ARE THEY EQUIVALENT?

(article begins on next page)

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In Search for Theories in Mathematics Education

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EXTENDING VALSINER'S COMPLEX SYSTEM: AN EMERGENT ANALYTIC TOOL FOR UNDERSTANDING STUDENTS' MATHEMATICS LEARNING IN PRACTICE

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This paper explores mathematics teachers' views about the impact of an intervention research project on primary mathematics teaching and learning. In order to account for the perceived effects, an extension of Valsiner's zones theory is proposed that incorporates other components of the school setting. The extension is felt to be more appropriate to understand the extent to which mathematics teachers constrain students' mathematics learning development. Prior to intervention, the mathematics teachers in the study seemed to have created a narrow non dynamic system for students. It is proposed that the intervention transformed/restructured the system only when other parallel transformations occurred on a number of components of the setting. Implications for qualitative transformation of student learning are drawn.

THE RESEARCH

Contemporary mathematics research paradigms call for espousing holistic and naturalistic approaches that help in understanding the complexity of mathematics teaching and learning (Kelly and Lesh, 2000). One of the most challenging research dilemmas is related to studying the extent to which students' learning is constrained, and consequently understanding to what extent students become active or passive participants within the teaching process (Mortimer and Scott, 2003; Nystrand, 1997).

This paper presents part of wider research examining the impact of implementing an intervention on primary mathematics teaching and learning which was conducted in primary Bahraini schools. The research methodology involved a design experiment (Cobb et al., 2003; Kelly and Lesh, 2000) and "multiple-embedded" case studies (Yin, 2003). A variety of data sources were used through one academic year, including classroom observations, video and audio recordings and interviews.

For this paper qualitative data of one case study obtained from one interview at the end of the intervention process will be presented. The interview was conducted with a mathematics classroom teacher (Laila) and her senior teacher (Afrah). In Bahrain, students of the second cycle (year 4 to year 6) are taught by mathematics specialist teachers and the senior teacher supervises the mathematics teaching and learning. The interview involves both teachers reflecting on the change that had taken place as a result of the intervention, which was to promote inquiry approaches (Jaworski, 2006) at both the classroom and the school level. To make sense of the statements made, an analytic tool is needed that can account for the extent to which a teacher structures and restructures constraints for students in the mathematics classroom. Initially

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Valsiner's (1998 and 1987) zones theory is outlined and used to analyse the data. Subsequently, Valsiner's notions are extended in order to provide an integrated interpretation of the presented data.

THEORETICAL ORIENTATION

In his study of the processes involved in child development, Valsiner (1987 and 1998) proposes two zones, the "Zone of Free Movement" (ZFM) and the "Zone of Promoted Actions" (ZPA). The ZFM characterises the child-environment relationship at a particular time and in a certain environment, by suggesting that his or her "freedom of choice of action (and thinking) is limited by a set of constraints" (Valsiner, 1987, p.97). The ZFM is a social construct that is created through mutual cultural interactions between the child and the adult (Valsiner, 1987 and 1998) which is set by, and based on, the adult's ongoing understanding and analysis of the child's capabilities in that context. As a result, as Valsiner points out, a ZFM is not fixed but is dynamic and can be altered when the adult feels it is appropriate (Valsiner, 1987 and 1998).

The second of Valsiner's (1987, p.99) notions is the ZPA, which refers to the "set of activities, objects, or areas in the environment, in respect of which the child's actions are promoted." The ZPA is normally a sub-zone of the ZFM. However, a ZPA can play a crucial role in development, as it can restructure a ZFM – by instigating a ZPA at the edge of a students' ZFM, the teacher's actions can enable students to cross the boundaries of their existing ZFM, thereby changing it.

Valsiner (1998 and 1987) contends that the two zones interact and form a complex system which "canalizes" the actions of the child in a certain direction. Valsiner (1987) gives an example of a very narrow ZFM/ZPA complex system in education, wherein the teacher focuses on rote memorization and leaves few choices to students other than doing and repeating what that teacher says.

These notions have been used in mathematics education. Blanton et al. (2005) used Valsiner's zone theory to interpret adults' mathematics and science practices through investigating classroom discourse. Goos (2005) employed the concepts of ZFM and ZPA to investigate instructional practices and beliefs for pre-service and novice teachers with respect to integrating technology in their secondary mathematics classrooms. Valsiner's notions allowed Goos to theorise teacher development as the construction of identity.

The remainder of this paper presents and interprets some teachers' articulations about changes in a classroom with regard to ZFM/ZPA complex systems.

DATA AND A DISCUSSION OF DATA

Extracts from an interview with Laila and Afrah conducted at the end of a course of study are presented and discussed (interpreted) below. Both were responding to an open question to talk about their experience of the intervention. Part of the intervention strategy was to follow a collaborative learning approach that allowed students to share their views and ideas in a whole class setting. In the following extract Laila has focussed on collaborative learning and makes it the main point of her comparison between the situation before and after the intervention.

1 Laila: I mean, at the outset I was not realizing that we can use collaborative
2 learning differently, for example (before) we give students exercises and
3 solve it, and the best group comes out and wins. We did not present their
4 work at all, this is the first time we do it, we present students' work and
5 discuss. Presenting, only one group comes out and solve, and if there were
6 more than one solution, others say no we did not solve in this way, we
7 solved in that way and discuss. But for those groups who have wrong
8 answers they say we have incorrect solutions and that's it. Not every
9 group comes out by itself whereas now every group comes out by itself
10 and discuss the very methods of solutions. I mean, there were interactions,
11 students by themselves as I told you, correct their mistakes, themselves
12 express their views, they wonder. During collaborative learning, they ask
13 questions, pose questions, before they ... might pose questions but during
14 the collaborative learning we direct them more. Now we provide them
15 with little help, I leave them to complete. Before when we had a
16 collaborative learning, I mean, we stay to assist the group itself we
17 provide assistance to arrive the solution...I Mean not just giving an idea
18 and leaving them alone to utilize it and solve in this way.
19 Afrah: Maybe I want to talk about group work... I understand about group work
20 after the explanation of the lesson, we employ it; we give them exercise as
21 group work and then an individual work as usual. Now... students started
22 learning... during the group work. Before ... I start with explaining the
23 concept or giving them a readymade concept. OK. Frankly speaking, I
feel
24 this is a big shift because the girl by herself forming the concept... hence
25 she is learning. She is wondering in order to attain the result…
…
30 Giving them free area for their learning. Mistakes let them make mistakes,
31 they will learn from their mistakes. Mistakes several times constitute a
32 spring-board for learning...

The above extract gives Laila's account of the two different states, before and after, of using collaborative learning in mathematics teaching. Laila's words (lines 1-2) indicate that in both situations, students were promoted to learn collaboratively. In both cases collaborative learning was available for them, but with different structures. In the prior situation, Laila set up narrow ZFMs, so that her students' movement within collaborative work was highly constrained compared to the latter situation. More precisely, students' freedom of choice in respect of mathematical actions and thinking during 'old style' collaborative learning was limited by a set of constraints: (i) restricting collaborative learning to solving exercises (line 2); (ii) marrying collaborative work with a competitive environment (line 3); (iii) not all groups presented their work, as only the "best group" presents its solution (lines 3-4); (iv) other groups can only tell whether their solutions were similar or not (lines 5-7); (v) groups with incorrect answers were not included in discussions (lines 7-8); and (vi) intervening during group work interactions (lines 15-18).

Although the teacher-established boundaries of the pre-intervention ZFMs enabled some student options in relation to mathematics learning during collaborative work, these boundaries structured rigidly students' means of accessing many of the aspects of their environment. For example, students had a restricted range of choices about new knowledge construction, suggesting a passive "epistemic role" (Nystrand, 1997) for students. They also had limited opportunities to advertise or to share different mathematical ideas, as well as being unable to negotiate or discuss their own solution methods irrespective of their correctness. Furthermore, Laila provoked students to compete with each other, which restricted student-student interaction.

Since each ZFM/ZPA system "canalizes" students' immediate and future actions in a certain direction, one can anticipate what sort of possible acts a student might produce. In the above case, it is claimed that students' participation in constructing new knowledge was restricted. Also, there was no reported place for discussions or for multiple perspectives within students' learning processes.

Laila's expression "I was not realizing that we can use collaborative learning differently" (lines 1-2) indicates a recognition of the establishment, prior to intervention, of some kind of ZFM/ZPA for her students, which, one might assume, was stable over a considerably long period of time. Laila suggests, in for example "we did not present their work at all" (lines 3-4), that there was little questioning or negotiations between her and her students, or analysis of students' capabilities as proposed by Valsiner. Indeed, the establishment of the constraints did not appear to be based on students' actual or potential capabilities; instead, they appear to have been based on teachers' *beliefs system* (Pajares, 1992) about mathematics teaching and learning and on school social and sociomathematical norms (Yackel and Cobb, 1996). Laila's comments about student and teacher actions after the intervention process was established (lines 12-18), however, suggest a transformation of her *belief system* and a recognition of being able to restructure the ZFM/ZPA in a way that gave students greater freedom.

We suggest that the intervention process led to an *epistemological shift* with regard to suitable tasks for students. In an interview prior to the intervention Laila had been asked about "drills for students" and responded 'Sufficient drills are the most important things in mathematics … what is so important in mathematics is training' (see also lines 19-24). We think that this prior perspective of Laila is important with regard to constraints on collaborative learning in classes in Laila's old regime; exercises vary in their influence on knowledge construction (Watson and Mason, 2004) but drilling exercises are likely to be at the lower end of the knowledge construction scale.

To return to the interview, Afrah emphasized the transformation of functions and purposes of collaborative learning. She commented that prior to intervention collaborative work was based on exercises following teacher introduction (lines 19-21). In the language of Valsiner this is a constraint, “giving them a ready-made concept”; in the language of Mortimer and Scott (2003), students just listen to Laila’s “authoritative” voice. After intervention, there was a “big shift” (line 24).

Both Laila and Afrah describe a different ZFM for the students after intervention. They still claim to promote collaborative learning but many of the previous constraints are perceived to have disappeared; and different areas and objects in the environment are made available for students (wider ZFM). They claim that students have more free movement to: (i) advertise and discuss their work and views (lines 9-10); (ii) correct their mistakes (line 11) and use “mistakes as spring-board for learning” (lines 30-32); (iii) initiate different kind of discourse during, and after, collaborative work (lines 10-13 and 25); and (iv) be more confident (lines 33-36) and less dependent on teacher interventions during the collaborative work (lines 14-15).

This speaks of a restructured ZFM, and much that was constrained became available to students. Multiple mathematical perspectives, experiences and individual/group solution methods were transformed from being constrained to becoming objects for discussions, negotiations and reflections for all students. As a consequence of this, especially in empowering students to conduct reflection, students had opportunities to correct and learn from their own mistakes. Moreover, within the new ZFM different kinds and patterns of discourse occurred. Laila’s statement “there were interactions”, and that students “wonder” and “ask questions” (lines 10-13), suggest that students had repositioned themselves with regard to the mathematics of the class and gained more freedom; in place of the control of knowledge there was, reportedly, greater student freedom to initiate different patterns of discourse. Lines 7-11 concern right/wrong answers in lessons prior and after intervention and suggest a perceived transformation of the teacher involvement, from being with the group in order to assist its members to get the right answer, to providing challenges for students when the teacher reduced her intervention and allowed groups to make their own decisions and accept the consequences – even if there were some mathematical mistakes; and of these mistakes becoming objects for student discussion and reflection.

The interview evidence suggests that the intervention process qualitatively transformed hitherto stable classroom environments and created new ZFM/ZPAs. This points back to the issue of how these complex systems are restructured. Valsiner (1987, p.99) states that ZFMs can be restructured by the adult’s (the teacher here) understanding of children’s capabilities within the environment, but we suggest that this does not take account of transformations in the adult/teacher. The restructuring in this study appears to have occurred because Laila internalised new ideas that reorganized her relationships with students and other mathematical objects in the class and hence restricted her ongoing acts also – in other words her own ZFM. The constraints on her were related to her system of beliefs about mathematics and mathematics teaching and learning as well as practices which were grounded in her school setting. In that sense, analytically, Valsiner's analysis fails to uncover how this transformation happened because his notions were not proposed to understand the development of the child in complex environments like school settings. We hold that the ZFM/ZPA complex systems of Laila's students were transformed because of a parallel transformation in Laila's ideas and her internalized ZFM. To clarify this argument, we provide the following extracts and discussion.

Afrah: Regarding the strategies, the strategies as I said to you at the outset it provide us a lot. We learned plenty. Starting from the way of lesson planning, our lesson planning is getting changed. First, to sit with a team and plan a lesson (collaboratively) this means it differs from previous planning when I do it alone. Within our planning procedure we carefully select the objectives, emphasising on high order objectives... For lesson implementation we were keen upon everything... In case I introduce this problem I anticipate what students will say to me, what I will reply to them, I mean Laila is eager: Afrah if they say to me like this, what I shall tell them, do you expect they will say? This means many, many things. Sometimes she calls me (laughing) and says to me now I thought Afrah that I gave them this problem and I asked them this way , if they tell me such and such. Do you think we have to change the task... So we were negotiating these issues. To be frank, this gives us a wide area to explore our potentials that were hidden... OK. Also the lovely thing is after lesson's implementation . We sit and discuss this lesson. It happened to change or modify something... Several times Mrs Laila says she prepares, as a matter of fact, she prepares several learning tasks. For instance, when she enters the class 6/2 and identifies this task to be inadequate, she goes to the other class and uses the second one. This means she does not rely on two or three tasks...

The teacher transformations reported in the above extract include: (i) lesson planning became a joint collaborative act (lines 71-75); (ii) mutual immersion in anticipating possible consequences of proposed instructional actions and decisions (lines 76-83); (iii) posing instructional conjectures, examining them in practice and then enacting modifications (lines 85-90); (iv) conducting reflection sessions after lesson implementation and exchanging observations and insights (line 84-85).

AN EXTENSION

The reported transformation of students' ZFM/ZPAs complex systems at the classroom level is interconnected with a parallel transformation at the teachers/school level and these two levels interact with each other. This transformation cannot be attributed to Laila alone without the presence of other "voices" in the school setting. Indeed, students' new ZFM/ZPA complex systems resulted from collective activities. This proposed analytic extension suggests related socio-cultural components play an essential role in creating new ZFM/ZPAs: mediational artefacts including teaching methods which are considered to be appropriate by the senior teacher and the school management; implicit and explicit cultural norms; teacher networking procedures; and instructional actions, decisions and tasks of practitioners who share responsibility for student learning. From this perspective, teachers' prior understanding of students' capabilities and the above components interact with each other to structure and restructure students' complex systems which draw the boundaries of the constraints on students' mathematics learning. This extension to Valsiner's notions can be viewed as an analytic tool for understanding how complex classroom systems are structured and restructured and, hence, understanding the development of students' mathematics.

CONCLUSION

Laila and Afrah described two complex systems related to before and after the intervention to develop an inquiry approach in the classroom in which emerging activities transformed students' learning. Afrah considered this to be a "big shift" in mathematics teaching and learning because students' participation had been transformed qualitatively by gaining more "epistemic roles", to use Nystrand's (1997) term. This evolution was successful for many reasons. One reason was Laila's endeavours in setting and examining pedagogical conjectures and modifying them when necessary. Another was the development of teachers' networking that utilizes collaborative planning and collective reflections. Student activity and norms within the mathematics classroom and between mathematics and senior teachers were also transformed. Thus, students' ZFM/ZPA complex systems were transformed through a parallel transformation of several components. We have argued that the transformation in student learning was interrelated with transformations in these other socio-cultural components in the school, for example the development of the teachers' networking, and the cultivation of different norms for mathematics teaching and learning. We feel that the qualitative transformation of students' learning depends on the extent to which all components are transformed.

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TEACHER EFFECTS ON THE PROBABILISTIC THINKING OF PUPILS IN ENGLAND

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Multilevel Rasch measurement methodology was used to analyse data collected when a 10-item instrument was administered to a sample of 754 pupils. A scale called "ability to overcome the effects of the representativeness heuristic" (hence Ability) was built to indicate the capacity of pupils to answer correctly to probability items. A "tendency towards representativeness errors" scale (hence Representativeness Tendency) was also built to indicate the tendency of pupils to exhibit answers affected by the representativeness heuristic effect. This study investigates the differential teacher effect on both Ability and Representativeness Tendency scales and investigates whether there is differential teacher effect for different heuristics. The class size affected the performance of the pupils on the Ability scale.

THE BACKGROUND OF THE STUDY

Researchers, like Williams and Ryan (2000), argue that research knowledge about pupils' misconceptions and learning generally needs to be located within the curriculum and associated with relevant teaching strategies if it is to be made useful for teachers. This involves a significant transformation and development of research knowledge into pedagogical content knowledge (Shulman 1987). Pedagogical Content Knowledge (PCK) "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman 1986, 9). Pedagogical Content Knowledge also includes the conceptions and preconceptions that students bring with them to the learning. If those preconceptions are misconceptions, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners. Many studies have found that teachers' subject knowledge and pedagogical content knowledge both affect classroom practice and are modified and influenced by practice (Turner-Bisset 1999).

Hadjidemetriou and Williams (2002) found that some teachers harbour misconceptions themselves. Godino, Canizares and Diaz (n.d.) conclude in their research that very frequently teachers do not have the necessary preparation and training in probability or statistics in order to teach efficiently; they also concluded that student teachers may have various probabilistic misconceptions themselves and this might affect their teaching.

Much research on pupils' mathematical knowledge exists (e.g. Hill, Rowan and Ball 2005), however this normally refers to general mathematical ability using rather blunt instruments, in comparison to a more focused test, like the one used in this study. Other recent research has shown teacher effects of a magnitude of around half a

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standard deviation of pupils' ability, but has also found a negligible correlation between the performance of the pupils and class size (Nye, Konstantopoulos & Hedges 2004). The main point is that there is much research that quantifies the so-called 'teacher effect', especially in the context of 'school effectiveness' research, however, very few – if any – research papers identify differential teacher effects on heuristics.

Therefore, the authors carefully formulated a research design where nested data (i.e. pupils nested within teachers, who are nested within schools) were collected using a dedicated instrument (a test measuring the impact of the representativeness heuristic on the probabilistic thinking of the pupils) and a specialized statistical method was applied (i.e. generalized mixed effects models). What is mostly important is that the responses of the pupils (and their teachers) on the same instrument were scored in two ways: (a) right/wrong responses, and, (b) responses indicating that the probabilistic thinking of the pupils was affected by the representativeness heuristic.

METHODOLOGY

The instrument and the dataset

Ten items were used to construct the instrument (reached at www.relabs.org). The items identify four effects of the representativeness heuristic; the recency effect, the random-similarity effect, the base-rate frequency effect and the sample size effect. Most of the items have been adopted with slight modifications of these used in previous research (Green 1982; Kahneman, Slovic and Tversky 1982; Shaughnessy 1992; Konold et al 1993; Batanero, Serrano and Garfield 1996; Fischbein and Schnarch 1997; Amir, Linchevski and Shefet 1999). Other items were developed based on findings of previous research.

The items were divided into two parts. The first part consisted of multiple-choice answers. In the second part the respondents were asked to give a brief justification for their choice ('Explain why'). The Ability scale was built to range from 0 to 10 (where 0 indicates no correct responses and 10 indicate fully correct responses to all multiple choice items). For the Representativeness Tendency scale 0 marks were awarded on a question to indicate 'response most likely not affected by the Representativeness Heuristic' whereas 1 mark means 'response most likely affected by the Representativeness Heuristic. Therefore, for the Representativeness Tendency scale 0 marks mean 'no representativeness effect identifies" and 10 means "all responses of the pupil seemed to be affected by the Representativeness Heuristic".

The instrument was administered to 754 pupils from schools in the NW England. In each case, the gender, the age, the Year group, the teacher's name and the school's name of the pupil were recorded.

Data Analysis

Using the lme4 package of the R software (http://www.r-project.org) we treated the items as a fixed effect and the pupils and the teachers as random effects in a binomial model to construct multilevel Rasch scales (Doran, Bates, Bliese & Dowling, 2007). Extending the single-level Rasch model (Wright & Masters, 1982) to accommodate for the nesting nature of the data is increasingly becoming common practice in educational research.

At first, an initial single-level Rasch model was fit on the data in order to compare the results of the lme4 package with the results of Bigsteps, a more 'traditional', also free, Rasch package. The result was to reconstruct the Ability and Representativeness Tendency scales mentioned in previous research using the same data (i.e. Afantiti Lamprianou and Williams 2002, 2003; Afantiti Lamprianou, Williams Lamprianou 2005). Then, increasingly complex models were used in order to investigate more subtle aspects of the issues under investigation.

Aims of the research

This research aspires to investigate whether there is a statistically significant and practically non-negligible teacher effect on (a) a probabilities Ability scale, (b) on a Representativeness Tendency scale. In addition, the effects of Age, Year group and Gender are investigated under the light of multi-level Rasch models that account for nested effects. The differential teacher effect on different heuristics is also investigated.

RESULTS

Fitting an initial multilevel Rasch model

An initial multilevel Rasch model was fit on the data, using the Teachers and the Pupils as random effects. In the case of the Ability scale, the teacher effect accounted for a large proportion of the variance, equal to half of the between-subject variance. However, for the Representativeness Tendency dataset, the Teacher effect appeared to be very small. The standard deviation of the Teacher effect (SD=0.186) was the one sixth of that of the Pupil effect (SD=0.767).

Modelling the Misconception Type by Year group interaction

Since the misconception (thus, 'Misconception' effect) addressed by each item is a repeatable factor and of interest itself, we modelled it as a fixed-effects term. The pupils, the teachers as well as the items are modelled as random effects. The Year group (years 6 and 7) was also modelled as a fixed effect. The items are modelled to be nested within teachers, since we are interested in identifying the main teacher effect, but also to measure the magnitude of the interactions between teachers and items (i.e. differential difficulty of items by teacher).

There seems to be a strong Teacher main effect (SD=0.496) since the variance attributed to teachers is three quarters of the variance attributed to individual pupils (SD=0.792). The variance of the teacher by item effect (SD=0.662) is twice as large as the variance of the main teacher effect. All in all, the total variance from the teacher main effect and the teacher interaction effects is larger than the variance because of pupil differential location on the scale. The item variance was 0.993 (SD=0.996). The results show that there is no statistically or practically significant Year group by misconception type interaction effect.

However, when a similar analysis is run on the Representativeness dataset, the results were, again, slightly different. The main teacher effect for the Representativeness Tendency dataset is almost negligible (SD=0.228), but there is much variance coming from the interaction between teachers and items (SD=0.709). In any case, the main Year effect seems to be negligible on the scale. The pupil variance was rather large (SD=0.799). One could also mention the Lilliputian main item effect (SD=0.103).

The interaction effect between teachers and type of misconception

If the usual hypothesis that pupils' learning is heavily affected by their teachers holds true, then a teacher by misconception interaction effect is useful to model our dataset. The aim of this analysis is to investigate whether the teacher effect is stronger for some misconceptions but not for others. A teacher by misconception interaction was added in the 3-level Rasch model.

```
  Random effects:
 Groups                Name        Variance Std.Dev.
 pupil                 (Intercept) 0.604072 0.77722
 teacher:misconception (Intercept) 0.509101 0.71351
 teacher               (Intercept) 0.042950 0.20724
 item                  (Intercept) 0.967565 0.98365
Number of obs: 7043, groups: id, 754; teacher:misconception, 114; teacher, 29;
item, 10
```

Table 1: The 3-level Rasch model with interactions (Ability scale)

The teacher by misconception effect seems to explain almost as much variance as the pupil effect. The main teacher effect is practically zero, although the item main effect is still very large.

The following model investigates the way the variance is partitioned for the Representative Tendency dataset. There is no 'Teacher' random effect and no 'Year' fixed effect because their presence in the model was not statistically significant in any way.

```
Random effects:
 Groups                Name        Variance Std.Dev.
 id                    (Intercept) 0.62366  0.78972
 teacher:misconception (Intercept) 0.73897  0.85964
 item                  (Intercept) 0.06644  0.25776
Number of obs: 7043, groups: id, 754; teacher:misconception, 114; item, 10
```

Table 2: The 3-level Rasch model with teacher by misconception interactions (Representativeness Tendency scale)

The most interesting comparison between the results of the Ability and the Representativeness Tendency scales is the slightly increased teacher by misconception interaction for the misconception scale, with a simultaneous very small variance due to item effects. This is difficult to explain, although one might assume that all the item variance in the Representativeness Tendency scale is absorbed by the misconception effect. Also, the Random-Similarity effect is not significant for the Ability scale, but is highly significant for the Representativeness Tendency scale.

When a three-way interaction Teacher:Misconce:Item was added in the model in order to investigate whether it was possible to see where the item variance goes, the results for both the Ability and the Representativeness Tendency scale were slightly different. For the Ability scale, the interaction had a small, but non-negligible effect (Table 3). However, for the Representativeness Tendency scale, the results were very clear: there is no measurable variability between items within misconception.

Ability scale				Representativeness Tendency scale			
AIC BIC logLik deviance				AIC BIC logLik deviance			
5370 5425 -2677 5354				4776 4830 -2380 4760			
Random effects:				Random effects:			
Groups	Name	Variance	Std.Dev.	Groups	Name	Variance	Std.Dev.
pupil	(Interc)	0.62933	0.79330	pupil	(Interc)	0.623657	0.78972
teacher:misconc:item	(Interc)	0.14685	0.38321	teacher:misconc:item	(Interc)	0.000000	0.00000
teacher:misconc	(Interc)	0.52951	0.72767	teacher:misconc	(Interc)	0.738967	0.85963
item	(Interc)	0.97624	0.98805	item	(Interc)	0.066442	0.25776

Table 3: The 3-level Rasch model with teacher by misconception by item interactions

This indicates that pupils whom probabilistic thinking 'suffers' by the Representativeness Tendency heuristic, tend to give uniformly right or wrong responses to items that measure the same heuristic.

The effect of the actual class size

The effect of the size of the class was investigated by correlating the random effect estimate of the classes with the number of the pupils in the class. The results were strikingly different for the two scales. On the one hand, there was an unusually high and positive correlation between the class size and the performance of the pupils on the Ability scale ($r=0.609$, $N=29$, $p=0.0001$). On the other hand, there was a much smaller and statistically insignificant correlation between the location of the pupils on the Representativeness Tendency scale and the class size ($r=-0.299$, $N=29$, $p=0.115$). It is reasonable for the second correlation to be negative, in the sense that more Ability means fewer Representativeness errors in the responses and the reverse.

It was therefore assumed that including the class size in the multilevel Rasch models as covariates might result in different variance component estimates for the teachers. This was true for the Ability scale, where the introduction of class size reduced the variance because of the teachers by 15% (from 0.49865 to 0.42312) but had no effect on the Representativeness Tendency scale. It was thus, deduced, that although the class size may have affected the general ability of the pupils to respond to probability questions, it does not seem to affect their chance to give responses 'suffering' from the Representativeness heuristic.

DISCUSSION

This study used multi-level Rasch models in order to investigate (a) the contribution of teachers on the variance of the Ability and the Representativeness Tendency scales, (b) a comparison between the results for the two scales.

The teacher effect was indeed strong for both scales. In the case of the Ability scale, the total teacher variance is practically equal to the pupil variance. In the case of the Representativeness Tendency scale, the total teacher variance is just larger than the pupil variance (0.68 to 0.62). This shows that being a pupil of a specific teacher may be more important (or equally important) than who you are and what previous knowledge or experiences you carry when you go to school.

The main findings of this research are in agreement with previous research, but in this case, the teacher effect seems to be much stronger than usually. For example, in a (typical) recent research (measuring general mathematical ability rather than a sub-domain ability like the present study) Hill, Rowan, Ball (2005) found that teachers' mathematical knowledge was significantly related to student achievement gains (in both first and third grades). However, the school effect in this case was significantly important than the teacher effect since for Grade 3 pupils the variance was 99.2 for the school component compared to 77.4 for the teachers; for Grade 1 pupils the variance was 24.4 for the school component compared to 79.3 for the teachers.

If one could consider the Ability scale to give a 'raw' measure of the representativeness effect (one could give wrong responses for other reasons as well as because of the representativeness effect), then the Representativeness Tendency scale would show a substantially larger teacher effect, BUT only if teachers are to blame for the persistence of the representativeness heuristics on the probabilistic thinking of the pupils. In addition to this, if we could assume that some teachers have a 'magic touch' that removes the representativeness effect from the probabilistic thinking of the pupils (either because they have more pedagogical content knowledge or more subject matter knowledge), then one would again expect a stronger teacher effect on the Representativeness Tendency scale.

All in all, it seems that teachers might be blamed for increased – or to praise for decreased – representativeness heuristic effect on the probabilistic thinking of pupils. But is this the case? Threfall (2004) focused on the teaching of probability to primary school pupils in England, and argued that

> "primary aged children did not learn anything about probability that could be reliably assessed, and so probability as a curriculum component did not contribute to some of the purposes of the National Curriculum". p1

He suggests that the teachers actually had no significant impact on the pupils, as far as the probabilities part of the curriculum was concerned. The results of this research do not seem to agree with him: the teacher effect is dominant; does this mean that the teachers 'pass' their misconceptions or errors to their pupils? Dole (2003) discussed the need to help pupils ‘unlearn’, in order to overcome errors and misconceptions in mathematics. It is important to know whether the teachers unknowingly pass their errors and misconceptions to pupils, but if this is the case, the new pedagogical/teaching strategies are definitely needed to address this situation.

Also, although the main item effect (the variance component because of items) is huge (dominant) in the case of the Ability scale, it is Lilliputian in the case of the Representativeness Tendency scale. Why is that? This may indicate the strength of the Representativenss effect. Once a pupil’s probabilistic thinking ‘suffers’ by the effect, then the variance between items measuring the same effect is almost negligible: responses to items measuring the same representativeness tendency effect are of similar nature.

In addition to the above findings, it seems that class size may affect the responses of the pupils on the ‘raw’ Ability scale, but not on the Representativeness tendency scale. This contradicts the findings from previous research, such as Nye, Konstantopoulos & Hedges (2004) who found class size to have negligible effect. This might mean that in a smaller class, the teacher might be able to explain to pupils how to avoid the most common general incorrect responses in a better way, or might mean that the pupils were more motivated in smaller classes. But, when it comes to de-coding the actual meaning of their responses in order to investigate the representativeness effect, it means that the class size is not important because the representativeness huristic can not be overcome by spending more time on each pupil individually. Most likely, it is the teacher himself, and his teaching approach and philosophy that counts, not how much time a teacher spends with individual pupils.

However, the elaborated methodology of this research has shown that the teacher effect is so large compared to the pupil effect, that some teachers (in some way) are certainly affecting pupils’ probabilistic thinking in England through their teaching to a large extend. Also, waiting has actually shown no significant impact, as the negligible differences between year 6 and year 7 pupils have shown. Therefore, spending more resources on helping teachers teach probabilities may be wiser than eliminating probabilities from the National Curriculum.

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PROMOTING CRITICAL THINKING ABILITIES VIA PROBABILITY INSTRUCTION

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Our attempt to promote the development of critical thinking abilities among students involved forty three subjects, who were exposed to teaching strategies designed to encourage critical thinking in probability lessons, by applying mathematics to real-life problems, encouraging debates, and planning investigative lessons. Prior to and following the experiment, students were asked to complete the Cornell Test of critical thinking. Assessment of the results as well as interviews and evaluation of written work collectively led us to the conclusion that critical thinking capabilities improved during the course of the study. These results indicate that deliberate and consistent encouragement of Critical Thinking is likely to improve these abilities among students.

INTRODUCTION

In the last two decades, there has been international acceptance of the need to change traditional methods of teaching. Developing different cognitive and metacognitive skills is considered to be of the utmost importance, and curricula increasingly incorporate skills requiring higher-order thinking, such as critical, deductive, creative and inventive thinking. In education, mathematics has been considered a field of thought which is suitable for promoting important educational skills, such as critical thinking. In the real world, we constantly need to make personal decisions based on complex situations. Hence, it is essential to instill in our students the ability to think critically. Critical Thinking (CT) is used in every profession, and it allows people to deal with reality in a reasonable and independent manner (Lipman, 1991). Furthermore, c.t.is an essential part of the education of future citizens in a democratic society, and if we want to prepare our students for life and not merely for their final exams, we need to help them learn how to transform their knowledge and abilities into positive, responsible actions and to make rational judgements in an era floded with information (Feuerstein, 2002; Perkins, 1992; Swartz, 1992). Research has investigated students' developing of thinking tools such as: evaluating, checking the truth of results, assessing a certain problem, comparing, generalizing, applying and defining, solution strategies and such (e.g., Avital & Barbeau, 1991; Akbari-Zarin & Gray, 1990). Our study, however, investigated students' development of abilities such as induction, deduction, value judging, credibility, assumptions, and meaning, according to the taxonomy of Ennis (1987), which we will elaborate on later. The purpose of our study is to determine whether critical thinking abilities can be developed through probability instruction.

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THEORETICAL BACKGROUND

Critical Thinking Definitions

There are many varied definitions as to what exactly critical thinking is. According to Schafersman, (1991) critical thinking is an ability that is learned . He claims that it must not be left to develop of its own accord, nor can it be taught successfully to students by an un-trained instructor. Both training and knowledge are necessary to promote these abilities and he believes that math and science teachers are well suited by their training and knowledge to carry out this task. Schafersman (1991) believes that critical thinking involves inquiry, asking questions, offering alternative answers, and questioning traditional and accepted beliefs. He suggests that because society does not welcome people who challenge authority, critical thinking is not encouraged. In his opinion "most people, therefore, do not think critically" (p.23). Another definition is that critical thinking is the ability and readiness to evaluate claims in an objective manner based on firm arguments (Wade & Tavris 1993). This research is based on three key elements: CT taxonomy that includes CT skills (Ennis, 1987;1989), The learning unit "Probability in Daily life" (Liberman & Tversky 2002) and the Infusion Approach between subject matter and thinking skills (Swartz, 1992).

Critical Thinking Skills by Ennis

Ennis defines CT as “a correct evaluation of statements". Twenty-three years later, Ennis broadened his definition to include a mental element. The improved definition is “reasonable reflective thinking focused on deciding what to believe or do” (Ennis, 1962;1987) **.**In light of this definition, he developed a CT taxonomy that relates to skills that include not only the intellectual aspect but the behavioural aspect as well. In addition, Ennis' (1987) taxonomy includes skills, dispositions and abilities. Ennis claims that CT is a reflective (by critically thinking, one’s own thinking activity is examined) and practical activity aiming for a moderate action or belief. There are five key concepts and characteristics defining CT according to Ennis: practical, reflective, moderate, belief and action.

Critical Thinking Abilities by Ennis Implemented in the Present Study

Seven aspects of critical thinking were considered as objective assessment criteria for evaluating the incorporation of critical thinking in students' mathematical education and they are induction, deduction, value judjing, observation, credibility, assumptions, and meaning. Although aspects of critical thinking are listed separately, overlap between them exists to a certain extent. For example, one might argue that deduction is involved in much induction, calling for the listing under deduction of the items listed under inductions. Similarly, one might also argue that observation and credibility judgements call for the implications of principles, a deductive process and should also be listed under deduction. Furthermore, one might argue that since basic deduction is simply the meaning of words and statements, everything classified under deduction could also be classified under meaning. This legitimate overlap is partially expressed by the items of the Cornell critical thinking of level Z (Ennis, Millman & Tomko 2005).

The Teaching Unit "Probability in the daily life"

In this learning unit, which is a part of the formal National Curriculum the student is required to analyse problems, raise questions and think critically about the data and the information. The purpose of the learning unit is not to be satisfied with a numerical answer but to examine the data and its validity. In cases where there is no single numerical answer, the students are required to know what questions to ask and how to analyse the problem qualitatively, not only quantitatively. Along with being provided with statistical tools students are redirected to their intuitive mechanisms to help them estimate probabilities in daily life. Simultaneously, students examine the logical premises of these intuitions, along with misjudgements of their application. Here, the key concepts are: probability rules, conditional probability and Bayes theorem, statistical relations, causal relations and judgment by representative (Liberman & Tversky 2002) .

The Infusion Approach

There are two main approaches for fostering CT: the general skills approach, which is characterized by designing special courses for instructing CT skills, and the infusion approach which is characterized by providing these skills by embedding them in the teaching of the set learning material. In according to Swartz (1992), the Infusion approach aims for specific instruction of special CT skills during the course of different subjects. According to this approach there is a need to reprocess the set material in order to integrate the teaching of thinking skills into the conventional instruction.

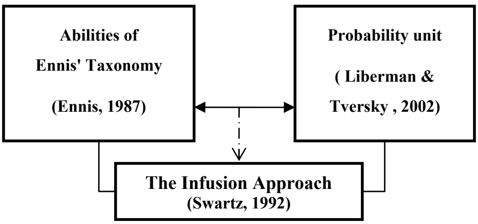

Diagram 1: The Infusion Approach

In this study, we will show how we integrated the mathematical content of "probability in daily life" with CT skills from Ennis' taxonomy, reprocessed the curriculum, tested different learning units and evaluated the subjects' CT skills. One of the overall research purposes was to examine the effect of the Infusion approach on the development of critical thinking skills through probability sessions. The comprehensive research purpose was to examine the effect of developing CT by the Infusion approach using the Cornell questionnaire (a quantitative test) and quantitative means of analysis.

OBJECTIVES AND METHODOLOGY

In our study we wanted to find out whether teaching conducted with the purpose of promoting higher-order thinking skills, through probability instruction, would improve the students' critical thinking abilities. The study was conducted over the period of an academic year (eight months) and involved fifteen lessons, each lasting ninety minutes. The teacher was one of the researchers.

Research Population

Forty-three children between the ages of fifteen and sixteen participated in an extra curricular program aimed at enhancing thinking skills of students from different cultural backgrounds and socio-economic levels. A teaching experiment was conducted in which probability lessons were combined with CT skills. The students who participated in this study were part of an after school program called Kidumatica, (designed and headed by the second researcher). The students are high achievers from diverse ethnicities and socio-economic background.

Data Collection and Method of Analysis

In order to assess the effects of the intervention, the Cornell Critical Thinking Test-Level Z (Ennis & Millman, 2005) was administered both prior to and following the study. In addition, the students were asked by their teacher to take tests, work on projects, and do activities covering the specific critical thinking abilities. The students' written products (papers, homework, exams etc.) were collected. Personal interviews were conducted with randomly selected students. Five students were interviewed at the end of each lesson and one week later. The personal interviews were conducted in order to identify any change in the students' attitudes throughout the academic year. All lessons were videotaped and transcribed. In addition, the teacher kept a journal (log) of every lesson. Data was processed by means of qualitative methods intended to follow the students' patterns of thinking and interpretation with regards to the material taught in different contexts. In order to check the development of the students' critical thinking abilities, the Cornell Test developed by Ennis and his colleagues was used (Ennis & Millman, 2005). The Cornell Critical Thinking Test-Level Z was chosen in order to adjust to the level of the group. The test includes general content with which most of the students would be familiar and it assesses various forms and correlates of critical thinking, such as induction, deduction, value judging, observation, credibility, assumptions and meaning. In the process of critical thinking, there is an overlap of these various forms as they are all dependent on each other. In the Cornell test, this inter-dependence is evident in the fact that frequently an item is assigned to several different aspects. It is important to note that both Observation and Credibility are evaluated according to the same items in the test (items twenty-two to twenty-five). It is a multiple-choice test with three choices and one correct answer. Although the test is meant to be taken within a fifty-minute period, we predicted that the students in the group would be unable to complete it within that time limit. For this reason we decided to give them eighty minutes in which to take the test.

Description of the Intervention Unit

As already mentioned, the probability unit combines CT skills with the mathematical content of "probability in daily life" (Liberman & Tversky, 2002). This probability unit included questions taken from daily life situations, newspapers and social surveys, Each of the fifteen session that comprised the intervention had a fixed structure. The lesson began with a short article or text that was presented to the class by the teacher. A generic (general) question relating to the text was then written on the blackboard. An open discussion of the question then took place in small groups of four students. Ten minutes were allotted to the discussion and there was no intervention by the teacher. Each group offered their initial suggestions about how the question could be resolved. This group discussion provided an opportunity to practice the CT skills. An open class discussion then followed. During the discussion, the teacher asked the students different questions to foster their thinking skills and curiosity and to encourage them to ask their own questions. The students presented their different suggestions and tried to reach a consensus. The teacher referred to questions raised by the students and encouraged CT, while instilling new mathematical knowledge: specifically, the identification and finding of a causal connection by a third factor and finding a statistical connection between C, A and B, Simpson's paradox and Bayes' Theorem. In general, the unit included critical thinking skills (abilities and dispositions) and mathematical knowledge (probability) using the infusion approach. Ennis made a clear distinction between abilities and dispositions. This study focuses only on the development of the abilities. A future study will deal with developing the dispositions. It is noteworthy that whereas abilities pertain to the cognitive component of critical thinking, dispositions relate mainly to the mental one. The mathematical topics taught during the fifteen lessons were: Introduction to set theory, probability rules, building a 3D table, conditional probability and Bayes theorem, statistical and causal connection, Simpson's paradox, and judgment by representative. The following CT skills were incorporated in all fifteen lessons: A clear search for an hypothesis or question, the evaluation of reliable sources, identifying variables, “thinking out of the box,” and a search for alternatives (Aizikovitsh & Amit, 2008). This reference also provides a full description of a lesson where the intervention unit was implemented.

RESULTS

In order to assess the effects of the infusion approach in the development of the students' ability to think critically, the Cornell Critical Thinking Test-Level Z was administered to the targeted students at the end of the intervention period. The two components where an **improvement** can be demostrated are those of **credibility and observation**. The former showed an increase from a pre-test mean of 0.257 to a post-test result of 0.801 ($t = 3.43$ and $p < .001$). The observation mean showed an similar increase. The parallel improvement in the credibility and Observation results can be clearly inferred from this histogram (Figure 1). Some improvement also characterizes induction. In general, a pattern of increase in the post mean relative to the pre-mean score characterizes all the separate components of critical thinking abilities tested in this study. These differences should be considered carefully, in accordace with the statistical significane of the findings. Whereas induction, credibility and observation increased significantly, the remining finding should be thought of more critically. The results of the pre-test and the post-test according to the different abilities are presented in figure 1.

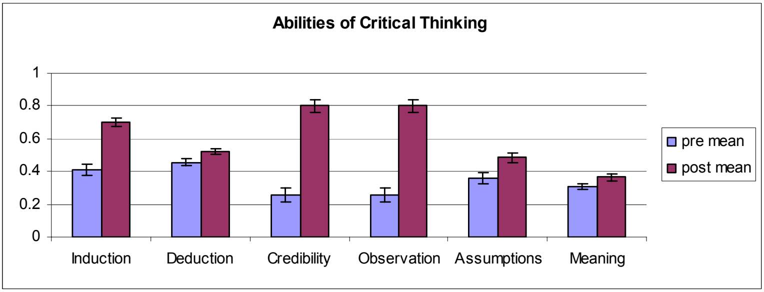

Figure 1: Results of pre- and post-test means of Cornell tests

DISCUSSION

In some of the Critical Thinking literature, we see that there is no significant improvement in most the sub-tests (Zohar & Tamir, 1993). This correlates with our finding on the Cornell Test results. In the sub-tests we did see a significant improvement in credibility and observation. This result can be explained by the familiarity of the students with the representational form in which the problem was posed. During this teaching unit, we repeatedly worked with tables and we therefore came to the conclusion that the familiarity of the students with tables enabled them to deal with the statements made and answer these specific questions more easily, namely the method of presenting the question seems to have an impact on the studets' understanding of the problem.The improvement in Induction was more difficult to explain and at this stage of our research, we do not have sufficient data to explain why there was an improvement at all, or why, if there was an improvement in induction, there was not a similar improvement in deduction.

Implications for Teaching

This study has shown that it is possible to incorporate activities into regular schools that will develop the students' critical thinking abilities. The subject matter was part of the high-school curriculum, therefore it would not take time away from the class syllabus. It would also not take the teachers' extra time or effort in order to prepare the unit and no special training is needed to accomplish the goal. It is essential that the teacher understands the importance of developing the critical thinking abilities in their students. To increase the generalizability of our results, we have expanded our research to several other schools. In these schools, the same unit will be taught by different teachers (and not by one of the researchers) to decide whether this study can provide an instructional model, which will promote critical thinking, initially in probability studies and perhaps in the future in other fields in mathematics.

