Mathematics Projects in Junior High School

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MATHEMATICS PROJECTS IN JUNIOR HIGH SCHOOL

A sequence of problems to promote creative problem-solving habits using basic tools of algebra 1.

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One of the ways in which students can demonstrate a certain mathematical maturity is if they are confronted with a mathematical situation whose scope is wider than they usually need in everyday learning situations. This means a single situation in which they can apply a variety of the mathematical topics, techniques, and different mathematical thought processes that they have experienced.

Final Projects: Background Thinking

About eight years ago, a law was passed in Israel under which the school system was changed from eight years junior school and four years high school to six years junior, three years junior high, and three years high school. One of the main purposes of this change was to effect a modernization in the school curricula. At present, about half of the school system has changed to the new system.

The mathematics group, part of the Israel Science Teaching Centre based in the Weizmann Institute, was formed to develop one of two junior high school curricula in mathematics. (Since Israeli education is centralized, junior high schools really only have a choice between the two approved curricula.) The time between the passing of the law and the beginning of the changeover to the new school system was exceedingly short. In consequence, there were a considerable number of difficulties, which contributed to the following weaknesses in our junior high school curriculum materials:

1. The curriculum has been developed in an almost entirely linear style; that is, the material was written chapter by chapter, without any attempt at a deliberate process of revision and reinforcement. When one topic is finished the next is begun, and the first is only recalled in so far as it is intrinsic to the second.

2. Almost all the exercises are directly based on the topic in hand and are of relatively routine nature.

3. The early texts in the series for the highest ability level contain few questions that can serve as a challenge for the very bright students. Also, open-ended questions, which involve "discovery" and mathematical thinking and allow students to reach reasonable conclusions at their different levels of ability, are relatively rare.

It was thus that we came to start work on what we have called "Final Projects for the 9th Grade."

The projects, one of which is described in detail in this article, are designed for the students to do on their own over a period of about a month, with occasional "sign-posting" by the teacher. The principal points that we had in mind while developing the projects are as follows:

1. To explore the level of maturity of mathematical thinking that ninth-grade students can achieve.
2. To encourage an exploratory approach to mathematical problems. In particular, we wanted to see if we could persuade students (and teachers) to attack a problem whose words they understood and to solve it, at least partially, even if they had not yet learned the "accepted" techniques for solving it.

3. To create situations in which students could revise and extend their knowledge of mathematical techniques and skills in an interesting way.

4. To create situations in which students are required to recall their mathematical knowledge, to organize it, and to apply it to the problem in hand. Whereas previously they had learned the material topic by topic, here they would be called on to use material from different topics in the same question.

Finally, a very important aspect of our work has been to create projects that are strictly related to the regular curriculum materials and to the core mathematics thereof. Too often, in our view, the project situation, large or small, is developed in topics that lend themselves easily to this approach—for example, combinatorial problems, statistics, descriptive and intuitive geometry, computing, and so on. Not only were these topics, on the whole, not represented within our curriculum at this stage, but we felt that it was important to develop something interesting specifically in the "dull and routine" areas of mathematics. We have adopted the same approach to developing mathematical games. Usually, commercial games have only a tentative connection with the core syllabus and rest only on claims to develop mathematical thinking. We are in no position to dispute such claims, but we feel that a game featuring the ordinary mathematics of the curriculum may prove more useful in the classroom. We hope to describe some of these games elsewhere.

Project: The Open Phrase $n^2 - 1$

The first project in the collection might be entitled "the mathematics of $n^2 - 1$." The text of the final project is given here with short explanatory notes following each section. We also include some comments on pupils' reactions and on points arising from the first use of this project in school. (The text of the project itself is in italics to distinguish it from the notes.)

Consider the open phrase $n^2 - 1$.

(i) Substitute various numbers into this expression and investigate what numbers you obtain as a result. (In the following sections you should be able to "guess" some of the answers by this sort of substitution.)

