSTEM TO GUIDE THE GEOMETRY TEACHER

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An expert system is a computer program that emulates the performance of a human expert in several kinds of tasks: giving advice, planning and suggesting decisions in specific subjects. The knowledge-base must be previously acquired from experts, and introduced in the program after it has been suitably structured (Gale, 1986). This kind of system has often been used in medicine.

The aim of this work is the development of such an expert system to guide the teacher of Geometry, according to the van Hiele theory (van Hiele, 1986). Depending on the van Hiele levels attained by the students in a class, the system can advise the teacher about the best strategies to adopt in the teaching of geometry to that class.

The teacher identifies the students' van Hiele levels through tests or interviews, and input the percentage of students in each level in the system. According to this information, the system immediately suggests the best level to be adopted for the teaching of Geometry to this class. If there are some students reasoning in a lower level than the recommended for the teaching, or who failed to follow the hierarchy of the levels, the system suggests extra activities to help them reach this recommended level.

The system also suggests what kind of tasks should or should not be given to this class, according to the descriptors of the van Hiele levels (Fuys, Geddes and Tischler, 1988), as well as examples of adequate exercises. For example, if the system recommends teaching at the second van Hiele level, some instructions will be given for the teaching, such as: "*students should be able to identify properties of shapes and classify shapes according to common features; students are not expected to give proofs or justifications, nor to grasp the inclusion of classes* ".

In this research, the system was developed regarding the geometry course for 8th-grade students (14-15 years) and for freshmen Mathematics undergraduate students. It is been tested by teachers of both age groups in Rio de Janeiro.

This expert system can be very useful, since it allows Mathematics teachers who don't master completely the van Hiele theory, or are not used to work with computers, to benefit from both of these tools in a simple, straightforward and time-saving way.

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GOING DEEPER INTO MATHEMATICAL ACTIVITY

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The work that we present is part of a wider investigation, the purpose of which is to give explanations of the use that students of 8° of E.G.B. (Enseñanza General Básica, 13-15 years) make of the mental images when they solve and make meaning of mathematical problems, and to analize if the role of the teacher has influence in these mathematical activities of the students.

The stages of the investigation, which started 3 years ago, are as follows:

a. Setting up the project during the speaker'stay at Florida State University (U.S.A.).

b. Collecting data in the school, La Corujera, Tenerife (Spain), during one year.

c. Analysis of the data and others factors, the present stage of the investigation.

In this study we used the ethnography method which was crucial in order to know more about what happens in a classroom.

We will describe and interpret here two case studies, their beliefs and their styles of mathematical understanding, the use or non-use that they make of mental images, inside and outside of the school, analizing the influence that the Math'teacher exercises in this activity.

Student A, who was repeating the course and was not considered to be academic, showed great confidence and creativity when it came to resolve mathematical problems.

Student B, who did not have academic problems, who had a good reputation with the teachers and high qualifications, showed little confidence in confronting mathematical problems.

Finally we will suggest some questions relating to our research.

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CURRICULUM MATERIALS IN MATHEMATICS EDUCATION REFORM

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This study explored the potential role of textbooks and teacher's guides in the current reforms in mathematics education. Using case study methodology, the author examined the experiences of two fourth grade teachers during their first year of using a reform-oriented mathematics textbook.

Because of their predominant role in American classrooms, textbooks have been viewed as potential levers of reform. Observers have been skeptical about the power of texts to transform teaching, citing evidence that teachers are influenced more by their beliefs and knowledge about teaching, learning, and the subject matter than by what is presented in texts (cf. Remillard, 1992; Stephens, 1982). Thus, underlying this study is the perspectives that in order for curriculum materials to lead to significant instructional change, they must foster learning on the part of teachers (Ball, 1995; Cohen & Barnes, 1993). A central question in the study was whether and how curriculum materials might contribute to such learning.

Findings illustrate how teachers' beliefs and the teaching context mediated their use of and learning from the textbook. They also reveal that significant learning occurred in both teachers when they engaged in the types of decision-making activities that are often the assumed purview of textbook writers. These findings have implications for changes in how curriculum developers interact with teachers through the materials they design. They also suggest that reform-oriented curriculum resources, without accomanying support, can be unpredictable mechanism to stimulate teacher change.

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Roots of Teacher Differences: Beliefs of Preservice Elementary Vs Secondary Teachers

It has been widely reported that teachers' beliefs greatly influence their classroom practices and that beliefs are extremely resistant to change. Weiss (1995) reports that responses reflecting beliefs of teachers in the 50 United States differ according to the grade level taught. This survey indicates that elementary grade level teachers are far less familiar with the NCTM Standards (1989) than are high school teachers; however, the elementary teachers espouse higher support for 17 out of 18 teaching strategies which are consistent with the Standards. This inverse relationship between familiarity with reform documents and belief in the reform goals is disconcerting as is the apparent lack of consistent teacher beliefs across the grade levels. A concern of this investigation is to what extent and in what ways do the beliefs of these two groups differ prior to their teacher training programs and classroom teaching experiences. Indications of prior beliefs or the roots of differences may assist educators in planning effective teacher preparation and inservice programs to achieve consistent support for the goals of the current reform movement in mathematics education.

Data on the beliefs and attitudes of undergraduate, preservice elementary and secondary teachers (N= 25 for each group) was collected by a 32 item written survey (Ebert, 1993) using a five point Likert scale. The teacher preparation program is a graduate program for these undergraduate students. The survey included statements on the nature of mathematics, the role of the teacher, methods of instruction, problem solving, etc. with positive and negative statements with respect to the mathematics education reform goals. A more qualitative measure was obtained from a written survey (O'Daffer, P., Charles, R., Cooney, T., Dossey, J. & Schlieack (in press)) of five questions on the nature of mathematics, attitudes about mathematics, confidence with mathematics, and influences on attitudes toward mathematics. Four of the questions were an open response format. The replies were analyzed for recurring themes and implied beliefs, with interrater agreement.

The beliefs of the preservice elementary and secondary teachers are quite different from each other, and each group differs from the goals of the reform movement in distinct ways. For example, an analysis of variance on the responses to the Ebert survey indicates that the two groups differ significantly at the $p=.05$ level on eight items including "using problems and applications as an excellent means to introduce new mathematical content." There was a strong tendency to differ on seven additional items. An analysis of the responses of the two groups to both surveys characterizes various beliefs prior to teaching experiences and illustrates areas of agreement and differences. A discussion of how educators might use this information in planning teacher preparation and inservice programs for consistency of teacher beliefs in support of the mathematics reform goals will conclude the presentation.

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COLLEGE STUDENTS' KNOWLEDGE AND BELIEFS

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This paper reports an attempt to describe mathematical knowledge and beliefs of students at undergraduate level.

The hypotheses underlying our research is that there will be a relationship between student beliefs, student knowledge and their prior experiences in mathematics. The purpose of the study was to describe student beliefs and their understanding of the concept of derivative and explore the relationships with student background from upper secondary school.

The target group for this investigation was students starting a study program in economics and business administration at Molde College in Norway in the fall of 1995. Data were collected by a questionnaire administered during one class session. 178 students filled in the form.

The questionnaire included questions on student beliefs about mathematics, tests on student understanding of the concept of a function and its derivative and information on student background from upper secondary school. Information on student grades at college courses was available from the college student records.

Students express a rather positive view on mathematics and its usefulness and support that it is important to understand mathematical concepts. Our results confirm findings in the litterature that most students have reached a rather poor understanding of the concept of derivative.

The preliminary analysis show covariation between student beliefs and knowledge as measured by the test on concept understanding and course grade in mathematics at the college. The data also give some interrelationship between beliefs and knowledge at college level and the number and grades of mathematics courses completed at upper secondary school.

We generally find our results to support a hypotheses that students' beliefs can be seen as an indicator of student prior experiences in mathematics.

The results so far seem to confirm that student beliefs play a role for the way they engage in mathematical activities and for the learning of mathematics.

THE OBJECT AND AIM OF TEACHERS AS REGARDS PUPILS' LEARNING IN MATHEMATICS.

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The aim of this study, is to find which intentions, explicit as well as non-explicit ones, that teachers have as regards to their teaching of mathematics. The study intends to answer the following questions:

1. What is it that teachers focus in their teaching, i. e. what are their objects and aims in mathematics?
2. Of what origin are these objects and aims, i. e. in what do they find their point of departure?
3. How do teachers understand the nature of matehmatical knowledge, skills and understanding that teachers wish their pupils to acquire through their teaching?

The study will be carried out within the phenomenographical research approach. Phenomenography is empirically studying and describing different ways in which people experience, perceive or are aware of the world around us (Marton, 1981; 1993, 1994).

The study includes five teachers of matehematics and their pupils from grade six and seven. The five teachers have been followed up in their teaching during a clearly defined section dealing with rational numbers. The teachers were interviewed before the actual teaching began, the aim of which was to find out what teachers focus before they start teaching. After that eight consecutive mathematics lessons were audio-taped where all the communication which took place in the class-room has been documented. After that a further interview has taken place.

As of today, the collecting of data is concluded and analysis is in progress.

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Mathematics Learning — Students' apropriation process of mathematical artifacts*

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In this paper I present some results of a research project on school mathematics learning of a small group of 8th grade portuguese students. In that research I tried to understand how the students' mathematical knowledge is structured and developed through the interaction with their everyday activities in the context of the mathematics classroom. The metodological option assumes a interpretative nature .

Vygotsky began, in 1930, the discussion about cognitive development under an almost individual perspective (how interpsychological becomes intrapsychological) but, in 1934, he was trying to understand how this development emerges from institutionally situated activities. Within the social environment where people learn, Vygotsky included people as well as tools and signs that mediate social interactions. But this cognitive change does not happen in a closed and determined system, but in *systems of social activity* which leads to the preference, within a sociohistorical approach to cognition, to talk about "individual-acting-with-mediational-means" instead of "individuals" (Wertsch, 1991, p. 12). We can find a similar perspective in Lave's "project" of looking to cognition as a *social antropology of cognition.* In this approach, Lave (1988) considered that "cognition observed in everyday practice is distributed — stretched over, not divided among — mind, body, activity and culturally organized settings" (p. 1). She does not accept knowledge acquisition as "context-free" and tries to do a (empirical and theoretical) "characterization of situationally specific cognitive practice" (p. 3). With some similarity with Vygotsky' ideas, she will argue that "a more appropriate unit of analysis it will be the whole-person in action, acting with the settings of that activity" (p. 17) as she conceives that "setting and activity connect with mind through their constitutive relations with the person acting" (p. 181). Under this perspective, mathematics is seen as an act of sense making which is socially transmitted and constructed. Learning to think mathematically means to develop a mathematical point of view and competence to work with the proper tools in order to understand and appropriate a mathematical sense making.