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TRIANGLES' PROTOTYPES AND TEACHERS' CONCEPTIONS

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During a workshop about triangles designed for in- and pre-service basic-school teachers, a diagnostic test was applied. The conceptions that teachers have on the subject are analysed using Hershkowitz's construct about prototypical figures.

FRAMEWORK

Geometry has a very special place in the research about mathematical conceptions of students and teachers, partly because the geometrical objects have a dual nature: they involve information of two kinds, graphical (or figural) and theoretical (or conceptual) (see e.g. Larios, 2007). This means that there are also sources of two kinds for the possible misconceptions, and the studies about them usually consider both.

Tackling this duality, Vinner and Hershkowitz (1980) have done research about the prototypical examples in the teaching and learning of Geometry. They have demonstrated that when a figure with certain characteristics is shown to the students, the concept image that they build (i.e., the total cognitive structure that is associated with a concept) is intrinsically associated with the figure (i.e., the figural component). As an example, they mention the triangles with a horizontal side and the rest of the figure "above" it, or that when speaking of the height of a triangle, the triangle used has all of its angles acute and therefore all of its heights interior. In the case of the triangle, the prototypical example is particularly significant, because the misconceptions that arise from it are born too early and may last too long.

The prototypical examples lead to prototypical judgements, which happen when a prototypical figure is used as a frame of reference. Two types are defined: Type 1 is when the visual judgement applies to other instances; for example, when subjects pretending to trace the altitude of a triangle draw an internal segment which is not a height. Type 2 is when the subjects base their judgements in the characteristics of the prototype itself and try to impose them to other examples of the concept (Hershkowitz, 1990).

Geometrical misconceptions have been studied in students and, although less frequently, in teachers. One such study has been thus summarized:

> Hershkowitz and Vinner (1984) reported on a study that included comparing elementary children's knowledge with that of preservice and inservice elementary teachers. They found that the teachers lacked basic geometrical knowledge, skills and analytical thinking ability (da Ponte & Chapman, 2006, p. 464).

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METHODOLOGY

Within the framework of a study about basic school teachers' content knowledge on Mathematics, a set of workshops on Basic Mathematics called TAMBA was designed, one of which is about triangles. It has been given twice. In 2007 it was offered to 36 teachers at the Conference of the Mexican Mathematical Society in the city of Monterrey (MR), and in 2008 it was offered to 31 teachers in a Teachers' Centre in Mexico City (MC). In both venues there were in- and pre-service teachers.

The workshop starts with a diagnostic evaluation, which consists of three items:

1. Four sets of three measures are given, and the participants are asked to say if a triangle can be built with them and, if not, why. The sets are (in cm) *a)* 20, 5, 8; *b)* 17, 12, 10; *c)* 5, 4, 3; and *d)* 15, 15, 40.
2. Three figures are given for a triangle ABC and its heights AH, BJ, and CK, with measures for the sides and the heights, and the participants are asked to say if the measures are possible or not, and why. Figure 1 shows one of them.

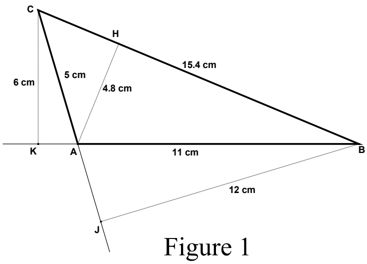

Figure 1

3. A discussion is given between three girls who must calculate the area of a triangle ABC (see Figure 2), after which the girls are supposed to ask the participant for his/her help, and the participant is asked to write down what s/he would answer to them. The discussion is:

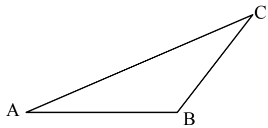

Figure 2

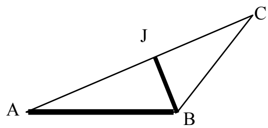

Figure 3

Emma:	The base has to be AC, because it is the largest side.
Fernanda:	No, the base is AB, because it is lying down.
Emma:	No, because besides that, you *can* draw the height from B to AC.
Fernanda:	No, the base is AB, also because that's how the triangle's name starts. The base is always AB.
Gaby:	You are both partially right. Fernanda is right in that the base must be AB because it is lying down, and the height is as Emma says, the one that we can draw, which is going out from B, like BJ (see Figure 3).

Note that in item 3, the conversation is planned as a catalogue of several prototypical ideas and misconceptions. In particular, the stress on the horizontality of the base can be linked both to Larios's (2007) construct of geometrical rigidity (the inability to mentally visualize a geometrical figure that is not in a standard position or to imagine what happens when it is moved or changed), and to the result that as much as 91% of children aged 11-12 draw a triangle with a horizontal base (Cutugno & Spagnolo, 2002). Also, Gaby's reasoning is a Type 1 judgement (Hershkowitz, 1990).

The teachers' answers to the evaluation were analysed and classified according to their correctness (for items 1 and 2) and the kind of geometrical criteria used. Some misconceptions were identified, which will be presented in the following section.

TEACHERS' ANSWERS. EXAMPLES AND ANALYSIS

Correct ideas

The correct answers to item 1 were all based on a correct use of the triangle inequality. These are some examples of reasons given for the impossibility of *a)* and *d)*. (The coding for transcripts is MR or MC for the workshop's venue, a 2-digit number for each participant, and the item number after a dash).

MR03-1: The sum of two sides must be larger than the other one.

MR10-1: [The sides] don't close.

Some of the correct answers to item 2 were based either on the comparison of the measures of perpendicular vs. non perpendicular segments or on the characteristics of a right triangle. These are some examples (they all relate to Figure 1):

MR24-2: If the line that is perpendicular to JC measures 12, AB should be larger.

MR30-2: The height must be smaller than side AC.

MC10-2: The hypotenuse must be larger than any of the sides, and this is not true for CK and BJ.

MR12-2: […] The area is different if you calculate it considering AC or AB. [He might have calculated 5×12/2=30 vs. 11×6/2=33].

As for item 3, the answers were not classified according to correctness, but many teachers expressed correct ideas. Here are some examples:

MR04-3: I would explain that each side of the triangle can be the base, and that each side that you choose as a base will have a height.

MR09-3: If we consider AB as the base, the height would be the opposite vertex "C" [*sic*]; for this we must extend AB and trace a perpendicular from C in order to know the height of the triangle. If we […] take AC as base, the opposite vertex would be B and the height would be BJ.

Misconceptions about the triangle inequality

The triangle inequality was needed to correctly solve item 1, and could not be used in item 2 (because some measurements were not provided, as AK or CH in Figure 1). Perhaps the most alarming misconception about this property was just a plain ignorance about it (9 out of 67 participants):

MC06-1: All [four triangles] are possible.

Some teachers applied the inequality but used it the other way around (as if the property stated that "the sum of two sides must be *smaller* than the other one"):

MR12-1 *b* and *c* are not possible because the measure of the third side (the largest) is not larger than the sum of the two shorter ones.

Some other misconceptions about the triangle inequality seem to arise from a vague knowledge of the property and probably the prototypical image that usually goes with its explanation, such as Figure 4; but it stays at a qualitative level and the actual numbers are not used to verify the property.

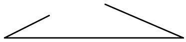

Figure 4

MC25-1: A triangle cannot be built with the measures in *a*, *b*, and *d*, because one side is shorter than the other one.

MR07-1: *b* is not possible because of the measure of its sides, since one measure is too long to build the triangle.

The last of these examples can be linked to a result found by Cutugno & Spagnolo (2002), in the sense that 48% of the children do not recognize triangles because "they have a too long side".

Misconceptions about the triangle's base(s) and/or altitude(s): prototypes

Several teachers incur in the prototypical judgements put in the girls' dialogue of item 3. For instance some think that each triangle has only one base, which must be horizontal; see the following examples:

MC03-3: Fernanda is right; the base is AB […].

MR21-3: The base is the straight [meaning horizontal?] line that is on a plane.

Other teachers do think that each triangle has three bases, but that the triangle must be rotated, and only when a side is horizontal can it be called a base. Although this is not exactly an error, it does show some degree of incomprehension about the nature of a triangle's base. This is a good example of geometrical rigidity.

MR14-3: To recognize the base: The base can be any side; it depends on the perspective […]. We could cut the triangle out in paper and rotate it; then we can name as a base each of its sides in turn [she draws Figure 5].

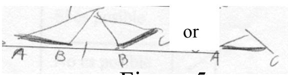

Figure 5

As for heights, there were several misconceptions. Some participants think that a triangle has only one height (and at that, preferably a vertical one):

MR01-2: [In Figure 1] the height is CK.

Some teachers think that it is not possible for a height to be "outside" the triangle, as in the following examples, which give reasons for the impossibility of Figure 1:

MR35-2: The segments of the height are not on the correct side, particularly BJ.

MR16-2: CK, AH, and BJ cannot be joined.

The previous instances show prototypical examples and geometrical rigidity. In their study, Cutugno & Spagnolo (2002) found that 39% of the children marked a vertical line for an altitude, and that 56% of the children marked the height inside the triangle.

Some think that an altitude is perpendicular to the base through one of its end points:

MC15-3: [The base is] AB […] and the height could start from B. [She draws Figure 6].

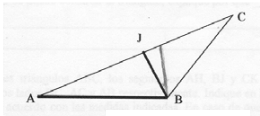

Figure 6

This teacher is making a Type 1 prototypical judgement similar to that quoted in the Framework: the drawing of an internal segment different from the altitude. Yet, she uses the property of perpendicularity, and thus she is also making a Type 2 judgement. She remembers that height has something to do with perpendicularity, but her concept image imposes an internal *and* vertical height.

Some participants clearly confused the height with the median:

MR25-3: I would say that the answer is incorrect because the height goes from the midpoint of the base, and here the height is traced from the vertex of the base to the midpoint of a segment of the triangle.

MR36-2: [Figure 1] is not possible because the height cannot be measured from a tilted side; you can from the vertex to the midpoint, but this one is not.

The confusion between height and median is a plain misconception that does not stem directly from a prototypical example, but is reinforced by two prototypical ideas: the one about internal heights, and the prototypical equilateral triangle used to teach the special segments of the triangle (in turn, these two reinforce each other, since in an equilateral triangle heights and medians coincide and are internal). Children also make these mistakes: In Cutugno & Spagnolo's (2002) study, 23% marked as height a line that divides in two parts the base of the triangle.

Evidently, some teachers incur simultaneously in several of the misconceptions mentioned above. See for instance the next case, where these prototypical ideas may be found: 1) the base is only one; 2) it is horizontal; 3) the rest of the figure is above it; 4) the height is internal. She also has the misconception of the height as a median.

MC04-3: Indeed, the base is AB because the figure is over it, and the height is JB because it is at the midpoint.

An interesting case of a Type 2 prototypical judgement is that of teacher MR26:

MR26-3: The base is what supports the triangle. The height goes from one vertex to the midpoint of one of the sides of the triangle, and that side is the base [She draws Figure 7].

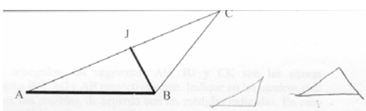

Figure 7

MR26 states two prototypical ideas: the base is horizontal, and the height is internal, which implies that it must be JB. Now she faces a conflict: If JB is the height, then the base must be AC and not AB. Apparently the only way to solve this contradiction is rotating the triangle: now the height is internal, AC is the base, and it is horizontal.

Misconceptions about the Pythagorean Theorem

Seven of the participants tried to use the Pythagorean Theorem in the solution of item 1 (where it can only be applied in case *c* because it is the only right-angled triangle, although no participants recognised it) or of item 2 (where it cannot be applied, even though there are many right-angled triangles, because the three measures were not provided). In some instances, the theorem was mistaken for the triangle inequality:

> MR06-1: *a*, *b* and *d* are not possible because the sum of the two smaller sides must be larger than the third, to comply with the Pythagorean Theorem.

In other cases, teachers tried to apply the theorem with non-right-angled triangles:

> MR06-2: [Figure 1] is not possible, because it does not comply with the Pythagorean Theorem. [She compares 25+121=146 with 15.4^2=237.16].

Some teachers undertook the addition of the squares of the hypotenuse and one leg, instead of the squares of both legs. In the next example this error could be put down to the prototypical image of a right-angled triangle with one horizontal leg:

> MR17-2: [Figure 1] is impossible, because 4.8×4.8=23.04 and 121+23.04=144.04.

Other geometrical misconceptions

Several teachers used a right-angled triangle terminology for a triangle that is not right-angled, as in the following example:

> MR15-2: [Figure 1] is not possible, because the hypotenuse BC is larger than the sum of the legs.

Another mistake made by several participants seems to stem from the idea that given a line and an external point, the segments joining the point with whatever point in the line all measure the same, or, alternatively, that the only valid triangles are isosceles:

> MC20-1: *d* is the only possible triangle, because it has two sides of the same size and one unequal side.

Some teachers decided to solve the area problem of item 3 in two parts, separately calculating the areas of the triangles ABJ and BCJ (see Figure 3). However, while doing this they all took for granted that BJ is the height corresponding to base AB (like the girl Gaby and like MC04-3 quoted in the previous page).

> MR31-3: They should calculate the area according to the formula A= (b×h)/2. Why over two? Because to be able to calculate [the area] we split the triangle in two parts A= (AB)(BJ)/2.

Finally, in item 3 oftentimes the explanations given by teachers to the girls revealed serious misconceptions. See the next example, as well as MR31-3 just quoted.

> MC15-3: In the first place, the base must indeed be AB, because that is how you start to calculate the area of a triangle […].

CONCLUSIONS

The expressions of teachers that we have quoted reveal both figural and conceptual misconceptions. Some figural misconceptions are those about the triangle inequality that involve a prototypical image (Figure 4), the idea that the base is necessarily horizontal (with the rest of the figure "above" it) and the height necessarily vertical and/or drawn from "the highest point", the idea that triangles must necessarily be isosceles, and the conception about altitudes needing to be internal. Some conceptual misconceptions are the ignorance or misuse of the triangle inequality, the idea that each triangle has only one base and one height, confusing the height with the median, the use of right-angled triangles terminology with non-right-angled ones, all the misconceptions about the Pythagorean Theorem and its applications, and the errors with the formula for the triangle's area. Evidently, both kinds of misconceptions can interact, as in MC15-3 (Figure 6), or when the formula for the area imposes figural misconceptions, or even when a figural misconception imposes errors on potentially correct concepts (such as when teachers need to rotate the triangle).

However, among these two kinds of misconceptions the former may be considered more critical, because when a prototypical image is present any conceptual aspect is overridden by it. As Larios (2007) states this (writing about high-school students), they use the figural aspect not as a heuristical resource but as a referential one.

With a slightly different perspective, Hershkowitz (1990) analysed the misconceptions' permanence. She identified several classes of misconceptions present both among young students and pre- and in-service teachers: (a) those that last from one grade to the next; (b) those that disappear with the acquisition of the concept; and (c) those that increase as the students advance throughout their schooling. Blanco & Barrantes' (2003) coincide with her: they assert that pre-service teachers repeat the misconceptions acquired during their schooling, and these misconceptions become implicit, stable and resistant to changes across their studies.

With regard to their knowledge about triangles, most of the teachers who participated in the two TAMBA workshops here reported behave like young students described in the literature. Their misconceptions are similar to those reported by Cutugno and Spagnolo's (2002) in their study with young children, and they range from Hershkowitz's classes (a) to (c) (such as those about the Pythagorean Theorem and the incorrect use of right-angled triangles terminology). Teachers also exhibit many prototypical judgments of Type 1 and very little analysis of the properties involved, and they use the figural aspect as a referential resource. The analysis of the relationship between the categories of misconceptions and some of the teachers' characteristics such as their gender, their teaching experience, and the educational level and the city where they work, is reported elsewhere (Alatorre & Sáiz, 2009); however, it can be said here that although the 67 participants are in no way a statistical sample, our experience in working with teachers leads us to consider them as a fairly representative subset of the Mexican schoolteachers.

In the last years there has been much discussion about what level of Mathematical Content Knowledge the teachers should have. Da Ponte & Chapman (2006) suggest that in order to conserve its meaningfulness, the research about teachers' mathematics knowledge should link it with other aspects of practice, more related to the Pedagogical Content Knowledge. Our TAMBA workshops are directed to both kinds of knowledge, as are also some elements of the diagnostic evaluation (such as item 3), but with a stress in the mathematical aspects. We claim that the mathematical knowledge is a *sine qua non***,** and that the only condition under which it can indeed come to a second level of priority is if teachers have a thorough understanding of, at least, the contents they must teach. Most of the teachers who participated in the two workshops here reported are not in this case; when the teacher training curricula ignores the existence of misconceptions, as is the case in Mexico, Blanco & Barrantes's (2003) assertion about the permanence of misconceptions is also valid for in-service teachers. When teachers have serious misconceptions, they not only perpetuate but also aggravate the ones of their students, and even more so when teachers incur in so-called explanations such as the last two quoted in this paper.

Therefore we sustain that the mathematical content knowledge of the elementary schoolteachers is something that urgently requires consideration, diagnosis, and attention, both in initial training and in professional development.

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THE ROLE OF DEFINITIONS IN EXAMPLE CLASSIFICATION

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This paper reports an empirical study of students' classification of sequences before and after meeting explicit definitions of 'increasing' and 'decreasing'. In doing so, it explores 1) students' interpretations of the definitions and 2) the appropriateness of this apparently straightforward context for teaching students about the status of mathematical definitions. In particular, it demonstrates that students' spontaneous conceptions in this context can be inconsistent with definitions, and it explores the extent to which exposure to formal definitions influences these conceptions. The results show an interesting pattern of modified classifications, which demonstrates increased consistency with the definitions but shows problems with some pivotal examples.

INTRODUCTION AND THEORETICAL BACKGROUND

Undergraduate students are often unaware of the status of definitions in mathematical theory. They may be unable to state important definitions, even after a substantial period of study, and many appear to reason about mathematical concepts using concept images instead of definitions (Tall & Vinner, 1981). This can be particularly problematic in Analysis, in which spontaneous conceptions, based on everyday use of terms or informal experience with concepts such as *limit*, can be at odds with the formal definitions (Williams, 1991; Cornu, 1991).

For success in undergraduate pure mathematics, it is vital to learn to use definitions correctly in making classifications and in constructing general proofs. It is therefore important for mathematics educators to study ways to help students achieve this, and this paper approaches the classification issue by analysing students' responses to a task that required them to classify examples spontaneously and then using (previously unseen) definitions.

To design the task, we first identified a context in which there is likely to be disparity between spontaneous conceptions (Cornu, 1991) and the extension of the formal definition. In Analysis, the obvious place to start is with the limit concept, since this is central in the subject and much work has been done in establishing common misconceptions (Williams, 2001). However, limit definitions are logically complex (involving three nested quantifiers) so any investigation of their use in classification is likely be confounded by difficulties in understanding their logical structure (see, for example, Dubinsky, Elterman & Gong, 1988). Thus, the research reported here used the concepts of *increasing* and *decreasing* for infinite sequences of real numbers. The definitions for these concepts are logically simple (they involve only

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one quantifier) and some classifications based on them are counterintuitive: for example, constant sequences are classified as both increasing and decreasing and sequences such as 0,1,0,1,0,1,0,1,… is classified as neither increasing nor decreasing. Specifically, the research addressed the following questions:

RQ1: To what extent are students' spontaneous conceptions about increasing and decreasing sequences inconsistent with definition-based judgments?

RQ2: When given a basic introduction to the definitions, can students work with these and correctly revise their judgments?

If there are sufficient inconsistent spontaneous conceptions and evidence that exposure to the definitions led to revisions, this simple context would arguably be appropriate for raising students' awareness of the way mathematical definitions are used to resolve ambiguity or disagreement by precisely specifying a concept.

In theoretical terms one might say that this research sets out to investigate the participants' *example spaces* and their ability to modify the structure of these to better mirror the *conventional example space* (Watson & Mason, 2005). Previous research has tended to use example *generation* tasks: Zazkis & Leikin (2007), for instance, considered what such tasks can reveal about the accessibility, richness and generality of individual's example spaces. However, in this case example generation was considered unlikely to lead to interesting results because new undergraduates' experience with sequences is likely to be limited to work with arithmetic and geometric sequences. Since we wished to gain insight into students' responses in counterintuitive as well as 'obvious' cases, we used an example *classification* task with the deliberate inclusion of examples such as those above. These examples were expected to be *pivotal* for at least some of the participants, in the sense that they might cause students to experience uncertainty and/or to recognise and question initial assumptions (Zaskis & Chernoff, 2008).

METHOD

187 students completed the task as part of a regularly timetabled Analysis lecture (within a standard lecture course not given by the researcher). All participants were in the first term of their first year of a mathematics degree at a high-ranking UK university. The entry requirements for the degree included grade A for both mathematics and further mathematics A-levels (or equivalent), effectively the highest possible pre-university mathematics requirement in the UK.

The task presented here was part of an intervention lasting approximately 25 minutes. The students were informed that the task would help the researcher understand their thinking, that they should work alone, that the responses would be treated as anonymous and that their lecturer would be given a summary but that the tasks would not influence their grade. They were also told that they could opt out by choosing not to hand in their paper. In the *spontaneous classification* phase, the students were asked to fill in a table to indicate whether they would classify each of the sequences in Table 1 as increasing, decreasing, both or neither, or whether they were not sure.

A: $0,1,0,1,0,1,0,1,\ldots$	F: $1,3,2,4,3,5,4,6,\ldots$
B: $1,4,9,16,25,36,49,64,\ldots$	G: $6,6,7,7,8,8,9,9,\ldots$
C: $1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8},\mathrm{K}$	H: $0,1,0,2,0,3,0,4,\ldots$
D: $1,-1,2,-2,3,-3,4,-4,\mathrm{K}$	J: $10\frac{1}{2},10\frac{3}{4},10\frac{7}{8},10\frac{15}{16},10\frac{31}{32},\mathrm{K}$
E: $3,3,3,3,3,3,3,3,\ldots$	K: $-2,-4,-6,-8,-10,\mathrm{K}$

Table 1: Classification task examples

The students were then shown definitions of 'increasing' and of 'decreasing', stated in notation consistent with that used in their course:

A sequence $\{x_n\}_{n=1}^{\infty}$ is <u>increasing</u> if and only if $\forall n \in \mathbf{N}$, $x_{n+1} \geq x_n$.

A sequence $\{x_n\}_{n=1}^{\infty}$ is <u>decreasing</u> if and only if $\forall n \in \mathbf{N}$, $x_{n+1} \leq x_n$.

These definitions were accompanied by a brief verbal explanation. In the following *definition-based classification* phase, the students were asked to fill in another table to show, *according to the definitions*, whether each of the sequences was increasing, decreasing, both or neither (without a 'not sure' option).

RESULTS

Spontaneous classifications

Table 2 shows the responses to the spontaneous classification task. The shaded cell in each row indicates the response consistent with the definitions (of course, at this stage it makes no sense to consider any responses 'incorrect').

	Inc	Dec	Both	Neither	Not Sure
A			86	93	4
B	187				
C		185	2		
D	1	4	154	23	5
E			0	187	
F	83		84	12	8
G	162	1		12	11
H	60		86	21	20
J	178	5	2	1	1
K	2	182	1	1	1

Table 2: Spontaneous classifications

This shows that, in many cases, spontaneous conceptions were not consistent with the definitions. Although it was the case that for sequences that are either increasing or decreasing but not both (B, C, G, J and K) a substantial majority gave a response consistent with the definitions, in all other cases, substantial numbers (often the vast majority) gave a response inconsistent with the definitions. In particular:

- Every participant classified the constant sequence as *neither*.
- Approximately half classified the sequence A: 0,1,0,1,0,1,0,1,… as *neither* (consistent with the definitions) and half as *both*.
- Only 23 participants classified the sequence D: $1,-1,2,-2,3,-3,4,-4,\ldots$ as *neither* (consistent), with a substantial majority classifying it as *both*.
- There was considerable difference of opinion regarding F: 1,3,2,4,3,5,4,6,… and H: 0,1,0,2,0,3,0,4,…. Only 12 and 21 respectively gave the response *neither* (consistent), with many more in each case selecting *increasing* or *both*.

It is worth noting that few students made use of the *not sure* option. It had been anticipated that more would do so, but with hindsight it seems reasonable that students who have not done much study of definitions and counterexamples would be comfortable making classifications in the absence of precise criteria.

Definition-based classifications

Table 3 shows the responses to the definition-based classification task. Again, responses consistent with the definitions are indicated by a shaded cell.

	Inc	Dec	Both	Neither
A		1	54	132
B	187			
C	1	186		
D		1	65	121
E	6		86	94
F	9		57	121
G	143		7	37
H	2		57	127
J	181	4	2	0
K		186	0	1

Table 3: Definition-based classifications

This shows that there were noticeable changes towards responses consistent with the definitions. Again, for the sequences that are either increasing or decreasing but not both, a substantial majority gave a response consistent with the definitions. Exposure to the definitions also meant that in all other cases but one, a majority responded consistently with these. However, these majorities were not overwhelming so it is not reasonable to say that the answer to RQ2 is an unqualified *yes*. In particular:

- The number classifying G:6,6,7,7,8,8,9,9,… as *increasing* (consistent) dropped from 162 to 143, with 37 participants now classifying this as *neither*. This could be due to misinterpreting the inequality to mean 'strictly less than' or perhaps becoming confused by the universal quantifier and believing the order relationship has to be the same for each pair of consecutive terms.
- Just under half (86) classified the constant sequence E as *both* (consistent). Almost exactly half (93) once again classified it as *neither*. Interestingly, 6 now classified it as *increasing*. It is possible that the latter found that it did satisfy the definition of increasing and assumed that this precluded being decreasing.
- For each of the four sequences A, D, F and H, approximately two thirds of the participants gave the classification *neither* (consistent), with approximately one third giving the classification *both*. The *both* response is in line with the most common spontaneous classifications, so could occur when participants simply do not change their minds. It is also in line with a misinterpretation of the universal quantifier in the definitions, for example classifying 0,1,0,1,0,1,0,1,… as *both* because it is true that for all *n*, either $x_{n+1} \geq x_n$ or $x_{n+1} \leq x_n$.

Individual responses

Analysis across all the participants shows that exposure to the definitions in this context did lead to a marked (but far from complete) move towards classifications consistent with the definitions. For each of the apparently counterintuitive cases (except for the constant sequence), about two thirds of the definition-based classifications were correct. Further examination of the individual participants' responses allows us to examine the question of whether this means that about two thirds of the participants 'got the idea' and gave entirely correct classifications. It also allows us to discern some internally consistent interpretations of the definitions that might indicate key misunderstandings. Table 4 summarises all the profiles of the definition-based classifications. These profiles account for over 80% of the participants and all the distinct profiles associated with four or more participants.

Response Profile	*n*	*Profile description*
N I D N B N I N I D	54	Correct
N I D N N N I N I D	30	Correct except *constant* classified as *neither*
N I D N N N N N I D	17	*Neither* applied to all *both* and all *neither*
B I D B N B I B I D	16	*Both* and *neither* switched
B I D B N B N B I D	9	*Both* and *neither* switched; 'steps' classified as *neither*
B I D B B B I B I D	12	*Both* applied to all *both* and all *neither*
B I D B B B B B I D	4	*Both* applied to all *both* and all *neither* and 'steps'

Table 4: Common definition-based response profiles

This data tells us that it is not the case that two thirds of the participants fully "got the idea". In fact, only 54 (29%) gave a correct set of classifications, although a further 30 (16%) were correct for all except the constant sequence and 17 (9%) gave the response *neither* for all both and all neither sequences (correct for all except the constant and 'steps' [G: 6,6,7,7,8,8,9,9,…] sequences). In addition, it tells us that 41 (22%) either switched around *both* and *neither* or applied *both* to the majority of the sequences. These latter profiles could indicate interpretations in which the universal quantifier is misunderstood or ignored, or could indicate that participants were thinking of sequences such as D: $1,-1,2,-2,3,-3,4,-4,K$ as two different sequences ($1,2,3,4,\ldots$ as increasing and $-1, -2, -3, -4,\ldots$ as decreasing). This thinking would be consistent with findings of Tall & Vinner (1981).

Because a relatively small number of participants gave fully correct definition-based classifications, we also examine the profiles for the spontaneous classifications. This allows us to investigate whether the students who gave correct definition-based classifications were already mostly correct in their spontaneous classifications, or whether some did reach a correct profile by making a substantial change in their interpretation. This is more difficult to do, because (unsurprisingly) there was more variation among the spontaneous classifications. Indeed, there were only two profiles given by more than eight students; we discuss each of these here.

42 participants (22%) spontaneously gave the 'both and neither switched' profile, suggesting that this is the most natural interpretation of combinations of the concepts *increasing* and *decreasing* for sequences. It is also internally consistent: these students apparently considered a sequence to be *both* if some terms are less than their predecessors and some are greater, and *neither* if all terms are the same. Of these 42 participants, for the definition-based classification:

- 12 changed to CORRECT.
- 9 changed to correct except constant classified as neither.
- 9 remained with both and neither switched.

The second prevalent profile, given by 28 participants (15%), also shows some internal consistency, although in a way that might not be recognised as such by a mathematician accustomed to precisely formulated property specifications. These participants classified sequences A:$0,1,0,1,0,1,0,1,\ldots$ and D:$1,-1,2,-2,3,-3,4,-4,K$ as *neither*, which is consistent with the definitions. They classified F:$1,3,2,4,3,5,4,6,\ldots$, G:$6,6,7,7,8,8,9,9,\ldots$ and H:$0,1,0,2,0,3,0,4,\ldots$ as *increasing*, perhaps indicating that the presence of a general 'upward trend' was enough to gain this classification, and without apparently experiencing the fact that H has infinitely many zero terms as problematic. Of these 28 participants:

- 12 changed to CORRECT.
- 6 changed to neither applied to all both and all neither.

This analysis of common spontaneous classifications gives some indication that there is not a straightforward relationship between those who gave correct definition-based interpretations and 'almost correct' initial responses. In particular, a sizeable proportion of those who moved to correct classifications did so from substantially different spontaneous classifications. We continue to analyse the data for patterns in the responses to the ten separate sequences.

PEDAGOGICAL IMPLICATIONS

Exposure to a broad range of examples may be appropriate in and of itself, since research indicates that at least some students do not spontaneously generate examples in response to definitions (Dahlberg & Housman, 1997), that one reason for students' difficulties with proof is that they do not have well-developed example spaces (Moore, 1994) and that at least some successful mathematicians use examples extensively to support their reasoning (Alcock & Inglis, 2008). A task such as that used here provides exposure to a deliberately broad range of examples, with the specific aim of including some for which there is likely to be conflict between spontaneous and correct definition-based classifications.

With this in mind, a lecturer might use the outcomes of this study in several ways. First, they might simply give extra attention to the issue of definition-based classification, since even simple definitions may not be applied reliably by students who are unaccustomed to this type of reasoning, and even an apparent move toward correct definition use might mask underlying misconceptions that are resistant to change. Second, this task could be used as an introduction to these concepts, with subsequent discussion focused on the common misinterpretations. Third, one could run a similar intervention in which students were allowed to confer with each other at some stage. In this case, examples A: 0,1,0,1,0,1,0,1,…, F: 1,3,2,4,3,5,4,6,… and H: 0,1,0,2,0,3,0,4,… have particular potential as pivotal examples, since they seem to be those for which there is considerable variation in initial classifications so that there would likely be disagreements to be resolved among the class.

Such suggestions depend, of course, upon the generalisability of the results presented here. It might be thought that because these students attend a high-ranking institution, they would better at all types of mathematical tasks than is typical. However, precisely because they are considered successful and capable, these students study Analysis in the first term of their first year at university. At many other institutions students do not study it until the second term or second year, and thus come to it with more experience of learning university mathematics, and more experience of working with definitions in subjects such as elementary set theory, linear algebra, etc. Overall, we do not see a strong reason to believe that the responses would be substantially different for those studying the same material at other institutions, though obviously more research is needed to establish this.

RESEARCH IMPLICATIONS

The main question of interest here is whether one or more interventions like this can have a positive impact on students' engagement with definitions in general and on their eventual attainment. That is, does repeated exposure to challenges to one's spontaneous conceptions, and subsequent work with definitions, lead to an underlying cognitive shift in the process of making classification judgements and possibly in the use of definitions in mathematics more broadly? Further study would be necessary to establish whether the changes observed in this study have a long-term effect, even in this restricted context. This could be explored in two senses: whether just this one exposure would have a lasting impact over a longer time frame or whether further exposure to similar tasks in the same context would have increased effect, even for those classifications that appear counterintuitive and resistant to change.

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IMPROVING ELEMENTARY STUDENT TEACHERS' KNOWLEDGE OF MULTIPLICATION OF RATIONAL NUMBERS

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The research results presented in this paper are only a small part of an action research performed with the main aim of improving primary school student teachers' understanding of mathematics. The re-teaching of mathematics was integrated with the teaching of pedagogy by asking student teachers (STs) to perform children's activities which have the potential to develop relational understanding of the subject. This paper presents some results concerning: (a) STs' knowledge of multiplication of fractions, (b) STs' difficulties in relearning multiplication of fractions and decimals, and some practical solutions proposed to ameliorate STs' learning difficulties.

SOME RELATED LITERATURE

According to Skemp (1976), relational understanding involves knowing both what to do and why it works, while instrumental understanding involves knowing only what to do, the rule, but not the reason why the rule works. My initial experiences as a novice mathematics school teacher and later as a teacher educator led me to think that both mathematics STs and primary school STs do not have enough relational understanding of the mathematics they are supposed to teach. The first student's question which made me aware that I did not have the kind of knowledge necessary for teaching was about multiplication of fractions. After three months I started teaching, an 11 year old was puzzled by the result of 1/2 x 1/3 = 1/6 and asked "1/6 is smaller than 1/2 and 1/3. Why do we get a smaller result number when multiplying fractions?". I could present my students with correct procedures, but could not answer most of their questions concerning the reasons for using certain steps in the procedures. These experiences led me to undertake an action research with the main aim of investigating ways of helping primary school STs to improve their understanding of mathematics in pre-service teacher education.

Studies of primary school STs' knowledge of rational numbers tend to show that it comprises mainly memorising a large repertoire of rules and algorithms with not much understanding of the underlying mathematical concepts and relationships (e.g., Graeber et al., 1989, Stoddart et al. 1993, Philippou and Christou, 1994, and Luo, Lo and Leu, 2008). Primary school STs could perform correct calculations with fractions, but many important connections seemed to be missing. This result was even more evident with multiplication and division of fractions. With respect to multiplication of decimals, Graeber et al. (1989) found that the misconception "multiplication always makes bigger" was held by some STs. They also had difficulty in selecting the appropriate operation to solve multiplication word problems

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involving decimal numbers smaller than one. Luo, Lo and Leu (2008) found that the hardest item to both U.S. and Taiwanese primary school STs was about multiplication of fractions.

Although multiplication of rational numbers is not used very often in everyday contexts, it is an operation that has important relationships with topics such as equivalence of fractions and proportion. Simon and Blume (1994) found that STs have difficulty in recognising ratio relationships. Similar to school students, they tend to select additive strategies when multiplicative strategies are appropriate. Some of the difficulties faced by the STs seemed to be connected to their weak pre-requisite knowledge about multiplicative structures (i.e., if the ratio is 3:2, the height is 1½ or 1.5 times the length of the base). In order to help future school students overcome the idea that "multiplication makes bigger", STs need to acquire enough knowledge to be able to teach operations with rational numbers earlier than is it is often recommended by curriculum developers, with a focus on practical work and games (Amato, 2008).

In the literature about teacher education, there are other results about STs' knowledge of multiplication of rational numbers, usually followed by a set of recommendations. However, I could only find two studies about teacher educators' efforts to improve STs' knowledge of the topic. These studies are reported in Stoddart et al. (1993) and Rule and Hallagan (2006). They both used multiple representations to improve STs' relational understanding of multiplication. This paper presents some results concerning: (a) STs' knowledge of multiplication of fractions, (b) STs' difficulties in relearning multiplication of fractions and decimals, and some practical solutions proposed to ameliorate STs' learning difficulties within the time available.

METHODOLOGY

An action research was carried out as part of a mathematics teaching course component (MTCC) in pre-service teacher education (Amato, 2004). The component lasts one semester (80 hours), is the only compulsory component related to mathematics offered to primary school STs at the University of Brasília. There were two main action steps and each lasted one semester. So each action step took place with a different cohort of STs. The third and subsequent action steps were less formal in nature, and involved less data collection. The main research question of the study was: "In what ways can primary school STs be helped to improve their relational understanding of the mathematical content they will be expected to teach?".

A new teaching programme was designed with the aim of improving STs' relational understanding of the content they would be expected to teach in the future. In the action steps of the research, the re-teaching of mathematics was integrated with the teaching of pedagogical content knowledge by asking the STs to perform children's activities which have the potential to develop relational understanding. These activities had four more specific aims: (a) promote STs' familiarity with multiple modes of representation for most concepts and operations in the primary school curriculum; (b) expose STs to several ways of performing operations with manipulatives; (c) help STs to construct relationships among concepts and operations through the use of versatile representations; and (d) facilitate STs' transition from manipulatives to symbolic mathematics. About 90% of the new teaching program became children's activities which are based on the notion that a deep understanding of mathematics can be achieved by involving learners in "activities that embed the mathematical ideas to be learned in five different modes of representation with an emphasis on translations within and between modes" (Cramer, 2003, p. 462). These modes will be referred as: (a) contexts (real-world situations), (b) manipulatives, (c) diagrams (part-whole diagrams), (d) verbal (spoken languages) and (e) symbols (written symbols). The main activities for multiplication in the actual programme are:

(1) Whole-class discussion about how to calculate ½ x ¼.

(2) Translating from symbols to manipulatives (paper strips already divided into parts with vertical lines). STs compare the sizes of ½, ¼, and the product 1/8. I describe my own initial difficulties and frustration in teaching multiplication of fractions.

(3) [Included in the second and subsequent semesters] Translating from context and verbal to manipulatives. STs cut two A4 sheets of paper (two "cakes") into thirds and show: (a) 6 portions of 1 third or 6 times 1/3, (b) 5 portions of 1 third or 5 times, […], (f) 1 portion of 1 third or 1 time 1/3, and (g) half of a third or half time 1 third (1/2 x 1/3).

(4) Translating from context and verbal to manipulatives. STs fold and colour sheets of A4 paper (rectangular "cakes") to represent: (a) 1/2x1/4 (b) 1/4x1/2, (c) 3/4x1/2, (d) 1/4x2/3 and (e) 3/4x2/3. In the tenth and subsequent semesters four more sums were represented in this way: (f) 0.8x0.1, (g) 0.4x0.2, (h) 0.2x0.4, and (i) 0.6x0.9.

(5) Translating from diagrams to verbal. STs were shown pictures drawn in A4 cards (flash cards). Each card had one face showing a fraction of a square coloured in yellow. The other face had the same picture, but it also included a fraction of the yellow part coloured in green. The class verbalise the sum and the product.

(6) Translating from context/verbal to manipulatives (circular "pizzas" made with plasticine). STs cut the pizzas to represent: (a) 1/3x1/4, (b) 1/3x2/4, (c) 2/3x3/4, (d) 1/4x1/3, (e) 1/4x2/3, and (f) 3/4x2/3.

(7) [Included in the second and subsequent semesters] Translating from symbols to manipulatives (paper strips already divided into parts with vertical lines). STs colour and draw horizontal lines to represent: (a) 4/5x1/2, (b) 3/4x5/8, (c) 0.3x0.1, and (d) 0.7x0.8.

(8) Exercises involving translations from symbols to square diagrams. STs colour a fraction of a fraction with (i) vertical and (ii) horizontal cuts. In the third and subsequent semesters STs were also asked to do similar exercises for a decimal of a decimal.

(9) Exercises involving translations from square diagrams for fraction of a fraction to symbols. STs write the sum and the product. Similar exercises for a decimal of a decimal will be included next semester.

(10) [Included in the tenth and subsequent semesters] Translating from context and verbal to manipulatives (rectangle "walls" made with paper squares with area of 1 cm2). Only halves and quarters are used because of the small size of the squares. For whole-

class discussions similar rectangle “walls” made with paper squares with area of 1 dm2 are displayed on a board with the help of pins. Halves, thirds, quarters and fifths are used.

(11) [Included in the fourth and subsequent semesters] Exercises involving translations from area diagrams (rectangles divided into unit squares with area = 1 cm2) to symbols and vice versa.

Four data collection instruments were used to monitor the effects of the strategic actions: (a) researcher’s daily diary; (b) middle and end of semester interviews with STs; (c) beginning, middle and end of semester questionnaires; and (d) pre- and post-tests. The questionnaires and interviews focused on STs’ (i) perceptions about their own understanding of mathematics and their attitudes towards mathematics before and after experiencing the activities in the teaching programme, and (ii) evaluation of the activities within the teaching programme. The tests involved open-ended questions, so that relational understanding could be probed through the context of teaching children. Each page of the tests contained three questions. The same heading was used for all the pages in the tests: “Answer the following questions as if you were introducing the concepts involved to primary school children. Describe briefly what you would do and say in each situation. Whenever possible draw pictures to illustrate your ideas.” Question F5 was about multiplication of fractions: “How would you explain the reason for the result of $3/4 \times 1/2$ ($2/3 \times 1/4$ in the post-test)?”. Much information was produced by the data collection instruments but, because of the limitations of space, only some results concerning multiplication of rational numbers, will be reported here.

SOME RESULTS

(a) STs’ knowledge of multiplication - Although the results presented in this section are mainly concerned with the second semester question about multiplication of fractions, the general patterns which emerged from the tests were similar in both the first and the second semesters. The STs who presented some relational understanding in the pre-tests were often those who had previously done a vocational teacher education course at school level and were already qualified as primary school teachers. They were seeking a second qualification at university level. The pre-test median mark was 10% and the post-test median mark was 70% in the second semester. The difference in the two medians indicates a considerable improvement in understanding, as judged by the tests. However, operations with rational numbers proved to be the most difficult topics to re-teach STs.

Only two STs showed a good relational understanding of multiplication in the pre-test. Both provided useful written explanations and diagrams. Seven STs wrote that they did not know how to explain the ideas behind multiplication of fractions. In the pre-test some drew part-whole diagrams, but were unable to relate the diagrams to the operation in any relational way. There were a number of unhelpful diagrams in both tests. Some STs represented: (a) only the factors (¾ and ½) with two separate part-whole diagrams (pre-test frequency = 3 STs and post-test frequency = 1 ST); (b) the factors and the product (3/8) in a third diagram (pre = 3 STs and post = 2 STs); and (c) represented only the product (pre = 2 STs and post = 2 STs). One ST expressed a misconception in her explanation: "I would say that when multiplying fractions we get a result which is even bigger than the addition of those two fractions". The pre- and post-tests responses of each ST were compared to investigate any changes in relational understanding which could be attributed to the teaching programme. An example of what I considered to be an improvement in relational understanding for multiplication of fractions is:

> ST203 (pre-test): I learned only as a rule.
>
> ST203 (post-test): [Drew a useful part-whole diagram] It is important the verbalisation in this case. Traditionally it would be said: 2 thirds times 1 quarter. What must be really said is: I wish 2/3 of 1/4. I got 1/4 and cut it into three parts. I was asked for 2/3. If 1/4 was transformed into 3 parts, the same it will happen to the other 3/4, one by one, which will give me a total of 12 parts (denominator). I take 2 of these parts.

The number of STs in the second semester was 44. More than half of the STs (n = 23) who finished the second semester (three gave up in the middle of the semester) did not show any changes in their understanding of multiplication of fractions. They continued to rely only on writing the steps in the algorithm. Perhaps the algorithm was too easy to need learning a new schema: ST216 (pre-test) "Multiplication [of fractions] is the easiest operation. It is only normal multiplication". Eighteen STs provided both useful diagrams and good verbal explanations in the post-test. Two of them had shown a good understanding in the pre-test. Four of them provided a correct result, a helpful diagram, and a good verbal explanation, but they did not answer the pre-test and so no comparisons could be made with the post-test. So, only 12 STs were considered to have had great improvements in their relational understanding from the pre- to the post-test.

These results did not fit well with the results of the post-questionnaire about understanding, where 81.6% of the STs said that the programme had improved their understanding of multiplication of fractions. Addition and subtraction of natural numbers were the topics which some STs reported not gaining much from the programme, probably because they thought they already knew enough about those topics before the MTCC. In another question in the post-questionnaire about understanding, some STs said they still needed to improve their knowledge of certain topics. As in the reconnaissance interviews with practising teachers (Amato, 2008), the STs' remarks were mainly about rational numbers, especially multiplication and division of rational numbers.