We feel that this is an important "play" section, especially for the weaker students, that helps them get a feel for $n^2 - 1$ and the sort of numbers it produces. It is, in effect, a hint to following sections: If you don't know how to start, "play" and see if you notice anything.

It is, of course, also a very realistic mathematical activity. Whenever we are confronted with a general problem, it is good sense to attack it through special cases, which in this case means the substitution of specific numbers.

(ii) Find the set of numbers which when substituted in $n^2 - 1$ give a negative number.

Many students at this stage find it difficult to develop a general argument. The first section should help them to get started. In our experience, some will get no further than isolated substitution. Here the teachers' discretion comes in: they can either be satisfied with an intuitive and incomplete solution or argument or if they feel the students can do better, encourage them to "prove" their results. From what we have seen of the students' work, even those somewhere near the top of the ability range tend to be satisfied with the first intuitive result they obtain. From a mathematical point of view, their arguments tend to be incomplete.

(iii) Find the set of numbers which when substituted in $n^2 - 1$ give a positive number.

A little relaxation after *(ii)* before *(iv)*.

(iv) Given that n represents an integer, find numbers which when substituted in $n^2 - 1$ give even numbers.

Again, "play" can be important for many until a general result is suspected—

therefore the casual wording. The better students can be prompted to prove. Typically, students would show that an odd number produces an even result, and then they would stop. Very few understood the necessity of showing that *only* odd numbers produce an even result. In general, results have to be chosen carefully so that they can be proved with the rather "primitive" mathematical tools available to the student in grade 9.

> *(v) Can the even number 6 be obtained by the substitution of an integer in $n^2 - 1$?*

This was inserted just to draw attention to the fact that not all the even numbers can be obtained in *(iv)*.

> *(vi) If you substitute all the odd numbers in $n^2 - 1$, you will obtain a certain set of numbers. Have these numbers any common factors? Prove your conclusion.*

In an earlier draft version of the project, this section was phrased exactly like section *(iv)*, with "odd" replacing "even." But those who had solved *(iv)* raced through it without effort or thought. However, while "playing" with *(iv)* some of the teachers (who attended day courses centered on these projects and their use in school) had noticed that the numbers obtained were not only even but multiples of eight. We thought that the general proof here was just a little more demanding and hence made this section a useful one for the better student. In fact, most of the students whose work we saw seem to have found it easy going and proved the result. This indicates one should not have too many preconceptions.

> *(vii) Determine the set of natural numbers which when substituted in $n^2 - 1$ give a prime number. Explain!*

For some reason not very clear to us, many teachers and students could not get beyond the stage of substitution and conclusion by conviction rather than proof, and even some of those moderately competent in other work made mistakes here. Perhaps they were not able to see the connection between the definition of a prime number and the factorization

$$n^2 - 1 = (n - 1)(n + 1).$$

Even those who did get this far did not usually see that $n - 1 = 1$ or $n + 1 = 1$ is necessary but maybe insufficient. Many got involved in very complicated arguments based on half-remembered facts about formulae for primes. Similar situations occurred again and again—fairly simple questions were buried by a mountain of half-digested and irrelevant mathematics. This trend is symptomatic of the fact that most exercises in the regular day-to-day class situation are solved within well-defined small areas in which the technique to be applied is evident from the context.

> *(viii) Try to find at least one further result similar to those in the preceding sections, and prove it.*

This section is quite obviously one of the high points of the whole project, and there is much to be said about it.

First, it is the nearest we come to the simulation of a real mathematical situation. Second, it was only too clear that most teachers (and students) did not want to have anything to do with it. They would hastily go on to the following sections and had to be coaxed back to attempt this section. In general, in spite of some general exhortations like "No prizes for the first one finished, but there are prizes for the best solutions," it took a long time at each new session to persuade them to get more out of a question than the first answer that came to mind. We began one teacher session before distributing the project with a short introduction and then the question "What questions can you make up about $n^2 - 1$?" The reply, as expected, was almost total silence, but it did have the effect of giving the project greater impact, and some good answers were received to this section, once the spirit of the project had been assimilated.