I will focus this presentation on the students' apropriation process of three mathematical artifacts (in Saxe's terms) in order to clear up some elements of the context that seems to take an important role on that process, such as: the social interactions with colleagues and teacher, the structure of practice and the students' individual motives and goals.

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THE INITIAL GROWTH OF PROSPECTIVE MATHEMATICS TEACHERS THROUGH PARTICIPATION IN A TEACHER DEVELOPMENT PROJECT

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The research presented here is part of an ongoing inquiry which investigates the influences and effects of the engagement of undergraduate mathematics majors in a continuous professional teacher development project while they are still enrolled in their teacher education program. The staff teacher development is an ongoing project at the Mathematics Institute at Universidade Federal do Rio de Janeiro in which collaborates university professors, secondary and elementary school teachers and undergraduate students. These persons from three different populations arrange themselves into small groups that work every week. Initially, the groups discuss and reflect about problems of teaching and learning school mathematics. Afterwards, with the help of research literature, they put into practice alternative ways to approach didactical problems into the school teachers' classrooms. In reality this is a collaborative project of action research that furnishes to all members opportunities to share expertise and knowledge as well as to question, rethink, reflect and investigate on-site the complexities of mathematics teaching, learning and assessment (Nunes, 1993; Raymond, 1994; Santos & Nasser, 1995). Having in mind the aim of investigating if the participation of preservice teachers in this collaborative enterprise is really leading to a professional preparation of future teachers, this author examined, analyzed and categorized data collected in 1994 and 1995. The sources of data were field notes of observations of undergraduate students' actions and attitudes in several moments as members of the small groups and/or whole group; their written questionnaires (e.g., their views of mathematics teaching, learning and assessment; their views of themselves as mathematics learners at the university level, their views of themselves as potential teachers, etc); their written reports about the work developed during the year in the project; their written reports of a mini-inquiry developed in a school setting; and their mathematics biography. In this study it is reported how these potential teachers have begin the development of their metacognitive awareness about mathematics teaching and learning throughout their participation in this continuous professional teacher development during the last two years. This work discusses excerpts of preservice secondary teachers' written and oral comments concerning their growth, awareness and perceived changes when engaged in a staff teacher development project that is an extra curriculum activity for them while undergraduate students.

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MATHEMATICS, FRACTALS AND ENCODING IMAGES

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The purpose of the learning experiments planned in this research in mathematics education is the study of the interactions a learner and a teacher establish and maintain in a problem-solving task in the context of a geometry microworld built with the MacBoxer programming environment (diSessa A, 1994).

The family of tools designed and the MacBoxer structures are the main components of this microworld which aims to help teachers and students to work collaboratively, to enrich their discussions and provide a deeper analysis through mathematical formalisations using basic Logo-like computer programming structures, in a class of tasks that facilitates constructing mathematical knowledge (Thompson P & Thompson A, 1994) and making connections in algebra, geometry and discrete mathematics (National Council of Teachers of Mathematics, 1991).

Three of the main benefits that can be found in this MacBoxer microworld are, firstly, its programming openness that can easily support collaborative forms of student-centered learning, secondly, its capitalization on Logo's strengths and powerful ideas, finally, its capability to let students learn to be fluent in some aspects of mathematical experience, for example, encoding images by simple geometric transformations, a technique that emerged very recently for processing images with great savings in computer memory (Peitgen & als.,1992).

The microworld enables an user to extract meaning about this process of encoding images with *iterated function systems* (IFS) either by experimenting the situations suggested, or by seeing the effects of changes in the tools' *doit boxes* or in the linear mappings' *data boxes* that define the process of attractors' generation, called *Multiple Reduction and Copy Machine*, (MRCM), a metaphoric designation due to Peitgen & als. Shortly stated, MRCM is like a copy machine that takes an image as input across three independent lens systems, each of which reduces the input image and assemblies the three reduced copies in some pattern, and then runs in a feedback loop this output as a new input, again and again. In our pilot learning experiments we use a Mac Quadra 650 computer, with a larger screen and 20 MB of RAM.

One of the implications of this research for practice can be the improvement of instructional materials to support teacher educators to enrich their teaching of inductive approaches.

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CONSTRUCTING KNOWLEDGE TOGETHER IN THE MATHEMATICS CLASSROOM: WHAT DO STUDENT TEACHERS LEARN ABOUT TEACHING MATH FROM THE PRACTICUM?

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The student teaching practicum is a significant feature of elementary pre-service teacher education. Mathematics educators often suspect that this practicum serves to reinforce the narrower, computation-oriented, approach to mathematics and doesn't contribute to reform efforts which put more emphasis on problem-solving and higher-order thinking strategies.This research was an attempt to document the connection between the way children were constructing mathematics in the class and the way the student teachers were constructing their knowledge base.

Two of the premises which inform the research are the following:
1) Learning mathematics is a social-cognitive process. It involves developing and enriching mental schemes and the connections among the schemes. (Carpenter, et al,1989; Cobb,et al, 1991; Hiebert & Wearne,1993)
2) Teachers' knowledge- subject matter knowledge and mathematical-pedagogical knowledge - influences, and is influenced by, what goes on in the classroom.(Cobb,et al, 1991)

The research was conducted in 1994-95. Two student teachers were assigned to second grade classrooms where the teachers agreed to teach mathematics from a constructivist perspective. Their program was significantly different from the conventional classroom - emphasis was on problem-solving, encouraging children's thinking strategies, and building the formal mathematics on children's intuitive knowledge. Two student teachers from the same pre-service program were assigned to second grade classrooms where the mathematics instruction was conventional.

The research focused on a qualitative documentation of the instructional approach, the children's mathematical understandings, and the student teachers' knowledge. The results of this case study provide compelling evidence that the student teachers in two different environments constructed their knowledge differently around two major issues: What is mathematics all about? What does it mean to teach mathematics in school?

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THE INFLUENCE OF "SUPPOSED OTHERS" IN THE SOCIAL PROCESS OF MAKING A MATHEMATICAL DEFINITION

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This paper explores the role of "supposed others" in the process of making a mathematical definition. A transcribed protocol was analyzed of a session in which two tenth grade students were videotaped while they were working on a task that asked them in pairs to classify and to make definitions of unfamiliar quadrilaterals. The focus of the analysis was on the social process of making a definition of "kite" by students. In an analysis of the protocol, the influence of the "supposed others" in the process of modifying the initial definition was identified. Based on this analysis, the author discusses the importance of suggestions and critiques from "others" for making a mathematical definition, and considers implications of this analysis for teaching.

The study is based on the analysis of a course described in "The Nature of Proof" (Fawcett,1938) that aimed to foster students' "critical thinking", in which definitions and propositions were socially constructed by students and the teacher. Fawcett's course does have a flavor of "an experiment in metacognition", as described by Crosswhite (1987), and suggests the importance of "critical thinking" fostered through the experiences of critiques from others. Drawing upon these sources, the study focused on the role of metacognition, as a critique from "supposed others" (Shimizu, 1993), in the social process of making a mathematical definition.

By attending to critiques from "others", it was found that students could distance themselves from the action of making a definition and were led to modify both the precision and constitutive elements of their definitions. It is one thing to learn a definition of a figure and it is quite another to appreciate the role of a definition in mathematics and how it is formed. The importance of "others" in helping students to appreciate the role and formation of a definition was suggested and some possible ways of facilitating this process are discussed.

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DEVELOPING NEW MODELS OF MATHEMATICS TEACHING: AN IMPERATIVE FOR RESEARCH ON MATHEMATICS TEACHER DEVELOPMENT

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New models of mathematics teaching that build on recent theoretical and empirical contributions are needed and essential for research on mathematics teacher development. Models of teaching are an essential component of conceptual frameworks for such research.

> A conceptual framework is an argument that the concepts chosen for investigation or interpretation, and any anticipated relationships among them, will be appropriate and useful, given the research problem under investigation. . . . conceptual frameworks are based on previous research and literature, . . . an array of current and possibly far-ranging sources. The framework may be based on different theories and various aspects of practitioner knowledge, depending on exactly what the researcher thinks (and can argue) will be relevant to and important to address about a research problem, at a given point in time and given the state-of-the-art regarding the research problem. (Eisenhart, 1991 p. 209)

Models of teaching, as part of the research framework, contribute both focus and justification for the foci chosen. Research on mathematics teacher development is intended to identify and describe the nature and extent of teacher growth (and the context in which that growth occurs) and key issues in and obstacles to that development. However, the range of teacher knowledge and activity is so vast that the challenge is to make useful choices about what to focus on. What constitutes development? Certainly, all teacher change would not be considered significant in a study of development.

Significant work in the development of new models of teaching is taking place. The development of Realistic Mathematics Education in Holland (Gravemeijer, , 1995) and the theory of situations in France (Brousseau, 1987) have provided models that can be adapted to guide research on mathematics teacher development. Ball (1993) and Lampert's (1992) studies of their own teaching and Cobb, Yackel, & Wood's (1992) teaching experiments in primary school classrooms have generated important constructs that are useful in characterizing new forms of teaching.

This short oral presentation will use an aspect of the author's emerging model of mathematics teaching (Simon, 1995) to illustrate the potential impact of a particular model on research on teacher development. The "hypothetical learning trajectory" characterizes important aspects of teacher thinking and decision making. As such it defines, particular knowledge and skills that are appropriate to focus in the study of the development of mathematics teachers.

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Teacher Beliefs and Practices in Primary Mathematics

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Research on teacher's subject knowledge (Wragg, Bennett & Carre, 1989) has highlighted the low proportions of primary school teachers in the UK who have specialised in mathematics. This is a cause for some concern in the light of evidence provided by Bennett & Turner-Bisset (1993) that subject knowledge has a powerful influence on teaching performance. Other studies also suggest that teachers' limited views about the nature of mathematics affect their classroom practice.

The research presented for this oral communication focuses on these issues. It arises from a study carried out in the UK investigating the process of organising and implementing mathematics tasks in primary classrooms (Brown & Simon, 1996). By focusing on teachers' intentions and expectations in relation to specific tasks, the research shows how primary teachers' views on the nature of mathematics, and their subject and pedagogic knowledge, are translated into classroom practice.

The research was carried out in four London schools and involved case-studies of the teaching of mathematics to children aged 7/8 and 10/11 in each school. The methods included interviewing and classroom observation.