(b) STs' difficulties in relearning multiplication - Activity (4) (see list of activities in the methodology) involved folding and colouring sheets of A4 paper. The class was asked to represent ½ x ¼ by: (a) imagining that the rectangle cake was baked for lunch, (b) colouring in yellow 1/4 of the rectangle and imagining that the yellow part was the amount of cake which was left from lunch, (c) colouring in blue 1/2 of the yellow part and imagining that the part which became green was the amount of cake eaten later as an afternoon snack, and (d) finding out which portion of the whole cake was eaten as a snack (1/8). Most STs did not have many problems in representing: (i) 1/2 x 1/4, (ii) 1/4 x 1/2 and (iii) 3/4 x 1/2. However, difficulties appeared when they were asked to represent multiplications in which the second fraction had numerator greater than one such as: (iv) 1/4 x 2/3 and (v) 3/4 x 2/3. Some STs could not represent 1/4 of 2/3 with only vertical cuts. The line separating the 2 thirds (yellow part) seemed to prevent them from imagining 1/4 of the yellow part. I suggested they cover the line separating the two yellow thirds and imagine it did not exist (Figure 1).

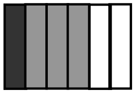
Figure 1

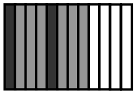
Figure 2


Figure 3

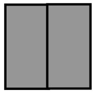
Figure 4

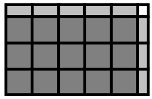
Figure 5

A ST suggested a second strategy which involves colouring 1/4 of each third (1/4 of 2/3 = 1/4 of 1/3 + 1/4 of 1/3 = 1/12 + 1/12) (Figure 2). Another ST suggested representing the quarters in a different direction (horizontal cuts) from the direction used to cut the thirds (vertical cuts) (Figure 3). When the class was asked to compare the three strategies, many STs agreed that the "vertical/horizontal cuts": (a) was easier, (b) provided the product obtained by the symbolic algorithm (2/12), and (c) the slices did not become too thin. I added that the vertical/horizontal cuts were similar to the lines in the area diagram used before for multiplication of natural numbers (Amato, 2005). I proposed STs represent 1/4 of 2/3 by hiding one of the thirds behind the two thirds (Figure 4). In this way their attention would be more focused on calculating a part of 2/3 and not on the initial whole "cake". Some STs also had problems in visualising the product. They thought that the result of 1/4 of 2/3 should be 2/8 and not 2/12 (Figure 3). They interpreted the yellow part (2/3) as the unit and did not refer to the original unit (the whole cake).

Multiplication of fractions was revised with activities involving circular "pizzas" made with plasticine (Activity 6). Again the cut separating the 2 quarters seemed to prevent some STs from representing 2/3 of 2/4. As only cuts through the centre are possible in order to get equal slices with circular units, the class agreed that vertical/horizontal cuts were not possible. Only the first two strategies would work. After the first semester I increased the number of activities for multiplication, especially to sums in which the second fraction has numerator greater than one (e.g., 1/4 x 2/3 and 3/4 of 2/3) and for decimals.

The paper strips used in activity (7) are already divided into parts with vertical lines. The only exception is the strip divided into 100 parts which has both vertical and horizontal lines (a 10 by 10 rectangle). ST241 jokingly commented after completing the activity: "There are too many pieces. My brain has melted. I have done much thinking". The STs' difficulties in colouring a decimal of a decimal were even greater than with a fraction of a fraction. Therefore, activity (8) was also extended to decimals in the third and subsequent semesters. Later, in the tenth semester, the same was done with activity (4). To save time in the case of decimals, I made copies of a sheet of paper with all the margins divided into ten equal parts. With large classes it proved to be difficult to have several activities involving different ways of representing the same operation. In activities 1 to 6, STs represent multiplication using cuts in the direction they wish. In the last activities (7 to 11) they are asked to draw (i) vertical and (ii) horizontal cuts. This convention made communication easier among the STs and helped them connect multiplication of fractions and decimals to the area representation used for multiplication of natural numbers (Amato, 2005).

While some STs only needed a few activities to understand rational numbers, others needed more activities. Some STs suggested increasing the teaching time for operations with rational numbers because they were much more difficult than operations with natural numbers (Amato, 2004). Besides, focusing more on operations with natural numbers during the first semester proved to be uninteresting for some STs. Apart from the difficulties in learning representations which were new to them, multiplication of fractions was one of the last topics in the first semester when STs missed more classes than usual in order to finish their end of semester assignments of other course components. Therefore, I decided to start the activities for operations with rational numbers earlier in the third and subsequent semesters.

With the short time available, the area representation for multiplication of fractions (e.g., Chinn and Ashcroft, 1993, p. 155) in activity (10) proved to be a very powerful and versatile representation. It provides STs with more experiences with multiplication and it helps them make important connections among: (a) multiplication and division of natural numbers, fractions, mixed numbers and decimals, (b) measurement of perimeter and area, and (c) addition of fractions. For example, through the "wall" in Figure 5, it is possible to visualise many related sums connected to area (i) [bottom left] 3x5, 15÷3, 15÷5, (ii) [top left] ½x5, 2½÷5, 2½÷½, (iii) [bottom right] 3x½, 1½÷3, 1½÷½, (iv) [top right] ½x½, ¼÷½, (v) [whole rectangle] 3½x5½ = 15+2½+1½+¼, 19¼÷3½, 19¼÷5½, and to perimeter (vi) 2x3+2x5, 2x½+2x5, 2x3+2x½, 4x½, 2x3½+2x5½.

SOME CONCLUSIONS

The strategic actions provided an appropriate solution to STs' instrumental understanding. They considered the children's practical and written activities to be important for their relearning (Amato, 2004). However, according to the practising teachers in the reconnaissance stage (Amato, 2008) and to the STs in the action steps, much more time in teacher education is needed to acquire the confidence and cognitive structure they need to teach rational numbers well. The data collected indicated that many STs improved their understanding of the topic, but operations with rational numbers proved to be one of the hardest topics in which to improve STs' understanding. Ideally, the activities for this topic should be spread over a longer period of time and more activities are needed. Therefore, another strategic action was to increase the teaching time dedicated to compulsory MTCCs, but such action involved institutional changes which proved to be very hard to achieve.

STs' difficulties were also related to "weaknesses in my teaching" and so my own pedagogical knowledge was an important social factor affecting their relearning of mathematics and pedagogy. Useful ideas for ameliorating unanticipated problems did not come to my mind immediately after observing these problems. The literature about teaching and learning mathematics does not always present solutions to very specific problems. Discovering weaknesses in my own teaching proved to be a slow process. As it may be noticed from the changes made in the programme, in some cases insight only came after much effort and thinking.

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HOW STUDENTS READ MATHEMATICAL REPRESENTATIONS: AN EYE TRACKING STUDY

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How do students, with different background knowledge in mathematics, read a mathematical text? How do they perform the transformations between the different representations (formulas, graphs and words) in order to grasp its meaning? We use eye tracking to highlight the ongoing process of making sense of mathematical representations during problem solving. Eye tracking is a method enabling a close examination of how attention is directed at a stimulus. Its use in educational settings is increasing because of its great potential in capturing various aspects of the learning processes. Our data indicate quantitative and qualitative differences between the novice and expert group. We discuss some implications for mathematics education in general, and the design of mathematics textbooks in particular.

THEORETICAL FRAME AND ISSUES

Mathematical objects are peculiar: they are not accessible by themselves but only through representations in suitable registers: "the only way to have access to them and deal with them is using signs and semiotic representations" (Duval, 2006, p. 107). For example, a function is accessible through its analytic equation, its numerical table, or its graph. This peculiarity has important consequences for mathematical activities and for the students who learn mathematics. In fact, in order to properly grasp and manipulate mathematical objects and their relationships, it is necessary to acquire competences in transforming representations from one register to another, and within the same register (Duval, *ibid.*).

In mathematics education it is not an easy goal to provide students with competences in transforming representations. Many pitfalls and mistakes in students' mathematical performances are due to feeble or missing competences of this kind: for some examples, see Duval (2006, pp. 115-124), or the notion of pseudo-structural students in Sfard (1991). This weakness is particularly evident when considering the differences between novices and experts in managing such transformations, for example in problem solving activities (Schoenfeld, 1985).

In this project, we look at the differences between experts and novices in the way they perform transformations between representations in different registers. This research problem can be studied using a variety of methodologies (e.g. analysing verbal or written productions). We use an innovative tool, namely eye-tracking, for recording the eye movements of a subject involved in mathematical tasks. The eye

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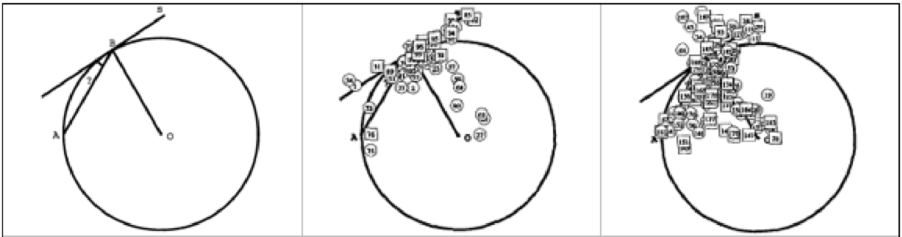
movements can give us information on cognitive processes of the subject (Yarbus, 1967, p. 190). Our research question and the method we use require competences not usually found in a single person. Therefore, our group consists of researchers in mathematics education, statistics, cognitive science and eye tracking methodology.

Eye-tracking has been used in many fields. Studies focusing on mathematics include students' interpretation of motion graphs (Ferrara & Nemirovsky, 2005; Ferrara, 2006), learners' interactions with multiple representations in a computer environment (San Diego et *al*., 2006), and subjects solving geometry problems (Epelboim & Suppes, 2001). To give an idea of how eye tracking can be used to highlight differences between performances of subjects, we report some results from Epelboim & Suppes (2001). They found that experts asked to find an angle on a diagram, marked by a '?' (fig. 1a), tend to move their eyes so as to "fill" a geometric entity not present on the diagram: namely, a triangle (fig. 1c). In comparison, novices seem to consider only the objects already existing in the diagram (fig. 1b).



a: the diagram | b: fixations of a novice | c: fixations of an expert

Figure 1. Epelboim & Suppes experiment.

From the patterns of fixations in fig. 1b and 1c, Epelboim & Suppes (2001) inferred significant differences between the cognitive processes of novices and experts. The fixations of the expert indicate that s/he imagines an element not present in the figure, namely the triangle (tracing with the eyes the lacking side), whereas those of the novice show that s/he only looks at the existing parts of the drawing.

In our study, we asked University students with different mathematical background to match a certain mathematical text, formula or graph to another one. The eye tracker was used to study their eye movements while solving the task; it provides us with information over time of the subjects' eye movements. Our overall aim is to describe differences in eye movement behaviour between experts and novices by combining both qualitative and quantitative analyses.

In this perspective, relevant questions for the research are: how do novices and experts examine the mathematical representations and the alternatives? Is it possible to outline different patterns in their eye movement behaviour?

The remainder of the paper is divided into two sections. The next section describes the experimental methodology and provides an initial analysis. The final part draws some first conclusions from the research.

EXPERIMENT AND METHODOLOGY

We collected data from 46 Swedish participants, all university students, but with two different knowledge background: 24 had no previous university studies in mathematics, while 22 had one year of studies at an engineer faculty. We will refer to the first group as "the novices" and to the second group as "the experts".

Each stimulus is a multiple-choice item made of an input and four alternatives. For each stimulus, the participants were asked to determine, among the four alternatives, the one equivalent to the input (one and only one of them was correct). The stimuli reflect the end of high school knowledge in advanced mathematics. Since we were interested in relationships between diverse representations (see above), three types of stimulus were taken into account: the input is a formula and the alternatives are texts, namely contain only words (ft type N=15 items); the input is a graph and the alternatives are texts (gt type N=12 items); and finally, the input is a text and the alternatives are formulas (tf type N=16 items). The first stimulus of ft type (labelled ft01) is shown in fig. 2*a*; the other types are similar in form as seen in fig. 2*b* and 2*c*.

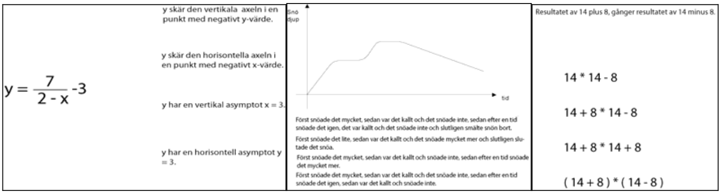


Figure 2. Examples of the stimuli: a) ft type, b) gt type, c) tf type.

The stimuli were different with respect to the number of steps required to solve them. For example, some stimuli may require a match between different ways of writing the same thing. Others may require multiple steps of reasoning for matching the correct alternative with the input.

At the laboratory, participants received a written instruction prior to the experiment and were then comfortably seated in the SMI HighSpeed eye tracker (fig.3).

Figure 3. SMI HighSpeed eye tracker.

This eye tracker has a precise accuracy and a sampling frequency of 1250 Hz. Stimuli were presented on a computer screen with the stimulus program called E-prime. In order to calibrate the accuracy of the eye tracker, participants looked at 13 dots in a specific pattern covering the whole screen. After completing the calibration, further instructions followed: a practice block was presented in order to familiarize participants with the test situation and the structure of the task. The task was to determine which of the four alternatives is equivalent to the input; to choose an alternative the subject had to click on it with the mouse. No paper and pencil were given to the participants. All stimuli (ft/gt/tf) were presented to the participants in random order. Each trial showed a mathematical representation (the input) for 5 seconds. Then all the alternatives were added to the screen for 40 seconds. After giving their answers, participants rated how sure they were of their answer on a scale from 0 (not sure) to 9 (sure), to control whether they guessed their answer or not (we are not addressing the guessing practice in this paper, but in future analysis). At the end, participants were asked if they could tell the purpose of the study. None could. Thus, we assume that students behaved according to the instructions given, and not in a way to strengthen or weaken our hypothesis by changing their behaviour accordingly. Participants were debriefed and left with contact information to the experiment leader.

As to our research questions above, we expect to find differences between experts and novices in terms of eye movements. In the next section, we investigate our expectations, analysing data by means of visualizations that illustrate processes over time.

DATA ANALYSIS AND PRELIMINARY RESULTS

Eye tracking data can be visualised and analysed in multiple ways. In this discussion we will use proportion over time graphs illustrated in fig. 4. Graphs show the proportion of subjects in the group looking at the input and at each of the four alternatives (on the y-axis) at a given time (time is on the x-axis). The input and each of the four alternatives are represented as five separate lines in the graphs. Data in this analysis contains the 40-second interval when both the input and the alternatives are displayed on the computer screen, but graphs show only the first 25 seconds.

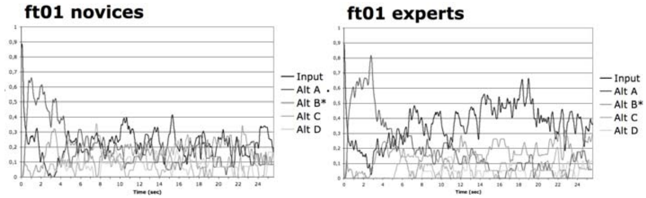


Figure 4. Proportion over time graphs for novices and expert for stimulus ft01.

We will look at four characteristics: overall *appearance, intertwining* between the input and the alternatives*, temporal order,* and *highest peak*. At a first glance on the graphs, we can see differences in overall *appearance* between the groups. For example, in the novice graph the overall appearance is fuzzy compared to the expert graph. The fuzziness is represented by the fact that the lines are overlapping during the majority of the time interval for the novices in fig. 4. Our interpretation is that novices frequently alternate between the five areas (input and the four alternatives), without any dominance for neither the input, nor the given alternatives. For the experts the lines are more clearly distinguished, suggesting that there is a dominating focus on one or a few areas represented by the input and the alternatives.

The second characteristic, *intertwining*, describes the relationship between the input and the alternatives. We observe that for more difficult stimuli, experts are looking at the input more than the novices. In the expert graph we can see that the line representing the input is separated from the lines of the alternatives to a higher degree compared to the novice graph. Our interpretation is that the experts are searching for clues to the answer in the input, instead of searching among the alternatives as represented by the overlapping pattern expressed by the novices.

The third characteristic is the *temporal ordering* of looking at the alternatives. For the experts, the pattern of an increase and then a decrease for alternative A, followed by an increase and a subsequent decrease in alternative B etc., suggests that the reading of the alternatives is ordered. Our interpretation is that the experts are more systematic than the novices, who do not show this pattern to the same extent. A final noticeable characteristic is the *highest peak* among the alternatives, which for the experts usually indicates the correct alternative; this is not the case for the novices.

To corroborate the qualitative analysis above, we used a statistical tool: the *autocorrelation function* (fig. 5). Autocorrelation allows finding the correlation between subsequent time intervals on the same line in the proportion over time graphs. For each instant t, it gives the correlation (on the y-axis) between the proportion of students looking at a certain part of the stimulus at t and the proportion of students looking at the same part at $t+k$ (k is a natural number representing a lag on the x-axis). It is a function of k.

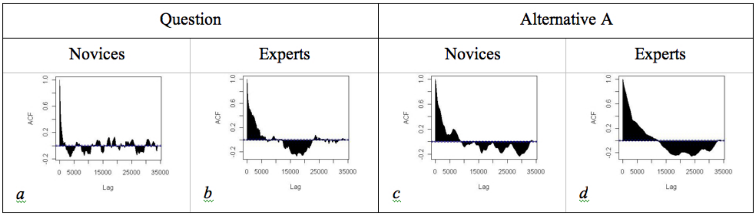


Figure 5. The autocorrelation function for stimulus ft01.

The patterns in the proportion over time graphs (fig.4) are reflected in the autocorrelation graphs in fig. 5. The many non-significant correlation peaks (close and around *x*-axis) for the novices in fig. 5*a* signifies a random average viewing pattern. Fig. 5*d* shows a clear and smooth decrease of the autocorrelation function (on the *y*-axis) over increasing lags (*x*-axis). It indicates two things: first, a moderate trend of simultaneous eye movements between areas among experts; secondly, a slight tendency to periodicity in the proportion over time graph. This tendency is shown by the dip of negative correlations (the region below the *x*-axis) in fig. 5*d*. Fig. 5*b* shows a similar but somewhat weaker trend compared to fig. 5*d*, while fig. 5*c* shows such a weak trend that it borders to random noise (like in fig. 5*a*).

DISCUSSION AND OPEN ISSUES

The example presented above suggests that there are two overarching strategies for the initial task: either spending a lot of time looking at the input, with excursions to the alternative; or spending the majority of time on the alternatives, with quick visits to the input. This provides an initial answer to the research question of how novices and experts examine the mathematical representations and the alternatives. The second research question concerns whether it is possible to outline different patterns in eye movement behaviour: our initial data analysis shows that this is possible. We want to further investigate whether this difference in strategies is characteristic of the two groups, or whether it reflects a difference in difficulty of the particular task, or the type of mathematical representation, or if it relates to whether the student gives a correct vs. incorrect answer.

In the near future, we also plan to analyse the minute movements between areas, in order to detect when in time overview looking on all areas occur, and when back-and-forth comparisons between pairs of areas take place. These two types of visual activities should be important indicators on phases in the cognitive processes. Detailed pair-wise comparison indicates a deep processing of the areas looked at. It is expected to be more common with experts, and also at later stages in the overall process, when it is likely that all alternatives have been read. Another important upcoming analysis looks at the visual behaviour just before a student has solved the task. Students being certain that their answer is correct should have another visual behaviour to areas just before giving it than students who give the wrong answer or happen to guess the right one. We expect to see long looks at the answer to-be-selected, possibly indicative of self-confirmatory looking behaviour, or last minute comparisons between alternatives of similar likelihood of being correct, that could be evidence of uncertainty.

Our initial finding of different eye movement behaviours between novices and experts has two direct didactic consequences. First, students have to learn *how* to read a mathematical task: they have to know where to direct their eyes in order to obtain relevant information for the given task. But this competence must be taught: in fact our data suggest that novices do not know how to read a mathematical representation, because their eye movements are different from those of experts. The role of the teacher with respect to this issue is fundamental. The competence of integrating different representations needs to be emphasized as an essential skill in reading mathematics. The relevance of such integrative competence is shown by Hannus' and Hyönä's (1999) study, in which low ability children did not look systematically between a scientific text with the corresponding picture nearly to the extent as high ability children. A second important consequence is that when teachers design mathematical tasks, care should be taken to point out and support *how* the task should be read. The precise design of text and picture combinations is vital for reading and integration, as shown by Holsanova, Holmqvist and Holmberg (2008): a serial presentation of text and pictures resulted in a higher number of gaze transitions between semantically related verbal and pictorial information, compared to a radial presentation of the same content. A proper design thus facilitates readers' construction of referential connections between text and illustration. Our initial analysis points to the importance of not only teaching the proper mathematical content, but also teaching *how* to read all the representations of a mathematical text.

Acknowledgement

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DIDACTICAL CONSEQUENCES OF SEMANTICALLY MEANINGFUL MATHEMATICAL GESTURES

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We consider the role of gestures as a semiotic resource in the learning of mathematics. Given the multimodal nature of communication in the classroom a key issue concerns whether the various semiotic resources exploited in teaching produce matching interpretations or not. In this study we used electroencephalography to investigate the effect of gestures on brain activity related to semantic processing. University students were presented with gestures followed by written words and indicated whether there was a semantic match or not. Results show that mathematical gestures produced similar brain activity to that elicited by other gestures and language, demonstrating that they are semantically meaningful and integrated into linguistic semantic processing. Possible didactical consequences for the mathematics classroom are briefly considered.

BACKGROUND

Many recent research studies have highlighted the *multimodal* nature of people's general cognitive processes (for an overview see: Granström et al., 2002) as well as of the learning of mathematics (Arzarello & Robutti, 2008). More precisely, analysis of learning processes in the mathematics classroom identifies a variety of resources utilised by students and teachers:

> Words (orally or in written form); extra-linguistic modes of expression (e.g. gestures, glances); different types of inscriptions (e.g. drawings, sketches, graphs,); various instruments (from the pencil to the most sophisticated ICT devices), and so on. All such resources, with the actions and productions they support, are important for grasping mathematical ideas: in fact they help to bridge the gap between the worldly experience and more formal mathematics (Arzarello et al., 2009).

In particular, gestures have been revealed to play an important role in thought processes (McNeill, 2005). The psychologist Susan Goldin-Meadow and her colleagues have analysed the relationship between gestures and thinking by considering everyday conversations and sometimes mathematical arguments developed by students and teachers (Goldin-Meadow, 2003). According to their findings, gesture plays a role in cognition (not just in communication) since it is involved in the *conceptual planning* of the messages. Therefore it is involved in speech production because it plays a role in the process of conceptualization. Gesture "helps speakers organize rich spatio-motoric information into packages suitable for speaking [...] by providing an alternative informational organization that is not readily accessible to analytic thinking, the default way of organizing information in speaking" (Kita, 2000, p. 163). Spatio-motoric thinking (constitutive of what Kita

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calls *representational gestures*) provides an alternative informational organization that is not readily accessible to analytic thinking (constitutive of speaking organization). Analytic thinking is normally employed when people have to organize information for speech production, since speech is linear and segmented (composed of smaller units). On the other hand, spatio-motoric thinking is instantaneous, global and synthetic, and not analyzable into smaller meaningful units. This kind of thinking and the gestures that arise from it are normally employed when people interact with the physical environment, using the body (interactions with an object, or an instrument, imitating somebody else's action, etc.). It is also found when people refer to virtual objects and locations (for instance, pointing to the left when speaking of an absent friend mentioned earlier in the conversation) and in visual imagery (for example, gesturing in the air for drawing the graph of a function not present on paper). Within this framework, gesture is not simply an epiphenomenon of speech or thought; gesture can contribute to creating ideas:

> According to McNeill, thought begins as an image that is idiosyncratic. When we speak, this image is transformed into a linguistic and gestural form. The speaker realizes his or her meaning only at the final moment of synthesis, when the linear-segmented and analyzed representations characteristic of speech are joined with the global-synthetic and holistic representations characteristic of gesture. The synthesis does not exist as a single mental representation for the speaker until the two types of representations are joined. The communicative act is consequently itself an act of thought. ... It is in this sense that gesture shapes thought. (Goldin-Meadow, 2003, p. 178; see also McNeill 1992, p. 246).

Many studies have shown that gestures play an important role in supporting thinking and communication processes, including in students who are learning mathematics (for some examples and references see Goldin-Meadow, 2003; and Edwards et al., 2009). In particular, Goldin-Meadow (2003) has pointed out the crucial role played by *matching* and mismatching between gesture and speech, namely when gesture and speech convey overlapping or non-overlapping information: "children who produce gestures that 'mismatch' their speech at a certain point during their acquisition of a math concept arrive at a deeper and longer-lasting understanding of that concept than children who don't" (Goldin-Meadow, 2003, p. 247).

A comparison between matching and mismatching semantic information is the basis of a common effect studied in the field of electroencephalography (EEG), the N400. EEG measures electrical signals generated by brain activity. The N400 is thought to index semantic processing, and manifests itself in a difference in the brain response to matching versus mismatching events. For example Wu and Coulson (2007a) employed this methodology to show that iconic co-speech gestures modulate conceptualization, enabling listeners to better represent visuo-spatial aspects of the speaker's meaning. Such studies concern mainly everyday conversation. As far as we know, there have not yet been studies of this kind concerning mathematical concepts. In this paper we present the results of an experimental study. They demonstrate that gestures, whether representing relatively abstract mathematical concepts or more concrete actions, are semantically meaningful and influence the semantic network accessed by written words.

The paper is divided in four parts: the first is the present chapter, which gives some general information on gestures and thinking; the second describes the experiment; the third its findings; the last one discusses the didactical consequences of our findings.

EXPERIMENTAL EVIDENCE

The N400 effect exploited in the current study is an event related potential (ERP) effect. Because the electrical signals generated by the brain are very noisy, the typical response to a stimulus event is calculated by averaging electrical signals from the brain over many presentations of the same type of stimulus. The N400 was first demonstrated by Kutas and Hillyard (1980), who showed that the ERP in response to reading the terminal word of a sentence was more negative approximately 400 msec post-stimulus for semantically incongruent endings (e.g. "I take tea with milk and dog") than for semantically congruent endings (e.g. "He spread the warm toast with butter"). This now well-studied effect shows a particular scalp distribution or topography (maximal over central parietal regions, and larger over the right hemisphere). A similar (possibly the same) N400 effect has since been found using other stimuli, such as pictures (e.g., Barrent & Rugg, 1990; Hamm, Johnson & Kirk, 2002; McPherson & Holcomb, 1999) and words primed by gesture videos (Wu & Coulson, 2007b).

In this study we examined the N400 to words which matched or mis-matched a preceeding iconic gesture, either depicting a mathematical or an action concept. We also included the classic N400 language task (Kutas & Hillyard, 1980, Johnson & Hamm, 2000, Hamm et al, 2002), for comparison. If gestures activate a common semantic system with words, then the topographies of the N400 effects in the different conditions should not differ. If, on the other hand, the topography of the N400 effect differs between conditions, we can conclude that the effects are generated by different underlying neural mechanisms.

The 12 participants in the study (9 male; mean age 21 years) at The University of Auckland were all right-handed, had English as their first language, and had a minimum of the final year of high school calculus, with most completing or having completed first year university mathematics. High density EEG (128 channel at a sampling rate of 1000 Hz) was employed.

In the gesture task, participants were presented with a video clip of a gesture, which lasted from 1500 to 2300 ms, followed by a probe word for 1000 ms, which either matched or mismatched the gesture video. Finally the words "match" and "mismatch" were displayed on the left and right of the screen in random order (to prevent prior activation of a motor response), and participants pressed a corresponding key to indicate the correct response.

Blocks of either action or mathematical gestures were alternated. Only the hand and torso was presented in the movie clips, which ranged in duration from 1580 ms to 2330 ms (see examples below).

	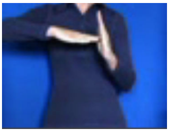	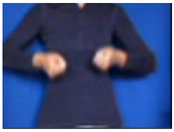	
Converging	Perpendicular	Lift	Punch

Figure 1. Examples of stills from mathematical and non-mathematical gestures video.

The action gestures were based on previous research (Ozyurek, Willems, Kita, & Hagoort, 2007): Drive, Drop, Give, Lift, Pull, Punch, Push, Stroke, Take, and Walk, while the mathematical gestures corresponded to the concepts: Converging, Decreasing, Diverging, Increasing, Linear, Maximum, Minimum, Parallel, Perpendicular, and Quadratic. Within each experimental condition each of the 10 gestures was presented 12 times (6 congruent and 6 incongruent), giving a total of 120 trials. Within-category incongruent pairings were pre-selected by the experimenters and remained the same throughout the experiment.

In the language task, sentences were presented at a rate of one word per second, with the last word either matching or mismatching, and a response cue as for the gesture tasks. There were 120 sentence stimuli (60 congruent, 60 incongruent) from Johnson and Hamm (2000).

RESULTS

The Language, Mathematical gestures, and Action gestures conditions showed 71, 90, and 87 significantly different electrodes (respectively) for matching vs. mismatching stimuli over the normal N400 time window (320-400 msec), confirming an N400 effect (see Figure 2). The Global Field Power (GFP), a measure of field strength, was used to investigate differences in amplitudes of waves. There was no significant difference between the three conditions. A topographic analysis of variance (TANOVA) showed no significant differences in the topography for the different conditions. These results suggest that gestures activate a semantic network similar to that accessed by visually presented words.

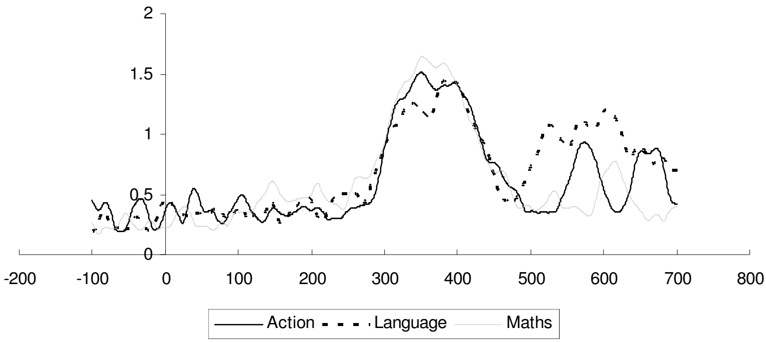

Figure 2. GFP of the three conditions (Action and Mathematics gestures and Language tasks).

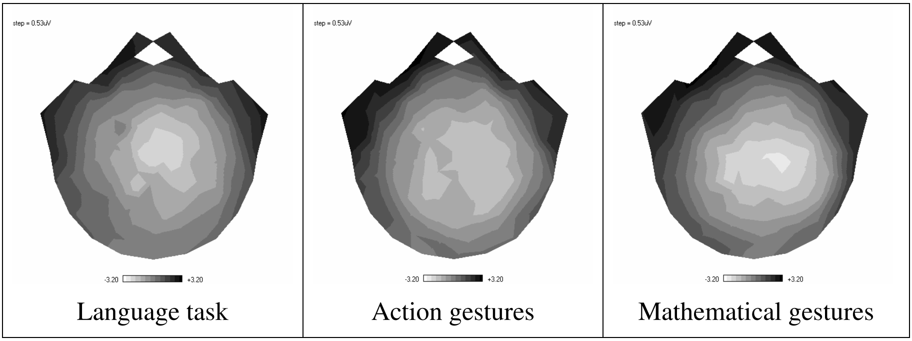

Figure 3. Topographies for language task, action and mathematical gestures.

Results should nevertheless be interpreted with caution because the amplitude of the N400 effect is known to be modulated by the cloze probability of the stimuli (Kutas & Hillyard, 1984) and stimulus repetition (Federmeier & Kutas, 2001), two factors that were equated between the sentence and gesture conditions.

Overall this study demonstrates that gestures, whether representing relatively abstract mathematical concepts or more concrete actions, are semantically meaningful and influence the semantic network accessed by written words. Hence, mathematical gestures can semantically prime a written word, to produce an N400 effect, compared with the standard priming of concepts via words. All of this leads to a clear indication that gestures could be usefully employed to assist in the teaching of mathematical subject matter. Possible ways in which this may be achieved are discussed below.

DIDACTICAL CONSEQUENCES

As indicated above, the complex activity of human communication in any form, including classroom teaching of mathematics, is reflected in multimodal semiotic resources that include gestures, classified by McNeill (1992) into beat, iconic, deictic, and metaphoric, with the final three roughly corresponding to the icon, index and symbol of semiotic signs. While metaphoric gestures are similar to iconic gestures in making reference to a visual image, the image pertains to an abstraction (Roth, 2001). Since mathematics has a primarily abstract content replete with symbols one would expect that metaphoric gestures would be commonly used. McNeill (1992) gives the example of a mathematician holding one hand steady and moving the other hand towards it until the two palms touch while discussing the concept of 'approaching a limit'. Hence, while the gestures used in this study may appear iconic, they can become metaphoric in nature when presented in a mathematical context, giving them a different referent. We may wonder though whether students can distinguish between the two. The fact that the participants in this study were able to relate the gestures to the words with precise, abstract mathematical meanings may support this, as well as the theoretical positions Roth (2001, p. 373) describes:

> Despite their differences, most current theories…suggest that iconic gestures emulate from a visual cognitive component that is semantically related to a concept or unit of discourse that corresponds to the gesture. It is this idea that remains paramount to most educational research on gestures because it suggests that gestures might actually provide some insight into the mind of the speaker.

How then might a teacher use gestures? While there is still limited research on the role of gestures in teacher-student interaction, there are at least two possible dimensions to this. Firstly, it may be that students who are in transition with respect to a particular concept, and are thus in a state of readiness to learn, may be identifiable because of mismatches between their gestures and speech (Perry et al., 1988), giving a discerning teacher insight into the student's thinking. However, it is not clear how such a skilled teacher might identify when a student experiences such a 'teachable moment', or how they would decide what type of assistance to offer them (Alibali et al., 1997). Conversely the student might employ the matching gesture and speech of the teacher and compare it with their own expectation. Secondly, a teacher might employ gestures as part of multimodal activity that can enhance communication of abstract concepts, by giving students additional resources to understand what the teacher is presenting. The teacher may use iconic (narrative function), deictic (grounding function) and metaphoric gestures. For example, iconic gestures can be used to add a perceptual dimension to a concept such as 'circle', although the abstract figural-concept requires the overlaying of properties on the image (Fischbein, 1993). Deictic gestures are a complementary resource too, possibly distinguishing examples from non-examples, such as pointing to one of a number of figures while saying 'this rectangle', or circumnavigating a circle on the board while orally distinguishing it from a disc.

One specific didactic strategy in which a teacher reflects a student gesture has been described as the 'semiotic game' (Arzarello & Paola, 2007; Arzarello & Robutti, 2008). In this game the teacher uses one of the semiotic resources shared with students (gestures) in communication in order to encourage more formal learning.

This activity has been observed and described by researchers, with Alibali et al. (1997, p. 190) noting "…both teachers and undergraduates frequently produced additions that could be traced to the gestures of the children in the vignettes. More than a third of these traceable additions were expressed in speech by the adults; the remainder were expressed in gesture." Arzarello and Paola (2007, p. 23) describe the possible nature of the semiotic game this way: "The teacher mimics one of the signs produced in that moment by the students (the basic sign) but simultaneously he uses different words: precisely, while the students use an imprecise verbal explanation of the mathematical situation, he introduces precise words to describe it…or to confirm the words." While Alibali et al. (1997, p. 192) question whether "…teachers, in fact, make different instructional choices after having implicitly acknowledged that a strategy is part of the child's repertoire (by reproducing the child's gestured strategy in their own gestures)", they note that we *can* infer that the strategy is active in the teacher's thoughts and thus may influence their choice of material to be taught or stressed, and hence this is a didactical strategy that may be explicitly pursued. Certainly this study has confirmed that teachers (and students) are able to understand the semantic meaning of the gestures they observe, and hence are thus in a position to respond to them. The phrase 'I see what you mean' may take on a literal meaning.

This discussion does raise questions for further study, aspects of which may be amenable to ERP methodology. For example, Can students distinguish iconic and metaphoric gestures in mathematics? Under what circumstances, and to what extent, do teachers use metaphoric gestures? How do these metaphoric gestures mediate students' construction of conceptual knowledge? Another somewhat unexplored area is the role of gestures in students' collaborative mathematical activity, where inherent problems due to inconsistencies and time delays, raise questions such as: "How do these inconsistencies mediate coordination and understanding within a group of learners?...How do temporal shifts mediate the communication comprehension in student-centred activity?" (Roth, 2001, pp. 380-381).

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STUDENTS' VIEWS OF THEIR MOTIVATION IN MATHEMATICS ACROSS THE TRANSITION FROM PRIMARY TO SECONDARY SCHOOL

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In this paper we explore students' views with respect to changes in their motivation, in classroom culture and of how their motivation in mathematics could be enhanced across the transition to secondary school. Eight students' were selected for semi–structured interviews, on the basis of the changes in their goal orientations across the transition. The analysis of the transcribed interviews indicates that students with a mastery orientation are more reflective about themselves, whereas students with a performance orientation focus less on their own role in motivation. Performance–approach orientation appears to be adaptive when a mastery orientation is also espoused, whereas the performance–avoid orientation leads to maladaptive outcomes such as psychological distress after the transition to secondary school.

BACKGROUND AND AIMS OF STUDY

The period surrounding the transition from primary to secondary school has been found to result in a decline in students' motivation in mathematics which appears to be related to certain dimensions of the classroom culture (Athanasiou & Philippou, 2006). Most researchers have so far examined motivational change for students as a whole group using a quantitative methodology assuming and inferring that the transition affects all students the same way. However, recent research in the area of students' perceptions of their classroom environments (Urdan & Midgley, 2003) supports the view that students perceive the same environment in variable ways related to the concerns they bring to the situation. Previous motivation research has not attempted to reconcile individual change with change of the group as a whole.

Very few studies examined motivational change of individual students through interviews. In the study of Demetriou et al., (2000) students' responses to the interviews reflected the difficulties some students have in sustaining their commitment to learning. Furthermore, after the transition students analysed friendships in terms of academic achievement and knew that if they were not with friends they would work better. Pointon (2000) examined the differences between primary and secondary school classrooms; the students reported that they liked moving from one classroom to another, but at the same time they complained that in secondary school they felt they had no space for their own. Furthermore the students reported that their mathematics room in secondary school was dense and stuffy and that that was the reason they did not like mathematics.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 65-72. Thessaloniki, Greece: PME.

The study of McCallum (2004) examined motivational change across the transition to secondary school using both quantitative and qualitative data. The quantitative analyses indicated the general decline in motivation across the transition, whereas the interviews indicated that students with different goal patterns focused on different aspects of the transition. Students with task and ego goals mainly focused on aspects about themselves and the students expressing social goals focused on issues concerning relationships. One limitation of this study was the lack of consideration of the approach-avoidance distinction in motivation. The literature reveals that a dichotomous model of goals (mastery vs. performance goals) is not sufficient in order to graph students' motivation (Covington & Müeller, 2001).

In this paper the analyses of eight students' responses to interview questions are presented. This study is a part of a longitudinal project examining students' motivational change across the transition to secondary school. The aim of the semi–structured interviews was to explore students' views of the change in their motivation, of the changes in classroom culture across the transition and of how their motivation in mathematics could be enhanced. Specifically, the main themes of the interviews were:

1. Students' motivational profiles in primary and secondary school and their motivational change stories after the transition. Students were asked questions about their motives during mathematics in both school contexts and about the change in their motivation across the transition (e.g., "What were the students' reasons for engaging in math work in primary school?"; "Did students perceive a change in their motivation in mathematics after the transition to secondary school?").

2. Students' perceptions of the differences between primary and secondary school. Students were prompted to highlight aspects that they thought were different across the transition through questions such as "What differences regarding the teacher did students perceive across the transition?".

3. Students' perceptions of how their motivation in mathematics can be enhanced after the transition. The questions addressed to students aimed to explore their views about how their self and the classroom environment should be in order their motivation to be increased (e.g., "What dimensions of the classroom contexts and activities do students perceive as enhancing their motivation?").

METHODOLOGY

Participants in the interviews were eight students who experienced the transition from primary to secondary school. The students were selected for the interviews on the basis of their gender and according to the changes in their motivational goal orientations as they were tapped by questionnaire data at two time-point measurements (one prior and one after the transition). More specifically, four couples of students were created which included one boy (B) and one girl (G) who experienced increase in their: (a) performance-approach goal orientation, that is in their performance in terms of demonstrating ability (B1, G1); (b) performance-approach and mastery goal orientations, that is in their performance perceptions and in their value of motivation for learning (B2, G2); (c) performance-avoid orientation, that is in their need not to demonstrate lack of ability (B3, G3); and (d) mastery and social goal orientations, that is in their value of motivation and in their perceptions of how much socially–oriented they were (B4, G4) after the transition to secondary school.

The students were interviewed once, four months after the transition. This time point was selected because it allowed the examination of motivational change as it unfolded without the immediate effect of the transition on students' motivation. A suitable interview time was arranged and the interview was conducted individually in a room at students' school. All interviews were audio taped and later transcribed.

RESULTS

The findings of the analyses of the students' responses are presented in this section for each of the four groups of students mentioned above. Characteristic extracts from the interviews are presented in order to illustrate the contexts of change more fully

Students with a performance-approach orientation after the transition

Prior the transition to secondary school B1 and G1 expressed high mastery and social orientations; they reported that they liked trying hard to improve their abilities in mathematics and they enjoyed working with other students. After the transition they expressed predominantly a high performance-approach orientation; they wanted to perform well to impress others.

> G1: I liked mathematics a lot when I was in sixth grade. I enjoyed learning new things. It was fun, especially when working with my friends.

> B1: Now I want my friends to recognize how smart I am. It is very important for me.

Both students experienced a decline in their mastery orientation after the transition since they did not focus on interest and doing ones best. They thought that being smart was the main ingredient for being good and getting high grades.

> B1: I don't want to study for a long time, unless I really have to. I think that I am smart, so I don't need to try hard.

After the transition B1 and G1 focused on the limiting aspects of the classroom and the teacher-student contexts and highlighted the negative aspects of the transition. Both students' mentioned that the classroom goal structure in elementary school was more mastery-oriented since the teacher emphasized trying and improvement whereas in secondary school the classroom environment became more performance-approach oriented since the emphasis was on getting good grades.

> G1: The teacher now wants us to answer all questions correctly. And he is telling us about our grades…In elementary school the teacher focused on improvement. And it was ok when we made mistakes. She was always telling us that mistakes were part of the learning process.

The two students also reported that in secondary school there was less interaction and support from the teacher, whereas there was much more press.

> B1: When we do mathematics this year we never talk to the teacher or to each other about anything else beyond mathematics. The teacher is not friendly. And sometimes he is pushing us to get high grades.

Both students mentioned that the competition among students was higher in secondary than in primary school. They reported that in seventh grade they actively made comparisons of the grades between themselves and their classmates.

> G1: When we have a test in mathematics we compare grades with classmates. Our teacher compares it so we do it as well.

B1 and G1 did not admit having difficulties in mathematics across the transition, although they were not looking forward to going to secondary school.

> G1: The transition was not so bad. I thought it was going to be worse. I was very worried whether the teachers of mathematics would like me.

In order to be more motivated in mathematics the two students would like the classroom organized in ways that enabled them to have status or impress particular people. These students were less reflective about themselves, since they seemed to think that teachers or the way the classroom was organized had the major role.

> B1: I think that I would be more motivated in mathematics if the teacher recognized the good work I do in mathematics. When we do team work, he could say to my classmates that I will be the leader because I can do the best work.

Students with high performance-approach and mastery orientations after the transition

Prior to transition students B2 and G2 endorsed a high mastery orientation; According to their statements their success in mathematics was the result of working hard. After the transition both students endorsed performance–approach as well as mastery orientations showing that a tension was developed between looking good and doing well with putting in too much effort.

> G2: I enjoy working hard this year as last year because I like getting high grades. That is why I am not thinking of anything else when I am solving problems.

The students expressed the differences between primary and secondary school in both positive and negative terms. They reported that their new classroom was more performance-oriented than in primary school, but they considered it as a characteristic of the secondary school. They also reported that there was less participation and interaction with the teacher but they emphasized the knowledge of teachers and thought that their secondary school teachers explained things better.