The responses to the section varied from relatively closed simple questions to ones of considerable generality, again illustrating the scope in such a project to coax from students that of which they, individually, are capable.

1. An example of a simple type of response is the following: Find the set of numbers that, when substituted in $n^2 - 1$, give a positive number less than 1.

2. An example of a much more interesting response is the following: Find the set of integers that, when substituted in $n^2 - 1$, give a multiple of 3.

3. One of those we got from students was as follows: If we substitute the odd numbers, in order, we obtain a sequence u_n such that the difference between u_n and u_{n+1} is $8n$.

Suggestion 2 came up quite frequently and seems to have been suggested by section *(iv)* above. The interest in the problem lay, however, not in itself or its solution but in the way it was used by the stronger teachers as a springboard for further generalization. They noticed that—

a) multiples of 2 (even numbers) are obtained from all numbers that are not multiples of 2 (odd numbers), and only those;

b) multiples of 3 are obtained from all numbers that are not multiples of 3, and only those.

So why not try a generalization—

c) multiples of k (k a natural number) are obtained from all numbers that are not multiples of k, and only those.

This led to a lot of fun, because even though it is false, one can always try to modify the generalization to get true results—and this is the stuff of mathematics. The teachers' role in this section, as a guide and assistant with the reasonable formulation of ideas, is indispensable.

The following sections of the project were phrased in terms of functions (rather than open phrases). Although the use of functions was in no way essential, we deliberately made the switch to encourage a freer use of alternative terminology.

In the following sections, we shall be concerned with five different functions: f_1, f_2, f_3, f_4 and f_5, all of which have the same rule:

$$n \to n^2 - 1$$

(ix) If the domain of the function f_1 is the set of natural numbers, what is the image set?

(x) If the domain of the function f_2 is the set of integers, what is the image set?

(xi) If the domain of the function f_3 is the set of noninteger rationals, what is the image set?

In the teachers' guide, which accompanies the projects for the students, we point out that although the functions f_i are, strictly speaking, distinct, it is a reasonable mathematical abuse to use the same symbol for all of them. A pedantic adherence to definitions, which so often characterizes the newer subjects in the curriculum, can obscure and interfere with the real mathematical content. Since our students in grade 9 are still at an early stage in handling functions, we mention the fact that f_i are different and simultaneously advise the teacher to avoid placing emphasis on it.

Section *(ix)* can be seen as a completion of sections *(iv)*, *(v)*, and *(vi)*. In our experience, a watertight description of the image set is not a trivial matter. Many notice that the sequence of images is

$$0 \quad 3 \quad 8 \quad 15 \quad 24 \quad 35 \ldots$$

and then that the sequence of (first) differences is

$$3 \quad 5 \quad 7 \quad 9 \quad 11 \ldots.$$

This leads to the suggestion that the image set is a sequence of numbers whose first differences form an arithmetic sequence beginning with 3 and difference 2, which still needs a little improvement.

Section *(x)* is again a little light relief. Section *(xi)* does not lead to a very satisfactory answer (as far as we know), and such a section is also worth having. Not all problems a mathematician dreams up have tidy solutions, if solutions at all! We can say that the image set is contained in the set of all noninteger rationals greater than -1. We can find examples of such rationals that do not belong to the image set: for example, 2/3. But so far we have failed to find a more precise description that is not entirely trivial.

(xii) *$\sqrt{3}$ belongs to the domain of f_4. What is its image under f_4?*

(xiii) *Given that 1 belongs to the image set of f_4, find an integer, or integers, which could be the origin (inverse image) of 1 under f_4.*

(xiv) *Given that the image set of f_4 is the set of natural numbers, find a possible domain for f_4.*

(xv) *Given that $4 + \sqrt{3}$ belongs to the domain of f_5, what is its image under f_5?*

(xvi) *The domain of f_5 is the set of all numbers of the form $k + \sqrt{p}$, where k and p are rationals, p is positive and $\sqrt{p}$ is irrational. Find a general form for the numbers in the image set.*