The mathematics teaching varied considerably from school to school, as did the expertise of the teachers involved. Many teachers were non-specialists, but whereas some used primarily whole-class teaching with little variety of style or content, others intuitively provided a range of challenging tasks for their children. However these latter teachers, and in consequence their children, were not fully aware of the potential provided by the tasks, and opportunities for developing children's mathematical strategies were not fully exploited. Specialist teachers observed in the study promoted the use of 'mental maths' in their schools, and children taught mathematics in this way were more challenged than most, experiencing different ways of solving mathematical problems.

Shulman (1986) suggests that teachers must not only be capable of defining for students the accepted truths in a domain, but they must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing and how it relates to other propositions. Our results illuminate how any such knowledge is activated in planning student work.

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CURRICULUM REFORM and TEACHERS' CONCEPTIONS OF MATHEMATICS

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This study investigated changes in teachers' conceptions of mathematics through the use of *Mathematics in Context: A Connected Curriculum for Grades 5-8* (MiC), materials developed from a realistic mathematics perspective (Freudenthal, 1991). Using a social constructivist framework (cf. Cobb, 1994), this case study explored changes in the mathematical knowledge and beliefs of three teachers over a 10-month period and the opportunities for and impediments to such changes.

Research has indicated that teaching for understanding depends on knowing mathematics well oneself (Wilson, Shulman & Richert, 1987). The procedure-oriented mathematics that most elementary teachers experienced as students in school has left many feeling inadequate and often fearful of or disinterested in the subject (Simon, 1993). Nonetheless, these teachers must teach mathematics to their students. This dilemma suggests that teachers need to continue learning mathematics, and that elementary teachers, in particular, may need to reconstruct their understanding of important mathematical concepts. Professional development or inservice experiences have been the common response to these needs, but curriculum materials may also offer opportunities for ongoing teacher learning (Ball, 1996).

This study reported that important changes occurred in the conceptions of mathematics held by three teachers as they worked through the units prior to teaching them, as they taught with the units in the classroom, and as they reflected on the knowledge of their students, the learning difficulties of individual students, and other aspects of learning to teach with the MiC units.

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MAKING CONNECTIONS: REPRESENTING AND UNDERSTANDING THE NUMBER SYSTEM

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The numeration system can be seen as a consistent and infinitely extendable system that facilitates complex mental and written notational forms of numerical representations (both whole numbers and decimal fractions). The fundamental basis of numeration is the notion of treating a group as a unit. Hiebert & Wearne (1992) described children's understanding of numeration as "building connections between key ideas of place value such as quantifying sets of objects by grouping by 10 and treating the groups as units... and using the structure of the written notation to capture the information about groupings" (p.99).

Clinical interviews were conducted by the researcher with a cross-sectional sample of 132 Grades K through 6 children. The interview tasks were developed from those piloted in an earlier study (Thomas, 1992) and were designed to probe understanding of numeration and encourage explanation of the problem solving process by the children. The connections between the components of the numeration system are explored. Is it possible to find links between these components so that a hierarchy of skills and processes that build up to understanding numeration can be established? Alternately is there a non-uniform nature to the way children develop understanding of numeration? Is there some outward expressions of understanding the numeration system that might serve as 'landmarks' (Rubin & Russell, 1992) in the child's developing understanding? It is argued that children must be helped to make connections between the key elements of counting, grouping, addition, multiplication, regrouping, place value, zero as a place holder and powers of ten in order to construct a sense of the numbers forming elements of a 'system'. The various representations of the numeration system, concrete, verbal and notational, must be linked and reflect the multiplicative nature of the numeration system.

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DECIDING ABOUT STUDYING MATHEMATICS AT THE SENIOR SECONDARY LEVEL OF SCHOOLING IN AUSTRALIA

Professor Ron Toomey, Professor John Deckers, Associate Professor Bob Elliott, Associate Professor John Malone, Richard O'Donovan

This is a report of work in progress on an Australian Research Council funded project entitled A Investigation of Factors Influencing Australian Students Choice of Mathematics Units at Upper Secondary School Level.

The literature on the factors which influence students decision making about the nature of their involvement with senior secondary mathematics has been reviewed and used to develop a conceptual framework to guide the study. Factors that have been shown to influence students decisions about their involvement with mathematics have been grouped as organisational (eg. school time-tabling arrangements - viz. Ainley et al 1994) and personal or individual (eg. self-esteem), confidence with mathematics - Kloosterman (1990), cognitive preference or learning style - Fennema and Paterson (1985). The framework has been used to develop a questionnaire to gather information from approximately 6000 senior secondary school students in three Australian states about aspects of the organisational, socio-cultural and personal contexts of the decisions they made about being involved with mathematics. The survey has been administered and a preliminary multi-variate analysis of the data suggests that it would be useful to examine more closely the extent to which cognitive preference or learning style is a predictor of the nature of the involvement students have with senior secondary mathematics

A second phase of the study involves developing approximately 40 case studies of individual students' involvement with mathematics across the same three Australian states. A mixed method design (Greene et al 1989), drawing heavily on grounded theory (Strauss and Corben, 1994) has been used to conduct a preliminary trial of this part of the study. Preliminary case study data have been gathered for twenty cases in two Australian states. The analysis of the data suggests that the views young adults have about mathematics is tied to the strength of their sense of identity (Weiss, 1994, Wexler, 1992, Marcia, 1993) and to the extent to which they have a well developed 'life-plan'.

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PROGRESS IN MATHEMATICS EDUCATION REFORM

Laura R. Van Zoest, Western Michigan University

Mathematics education in the United States is in the midst of an exciting time of reform. This reform has been spurred on by documents such as *Everybody Counts* (National Research Council, 1989) and guided by the *Curriculum and Evaluation Standards for School Mathematics* and the *Professional Standards for Teaching Mathematics* (NCTM, 1989, 1991). Central to the success of this reform is the classroom teacher's ability to make significant changes in his or her approach to the teaching and learning of mathematics.

The current study reports on the successes and obstacles of a group of thirty high school mathematics teachers who are midway through a three-year program focused on providing mathematical and pedagogical knowledge to assist them in reforming their classrooms. Data were collected through individual interviews with the teachers, observations of the teachers' classrooms, and writings the teachers completed as part of the academic component of the program. The information was analyzed to search for patterns of successful implementation of reform and common obstacles faced by the teachers.

The aspects of reform that appeared to be easiest to implement dealt with issues of curriculum and technology. Changes were made in these areas primarily through adopting textbooks based on reform ideas and using graphing calculators in the classroom. The teachers also found it reasonably easy to move to a small-group seating arrangement and to implement isolated pedagogical and assessment changes such as journal writing, projects, and occassional group tests. The most difficult area of change was in shifting the focus of the classroom, specifically the mathematical thinking, from the teacher to the students. All of the teachers expressed a strong to very strong commitment to making this shift but were significantly less successful at carrying it out in their classrooms.

Too frequently classrooms have the appearance of reform but not the substance. This points to the difficulty, and importance, of communicating to teachers and other vested parties that achieving this reform requires more than changes in the way a classroom is run and the tools that are used. Achieving true reform requires a dramatic shift in the approach both teacher and students take to the teaching and learning of mathematics.

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POSTERS

$\sqrt{2}$ litre

Julián BAENA: Centro de Profesores. Granada. Spain.
Moisés CORIAT: Dpt..Didáctica de la Matemática. Universidad de Granada. Spain.
Pedro NIETO: Centro de Profesores. Sevilla. Spain.

Overview[1]

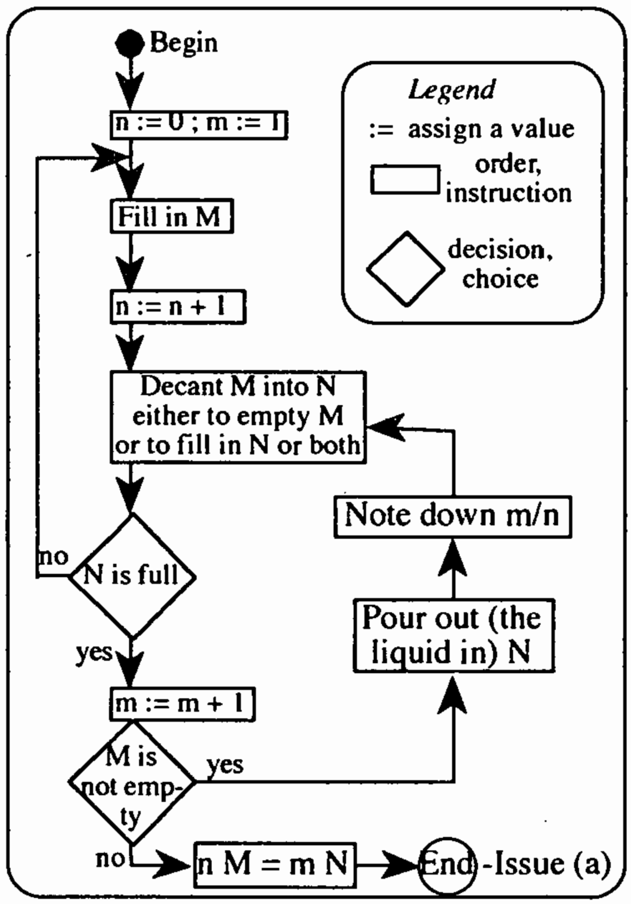
In order both to help pupils (14-15 years old) understanding irrational numbers, and to enhance a didactic phenomenology of real numbers, we present a measurement process based on Archimedes' axiom[2]. This process does not call at all for unit fractions; instead, it conveys a recurrent (ascending) calculation that is intended to make sense of endless decimals. The main goal is to obtain a capacity ratio. Capacity is the sole involved magnitude -we need no length nor volume. Therefore, the problem is *to measure a given quantity* K *by using a given unit* U.

Measurement process: By applying Archimedes' axiom to U and to successive multiples of K, we obtain a set defined by: $a_0 = 1$; $a_n = x$, x being the least integer verifying: $xU \geq nK$. To complete this definition, we allow for two mathematical issues. Either:

(a) K is *commensurable* with U. In this case, {an} is defined as being finite: $\exists$ p, q $\in$ N / $a_q = p$. $(pU=qK)$[3] .

Or:

(b) K is *not commensurable* with U. Then $\{a_n\}$ is infinite and $\{\frac{a_n}{n}\}$ converges to the irrational number defined by the ratio $\frac{K}{U}$.