> B2: This year the teacher is telling us how important is to get good grades. In elementary school we never thought of getting high grades just doing well. But I think that when you go to secondary school it is pretty logical for that to happen. It is just the

way things are anymore. We will have exams at the end of the year so we have to perform well.

G2: We do not participate in class so much as in sixth grade. The teacher usually lectures. But I know now that my teacher is an expert in mathematics, so I do not mind listening to him when he teaches.

Both students reported that in seventh grade they were more competitive with their classmates than in sixth grade, whereas they perceived that the increase in the competition environment was contributing to working hard and trying to improve.

B2: I compare my grades with those of my friends. If my grade is lower, I am trying harder because I want to succeed in mathematics.

The two students admitted facing difficulties during the transition and they expressed them in terms of the self and their response to the new situation. Both students were aware of the change in their motivation and described it in positive terms.

B2: The transition was a bit difficult. At first I did not know what the teacher expected from me. I did not know how to study in mathematics.

G2: I know that this year I am thinking much more about my grades than last year. I see it as a means of being more concentrated in order to succeed.

B2 and G2 students focused both on themselves and on the school and classroom environment as possible means to enhance their motivation in mathematics.

B2: How could I be more motivated? By reminding to myself that I have to try hard.

G2: I think I would be more motivated if I could work much more with my friends.

Students with a performance-avoid orientation after the transition

B3 and G3 were the students that prior the transition endorsed high mastery and performance-approach orientations since they reported that they demanded challenging tasks and enjoyed mathematics. After the transition students' orientations changed dramatically since they endorsed high performance-approach and avoid orientations. They wanted to perform well in order to gain social status yet at the same time they expressed an intense concern about not appearing dump.

G3: I always liked mathematics in elementary school. It was so exciting… The time went by so quickly.

B3: This year I want to be good at mathematics by getting good grades. But I am very much afraid of failure, because I do not want my friends to get the impression that I can not do mathematics.

The two individuals construed the achievement setting as a threat and tried to escape the situation if possible. The prospect of failure elicited anxiety and disrupted concentration and task involvement.

G3: Often I think of the problem as a monster I have to fight with and I must win to look good. Most of the times I want to run away. I want to try but failure is in my mind all the time and I cannot think of how to solve a problem.

B3 and G3 were extremely critical of the teacher and the classroom environment after the transition. They reported that the classroom goal structure in secondary school was more performance-oriented than in elementary school, whereas they perceived less interaction with the teacher and thought that the teacher was less friendly and supportive in secondary school than in primary school.

> G3: Our teacher is not friendly. She usually does not allow us to participate in class or to express our ideas in mathematics… She is just pressing us to do good work and find the correct answers but she does not help us.

The students mentioned that the classroom environment in secondary school was more competitive than in sixth grade. The two students compared grades with other classmates and expressed a relief when other students performed worse than them.

> B3: This year the environment is more competitive…I usually see my classmates' grades. And I am really happy when I am performing better than them.

The two students admitted facing difficulties over the transition to secondary school although they both expressed that they kept them for themselves.

> G3: Math is more difficult this year. There are a lot of new things that I have a difficulty understanding them. But I never told anyone that I had difficulties.

Both students were not thinking that improving their motivation in mathematics was basically dependent on themselves, since they focused on what the teacher could do for them and they wanted their teachers to motivate them in order to respond.

> B3: I do not think that there is much that I can do to enhance my motivation. I think that this is a task for the teacher. My mathematics teacher must motivate me.

Students with mastery and social orientations after the transition

Prior the transition B4 and G4 endorsed high mastery and social orientations; they saw poor result in a test or their inability to understand the new material as a signal to work harder and they enjoyed working with friends when solving problems. Students' motivational profiles did not change after the transition since both orientations predominated in seventh grade. Both students thought that improving was basically dependent on themselves and were aware of the effort they applied.

> G4: I was always thinking that trying is important in math. If I did not understand the new material taught in class I studied harder. I knew that the hard work would eventually help me understand… I liked working with my friends. We worked together a lot in sixth grade, in investigations.

After the transition G4 and B4 reported that the two school contexts were different and they expressed these differences in positive terms. The two students experienced an incline in the performance-approach classroom goal structure but they reported that the mastery goal structure was evident as well.

> G4: I know that there is an emphasis this year to be good. But to be good you have to try and work hard… These two aspects exist together anymore.

The students focused on the positive aspects of the teacher-student interactions and did not perceive the teacher to be less supportive and friendly, whereas they emphasized teacher's knowledge. They perceived the classroom environment in seventh grade to be more competitive than in sixth grade but they reported that this encouraged them to be competitive with themselves and not with others.

> B4: I think that my math teacher this year is as friendly and supportive as the teacher I had last year. When we need help he is always willing to help us.

> G4: We show the grades we receive at the tests to the rest of the class. But I do not compare them. I focus on myself and that is all.

Both students admitted having difficulties in mathematics after the transition expressing them in terms of the self and their response to the new situations. They believed that their motivation in mathematics can be enhanced by focusing primarily on themselves and with working with friends.

> B4: The transition was really difficult... At first I did not know what the teacher expected from me. I did not know how I was supposed to work in mathematics. So at first I tried really hard…Now I know.

> G4: I am always willing to try hard in order to learn new things and be a good student.

> B4: I believe that I could be more motivated in mathematics if I had the chance to work with my friends a lot. By working together I think that I would be more concentrated and try harder.

DISCUSSION

The main aim of this study was to examine the contexts of motivational change of eight students with different patterns of goal orientations across the transition to secondary school. The students grouped together according to their goal orientations, responded in ways that would be expected from their goal orientation emphases, without any differences according to gender. For students like B3 and G3 the intense concern about not appearing dump was foremost, whereas for students like B2 and G2 a tension was developed between looking good and doing well with putting in effort. Finally B4 and G4 represented the mastery–oriented learners, whereas B1 and G1 appeared to be the performance-oriented confident achievers.

All the students appeared to be aware of the changes in their motivation across the transition, although they were not all willing to express them. Students with a high mastery orientation were the only ones who admitted facing difficulties expressing these concerns in terms of the self and their response to the new situation, whereas students with a predominant performance orientation expressed an intense concern about keeping the difficulties for themselves. Previous studies (e.g. MacCallum, 2004) yielded the same results since the students who endorsed ego goals actively made comparisons between themselves and their classmates, whereas the students who espoused mastery goals tended to focus on themselves and tried to find strategies to work things out.

Although the students mentioned the same differences between primary and secondary school, they were not seeing them in the same light. More specifically, students with a high performance orientation tended to be extremely critical of the classroom environment, whereas students with a high mastery orientation expressed the differences in positive terms. Also the students with a more dominant mastery orientation tended to be more reflective about themselves and focused more on the importance of their own role in motivation, whereas performance-oriented students focused less on their own role in motivation. In the literature, mastery orientation was put forward as the most adaptive form of motivation that could lead to a better quality of learning (MacCallum, 2004). In the present study, mastery-oriented students appeared to have learned how to make the most of any environment and had a better fit in secondary school than the students with other predominant orientations.

Some studies have suggested that although a performance-approach orientation is sometimes associated with maladaptive patterns of learning it may also be associated with some positive outcomes especially when a mastery orientation is also high (MacCallum, 2004). In contrast performance-avoid orientation is associated with maladaptive outcomes with no evidence of positive effects (Covington & Müeller, 2001). This study provided evidence supporting the above findings since it showed that the performance–approach orientation is adaptive for certain students when a mastery orientation was also espoused. The results of the study also indicate the maladaptive nature of the outcomes associated with the performance-avoid orientation. Specifically, the students with a high performance-avoid orientation showed a significant psychological distress - that was not gender exclusive - that lead to less enjoyment of mathematics.

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ABSTRACTION THROUGH GAME PLAY

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This paper examines the computer game play of an 11 year old boy. In the course of building a virtual house he developed and used, without assistance, an artefact and an accompanying strategy to ensure that his house was symmetric. We argue that the creation and use of this artefact-strategy is a mathematical abstraction. The results add to knowledge on mathematical abstraction: of non-traditional knowledge; without teacher intervention; through game play.

We examine the game play of an 11 year old boy, Costas, who built virtual houses in the computer game *The Sims 2*. We focus on an artefact and an accompanying strategy he developed to enable him to build a symmetric virtual house and argue that the creation and use of this artefact and strategy is a mathematical abstraction in the sense of Schwarz, Dreyfus and Hershkowitz's (2009) model of *abstraction in context*. Issues explored in this paper other than abstraction are learning without teacher intervention, non-traditional mathematical content and learning through game play. The paper is structured as follows: an integrated review of literature and presentation of the theoretical framework; the setting and methodology of the study; results pertinent to the focus of this paper; a discussion of issues arising.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

We review literature on abstraction, with special regard to *abstraction in context* and to learners working without teacher intervention, and present selected literature on learning through game play.

There are many schools of thought about what a mathematical abstraction is. Boero et al. (2002) and Mitchelmore and White (2007) both provide summaries of different accounts of abstraction but most accounts can be regarded as belonging to one of two schools of thought, empirical and socio-cultural. Empirical views consider, with various levels of refinement, that abstraction involves generalisation arising from the recognition of commonalities isolated in a large number of specific instances. Socio-cultural views consider, in various forms, the development of an abstraction, through the use of mediational means and social interaction, from an initial rough idea to a refined construction that can be used in doing new, for the abstracter, mathematics. *Abstraction in context* is a socio-cultural account which views abstraction as vertically reorganising (vertical in the sense of Freudenthal, 1991) previously constructed mathematical knowledge through three nested epistemic actions: recognising (familiar mathematics), building-with (elements of familiar mathematics) and constructing (new mathematics). Familiar and new mathematics will depend on the person – what is new to one person may be familiar to another. Three abstraction

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stages are posited: a need for new mathematics; the construction of new mathematics; the consolidation of the new mathematics, i.e. its use in further mathematical activity. Schwarz, Dreyfus & Hershkowitz (2009) provide a comprehensive review of *abstraction in context* including many PME papers.

Socio-cultural education researchers view mediation as paramount. This can be reflected in a view that teaching is essential for learning, e.g. “learning physics will not happen without teaching and the mediation of adult and of sign” (Tiberghien & Malkoun, 2009). The veracity of this statement, we feel, depends on what one means by teaching. If teaching refers to a teacher, then we feel that this is almost always, but not always, the case in classroom learning. If teaching means mediation, then we agree with the statement and note that learning, as we argue in the case of Costas, can be computer mediated. Two contrasting *abstraction in context* studies report on learning without teacher intervention. Dreyfus & Kidron (2006) reports on a solitary learner, an adult mathematician, solving a problem concerned with bifurcations in dynamic processes that she had set herself and which required new, to her, mathematics. Abstraction was assisted by books, internet resources and *Mathematica*™. Williams (2007) reports on “spontaneous” learning by individual school students in working on mathematics set by adults; Kerri, for example, worked on a problem in a test on the equation of a straight line. In both studies the abstractions concerned scholastic mathematics, i.e. mathematics that might be formally taught. In Costas’ case his abstraction concerned a symmetry technique specific to the task he set himself in a specific computer game. Such mathematics may not be visible to all or publicly valued but it is, we claim, still mathematics.

Mathematics educators often recommend mathematical game play, “Mathematical games can foster mathematical communication ... can motivate students and engage them in thinking about and applying concepts and skills” (NCTM, 2004). An early study on games in mathematics (Bright, Harvey & Wheeler, 1985, p.133) concluded “games can be used to teach a variety of content in a variety of instructional settings ... there is no guarantee that every game will be effective ... But many games are effective”. Bragg (2006, p.233), however, in a study of students’ perceptions of game play concludes that “it appears that assumptions that students will see the usefulness of mathematics games in classrooms are problematic”. It depends, to us, on the game, the students and the context of game play. *The Sims 2* (2006) is a popular life simulation computer game that allows players to control the lives and relationships of game characters and create houses and neighbourhoods for them. Building a house in *The Sims 2* requires virtual money. Prensky (2006) refers to how players in *The Sims*, the precursor to *The Sims 2*, can learn how to resolve social and financial household issues through game play. Although the game play we report on did involve financial considerations the learning we report on is not primarily financial in nature.

SETTING AND METHODOLOGY

This paper reports on an exploratory case study (Yin, 2003) of one 11 year old Cypriot boy, Costas, who satisfied three *a priori* criteria: prior experience of computer game play but not of *The Sims 2*; ability to read and understand English (the language of the game); perceived willingness and ability to express himself to the researcher (first author, a Cypriot). Costas was told that the game play was for research purposes but was not informed of the mathematical focus until after data had been collected. The study was conducted in Costas' bedroom and on his computer; case studies should be conducted "within its real-life context" (Yin, 2003, p.13). Five meetings took place. The first was *off the record* and the other four were recorded (procedures are explained below). Approximately three hours of data was recorded. The intention was for Costas to build two houses: the first to be built without budget constraints, the second to be built for a specific family and to a strict budget. The first meeting, for which researcher's post observation notes were the only data, was free play familiarisation, assisted by the researcher, on any aspect of the game. At the end of this meeting Costas, fortuitously, asked to build his own *Sims* house as his task.

A note on building a house in *The Sims 2*. Figure 1 shows Costas' partially completed second house with swimming pool. The player is constrained to work in a rectangular grid and with predetermined building tools and extras (doors, windows, swimming pool) but is free to choose how to assemble these. Floor space, wall units and extras cost *Sims* money. Knocking down and rebuilding a part of a house costs money because the refund on floor units (called "cubes" in *The Sims 2* and, hereafter, in this paper), wall units and extras is less than the original cost.

Figure 1. Central swimming pool

The next four meetings were recorded using *BB Flashback* screen capture recording software (http://www.bbsoftware.co.uk/bbflasback.aspx) which recorded all screen activity and discourse; this formed the primary data for analysis. The researcher acted as *observer participant* but all decisions regarding the house were made by Costas, the researcher simply encouraged Costas to express his thoughts and occasionally helped him with purely technical matters. Costas built, at his request, three houses; the first two without constraints. He called the third house his "dream house" for his family and he had a modest budget of 40 000 in *Sims* money which he regarded as Cypriot pounds. He was extremely motivated – the house was to be perfect and the meetings ended when he said, with satisfaction, "Seems good. OK, I'm done. I think the family should take it from here."

Data analysis was conducted in three stages. The first author carried out the data analysis and the second author conducted independent analysis on selected data. The first stage produced open codes á là Strauss and Corbin (1998) with regard to Costas' actions during game play. This, we felt, was useful starting point to see what categories emerged. The second stage of data analysis, isolating problems, arose from an observation in conducting the first stage analysis – a pattern of work was detected, Costas usually planned to include a feature, e.g. a door, then executed his plan using calculations and then considered the appearance of what he had done. We referred to these sequences as *mini-problems.* Many of these mini-problems were nested. The third stage of data analysis arose from a question in stage two, how were mini-problems initiated and how did they end? In the course of addressing these questions we looked at the goals Costas needed to accomplish to build his house. Some mini-problems had a single goal but some had several goals.

RESULTS

We present selected results from the three stages of data analysis (selected to illustrate results but also to prepare the reader for the Discussion section) and a description of an artefact Costas created to ensure symmetry.

The first stage of data analysis produced 14 categories. Space does not allow us to list them all but mathematics related categories were: calculations (+, - , x, ÷ and counting); symmetry; size comparisons; money matters (*275 pounds for a bar, no way*); mathematical terms. The second stage of data analysis isolated 42 mini-problems. We illustrate the nestedness of many mini-problems with an important, for Costas, episode in getting a door centrally placed. Mini-problem 11 was *How to put the front entrance door in the middle*? but the front side of the house was 15 cubes long and the door took two cubes – he realised that a door in the middle was not possible, so he put mini-problem 11 aside to tackle mini-problem 12, *What can be done to the front side of the house to allow a central door?* He resolved mini-problem 12 by deleting a front cube but this cost money. These problems occurred in building the second house and the loss of money was a reason it was discarded and for designing the symmetry tool. The third stage of the data analysis isolated 55 goals. *The Sims 2* was important with regard to goals: seven goals started and 13 ended as a result of its features and a further 44 started (34 ended) as a results of a combination of *The Sims 2* features, mathematics and social knowledge; for example the goal of avoiding losing money by deleting cubes was initiated by the fact that the game did not return the full money value of deleted cubes but this is only important if this financial loss (mathematics) is regarded as important (social knowledge). With regard to goals and the final house the experience of losing money by deleting cubes was clearly important to him. His first mini-problem in building the third house was *How to build the foundations of the house?* He had two goals: to make the foundation 18 x 18 cubes; to ensure that this would allow a centrally placed door. We use the term *episode* to describe a set of related (including nested) mini-problems and goals.

Thirteen of the 55 goals were related to making his house symmetrical. We report on two episodes, the first where he created and used a tool to ensure symmetry, the second where he used this tool again.

Getting the house in the middle of the plot (house 3)

Costas wanted to create a foundation of 18 x 18 cubes for his house. He also wanted to know where the middle of the foundations would be in advance (to avoid deleting cubes) so he created a two cube artefact and said: “the middle is the line between those two cubes”. He used this artefact as a central point of reference to build the foundations; he added a row of 8 cubes starting from the left of the artefact and another row of 8 cubes starting from the right, so that he could get 8+2+8 = 18 cubes overall, which was the length of the foundations that he wanted. In this way he had marked where the middle of the house was.

Making the swimming pool in line with the middle of the house (house 3)

Costas wanted the swimming pool in line with the middle of the house and said: “Since the other houses were too big when I added extra rows for the pool, I am thinking of cutting the [unwanted] cubes differently this time. I think I will draw a line in the middle like I did with the cubes [he meant the artefact] before, and then start cutting from left and right”. He counted the cubes starting from left to right until he reached the 9^{th} cube and said: “the middle is the 9^{th} and 10^{th} cube together, because it’s 18”. He then painted the artefact black, to see what to cut. He used the black cubes as an outline of what he would cut, in order to get the swimming pool in the middle of the foundation.

DISCUSSION

We first discuss the interplay of context, task ownership and social and scholastic knowledge. We then argue that Costas’ construction of and use of his two cube artefact is a mathematical abstraction in the sense of *abstraction in context*. We end with a discussion of game play, non-traditional mathematical abstractions and mediation.

The interplay of context, task ownership and knowledge

Our primary intention in writing this subsection is to provide evidence that Costas took possession of the task and that no teaching took place. A secondary intention is to view this appropriation of the task with regard to physical context and knowledge.

Although it was the researcher’s intention to ask Costas to build a house, at the end of the familiarisation session Costas requested this without the researcher asking him to do this. The researcher said that he could build a first house without financial constraints prior to building a house to a given budget. Costas readily accepted this and referred to the final house as his “dream house”. Costas abandoned (left incomplete) two houses, not just one, in order to get his dream house perfect. We regard the above as evidence that Costas appropriated the task of building his dream house to a budget as his own. The researcher introduced Costas to *The Sims 2* and occasionally suggested technical help but this, we feel, merely accelerated game play as *The Sims 2* is an internationally popular game that many children play with only virtual assistance. The researcher provided no assistance on the direction of game play or on any mathematics – and Costas was not aware, during game play, that the researcher was interested in his mathematical actions.

The game was played in his house not in a classroom. We contrast this with Monaghan (2007) in which students worked on a task that *was set* by a company director and carried out *in a classroom*. Monaghan claims that the fact that students were working in a mathematics class mattered, as students stated that they expected to do school mathematics in such a classroom. Monaghan also claimed that students did not address the company director's task but transformed the given task. So we feel that the physical context was important, that the task may have been appropriated differently in, say, a school mathematics class physical context.

As noted in the Results, most goals were initiated and terminated as a result of a combination of computer features, mathematics and social knowledge but this interrelated combination went beyond just goals and permeated Costas' work. We present an extract from Costas in the building of the third house to illustrate. NB he made an arithmetic mistake:

> "What, 350 pounds for the door [*Sims 2* glass-door] Oh … that's expensive… well… there are more expensive ones, but… there are also cheaper ones… I want them to see the pool from the living room. Well, it's three doors for the lower floor and one for the master bedroom upstairs… That's up to 1500 pounds (he sighs). I guess it's OK."

The construction and use of the two cube artefact is a mathematical abstraction

Abstraction in context, as mentioned above, posits three stages: a need for new mathematics; construction of new (vertically reorganised) mathematics; consolidation (use in further mathematical activity) of the new mathematics. We attend to all three stages in this subsection.

Costas was obsessed with the idea of making his house symmetric but this is not absurd as the practice of building houses often involves symmetric shapes. His desire to create the swimming pool in line with the middle of the house and the door in the middle of the wall reveal aspects of his understandings of middle and of symmetry. Costas could find the middle of an odd number of cubes (the middle of a five-cubed wall was the third cube) but when he encountered an even number of cubes he had to modify his strategy of finding the middle – he needed a cost efficient strategy for his dream house. When the problem of placing a central door occurred in the second house Costas deleted a column of cubes in order to have an even number of cube wall he said that the door should be put after the seventh cube (presumably dividing 14 by 2). This was a small but important expansion of knowledge about the middle. It was computer-game-play-mathematics knowledge expansion, he was dealing with cubes, not numbers: the middle of 14 is 7 but the middle of the 14 cubes was "the line between those two cubes", "the 7th and 8th cube together". This *visual middle*, the 7th and 8th cubes is a representation "…specific information is contained in representations … It is specific information that allows subjects to control for the meaning and reasonableness of their answers in problem situations" (Nunes, Schliemann & Carraher, 1993, p.147) – Costas' understanding of specific information was held in the representations, which in this case were the cubes. Costas constructed his two cube artefact, together will a counting cubes mode of using the artefact (his strategy) from these representations. Prior to using the artefact Costas did count cubes from left and right but, as noted he experienced problems. Costas' constructed artefact-strategy was a vertical reorganisation of prior knowledge. As the second episode at the end of the Results section showed, he went on to use this artefact-strategy to make the swimming pool in line with the middle of the house.

Need, construction and consolidation with regard to an artefact-strategy for cost-efficient building of a house are all present in Costas' actions. The artefact-strategy is an abstraction in the sense of Schwarz et al. (2009).

Game play, non-traditional mathematical abstractions and mediation

Costas' mathematical activity was greatly influenced by *The Sims 2*. The video game's features facilitated the interrelation of his mathematical and social knowledge and the majority of Costas' goals were initiated and terminated, at least in part, by features of *The Sims 2*. The computer game was a means by which Costas' mathematical ideas and meanings became visible to him. We do not make general claims from this case and, indeed, believe that Costas may not have formed his abstraction if the researcher present had said "it does not matter if you exceed your budget" or if Costas had not been so intent on building his dream house.

The mathematical abstraction that Costas constructed and used in the course of game play is not a part of scholastic mathematics – it is not privileged mathematical knowledge. It is, however, to us, certainly mathematics in that he engaged with relationships between objects (even if these object were, to him, cubes and not numbers). Further to this we view that Costas engaged in vertical mathematisation in the sense of Freudenthal (1991). With regard to claims that *learning will not happen without the teacher*, we feel these may be best kept to learning privileged knowledge. But Costas' abstraction would not have come about without artefact (*The Sims 2*) mediation, so we are happy with a claim that *learning will not happen without mediation by person or artefact*.

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ARE THEY EQUIVALENT?

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This paper examines the enactment of a written lesson, which centres on determining and justifying equivalence and non-equivalence of algebraic expressions. It studies the ways the main ideas in the lesson were offered to students (1) by two different teachers, and (2) in two different classes taught by the same teacher. The findings show differences between the two teachers, and between the two classes taught by the same teacher, regarding the occurrence of each idea and its connections to the other ideas, the extent to which the ideas were explicit, and the contributions of the teacher and the students to their development. The differences illustrate the complex interactions among curriculum, teachers and classrooms.

INTRODUCTION

Equivalence of algebraic expressions lies at the heart of transformational work in algebra, providing students with theoretical foundations of their manipulative work (Kieran, 2007), allowing the replacement of an algebraic expression by another when solving a problem (Nicaud, Bouhineau & Chaachoua, 2004). School algebra has commonly focused on learning procedures that preserve equivalence (e.g., simplifying expressions), without attention to conceptual understanding (Kieran, 2007). Recently, attention is given to developing students' understanding of the notion of equivalence, as reflected by carefully designed experiments that focus on theoretical aspects of equivalence of expressions (e.g., Mariotti, 2005). Within this trend, engaging students with proving equivalence and non-equivalence is emphasized (ibid). Yet, missing are classroom studies that examine this in non-interventional situations

The aim of this study is to examine the enactment of a written lesson, which centres on determining and justifying equivalence and non-equivalence of algebraic expressions. The study focuses on ways important mathematical ideas were offered to students, the extent to which they were explicit in the lessons, and the contributions of the teacher and the students to their development. Recent research suggests that different teachers enact the same curriculum materials in different ways (Manouchehri & Goodman, 2000), and that the same curriculum materials may be enacted differently in different classes taught by the same teacher (Eisenmann & Even, 2008). Thus, we chose to focus here on the ways the main ideas in the lesson were offered to students (1) by different teachers, and (2) in different classes taught by the same teacher. This study is part of the research program *Same Teacher – Different Classes* (Even, 2008) that compares teaching and learning mathematics in different classes taught by the same teacher as well as classes taught by different

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teachers, with the aim of gaining insights about the complex interactions among curriculum, teachers and classrooms.

THE WRITTEN LESSON

The lesson appears in a 7th grade curriculum program developed in Israel in the 1990s (Robinson & Taizi, 1997). The curriculum program includes many of the characteristics common nowadays in contemporary curricula. One of its main characteristics is that students are to work co-operatively in small groups for much of the class time, investigating algebraic problem situations. Following small group work, the curriculum materials suggest a structured whole class discussion aimed at advancing students' mathematical understanding and conceptual knowledge. The curriculum materials include suggestions on enactment, including detailed plans for 45-minute lessons.

The lesson "Are they equivalent?" which is the focus of this paper, is the 6th lesson in the written materials. Prior to this lesson, equivalent expressions were introduced as representing "the same story", e.g., the number of matches needed to construct a train of *r* wagons. The use of properties of real numbers (e.g., the distributive property) was mentioned briefly as a tool for moving from one expression to an equivalent one, but it was not yet presented explicitly as a tool for proving the equivalency of two given expressions. Three ideas are explicit in this lesson:

Idea 1: Substitution that results in different values proves that two expressions are not equivalent (could be regarded as a specific case of refutation by a counter example as mathematically valid).

Idea 2: Substitution cannot be used to prove that two given algebraic expressions are equivalent (a specific case of supportive examples for a universal statement as mathematically invalid).

Idea 3. This idea addresses the problem that emerges from idea 2: the use of properties in the manipulative processes is a mathematically valid method for proving that two expressions are equivalent.

These ideas are known as difficult for students (e.g., Booth, 1989; Jahnke, 2008).

The lesson is planned to start with small group work aiming at an initial construction of Ideas 1 and 2. Students are given several pairs of expressions; some equivalent and some not. They are asked to substitute in them different numbers and to cross out pairs of expressions that are not equivalent. After each substitution they are asked whether they can tell for certain that the remaining pairs of expressions are equivalent. Finally, students are instructed to write pairs of expressions, so that for each number substituted, they will get the same result.

Then small group work continues, asking students to write equivalent expressions for given expressions. The aim is to direct students' attention to the use of properties in relation to equivalence of algebraic expressions, which is relevant to idea 3.

The whole class work returns to idea1, and moves, through idea 2, to idea 3, aiming at consolidating these ideas, by discussing questions, such as: How can one determine that expressions are not equivalent? that expressions are equivalent? By substituting numbers? If so, how many numbers are sufficient to substitute? If not, what method is suitable? Finally, the teacher guide recommends that the teacher demonstrate the use of properties for checking equivalence, and together with the students implement this method on several pairs of expressions in order to check their equivalency.

Ideas 1, 2, and 3 are connected to three other ideas, none of which appears explicitly in the first six lessons in the written materials:

<u>Idea 4</u> justifies Idea 2: There may exist a number that was not substituted yet, but its substitution in the two given expressions would result in different values, thus showing non-equivalence.

<u>Idea 5</u> justifies Idea 3: The use of properties of real numbers in the manipulative processes guarantees that any substitution in two expressions will result in the same value, thus showing equivalence.

<u>Idea 6</u> is the underpinning for Ideas 1, 2, and 3, as well as for Ideas 4 and 5. It defines equivalent algebraic expressions: Two algebraic expressions are equivalent if the substitution of any number in the two expressions results in the same value.

METHODOLOGY

The data source includes video and audio tapes of the enactment of the written lesson in four classes, each from a different school. Sarah taught two of the classes, S1 and S2; Rebecca, the other two classes, R1 and R2. The talk during the entire class work was transcribed. The transcripts were segmented according to focus on the three ideas, yielding 3-4 more or less chronological parts in each class. Next, the collective discourse in the classroom was analyzed by examining the contributions of the teacher and the students to the development of the ideas in each enacted lesson. We compared how the teachers structured and handled the ideas in each lesson in different classes of the same teacher and in the classes of the two teachers.

THE ENACTED LESSONS

In both Sarah's and Rebecca's classes, the above three activities were enacted. However, there were differences in the ways the ideas in the lesson were offered to students between the two teachers, and between the two classes of the same teacher.

Sarah's classes S1 & S2

The treatment of the ideas in both Sarah's classes was similar, with one exception at the end. In line with the written curriculum materials, the whole class work in both classes included an overt treatment of Idea 1. However, contrary to the recommendations in the written materials, both classes performed substitutions in pairs of algebraic expressions from the first activity since the teacher requested them to do so, and not as a way of addressing the problem of determining non-equivalency.

When the substitutions resulted in different values, the classes concluded that the two expressions were not equivalent. In both classes it was Sarah who eventually presented Idea 1 explicitly, attending to the specific context of non-equivalence of expressions, with no reference to the general idea of refutation by using a counter example as mathematically valid.

After working on non-equivalence, the two classes proceeded to work on equivalence of algebraic expressions. In both classes, Sarah presented Idea 2, that substitution cannot be used to prove that two given algebraic expressions are equivalent. She explicitly incorporated in the presentation of this idea its underlying justification that there might exist a number that was not yet substituted, but its substitution in the two given expressions would result in different values (idea 4). For example:

> We saw that with substitution, it is always possible that there is a number that I will substitute, and it will not fit. We can substitute ten numbers that would fit, and suddenly we will substitute one number that will not fit, and then the expressions are not equivalent… We have to find some way other than substitution, which will help us determine whether expressions are equivalent.

Sarah presented Idea 2 as a motivation for finding a method to show equivalence, and immediately proceeded to work on using properties in the manipulative processes as a means to prove equivalence (Idea 3). Led by Sarah, S1 and S2 searched for properties that show that the expressions they produced were equivalent. Sarah then stated that the use of properties is the way to prove equivalence, not substitution.

When introducing Idea 3 in S2, Sarah explicitly connected it with Ideas 5 and 6. However, no such connections were made then in S1. Only later on, in her concluding remarks in S1, when summarizing both ways of proving equivalence and non-equivalence of expressions, Sarah explicitly proposed Idea 6.

Rebecca's classes R1 & R2

The treatment of the ideas in Rebecca's classes also had similar features, but more differences were found between the two classes. Rebecca's treatment of Idea 1 in both her classes was similar to that of Sarah's: The students performed substitutions, and when resulted in different values, the class concluded that the expressions were not equivalent. Yet, when a student asked for the meaning of equivalent expressions at the beginning of this activity in R1, the definition of equivalent expressions as expressions that the substitution of any number in them results in the same value (Idea 6) was introduced explicitly. As in Sarah's classes, it was the teacher who eventually presented Idea 1 explicitly, attending only to the specific context of non-equivalence of expressions, with no reference to the general idea of refutation by using a counter example as mathematically valid.

After working on non-equivalence, the classes proceeded to work on equivalence of algebraic expressions. Idea 2, that that supportive examples (i.e., substitution) cannot be used to prove that two given algebraic expressions are equivalent was dealt with differently from Sarah's classes. In general, in both classes Rebecca pressed on finding a method that works, rather than evaluating the method of substitution, which does not work. However, the issue of substitution continued to be raised. In R1, following students' proposal, the initial focus was on rejecting substitution because of the inability to perform substitution of all required (infinite number of) numbers. Idea 2 was not dealt with in R1. Rather, it seemed to be taken as shared. Repeatedly, after substituting numbers in pairs of expressions and receiving the same value, the class concluded that the pairs appeared to be equivalent but that it was impossible to know for certain. For example,

T: So, does it mean that they are equivalent?

S: Yes. Ah, no, not necessarily.

T: Why? Do you have a counter example?

S: We don't know that they are equivalent.

Still, there was no explicit rejection of substitution for proving equivalence. Instead, Rebecca changed the focus of the activity to looking for a connection between the two algebraic expressions in each pair, as a transitional move towards Idea 3.

In contrast with R1, R2 embraced the idea that substitution is a valid means for determining equivalence of algebraic expressions. Unlike R1, where after several substitutions that resulted in the same value, students claimed that they still could not conclude that the two expressions were equivalent, in similar situations R2 students claimed that the expressions were equivalent because all the numbers they substituted resulted in identical numerical answers. This happened even after Rebecca offered idea 4, that there may be a number, which was not yet substituted, but its substitution in the two given expressions would result in different values. For example,

T: So, what do you say, what should I do, check all the numbers; maybe there is a number that won't fit here?... Or will it always fit?

S: Always.

…

T: Why are they equivalent? Why do I say that these are equivalent…?

S: Because we checked at least thirty.

…

T: Because you checked, but we said that maybe there is one number that you did not check.

S: But we checked almost all the [inaudible].

Eventually, Rebecca changed the focus of the activity to looking for algebraic expressions that are equivalent to given expressions, aiming at Idea 3. Thus, unlike Sarah, who used the brief mentioning of Idea 2 (and 4) as a motivational transition from Idea 1 to Idea 3, in R2, Rebecca did not motivate the search for a method different from substitution.

R1 started to work on Idea 3 by searching for connections between pairs of expressions that were not crossed out as non-equivalence after several substitutions. The class quickly embraced the discovery that by using properties, it was possible to move from one expression to another and show equivalence. Rebecca then explicitly introduced Idea 3. Like in S1, no connections were made to Ideas 5 and 6.

R2 had a different starting point for treating Idea 3 because the class was confident that based on the substitutions they performed they could infer that the remaining pairs of expressions were equivalent. Rebecca then asked the class to find new expressions that would be equivalent to the given ones. Eventually, R2 embraced the idea that equivalence can be determined by manipulating the form of expressions, using properties. In R2, too, no connections were made with Ideas 5 and 6. Moreover, Idea 6 was not proposed at all.

Figure 1 depicts the teaching sequences of the ideas as offered during the whole class work, in the written materials, as well as in the four classes. The figure clearly demonstrates that only Sarah explicitly proposed the sequence of the three ideas (1, 2, and 3) that were explicit in the written lesson. Rebecca explicitly proposed only Ideas 1 and 3. Moreover, connections between these three ideas and the other three ideas (4, 5, and 6), which did not appear explicitly in the written lesson, were made only in Sarah's classes: Idea 2 was connected to its underlying justification, Idea 4, in both of Sarah's classes, whereas Idea 3 was connected to its underlying support, Ideas 5 and 6, in S2 only. Rebecca offered idea 4 in R2 with no explicit connection to Idea 2, and Idea 6 was offered in S1 (at the end of the lesson) and in R2 (at the beginning of the lesson), with no explicit connections to other ideas.

FINAL REMARKS

Sarah and Rebecca taught the written lesson "Are they equivalent?" using the same written materials, which included a detailed lesson plan. Thus, it is not surprising that the mathematical problems enacted in all four classes were similar. However, the ways the main ideas in the lesson were offered to students differed to some degree from what was recommended in the written materials. There were also differences between the two teachers, and between the two classes of the same teacher. One of the main differences is related to offering Idea 2. This idea is central in the written materials. However, Sarah only briefly mentioned it in her classes, just as a transition to Idea 3. In R1 this idea was taken as shared, never made explicit. It was not made explicit in R2 too, which strongly embraced the opposite idea. Another central idea in the written materials is Idea 3. The way that the written materials deal with Idea 3, without making Ideas 5 and 6 explicit, seemed to make teaching it a challenge. Eventually, each teacher handled this idea somewhat differently in each of her two classes.

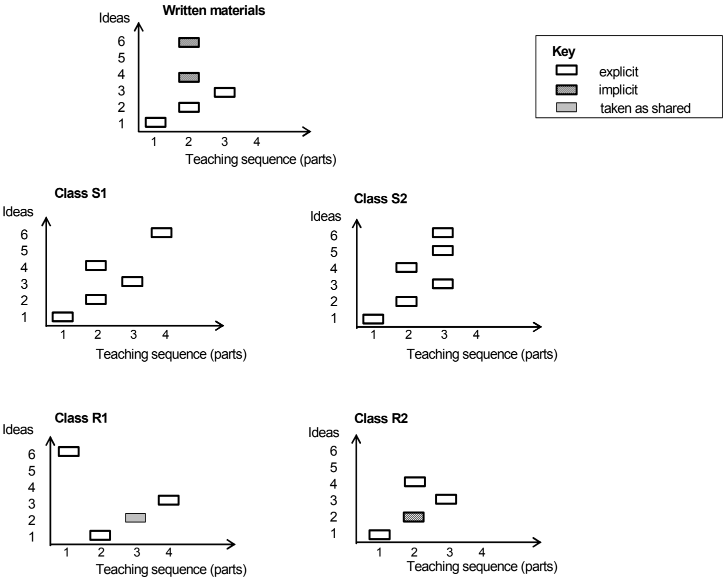

Figure 1: Teaching sequences of the ideas, as offered in the whole class work, in the written materials, as well as in the classes

These differences seem to be related to differences in teaching approaches. Sarah tended to make clear presentations of important ideas. Rebecca hardly made presentations, but instead, attempted to probe students, expecting them to explicate these ideas. Thus, some ideas were never made explicit, in one class more than the other, because of differences in students' mathematical behaviour and performance.

These findings illustrate the complexity of the interactions among teachers, curriculum and classrooms (Even, 2008). Rebecca faced serious challenges in her attempts to make students genuine participants in the construction of mathematical ideas, as was recommended in the written materials – more so in one of her classes – challenges that lie at the meeting point of the specific teacher, specific curriculum and specific class. Sarah, who chose to make clear presentations of the mathematical ideas, faced different challenges, even though she used the same materials.

The mere fact that different teachers offer mathematics to learners in different ways, even when using the same written materials, is not entirely surprising, and has been documented by empirical research (e.g., Manouchehri & Goodman, 2000). Nonetheless, the nature of the differences is important because what people know is defined by ways of learning, teaching, and classroom interactions, as documented by Boaler (1997). Consequently, Sara'h and Rebecca's students were offered somewhat different ideas that are central to conceptualizing equivalence and to proving in algebra and in mathematics in general. Furthermore, when instead of focusing solely on the comparison between teachers, different classes taught by the same teacher were also compared, important information was revealed about the interactions among curriculum, teachers and classrooms**.**

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DIFFERENT PERCEPTIONS OF INVARIANTS AND GENERALITY OF PROOF IN DYNAMIC GEOMETRY

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Literature on research in dynamic geometry environments (DGEs) addresses the importance of the perception of invariants in open problem investigations. In this paper, in order to analyse students' processes during exploration, conjecturing, and proving in DGEs, we introduce a framework that distinguishes different types of invariants. Students' interpretation of these invariants seems to be strongly rooted in the processes of their discovery more than in the generality of theorems and proofs.

INTRODUCTION

The importance to mathematicians of visualization is widely recognized, and recently the appearance of Dynamic Geometry Environments (DGEs) has led educators to reconsider the issue of imagery in mathematics education (for example, Goldenberg, 1995; Presmeg, 2006). Certainly DGEs have revolutionized the approach to understanding the complex relationship between images and concepts in Geometry education. In particular, DGEs have been expected to enhance geometrical reasoning in problem solving, promoting visual exploration and discovery (Dreyfus, 1993; Goldenberg, 1995; Goldenberg & Cuoco, 1998). Laborde, speaking of a specific DGE, Cabri géomètre, states (Laborde, 1993, p. 56):

> "The nature of the *graphical experiment* is entirely new because it entails movement. The movement produced by the drag mode is the way of externalising the set of relations defining a figure. The novelty here is that *the variability inherent in a figure is expressed in graphical means of representation* and not only in language. A *further dimension is added to the graphical space as a medium of geometry:* the movement."

However, the complexity between spatial-graphical and geometrical aspects that is intrinsic of geometrical reasoning, cannot magically dissolve. If one hand DGEs seem to foster the development of a link between spatial-graphical and geometrical aspects; on the other hand, they do not seem to foster achievement of the theoretical control over the relationship between purely spatial-graphical properties and theoretical properties of the figures represented (Duval, 1993; Laborde, 1993).

The intrinsic complexity of geometrical reasoning is nicely expressed by the notion of *figural concepts* introduced by Fischbein (1993). Geometrical figures are mental entities that simultaneously possess both conceptual properties (general, abstract relations, deducible in the Euclidean theory) and figural properties (shape, position, magnitude). In solving geometrical tasks, the interaction between figural and conceptual components of external representations (drawings) may explain productive reasoning leading to correct solutions, but it may also explain mistakes

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 89-96. Thessaloniki, Greece: PME.

and difficulties that could be due to an incomplete fusion between the two components (see, for example Mariotti, 1993; Mariotti & Fischbein, 1997). In the case of tasks to be accomplished in a DGE, figures, that is drawings produced through a sequence of commands chosen by the user, are drawings with their own intrinsic logic. The geometric properties defined by the commands control the appearance of invariants under dragging, so the intrinsic logical dependency among the elements of the dynamic figure may affect the interplay between the figural and the conceptual components involved in the solution of a task.

This study is part of a greater project in which we analyse some cognitive processes that students activate when solving open problems requiring the formulation of a conjecture in a DGE. The study involves students in grades 9, 10, and 11 who have been using a DGE in the classroom for at least one year. During clinical interviews the students work in pairs or singularly while being audio and video recorded. The paper aims to propose a classification of different types of invariants that seems to be powerful for analysing and explaining some difficulties encountered by the students.

GEOMETRICAL INVARIANTS

In the literature on research in DGEs, the terms “invariants,” “geometrical invariants,” “invariant properties” of a figure have been used to refer to certain properties (e.g. Yerushalmy et al., 1993; Goldenberg et al., 1998; Hadas et al., 2000) that are maintained when some transformations on the figure are performed. As Laborde describes it:

> “A geometric property is an invariant satisfied by a variable object as soon as this object varies in a set of objects satisfying some common conditions.” (Laborde, 2005, p. 22).

In the work we have so far accomplished for the study, we have noticed recurring student-behaviours that are not consistent with a correct mathematical interpretation of the situation. As noticed by other researchers (for ex. Laborde, 2005), we also are finding that students encounter difficulties in dealing with dependency relationships of points and interpreting invariants of a figure. In order to analyze students’ behaviours, we find it useful to distinguish different notions of invariance. In order to define them, in a more precise manner, we first focus on the dragging of points, and make the distinction between base-points and constructed points. A base point is a point constructed freely, or semi-freely (for example, a point chosen on a circumference, which can therefore only be dragged along the circumference itself) on the screen, upon which other objects of the construction depend. A base point is also a free point, in that it can be dragged anywhere on the screen (or freely along the curve it is linked to). On the other hand, dependent points are points built as the intersection of constructed objects, and consequently they cannot be dragged directly.

We call *construction-invariant* a geometrical property of the figure which is true for any choice of the base-points. In Cabri an invariant of the construction is a property that is maintained for dragging of any base-point (which is also free) of the figure. It is useful to consider the set of all construction-invariants, which we will call *I*.

We may also consider a geometrical property that is true for any choice of one particular base-point of the construction, while the other base-points are fixed. In fact, only one point at a time can be dragged, so students actually perceive properties as being invariant for the dragging of the specific point they are dealing with at the moment. In this case we have a *point-invariant*. If the particular base-point considered is P, we will call such invariant a *P-invariant*. It is useful to consider the set of all P-invariants, which we will call I_P. It is clear that I is contained in I_P for every base-point P of the construction.

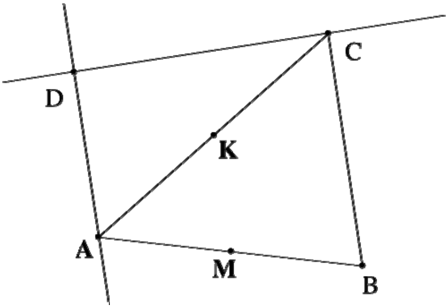

Fig 1: A, M, K are base points and free in this construction; "ABCD is a right trapezoid" is a construction-invariant; "the length of BC" is an A-invariant.