(xvii) *Does the number $1 - 3\sqrt{2}$ belong to the image set of f_5?*

(xviii) *What can you say about the image set of f_5?*

These sections have "technical manipulation" as one of their objectives—for example, the evaluation of expressions of the form $(a + b)^2$ when one term involves a square root. Sections *(xii)* and *(xiii)* are simple. Section *(xiv)* is intended, for the better student, to lead to a discussion: there cannot be a unique answer to such a question. In fact, the question is somewhat ill-defined in the sense that one cannot work backwards to a domain. However, this is not an unusual mathematical situation, and it leads to a reconsideration of the role of domain in the definition of a function.

Section *(xv)* is pure technique and leads straight into the next three sections. We find that the image of $k + \sqrt{p}$ is of the form $q + e\sqrt{p}$, where q and e are rational and $q > -1$. But not all numbers of this form are images.

We can continue the investigation in response to section *(xviii)* and notice that there is a connection between q and e, i.e.,

$$q = \frac{1}{4}e^2 + p - 1.$$

We can plot q against e, and for each p we get a parabola that cuts the q-axis at $p - 1$. All the rational points (e, q) on these parabolas form the image set (fig. 1).

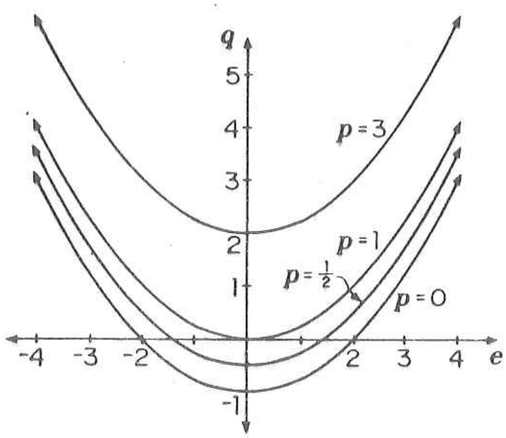

Fig. 1

This takes us to the edge of the curriculum, which is a study of the parabola and quadratic functions.

The last two sections are purely for fun (partial fractions are not mentioned in the curriculum). Interestingly, the students managed the first section quite well (as opposed to our experience with teachers), but few managed to apply the result in section *(xx)*.

(xix) *Consider the fraction*

$$\frac{1}{n^2 - 1} \quad (n \neq \pm 1).$$

Find two numbers a and b such that for all $n \neq \pm 1$,

$$\frac{1}{n^2 - 1} = \frac{a}{n - 1} + \frac{b}{n + 1}.$$

(xx) *Find a simple way of verifying the following:*

$$(a)\quad \frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}=\frac{1}{5}$$

$$(b)\quad \frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}=\frac{5}{11}$$

$$(c)\quad \frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}+\frac{1}{120}+\cdots+\frac{1}{288}+\frac{1}{360}=\frac{9}{40}.$$

In this project we have tried to show how it is possible to create "mathematics" from the relatively simple mathematical expression $n^2 - 1$. Clearly, it is possible to extend the mathematics further. One can add any number of further sections. For example, it is possible to "discover" that $n^2 - 1$ is the sum of the odd numbers from 3 to $2n - 1$, and so on. Undoubtedly, some students will continue, once they have caught the bug of not only creating solutions, but also creating problems.

Conclusion

The project described above, and other similar ones that we have developed, certainly interested the teachers and students. Many of them had a real mathematical experience probably for the first time in their lives. By real mathematical experience we mean inventing a problem and solving it. By being introduced to so many sections on the same basic topic, they first experienced the type of lateral thinking that develops new problems from old by analogy, and then they were encouraged to try it themselves. We hope that if you haven't tried it yet you will also be encouraged to have a go and then to encourage your students likewise. Even the apparently dullest mathematical situation can be made rich if one wears the right spectacles.

> Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking. [Polya 1946, Preface]

REFERENCE

Polya, George. *How to Solve It.* Princeton, N.J.: Princeton University Press, 1946.