Activity: Given two containers, M and N, follow the algorithm (see Flow Chart) to find the ratio:

(Capacity of M)/(Capacity of N).

Study the case

(Capacity of M)/(Capacity of N)=$\sqrt{2}$.

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[3] Dhombres, J. (1978): **Nombres, mesure et continu.** Paris, CEDIC-NATHAN.

[4] Euclides. **Elementos.** Libros V-IX. Spanish trans. Puertas, M. L. Madrid: Gredos (1994).

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[1] We follow guidelines proposed by some of us [1].

[2] «Given two quantities A and B (A<B) there exists an integer n such that nA>B.» Several authors ([2] (p.25); [3] (p.33)) consider as equivalents this axiom and Euclides' definition 4, V ([4] (p. 10).

[3] The last term is a_q; q is the least integer verifying that U divides qK.

EVALUATING AN INTERACTIVE CD-ROM DESIGNED FOR PRESERVICE TEACHER EDUCATION

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Mathematics educators have been involved in designing and implementing hypermedia instructional materials for the past several years (cf. Barron & Goldman, 1994; Lampert & Ball, 1990). The underlying goal of many of these projects has been to provide students with opportunities to explore the teaching and learning process by integrating video from real classrooms with interactive text and graphics. The question we are now considering is, "How can such a program be evaluated?" This question will be discussed in the context of our present efforts to evaluate the *Investigations in Teaching Geometry* CD-ROM. The poster contains screen captures as well as qualitative and quantitative data describing our efforts to evaluate how this program has been received by preservice teachers in five different methods courses taught by three different instructors at Vanderbilt University.

The overall intent for using this CD-ROM has been to focus on the ways in which mathematical meaning was established during the classroom discourse that is featured on the digitized video. That is, rather than presenting the segments as "exemplary teaching, we wanted the preservice teachers to investigate the ways in which the classroom teacher accommodated and built on her pupils' contributions. Based on this goal, our evaluation focuses on the depth of the preservice teachers' insights into the role of the teacher and the importance of interpreting pupils' mathematical conceptions. These evaluation data were derived from the preservice teachers' oral and written presentations and their open-ended comments regarding the CD-ROM itself. For example, comments such as:

> *This advanced computer technology allows us to set a foot inside a classroom and observe teaching procedures and student responses. This observation has already proven helpful. In preparing my own lessons, I attempt to anticipate children's responses, and viewing this video helped me to watch how and why children might respond as they do.*

indicate that students were beginning to challenge their current beliefs and attempted to apply these reflections in their student-teaching experiences. These evaluation insights are being used to inform our future development and research efforts.

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THE TEACHER AS PROBLEM SOLVER AND HIS/HER CONCEPTION ON PROBLEMS SOLVING IN THE MATHEMATICS CLASSROOM

Luis C. Contreras, José Carrillo and Fernando Guevara
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Teachers are demanded by the present curricula for putting into practice a methodology based on problem solving. In order to make that possible, we have to adequate teachers' capability concerning problem solving both as solvers and as teachers. That is why our research focuses on the consideration of different conceptions of problem solving in the classroom and, at the same time, on the distinction of several levels in relation to problem solving performance among teachers.

Starting from our previous research on teachers' conceptions of mathematics and its teaching, and, to be more exact, starting from our instrument for a qualitative analysis (Carrillo & Contreras, 1994, 1995), designed in order to obtain a closer understanding of mathematics teaching conceptions, we have developed a theoretical instrument that (we think) can characterize the role teachers give to problems in a classroom, depending on their teaching conceptions (or viceversa, it is very probable that such a role defines in a big way the abovementioned conception).

On the other hand, we have developed an assessment instrument in order to analyse the problem solving styles, distinguishing some categories and descriptors, and, what is more important, this instrument includes a valuation scale that gives us the possibility of making up a qualitative report (Carrillo & Guevara, 1996).

Finally, we have gained access to teachers' behavior and attained some profiles that highlight the lack of mathematical preparation, at least in relation to problem solving. In other words, we should be conscious that to put into practice a methodology based on the resolution of problems should not be supported in the teachers' weak capability for problems solving, which we have already observed (in our case study).

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THE IMPORTANCE OF RELATIVE CODE TO YOUNG CHILDREN'S UNDERSTANDING OF THE DIVISION CONCEPT

Jane Correa and Peter Bryant
Federal University of Rio de Janeiro; University of Oxford

In previous study (Correa and Bryant, 1994), 6- to 7-year-old children were asked to make judgements about the relative size of the quotients in non-computational division tasks. The children were only asked relational questions (same, more or less). In as far as the children were successful in using the inverse divisor-quotient relationship in these tasks, we can now ask whether they would be able to work out the actual absolute value as well. Based on Bryant's (1974) theory about the importance of relative codes for children's reasoning, our assumption was that young children are able to reason about the elementary relations between the number of divisors and quotient in the process of division although they find difficult to work out the solution to those division problems by computational means. If this is right, children of the same age as those who succeeded in the relative tasks of our previous study might find a computational version of the task much harder.

In order to test our basic assumption about the importance of relative codes in children's understanding of the relations between the number of divisors and quotient in elementary division tasks, two experiments were designed to investigate children's ability to work out the cardinal value of the quotient in partitive and quotitive division tasks respectvily. We presented children with tasks which not only involved the same numbers for dividend and divisor but also would reproduce the situation in which the children would have faced in our previous study if they were trying to use their computational skills to solve our non-computational partitive and quotitive problems.

An inspection of the percentage of 6- and 7-year olds who carried out the computational solution for tasks which involved the same quantities presented in our non-computational partitive and quotitive tasks was considerably lower than the percentage of children of the same age group who succeeded in the equivalent non-computational tasks in our previous study. Young children performed better in division tasks which involved relations which they could easily understand (e.g. same, more/less than) than in tasks where they were asked to make use of their computational skills.

These findings encourage the use of a non-computational approach not only in future research on the development of the arithmetical concepts in young children but also in teaching arithmetic in elementary school.

Correa, J and Bryant, P (1994). Young children's understanding of the division concept. Proceedings of XIIIth Biennial Meetings of ISSBD.
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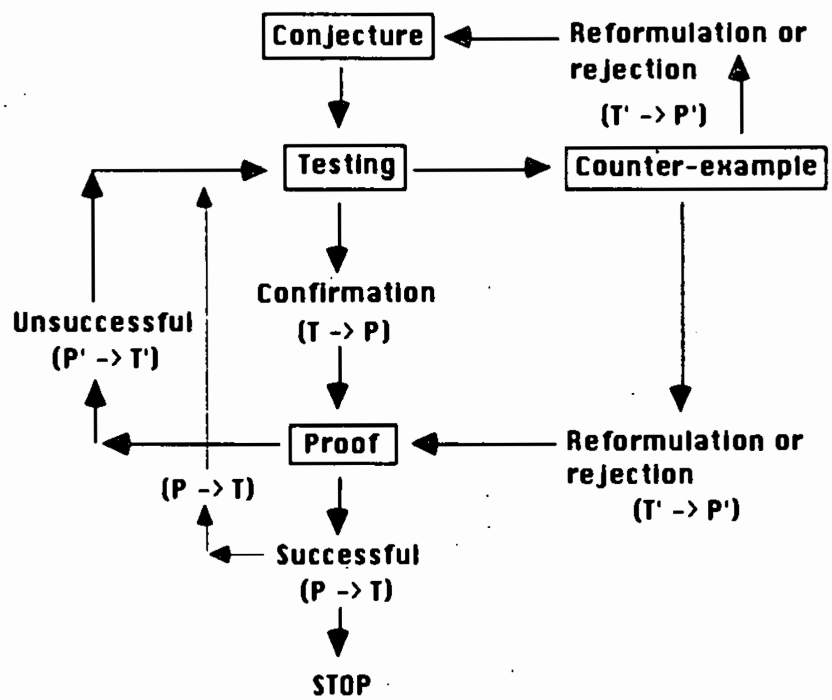
SOME ADVENTURES IN EUCLIDEAN GEOMETRY

Michael de Villiers

University of Durban-Westville, South Africa

The author has recently published a book by the above title which is addressed primarily at gifted high school mathematics pupils and teachers who are looking for enrichment material, and university or college lecturers involved in the under-graduate or in-service training of mathematics teachers. The purpose of this book is to actively involve the reader in the heuristic processes of conjecturing, discovering, formulating, classifying, defining, refuting, proving, etc. within the context of Euclidean geometry. Extensive attention is also given to the classification of the quadrilaterals from the symmetry of a *side-angle* duality, and covers convex, concave and crossed cases.



The poster will present and briefly explain a model for how new mathematics is sometimes discovered and proved, and which formed an important guideline in the writing of the book. Also presented will be a classification scheme of some of the quadrilaterals dealt with in the book.

DOES LANGUAGE AFFECT PROPORTIONAL REASONING?

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Both ratio and fraction involve the idea of proportionality, often shown to be rather difficult for children. Ratio and fraction are so closely related that the same situation can be presented either as a fraction (e.g., a mixture of 1/3 white and 2/3 red paint) or as a ratio (1 can of white paint to 2 cans of red) problem. Nunes and Bryant suggested that these differences in the language of presentation have a significant impact on children's problem solving: ratio problems can be connected to the one-to-many correspondence schema whereas fraction problems are likely to be connected to a sharing schema. Considering the developmental findings about these two schemas, it can be predicted that problems presented in ratio language result in greater rates of success than those presented in fractional language.

This study investigated children's success in solving the same problems as a function of the language of presentation and the availability of concrete props to support problem solving activities. The subjects (120 English children, age range 8 to 10 years) were randomly assigned to the different testing conditions and answered four different problems involving proportionality. Problems referred to mixtures of liquids (paints of different colour and water-orange concentrate) and distribution of money (proportional to amount of work done or ice-cream consumed).

Younger children performed significantly better when problems were presented in ratio rather than in fractional language whereas the difference was not significant for the 10-year olds. Availability of concrete materials did not affect performance significantly. Children's justifications in the ratio condition were significantly more advanced in terms of proportionality judgements than those in the fraction condition.

Thus differences in the language of problem presentation appear to result in the use of different interpretation schemas related to the concept of proportionality; this may explain to some extent the variation in pupils' performance across situations.