For example consider the following construction. Let A, M, K be three base-points, and construct B as the symmetric point of A with respect to M, and construct C as the symmetric point of A with respect to K. Construct the parallel line l to BC through A, and the perpendicular line r to l through C. Let D be the intersection of l and r.

It is easy to prove that ABCD is a right trapezoid for any choice of A, M, K. By construction DA lies on l, which is parallel to BC, and CD lies on r, which is perpendicular to l. Consequently, r is also perpendicular to BC (for a known theorem of Euclidean geometry). Therefore the fact that ABCD is a right trapezoid is a *construction-invariant*. Moreover, the fact that BC is twice MK and parallel to it is also a *construction-invariant*. This is because the triangles AMK and ABC are similar with ratio of proportionality 1:2 (since AC is twice AK, and AB twice AM). However, the fact that the length of BC is constant is not a *construction-invariant* because, for example, choosing a different M (or dragging M) leads to a variation of it. Instead, the length of BC is an *A-invariant*, because for any choice of A (or movement of A through dragging), MK is fixed and therefore the direction and length of BC are constant (even if the points B and C change as a consequence of the new choice of A).

We will now show how the notions of construction-invariants and P-invariants, together with that of figural concept, can be efficiently used in our analysis. In the following we consider the representative case of a pair of students engaged in the solution of an open problem.

ANALYSIS OF A TRANSCRIPT: THE CASE OF GIULIO AND FEDERICO

Below are some excerpts from the transcript of an interview of two students, Giulio and Federico. The task is based on the configuration of the example above. After accomplishing the construction, the students are asked the following: "As points A, M, K vary, formulate conjectures on the types of quadrilaterals that ABCD can become, trying to describe *all* the ways in which it is possible to obtain a certain quadrilateral." Giulio and Federico are in the second year at an Italian high school (grade 10) and have used dynamic geometry in class during their previous year.

The two students have been looking at the screen while Federico has been dragging point A randomly. In the excerpts below, I indicates the researcher.

Excerpt 1

1	F:	Good, so we can say that the quadrilateral, as A varies, uh, we always have a trapezoid.
2	G:	…a right trapezoid.
3	I:	It's a trapezoid.
4	G:	a right one.
5	F:	a right one! Yes, it is a right trapezoid.
6	I:	Ok, it's even a right one.
7	F:	So, dragging A…it's a right triangle…yes.

While Federico is dragging point A, the two students observe that ABCD is "always" a right trapezoid. Therefore, we can say that they conjecture that the property "ABCD is a right trapezoid" is an *A-invariant*. This also emerges from Federico's words every time he formulates the conjecture ("as A varies" (1), "dragging A" (7)).

In the following excerpt the students are involved in the production of the proof of the conjecture.

Excerpt 2

13	F:	Yes, let's prove it. Let's prove this one [conjecture] right away. So, by hypothesis we have that CD is perpendicular to AD.
[…]		
16	G:	Yes, and we also have that by hypothesis AD, since it lies on line *l*, is parallel to CB.
17	F:	Yes…
18	G:	uh, yes, because it's written [referring to the text of the problem].
19	F:	Yes.
20	G:	Construct line *l* parallel [G rereads the text to F] to BC.
21	F:	Good, so…
22	G:	So we know that …[speaking together]
23	F:	Well, we have that AD e BC are parallel *exactly* by hypotheses.
24	G:	So ABCD is a trapezoid….immediately.
25	F:	Yes and ABCD is a trapezoid, exactly! Then we already said this, AD [indicating with the pointer] is perpendicular to CD, but AD is parallel to BC, so AD is perpendicular to DC, but BC is also perpendicular to DC.

26	I:	Yes…
27	F:	So, it's a theorem!
28	I:	you don't need to prove the theorem [smiles]
29	F:	uh, we did it [in class].
30	I:	What is it that the theorem says?
31	F:	Uh, that if there are two parallel lines…
32	I:	Yes…
33	F:	uh, if we draw a perpendicular to one of them…
34	G:	uh, there are alternate interior angles…in the end
35	F:	Exactly.
36	G:	Because these two have to be supplementary.
37	I:	there we go, supplementary. I agree.
38	G:	Yes, therefore…
39	F:	uh, it has to be that if one is 90, the other, too, has to obviously be 90. Ah, so then it is proved, so it is a right trapezoid.
40	I:	Alright.

Giulio and Federico prove their conjecture correctly without making use of the dragging tool and, in fact, while the proof is produced, the figure remains static on the screen. Moreover, no reference to a particular choice of the base-points appears in the proof and its generality is assured by the theory of Euclidean Geometry. Therefore, the students have proved that "for any base points A, M, K ABCD is a right trapezoid". According to the notions of invariants proposed above, we can say that Giulio and Federico proved that the property "ABCD is a right trapezoid" is a *construction-invariant*. After the proof one would expect that the *A-invariant* should have changed its status for the students, becoming a *construction invariant*. However, from a cognitive point of view, the generality of the proof does not seem to have such effect on Federico's interpretation of the invariants, as the following excerpt shows.

Excerpt 3

41	F:	Now let's try something else…Let's do free dragging…
42	I:	Try to see if it can be something else…like other quadrilaterals.
43	F:	Still dragging A?
44	I:	Well, it says as A, M, K vary. So you can drag…
45	F:	So A, M, K [looking at them on the screen]…also M and K?
46	I:	Also M and K.
47	F:	Let's try…dragging M [F drags M freely] they all vary…yes, all the ...the sides…
48	I:	uh huh…
49	F:	Let's see…what can we say?…[he continues to drag M]…well, it seems to be again a trapezoid.

Federico proposes the same conjecture they have already proved as if it were a new property. In fact, even if this conjecture is now a mathematically proved proposition, we claim that for Federico this is actually a new conjecture (49: "It seems to be again a trapezoid"): according to our framework, he is saying that the property "ABCD is a trapezoid" is an *M-invariant*. We know that every *construction-invariant* is a *point-invariant* but, for Federico, this property does not seem to be a *construction-invariant*, even if a general proof has been constructed. For him, the property is only an *A-invariant* and therefore it might not be a *P-invariant* for other points P. For this reason the fact that it is an *M-invariant* is surprising to him. In the last part, Giulio's different interpretation of the invariant appears:

Excerpt 4

50	G:	Well, it is always *anyway* a trapezoid.
51	F:	Always a trapezoid, exactly.
52	G:	because by hypothesis, basically.
53	F:	It seems to always be…

Giulio is convinced that "ABCD is a trapezoid" is a *construction-invariant* ("it is always anyway a trapezoid"), therefore it is obvious for him that it is an *M-invariant*.

During the whole interaction, between Federico and Giulio there seems to be an underlying non-complete understanding that can be explained by the different way in which the two students treat the property "the quadrilateral is a right trapezoid." Federico treats it as an *A-invariant*, while Giulio treats it a *construction-invariant*. In this regard, we can notice the expressions that the two students use to talk about the conjecture: while Federico specifies that the quadrilateral is a trapezoid "as A varies" or "dragging A", Giulio never associates the claim of the fact with dragging point A (or any other), and in 50 he probably tries to underline his belief by adding "always *anyway*" [in Italian: "sempre *comunque*"] to his statement. Giulio seems to be convinced of this fact from the beginning of this sequence (after the initial dragging of point A that Federico does), even before he and Federico prove it.

Moreover, the interaction between the two students can also be explained well through the different interpretation that Federico and Giulio give of the observed invariant: their different interpretations of invariants lead to the astonishing (to us) exclamation of Federico ("It seems to be again a trapezoid") and to the, almost irritated, answers of Giulio ("It is always *anyway* [sempre *comunque*] a trapezoid").

CONCLUSIONS

As described elsewhere (Laborde 1998, p. 190) in the case of construction-tasks, the dynamic possibilities of Cabri may introduce a new level of complexity due to the possible variations of the given elements of the problem and their interrelations. Of course this consideration applies to other explorations. The previous examples show how such complexity emerges in the solution of an open problem and may affect the heuristic phase. The classification of invariants we have proposed seems to be a useful tool to describe and explain such complexity. On one hand exploring by dragging may lead to noticing an invariant, but the geometric interpretation of the invariant may be strongly linked to the dragging process that lead to its discovery, thus it will be conceived as a point-invariant. On the other hand, the production of a proof may have different effects with respect to the generality of the geometric property related to the observed invariant. While Giulio seems to have grasped the generality of the theorem (the property for him becomes a construction- invariant), Federico does not. The link between the dragging process and the interpretation of the invariant is so strong that not even the production of a correct mathematical proof can induce Federico to change the status of the observed invariant. It may not be a chance that Giulio is the one who shows awareness that the invariant is independent of the dragging of any point, because he was not doing the dragging. It might be that the difference in the perception of the figure could depend on the type of control that each student has over the figure itself. Further research is needed to explore this idea.

Thus it seems that the basic claim that "spatial location and muscular movement in space are linked to variance and invariance, which lies at the heart of awareness of generality" (Mason & Heal, 1995, p. 298), has to be refined. The frame of Figural Concepts helps us to articulate the complexity of this process. Consider the geometrical interpretation of the observed invariant: the figural aspect is linked to the invariant spatial properties that are perceived under the constrains of the dragging experience. Thus it is not surprising to find out how persistent the effect of the dragging action might be. It is worth noting that the contribution of perception has to be considered in its complexity, "spatial location and muscular movement".

In line with findings of previous studies (for instance, Fischbein, 1982; Balacheff, 1988; Chazan, 1993) in a traditional paper and pencil setting, our students encounter difficulties in capturing the "generic" in a proof. According to our interpretation, once the proof is produced a new relationship between the figural and the conceptual component has to be elaborated. In the interpretation of the invariant, the point dependence of the property has to be overcome in order to achieve the generality stated by the proof. It may happen that such new elaboration is not accomplished and the generality of the property is not recognized. In terms of figural concepts there is a break between the figural and the conceptual aspect that needs to be recomposed.

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LEARNERS VOICES ON ASSESSMENT FEEDBACK

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In this article we report on learners' understanding of teacher (educator) assessment feedback. The study was conducted with five Grade 9 mathematics learners. Data was generated from interviews, seven journal entries and video taped classroom observations. The learners identified several purposes of assessment feedback that teacher should provide and displayed strong feelings about the manner in which the feedback should be communicated to them

INTRODUCTION

The purpose of this article is to describe and share our findings about a classroom based study which explored 5 learners' meaning of educator assessment feedback in a Grade 9 mathematics classroom in KwaZulu- Natal in South Africa.

In 1994, educational reform in South Africa was heralded by the introduction of the new curriculum framework, Curriculum 2005 (C2005) with its Outcomes Based approach. This approach aimed "to equip all students with the knowledge, competence and orientation needed for success" after schooling (Pretorius, 1998). As part of the reform process, it became necessary for assessment techniques to be reviewed. In analyzing the relevant curriculum documents (DoE, 2002a; DoE, 2002b; DoE, 2005) it is clear that the curriculum policies endorse that the assessment of learners' performance should enhance individual growth and development and facilitate learning. Mathematics assessment is a mechanism for the construction of learners' mathematics understanding. Educator feedback to learners is a crucial component of this mechanism, and it is on this aspect of assessment that the study is focused.

FOCUS AND RELATED LITERATURE

Magieambal Naidoo, one of the authors, was concerned that the issue of educator feedback did not feature strongly in the policies. Her concerns about the effectiveness of educator feedback, stemmed from observed instances of learners repeating their mistakes in several tasks; even though feedback on how they could remediate their mistakes was provided. Her experience is supported by Sadler's (1989) observations: "…when feedback is given, it is often ineffective as an agent for improvement. Students seem to show the same weaknesses again and again", (p.73). The authors also speculated about why learners keep making the same error despite educators'

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best efforts. We were adamant that any research that focuses on assessment feedback needs to explore learners' understanding of it. The above comments lead us to ask, what do learners consider as helpful? It is for this reason that the research question of the project, that is reported in this article is: What are learners' views about educator assessment feedback?

It is necessary to clarify our understanding of the term "feedback". The Wordsworth Concise Dictionary defines feedback as a "response or reaction providing useful information or guidelines for further development" (1998, p.354) which we think encompasses the main ideas in this study in a succinct manner.

Wiggins (1998) explained the purpose of feedback as addressing discrepancies in knowledge that learners display, namely the difference between their current status and the desired end and "providing feedback in the middle of an assessment is sometimes the only way to find out how much a student knows" (p.60) in terms of the 'desired end'. Wiggins' views are similar to that of Vygotsky's (1978) Theory of Cognitive Development, where the gap between actual development and potential development, that is, between what a child can do unaided by an adult and what he/she can do "under adult guidance" (p.86) is termed the zone of proximal development (ZPD). Through feedback processes, the "hints and prompts that help children during assessment could form the basis" (Slavin, 2003, p.48) for children to work in their ZPD. This intervention (feedback) assists learners in crossing their ZPD (Clarke, 2000).

In a small-scale quantitative research project that looked at students' responses to feedback, Young (2000) drew attention to the emotional impact of receiving marks. His research focused on how students react to feedback and he found that there are "psychologically vulnerable students" in all classes (Young, 2000, p.409). From Young's study (2000), there emerged a relationship between how students responded to feedback received and their self-esteem (judgement of self-worth). Students with a high self-esteem displayed acceptance of feedback received. Those students with low self-esteem were vulnerable to unfavourable judgements.

METHODOLOGY

The research reported here formed part of a larger study for Naidoo's (2007) Masters degree in Education. This study was conducted at a co-educational secondary school in KwaZulu-Natal with an enrolment of 1107 learners. The five Grade 9 participants in the study were Riva(male), Cleme (female), Edi(female), Chri (female) and Mabu (male). Pseudonyms are used to protect the identity of these learners. The learners' socio-economic backgrounds range from lower to upper middle class.

A naturalistic inquiry was used as it has an emphasis on interpretive dimensions where the goal of the researcher is to understand reality (Cohen, Manion & Morrison, 2000). A qualitative case study approach provided the opportunity to concentrate on a specific instance or situation (Cohen et al., 2000), namely the learners' meaning of educator feedback. Seven journal entries of each participant were analysed. Through journal writing, students "are encouraged to think freely in writing … in their own words" (Mett, 1987, p.534). An additional source of data was from a group interview conducted with the 5 learners because we were mindful of the words by Cohen et al. (2000) that "group interviews of children might also be less intimidating for them than individual interviews" (p.287).

RESULTS

Riva

Riva is a fifteen year old male learner whose home language is English and has an above average mathematical competence. He stated that "feedback refers to a teachers report back on any kind of work that is done". "Report back" suggests that he sees it as the teacher's responsibility to respond to the learner's work. Riva also made a comment about the importance of the teachers' attitude. He said that he prefers feedback that encourages learners and not negative feedback that belittles learners. In his words: "I like feedback that is encouraging and motivating…not [to] ridicule or mock you"

Edi

Edi is a 15 year old female learner, whose home language is isiZulu and has a very good command of English. Her mathematics ability ranges from average to above average. She defined feedback as when an educator has given learners a task then he/she "will mark the work and will point the mistakes" to the learners. Edi's definition of educator feedback focused on the identifying of mistakes. Edi also noted that through feedback once the educator identifies learners' mistakes, the educator explains "what [the learner should] look out for when given a task". To her, through feedback, learners are conscience of not repeating their mistakes. She explained that: "feedback helps the student to recognize their mistakes and where they tend to go wrong and they rectify the mistake the next time they do a task". So Edi sees the teacher as identifying the mistake, so that the learner can then correct her mistake. The idea of feedback identifying mistakes was extended to the educator "giving advice or guidance on how you're doing … she'll give [advice or guidance] on how to correct your work". A comment that is of concern was that the teacher should not embarrass the learner when offering verbal feedback. She commented in the interview that "it is good …when the teachers won't embarrass you…" by making comments such as "you're so stupid". She remarked, "if the teacher explains the maths…its okay…it must not be personal"

Cleme

Cleme is a 15 year old male learner whose home language is English and he was considered to be a high achiever in mathematics. In his first journal entry his perception of feedback reads; "feedback is saying something from the teacher's point of view". The phrase "… saying something …" was elaborated upon when he added that feedback "… is also a good way to communicate". The reason that Cleme provided for this was for "people [to] know what the teacher really wants". Feedback was portrayed as means of communication of the teacher's view in order to learn what the educator requires. This perception was further elaborated on during the group interview. Although he maintained the notion of feedback being a form of communication, he accepted different forms of how feedback was being aligned to communication because "rings, crosses … [are] a form of communication with the student". Cleme places great importance on the teacher's point of view and the need for the learner to know what the teacher is looking for so that the learner can align himself with the teacher's expectations. Cleme seemed to be concerned about shy learners who would be afraid to ask the teacher for help. He said that there were some learners "that are afraid to ask the teacher [for] help".

Chri

Chri is a 14 year old female learner whose home language is English and could be viewed as an average achiever in mathematics assessments. In her first journal entry Chri recorded her definition of feedback as a teacher's task of informing learners of their performance. She wrote, feedback "is a task that the teacher does to inform us of how we performed". She stated two purposes of feedback as "to ensure that we do good and understand" and "it also tells you where you went wrong…what to do to get better…ways you can improve your working ability." According to Chri, the first purpose of feedback is to ascertain that learners understand the work and do well. The second is to reveal to learners errors so that they can identify where they went wrong and hence establish ways to improve their ability. During the group interview, she maintained "it [feedback] makes you a better person … meaning in your schoolwork". The importance of the individual in learning, is clearly illustrated in this comment. This importance in improving learning was reiterated in her journal entry and during the group interview.

Chri also saw feedback as providing a challenge. This was evident from her interview when she stated, "… feedback is also very challenging", and her journal entry where she wrote "it is very challenging". The explanation that she gave for the challenging nature of feedback was linked to "when a teacher comes to you…say that you need to do better; you need to improve". She perceives the challenge posed through feedback as inciting competitiveness amongst learners. "…You want to compete with your friends", "…I like competing with my friends. It is quite fun…" Chri was also concerned about shy learners being sidelined because they were scared to ask the teacher questions. "People that are shy … do not like to ask the teacher questions about maths."

Mabu

Mabu is a 16 year old male learner, whose home language is isiZulu. He is reserved and works mostly by himself. His mathematics competence is average. Mabu understood feedback as checking and reporting on work as evident in the following "The work…is checked and a report that's feedback is given". Mabu's journal entry also stated "when you don't understand something…the teacher explained it back to you she/he is doing the feedback to you". From the former statement, it is apparent that he perceives feedback to serve as remedial teaching. In addition, during the group interview he stated: "if…there's something…I don't understand…she will point it to you. She's giving you feedback".

Mabu articulated a concern about sometimes getting lost because of the language used by the teacher. In his interview he stated, "…maybe the teacher can say the higher language…the mathematical language that I won't understand". He further stated that although he "can understand …even mathematical words" in its written form, sometimes "I get lost I…in mams language sometime".

ANALYSIS

The results of this study indicate that learners have developed meaningful perceptions of the concept of 'educator feedback'. There are three points, about the learners' perceptions about the role of the teacher in providing feedback, which we wish to make in this discussion. Firstly the learners' understanding of educator feedback conveys a broader perception of the meaning of the term 'educator feedback' and they show insight into the value and purposes of feedback. Interestingly, none of these learners described educator feedback as the marking of answers right or wrong. Riva stated that "feedback refers to a teacher's report back on any kind of work". Mabu's definition of feedback was boarder than Riva's. Mabu perceived educator feedback as more than a mere report. To him educator feedback is when "work [is] checked and a report … [is] given". The word checked suggests that the work should undergo some scrutiny or inspection before being reported on.

Edi's definition of educator feedback related to when the teacher "will mark and will point … mistakes". Her definition of educator feedback highlighted the role of feedback as diagnostic. She further explained that the diagnosis of errors through feedback result in learners "rectify[ing] the mistake the next time they do a task". Apart from diagnosis of errors, Edi expected educator feedback to provide "advice and guidance on how … to correct your work". Chri also perceived feedback as diagnosis of errors when she wrote; "it tells you where you went wrong". Like Edi, Chri also anticipated feedback to "tell you where you went wrong" and to suggest "what to do to get better … improve your working ability". Furthermore, Chri identified the effect of feedback that diagnoses errors and recognizes room for improvement. According to her, this is when feedback presents a challenge to the learner to improve. She also believes that this challenge urges learners to be competitive. The diagnostic purpose of feedback was further reiterated by Mabu during the group interview. He said if there was anything that he did not understand, the educator "will point it [out]". However, unlike Edi and Chri who expect guidance on how to improve, Mabu expects feedback to serve as remedial teaching. He wrote when the educator "explained it back to you she/he is doing the feedback to you".

Learners are insightful when they note that the purpose of feedback is a tool that can improve their understanding. This is confirmed by Black, Harrison, Lee, Marshall and William (2003) who stated that feedback provides information about 'gaps' in learning. From the data there emerged a parallel view where learners believed that feedback identifies or points out their errors. It is significant that learners meaning of feedback have resonance with Vygotsky's theory about the ZPD (Vygotsky, 1978), which they most probably have not heard of, yet they display an intuitive understanding of what is good for their learning experiences.

Secondly, one learner thinks that the purpose of feedback is to convey the teacher's point of view. Cleme sees the teacher as being right and that the learners' job is to figure out what the teacher is looking for. He portrayed feedback as communicating the teacher's view, by offering clues that would lead him to the teacher's goal. The learners who look towards the teacher's feedback for clues about the teacher is looking for, show that they see the teacher as being in control and who knows where they need to go. In contrast, Clarke (2000) defined effective feedback as when "the teacher must give feedback against the focused learning objectives of the task (whatever the child was asked to pay attention to), highlighting where success occurred against those objectives and suggesting where improvement might take place against those objectives" (p.37).

A third theme that emerges from the data is the role that feedback plays in building or breaking a learners' self confidence. All the learners expressed a wariness, about shy learners not being confident enough to approach the teacher and learners being insulted by the teacher. In fact one learner used the words " you are so stupid" to describe what she did not want to hear. The learners felt that the teacher may belittle learners or cause certain learners to lose their confidence because of the teachers' negative comments. On a similar note Moodley's (2008) study on South African learners' self efficacy beliefs about mathematics reported that most learners felt that their mathematics teachers displayed a negative attitude towards them. Her sample consisted of 32 Grade 11 mathematics learners. For example 91% of her sample indicated that the teacher ignored them when they asked questions and 93% indicated that the teacher made them feel silly when they asked questions in the maths class. Some of the learners' comments that were made include: "He tries to be funny but he doesn't know that he actually embarrasses and hurts people" and "You know you afraid to ask questions. Maybe the teacher will make you feel stupid" and further, "I hate being looked down upon". These comments support the contention made by the learners in our study that some teachers sometimes make negative comments to learners and when this happens, they get embarrassed and feel belittled by these disparaging comments.

These findings are supported by Young (2000, p.414), who explained that the "most powerful and potentially dangerous dimensions of students' feelings about feedback is the extent it impacts on themselves as people". In his study, verbal comments that are derogatory are viewed as "absolutely annihilating" for the learner in the learning experience (p.414).

CONCLUDING REMARKS

Although South African educational policies have been well received, a gap in the policy is the provision of detailed guidelines on ways of providing feedback to learners. The learners in this study have revealed that they have demanding expectations from the teachers and they view the teachers' role in assessment as going beyond mere implementation of different techniques. They expect the teacher to provide meaningful feedback to their work. They expect the teacher to diagnose their errors and to show them how they could close the gap. They expect the teacher to provide feedback which will improve their understanding. Furthermore they do not welcome derogatory comments about their abilities from their teachers, they view this as personal. The study has revealed that learners prefer teachers to provide feedback that makes a difference to their understanding in a manner that is supportive of them. It is vital for education authorities to acknowledge that the stipulation of various assessment methods is not sufficient. Teachers themselves have to also acknowledge the role they play in developing or annihilating their learners' confidence. Opportunities for teachers to improve their skills and sensitivity in feedback practices need to be made available in order to lead to sound reform of assessment in South African education.

The study has also demonstrated the need for more research that focuses on learners' voices on education - about what they view as important or not, what they value or do not, what they need and what they do not need. Research that focuses on learners' voices can help us identify the gaps between the vision of policy and the actual classroom implementation of the policy.

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LINKING PROPERTIES OF DIFFERENTIABLE FUNCTIONS TO GRAPHS: AN EXPLORATORY STUDY

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The research in mathematics education highlights the role of visual reasoning in the learning processes. In this frame the paper presents an investigation of the role of graphs in the conceptualization of the derivative of real valued, differentiable functions. The study is carried out with a sample of Italian Science freshman students. The study displays a lack of coordination of the semiotic systems involved in the representation of derivatives and in particular the occurrence of pragmatic aspects related to the use of graphs.

INTRODUCTION

This work deals with role of visual representations (graphs) in the learning of concepts such as monotonicity and derivatives of functions, which are generally included in the curricula of introductory mathematics courses for freshmen of scientific faculties. In particular I refer to examples of real valued, C^1 functions in real domains. The main purpose of this paper is to highlight and interpret some obstacles students meet when dealing with such topics, in an educational setting where graphs are widely used both in teaching and in examinations, and to understand if and how graphs could improve the teaching of introductory calculus. I will show that graphs are a powerful tool in order to both verify and improve the understanding of the notions above, although their use might create specific obstacles, since research has shown a poor use of visual representations in the teaching of mathematics (Presmeg 2006). In particular, as observed by Tall (1991 p.1) 'Research into mathematics education shows that students have very weak visualization skills in the calculus, which in turn leads to lack of meaning in the formalities of the mathematical analysis'.

THEORETHICAL BACKGROUND

Semiotic systems and learning

First of all, I adopt A. Sfard's claim that *"learning mathematics may now be defined as an initiation to mathematical discourse ..."* (Sfard (2001, p.28), and so languages are to be regarded not just as carriers of pre-existing meanings, but as builders of the meanings themselves. From this perspective, the linguistic means adopted in communicating mathematics are crucial also in the development of thinking. This holds not only for verbal language but for other semiotic systems too. As the study I

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 105-112. Thessaloniki, Greece: PME.

am reporting deals with the interplay between different semiotic systems (formulas, verbal texts, visual representations), R.Duval's framework[1] has proved appropriate.

According to Duval (1993) the cognitive functioning of human thought needs multiple semiotic systems. We recall that sémiosis denotes the production of a semiotic representation and noésis denotes the conceptual learning of an object. We share Duval's idea that there cannot be noésis without sémiosis. In this frame the coordination of at least two semiotic systems is basic for the learning of a mathematical concept. Duval makes a distinction between treatment of a representation, which is a transformation (manipulation) within the same semiotic system, and conversion (translation) between different semiotic systems. For example computing the sum of two fractions is an example of treatment, whereas translating a fraction into an equivalent decimal expansion is an example of conversion. Moreover, conversion must not be mistaken for *codification*, which is a pointwise transcription of a representation into another one by means of substitutions given by specific rules. The fundamental difference is that codification does not require knowledge of the internal rules of the target semiotic system, but only knowledge of the rules of transformation. The literal transcription of a spoken text into a written one is an example of codification which is not conversion, as the resulting text does not comply with the usual properties of written texts.

Language and context

Although here I am not focusing on students' verbal texts, I adopt some ideas from pragmatics (and in particular, functional linguistics) applied to mathematics (see Pimm 1987, Morgan 1998, Ferrari 2004). In particular I refer to the notion of *register* as a linguistic variety related to use and to the basic pragmatic assumption that any form of human communication requires some form of cooperation to achieve the goals of the exchange. Ferrari (2004) argues that the registers customarily adopted in mathematics share a number of features with literate registers and may be regarded as extreme forms of them. In particular, mathematical registers systematically violate cooperation principles. For example, denoting a square with the word 'rectangle' or a straight line with the word 'curve' is mathematically correct but violates cooperation principles, as the words 'square' or 'line' would sound more appropriate.

The question of which pieces of information can be drawn from a graph is not a simple one. A graph can represent a portion only of a function defined on real numbers, and properties such as continuity, differentiability and many others, as they cannot possibly be recognized by the graph only, should be either explicitly stated or inferred by the assumption that the graph is cooperative. The use of graphs is possible only recognizing that students generally assume they are strongly cooperative (Ferrari, 2004). Some more examples are provided below.

THE STUDY

Methodology

The participants were 123 Italian Science freshman students attending an introductory Mathematics course focused on real functions and differential and integral calculus. The research has been carried out at the Università degli Studi del Piemonte Orientale "A.Avogadro" in Alessandria. Contrary to standard teaching practices in this course graphs have been largely employed, both in face-to-face lessons and examinations. The data have been collected from students' answers to written tasks based on diagrams. The tasks were part of written midterm and final examination tests. All the answers were to be explained by means of written verbal texts. We have focused on students' strategies, trying to understand the reasons for their wrong answers. We have taken into account some linguistic properties (mainly the register adopted) of the written texts provided as explanations.

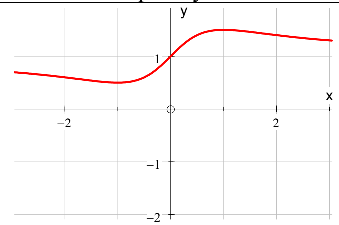
Tasks

We present here a translation of four typical problems among those given to students:

Consider the function f defined as $f(x) = \dfrac{x}{x^4 + 1} + 1$

a) Compute $f'(x)$
b) Compute $f'(0)$
c) One of the graphs A, B below does not correspond to the derivative of f. Find it and explain your answer.

A



B

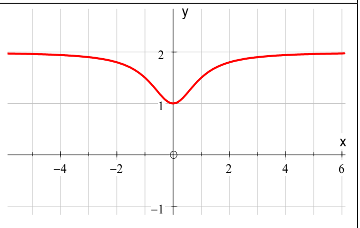
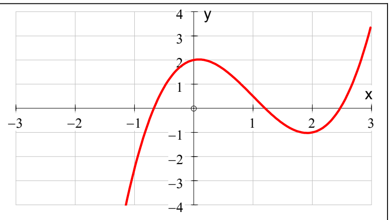


Table 1: Problem 1.

Consider the function h (a portion of) whose graph is on the right.

One of the graphs A, B below does not correspond to the derivative of h. Find it and explain your answer.



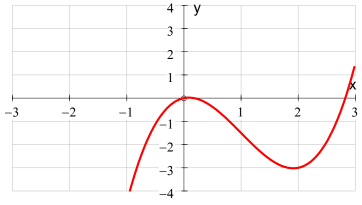
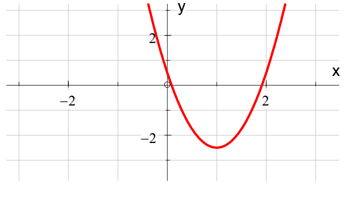
A 	B 

Table 2: Problem 2.

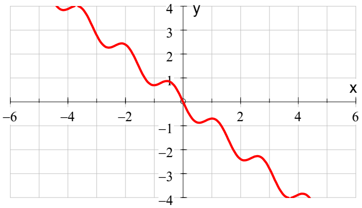
Consider the function g (a portion of) whose graph is on the right. Consider the statement: "$g'(0) \le g'(-1)$" and tell if it is true, false or you cannot tell. Explain your answer.	

Table 3: Problem 3.

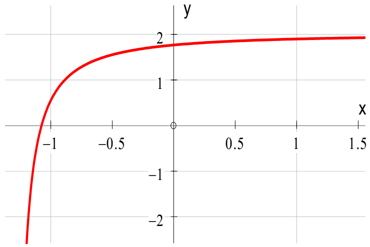
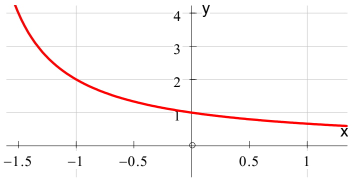
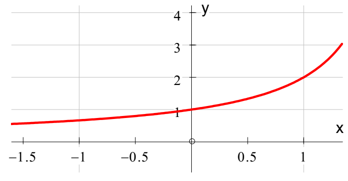
Consider the function u (a portion of) whose graph is on the right. a) Consider the statement: "$u'(0) \le u'(-1)$" and tell if it is true or false. Explain your answer. b) One of the graphs A, B below does not correspond to a derivative of u. Find it and explain your answer.	
A 	B 

Table 4: Problem 4.

All of the problems have been given in order to assess the learning of the connections between differentiable functions, their derivatives and their graphs. This requires the ability at gathering data from graphs. The graphical representations are of a conceptual nature, but at the same time they have all the limitations of real drawings. Let's see an example. It is impossible to tell on a perceptual basis only if a given graph actually passes through a point (x_0, y_0). If a graph looks like it passes through (x_0, y_0), to infer that it actually passes through (x_0, y_0) some cooperation principle is needed. These remarks have guided the formulation of the questions of the tasks. For example, we never ask to find the graph that corresponds to some formula, but only those that do not, and so on. The use of graphs itself can be a further source of errors. Students have to distinguish what they can and what they cannot read in a graph i.e. they must distinguish the conceptual nature from the perceptive (realistic) one. This difficulty can arise from a lack of coordination of semiotic systems (in this case involved in the notion of derivative) or from the misunderstanding of the construction rules of the representation.

RESULTS AND DISCUSSION[2]

First of all we notice that most students succeeded in solving Problem 1 (part *a*),*b*) 76%, part *c*) 55%) and Problem 2 (81%). Problems 3 (36%) and 4 (part *a*) 43%, *b*) 43%), on the contrary, proved more troublesome. The previous percentages of successes takes into account the explanation of the answers (bad or missing explanations are always classified as wrong answers). In what follows I present some excerpts that represent some of the most common students' behaviours. The errors have been classified into two categories. The first category refers to errors due to the concept of derivative and to the mental models students have developed on this topic. The second refers to errors related to how students extract information from graphs i.e. how the representation influences students' answers. Anyway one must not consider these categories as clearly distinct, but they influence simultaneously and continuously each other. Our thesis is that both these classes of errors can be explained as a lack of coordination of semiotic systems.

Knowledge and semiotic systems

Let's start with Problem 1). The most common approach to question *c*) considers a point (x_0, y_0) which belongs to graph A but not to graph B or vice versa and in checking whether or not x_0, y_0 satisfy the equation $y = f(x)$. This approach is also used by most of the students providing appropriate answers to questions *a*) and *b*). Very few people exploited answers *a*) and *b*) to solve *c*). The prevalence of this kind of answer is due to the fact that it consists in a codification in the sense of Duval (1993), requiring no other knowledge on the relationships between the symbolic representation of the function and its graph. The use of the derivative in 0, on the contrary, requires a more global conversion process, involving the knowledge of properties linking the symbolic semiotic system to the visual one. Indeed students usually choose to compute $f(x_0)$ rather than to use the result available on $f'(0)$.

Consider now the other problems. We want to explain why problem 2 has had a high number of correct answers and the other two have not. In problem 2 students apply the monotonicity test which connects the monotonicity of a function and the stationary points with the sign of its derivative. These topics are understood well enough to give correct answers to problem 2, sometimes to problem 3 and in a few cases to problem 4 too. Consider the following excerpt of an answer of student A in problem 3 and notice that the same student succeeded in solving Problem 2.

A: One can not answer since in the problem there is neither the graph of the derivative function g' nor the values of the derivative g' in points 0 and -1.

Consider now the behaviour of student B. He answers to problem 2 correctly using the above rule and succeeds in answering also to Problem 3:

B: In 0 the function is Thus *g'(-1)* is larger than *g'(0)* and the statement is true.

But student B gives the following answer to point *a*) of Problem 4:

B: Since the function *u* is increasing *u'(0)* and *u'(-1)* take positive values. It is impossible to determine which is the greater between the two.

Answers like A's and B's are very common in the whole group.

Another student C in Problem 3 tries to find a formula for g satisfying the condition $g'(0) \leq g'(-1)$ but he admits that such formula does not represent the graph given. Also this student succeeded in solving problem 2.

Behaviours like those of A, C denote that these students cannot link their knowledge of the derivative of functions to the figural properties of the graphs. Their answers explicitly show lack of coordination of different semiotic systems. The situation of B is a little bit better since he can realize that there is a neighbourhood of the origin where *g* is decreasing and then its derivative will assume negative values (exploiting the monotonicity test) and hence also *g'*(0) will take a negative value. But also in this case the student seemingly does not know what does *g'*(0) represent (i.e. the slope of the tangent line for *x*=0). Behaviours like those of A, B, C mean that the notion of derivative is considered as a global property i.e. a property referred to a whole interval and not as a pointwise one. Given a function *f(x)* on a domain D, its derivative *f'(x)* is seen as a function that one can compute just having the formula of *f(x)*. In these cases the *concept image* (Tall & Vinner, 1981) evoked by students is not related to any formal or informal *concept definition*, for example topics such as the slope of the tangent of a curve or the limit of the difference quotient. These students properly use the property which relates the monotonicity of a function with the sign of its derivative i.e. they have learnt the codification increasing (decreasing) function – positive (negative) derivative and that null derivative corresponds to stationary points. It comes out that such properties are considered by students as rules which allow them to solve problems like 2, and in some cases also like 3 but not like 4. Moreover, very few students used the slope of the tangent line to solve the problems.

Notice that also the answers to problem 1 confirm that derivative is not seen as a pointwise object. Here students who choose to use derivative arguments do not use the derivative at point 0 computed to answer *b*) but they study the sign of the derivative or search stationary points exploiting the formula previously computed in order to answer question *a*). Also in this case few students use the derivative at point 0 computed to answer *b*).

Pragmatics and semiotic systems

Some wrong answers can be ascribed to poor interpretations of the graph (regarded as a representation in a semiotic system). Sometimes the students read from the graph pieces of information that should not properly be read. This might be ascribed to pragmatic factors: the students often implicitly assume that a graph (rather than a piece of text or a formula) has to be highly cooperative. So from a graph like B of problem 4, plenty of students are ready to learn that that function is increasing on the whole real numbers. This behaviour is the consequence of the application of principia that usually work in colloquial registers, since graphing a non-increasing function in an interval where it looks as increasing would break cooperative rules. This is the case of student S when solving Problem 4*b*):

> S: Since *h* is an increasing function its derivative will assume positive values. Graph A does not represent *h'* since it tends to have negative values.

Student S seems to assume that h is defined and increasing on all the reals. Moreover, since *h'* is decreasing, he infers that it will eventually take negative values even if this is not shown in the portion of the graph actually represented.

We found a similar behaviour in some students when addressing problem 1). They use limits ($x \to \pm\infty$) to answer the question 1*b*). Also in this case they seem to believe that the graph represents the behaviour of the function on all the real line rather than in a portion of it only. Look at the following answer to problem 2:

> T: Graph B does not correspond for sure to the derivative *h'* because it does not pass through the origin. In fact *h* has a maximum at (0,2). Moreover in the interval [0,1] *h* is decreasing so *h'* must take negative values.

In this case the error could be interpreted as misleading perception. In the last three kinds of answers the mistakes are due to the graphical system. Graphs hinder the resolution of the tasks but anyway this would not happen if the students had a firm grasp on the concept of derivative.

This research highlights poor visual reasoning skills related to the concept of derivative in freshman students and points out their difficulty in connecting different aspects of the same topic. It seems that the statement of a problem evokes some parts of the related concept image and students do not mind to use other resources that they could gather from that. In particular this seems related to poor skills in the conversion of different semiotic representations. Indeed some errors occurring in the tasks are due to the relationships of the graphs with the context. We think that problems like those presented in this paper can help students to improve the use of their concept image and hence to develop links between their different cognitive resources.

[1] In contrast to Duval, I do not use the term ‘register’ to denote a semiotic system, as it is more generally used to denote a linguistic variety related to use.

[2] Since the translation into English of a text written in another language (Italian in this case) may affect some linguistic properties of the original text we report here the translation of just a few parts of protocols in order not to change the features for the analyses of data.

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HIGHSCHOOL TEACHERS' KNOWLEDGE ABOUT ELEMENTARY NUMBER THEORY PROOFS CONSTRUCTED BY STUDENTS

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This study investigates changes in teachers' knowledge regarding their students' construction of correct and incorrect proofs within the context of elementary number theory, before and after a professional development course. Twenty high school teachers were requested twice to suggest correct and incorrect proofs their students might construct, at the beginning and at the end of the 15 week long course. The suggested proofs were analysed according to modes of representation. Results indicate that the teachers' suggestions of correct and incorrect proofs students might construct increased both in number and variety.

THEORETICAL BACKGROUND

Recent reforms in mathematics education recommend including proofs as a key component in school mathematics (Australian Educational Council, 1991; Israeli Ministry of Education, 2004; National Council of Teachers of Mathematics, 2000), because of the vital role proofs play for validation and for refutation in mathematics (Aigner & Ziegler, 1998; Thurston, 1994). Different types of proofs require the use of different methods. For a universal statement a general proof, covering all relevant cases is necessary to validate the statement while a single counter example is sufficient to refute such a statement. For an existential statement a single supportive example is sufficient to prove the statement, while a general proof, covering all relevant cases, is necessary to refute the statement. In addition, a proof may be given in various modes of argument representation (Stylianides, 2007), such as verbal, numeric or symbolic representation.

Teachers are responsible to incorporate proofs and proving in everyday school practise. What are the types of knowledge a teacher needs to implement proofs in his class? "Teachers' subject matter conceptions have a significant impact on their instructional practices" (Knuth, 2002, p. 63), and hence, analysing teachers' knowledge with respect to proof and proving is important. Knowing to verify valid statements and to refute invalid ones is an essential component of teachers' subject matter knowledge. Yet, it is not the only requirement. Hill, Ball, Sleep and Lewis (2007) rhetorically ask "Is the knowledge mastered by someone who majors in mathematics sufficient content knowledge for teaching?" (p. 112). Teachers need to be able to evaluate students' suggested proofs to various statements. In addition, teachers need to be familiar with the ways students correctly and incorrectly justify a variety of statements. This latter type of knowledge is the focus of the present study.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 113-120. Thessaloniki, Greece: PME.

What does the literature tells us about teachers' familiarity with the ways students justify statements? Not much. We found several studies relating to secondary school teachers' evaluation of students' given justifications (e.g., Dreyfus, 2000; Healy & Hoyles, 2000). However, we did not find studies in which teachers were asked to provide correct and incorrect justifications that students are likely to construct for various statements.

Hence, we looked for studies related to students' knowledge of proofs and proving. Studies have shown that students are not always aware of the necessity for a general, covering proof when proving the validity of a universal statement for an infinite number of cases (e.g., Bell, 1976). Healy and Hoyles (1998, 2000) found that 14-15 year olds have difficulties constructing a complete proof based on deductive reasoning. Balacheff (1991) found that students relate to counter examples as bizarre instances and do not always recognize a counter example as being sufficient to refute a universal statement. Regarding the types of representations used by students when constructing proofs, Bell (1976) found that none of the 36 high school students in his study used an algebraic proof when proving a numerical, universal conjecture. Healy and Hoyles (1998, 2000) found that students preferred verbal explanations over other kinds of representations.

In the present study we ask two questions: (1) Are high school teachers familiar with correct and incorrect justifications that students may construct for various elementary number theory statements? (2) To what extent did a professional development course contribute to teachers' knowledge of students' justifications?

SETTING AND METHOD

Twenty high school teachers participated in the study. All of them were studying towards a master degree in mathematics education.

The participants participated in weekly meetings, two hours long, of a 15 week professional development course. The course aimed at enhancing participants' knowledge with respect to mathematical aspects of proofs and proving, as well as with respect to didactical aspects of teaching proofs in secondary school. The course included the design, by the participants of a learning unit (2-4 lessons), in the domain of elementary number theory (ENT). The ENT context was chosen as the related concepts were thought to be familiar to teachers and to middle grade students, enabling the teachers to focus on proving the statements and minimizing difficulties that may have arisen due to problems with the content domain and misunderstanding of terminology. The participants implemented the unit they designed in their own classes, and reported back on these implementations during the course sessions.

Tools

At the beginning of the course, a set of three questionnaires, consisting of six ENT statements was administered to all participants (Table 1). The validity of each statement is determined by a combination of the predicate (always true, sometimes true, or never true) and the quantifier (universal or existential). The teachers were asked to answer a three part questionnaire: first to produce a proof (or a refutation) for each of the six statements, which they consistently did correctly (part 1). Next, for each of the six statements, teachers were asked to suggest as many correct and incorrect proofs that, in their opinion, students would give for these statements (part 2). Finally, the teachers were presented with different, correct and incorrect justifications for each statement (part 3). In this paper we limit the discussion to teachers' answers to the second part. (A discussion on the findings relating to part 3 can be found in Tsamir, Tirosh, Dreyfus, Barkai and Tabach, 2008).

Predicate Quantifier	Always true	Sometimes true	Never true
Universal	**S1**: The sum of any 5 consecutive natural numbers is divisible by 5. *True*	**S2**: The sum of any 3 consecutive natural numbers is divisible by 6. *False*	**S3**: The sum of any 4 consecutive natural numbers is divisible by 4. *False*
Existential	**S4**: There exists a sum of 5 consecutive natural numbers that is divisible by 5. *True*	**S5**: There exists a sum of 3 consecutive natural numbers that is divisible by 6. *True*	**S6**: There exists a sum of 4 consecutive natural numbers that is divisible by 4. *False*

Table 1: Classification of statements

Data Analysis

All proofs presented by the teachers were categorized according to their modes of representation (Stylianides, 2007). This analysis resulted in three modes of representation: numeric, symbolic, and verbal.

RESULTS

We first present overall results about correct and incorrect justifications that the teachers suggested as students' constructs. We follow with an analysis of modes of argument representation for the correct justifications with examples of teachers' suggestions, and finally a similar analysis with examples for the incorrect justifications.

At the beginning of the course, our participants suggested a total of 291 correct and incorrect justifications that students may give to the six statements in Table 1. At the end of the course, the total number of justifications had increased by 51%, resulting in 440 justifications. At the beginning of the course, the teachers suggested 169 correct and 122 incorrect justifications and at the end of the course 255 correct and 185 incorrect justifications. Tables 2 and 3 present the numbers of correct and incorrect justifications (respectively) suggested by the teachers as possible justifications that students will construct to each statement. Overall, it seems that they found it easier to think of correct justifications that students will construct than of incorrect justifications. A more detailed examination of the tables reveals, however, that this is the case only for statements S2-S5; for statements S1 (universal, always true) and S6 (existential, never true), on the other hand, the teachers provided more incorrect justifications then correct justifications before the course. These two statements require a general-cover proof. This is in line with research findings about students difficulties with general-cover proofs (Bell, 1976). However after the course, the number of correct and incorrect justifications in the case of these two statements was almost the same.

	S1	S2	S3	S4	S5	S6	
	---------- Universal ----------			-------- Existential --------			
True	(Always)	(Sometimes)	(Never)	(Always)	(Sometimes)	(Never)	Total
Before	25	33	33	31	31	16	169
After	47	50	44	42	42	30	255

Table 2: No. of justifications teachers provided as students' **correct** justifications

The number of suggestions for incorrect justifications to statements S3 (universal, never true) and S4 (existential, always true) was significantly lower than the number of suggestions for the other statements, both before and after the course. The teachers' common remark regarding S3 was that since any numerical example of the sum of four consecutive numbers is not divisible by four, students will not err in this case. Similarly, with respect to S4, teachers claimed that since the sum of any five natural consecutive numbers is divisible by 5, students will not err in this case either. Teachers' claims in these cases are in line with reports about students' tendency to start with examples (Healy & Hoyles, 1998).

	S1	S2	S3	S4	S5	S6	
	---------- Universal ----------			------ Existential ------			
True	(Always)	(Sometimes)	(Never)	(Always)	(Sometimes)	(Never)	Total
Before	29	33	10	5	24	24	122
After	46	43	17	9	37	33	185

Table 3: No. of justifications teachers provided as students' **incorrect** justifications

It is notable that the change from pre test to post test in the number of justifications provided for each statement varied considerably. The largest change was found in correct justifications for statements S1 and S6, which means that the participants could provide more correct general-cover proofs after the course. Also, the participants could provide more incorrect proofs for statements S3 and S4.

We now turn to the modes of argument representation suggested by the teachers for the correct justifications of each statement before and after the course (Table 4).

	S1		S2		S3		S4		S5		S6	
	---------U n i v e r s a l---------						--------E x i s t e n t i a l--------					
True	(Always)		(Sometimes)		(Never)		(Always)		(Sometimes)		(Never)	
	Before	After	Before	After	Before	After	Before	After	Before	After	Before	After
Symbolic	23	31	15	24	14	21	13	19	9	17	15	23
Verbal	2	16	0	3	0	4	0	0	0	0	1	7
Numeric	0	0	18	23	19	19	18	23	22	25	0	0

Table 4: Categorization of the correct justifications that teachers suggested as students' justifications according to the mode of argument representation

In each mode of argument representation we can see an increase in the number of suggested correct justifications. After the course the teachers suggested more correct symbolic justifications; all the participants provided the symbolic justification where x represents the first of the consecutive natural numbers for each statement, including statements S2 – S5, for which supportive or counter examples suffice as proof. A few other symbolic justifications were suggested – mainly after the course, representing the middle element as x, using mathematical induction, or using the formula for the sum of an arithmetic sequence. Many more verbal justifications were presented after the course than before (30 vs. 3), especially for statements S1 and S6, which required a general-cover proof. An example for a verbal justification to S1 before the course: "*One of five consecutive numbers is divisible by five. The other numbers, when divided by five, will give the remainders 1, 2, 3, 4. The sum of these remainders is ten, which is divisible by five*". After the course, additional justifications were given, many of them were of the kind: "*I will check the sum of the first five consecutive numbers: 1 + 2 + 3 + 4 + 5 = 15, divisible by five. The sum of the next five consecutive numbers can be obtained by adding five to this sum (since each addend is larger by 1, hence the sum is larger by five). In each step we add five to a number that is already divisible by five; hence the sum is always divisible by five. The statement is true*". While twenty correct verbal induction type justifications were presented after the course, only seven induction justifications were presented symbolically. With respect to numerical justifications, before the course the teachers provided justifications which included one numerical example – either as a supportive example or as a counter example. After the course, there was a general small increase, especially in the category of several numerical examples (for example, the student will choose a supportive example, and also a counter example as justification). This is in line with research findings that indicate students' tendency to use more than one example as justification (Bell, 1976).

The modes of argument representation in the case of incorrect justifications provided by the teachers as their students' construction can be examined in Table 5.

	S1		S2		S3		S4		S5		S6	
	---------U n i v e r s a l---------						--------E x i s t e n t i a l--------					
True	(Always)		(Sometimes)		(Never)		(Always)		(Sometimes)		(Never)	
	Before	After	Before	After	Before	After	Before	After	Before	After	Before	After
Symbolic	4	8	7	12	6	7	2	6	10	12	4	6
Verbal	4	1	2	1	4	3	2	1	2	1	3	2
Numeric	21	37	24	30	0	7	1	3	12	24	17	25

Table 5: Categorization of the incorrect justifications that teachers suggested as students' justifications according to the mode of argument representation

For the incorrect justifications, as for the correct ones, we note an increase in the number of the symbolic justifications for each statement. It seems that the participants expand their repertoire of students' errors. Two categories of symbolic lapses were identified in teachers suggested justifications. Generality lapses related to cases were the symbolic representation of the consecutive numbers was wrong, like *x*, *x*, .. or *1x*, *2x* …, and hence the generality of the justification was violated; and inference lapses like an invalid chain of inferences ("*The sum of five consecutive numbers: x + x+1 + x+2 + x+3 + x+4 = 5x + 10, 5x = -10, x=-2. There is a solution and hence the statement is true*"). In some cases the wrong inference was to transform an expression into an equation, while in others the expression that represents the sum of the consecutive numbers was given a wrong interpretation. A decrease was found in the verbal mode of argumentation for all cases of incorrect justification. This is not in line with the increase evident for the correct justifications (Table 4). It seems that intuitive verbal justifications (like, "*The sum of any five consecutive natural numbers is divisible by five, hence the sum of any four* *consecutive natural numbers is divisible by four*"), was abandoned by the participants. With respect to the numerical mode of argumentation, we note an increase for each statement. The teachers seem to have become aware of a variety of errors with respect to numerical examples as justifications, like providing several examples and concluding that a statement holds for all cases, or checking a "small" numerical example and a "large" numerical example, and concluding that the statement holds for all cases. This is in line with research findings about students' incorrect justifications (Harel & Sowder, 2007).

CONCLUDING REMARKS

In the present study we asked two questions: (1) Are high school teachers familiar with correct and incorrect justifications that students may construct for various elementary number theory statements? (2) To what extent did a professional development course contribute to teachers' knowledge of students' justifications?

As the range of teachers' suggestions for correct and incorrect justifications suggests, the participants in our study think that students' use of the symbolic mode of argument representation is extensive. Also, it seems that at least some of the participants showed no awareness of students' preference for the verbal mode of argument representation. While the participants' tendency to suggest symbolic justifications did not decrease after the course, their suggestions for verbal correct justifications in the case of general cover proofs increased considerably.

The professional development course influenced both, the number and variety of high school teachers' responses when asked to present students' correct and incorrect justifications to six ENT statements. In particular, we point to the increases in the number of correct verbal justifications and the number of incorrect numerical justifications; indeed, the literature indicates that students tend to use verbal justifications for proving correct ENT statements, and numerical examples when constructing an incorrect justification (Harel & Sowder, 2007). The participants also expanded their repertoire of symbolic lapses, which may help them identify such justifications when presented in their own classes.

Three factors may have contributed to the observed changes: The professional development course itself has probably had an effect by means of its discussions on various justifications for different types of statements. The learning unit, which the teachers planned and implemented, may have provided opportunities for confrontation with students' actual justifications and thus contribute to the teachers' repertoire of justifications. Finally, it is possible that the test given at the beginning of the course, which included suggestions for students' justifications, influenced the teachers' knowledge.

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IN SEARCH FOR THEORIES: POLYPHONY, POLYSEMY AND SEMIOTIC MEDIATION IN THE MATHEMATICS CLASSROOM

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This theoretical report addresses the theme of the PME Conference ("In search for theories in Mathematics Education"). The history of two interlaced research programs (Mathematical Discussion and Mathematical Machines) headed by the author is outlined, together with the merging of both, combined with studies on information and communication technologies. They are the roots of the theoretical framework of semiotic mediation after a Vygotskian approach (Bartolini Bussi & Mariotti, 2008). May this framework answer the present needs of focusing cultural historical issues and the teacher's role in the teaching-learning process within the mathematics classroom?

INTRODUCTION

This theoretical report reconstructs the scientific and cultural roots of a theoretical framework about the relationship between artifacts and signs in the classroom process after a Vygotskian approach, with emphasis on the teacher's guiding role (Bartolini Bussi & Mariotti, 2008). The narration outlines the development of two research programs (called the *Mathematical Discussion program* and the *Mathematical Machines program*) and offers an annotated bibliography of both. The two programs, headed by the author, were developed independently from each other for years before giving rise to a shared theoretical framework as a result of a dialogue between empirical and theoretical issues. Mariotti is credited for joint elaboration of this framework, mainly (but not only) thanks to her expertise in information and communication technologies (ICT, e. g. Mariotti, 2002). All this was developed within the paradigm of *Research for Innovation in Mathematics Education*, as empirical-theoretical classroom research (Arzarello & Bartolini Bussi, 1998).

THE MATHEMATICAL DISCUSSION PROGRAM

In the 80s, the author and a group of mathematics teachers (grades 1 - 8) started to study the conditions for realizing effective whole class interaction. Within the European tradition of teaching and learning, we felt uncomfortable with the one-sided focus on learners' activity and on peer interaction that characterized the constructivist approach (dominant in those years in the field of mathematics education). We did not consider this focus respectful of the cultural role of teachers: hence a cultural historical perspective was assumed (Vygotsky, 1978, 1981).

Vygotsky: obučenie. The idea of *obučenie,* i.e. the bilateral process of transmission and appropriation of knowledge, skills and methods, carried out by the teacher and the learner together (Vygotsky, 1990, p. xx), was a warning against reductionist

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 121-128. Thessaloniki, Greece: PME.

approaches. We were especially interested in the Vygotskian asymmetry between the expert adult and the young children in the *zone of proximal development* and in the process of *internalization* (Bartolini Bussi, 1998a).

Leontev: activity, actions, operations. We focused long term processes, which last weeks, or even months. This depends, first, on the organization of the Italian school system (where a teacher teaches the same classroom for three or even five years) and, second, on the belief that many relevant changes can be observed only in the long run. Leontev activity theory (1978) offered the distinction between the three levels of *activity* (collective and directed towards an object-motive), *actions* (goal direct processes) and *operations* (the way of carrying out actions in variable concrete circumstances, Bartolini Bussi, 1996, p. 15ff). Because of the focus on the teacher's role, actions and operations were studied mainly with reference to this acting subject. As it is impossible to design a priori the complexity of interactional processes, the study of teacher's on the spot improvisation was needed.

The Italian comedy of art: anticipated improvisation. To approach the issue of teacher's operations, we considered the idea of *improvisation*, as emergent in the Italian comedy of art (Fo, 1987). Actors do not always invent cues; rather they often choose them in a repertory that has been studied diligently according to the different situations which may occur. Hence, our effort was directed towards eliciting goals of the teacher from specific classroom situations and towards collecting "constellations" of communicative strategies (cues) which had shown effective empirically in fulfilling the goal, in order to construct a repertory to be learnt (Bartolini Bussi, 1998b). A recent and more complete work has been carried out by Falcade (2006).

Bachtin: polyphony. The metaphor of *polyphony* (Bachtin, 1968) was adopted to consider the system of utterances produced by students and teachers or by evoked authors of texts (e.g. historical sources, textbooks). We used the word *voice* after Bachtin to mean a form of speaking and thinking, which represents the perspective of an individual, i.e. his/her conceptual horizon, his/her intention and his/her view of the world (Bartolini Bussi, 1996).

The construct of Mathematical Discussion orchestrated by the teacher. After some years of empirical work in the mathematics classrooms from grades 1 to grade 8, we described in a precise way the specific form of classroom interaction we were working on, i. e. the *Mathematical Discussion orchestrated by the teacher.*

> The Mathematical Discussion [orchestrated by the teacher] is polyphony of articulated voices on a mathematical object (e.g. a concept, a problem, a procedure, a structure, an idea or a belief about mathematics), that is one of the motives of the teaching-learning activity (Bartolini Bussi, 1996, p. 16).

The recourse to musical metaphors was not accidental. Besides borrowing the words polyphony and orchestration from Bachtin we wished to emphasize also the importance of imitating voices in counterpoint. This position was very strongly influenced by Vygotsky's emphasis on *intellectual imitation* as one of the basic paths of cultural development of the child (Vygotsky, 1978). In other words, we stated very firmly that imitation is essential in the teaching-learning process and not opposed to creative thinking (for a contrast with constructivist perspectives on mathematical discussion see Bartolini Bussi, 1998a, p. 14 ff.).

Several experiments were carried out in the following years in grade 1-8 classrooms (Bartolini Bussi, 2007; Bartolini Bussi & Boni, 2003; Bartolini Bussi et al., 1999, 2005, 2007). In parallel, Mariotti implemented systematically mathematical discussions in ICT environments (e. g. Mariotti, 2002; Cerulli, 2004).

THE MATHEMATICAL MACHINES PROGRAM

Most of the research studies quoted at the end of the previous section concern concrete artifacts taken from the history. This is consistent with one of the major tenets of cultural historical school. Actually Vygotsky did not study only language but also the role of artifacts in the cognitive development (Bartolini Bussi & Mariotti, 2008, p. 751) and suggested a list of possible examples:

> various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs, etc" (Vygotsky 1981, p. 137).

In the Laboratory of Mathematical Machines (www.mmlab.unimore.it) headed by the author, more than 200 artifacts have been reconstructed drawing on the historical phenomenology of geometry, from the classical age to the 20th century. A *mathematical machine is a tool that forces a point to follow a trajectory or to be transformed according to a given law.* The most common mathematical machines are the pair of compasses (that forces the point with the graphite lead to draw a circle). Since the 80s empirical classroom activity was carried out at high school level (grades 9 - 13) by the members of the Laboratory team. In all the experiments, small group work with a mathematical machine was realized before whole class discussion of findings. The historic epistemological analysis was in the foreground whilst the study of classroom organization and processes came later (Bartolini Bussi & Pergola, 1996; Bartolini Bussi, 2005; Bartolini Bussi & Maschietto, 2006).

Rabardel: artifacts and instruments. Rabardel's instrumental approach (1995) is based on the distinction between artifact and instrument: the artifact is the material or symbolic object per se whilst the instrument is a mixed entity made up of both artifact-type components and schematic components (utilization schemes). The utilization schemes are progressively elaborated by the user during artifact use to solve a task; thus the instrument is a construction of an individual, it has a psychological character and it is strictly related to the context (Bartolini Bussi & Mariotti, 2008, p. 748 ff.). The elaboration and evolution of the instruments (*instrumental genesis)* can be articulated into two processes: *instrumentalisation*, concerning the emergence and the evolution of the different components of the artifact, e.g. the progressive recognition of its potentialities and constraints; *instrumentation*, concerning the emergence and development of the utilization schemes. The two processes are outward and inward oriented, respectively from the subject to the artifact and vice versa, and constitute the two inseparable parts of instrumental genesis. Both analyses have been applied to mathematical machines (Bartolini Bussi & Maschietto, 2006, ch. 4; Martignone & Antonini, in press).

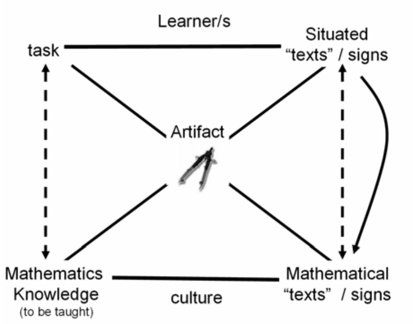
Wartofsky: polysemy. Wartofsky (1979) analyses artifacts from epistemological perspective. According to him, the term artifact has to be meant in a broad sense (Bartolini Bussi & Mariotti, 2008, p. 760 ff.), including tools (*primary artifacts*), representations (*secondary artifacts*) and theories (*tertiary artifacts*). Consider the case of the pair of compasses. One may use it to draw round shapes (primary artifact), to find a point at a given distance from two given points, according to Euclid's definition of circle (secondary artifact) or to evoke the Euclidean geometry (tertiary artifact). The introduction of an artifact in a classroom does not automatically determine the way it is used and conceived of by the students (i. e. *polysemy* emerges) and may create the condition for generating the production of different voices. For each artifact, one may analyse a priori the *semiotic potential* that links the meanings emerging from its use, aimed to accomplish a task, and the mathematical meanings evoked by that use. This analysis suggests ways of starting, monitoring and managing *polyphony* in classroom interaction.

Wartofsky's analysis offered an epistemological perspective, whilst Rabardel's instrumental approach offered a cognitive perspective on the use of mathematical machines.

SEMIOTIC MEDIATION: THE MERGING OF TWO RESEARCH PROGRAMS

In the early years the didactical analysis in the field of experience of Mathematical machines looked still weak, whilst in the Mathematical Discussion program it was stronger. The merging of two programs aimed at deepening the construct of *semiotic mediation*, as conceived by Vygotsky (1978, p. 39-40). Polyphony caught the linguistic, whilst polysemy caught the instrumental aspects. Moreover the presence of concrete artifacts emphasized the importance of other semiotic systems in addition to language (e. g. gestures, drawings; Maschietto & Bartolini Bussi, 2009).

It is beyond the aim of this short report to present the resulting theoretical framework (Bartolini Bussi & Mariotti, 2008), that encompasses all the issues above and includes also ICT. Two schemes may recall and outline the framework. The first (fig. 1) represents the system of semiotic activity that hints at the teacher's roles; the second (fig. 2) represents the didactical cycle that hints at long term processes.



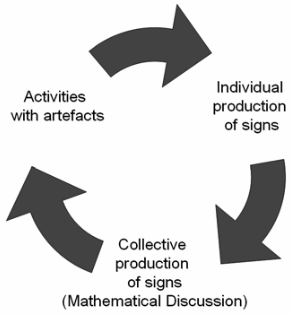


Figure 1: Artifacts in semiotic mediation. Figure 2: The didactical cycle.

The fig. 1 represents what happens when a student (or a small group of students) is given a task, that, according to teacher's intention, is related to both a piece of the mathematics knowledge to be taught and the use of an artifact (e. g. a pair of compasses). A solution of the task, although correct, might be only a technical solution (where the artifact is used as primary one), with situated "texts" (signs) and without any awareness of mathematical meaning. On the right side of the diagram, it is represented the teacher's aim to transform the above situated "texts" (signs) into mathematical "texts" (signs) which might be easily related to the piece of mathematical knowledge to be taught. The teacher's main roles are the following: (left) to construct suitable tasks; (right) to create the condition for polyphony, eliciting the polysemous feature of the artifact and to guide the transformation of situated "texts" (signs) into mathematical "texts". In this way the teacher mediates mathematical meanings, using the artifact as a tool of semiotic mediation. Without teacher's intervention, there might be a fracture between learner/s (top) and culture (bottom) planes, hence no learner's construction of mathematical meanings.

This is not a short term process. The fig. 2 shows the articulation of classroom processes over time, from individual or small group solution by means of the artifact, together with the individual production of "texts" (signs), to collective production of "texts" (signs) in mathematical discussion orchestrated by the teacher (Bartolini Bussi & Mariotti, 2008). Fine grain analysis of the teacher's role in this long term process has been carried out by Falcade (2006) and is still in progress in specific situations.

As motives of activity include "existing" mathematical meanings (often expressed in a crystallized form), a critical reader might object that such a teleological approach contrasts Bachtin's idea of polyphony. On the contrary, we claim that, according to the Vygotskian construct of internalization (1978, p 56), what is internalized by students is not the crystallized result but rather the interpersonal process of construction, that is truly polyphonic. There are always traces of polyphony in students' protocols, which may involve other semiotic systems, e. g. gestures (Maschietto & Bartolini Bussi, 2009). As authors end novels, teachers lead students to construct/appropriate 'existing' mathematical meanings: it is not a finish but rather a new start. As Vianna & Stetsenko (2006) claim, from a Vygotskian perspective:

> Present generations never invent their world and themselves from scratch but inevitably continue their past, even if by completely breaking away from it. However, it is also clear that the past does not simply evolve in the present but is enacted by each generation of people each time anew and in view of the present and the future (which is flexible too), through innovative and bold contributions to it. This mutual interpenetration of past and present can be well captured by the metaphor that the present without the past is blind, but the past without the present is powerless (p. 100-1).

CONCLUDING REMARKS

The main contributions of this comprehensive framework concern the role of history and culture and of tools (artifacts from both history and ICT) as mediators under the guide of the teacher, and the study of teacher's asymmetric role (as a guide) in the mathematics classroom.

When the Mathematical Discussion program was started, it was not easy to find interlocutors in the scientific community, because the focus on teaching was mistaken for neglecting learning. After more than twenty years the situation is expected to be different. According to Sfard (2005), in the 21st century, from the era of the learner we have entered the era of the teacher. Yet, is it real? Vianna & Stetsenko (2006) have carefully analysed some research programs from US and UK and have found a pivotal difference in views on the role of history and culture and of teaching in development, in comparison with Vygotskian tenets. This difference seems to draw on deep roots. As Clarke (2007) observed

> Whether we look to the Japanese “gakushushido", the Dutch “leren” or the Russian “obučenie”, we find that some communities have acknowledged the interdependence of instruction and learning by encompassing both activities within the one process and, most significantly, within the one word. In English, we seem compelled to dichotomise classroom practice into Teaching or Learning (p. 23).

Yet, now the world of mathematics education communities is expanding and attitude is changing: *may the theoretical framework of semiotic mediation after a Vygotskian approach, as developed by Bartolini Bussi & Mariotti* (2008), *be assumed as a useful theory for research on teaching - learning in the mathematics classroom?*

ACKNOWLEDGEMENT

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MATHEMATICAL TASKS TO PROMOTE STUDENT LEARNING

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This paper examines teachers' selection and resolution of function problems and relates this to their students' understanding of the concept. Focusing on the way in which tasks are presented and resolved, the paper indicates that the models teachers present to students to obtain solutions may lead to two contrasting outcomes. On one hand the teacher's emphasis may confine the students to a limited way of thinking about the core ideas and contribute towards misconceptions. In contrast, a model that emphasizes connections between the ideas and the representations can encourage a conceptually rich knowledge of function and minimizes the occurrence of misconceptions.

INTRODUCTION

Tasks shape the way students think about the subject matter and, thus, influence their learning (Doyle, 1983; Stein & Lane, 1996). NCTM (2000) recognized the importance of using worthwhile tasks in teaching mathematics suggesting that "*…tasks should be intriguing; with a level of challenge that invites speculation and hard work*" (pp. 16-17). Stein et al (1996) considered a 'mathematical task' as a classroom activity with the purpose of engaging students with a concept or algorithmic skill. They suggest that a mathematical task passes through three phases until it becomes a learning outcome. The first is concerned with designing and presenting the task as it appears in the instructional materials. The second, *the set up phase*, entails the teacher's introduction of the task in the classroom, whilst the third stage, *the implementation phase*, embraces the process in which the task is performed by the teaching-learning community. Stein et al (1996) discuss two dimensions of the mathematical task: *task feature* and *cognitive demand*. Task feature may require using more than one solution strategy or it might request recalling pre-presented rules and procedures and applying them to the problem at hand. Cognitive demand entails the thinking process within which the teaching-learning community engages when implementing the problem (ibid).

Stein and Lane (1996) reported that there was an increase in the students' understanding when the teachers engaged them with the conceptual tasks without reducing the task demands. They call for new qualitative studies suggesting that these studies should consider the cognitive processes set into the tasks and the quality of teaching discourses that support or inhibit student engagement with them. This paper contributes to a growing body of research in the field by examining two Turkish teachers' selection and implementations of function problems and relates it to their students' understanding of this notion. In this paper, 'task' refers to 'function

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problems' whilst 'task condition' refers to 'teaching discourses' that the teachers displayed when solving these problems.

RESEARCH DESIGN

This paper builds upon a study (Bayazit, 2006) which investigated the influence of classroom teaching on student understanding of the function concept. The participants were two experienced teachers (Ahmet: 25 and Burak: 24 years) and their 9th grade students. The study employed a qualitative inquiry (Merriam, 1988) and used a purposeful sampling strategy to control teacher/student related factors (e.g., students' initial achievements). Teachers' selection and implementation of function problems was explored through classroom observation and document reviews. Each teacher was observed teaching all aspects of the function concept. Lessons were tape-recorded and field notes were taken. Students' learning was investigated through pre and post tests which encouraged them to provide reasons for their answers. Clarification interviews with three students from each class were carried out after each test.

Theoretical Frameworks and the Data Analysis

The methods of discourse and content analysis (Philips & Hardy, 2002) were used to analyze the qualitative data. These methods aimed to discern meaning embedded in the written and spoken languages and to construe them in the surrounding conditions. Literature associated with the cognitive processing and teaching of the functions (see, for instance, Breidenbach et al, 1992; Vinner, 1983) and the notions of 'task feature' and 'cognitive demand' (Stein et al, 1996) were used to analyze the task quality and teaching discourses displayed during the task implementation. Initial codes were assigned to a data base of 308 tasks presented to the students during the observations (Ahmet: 158, Burak: 150). Since there was no difference in the teachers' selection and implementation of function problems using set-diagrams and ordered pairs, these problems were not considered here. In the second phase of analysis the focus was upon tasks with either an algebraic (Ahmet: 103, Burak: 115) or graphical form (Ahmet: 40, Burak: 15). Codes such as 'connection needs to be established…' and 'addresses the univalence...' were established for each problem. Repeated on different copies of the texts this eventually led to the creation of three major categories: 'procedural tasks', 'conceptual tasks' and the 'others'. Aspects of each of these categories are illustrated in the result section. Concerning the teachers' task implementations, lessons were fully transcribed and considered line by line whilst annotated field notes were used as supplementary sources. The first phase of analysis produced 47 categories (Ahmet: 25, Burak: 22), for instance, Ahmet: '… *always refers to the concept definition*', Burak: '… *does not establish connections between the representations*). Repetition of this process produced 6 major categories for each teacher, and these are illustrated in the coming section as well.

Quantitative methods were used to provide descriptive statistics of the students' test results. The notions of action-process conceptions of function provided a framework to interpret the students' understanding of the concept. An action conception entails the ability to insert an element into an algebraic function and calculate its image through step-by-step manipulations (Breidenbach et al, 1992). A process conception is attained through internalizing actions associated with the previous step. It enables one to think of a function process in terms of inputs-outputs (ibid) and to construe the process in light of the concept definition. A possessor of a process conception recognizes an 'all-to-one' transformation in the algebraic and graphical contexts. The notions of action-process conceptions were also utilized to identify key features of the teachers' teaching discourses which could encourage their students' understanding towards a process conception of function or confine it to an action conception of function. Lastly, cross-case analysis (Miles & Huberman, 1994) was used to establish the relationship between the cases. Instances where the students displayed noticeable differences in their understanding of the function concept were identified and cross referenced to corresponding variables in the teachers' selection and implementation of the function problems. This was also associated with a reverse analysis – teachers' selection and implementation of the function problems and student learning.

RESULTS

An analysis of the data base indicated that the teachers differed considerably in their tendencies to use conceptual or procedural tasks (Table 1).

Task profiles	Ahmet	Burak
Conceptual tasks	63	25
Procedural tasks	20	75
Others	60	30
Total (n)	143	130

Table 1: Task profiles used by the teachers.

Procedural tasks were implemented through the application of rules and procedures and they had the potential that students could develop misconceptions such as the notion that a function is an algebraic expression in an equation form (Vinner, 1983). Conceptual tasks were considered to pose cognitive demands on the students and engage them with the concept of function, its properties, and related sub-ideas. They could encourage the development of a process conception of function. The problems that fell into the category of 'others' did neither have a clear focus nor cognitive demands as indicated above. Such problems could be manipulated procedurally or they could encourage students' conceptual understanding but this depended upon the teachers' approach. Table 1 illustrates that Ahmet prioritized conceptual tasks over the procedural ones in the ratio 3:1 whilst Burak did the reverse in the same quotient. For instance, all the graphs Burak presented to his students were smooth and continuous lines or curves. This limitation is likely to cause, and did so, students to develop a continuity misconception. In contrast, Ahmet presented his students with many problems that were conceptually focused and cognitively demanding, for example partitioned graphs, and tasks that encouraged his students to spontaneously reflect upon a function process and the inverse of this process without loosing the sight of univalence. The distinction continued in the teachers' task implementation (See table 2).

Ahmet	Burak
• Prioritizes concepts over the procedures. • Establishes connections between the representations and between the ideas. • Implements procedural tasks in a conceptual way. • Encourages students to visualize the graphs of functions. Enforces their flexibility at shifting between algebraic and graphical forms of the function. • Attentive to the continuity and consistency in the task demands performed one after another. • Displays multiple perspectives on a task (e.g., identifies situations in which an algebraic or graphical relation did or did not represent a function).	• Prioritizes procedures over the concepts. • Does not establish connections between the representations and the ideas. • Implements conceptual tasks in a procedural way. • Makes interference: The teacher diverted the students' attention from the concept of function and engaged them with procedures or other mathematical ideas. • Does not care continuity and consistency in the task demands performed one after another. • Oversimplifies the task demands (e.g., totally ignores the task demands and manipulates the functions like an ordinary algebraic expression).

Table 2: Key features of the teachers' task implementation.

Irrespective of task quality Ahmet created task conditions that encouraged his students to develop conceptually rich knowledge of function. His implementation of tasks had six crucial aspects each of which acted as a scaffold promoting his students' progress towards a process conception of function. Burak's task implementation included six constraints each of which caused reductions in the task demands and apparently played a major role in confining his students' understanding to an action conception of function.

To illustrate this, an example is presented from each teacher. Ahmet (the teacher of Class A) presented the following to his students:

What are the values of ‘a’ and ‘b’ for which $f: R \to R$, $f(x)=(a-2)x2+(b+1)x+5$ represents a constant function?

After explaining with the aid of a set-diagram the idea that a constant function – an all-to-one transformation – satisfies the univalence condition Ahmet continued (Episode A):

Ahmet: This expression involves something that does not allow the transformation of all the reel numbers to one and the same element. What should we do so that this function produces the same element…whatever we put into the *x*?

Student: The value of ‘a’ is 2 and the value of ‘b’ is -1.

Ahmet: How did you find out?

Student: The expression must involve just 5 so that it matches all the values of *x* to 5…

Ahmet: If the rule of a function involves an independent variable like *x*, that function produces different outputs… We should fix the value of *y*, the image. We ensure it as we remove the terms containing *x*’s. … No matter what put into the *x*, say -5, 0, 4…, all goes to 5 [under this function]…

Using the definition of the constant function Ahmet encourages his students to find out the idea that the terms containing x must be removed form the expression so that it transforms every input to one and the same output. Burak (the teacher of Class B), when implementing epistemologically the same problem, brought an algebraic description $f(x)=a$ ($a \in R$) to the students’ attention and repeatedly emphasized factual knowledge – a constant function does not involve *x* – but he did not illustrate the underlying meaning. His students were asked to:

Work out the precise form of the constant function $f(x)=(4n-2)x+(2n+3b)$.

Burak explains (Episode B):

Burak: … Let’s remember the algebraic form of the constant function; it will help us so much. …we represented it as $f(x)=a$, $a \in R$. So, could we say that a constant function involves just a number… In this expression there are two terms; one is the constant term, 2n+3, and the other is a term involving *x*, (4n-2)*x*. … So, first of all we should get rid of the term containing *x*…if this is the constant function…it must not involve *x*. How can we do that…?

Student: We would equalize the coefficient of *x* to 0.

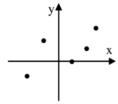
Burak: …This is what we must do… We should equalize the coefficient of *x* to 0…

Through these two examples we can see that Ahmet prioritizes the concept over the procedure whereas Burak does the reverse.

Learning Outcomes

Pre-test given to the students within each class indicated that there was almost no difference in their informal knowledge of function: their conceptual understanding (e.g., an understanding of dependence between two varying quantities) and their procedural skill (e.g., manipulating algebraic expressions). Nevertheless, after teaching the two groups displayed considerable differences in their understanding of the function concept. To illustrate this we consider their responses to two of the tasks presented to them.

1: Does the graph made up five discrete points represent a function? Give your answer with the underlying reason.



2: Does $y=7$ represent a function on R? Give reason to your answer.

In response to Task 1, 61% of Class A students specified the domain and illustrated the transformation over the graph. Only 30% of Class B students did this. The largest group within Class B (44%) disclosed a continuity misconception by linking the points with curves or broken-lines and then claiming that the graph they had sketched represented a function. This misinterpretation was evident amongst only 14% of Class A students. Interestingly, 82% of both classes provided a correct response to the second question. However, here the similarity ended – concept driven explanations dominated reasons given by Class A students (90% of correct answers) and exceeded those given by the students of Class B in a ratio of 2:1. In the interviews, two students from Class A (Okan and Demet) and one from Class B (Aylin) displayed a strong process conception of function both in the algebraic and graphical contexts. All three identified the circumstances where the graph did or did not represent a function and they articulated the idea that a constant function transforms every input to one and the same output. Okan's response to Task 1 is typical:

> …if the domain contains only these five elements [marks the pre-images on the x-axis and illustrates the transformation over the graph], this graph represents a function… Otherwise it does not, because it leaves elements in the domain. …

Erol (Class A) and Serap (Class B) appeared to be in transition from an action to a process conception of function in the graphical situation. They rejected the graph arguing that it did not meet the univalence condition, yet they failed to recognize the function process defined on five split domains. Serap was moving towards a process conception of function in the algebraic context. Although she construed $y=7$ as a process doing a transformation, she could not recognize an 'all-to-one' transformation in the situation. Erol's response to $y=7$ suggested that he had developed a process conception.

> …no matter what we give for x, it goes to 7…f of 1 is 7, f of 2 is 7… This function matches all the elements in the domain to a single element [7]…

Belgin had an action conception in the algebraic and graphical situations. Although she had an idea that a graph of function transforms elements from the x to the y-axis she could not illustrate how and why this transformation occurred. Acting with her concept image she disclosed a continuity misconception: "*…I have not seen any graph like this. I must join them in some way…*". She recognized the expression *y=7* from memory but displayed no understanding of the 'all-to-one' transformation that the function does.

Table 3[1] summarizes the interviewees' development of the function concept and compliments the class difference identified through post-test questionnaire.

Representations	Class A			Class B		
	Okan	Demet	Erol	Aylin	Serap	Belgin
Graphical task (1)	P	P	A→P	P	A→P	A
Algebraic task (2)	P	P	P	P	A→P	A

Table 3: Summary of the interviewees' development of the function concept.

DISCUSSION AND CONCLUSION

The purpose of this paper was to illustrate the impact of the teacher's task selection and implementation on their students' understanding of the function concept. The findings suggest that procedural tasks, when implemented with little connection to underlying meaning, are likely to confine students' understanding to an action conception of function and create misconceptions. Almost 50% of Burak's students, drawing upon his emphasis upon smooth and continuous graphs, revealed a continuity misconception with their desire to link graphs formed from discrete points with curves or broken-lines.

Tasks shape the way students think about the subject matter and, thus, can influence their learning (Stein & Lane, 1996). Our evidence suggests that though they may promote thinking, it is the conditions promoted by the teachers through their models of implementation that are more influential in supporting student learning. We can see in Episodes A & B that the two teachers implement epistemologically the same problems but emphasized different things. Ahmet engaged his students with the notion of constant function using the definition as a cognitive tool. Unlike Burak, he does not set up an easily accessible goal (get rid of the terms with *x* from the expression) but prompts his students' thinking by providing concept-driven explanations: "*…there is something that does not allow the transformation of all the real numbers to one and the same element*". In contrast, bringing $f(x)=a$ ($a \in R$) to the students' attention, Burak emphasizes factual knowledge, *a constant function does not involve x*, but he does not encourage his students to establish the underlying reason for such knowledge. We can see the impact of these differences on students' learning. Although 82% of each class identified *y=7* as a function, almost all of

[1]**Abbreviations: A:** An action conception of function; **P:** A process conception of function; **A→P:** Transition from an action to a process conception of function.

Ahmet's students provided concept-driven explanations whereas less than 50% of Burak's students did so. This was complimented through the interviews in which three students from Ahmet's class indicated a full understanding of the transformation in $y=7$ whilst only one from Burak's class did so.

In conclusion, the evidence suggests that tasks should not be seen as a panacea. It is the conditions associated with task resolution that engage students with the subject matter and, therefore, may help them make progress in learning. In the context of functions, these conditions appear to include using process-oriented language consistent with the epistemology of the function concept, establishing connections between the ideas and the representations, applying continuity and consistency in successive task demands, encouraging students' visual thinking, displaying multiple perspective on a task, and using the definition of function as a cognitive tool when resolving function problems.

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INTERPRETATION AND TEACHER IDENTITY IN POST-SECONDARY MATHEMATICS

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Over a six-month period conversations were held with mathematics graduate students exploring their experiences and perspectives of mathematics teaching. Using hermeneutic inquiry and thematic analysis, the conversations were analysed and interpreted with attention to themes and experiences that had the potential to influence the graduate students' ideas about and approaches to teaching. The purpose of this research project was to uncover issues and difficulties that come into play as mathematics graduate students develop their views of their roles as university teachers of mathematics. It is hoped that this research will contribute to the understanding of teaching and learning in post-secondary mathematics as well as provide guidance in structuring post-secondary teacher education in mathematics.

INTRODUCTION

Within the field of mathematics teacher education, mathematics graduate students have recently become subjects of investigation. While research tends to focus on pre-service elementary and secondary mathematics teachers, and undergraduate experiences in mathematics, little has been done to examine prospective university teachers of mathematics and their understanding of its teaching and learning (Bass, 2006). As a result, the experiences of mathematics graduate students and the development of their teaching practices are not well understood.

Almost seventy-five per cent of mathematics PhDs will become professors at post-secondary institutions dedicated to undergraduate education rather than research (Kirkam et al., 2006). Since much of their careers will be spent in the classroom, attending to the manner in which mathematics graduate students develop their teaching practices is crucial in preparing them for their future profession. Moreover, as mathematics graduate students and professors represent the last models of mathematics instruction for future elementary, secondary, and post-secondary mathematics teachers, university mathematics teaching has a far-reaching influence on teaching at all levels (Golde & Walker, 2006; Shulman, 2004).

The most recent research into mathematics graduate students' teaching has examined their classroom practices and possible connections between their practices and beliefs about teaching and learning. Researchers concluded that newly acquired positive attitudes and beliefs about teaching mathematics did not produce hoped for changes to graduate students' teaching practices (Belnap, 2005; Speer, 2001). Even when

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graduate students in mathematics could speak of teaching using reform-oriented terminology (e.g., teaching for understanding, group learning), these students also reported that they maintained a traditional lecture style form of instruction (Belnap, 2005). Other research has shown that enrolment in a course in pedagogy did not produce expected changes to mathematics graduate students' teaching practices (DeFranco & McGivney-Burelle, 2001). Thus, as these studies have found that informing mathematics graduate students of different approaches to pedagogy, student learning, and curriculum reform did not change classroom practices, something remains to be explored.

ORIENTATION OF THE RESEARCH

While previous research reports that mathematics graduate students receive very little preparation for teaching, one could argue that they have essentially received years of instruction in teaching mathematics through their experiences as students. As well, through involvement in the routines of a department of mathematics, graduate students' views of the discipline and its teaching are shaped (DeFranco & McGiveny-Burelle, 2001). Indeed, graduate students in mathematics encounter many texts and contexts that have the potential to be interpreted as having implications for how they should live their lives and convey their work as mathematicians (Austin, 2002). In addition to the experiences in departments of mathematics that have the potential to interfere with graduate students' teaching, their own ideas and beliefs also appear to have an influence on their teaching (Speer at al., 2005).

Therefore, a more thorough investigation of graduate students' lives in mathematics is needed in order to understand the role each of these phenomena has in the development of graduate students' teaching practices. Within the graduate students' lives in mathematics exists a complex and intricate interplay among the structures that mathematics graduate students encounter, their feelings about mathematics and themselves, their interpretations of the nature of mathematics, and their sense of their new role as teachers. The bearing that these experiences have on mathematics graduate students' teaching practices must be explored in order to gain some understanding of how teacher education for post-secondary teachers of mathematics might approached and developed.

To this end, in an effort to inform teacher education programs for mathematics graduate students, the intent of my research is to gain a deeper understanding of mathematics graduate students' experiences in mathematics and how these experiences may be interpreted as having meaning for their teaching practices.

THEORETICAL FRAMEWORK

As my questions are concerned with various texts in mathematics (e.g., textbooks, directions for producing mathematics), the structures that graduate students encounter (e.g., department structure, teaching assistantships), and interpretations within the lived experiences of graduate students and their teaching, I am drawn to hermeneutics as a way to seek an understanding of these phenomena. Hermeneutics helps one to unearth the ways and the whys in which we understand life, and how we can create and find meaning through experience and social engagement (Brown, 2001; Smith, 1991). Hermeneutics also recognizes the place of language and the implied importance of text in human experience.

Hermeneutic inquiry also acknowledges the significance of non-textual phenomena such as culture, human existence, and being itself (Gallagher, 1992; Ricoeur, 1976). With this perspective in mind, the non-textual phenomena that mathematics graduate students encounter include their professors' teaching, departmental expectations, and so on, all of which have interpretive implications for what graduate students make important in their lives. As students' knowledge of their future worlds develop "as a consequence of their encounter with the department: semi-automatic, barely conscious interpretations of what teachers say and do" (Gerholm, 1990, p. 264), hermeneutics opens up a space for understanding interpretations within all forms of interactions and experience.

For this research project, hermeneutics affords attentiveness to the questions: What variety of experiences do mathematics graduate students encounter as they progress through their graduate programs? Within their experiences, what in particular is taken as having meaning for who and how they should be as mathematicians and as teachers of mathematics? How are these experiences interpreted to have meaning for how one lives a life in mathematics? Finding the answers to these questions will help to deepen the understanding of teaching and learning in post-secondary mathematics and provide guidance in structuring post-secondary teacher education in mathematics.

THE RESEARCH STUDY

Graduate students in mathematics from an urban, doctorate granting university were approached to be participants in this study. Six students agreed to participate. The group was quite diverse in their backgrounds: three were master's students and three were doctoral students; they ranged from a first semester master's student through a final year doctoral student; four were men, two were women; their ages ranged from 22 to 33 years; and there were four nationalities among them. During their graduate programs in mathematics, all of the participants had been assigned to teaching assistantship duties such as tutoring workshops where they helped students one-on-one with homework exercises, marking homework and exam papers, or one-hour tutorial sessions during which they presented mathematical topics similar to those in the affiliated lecture section of the course.