Making sense of science and mathematics through classroom dialogue

B. A. Doig, ACER, Melbourne, Australia.

S. C. Groves, Deakin University, Melbourne, Australia

J. S. Williams, University of Manchester, Manchester, UK

This poster and video segment will present and analyse a dialogue between four children and a researcher about an activity which is part of an experimental maths~science programme in years 5-7 classrooms[1]. The notion of dialogue is borrowed from the "Children's Philosophy" movement, where it is contrasted with both conversation and discussion. The classroom dialogue is constituted in a 'community of enquiry', essentially as a form of Socratic dialogue[2]. The scientific~mathematical context is established through the problem posed, the rules of evidence applied and the teacher's focussing contributions. For us the dialogue plays a pivotal role in the social psychology of the classroom: the individual children make contributions which express their sense of a shared activity, and the dialogue builds on the most constructive elements of these contributions through a group process of critical consensus-making. This consensus then provides each individual with an opportunity to 'make sense' afresh, that is, to learn from the dialogue.

Part of a dialogue shown on the video was analysed from three points of view. First we examined the progress of ideas in the dialogue, which focusses on the relationship between the children's science and their mathematical analysis of the data collected in the activity. Second, we examined the contribution from each child to the dialogue, and their 'opportunity to learn' from it. Third, we focussed on the role of the leader of the dialogue, in this case the researcher.

Finally, we draw inferences about appropriate leadership for dialogue. In most classroom dialogues the leader will usually be the teacher, who has additional responsibilities as the most expert in the subjects involved. Their primary role is to encourage listening, encourage references to previous contributions (which make sense of what has been said) and encourage the use of reasoning to support arguments. A secondary role for the expert in leading scientific and mathematical inquiry is in focussing on ideas which may lead to productive lines of argument, summarising the apparent consensus of the group, and validating scientific ideas and reasoning contributed that can help the dialogue make progress. The primary role elicits diverse views and personal sense–making. The secondary role encourages moves towards productive scientific ideas and thought.

References:

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THE ASSESSMENT OF ALGEBRAIC PARTIAL KNOWLEDGE: MODEL-ELICITING ACTIVITIES FOR ALGEBRA

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University of Granada. Spain

This work tries to assess, in a way, students' *partial or incomplete knowledge* of problems without resorting to other formulas such as solved/failed, right/not right, whole/nothing, 0/1.

Certain kind of activities are proposed, i.e. model-eliciting activities which elicit Secondary Education students' partial knowledge of school algebra. It is intended to identify the different representation's systems or semiotic representations displayed by students when solving one of these activities, according to their complexity and depth.

The application of these model-eliciting activities are, at the same time, part of a formative and epistemological assessment. In this case, instruction and assessment are inseparable.

Within all those stages of word problem solving model-eliciting, the most important thing is to analyze the correct translation (also the integration) of the problem statement into a symbolic language through representation's systems.

We have found four main correct representation's systems to solve an algebraic problem model-eliciting, which different levels of complexity and profundity of algebraic thought are shown; these systems are: 1) trial and error or testing out (numerical representation); 2) graphic (using drawings to represent the relationships within problem);3) graphic-symbolic (using alphabetical symbolism but with the necessary drawings to support the relationships); 4) symbolic (using alphabetical symbolism, the possible drawings is irrelevant for the relationships).

When assessing, teachers should not only consider as correct some representation's systems, but any valid procedure.

***Activity**: You have a long wooden strip. You want to split it in two parts such that one is four times as long as the other. How should you do it?.*

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TWO CASE STUDIES OF FUTURE TEACHERS' CONCEPTIONS ABOUT MATHEMATICS, ITS TEACHING AND LEARNING

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During an introductory teaching practice course for teachers of Mathematics at a Secondary level, we have realized two studies of cases in order to detect the beliefs and conceptions of two student teachers using content analysis of their essays and with a very detailed interview.

So as to categorize the diffferent units of information, we have devised a two-dimension grid, according to *planes* and *stages*. The planes represent different levels of reflection on the educational process.

We distinguish five *planes*: *1- Epistemological plane.-* The reflection on the relationship between the researcher, the outside world and the mathematical community. *2- Psychoepistemological plane.-* Internal processes that occur in the researcher. *3- Psychodidactic plane.-* Internal processes that occur in the student. *4- Didactic plane.-* Relationship between school mathematical knowledge, teacher and student, and school community. *5- Epistemic-Didactic plane.-* The reflection that theorists on Mathematics education realize about the educational practice.

The *stages* describe the different phases that knowledge goes through until it is constituted. We consider three stages: *1- Gnoseological stage.-* The specific activities that occur in the subject when he/she faces a problem and tries to solve it. 2- *Ontological stage.-* Essence of the interactions among the subjects implied in the corresponding level. *3- Validity stage.-* Criteria and agents that validate knowledge.

The content analysis of the two students' essays has lead us to design two profiles of their beliefs and conceptions about Mathematics and its teaching and learning. These two profiles are classified into four categories: a) Ways of conceiving mathematical knowledge and their attitude towards this knowledge. b) Description of teaching. c) Teacher training process. d) Expectations as teachers.

These profiles show us that the student Luis finds himself in a stage of external authority which makes him need and trust external elements to verify knowledge and ways of teaching. The student Eva finds herself in a stage of integration of authority which places her in different levels with regards to mathematical knowledge and educational knowledge. For her, the acceptance of mathematical knowledge needs the debate among mathematicians in a historical process. Even though she coincides with Luis in considering the educational knowledge in a relative way, she is willing to discuss about the educational qualities of the teaching methods.

LANGUAGE AND NUMBER NOTATION: A STUDY OF DEAF CHILDREN

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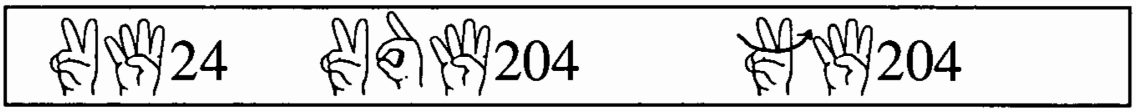
Dept. of Developmental Psychology - Institute of Educational Sciences - University of Barcelona

Children commit errors when writing and reading numerals regardless of language. Errors are considered lexical when only the digits are mistaken and are considered syntactic when the overall structure of the numeral is mistaken. Explanations for these errors were given taking into account the lack of congruence between oral language and notational systems. The *transparency hypothesis* predicts that the more congruent language and notation are the less errors will be made. In support for this Chinese, Japanese and Korean children were found to produce less errors than children speaking European derived languages since in the former, the number words are transparent regarding the base ten number system. The Catalan Sign Language (CSL) provides an excellent oportunity to test this hipothesis since CSL, with some exceptions, is transparent regarding the correspondence between signs and number notation.



Subjects were seven profoundly deaf girls and boys eleven to fifteen years old. They were in the 6th and 7th grade and use sign language although they have been educated in regular oral classes. They were asked to read arabic numerals, orally or by signing, and they also had to write following the signing of their teacher. It was predicted that the children would produce less errors in writing than in reading.

Percent of errors produced by each child in reading and writing

	Pau	Vanesa	Mireia	Marc	Miquel	Cristina	Ismael
Reading	41,66	8,33	-	48,33	58,33	11,66	38,33
Writing	25	5	15	30	25	0	25
Difference	39,99	39,98	-	37,93	57,14	100	34,78

Regardless of the subject´s level of knowledge, it is always more difficult to read than to write. The difference between reading and writing errors is similar for each individual. To write from signing one must just note down the face value of the digits whereas to read from conventional notations requires not only to sign the face value but also to show the positional value by a multiplier sign. Qualitative analysis showed, however, that errors in writing were as a rule, syntactic. For example, when asked to write ten-thousand-seventy-eight, children wrote 10.78. The difficulty of reading together with the kind of mistakes produced in writing prevent us from taking success in writing as reflecting proper understanding of the numerical system. Such understanding might be linked to a deficient linguistic development related to late access to schooling in sign language.

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AN APPROACH TO ANALYZING THE MATHEMATICS TEACHER'S PROFESSIONAL KNOWLEDGE: THE CASE OF THE CONCEPT OF FUNCTION

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Departamento de Didáctica de las Matemáticas
Facultad de Ciencias de la Educación
Universidad de Sevilla

The present study forms part of a broader project whose purpose is to describe the mathematics teacher's professional knowledge. The aim is to construct several case studies centred on Secondary-level Mathematics teachers.

An adaptation of the Repertory Grid Technique was made for the present study. Several semistructured interviews were designed for each of the different aspects considered. The exploration of the teacher's system the constructs in relation to the mathematical concept of function was centred in two domains: (i) functions as a curricular topic and their different modes of representation, and (ii) the concept's relationship with other educational mathematical topics. A first interview focused on obtaining the representative elements, with it being decided that the teachers should elicit their own elements. A procedure of triadic comparison was followed in a second interview. In this manner, two linked meanings are obtained for each triad of elements used. The constructs can be considered as a continuum whose extremes are the two meanings elicited.

We here present the results of the cluster analysis of the constructs elicited by the teacher in one of the domains. The constructs were focused on characteristics of the mathematical content of the elements. To a certain degree, rather than simple qualifiers, the constructs contribute new aspects of mathematical content in relation to the teacher's cognition concerning functions as material to be taught in Secondary Education. What the clusters indicate is that the "effects" of using the constructs to score the set of elements of this domain are in relation, but not necessarily that the constructs'semantic domains are isomorphic.

On the basis of this analysis, we shall try to infer features relative to aspects of a teacher's professional knowledge with respect to the content and interpretative processes of the teacher. The results are relevant to note the integration of aspects relative to mathematical content with aspects relative to the relationship between the student and the mathematical content.

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MATHEMATICS TEACHERS' BELIEFS CONCERNING TEACHING AND LEARNING

F. Gil; M. F. Moreno (Almería University. SPAIN)

The aim of the present study is to describe the structure of ideas and valuations which shape the beliefs help by Spanish mathematics teachers concerning the teaching/learning process. It is a continuation of a papers by Rico, L. et al. (1995) in which a system of 41 categories provides a structure for the ideas and valuations related to assessment which mathematics teacher of the aforementioned nationality hold.