Carson (1986) and Van Manen (1997) propose conversation as a mode of doing research within hermeneutic inquiry to explore and uncover one's own and others' interpretations and understandings of experience. In consideration of this, over a period of six months, a series of five audio-recorded conversations were conducted with the research participants. The first two meetings and the final meeting were conducted with each participant individually, each meeting lasting approximately one hour. The third and fourth meetings were conducted with all participants present, each lasting just under three hours. A recursive process was used in which the topic of subsequent conversations was based upon themes from previous conversations. Throughout the project, the research participants had the opportunity to review the analyses in a collaborative effort to refine, augment, and improve the reporting of their experiences.

Each conversation was transcribed by the researcher, who listened for the topics of conversation and the language used by each of the research participants. Notes were made of the congruence among the research participants. These similarities were not limited to broad categories of their lives, such as how they each had to attend to their teaching assistantship duties or their graduate level course work. Rather, it was opinions and perspectives about various aspects of their experiences that appeared to be in common. These similarities were grouped into themes using the guidelines of thematic analysis described by Braun and Clarke (2006). The themes and the participants' comments within each theme were then assembled and analysed using a hermeneutic, interpretive lens to understand what facets of their lives in graduate school were taken as having meaning for their identities as mathematicians and teachers of mathematics.

FINDINGS

There were several experiences and perspectives that the mathematics graduate students voiced as having an influence on their teaching practices. These included observable structures, such as their teaching assistant duties and the physical spaces in which they worked. Some of the influences on their teaching were not as tangible; for example, their views on the role of a professor. These and other influences are described below.

The structure of their teaching assistant work

The time and physical structures of the graduate students' work as teaching assistants were said to prevent them from being able to engage in meaningful experiences with undergraduates. In the tutoring centre, the number of students waiting for help and the hours spent helping students repeatedly with the same questions quickly diminished the graduate students' ability to provide meaningful learning experiences. The frustration and exhaustion within the tutoring centre was common among the graduate students. There was also a sense of disappointment of how things took place over time. In this regard, the graduate students weren't able to observe the undergraduate students' progress and understanding of concepts over time, and so the act of tutoring in the lab situation was felt as an unrewarding and tiring experience. One participant described how the lab situations became "how fast can you turn them over." Rather than being able to provide the undergraduate students with an in-depth learning experience, when there were many undergraduate students waiting for help, it became "a lot faster to plug and chug."

Interpretations of undergraduates' behaviour

One expected outcome for the research was that the mathematics graduate students' teaching practices would be mostly shaped by their own professors' teaching styles and approaches to mathematics. However, what appeared to be most influential to the graduate students' interactions with undergraduate students was the interpretation of undergraduate students' behaviour. In particular, if an undergraduate student approached a graduate student asking for quick help with a particular problem, their actions were interpreted to have the following meaning – that the undergraduate student was not interested in mathematics, not interested in graduate student and what they had to offer, and that the undergraduate was not a motivated student in general. These opinions were based on short interactions when helping students one-on-one in the tutoring centre. This interpretation caused the graduate students to not engage in a potentially educative moment by limiting their interaction with the undergraduate. This limited contact consisted of providing students with brief communication, showing only how to calculate answers rather than understand the conceptual processes and meaning in the mathematics. The graduate students' ideas of how undergraduate students should learn and be in mathematics, a sense for 'what it takes' to be in mathematics, was one of the leading obstacles in understanding undergraduates' struggles with mathematics and embracing alternate modes of engaging with learners.

The problem is in k – 12 mathematics teaching

During our conversations, the mathematics graduate students expressed frustration about what they perceived as the low quality of learners in first year university mathematics courses. They conveyed strong opinions about "fixing" teaching at elementary and secondary levels; that if students were taught mathematics properly at an earlier age, then teaching mathematics at the university level would become unproblematic. When asked whether changes could be made to university level mathematics teaching, the graduate students expressed a powerless to attempt new approaches themselves because of a perceived set curriculum and expectations for teaching at the university, which is reminiscent of oft-heard arguments given by school teachers for not adopting reform-oriented practices. Moreover, when discussing whether they could make changes to their teaching practices, the graduate students expanded the issue from a local to a global problem – describing how one would need to change the "entire system" (meaning the university, the provincial and national boards of education, as well as k – 12 education) before they could enact new ways of teaching mathematics at the university.

Calculus – How many ways can you skin a calculus class?

It is important to include calculus as one of the many structures graduate students encounter in their first experiences in teaching mathematics. Mathematics graduate students are most often assigned teaching assistantships in first-year calculus courses.

While it may have been years since they were students in a calculus course, they have fixed ideas of how to teach calculus. As one participant remarked:

> It's easy to keep teaching calculus like this. We've done it forever. We know exactly what we have to do. Almost everyone does it the same way. I mean by the time you have your PhD, you've probably been teaching calculus three or four times. You've taken it, you've TA'd for it. I mean, you know the problems, you know the classic examples. You almost don't even need a book. You can just walk up there and start teaching.

Another participant asked "How many ways can you skin a calculus class?" denoting his view that there is only one way to teach calculus. There was the sense that there is nothing left to learn about teaching calculus. In contrast to this, the participants also described how "you have to teach these tools just because that is the way the course is set up." While the participants initially spoke of teaching calculus as something effortless, here they recognized the imposition of the structure of calculus courses; that in teaching calculus, they had to maintain the predefined structure. Finally, they shared their frustration about how calculus had become representative of all mathematics to those outside of the discipline and they expressed a desire to teach other "flavours of mathematics."

Teacher versus professor

The graduate students described a difference between teachers and professors, as well as a difference between students' roles in high school versus the university. They held tightly to the notion that they were professors, not teachers. While they described the expectation for a teacher to engage and guide students in their kindergarten through grade twelve learning, they saw the role of a university professor as solely presenting material with the students' role being to teach themselves. The professor's task, then, was to provide the mathematics to their students through lecturing. Beyond this, however, the professor had no responsibility in assisting the students further, in helping students to understand the mathematics, or in motivating students to learn.

CONCLUSIONS

In light of the recent research of teacher education programs for mathematics graduate students (Belnap, 2005; Speer, 2001), the purpose of this research project was to uncover issues and difficulties that come into play as mathematics graduate students develop their views of their roles as university teachers of mathematics. It is clear that there is an intricate and complex interplay of experiences and perspectives that have the potential to work against teacher education programs in post-secondary mathematics. In their initiations into teaching, both new experiences and previously formulated ideas influenced their views about teaching and how they viewed themselves as teachers.

The graduate students' experiences in the tutoring centre and their opinions about undergraduates' behaviour shaped the images they were developing about undergraduates. Specifically, what was observed in the conversations with the graduate students was the emergence of a disapproving and negative attitude toward the students they were expected to teach. The tendency to view learners of mathematics in these ways has the potential to be an obstacle for teacher education programs in post-secondary mathematics. Specifically, because they viewed their students as poorly prepared, the graduate students felt they should not have to assist their students beyond a rudimentary level. If alternate modes of engaging with learners of mathematics are not seen as useful or necessary, as a consequence, it is unlikely that alternate modes promoted through teacher education programs will be embraced. Further, these negative attitudes about undergraduates have the potential to be carried forward into the graduate students' future work as professors.

In the graduate students' experiences as teaching assistants for first-year calculus courses, they described teaching calculus as easy, something that could be and had been predetermined. Not only did calculus represent their first experience with teaching students mathematics and much of what teaching entails, such as creating and marking exams and marking homework assignments, it also appeared to have a large influence on how they felt about teaching. The sense that teaching was preset extended to other courses as well with one participant remarking "we teach the best possible scenario." This perspective led to inflexibility in considering alternate practices not only for calculus, but also for the courses they wished to teach when they became professors. Thus, if teaching of mathematics is already fixed and does not need to be changed, there will be resistance to learning alternate ways of teaching mathematics offered in teacher education programs.

Within the graduate students' positions that they are professors, not teachers, and the view that issues in mathematics education rest solely in kindergarten though grade 12 education exists a disassociation from their role as teachers of mathematics. These views represent a distancing from any responsibility or role they might have in the teaching and status of mathematics. This perspective makes changes to teaching practices difficult when one's role is seen in this way – as one who presents mathematics, but does not teach mathematics. Since the graduate students did not view themselves as teachers, there would be little impetus for them to engage in new ways of teaching or connecting with learners of mathematics.

To conclude, the mathematics graduate students encountered several issues in their graduate programs, yet did not have a forum or support network to assist them in understanding their experiences. In their attempts to understand these issues without guidance, many of their pre-existing notions about learning mathematics and new experiences had significant influence on their views of teaching. The understanding of their experiences in this research report has the potential to inform university level teacher education. In particular, this research can help to either establish new or inform current teacher education programs in university level mathematics, as well as offer support in developing mathematics graduate students' teaching practices.

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A VERBAL FACTOR IN THE PISA 2003 MATHEMATICS ITEMS: TENTATIVE ANALYSES

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This study uses a statistical method to identify verbal items among mathematical items from PISA 2003. The verbal items are preliminary analysed and compared to the non-verbal items concerning number of text lines, response types, cognitive level, and competences measured. The results show that the verbal items, to a higher percentage than the non-verbal items, measures the reproduction competency, are straightforward, and of open constructed-response type. These results and proposed further analyses are discussed.

INTRODUCTION

This report presents a study that is part of a larger research project. The purpose of the overall project is to develop new knowledge of, as well as tools for, validation of assessment in mathematics and science with a main focus on the importance of the written language. One aim of the project is to identify and describe *critical language traits* that have an impact on the tasks' difficulty levels. The purpose of the study presented in this paper is to identify PISA mathematics items for which student performance is influenced by reading ability. The purpose is also to perform a few statistical analyses in order to characterize these items in search for possible explanations of the verbal factor. The study is an initial and preparatory study, as further analyses of the verbal items will follow.

BACKGROUND

Most assessments of mathematical ability, in particular international comparative studies, consist of paper and pencil tests. According to Messick (1989), a threat to the validity of evaluations of educational achievement is all types of construct-irrelevant variance. A source of potential threats to the validity of a written mathematics test is therefore the readability of the language used to formulate the tasks. On one hand, tests which are intended to measure achievement in mathematics should not measure reading ability. In the framework for PISA it is written that: "The wording of items should be as simple and direct as possible." (PISA, 2006, p. 108). On the other hand, the relation between reading and knowing mathematics is complex. You have to be able to read and write in order to pass a paper and pencil test, and also, communication is one of the competences brought forward within frameworks describing school mathematics worldwide (e.g. NCTM, PISA). It is a matter of judgment to evaluate the existence of improper or irrelevant language influence on achievement in mathematics. A summary of existing research points at several

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linguistic features that might make mathematics items difficult. These features are presented below.

Difficult vocabulary. The mathematical vocabulary is highly technical and includes both mathematical words (sum, fraction) and words with particular meanings in mathematics (borrow, product) (Mitchell, 2001; Schleppegrell, 2007; Shorrocks-Taylor & Hargreaves, 1999). In order to read texts in mathematics it is necessary to be able to recognise which category words belong to in order to be able to interpret them correctly. Words with multiple meanings seem to cause difficulties for English first language and second language learners, who confuse the meanings used in different contexts (Wellington & Osborne, 2001, in Dempster & Reddy, 2007).

Multiple semiotic systems. One challenge in mathematical texts is the multiple semiotic (meaning-creating) systems used within mathematics: symbols, oral language, written language, and visual representations (graphs, diagrams). In a written mathematics test the oral language is not used, but there are still at least three semiotic systems that the students need to be able to interpret (Kress, 2003; Schleppegrell, 2007).

Grammatical patterns. Another difficulty with the mathematical register is the existence of various grammatical patterns. Liberg, Folkeryd et al. (Liberg, Folkeryd, af Geijerstam, & Edling, 2002) state that a text is built using different forms of connective markers and genre patterns, and these are important in order to make it possible for the reader to build a cohesive, coherent understanding of the text. There are some typical – and sometimes problematical – patterns for texts in mathematics. One characteristic is the use of *passive voice*. A subject of a verb in the passive voice corresponds to the object of the same verb in the active voice – “the ball was thrown by John” (Dempster & Reddy, 2007). Another is the specific use of logical connectives, such as if, unless, although, whenever, therefore (Dempster & Reddy, 2007; Schleppegrell, 2007). Students had in particular problem with items for which “when” was used as a logical connective rather than to start a question (Dempster & Reddy, 2007). A third pattern is the use of *long dense noun phrases* to construct concepts (Schleppegrell, 2007). One other characteristic that has been identified as problematic is *nominalisation*, the turning of a verb into a noun (Dempster & Reddy, 2007).

Other traits. Besides the properties mentioned above there are other characteristics of texts that are important to consider when it comes to reading comprehension, e.g. *sentence complexity* (Dempster & Reddy, 2007; Liberg, Folkeryd, af Geijerstam, & Edling, 2002), the use of *many qualifiers* (adverbs, adjectives, and prepositional phrases) (Dempster & Reddy, 2007), and the *content* of the text (Liberg, Folkeryd, af Geijerstam, & Edling, 2002).

PISA 2003

The Programme for International Student Assessment (PISA) is an internationally standardised assessment jointly developed by participating countries and administered to 15-year-olds in schools. PISA measures mathematical literacy, reading literacy, scientific literacy, and problem solving. The mathematics items in PISA 2003 are of five different types when it comes to the response the students make: multiple choice (MC), complex multiple choice (CMC), short response (SR), closed-constructed response (CCR), and open-constructed response (OCR) (PISA, 2006). Multiple choice items (MC and CMC tasks) give the students a number of alternatives to choose their answer from. In the complex multiple-choice format items, the students are required to select a response from given optional responses, e.g. by marking true or false to each of a complex of statements. The SR tasks are formulated so that the students are required to construct a short response in their own words, often a single word or a calculated quantity. Constructed response tasks (OCR and CCR tasks) require the students to construct a solution (and formulate an answer) rather than select an answer. Open-constructed response tasks usually support more than one solution process and a wider range of possible responses. A closed-constructed response item is very much like traditional fill-in-the blank questions. There is only one correct answer and the item usually requires simple recall of information. About one third of the items in PISA are open constructed-response items, one third are closed constructed-response items, and one third multiple choice items. (PISA, 2006). The PISA mathematics items are also divided into three competency categories: connections, reflections, and reproduction (PISA, 2006).

METHOD

The study consists of two parts. First, a statistical method is used to identify so called *verbal items* for which the students' reading comprehension abilities statistically explain some part of the variation of the students' results. Second, statistical analyses of the differences and similarities between the verbal items and the non-verbal items concerning number of text lines, response types, cognitive level, and competences measured is performed.

Collection of data

The object of analysis in this study is the available data concerning all of the Swedish versions of the mathematics items from PISA 2003. There are 84 items and 4624 Swedish students in the datafile. Each student has encountered much less than 84 tasks, but the items were distributed according to a statistical model in such a way that so called *plausible values* of the students' abilities could be calculated (PISA, 2006). In addition, the plausible values from the reading literacy part of the assessment are used as a measure of the students' reading abilities.

Method of analysis

Each item was analyzed using an ordinal regression model with item score as the dependent variable, and PISA plausible values (PVs) in both mathematics and reading literacy as covariates. Items for which the reading covariate gives a significant contribution (<0.05) to explaining the item difficulty, while subject-specific achievement level is taken into account, were identified and subjected to further analysis (Nyström, 2008). More concretely this means that for the verbal tasks, if a group of students had the same plausible values in mathematics, the students with higher plausible values in reading literacy had a higher success rate on the item. These items can therefore be said to measure reading ability to a higher extent than other items. Since several items were on the border of having the reading covariate giving a significant contribution, the items were divided into three groups: *verbal items*, i.e. tasks for which the reading literacy plausible variable explained the variation of the students' performance to a significance level less than 0.03, *non-verbal items,* i.e. tasks for which the reading literacy plausible variable did not significantly explained the variation of the students' performance (significance level over 0.065), and *borderline items*, i.e. tasks for which the significance level is between 0.03 and 0.065 (all borderline items had significance levels between 0.042 and 0.062).

When identified, existing statistical information of the items was used to compare the verbal and the non-verbal items: the number of text lines in the items, the items response type, their cognitive levels, and the competence measured according to the PISA framework (PISA, 2006). The purpose of this comparison was to try to explore possible explanations of the verbal factor of these items. Some of the items might have a particularly complex sentencing or might contain very difficult words, but the verbal factor might depend on other aspects of the items. In order to not jump to conclusions on what is difficult for the students, it is important to systematically examine the design, structure, and contents of the items. Therefore, the verbal items were compared to the non-verbal items and to the set of all items (including borderline items) from several aspects: the number of text lines in the item, the item response types, the cognitive level of the item, and the competency measured by the item. The item response type and the competence classes are defined within PISA (see the section on PISA above). The number of text lines was counted and the cognitive level was determined within a previous research project (Lindström, unpublished). The cognitive level was defined as the number of steps required in order to solve the item. The classification uses the following three categories: *straightforward* application of learned material, application of a definition in *one step* to make a conclusion, and application of one or several definitions in one or *several steps* to make a conclusion.

Results

In PISA 2003 there were 84 mathematics items and 19 of these (23%) were identified as verbal items according to the method described above. There are also 6 borderline items and 59 non-verbal items. The first statistical comparison between the verbal and the non-verbal items was based on a calculation of the number of lines of text in the items. Both verbal and non-verbal items had in average 3.8 lines of text, so this first shallow analysis did not explain anything of the verbal factor in the items.

Response types

For each response type the percentage of verbal and non-verbal items varied. The largest difference was between the percentage of verbal items that are open constructed-response items (OCR), 37%, and the percentage of non-verbal items of the same type (17%), see Figure 1. These verbal items of OCR type may have been identified as verbal items *not* because they are difficult to read or to understand, but because the students' have to formulate their responses themselves. It is likely that the students' reading ability is closely connected to their ability to write and therefore to their ability to formulate solutions. These items might still have a verbal component in the sense that they are difficult to read, but when searching for verbal traits in the formulations of the items it is necessary to be careful when drawing conclusions.

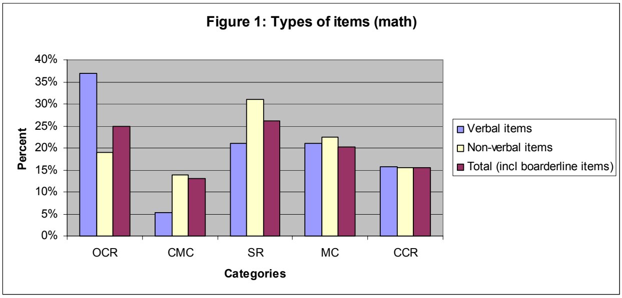

Figure 1: Distribution of verbal and non-verbal items over response types

The non-verbal items have however a higher percentage of the more complex multiple choice items (CMC) and short response items (SR) than the verbal items. It is difficult to determine why the CMC items are "less verbal" than other types. Perhaps a third factor, e.g. some type of logical thinking, is more significant when it comes to explaining the variation of the CMC items. Or perhaps the item authors choose to use simple language when the item format is complex in it self. A closer analysis of the wording and context of these items is necessary in order to find possible explanations of this result. A closer analysis of the SR items is also necessary.

The regular type of multiple choice items (MC) and the closed constructed-response (CCR) items are almost equally represented among verbal items and non-verbal items. It seems that even if some items demands a higher level of reading ability, there is no obvious inherent verbal factor in these items due to their format.

Competences and cognitive levels

The PISA mathematics items are divided into three competency categories: connections, reflections, and reproduction. The verbal items are to a higher extent of the category *reproduction* than the non-verbal items, see Figure 2.

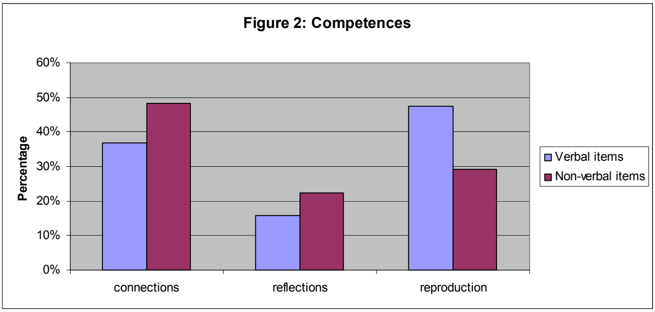

Figure 2: Distribution of verbal and non-verbal items over competence class

This is to some extent in line with the results concerning the cognitive levels, see Figure 3. As mentioned in the method section, a classification of the items resulted in a partition of the items into three groups: items with solutions containing a *straightforward* application of learned techniques and/or methods, application of a definition in *one step* to make a conclusion, and application of one or several definitions in one or *several steps* to make a conclusion.

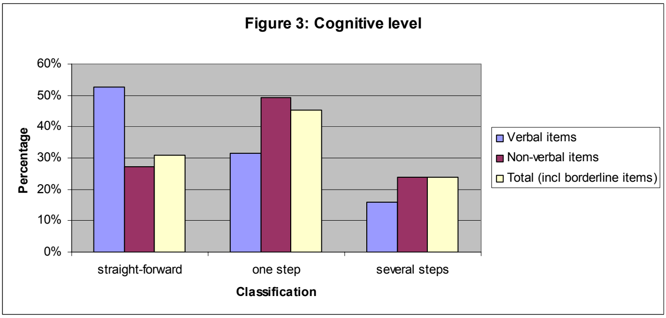

Figure 3: Distribution of verbal and non-verbal items over cognitive level

The verbal items are classified as straightforward to a higher degree than the other items. It is difficult to explain these results without further analysis of the items.

DISCUSSION

The purpose of this study is to identify PISA mathematics items for which student performance is influenced by reading ability and to perform a few statistical analyses in order to characterize these items in the search for explanations of this verbal factor.

Discussion of method

There are some weaknesses in the presented method of identifying verbal items. If the students' mathematics and reading abilities have a high correlation, the identification of the verbal items is less reliable. The same is true for items that have a very high or very low passing rate. Continued studies will further examine the correlations and the passing rates in order to enhance the reliability of these results. There is also a difficulty in the use of plausible values (PVs) to compare students' abilities. The PVs (PISA, 2006) are meant to be used in order to compare groups of students, not to measure individual students' abilities. A few items were analysed using the second PV for each student instead of the first (each student has five) which resulted in the same statistics. This indicates that the PVs are possible to use for the type of analysis produced here. Further work in order to confirm this is however necessary.

Discussion of results

It is not surprising that verbal items are often found among the open constructed-response items. Items for which the students are supposed to formulate a solution in writing are probably more sensitive to the students' reading abilities than e.g. multiple choice items, since the ability to read is closely connected to the ability to write. This will however make it difficult to identify language traits that make these items difficult, since there might not be any. This is a general problem when it comes to analysing test items.

For some items, there is an obvious limit between reading an item and solving the task presented in the item. An example is the famous four colour map problem: How many colours are needed in order to colour a (plane) map in such a way that any two adjacent regions have different colour? This is a question that many people would be able to read and to understand, but very few could solve. The problem of reading is quite clearly separated from the problem of solving, in this case. In the following task, the situation is different: If Anne was half Mary's age, when Mary was 14, then how old will Mary be when Anne is 50? The reading and understanding of this task is much closer to the solving of it, than it is for the four colour map problem. In fact, once we have read and fully understood it, the solution is right at hand. The difference between these two tasks indicates that it is possible to analyse (and perhaps even classify) mathematics items from this perspective: how closely *reading* the item is related to *solving* the item.

Classifying the items into *consistent* and *inconsistent* items (Hegarty, Mayer, & Green, 1992), or items demanding *imitative* or *creative* reasoning (Lithner, 2008) might also spread some light on the relation between reading and solving mathematical tasks. Research indicate that students with a good ability to solve consistent tasks (tasks that are possible to solve using a keyword procedure) but low ability to solve inconsistent tasks (unsuccessful problem solvers) do not read the tasks they encounter very thoroughly (Hegarty, Mayer, & Green, 1992). It is therefore possible that unsuccessful problem solvers are less affected by verbal factors in items than successful problem solvers. How this might influence the statistical identification of the verbal items will be looked into within the project. The consistent items presented by Hegarty et al. have a lot in common with tasks solvable with imitative reasoning as presented by Lithner (2008). One possibility is that imitative items are not identified as verbal at all, but that is not in line with the results concerning the high percentages of verbal items that are of the type straightforward and reproduction. Further analyses of the large quantities of existing data from PISA are both possible and important.

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SCHOOL MATHEMATICS AND MATHEMATICIANS' MATHEMATICS: TEACHERS' BELIEFS ABOUT MATHEMATICS

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There is broad acceptance that mathematics teachers' beliefs about the nature of mathematics impact the ways in which they teach the subject. It is also recognised that mathematics as practised in typical school classrooms is different from the mathematical activity of mathematicians. This paper presents evidence that some mathematics teachers hold differing beliefs about nature of mathematics, as a discipline and as a school subject and suggests possible implications for practice.

INTRODUCTION

Research aimed at describing teachers' beliefs concerning the nature of mathematics, as well as theoretical analyses of the same, have been based on the assumption that a teacher's idea of what mathematics is will influence the way in which they teach the subject (Sullivan & Mousley, 2001). Thompson (1992) quoted Hersh (1986, p. 13):

> One's conception of what mathematics *is* affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it ...The issue, then, is not, What is the best way to teach? but, What is mathematics really all about?

Since mathematics is what mathematicians do and create, answering Hersh's essential question demands a consideration of the mathematical activity of mathematicians; activity that has been contrasted with that which typically occurs in school mathematics classrooms (Burton, 2002).

SCHOOL MATHEMATICS AND MATHEMATICIANS' MATHEMATICS

Ernest (1998, cited in Burton, 2002) suggested differences between mathematics classrooms and the work of research mathematicians in relation to a) whether knowledge is created or existing knowledge is learned, b) who selects the problems to be worked on, c) the time frames over which problems are worked on, and d) the purpose of the learning (for personal achievement or to add to public knowledge). From a constructivist view of learning, such as taken in this paper and adopted by Burton (2002), learning is inherently creative but the others differences remain.

Knoll, Ernest and Morgan (2004) described contrasts between the activity of pure mathematicians and school classroom mathematical activity as "sharp" and assumed that this should not be the case. In describing mathematical research, they drew attention to its creativity and the use of strategies such as the search for examples and counter-examples, cases and constraints, patterns and systems of rules, the use of

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justification and proof, and the framing of problems. These are all things that could and arguably should be part of school mathematics. Burton (2002) identified features of the practice of mathematicians that represent desirable commonalities with school mathematics. These included the notion that both contexts constitute communities of practice, the search for connectivities among mathematical ideas, an appreciation of mathematical aesthetics, and the role of intuition in mathematical work.

One difference that appears beyond reconciliation relates to the purpose of mathematical work in classrooms and for mathematicians. Ernest (1999) pointed out that the community of mathematicians requires warrants of new mathematical knowledge, whereas, in education, learners may justify their mathematical ideas, but ultimately teachers require evidence that the student has in fact constructed the desired knowledge (which itself is not contentious in this context). In essence, mathematicians assess mathematics but educators assess learners.

In spite of this, it appears that the differences between school and mathematicians' mathematics could be greatly reduced by an increased emphasis on the use of practices associated with research mathematics in school mathematics classrooms, and there appears to be consensus that this is a worthy goal. In Burton's (2002, p. 171) words it is necessary for the "creation of a mathematically aware citizenry able to appreciate the joys of mathematics, as well as its usefulness".

TEACHER BELIEFS ABOUT THE NATURE OF MATHEMATICS

Reconciliation of school and mathematicians' mathematics requires that teachers must have an appreciation of mathematics that is akin to that of mathematicians. Ernest (1989a) described three categories of teacher beliefs about the nature of mathematics. The first is the Instrumentalist view that sees mathematics as, "an accumulation of facts, skills and rules to be used in the pursuance of some external end." (Ernest, 1989, p. 250). According to this view the various topics that comprise the discipline are unrelated. The second is the Platonist view in which mathematics is seen as a static body of unified, pre-existing knowledge awaiting discovery. In this view the structure of mathematical knowledge and the interconnections between various topics are of fundamental importance. Ernest's (1989a) third category is the Problem-solving view in which mathematics is regarded as a dynamic and creative human invention; a process, rather than a product (Ernest, 1989).

Beswick (2005) presented Ernest's (1989) categories of beliefs regarding the nature of mathematics, an adaptation of his categories with respect to beliefs about mathematics learning, and Van Zoest, Jones and Thornton's (1994) categories of beliefs about mathematics teaching as in Table 1. The rows comprise theoretically consistent views whilst the columns have been regarded as continua by some researchers (e.g., Van Zoest et al., 1994). It is recognised that few individual teachers would hold beliefs about mathematics that fall neatly into a single category and hence hold beliefs about teaching and learning mathematics that are described solely by one of the corresponding categories. In addition, this paper presents evidence that some teachers view school mathematics differently from the discipline of mathematics and hence have beliefs about teaching and learning mathematics that are not well represented by those in Table 1 either alone or in combination. An elaborated framework that accounts for this is proposed.

Beliefs about the nature of mathematics (Ernest, 1989)	Beliefs about mathematics teaching (Van Zoest et al., 1994)	Beliefs about mathematics learning (Ernest, 1989)
Instrumentalist	Content-focussed with an emphasis on performance	Skill mastery, passive reception of knowledge
Platonist	Content-focussed with an emphasis on understanding	Active construction of understanding
Problem-solving	Learner-focussed	Autonomous exploration of own interests

Table 1. Categories of teacher beliefs (Beswick, 2005, p. 40)

Adequate consideration of the context in which beliefs are articulated and/or enacted is key to reconciling apparent inconsistencies among teachers' beliefs (Beswick, 2005). Beswick (2005) included in context the individual's entire system of beliefs which can usefully be considered in terms of Green's (1971) description of belief systems. Of particular relevance here is the notion of clustering of beliefs. Disjoint belief clusters are likely to develop when the relevant beliefs are formed in different contexts (place or time) and may be contradictory since beliefs in separate clusters are not, in the normal course of events, juxtaposed to highlight their inconsistency (Green, 1971). In particular, teachers may hold beliefs about the discipline of mathematics in isolation from their beliefs about the school subject.

Examples from the literature

With the notable exception of Thompson's (1984) seminal study of the beliefs of secondary mathematics teachers that influenced the development of Ernest's (1989) categories, relatively little attention has been paid to teachers' beliefs about the nature of mathematics, and there are few reports of teachers holding different views about the discipline and school mathematics. This could be a consequence of the lack of in-depth research in the area or because such inconsistencies are rare. It would be reasonable to assume that the phenomenon would be uncommon among secondary mathematics teachers if they are required to have study mathematics to a high level - Moreira and David (2008) suggested a mathematics major is a usual requirement. However, given the worsening shortage of mathematics teachers in many countries, this is likely to become less and less the case. In the study from which the case reported later in this paper is drawn, eight of the 25 secondary mathematics teachers had studied mathematics to third year university level and, of these, just three claimed to have majored in mathematics. Both instances found in the literature and described below involved primary preservice teachers.

Mewborn (2000) used Green's (1971) ideas of beliefs systems to describe the beliefs and practice of a preservice primary teacher, Carrie, progressing through her course. According to Mewborn (2000), Carrie began with a fully integrated set of beliefs about students, teaching and learning, but held negative beliefs about mathematics and was consequently unsure of how she could teach that subject effectively. As a result of working with an experienced teacher, Carrie realised that mathematics could be taught in accordance with her beliefs about students, teaching, and learning and hence was able to adopt mathematics teaching practices that were consistent with them. Mewborn (2000) acknowledged that Carrie's beliefs about mathematics as a discipline did not appear to change but did not mention as problematic the continuing isolation of Carrie's beliefs about the discipline of mathematics.

Schuck (1999) found that many preservice primary teachers held beliefs about the importance of making mathematics enjoyable, but did not believe that their own mathematical knowledge was important to their ability to teach it well (Schuck, 1999). The belief that mathematical ability is not requisite for effective mathematics teaching allowed them to maintain belief in themselves as effective teachers (Schuck, 1999). These teachers may have taught mathematics in ways superficially consistent with a Problem-solving view of the discipline but from motivations having nothing to do with an appreciation of the aesthetic appeal of the discipline or understanding of what a mathematician might mean by doing mathematics.

THE CASE OF SALLY

Sally had been teaching secondary mathematics for 18 years. She had studied tertiary mathematics for 3 years and had since completed an M.Ed. Some years earlier she had spent 3 years as the district Senior Curriculum Officer (SEO) (Mathematics), for the Education Department. Sally was currently teaching a grade 7 mathematics class and a combined grade 9 and 10 class, in which the students were studying non-compulsory advanced mathematics courses. Sally was a senior teacher with responsibilities including providing leadership in mathematics.

Instruments and procedure

Data concerning Sally's beliefs were collected using a survey requiring responses on a five-point Likert scale, to 26 items, taken from similar instruments devised by Howard, Perry, and Lindsay (1997) and Van Zoest, Jones, and Thornton (1994), relating to beliefs about mathematics, its teaching and its learning, and also from an audio-taped, semi-structured interview of approximately 1 hour's duration. Among other things the interview asked her to: describe an ideal mathematics classroom and compare this with the reality of her own mathematics classes; and respond to 12 statements about each of the nature of mathematics, and the teaching and learning of mathematics, based upon the findings of Thompson's (1984) case studies. Those about the nature of mathematics comprised four corresponding to each of Ernest's (1989) three views of mathematics, and the statements relating to the teaching and learning of mathematics were representative of the corresponding views of mathematics teaching and learning shown in Table 1. Sally completed the beliefs survey during the first few weeks of the school year. The interview was conducted several months later.

Sally's beliefs about the nature of mathematics and mathematics teaching

Sally's responses to the beliefs survey suggested that she held beliefs consistent with a Problem-Solving/Learner Focussed/Autonomous Exploration of Own Interests orientation to mathematics and its teaching and learning, and there were no apparent contradictions among her responses. For example, she (strongly) agreed with:

> Effective mathematics teachers enjoy learning and 'doing' mathematics themselves.
>
> Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking.
>
> Justifying the mathematical statements that a person makes is an extremely important part of mathematics.

and she (strongly) disagreed with such items as:

> Telling children the answer is an efficient way of facilitating their mathematics learning.
>
> Mathematics is computation.

Sally had been influenced by the reform agenda in mathematics, and by the 3 years that she had spent as a district SEO. She described how during this time she had presented workshops for primary and secondary teachers, and that this provided both the stimulus and the opportunity to think about new ideas in mathematics education.

When asked what sprung to mind in response to the word "mathematics" Sally answered in terms of the strands of the curriculum, describing this as much broader than her own and the general view of mathematics in earlier times. She believed that mathematics was now, "much more exciting and certainly less boring and academic" than it had been before. Similarly, when responding to statements about the nature of mathematics, Sally frequently answered in terms of school mathematics and had difficulty considering mathematics as a discipline that extended beyond this context. This was so even when she was prompted to consider the discipline as a whole and not just mathematics taught in school. For example, in response to the statement, "The content of mathematics is 'cut and dried'. Mathematics offers few opportunities for creative work", she said:

> … I disagree quite strongly I think … I don't think that the content of maths is cut and dried. I think a lot of the professional development that's gone on in the last 10 years … I think has opened up huge opportunities for creative work ...

Sally had difficulty conceiving of what mathematicians might do and was unable to say whether the change in emphasis that she had described in school mathematics better reflected their activities. In discussing the origins of mathematical content, she was comfortable with the sciences and other practical needs as sources of mathematics, but with regard to mathematics being self-generating she acknowledged that it could be so, but said, "I'm not really sure why, … it's a bit like a brain exercise for some people", and also spoke in the third person when she agreed that some people enjoy mathematics, saying, "They enjoy it for itself".

Sally agreed that, "Mathematics is a challenging, rigorous and abstract discipline whose study provides the opportunity for a wide spectrum of high-level mental activity", and again related it to school mathematics, and particularly to what she believed was an ongoing shift from pure to applied mathematics in that context. Sally saw this change in emphasis as being most apparent in primary schools, and least in senior secondary classes. For students not likely to study tertiary level mathematics, she regarded mathematics as making fewer demands in terms of high-level mental activity than it had in the past, because it was more "applied".

Sally indicated that she suspected that the results of mathematics were not tentative but rather in most cases they were "sort of law". She agreed with the statements, "Mathematical content is coherent. Its topics are interrelated and logically connected within an organisational structure or skeleton", and, "Mathematics is an organised and logical system of symbols and procedures that explain ideas present in the physical world", adding that it was the logic of mathematics that appealed to her.

In responding to the statement, "Mathematics is a collection of skills and procedures that are useful in meeting basic needs that arise in everyday situations", Sally indicated that this probably was true of what she described as "mathematics in a very pure sense", but that her understanding went beyond this. She said:

> … I think if you're talking about mathematics in a very pure sense then it probably is a collection of skills and procedures that help you meet sort of basic needs. But if you try and think about it in a broader sense then I can see it's much more than that …

And then, in elaborating on the distinction between numeracy and mathematics:

> … I would see maths very much the pure maths part of it whereas once it becomes more applied and how you use it everyday and how you justify what it is and whether you know that the person who's delivered your wood, whether they've given you the correct amount, the ability to be able to work something like that out or, to me is a more numeracy type of understanding than a mathematics one …

Sally described students in an ideal mathematics classroom as engaged and motivated, involved in practical problem solving investigations that originated from their own interests and questions, and working either as individuals or in groups perhaps for extended periods of time. The class would include a range of ability levels and the tasks with which they engaged would be accessible to all. They would have access to computers, and their activities would be characterised by hypothesising, describing patterns, testing hypotheses, and discussing their ideas.

Sally was consistent in her disapproval of teaching procedures without meaning and likened the use of rote-learned algorithms to the performance of a trick. She believed that, "Students should not be satisfied with just carrying out mathematical procedures; they should seek to understand the logic behind such procedures". Sally's responses revealed a firm belief in the importance of recognising that there could be many appropriate ways to solve a mathematical problem and that the teacher should neither prevent students from using alternative methods that were meaningful for them, nor convey the idea that a particular way is necessarily the only way.

Sally described content arising from the students' interests as more likely to "make an impact" and agreed that, "The teacher should appeal to students' intuition and experiences when presenting material in order to make it meaningful", describing this as an important way of engaging them in mathematics.

Sally expressed quite student-centred views of mathematics teaching, but did not believe that the teacher's role was insignificant or passive. She said:

> … the teacher has a fairly important role, to facilitate, or suggest to them, or guide them in terms of reaching a solution or an answer, … I think you have to have a balance … you can't just be so open-ended that they never get there …

IMPLICATIONS AND CONCLUSION

The cells in Table 2 suggest ways in which distinct clusters of beliefs about the discipline and school mathematics could interact to influence beliefs about maths teaching/learning.

		Beliefs about the nature of mathematics (the discipline)		
		Instrumentalist	Platonist	Problem-solving
Beliefs about the nature of (school) mathematics	Instrumentalist	No conflict	School mathematics is about learning basic skills that will allow understanding of higher level, more interesting mathematics later	Mathematics can be creative but you need to have a set of basic skills first. Mathematical creativity is not for school.
	Platonist	School mathematics is a body of hierarchical interconnected knowledge that needs to be learned so that it can be applied to practical situations	No conflict	School mathematics is a body of hierarchical interconnected knowledge that needs to be learned so that the gifted few can eventually be mathematically creative
	Problem-solving	Learner/process focus is aimed at motivating students to learn the skills they need in everyday life	*Learner/process focus is aimed at motivating students so that they come to understand more of the body of hierarchical interconnected knowledge that is mathematics*	No conflict

Table 2. Possible teaching/learning orientations resulting from different beliefs about school and mathematicians' mathematics.

Unlike the cases reported in the literature (Mewborn, 2000; Schuck, 1999), Sally was an experienced teacher with a relatively strong mathematics background albeit not a mathematics major. She appeared to have a Problem-solving view of school mathematics but, to the extent that she could conceive of the broader discipline, she seemed to have Platonist view of it. A teacher such as Sally could teach in ways consistent with a Problem-solving orientation according to Table 1 but not because this is the way in which she views the discipline (See italicised cell in Table 2). Further research, ideally involving classroom observations, is necessary to confirm or contradict these ideas, but recognition that at least some teachers have different beliefs about school and mathematicians' mathematics may go some way to explaining apparent inconsistencies among teachers' beliefs about mathematics and its teaching and learning. It also suggests that attempts to influence teachers' practice should address the both their beliefs about school mathematics and the discipline.

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CONTINUITY IN MATHEMATICS LEARNING ACROSS A SCHOOL TRANSFER

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This paper reports on the school transfer of fifteen students identified as mathematically gifted and talented. The results are drawn from students', teachers', and parents' perspectives pre and post school transfer. Some of the schools practised 'fresh start' and so there was no sense of mutual trust between sending and receptions schools about information of students' levels of achievement in mathematics. Most students felt academically well prepared but for some there was a lack of mathematical challenge in their new schools. Curriculum continuity, pastoral care, and home-school communication were recognized as issues.

INTRODUCTION

Students face a variety of challenges and potential problems when they make a school transfer. When students make a school transfer it is at two levels─the macro level of the school's physical structures and organization and at a micro level in the classroom. This transfer at the micro level means a new mathematics teacher and a teacher who may use different teaching approaches. School transfer coincides with the formative adolescent years which can also be problematic.

The international literature on school transfer provides evidence of dips in student progress at each point in transfer, primary to middle school or middle to junior high (Anderson, Jacobs, Schramm, & Splittgerber, 2000; Galton & Morrison, 2000). Noyes (2006) raised the issue, specifically using the context of mathematics, of trajectories and how school transfer acts like a prism diffracting the social and academic trajectories of students as they pass through it. According to Demetriou, Goalen, & Rudduck (2000) students showed signs of anxiety and excitement at the prospect of moving to a new school which is often a much larger school and some students expressed difficulties with sustaining commitments to learning and in understanding the continuities in learning. However, for the majority of students any fears largely disappeared after the first term. The main problem that typically remained was a lack of continuity across the curriculum. Students were faced with revision and a lower level of task demand which led to boredom (Galton, Morrison, & Pell, 2000), decline in motivation (Anderman & Maehr, 1994; Athanasiou & Philippou, 2006), disengagement from school (Anderson et al., 2000). In a recent PME paper, Athanasiou and Philippou (2008) highlighted the developmental differences between the actual and preferred classroom environments in mathematics as perceived by students, pre and post transfer.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 161-168. Thessaloniki, Greece: PME.

Transfer or transition is a difficult topic to address internationally because of both terminology in the literature and the different systems that exist within countries. However, there are likely to be commonalities in the challenges for students, teachers, and parents in addressing students' mathematics education pre and post school transfer. In this paper the term transfer will be used to define the move from one stage of schooling to another (Demetriou, et al. 2000). Anderson, et al. (2000) suggested three major concepts for understanding and improving school transfer and success. These concepts were preparedness, support, and transitional success or failure (which is implicit). According to the writers, preparedness is multidimensional and includes academic preparedness, independence and industriousness, conformity to adult standards, and coping mechanisms. Support from others, be it informational, tangible (resources), emotional or social, facilitates successful transfer. This support may come from peers, teachers, or parents. Transitional success or failure can be judged by factors such as grades, appropriateness of a student's post-transfer behaviour, social relationships with peers, and academic orientation. These indicators are what are commonly commented on report cards namely: achievement, conduct, and effort. This framework has been recognized as useful for addressing transfer problems (Galton & Morrison, 2000).

The focus of this paper is on student placement, mathematical preparation and expectations across school transfers, pastoral care, and home-school communication. The paper presents findings from a variety of perspectives─students, teachers, and parents.

METHODOLOGY

This paper is extracted from a larger study that examined how a school transfer was managed for a group of students identified as mathematically gifted and talented. The research paradigm that guided the research was essentially interpretive although aspects of naturalistic inquiry were blended into the study. The main study was designed using a case study approach and predominantly qualitative methods. These included student, parent, and teacher interviews, documents (school policies, teacher plans, student workbooks), and classroom observations. Data were gathered over a two-year period in which 15 students (10 to 13 year olds) were tracked across school transfers. The focus of the study was on the learning and teaching experiences in mathematics. The students had been identified by their schools as gifted and talented in mathematics (purposive sampling) and originated from three different schools. In the second year of the study, the students transferred to nine different schools. Ten of the students made the transfer from a primary school to an intermediate school (Years 7 and 8) and five students transferred from intermediate to secondary school.

The data for this paper are drawn primarily from the interview transcripts. The first level of coding was based on the conceptual framework provided by Anderson and colleagues (2000); these include the categories of preparedness, support, and transitional success or failure. The next level of coding came from the common threads in the participants' accounts such as the concepts of 'fresh start' and 'little fish in a big pond'. Interesting points of difference were also noted as part of the memoing process. Data were triangulated through the multiple perspectives (students', teachers', and parents), documents, and classroom observations.