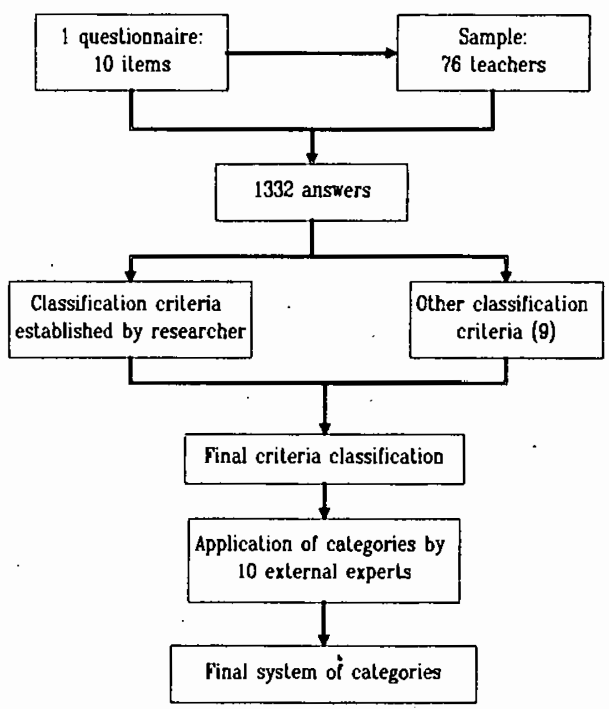
The study was based in the field of descriptive methodology, andconsisted of a cross-sectional study carried ont by means of a small-scale survey (N=76). A questionnaire with ten consecutive open questions was used (fig. 1) and was distributed in the second half of 1995. The teachers took roughly an hour to answer the questions and gave several responses to each one. The researcher worked ont a cllassification system in order to ascertain the different concepts contained within the 1332 answers obtained. Simultaneously, the list of responses was sent to 9 experts who independently designed their own system of categories. A categories system resulted from a systhesis of these and this was in turn submitted to a validation process by 10 independent experts (fig. 2).

A system of 45 categories has been obtained. This provides the frame for the beliefs held by the Spanish mathematics teachers. More especially, there is a system of categories for each question (ranging from 3 to 8 categories) which allows us to draw up a "closed" questionnaire relating to beliefs.

Rico, L. et al (1995). Teacher's conceptual framework on mathematics assessment. *P.M.E. XIX* (pp 2/130-137).

Fig. 1 Questionnaire

1) What process do you use prepare material for you pupis?
2) What gives you the feeling that you have made a good job of teaching your pupils?
3) What for you is a "good" maths student?
4) How could the professional qualifications of secondary school mathematics teacher be improved?
5) Why should mathematics be studied at Secondary School level?
6) How is mathematics learnt?
7) Which are the most important (contents) in the teaching/learning of mathematics?.
8) What kinds of activity are most recommended for teaching mathematics?
9) What are the difficulties of mathematics teaching in Compulsory Secondary Education?
10) What role does error play in maths teaching at Secondary School?



THE AFFECTIVE PROCESSES.
A MATHEMATICS LEARNING PROGRAM IN A SOCIO-CULTURAL CONTEXT

Inés Mª Gómez Chacón
Instituto de Estudios Pedagógicos Somosaguas. Spain.

In the past few years the importance of the affective dimension in learning and mathematical instruction has become obvious. The improvement in Mathematical Education will call for changes in the affective reactions both in boys and adults alike (McLeod, 1992, Lafortune 1992, 1994).
The recent influence of anthropological approaches on educational research is beginning to have a significant impact on research related to affect. Hart and Allexsaht-Snider (Mcleod, 1994) argue that the sociocultural context of learning has received too little attention in research on affective issues in mathematics education. We detach the studies of Abreu (1993) that analyses children's beliefs and attitudes towards different mathematical practices and the socio-cultural organisation.
The purpose our study is to develop different alternatives in teaching and learning mathematicas, which take into account the experiences that the students have in their professional enviroment or in the schools where they get a professional training. The learning environment within which the curriculum is established is instrumental in assisting students to handle their emotions and develop positive attitudes.
We look for the conections between the affective issues and the cultural influences in the mathematical learning. Both generate in the student certain beliefs, emotional reactions, and it would be interesting to know to link differences in achievement to beliefs that are connected to cultural influences.
These alternatives would deal with how mathematics are used in their context, in their professional enviroment, in their lives, and how this can be incorporated in the academic curriculum. In this research project we follow young people (15 - 19 years old) who dropped the Spanish school system and joined an "Centro-Taller" Center, a few years later. The purpose of center is technical education.
We try to explore the different approaches to the learning process in the classroom and the one in the cabinetmaking's workshop, and find out if the way the students experience this interrelation influence their knowledge and beliefs, attitudes about mathematics. To illustrate both approaches we would like to describe their thinking strategies both in the classroom and in the cabinetmaking's workshop.

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Algebraic Thinking: Using Magic Squares with 5th Grade Students

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Graduate Program in Mathematics Ed. Santa Ursula University-Brazil

The purpose of this investigation is to better understand how students at 5th grade develop and deal with algebraic thinking.

This study took place in a 5th grade mathematics classroom in Rio de Janeiro, Brazil. The students were divided in groups of four children each, one group was selected for observation. This group was videotaped while working with Magic Square activities and interviewed during this process.

In Brazil, on 5th grade, Algebra does not figure in the content list of text books, consequently no algebra-activity is done in classrooms. The text book starts with a review on Natural numbers so we decided to elaborate activities based on Magic Squares. Those activities were influenced by Arcavi's article named Symbol Sense (Arcavi 1995).

- Activity 1: An empty table 3x3 was given to the students fill in with numbers that add 15 horizontally, vertically and diagonally.
- Activity 2: Fill in the blancs in order that any row, any column and any diagonal add up to 9

	3	
1		4

- Activity 3: The same as above, changing the number in the middle to 2 and the sum to 6.
- Activity 4: The same as A2 changing the middle number to 4 and the sum to 8.

During the three first activities, the students were able to find the numbers to fill in, by playing with numbers for a while. Regarding activity A4 they tried without success. At that point, the teacher intervened by asking how they could afirm that it is an impossible task or to afirm that there is a solution. The students started to build on their own language ways of talking about this fact and they started to use symbols to represent the variable numbers.

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TEACHERS RESPONSES TO PUPILS' ERRORS

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Instituto Superior de Psicologia Aplicada

This research intends to contribute to the understanding of the way teachers deal with pupils' errors and misconceptions in schools. As errors are fundamental elements for the fostering of cognitive development (Bickhard, in press, Vergnaud, 1990) and in the supplying of scaffolding (Bruner, 1985), we meant to study the ways teachers cultivate to react to it, and how they impact on their planning of daily activities.

Our work was restricted to 25 5th and 6th grade teachers and the subject matter that was covered decimal numbers. We first elaborated a pilot instrument and administered it to a group of students in order to detect patterns of frequent errors in the included items. We then constructed a worksheet with four groups of questions that were answered with a consistent pattern of error in each group.

This worksheet was given to the teachers and we asked for their comments in two different scenarios: i) designing a plan for the next class in order to help students to overcome their misconceptions, having in mind that there had been a large percentage of errors of the same kind; ii) producing a written comment to a hypothetical student who had originated that kind of answers.

In our poster we will show the different ways found by teachers whilst answering this request.

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A COMPARISON OF THE GEOMETRIC PERCEPTIONS OF AUSTRALIAN AND AMERICAN STUDENTS.

Christine Lawrie
University of New England

In the early 80s, Mayberry (1981) developed a diagnostic instrument to be used in an interview situation, to assess the van Hiele levels of pre-service teachers. The Mayberry study has been replicated under Australian conditions in a written format, testing sixty first year primary-teacher trainees. This poster presents the results of the study, comparing them with the results of the Mayberry students, and relating them to their geometric backgrounds. All results are given as percentages to facilitate comparison.

Highest van Hiele level reached by the Australian and the Mayberry students for each concept

	*No Level		Level 1		Level 2		Level 3		Level 4	
Concept	Aus	May	Aus	May	Aus	May	Aus	May	Aus	May
Square	0%	0%	3%	11%	84%	32%	7%	26%	7%	32%
Right triangle	3%	26%	19%	21%	55%	21%	19%	16%	3%	16%
Isosc triangle	7%	26%	27%	16%	43%	11%	20%	26%	3%	21%
Circle	0%	5%	13%	11%	19%	16%	52%	21%	16%	47%
Parallel lines	0%	26%	17%	16%	80%	16%	0%	37%	3%	5%
Congruency	0%	0%	32%	21%	35%	32%	3%	21%	29%	26%
Similarity	0%	5%	43%	42%	40%	5%	10%	21%	7%	26%

* No level indicates students who failed to identify a concept

For both studies, the results above show that the majority of students were assessed as having no better than van Hiele Level 2 understanding, i.e., they were comfortable recognising concepts, and listing the asociated properties, but did not understand the relationships between the properties. The table below shows that many of the students who had completed a recognised senior secondary geometry course in which the instruction is at Level 3 or higher, could not display better than Level 2 understanding in their responses.

Comparison of highest levels reached by Australian and Mayberry students in relation to their senior secondary geometric background

	Senior geometry background		No senior geometry background	
Highest level reached	Australian	Mayberry	Australian	Mayberry
0	1%	13%	3%	12%
1	12%	16%	33%	26%
2	50%	12%	55%	33%
3	19%	29%	9%	14%
4	18%	30%	1%	14%

The apparent difference between the results from the two countries needs to be investigated further. Were the levels of responses acceptable for both studies the same, and, does the offering of geometry as an elective course, as in the USA, result in a more effective teaching/learning situation?

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A DIDACTIC ENGINEERING FOR A NOTION OF LIMIT OF FUNCTION

Ana Lúcia Manrique - PROEM/PUC-SP and USJT
Saddo Ag Almouloud - PROEM/PUC-SP

When the students, in Brazil, are in the first grade of College, they learn the concept of limit of function during many courses about Exact Science. Along these courses, the student should have the concept of limit: the formal and the spontaneous one. The notion of limit is concerned to a central position in the Analysis, as the basement of the Continuity and of the Calculus development.

Based on the theory of R. Douady [1], about Tool-Object Dialectics, we elaborated a didactic engineering where the notion of limit shoud be used to solve a specific problem. At first, be used implicitily and informally as a tool to get cognitive experiences, before the problem becomes the exact focus of attention as a mathematics object.

To improve the development of the research and evolve the student's concept of function's limit, we will use play of charts. The change of charts is a way to obtain different formulations about a problem that, without being necessarily equivalent, may give a new approach to the difficulties faced, and begin the tools and techniques that didn't appear during the first formulation.

In the problem-situation ellaborated, we didn't suppose the students would know what was a tangent to a curve on a point of this curve. The works of A. Sierpinska [2], S. Vinner [3], J. Robinet [4], P. Perrin [5] and a research of scope we applied on graduate students from Brazil, shows that the students do some associations about the tangent, as well as a tangent to a circunference, that only sever the curve on/at only one point, can't sever a curve at other points.

The results of this didactic engineering will be presented on this poster.

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SUPPORTING TEACHERS OF MATHEMATICS AT A DISTANCE: FIFTEEN YEARS OF PRACTICE IN THE UK

John Mason, Eric Love, Christine Shiu, David Pimm
Centre for Mathematics Education, Open University, Milton Keynes, UK

For fifteen years since its foundation, our Centre has supported the teaching of mathematics at all levels. The basis of our approach has been

to draw upon research and our experience,

in order to offer teachers ways of working on their experience;

presented in the form of undergraduate–in-service courses of study at a distance

and in the form of materials for teachers to work on

both in courses run by other agencies and individually.

PARTICULAR STRENGTHS

In our materials and workshops we focus as much on ways of working as on mathematical or pedagogical content.

We usually start by inviting teachers to refresh their own experience, reflect on this, and only then consider implications for their classroom (summarised as Adult–Process-Classroom).

We offer distilled frameworks (such as Adult–Process-Classroom) linked to current and recent experience (including videotape of classrooms) to inform future practice.

We offer collections of pertinent articles (reprinted and commissioned) as readers for our students and for other institutions to use.

We have offered videotapes of classroom interactions, and audiotapes of comments by reflective practitioners and researchers.

Every institution concerned with mathematics education in the UK has been influenced by our approach, through staff either tutoring or studying our courses, and-or using our materials for themselves or with their students.

HISTORICAL CHANGES

As our own children have grown up our interests have tended to follow them.

As Mathematics Education has matured as a discipline, our courses have become more theoretical and less classroom based.

CURRENT PROJECTS

Studies of the experiences of students (and their tutors) on an entry-level course in maths taught at a distance through the OU.

Collection of informative frameworks, glossary of technical terms, and descriptions of ways of working on Meaning Enquiry in Mathematics Education. for use by teachers.

Conducting short Graphics Calculators workshops for teachers (includes materials).

Franchising and evaluating 10-day and 20-day courses for primary teachers using our materials.

Collection of word-problems on the boundary of arithmetic and algebra as resource for teachers in an investigation in the expression of generality as the principal root of school algebra.

ARITHMETIC GAMES WITH STRATEGIC COMPONENTS IN THE EARLY PRIMARY MATHEMATICS CLASSROOM

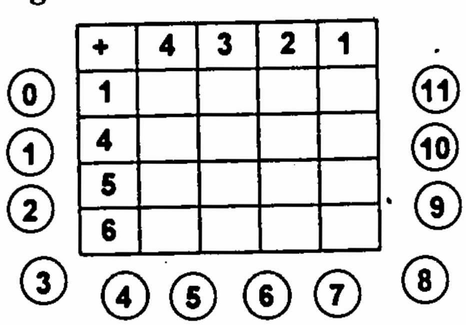
Doris Mosel-Göbel, Andrea Peter, Peter Sorger
Institute for Didactics of Mathematics
University of Münster, Germany

The value of using learning games in the primary mathematics classroom is widely acknowledged. Nevertheless, until now few empirical studies have investigated their effectiveness.

This poster reports some first attempts to categorize and analyze the game performance of German first and second grade students playing arithmetic games that contain strategic components.

These games (see Figure 1 for an example) are designed to promote an active and discovery-oriented arithmetical learning process. They allow children to compare sizes of numbers and to discover relationships and patterns such as the commutative law of addition and multiplication.

Figure 1



Rules:

1 . Players in turn place their tokens on free elements of the matrix, calculate the sum and place another token of the same colour on the correct result in the circles around the matrix.
2 . In case a result circle is already occupied, the player is allowed to replace the token of the opponent with one of his/her own.
3 . The game is over when all elements of the matrix are occupied. The winner is the player with the most occupied result circles.

They also embody frequently neglected strategic components such as recognizing multiple ways (elements in the matrix) of achieving a score (numbers in the circles). Optimal strategic behaviour requires far-sighted and deductive thinking and reasoning.

First results indicate that the players initially seek only to substitute their tokens for their opponents' tokens on the result circles. There was no evidence that the first grade children were thinking ahead and planning subsequent moves. Second grade children were the first to show signs of this behaviour. Occasionally they were able to identify those matrix elements which gave a non-repeatable result. However, the strategic advantages of matrix elements which yielded the same result three times were not fully comprehended by those second grade students.

NUMBER INITIATION-AN AFRICAN PERSPECTIVE

AUTHOR'S NAME: NGWA, ROSEMARY KONGLA (MRS.)
AFFILIATIONS:- MATHEMATICS TEACHER'S ASSOCIATION (MTA)
- THE EDUCATIONAL RESEARCH GROUP (ERG)

CONTENT OF POSTER: Local materials e.g., bean seeds, cowries, melon seeds, raffia-nut scales and soft-drink corks, which are abundant in the environment and are commonly used to teach counting and additive structures will be glued on the poster to represent each number from 1-12 with the numeral written under it. Numbers will be presented in two different arrangements. These materials have cultural and social significance.

RATIONALE: Structuring of Numerical Knowledge

Cognitive Psychology today recognises that higher mental processes are involved in the learning of early arithmetic. Number is viewed as a concept scheme, i.e., a network of related knowledge together with all the problem situations in which it can be used (Bergeron & Herscovics, 1990). The child's ability to count is inevitably based on the acquisition of the number-word sequence. Most Anglophone Cameroonian children (English as a second language) below the age of 7 are working on learning the number-word sequence to ten informally, since more than half of these children live in rural areas and do not have the opportunity of attending Kindergartens.

The incorrect sequence produced by children before they have learned the standard sequence have a characteristic structure (Fuson, et al, 1982, Fuson, 1988). Crucially important number-word sequence learning continues long after the child is able to produce number-words correctly. This continued learning manifests itself in an orderly succession of new abilities, groups of which require a representation of the number-word sequence that differ qualitatively from the representation of the sequence at other levels; these have been designated as belonging to five different levels. (Fuson et al; 1982). This elaboration of the number-word sequence is a lengthy process ranging from age 4 to age 7 or 8.

The five levels of elaboration are a) string level; b) unbreakable list level; c) breakable chain level; d) numerical chain level; and e) bidirectional chain level (Fuson 1988, Fuson, et al; 1982). These different levels are marked by increasingly complex sequence abilities, becoming able to start and to stop counting at arbitrary number words, to count up a given number of words, and to count backwards starting and stopping at arbitrary number words, or counting down a given number of words. Children also increase their ability across these levels to comprehend and to produce other relations on the words in the sequence.

This study will examine instructional practices carried out in the teaching of number-word sequences in the GTTC Practising Nursery School Bamenda, in an attempt to show how these practices may affect the manifestation and development of these elaborative levels. The results of this study will be presented in a longer version for distribution at the PME 20 conference.

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Kites makes mathematics fly

Isolina Oliveira - Escola 2, 3 Damião de Góis - Portugal
Isabel Branco - Escola Secundária António Arroio - Portugal
Judith Silva Pereira - Escola 2,3 Marquesa de Alorna - Portugal

Students come to school with some knowledge that was provided by their own experiences in daily life. It is common to listen people saying that experience is the greatest master, in the sense that the situations that provide involvement to the individual makes her/him learn. Learning is a process of making sense of situations in class or in different environments, involving people in social interaction. It is shown by educators and researchers that students have difficulty to use the knowledge that they bring to the class and make connections with academic tasks. Brown & Al (1988) reinforce the idea that knowledge is not independent but in part "a product of the activity, context, and culture in which it is developed".

Guided by some concerns that are not solved in theory and practice as:

- How to create educational situations that make students establish relations between academic and non academic mathematics knowledge in the way that they can make sense and use it? How to build knowledge that is meaningful to students?

and also reinforced by the perspectives of the Portuguese mathematics curriculum introduced by the Reform in 1990, we developed some educational situations in mathematics class that incorporate students experiential knowledge. In this poster we will present an interdisciplinary approach developed in two schools (middle and high school) and the reflection that it originated.

Student Decision Making in a Game of Chance and Misconceptions in Probabilistic Reasoning

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Abstract

This research determined whether a group of 50 Year 9 students playing a card game that involved probabilistic reasoning demonstrated a type of misconception in the selection of strategy they employed. Earlier research into misconceptions in probabilistic reasoning by the author identified widespread use of the heuristics of availability and representativeness by Year 11 students. (Peard, 1991, 1994). The present research identified a misconception of a different nature relating to the concept of mathematical expectation.

Background

A card game has been used by the author in the classroom to introduce elementary concepts in probability (See Peard, 1990). In the play, students demonstrated a preference to take the part of a player over the part of the dealer. Athough the game favoured the dealer if the players do not employ any game strategy, all players were able to develop a simple game strategy which improved the player's chances. A complete analysis of the game (Pedler, 1992) showed that the odds are clearly in favour of the dealer.

The research examined whether the students, after playing the game, failed to recognise that the odds favour the dealer and the criteria used to decide on their preference for taking the part of either player or dealer. The research sample for the study consisted of 51 Year 9 students (two classes) at a metropolitan State high school in the Brisbane region. The prevalence of social gambling at this school had been previously established (Peard, 1991).

The poster presentation will visually demonstrate the play of the game and the pupil strategies.

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THE USE OF COGNITIVE MAPS FOR ANALYZING THE UNDERSTANDING OF TRANSFINITE NUMBERS

CARMEN PENALVA MARTÍNEZ — Universidad de Alicante. Spain.
CLAUDE GAULIN — University of Laval. Canada.
ANGEL GUTIÉRREZ — Universidad de Valencia. Spain.

ABSTRACT

Cognitive maps constitute a powerful tool for analysing students´understanding of mathematical concepts. In her recent doctoral dissertation, the main author has used them to conduct case studies about the comprehension of transfinite numbers by university undergraduate mathematics students. Illustrations taken from one case study will be presented and commented here.

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INTRODUCTION OF THE CONCEPT OF PROBABILITY TO TEENAGERS -(12/13 YEARS OLD)-

Cileda de Queiroz e Silva Coutinho - PROEM/PUC-SP, USJT e UniSantos.
Inês Esteves - Fundação Lusíada/Santos.
Michel Henry - IREM de Besançon.
Tânia M. M. Campos - PROEM/PUC-SP.

The teaching of Probability goes, more and more, towards the experimental approach about the stability phenomena of a certain event's frequency, during a random experience.

Coutinho[1], in a study carried out with young people (16/20 years old), concluded tha a precocious teaching of Probability, could be profitable in line with the theories of Fischbein and al[2]. According to Piaget and Inhelder[3], children are ready to the learning of Probability when they are 11/12 years old. Their conceptions are affected by their knowledge, defined by Amir and Williams[4] as an association of their language, their experiments and their believes.

Along this study, we analysed the pre-conceptions of brazilian tennagers (12/13 years old), through the comparative analyse of two tests. The first one was applied in the beginning of August/95, before the beginning of the didactic sequence, and the last one, in November/95, three months after this engineering.

Two questions in both tests asked about a box full of red and blue balls, without being mentioned the quantities. The students were invited to give their opinion about the next draw, and as an information, we wrote the results of 100 draws with replacing. An other question was about the prediction of a result of a "head and tail", using information on the last hundred results.

We found that many of the students who presented a misconception in the first test, presented a right conception, after the didactic sequence, but we could also find some of them doing the same mistake as in first test. One of the most resistent misconceptions found, which will be exemplified in this poster, is: "From the lack of information about the conditions of the random experience, students draw the conclusion of the equiprobability of the results".

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Assessment of University Students' Mathematical Generalization and Symbolization Capacities

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Juan D. Godino, *Universidad de Granada*

In this paper the symbolization and generalization capacities in students beginning their university studies are analyzed.

During their mathematics classes, a written questionnaire was given to a sample of 429 first-year students from different university specialties at the University of Córdoba (Spain). The questionnaire included an arithmetic and a geometric problem, whose solution required symbolization and generalization capacities, even when only elementary mathematic notions were involved.

The results confirmed our previous assumptions about the low level of students' achievements: Most students only checked the given mathematical properties for particular cases. Moreover, 44% of the students were unable either to express symbolically the problems, or to formulate and prove their solutions for the general case. Only 32.8% of the students solved both problems correctly.

These results, as well as the importance of generalization and symbolization for mathematical activity (Freudenthal, 1991; Dörfler, 1991), point to the interest of continuing our research at secondary school level. Therefore, a wider research project on "Assessing and developing symbolization and generalization capacities in secondary school students" has been started, in which we intend to study these problems and to experiment didactic proposals for promoting the development of these basic mathematics capacities.

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STRATEGIES USED BY MATHEMATICS UNIVERSITY STUDENTS IN SOLVING COMBINATORIAL PROBLEMS

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University of Granada

SUMMARY

Although it has been argued that combinatorial reasoning skills develop with formal capacities (Inhelder and Piaget, 1955), recent research shows that solving a combinatorial problem is not always an easy task. In particular, Navarro-Pelayo (1994) and Batanero et al. (in press) found that 14-15 years-olds students, even after instruction in the topic, generally experienced difficulty in solving combinatorial problems. In this paper we analyze the processes followed by four students in the final year of their Degree in Mathematics when solving simple and compound combinatorial problems. These students, selected from a sample of 29, had the best and worst results when they were given 13 combinatorial problems to solve in a written questionnaire. An in-depth interview with each student explored his/her processes to solve the problems. The results show that some students, in spite of their high mathematical preparation, had a great difficulty with the problems. The better problem solvers were characterized by their ability to identify the combinatorial configuration, their understanding of the relevancy of order and repetition in the statement of the problem, their systematic enumeration, their recursive and generalizing capacity and their identification of either the adequate combinatorial operation, or the equivalent series of arithmetic operations. On the other hand, the main causes of failure were confusion about order and repetition, misunderstanding of the type of element to be combined, lack of enumeration ability and changing the arithmetic or combinatorial operation needed to solve the problem.

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ADITIVE OPERATORS IN NUMERICAL TABLES

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This research attempts to investigate about the didactic possibilities of some numerical tables (LITWILLER, B.H. et al., 1980), having the Numerical Thinking as a theoretical framework. This paper gives a classification of activities in the a hundred numbers table as well as a mathematical structure and didactic applications of several visual patterns. The study is being developed with university students (third course of Teacher Training for Primary Education).

Searching and interpreting visual patterns in the hundred numbers table leads us to state the concept of **the group of additive operators** associated to some geometrical patterns. For that purpose, we consider the set **P** of the patterns that we call **chain-patterns.** We associate to each chain-pattern an additive operation given by the difference between both ends of the chain. An equivalence relation in **P** , given by:

"$\forall\ p_1, p_2 \in P;\ p_1\ R\ p_2 \Leftrightarrow$ they produce the same additive effect",

leads us to the concept of **additive operator set O**, as the quotient set **P/R.** If we define the composition of additive-operators ($\oplus$) as follows:

$\forall\ [x],[y] \in$ **O**; $[x] \oplus [y] = [x+y]$,

(Figure 1, shows the composition of [19] and [21]), we have the structure of **abelian group of the additive operators (O, ⊕)** associated to the chain-patterns. This group can offer a two-dimensional approach of the table from an additive point of view.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

[19] ⊕ [21] = [40]

Fig. 1

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MATHEMATICS TEACHERS' PROFESSIONAL KNOWLEDGE AND THEIR PROFESSIONAL DEVELOPMENT

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Academics' knowledge can improve teachers' professional knowledge but this one demands the use of other sources such as i) the intuitive understanding of the situations, ii) the capacity of articulate the thinking and the action, and iii) the understanding of the personal relations and the self confidence. Ponte (1994) says that to understand the professional practice it is required of us that we take it as the starting point and not just as the place where we apply theory.

It is important not just to know and understand what the teachers do in their classes but as well the way they do and learn it, and the meanings they give to their professional practice. To Brown and McIntyre (1993), it is necessary to understand how the teachers build and assess their own teaching and how they make decisions in dificult situations. It's also urgent to gain the teachers into an analitic contribution in research, and its results have to be built with the teachers, if we want a real change. As Cooney (1994) says, the changes seem to be more close to the teachers' perceptions as professionals than any particular structure in the teachers' training. Thus, it is necessary to develop reflexive collaborative research teams in which the interaction of different interpretations about the practice (a commum context), as Steinbring (1994) says, enables teachers and researchers to share situations in such a way that different interpretations can be created according to the objectives of different professional domains.

This poster will report one project where the reflexive collaborative triangle (Teachers, Researcher and Practice) is fundamental to understand the Mathematics Teachers' Professional Knowledge and their Professional Development.

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COGNITIVE MAPPING: A STUDY ABOUT THE CONCEPT OF FRACTIONS IN STUDENT TEACHERS AND ELEMENTARY TEACHERS (GRADES 1-4)

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The research reported here explores the concept of fractions presented by student teachers and elementary teachers (first to fourth grade) in a private school.Based on the cognitive view of teaching and learning developed by Ausubel, Novak and Hanesian (1978) and using concept mapping in the way proposed by recent research (Moreira, 1985; Heinze-Fry and Novak, 1990; McCagg and Dansereau, 1991; Mason, 1992; Gliessman and Pugh, 1994; Huerta, 1995) the study investigated the concept of fractions presented by 7 elementary teachers and 19 student teachers.

The initial hypothesis was that the student teachers present the concept of fractions not completelly developed but they improve the concept with practice (that means, teaching the concept, attending improvement courses and knowing new material). According to this, it was supposed that elementary teachers would present the concept of fractions in a significant, well-established and more complete way while the student teachers would not present the concept of fractions in a similar way. The data was collected through a questionnaire, a mathematical test about fractions and a construction of a concept map. The data analysis was done using Item Response Theory (IRT).

It was found that the students had a better performance on the cognitive mapping and on the mathematical test whereas the elementary teachers presented a lower performance on cognitive mapping and passed over questions on the mathematical test.In addition, results denoted that years spent on teaching, improvement courses and the knowledge of material are not determining factors for concept improvement. The students teachers are able to apply "techniques" when they are solving problems with fraction, but apparently they know only the sequential arrangement of the learning task. Further studies using a more controlled sample is recommended.

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RESEARCH OF WHAT-IF-NOT STRATEGY IN PROBLEM POSING

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The What-If-Not strategy proposed by Brown & Walter is the strategy of the challenging phase of problem posing. But, it is not clear how this strategy is used to accomplish the aim of the phase. The purpose of this poster is offer a new interpretation of the What-If-Not strategy by clarify the aim of the challenging phase.

Brown & Walter(1983) proposed two phases of problem posing. Namely, in the accepting phase, once this task is occupies, the given object is made clear. In the challenging phase the essence and significance of the object is clarified from a multiple perspectives. For Brown & Waler, the challenging phase means coming up with a new idea, finding an appropriate image to enable us hold on to an old one, evaluating the significance of an idea we have already learned, or seeing new connections(p.32). However, it is not clear how the five levels which constitute the What-If-Not strategy manage to accomplish the aim. The four tasks above can be subdivided into two main categories. By subdividing the Four tasks above into two groups, it becomes clear that the accomplishment of the tasks in the second group is made possible by reconsidering the given from the perspective gained through the first task. I, consequently propose a new interpretation of the What-If-Not strategy, which includes the process of comparative analysis.

The five levels are as follows. Level 0 starts from concluding what is given. Level 1 lists some attributes of the given. In Level 2, the selected attributes are modified by asking what if each attribute were not so; what could it be then? Level 3 consists of questioning the modified attributes and posing the problem. Level 4 has two stages, in level 4-1, the problem posed in level 3 is solved. In level 4-2, by comparing unified problem of level 4-1 with the given, a new problem posing becomes possible. The proposed interpretation provided clear method for accomplishing the aim of clarifying the essence of the given.

L 0: Choosing a Starting Point		L 0: Starting from Concluding what is Given
L 1: Listing Attributes		L 1: Listing Attributes the Given
L 2: What-If-Not-ing	→	L 2: Selecting Attributes and What-If-Not-ing
L 3: Problem Posing		L 3: Problem Posing about Attributes modified
L 4: Analyzing the Problem		L 4-1: Solving the Problem
		L 4-2: Comparing the Problem and the Given

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- [x] Other Reason (Specify): Lost; should have been in in 1997
- [ ] PRIORITY status (specified by ERIC Program Office)

PROCESSING DATA *(Required for PRIORITY Documents only)*:

- Date Acquired by Facility: ______
- Date Shipped to Clearinghouse: ______
- Date Received by Clearinghouse: ______
- **Date Required back at Facility:** ______
- Date Bibliographic Data Transmitted: ______
- Date Document Mailed Back by Clearinghouse: ______
  Clearinghouse Accession #: ______
- Date Received Back at Facility: ______
  RIE Issue ______ **ED #** ______

SPECIAL PROCESSING INSTRUCTIONS:

PLEASE RUSH!!

— See over for general *FAST TRACK* instructions —