RESULTS

This paper focuses on aspects of the transfer process; these are: student placement, academic preparedness and expectations, pastoral care, and monitoring systems in mathematics. The findings are presented from the perspectives of students, teachers, and parents.

The placement of students was a key part of the transfer process. The students faced a change from being 'at the top' class level of their sending school to the 'the bottom' of the reception school. For most students there was also a change from essentially one class teacher for all core subjects to a specialist teacher for a subject such as mathematics. There was also a change for most of the students (n=11) to a bigger school. Most of the students felt that they were prepared for the systemic and/or organizational changes through school visits to the reception school, prospectus information, or siblings answering questions. The students had to face the change of different mathematics teachers and programs. Bear in mind that these students had all been in special classes or programs for the mathematically gifted and talented and so the previous teachers, parents, and students were keen to see if these abilities were recognized by the new school. All of the teachers from the sending schools passed on written information to the reception schools. It seems, however, that not all messages from the sending schools were understood, trusted, or acted upon. One teacher admitted that he had not looked at the mathematics records for the four students from this study placed in his class. The teachers from the reception schools were, on the whole, less interested in the transfer process and the student information than the sending school although the majority of the reception schools conducted interviews with teachers from the sending schools. Most of the students sat tests, either on their visits to the school in the preceding year or early in the year at their new school. They were then placed in classes based primarily on these academic results gained from the new school. There was hope from the parents that, given the students had come from extension/enriched/accelerated programs in mathematics, they would achieve 'good' results in any pre-selection tests, and that special teaching programs would ensue. There was also an expectation from parents and teachers from the sending schools that there would be an exchange of academic information between the schools and this would aid in the transfer process. They expected this to include details about their achievements in specialized mathematics programs for gifted and talented students and results in mathematics competitions.

The issue of identifying mathematically gifted and talented students by the reception schools was problematic for some students in this study. With limited identification methods (most used tests only) for mathematics in the majority of the schools, some students did not experience a smooth transfer. Eric (pseudonym), for example, was not selected for the gifted and talented mathematics program in his new school. This school practised *tabula rasa* or 'fresh start' and did not refer to information from the sending school. He had performed poorly in a school placement mathematics test and so missed out on selection for the gifted and talented mathematics class. After several months the parents raised their concerns because of Eric's changing attitudes and interest in mathematics. However, it was not until six months later and he had performed exceptionally well in an international mathematics examination that Eric was moved to a class for gifted and talented mathematics students. Eric's parents, like others in the study, expected a continuation of challenging programs, that their children would be seen as high achievers, continue to grow in their mathematical development, and not lose their love for mathematics.

So what were the students' expectations? The majority of the students anticipated with enthusiasm the prospect of being taught by a mathematics specialist teacher. These students talked primarily about the subject, the teacher, and the level of academic challenge. The students wanted a teacher who "knew tons of maths", enjoyed mathematics, and recognized their individual differences. Students repeatedly mentioned flexibility, in relation to contexts such as time, organization, responding to students' needs and in the use of resources. They wanted a teacher who appreciated and provided them with "challenging mathematics". Challenging mathematics invariably meant "not too easy and not too hard". One student explained it as "outside the square, outside the octagon…to think in ways that you wouldn't normally think in". Some of the students were concerned about subject continuity and how well prepared they felt in certain topics such as algebra and geometry. A few were not prepared for the feeling of being a 'little fish in a big pond'. With a move, in most cases, to a larger school, the students found themselves in a bigger pool of mathematically gifted and talented students. This realization that they were one of many high achievers was expressed by both the students and parents. Some of the students were also challenged more in their knowledge and skills in mathematics; the mathematics was not so easy. One student explained that it was quite a lot harder and that she had gone from being in "the top group in the class" to a class where "I'm at the bottom". However, most of the students believed, academically, they were well prepared, no gaps in mathematical knowledge had surfaced, and they were coping well compared to other students in their classes.

The teachers from the sending schools had clear expectations for their gifted and talented students including a continuation of advanced levels in mathematics, that the level of mathematics would be suitably challenging, and the teaching approach encourage open-ended investigations and opportunities for self-directed learning. They understood that there would be an information sharing process so the reception schools would know that the students had been identified as gifted and talented and had been involved in special mathematics programmes. They therefore expected that the students would be placed in appropriate classes. There was also an expectation that the students would be encouraged to take greater responsibility for their learning and "keep on pushing themselves". One teacher explained that she articulated to her students that she expected them to be in accelerated programmes at secondary school and that their pathway was to go on to university. Although Miss L saw her role as giving her students a good grounding and kick start to secondary school, she believed that these students had a vision for themselves "so that they can choose the pathway, rather than have the pathway choose them".

Seven of the parents acknowledged the positive support their children had received from the reception school and in particular the students' form class and/or mathematics teachers. This support included helping students develop skills in setting goals, maintaining consistent work standards, and clarifying expectations in mathematics. The parents appreciated approachable, supportive teachers who recognized their children's talents. Not all of the parents felt that the school had aided the transfer process. Two parents were not impressed with the level of pastoral care and guidance provided for their children especially in relation to the accelerated programmes. One father explained that although the students were "doing serious level maths" they were "tender in years in maturation" and so maybe there should have been more initial pastoral care to support the adjustment to secondary school. Three parents recognized the pressure that their sons were now under at secondary school; they were with "the cream of the cream" and were under pressure to perform well and maintain their place in the top streamed accelerate class. These were students who were used to getting very high scores in tests, and were very confident in their abilities in mathematics. Suddenly, their numerical scores had dropped and they were with a wider pool of gifted and talented students. Two parents, with children from the same school, believed that their sons were not well prepared in mathematics for the year ahead. They felt that there were gaps in their children's mathematics education, particularly in algebra.

Once the students had settled in to their new schools the parents were interested in monitoring their children's academic progress particularly in mathematics. Parents were questioned about how they were informed of their children's academic progress in their new schools and what communication there was between home and school. The schools all had regular reporting systems. Early in the year, for most schools, it was one-way information sharing, essentially a 'meet the teacher' opportunity. Several parents commented that they felt comfortable about approaching the teacher or school if they had concerns about how well their children had settled in to the new school. Five parents, out of the fifteen parents in the study, were not impressed with their children's progress in mathematics at their new schools. They seemed reluctant to step in and question this lack of progress and the level of challenge in the mathematics programmes. They were not acting as 'pushy' demanding parents but took a 'wait and see' approach. This 'wait' was usually for the interviews later in the year where they felt they had an opportunity, supported by a written school report, to raise their concerns. The concerns were based on their children's attitudes, and decreased levels of effort, motivation, and enthusiasm for mathematics. A few of the parents felt uninformed about their children's progress and the mathematics curriculum. The schools had systems in place to inform parents on a regular basis but it would appear that the quality of information communicated about students' progress and the mathematics curriculum was limited. Formal written reporting occurred later in the year when teachers had accumulated a reasonable amount of assessment data. The parents wanted earlier opportunities to communicate with teachers and for this to be initiated by the school.

DISCUSSION

Students were prepared for this transfer in a variety of ways. The majority of the students had been given information about their new schools and some had attended an orientation visit. These visits, prospectuses, and discussions with peers and teachers meant that the students felt reasonably well prepared for the organizational aspects of the new school. The students knew to expect the systemic changes and possibly different teacher expectations. All of these go some way towards enhancing the transfer process (Simpson & Goulder, 1998).

The academic focus of ensuring curriculum continuity and the learning and development of individual students in mathematics were not evident for all students in this sample. The practice of 'fresh start' was evidenced in three schools that based placement on their own selection tests. Justification of this 'fresh start' policy has been argued, according to evidence reviewed by Galton et al. (2000) on the reasoning that a secondary school's objectives are more academically specific and the secondary specialist teachers can better ascertain a student's ability in a subject such as mathematics. Accordingly, previous school records were not always taken into account. Galton and Hargreaves (2002) write that it is questionable therefore, whether curriculum continuity is taken seriously and is an achievable goal.

There was an expectation, by the majority of the teachers in the study, that as gifted and talented mathematics students they would be independent learners and therefore programmes designed for these students would encourage independent work. Only one of the teachers from the sending schools spoke specifically about helping students develop skills in self and time management, studying, gathering, and using information, communication, decision-making, and conflict resolution. There was little evidence, in this sample of students, that they were specifically taught skills for coping and being independent learners. These skills are recognized as making a transfer across systems more successful (Schumaker & Sayler, 1995). This lack of focus on preparedness by teachers concurs with Hawk and Hill's (2001) study that found "many teachers are so focused on curriculum coverage that they do not take the time to incorporate these [self-management, time management, study skills etc.] into the programme" (p. 31).

With their children moving to a new school, the parents wanted to monitor not only their children's social-emotional well-being but also their academic progress in mathematics. This is important given that the literature has shown that most students experience dips in achievement post-transfer (for example Anderson, et al., 2000; Galton & Morrison, 2000). Most parents found this difficult; there was limited opportunity for communication and progress reports came later in the school year. The parents were reluctant to be involved in the new schools for several reasons such as time, adolescent independence, their own level of mathematics, and a belief that they would be seen as ‘pushy parents’. Some parents expressed concerns about their children’s attitudes, efforts, motivation, and levels of achievement in mathematics. The provision of academic continuity in pupils’ experiences is viewed as a vital component in a successful transfer (Simpson & Goulder, 1998) yet, it is disconcerting how many of the students and their parents mentioned the lack of challenge in their new mathematics programs. This result was also surprising given that the students were taught post-transfer by teachers who had greater expertise in mathematics than their previous teachers. There were expectations from the parents related to identification and recognition of interest and abilities, a desire for a teacher who continued to challenge their children, and to be informed of their children’s progress.

Students experience several transfers during their years of schooling; each one of these is important. However, just because a student has successfully negotiated one transfer does not mean that he or she will successfully negotiate the next one. The nature and extent of change is dependent on many factors, some of which have been outlined in this paper. For the many students in this study, the transfer was relatively smooth and unproblematic although there are several implications to be considered from the findings.

School transfer can be a daunting process for any student; these students showed that they had high expectations in terms of teacher qualities and curricular challenge. Dips in academic achievement commonly occur post-transfer but consideration needs to be given to factors including those briefly outlined in this paper so that schools can eliminate any hiatus in progress. If secondary teachers do not pay attention to information from sending schools because they perceive them to be unreliable, this implies a potential level of distrust. It gives rise to the question as to whether teachers should spend considerable time and effort on compiling, improving, and refining records when this study showed that many secondary schools could not offer reassurance that they would use these records. Aspects related to pastoral care could also be raised earlier in the school year rather than parents waiting until formal reporting systems later in the year. Students also need to have opportunities to raise their concerns in a safe and supportive environment early in their days at the new school. Schools could also consider strengthening their contribution to students’ transfer with access to mentors and learning counsellors. An earlier three-way conference (student-teacher-parent) might go some way towards capturing sooner, concerns about students’ motivation, interest, and academic progress in mathematics post transfer. If we are to help young people sustain, through primary and secondary schooling, an enthusiasm for the learning of mathematics, confidence in them as learners, and a sense of achievement and purpose, then we should pay attention to the transfer process.

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PHENOMENOLOGICAL PERSPECTIVE ON MATHEMATICS EDUCATION

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This is a philosophical essay on a phenomenological way to understand and to work out Mathematics Education. Its philosophical grounding is the Husserlian work, focusing on its key word "going to the things themselves" in order to keep us away from the theoretical educational truth, took as the unique one. We assume the attitude of being on the life-world with the students and Mathematics as a field of research and practice that show and express themselves through lived experiences and through language. We assume to be in search of understanding of education, learning and Mathematics, as we take care, consciously, of what we are doing and saying in the same movement of saying and doing it.

Key words: Phenomenology. Lived Experience. Mathematics Education.

A STATEMENT ABOUT THE PHENOMENOLOGICAL PERSPECTIVE ON MATH EDUCATION.

It is important to clarify what the phenomenological approach pointed out in the title of this essay is about, in order to defend this posture in Mathematics Education, being seen from the perspective of its pedagogical practice and from a perspective of researching on its own themes.

Perspective, within this context, is about a comprehensive view, assumed from a position about a determined theme. In this case, the theme does not show itself as being the phenomenology, but taken in a qualifying dimension, as it qualifies the perspective.

Phenomenology[1], a philosophical school of thought, has got in its kernel the meanings, whether physical or not, we build about things that are around us in the life-world's horizon. It is this search that makes difference and puts itself as meaningful, especially in the Education's context. In case of Mathematics Education, there is difference between taking Mathematics as a fact, in other words, as data, expressed in scientific terms and understanding the meaning of this fact or of what was expressed.

[1] *Phenomenology* is a word composed by *phenomenon + logos*. *Phenomenon* means what shows itself, what turns up, and *logos* is understood as thought, reflection, and articulation. Therefore, *Phenomenology* can be taken as the articulation of the meaning of what shows itself, or as reflection on what shows itself.

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In the first case, it matters working with arithmetic and geometry, for instance, in terms of language, propositions, methods of construction, inductive and deductive ways of reasoning, ways of products' creation, ways of operating their standard units, possibilities of application and so on. In other words, it is important to teach contents, operations and possible applications. This is a scientific posture, based on the way of making science, in this case, Mathematics. The fact is the kernel. In the second case, it is important to search for the meaning that arithmetic and geometry, within their ways of being, does to the person and to the world-life where he lives, as well as to the World of Mathematics and Science and technology in general.

That is the difference between what we name as a positivistic approach, or, in Husserl's words, naturalist, and the phenomenological posture. The positivistic approach works with facts. That author names it as natural attitude because, in the sample focused, it is not questioned what a mathematical operation is. It is just made. In the phenomenological approach, the mathematical operation is perceived and can be comprehended in the acts actualized in the consciousness' movement, in an attentive way, conscious or just in a *passive synthesis* (HUSSERL, 1998), by the person who has done it. This synthesis says about the movements whose we articulate perceptions without realizing it. It is previous to the perception of the perceived. As Mathematics educators, we hope that what we work with students is meaningful. Right, it is needed for us (teachers and students) to know what we are doing. Therefore, we need to know the operations carried out, the discourse of the mathematical text and its propositional language and technique, as well as its applications. Furthermore, we chase the meaning that this knowledge does to us, people present at the situation of learning and teaching, and at the scientific inquiring region, in other words, the meaning that is revealed in the investigation of its historical grounding.

PHILOSOPHICAL FRAMEWORK APPROACH

The philosophical framework of this essay is the phenomenological thought, whose production was initialized by Husserl. His motto 'going to the things themselves' (HUSSERL, 1970; 1931) refuses to understand the world by means of scientific theoretical lenses, revealing the phenomenon meaning and the radical criticism. By using this reasoning, Husserl aims to comprehend the European Science, doing a criticism that covers different scientific enquiring regions, especially Mathematics and Psychology. Within Mathematics, (HUSSERL, 1970, 2003; MILLER, 1982), he interrogates the constitution of this science, and the persistence of its objectivity that goes through the history of cultures, from a socially and geographically point of view, although assuming that it had been created from evidences that happen in the subjectivity sphere (HUSSERL, 1970; ALES BELLO, 1986).

Within Psychology, his criticism begins interrogating the Cartesian *ego*, as once it had been found, it became completely forgotten, because Descartes and his pupils started to work out just with the thought by the *ego*, looking only to the produced by the thinking (HUSSERL, E.1970, 1977). Maintaining himself in his line of thought, he goes to the thing itself, what means, in this case, to leave out theories and approach the phenomenon manifestations – the psychological. In this search, gets close, as student, to Franz Brentano, who helps him to understand the meaning of intentionality. Along his work, he expresses the psychological as one of a person's dimension (ALES BELLO, 1986).

During some time, he argues about the origin of arithmetic, as if it were constituted in the psychological sphere (HUSSERL, 2003; MILLER, 1982).

Along his life, he gets through those theoretical obstacles, focusing his study on questions about intentionality, which materializes the embodied body (BICUDO, 1991); on the constitution on subjectivity, intersubjectivity and objectivity, in a different way from the naturalist attitude, assuming it phenomenologically; on the life-world (HUSSERL, 1970); on the language (HUSSERL, 1970; MERLEAU-PONTY, 1978); and on the horizon of comprehension (HUSSERL, 1970).

Intentionality, characteristic of consciousness, is understood as the movement of self-expanding. It means way of being intentional. Intentional means to tend towards a direction, to extend, to become attentive, to sustain, to give intensity, to assert strongly. So, *consciousness* is not conceived as a physical place where the principles of value are, nor a container where judgements are put in, etc. The consciousness is like a point of convergence of human operations, which allow us to say what we are saying or to do what we do with human beings (ALES BELLO, 1986). Due to its characteristic, the intentionality is also understood as self-extending to something, tying and bringing it to itself, in order to advance, through its acts, and to express them by means of an articulated meaning. In that way, we have perceptive acts that give us a first level of consciousness. They are like an opening for the meaning with a possibility of a more elaborated and reflected comprehension, which could be opened by reflective acts, understood in a second level. Those acts are done on the performed action. It is an act that is done in an embodied way, in the materiality of the body that is considered an intentional unit.

The consciousness, understood as convergence of human operations, is a movement that actualizes and carries out the acts, articulating their meanings, in other words, carries out the reflective process. It is the movement of getting award, of being attentive to what one does and to what happens. It is a movement that ties the act of perceiving, as well as the act of reflecting, opening space to reflect on itself, in other words, on the human being and on the ways which the operations product of those acts are communicated, i.e., about communication between people. It covers empathy and language.

The statements above involve a net of arguments and of explicitations that also talk about the subjectivity, in a net shape, bringing to its kernel the constitution of the intersubjectivity and objectivity.

Subjectivity is characterized it by presenting acts with differentiated qualities. There is the embodied dimension, a first one, which is carnal and carries out movements, experiencing the space and perceiving close and far locations and physical obstacles, for instance. The ways of being embodied open themselves to the world, through sensory acts whose touch allows us to register the limits of the own-body and the others' bodies, of people or not. The perception of having a body is based on the analysis that we carry out. It conducts us to the psychical dimension, when it is possible to focus on the psychical acts[2], with psychological characteristic, opening a huge field of investigation, which demand special attention. They are acts related to registering actions, such as comparing, fantasising, abstracting, touching, listening, and so on. Besides them, there are acts that also differentiate themselves from the psychical ones and are still subjective as, for instance, the self-perception of perceiving, in other words, the self-perception of being in action. That means there are acts that reflect on the acts that have been being actualized. One walks, within this thinking movement, to the dimension of reflective acts. The reflection experience installs the act of getting award of us, of what we are doing, and carries out acts of decision and of evaluation. This is the spiritual dimension. The spiritual acts are carried out by the own-body, being, therefore, as already mentioned in this chapter, embodied. In phenomenology, when one talks about subjectivity, one covers the own-body dimensions, and the psychical and spiritual acts. The subjectivity, however, throws itself and covers the neighbourhood of what is in the life-world, including the other, the 'not-I' perceived as the other person, once it is not my body that I perceive walking, feeling, acting. Not being closed in itself, the subjectivity builds at self-expanding to the aimed and to the acts carried out that show themselves as tentacles, which take and bring the perception and the perceived. The act that actualizes other's perception is called empathy or entropathy. As perceptive act, empathy gives us the comprehension of other's existence. It is not an act of feeling affection, but of perceiving other human being, as a being who lives in the same dimension that we live: as intentional consciousness. The empathy is the act that opens us to *intersubjectivity*. That is a complex world: it is the world of culture and history that opens itself. Empathy does not cover all the acts required by the world of *intersubjectivity*. There is a demand of expressing the articulated within the intentional acts, through a way of communication that is not limited by affective or unpleasant acts, for instance, and that is enough structured to manage to keep the communication in the cultural and historical dimension. This structured way is built through experiences lived among human beings in a space of co-subjects, mates in situations and that, through comprehensions and its well succeeded pieces of communication, establishes ordinary ways of expressions and of communication. So,

[2] The psychical acts are understood in the context of Franz Brentano's classes, attended by Husserl, and basis to his work about the importance of those acts in the constitution of arithmetical knowledge. Later, during his life, he comprehends that arithmetic cannot be based just on those acts. He goes further, as we will see, with the intersubjectivity and objectivity subjects, the abstractive acts, the idealities' constitution, in a way that he understands, in a more elaborated and wider mode, the constitution of the number and of other mathematical idealities.

the world of language opens, transcending the gestural and directive language, getting into the language logical structures, in other words, the communication sustained by a linguistic structure and by a propositional language. Empathy and language are the communication's kernel and, therefore, the possibility for the world to keep the way it maintains all its history, its tradition, its culture, and that social organization finds space and sustentation. Within language, the expression of consciousness acts, through signal acts, finds ways to self-manifest. In that way, the world of *objectivity* opens itself.

According to the phenomenological approach, the objectivity is build in the dialectic subjectivity/intersubjectivity, whose movement happens in the life-world grounding, which is historical, cultural and mainly based on the communication between co-subjects. The canal of this communication is open in the perception of the other by the empathic act, and the communicated is sustained in the linguistic structure. Being a constituted objectivity, its interpretation occurs when one focuses on it consciously, searching for meaning. It is an objectivity structured on comprehensions, and on historical and cultural interpretation, that is maintained in the language, conducted by the tradition.

REFERENCE TO RELATED LITERATURE

This is an essay arisen from studies that aimed to understand Phenomenology, especially the Husserlian one, from the Mathematics Education perspective. In that way, the literature related to this theme can be found in that own author and in those who tried to interpret him and go ahead with ways of working Mathematics Education phenomenologically. From this perspective, we have opted to mention subjects and present relevant references.

The theme science, and in particular Mathematics, can be found especially in Husserl' works (1970), Ales Bello (1986), Tierszen (1996), Bicudo (2000) and Kluth (2005). The theme language can be found in Husserl (1970), Merleau-Ponty (1978) and Bicudo (2000). The theme person and empathy is worked in Husserl (1970) and Ales Bello (1986). Mathematics Education has got also many important works used in this paper, such as Blair (1981), Ernest (1991), Bicudo (1999, 2000), and Bicudo and Garnica (2001).

AUTHOR'S POSITION ON PHENOMENOLOGICAL MATHEMATICS EDUCATION

We see the phenomenological approach assumed at doing and to do Mathematics education as a turning point in the perspective from where one looks at the student, at the school, at Mathematics and at ourselves as teachers. This change of focus is related to clarity that one always sees and comprehends conceptions from a perspective given by the position that we assume, in an embodied way in the world. Comprehension that materializes in ways of acting and of addressing questions, of self-perceiving existentially being in the life-world, grounding of lived experiences and those ones of science. Within this, the centre of the comprehensive view is my own-body, however, never closed in itself, in the psychological subjectivity, but always self-extending to what is around it. This view covering happens through the effectuation of the intentionality that, at self-expanding covering the other, gives itself to the other and, at the same time, brings the other towards. A relation between the seeing and the seen is created. The turning point is the comprehension that everything that is brought comes over under lenses used by who sees, and so, the seen is never an empirically given fact, but always perceived. On the other hand, the perceived is not isolated from a historical context, but is always perceived in a grounding outlined as background of a figure that highlights. The other is given to me in its materialization, manifestation of its own body that is there, in the spatiality and in the temporality of the life-world, at the same time as an equal, because in this way I recognize it by the empathic perception, and as a different, because it is not me that is there, but the other, the alien. In the situation established in the actions of teaching and learning, located in a specific context – we can think of the school, for instance -, phenomenology shows us that we are there with the other, our students, and for each of those, we are also an other, constituting figures and background that constitute themselves in a dynamical flow of actions. Those actions are intentional and are driven to a convergence, which is gradually outlined as educational. The background gets gradually tied, in a way that constitutes a grounding of lived experiences. We consider that this is the scenery where we should move around with students and with Mathematics, when we have this science and its practice as the content to be worked. We are not talking about a science that puts itself "above suspicion", in other words, like a shiny being sustained by a truth, which conducts all processes of thinking and of producing knowledge. But as a science that says about the life-world, showing some aspects that constitute it, and that is historical, showing itself in perspectives. It does not mean that there is no common kernel in this science, expressed by means of ways of saying – language -, by an structure that organizes what was said – logic -, and by doing, based on procedures considered suitable – devices of construction and of application of the constructed. On the contrary: there is. But it is not a rigid kernel. It gets constituted by self-tying and self-untying according to the tentacles that it is bowed by. For this reason, it is possible to see it under different perspectives. At the same time, its comprehension does not happen just in a subjective and casuistic way. There is a conducting line of the historical grounding that articulates the productions and the intentionality of that one in position of questioning. There are others, mates who one is with that can also question, opening oneself to the possibility of comprehensive dialogues.

The work that has as background the phenomenological attitude assumed by the teacher – and here we can also say by the school, when it is included in its pedagogical project – is manifested in the respect and in the attention to the other, in the preponderance of dialogues among co-subjects that become possible due to empathic opening, in the language dimension.

POSSIBILITIES OF FORWARD RESEARCHES

The phenomenological posture opens horizons of comprehension related to science, of constitution of Mathematics, of encounters of comprehensive horizons, of history and meta-comprehension of knowledge already produced or in process of production. They are nuclear aspects for the educative action, which is expanded to teaching and learning acts, always seen in a contextualized way.

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CREATING CONDITIONS FOR DIFFUSION OF ALTERNATIVE ASSESSMENT IN PRE-UNIVERSITY MATHEMATICS EDUCATION

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The goal of the presented study is to characterize the process of creating conditions for adopting alternative assessment in a pre-university institution adhering to traditional assessment in mathematics. In accordance with Roger's model of decision-making during diffusion of innovations, we identified the needs of such an institution that can be addressed using alternative assessment. We then explored which types of research evidence can convince the academic staff to implement alternative assessment. The most convincing evidence came from the statistical analysis. It showed that some of the traditional exams used at the institution had low predictive validity, and thus, could be substituted with alternative assessment tools with no risk of decreasing the predictive validity of overall grades.

THEORETICAL FRAMEWORK

The study presented in this paper is part of an on-going research project aimed at exploring various effects that incorporates alternative assessment on teaching and learning mathematics at a pre-university centre of a technological university. The idea of using alternative assessment in mathematics education is broadly substantiated in the research literature (e.g., Brookhart et al., 2004; Topping, 2003). It is well known that introducing alternative assessment in an institution adhering to traditional assessment strategies is not an easy endeavour (e.g., Lesh et al., 1992; Sullivan, 1997). Similarly to any other innovative idea, it may raise and fall depending on many factors, and, in particular, on the readiness of the academic staff to consider alternative assessment as a possible way for addressing limitations inherited in traditional assessment practices (e.g., Watt, 2005).

The goal of this paper is to characterize the process of creating conditions for adopting particular alternative assessment tools. We pursue this goal based on a model of decision-making during diffusion of innovations suggested by Rogers (1995). According to the model, the first stage in adopting an innovation is awareness of the rising need for an innovative solution. Only when the awareness reaches some critical point, the decision makers can advance to the second stage: assessment of potential advantages of the suggested innovation. Next, only in the case of positive assessment of the innovation, they enter the third stage: the decision to adopt and act on the implementation of the innovation.

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The literature on alternative assessment in mathematics education corresponds mainly to the second and the third stage of the Rogers' model. Indeed, there are many publications describing positive effects of alternative assessment on students and teachers. Many of them can be used to convince decision makers to assess the potential benefits of alternative assessment in their institution and guide them when (and if) the decision is made. For instance, we learn from the literature that alternative assessment opens valuable learning opportunities for students and encourages them to discuss, explain, listen and challenge each other (Topping 2003; Zevenbergen, 2001; Zariski, 1996). We also learned that alternative assessment can help students to increase responsibility for their own progress (e.g., Bedford & Legg, 2007; NCTM, 1995). In addition, it provides teachers with an excellent opportunity to assess skills which are hard to measure by traditional tools (Stenmark, 1991; Watt, 2005) and opens a window into student mathematical thinking (e.g., Blumhof & Stallibrass, 1994).

However, all these studies and projects are rather silent with the rules of engagement of decision makers and teachers in alternative assessment, or, in terms of the Rogers' model, are silent about how to pass the first stage in the process of diffusion of the innovation. Moreover, there are studies that report high levels of satisfaction from the traditional assessment in mathematics among teachers and decision makers (e.g., Watt, 2005). In retrospect, most of the studies reporting the benefits of alternative assessment start from the point where the decision to welcome alternative assessment is made in an institution, and do not explain how the decision makers come to the decision. Our study elaborates on this important issue, and, specifically, addresses two interrelated research questions:

1. What are the needs, related to mathematics education, of a pre-university institution that can be addressed by incorporating alternative assessment tools?
2. What types of research evidence can convince academic staff of the institution to act upon incorporating alternative assessment in teaching and learning mathematics?

THE STUDY

The research field

In our study, the above questions are explored in the context of the Centre for Pre-University Education (CPE) of the Technion – Israel Institute of Technology. Such centres, sometimes also called preparatory departments, exist in many universities worldwide. They serve to close gaps between students' knowledge and prerequisites for academic study. They also serve as a channel for selecting the future students, in addition to traditional application procedures. Typically, CPE students intensively study mathematics during an academic year; their progress is systematically assessed by "traditional" exams. Eventually, the CPE students' entrance to the university is a function of their scores at various mid-term and final exams. The Technion is particularly interesting and challenging with respect to the possibility to incorporate alternative assessment strategies since the Technion's CPE puts great effort in developing high quality traditional exams.

The study at the Technion CPE is divided into three trimesters, and at the end of each trimester the students are tested in mathematics. The teacher decides only 10% of a student's final grade based on his or her impression from the student's effort and ability expressed during the lessons. The rest of the final grade is determined by the student's achievements in the trimester exams. Traditionally, the weights of the first and second trimester exams are 20%, and of the third, final, one - 50%.

Sources

Members of the CPE academic staff and students' files were the data sources in our study. Specifically, the data were collected from the head, the academic advisor and 6 out of 10 mathematics teachers of the CPE. In addition, we analyzed about 250 randomly chosen students' files representing about 600 files of students who had studied at the CPE in the past and then were accepted to the Technion.

The power to make a decision about incorporation of alternative assessment in the CPE was in hands of the head and the academic advisor of the CPE, who discussed the topic with the teachers. The department head and the academic advisor are referred to henceforth in this paper as "decision makers."

Method

The Decision Makers were interviewed using a conversational interviewing methodology, and the teachers – using a semi-structured interviewing methodology (Patton, 1990). These open-ended interviews were chosen as they appropriate to the situation and fit our purposes: to learn about the needs of CPE academic staff related to teaching and assessing mathematics and, by the end of the interviews, to deliver initial information about the project and ways of alternative assessment. All the interviews included the following questions:

What are the goals of trimester and final exams in teaching and learning of mathematics? How are they prepared? How do the exams assist you in making teaching decisions? Are you satisfied with the way the CPE assesses the students' knowledge and progress in maths? How can the CPE help students to improve their learning skills? What are the roles of the teachers' grades? How are they determined? Why do they constitute 10% (and no more or less), of the final grade?

In order to check the reliability of the interview data, a multiple-choice questionnaire based on a sample of the teachers' responses was constructed. The questions were formulated in the format "To which extent you agree with the following statement..." This questionnaire was filled in by four interviewed teachers and two additional teachers. The questionnaire was not intended to be used as an independent research instrument because of the small number of participants. Instead, it was used to check how many teachers support particular statements. In the rest of the paper, statements supported by three or more teachers are referred to as "typical".

Next, the authors of this paper considered 10 randomly chosen students' exam notebook to see how they are scored. This was followed by an observation of 10 mathematics lessons of an experienced teacher and a series of follow-up conversations with the teacher. All the lessons were videotaped. These research activities were needed in order to understand to which extent the lessons are directed towards the preparation for exams, to which extent exams match what is taught and how the teacher determines his 10% of the students' final grades.

Third, about 250 individual student files were statistically analysed in order to reveal how trimester and final mathematical exams predict success of the students' study at the Technion. This was decided based on the interviews with Decision Makers, who emphasized that CPE assessment should support prediction of the students' success in the Technion (see *Findings* Section). The data analysed consisted of background information of the students (gender, IQ, final high school grades, levels of mathematic study when in high school), the full set of their CPE grades and their first semester Technion grades.

Analysis

Five out of six interviews were audio taped and transcribed; the interview with the head of CPE was documented immediately when it was finished. The interviews were analyzed in accordance with the principles of grounded theory (Strauss & Corbin, 1990). Namely, anything that seemed relevant or interesting was marked in the interview protocols. Then the marked parts were clustered, and the clusters were categorized by their content, with respect to the research questions.

The videotaped lessons were examined for patterns of interactions between the teacher and the students (Dinur & Leikin, 2007) and for the types of mathematical tasks taught in the classroom.

Students' files were analysed using backward step-wise regression analyses, with several dependant variables: the final average grades of the first semester at the Technion and the final grades in calculus and linear algebra, which are compulsory first semester courses. The CPE grades and the quantified background info were considered as independent variables. This was done since the regression analysis enables the making of conclusions about the predictive power of every independent variable, with respect to a chosen dependant variable (Guilford & Fruchter, 1973).

FINDINGS AND DISCUSSION

In this section we present, in some detail, findings related to the first research question and outline findings related to the second question. The quotations from the interviews with the teachers represent typical statements, in the sense described above.

What are the needs of a pre-university institution related to mathematics assessment?

Five categories emerged from the data.

1. *The need to deal with the gap between the exams and what is taught in class*

All the teachers remarked that they had no part in writing/preparing the exams and that this fact causes disparity between the exams and what had been actually studied in class. In words of one of the teachers, "sometimes there are unfair exercises in the exams … [For example], there was a question about inverse trigonometry functions which was at the margins of the material. There is no doubt that it [the question] does not reflect their knowledge, but nonetheless it cost them 8-10 points." Another typical remark: "Often we hear from students: 'but I know much more than what I got in the exam'…Sometimes we believe the students [who say this], but we have no way of checking it without giving more and more exams."

2. *The need to have more information about students mistakes*

The teachers remarked that the exam notebooks do not reach the teachers; they do not check them so they receive the analysis of the results only from a third party. They claimed that this analysis is not very useful in making decisions about how to refine their teaching. One of the teachers said: "I know how many points they get, but not their mistakes." This finding was supported by what was found in 10 randomly chosen students' exam notebooks. Another teacher said: "In the trimester exams they do not compile the common errors that occurred and there is no teachers' staff meeting from which it would be possible to learn how the students reached the wrong conclusions."

3. *The need to improve learning atmosphere, especially at the beginning of the study at the CPE*

The teachers remarked that difficult exams harm learning atmosphere in the CPE. As one of them said: "Difficult exams at the beginning ... [cause] an atmosphere of depression. It affects them greatly. Only the strong ones survive". A complementary finding emerged from the interviews with the Decision Makers. They both were concerned about drop-out level at the beginning of study at CPE, due to traditionally low achievements at the first trimester exam. They were also concerned about the learning atmosphere at the CPE and felt the need to strengthen the learners' responsibility for their outcomes. The academic advisor said: "They [students] gradually understand that they have to study by themselves...The intention is to reach the situation where they are able to study by themselves. We don't really succeed in reaching it, but I would have liked them to reach it gradually... This is the best thing we could teach here." An apparent gap between students with stronger and weaker mathematical background is an additional source of frustration for the weaker students. The lesson observation data support this finding.

4. *The teachers' need for taking more responsibility for the assessment*

Three teachers would like to consider raising the percentage of the teacher's grade from 10% to 20%. One of them expressed it in this way: "I think all the teachers here are for increasing the percentage of the teacher's assessment, because they are the ones who know them [the students] the best". However, two teachers were against raising their part in the students' grades, just because they did not know how to assess students' knowledge without exams. A complementary finding came from the Decision Makers who expressed their interest in strengthening the teachers' responsibility for the learning outcomes by raising the weight of the teachers' part of the grade.

5. *The need to have assessment tools with high predictive validity*

This need was strongly expressed by the head of the CPE. In particular, he said: "The Technion is our customer, and we should let in only students who can succeed..." As to the existing assessment tools, he noted: "I think that our assessments are good and genuine. A proof is that our students [successfully] continue their studies at the Technion." The teachers also felt that the CPE exams are good predictors. This finding is consistent with what is reported by Watt (2005).

What types of research evidence can convince the academic staff to act upon incorporating alternative assessment?

An innovation can be adopted only when it addresses truly important needs, and the convincing strategy should be built accordingly (Rogers, 1995). Convincing the decision makers is an especially important part (e.g., Fullan, 1998), and it is crucial that they would be privy to the findings and conclusions at every stage.

We could convincingly argue in front of the Decision Makers that needs 1-4 (above) can naturally be addressed by incorporating such assessment tools as peer assessment and self assessment. Indeed, the literature sources mentioned in *Theoretical Framework* section, as well as many additional sources, provided us with appropriate arguments. However, we did not have arguments related to crucially important need 5, as we could not claim that peer assessment and self-assessment would also lead to better (or at least, no worth) prediction of the students' future success at the Technion. It was also clear that the Decision Makers could not put at risk the CPE students by allowing us to conduct an experiment that, in case of success, would provide us with the missing arguments. At this point, we decided to examine the existing quantitative data about students' achievements at the CPE and at the Technion.

Briefly speaking, the regression analysis resulted in the following. First, the teachers' grades are good predictors of the students' success. Second, the first trimester exam does not predict any grades gained by students during the first semester of study. Again, this exam is particularly discouraging for students. Third, the second and the third exams are fairly good predictors of the students' success.

These findings were presented to the Decision Makers and teachers, along with the findings from observations and interviews. The statistical findings were found especially surprising, interesting and convincing as they destroyed the myth of "our assessments are good and genuine". Actually, it became evident that part of the trimester exams did not achieve their goal as perceived, in spite of great investment of resources and time in their development. Eventually, this (negative) result gave support to incorporating alternative assessment.

The evidence that came from the teachers' interviews was helpful. The Decision Makers were surprised to discover that the teachers were dissatisfied with the traditional way of assessment. They were also surprised that some teachers were interested in being more involved in CPE student evaluation processes.

It should also be mentioned here that the first author conducted many informal conversations with the teachers, mainly in the teachers' room, whose aim was to expose them to think how to implement alternative assessment in their classes. These conversations played an important part in the convincing campaign, a fact that became evident from teachers suggesting their classes to take part in our project. The data from the lesson observations was not particularly convincing for the teachers and the Decision Makers.

Eventually, the decision was to gradually change the assessment scheme. The first step is to develop and try in selected classes a set of peer assessment activities based on the CPE mathematics syllabus. The second step is to use peer assessment grades instead of the first trimester exam grades in a computer simulation (regression analysis) in order to see how this replacement would affect the predictive power of CPE final grades. In case of success, peer assessment will actually substitute the first trimester exam. It is also planned to increase the weight of the teachers' grades. The optimal weights of different components of the final grade will be found by means of computer simulations, and then the new assessment scheme will actually enter the CPE.

SUMMARY AND CONTRIBUTION

According to the Rogers' (1995) model, incorporating an innovation is an organizational process in which innovation is diffused through various channels of communication, over time, to the members of the organization. Incorporation of alternative assessment in an educational institution is not an exception. This research shows how this can be done in an institution of pre-university education centre with strong traditions of traditional assessment. We believe that the described convincing campaign is instructive and can be adapted to other institution. The major elements of the process identified the needs that can be addressed by the alternative assessment and provided the Decision Makers with research evidence about what can (and should) be improved in their institution. The most convincing evidence came from the statistical analysis. It showed that some of the traditional exams used at the institution have low predictive validity, and thus, can be substituted with alternative assessment tools that can address most of the identified needs.

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TOWARDS A COMPREHENSIVE FRAME FOR THE USE OF ALGEBRAIC LANGUAGE IN MATHEMATICAL MODELLING AND PROVING

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In this paper we consider the use of algebraic language in modelling and proving. We will show how a specific adaptation of Habermas' construct of rational behaviour allows to describe and interpret several kinds of students' difficulties and mistakes in a comprehensive way, provides the teacher with useful indications for the teaching of algebraic language and suggests further research developments.

INTRODUCTION

According to Habermas' definition (see Habermas, 2003, Ch. 2), a rational behaviour in a discursive practice can be characterized according to three inter-related criteria of rationality: *epistemic* rationality (inherent in the conscious control of the validity of statements and inferences that link statements together within a shared system of knowledge, or theory); *teleological* rationality (inherent in the conscious choice and use of tools and strategies to achieve the goal of the activity); *communicative* rationality (inherent in the conscious choice and use of communication means within a given community, in order to achieve the aim of communication).

In our previous research we have dealt with an adaptation of Habermas' construct of rational behaviour in the case of conjecturing and proving (see Boero, 2006; Morselli, 2007; Morselli & Boero, 2009 - to appear). In this paper we focus our interest on the use of algebraic language in proving and modelling. Algebraic language will be intended in its ordinary meaning of that system of signs and transformation rules, which is taught in school as a tool to generalize arithmetic properties, to develop analytic geometry and to model non-mathematical situations (in physics, economics, etc.). In particular, for what concerns modelling (see Norman, 1993, and Dapueto & Parenti, 1999) algebraic language can play two kinds of roles: a tool for proving through modelling within mathematics (e.g. when proving theorems of elementary number theory) - *internal modelling*; or a tool for dealing with extra-mathematical situations (in particular to express relations between variables in physics or economy, and/or to solve applied mathematical problems) - *external modelling*.

Our interest for considering the use of algebraic language in the perspective of Habermas' construct depends on the fact that our previous research (Boero, 2006; Morselli, 2007) suggests that some of the students' main difficulties in conjecturing and proving depend on specific aspects (already pointed out in literature) of the use of algebraic language, which make it a complex and demanding matter for students.

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In particular, we refer to: the need of checking the validity of algebraic formalizations and transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use; the goal-oriented character of the choice of formalisms and of the direction of transformations; the restrictions that come from the need of following taught communication rules, which may contradict private rules of use or interfere with them. In this paper, we will try to show how framing the use of algebraic language in the perspective of Habermas' theory of rationality: first, provides the researcher with an efficient tool to describe and interpret in a comprehensive way some of the main difficulties met by students when using algebraic language; second, provides the teacher with some useful indications for the teaching of algebraic language; third, suggests new research developments.

ADAPTATION OF HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOUR TO THE CASE OF THE USE OF ALGEBRAIC LANGUAGE

Epistemic rationality

It consists in:

- *modelling requirements*, concerning coherency between the algebraic model and the modelled situation: control of the correctness of algebraic formalizations (be they *internal* to mathematics - like in the case of the algebraic treatment of arithmetic or geometrical problems; or *external* - like in the case of the algebraic modelling of physical situations) and interpretation of algebraic expressions;

- *systemic requirements* in the use of algebraic language and methods. In particular, these requirements concern the manipulation rules (syntactic rules of transformation) of the system of signs usually called algebraic language, as well as the correct application of methods to solve equations and inequalities.

Teleological rationality

It consists in the conscious choice and finalization of algebraic formalizations, transformations and interpretations, according to the aims of the activity. It includes also the management of the writer-interpreter dynamics (Boero, 2001): the author may write an algebraic expression under an intention and, after, interpret it in a different goal-oriented way, by "seeing" new possibilities in the written expression.

Communicative rationality

In the case of algebraic language we need to consider not only the communication with others (explanation of the solving processes, justification of the performed choices, etc.) but also the communication with oneself (in order to activate the writer-interpreter dynamics). Communicative rationality requires the user to follow not only community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions.

Some comments

We are aware of the existence of several analytical tools to deal with the teaching and learning of algebraic language. In our opinion, Arcavi's work on Symbol sense (Arcavi, 1994) offers the most comprehensive perspective for the use of algebraic language. With different wordings, it includes concerns for teleological rationality and some aspects of epistemic rationality. Comparing our approach with Arcavi's elaboration, we may say that we add the communicative dimension of rationality. We will see how it will allow us to account for: the possible tension between private rules of communication in the intra-personal dialogue, and standard rules; and the interplay between verbal language and algebraic language. Moreover we will see how our distinctions between the epistemic dimension and the teleological dimension, and between the modelling requirements and the systemic requirements of epistemic rationality allow to deal with the tensions and the difficulties that can derive from their coordination.

In order to justify a new analytic tool in Mathematics Education it is necessary to show how it can be useful in describing and interpreting students' behaviour, and/or in orienting and supporting teachers' educational choices, and/or in suggesting new research developments. We will try to show it in the following Sections.

DESCRIPTION AND INTERPRETATION OF STUDENTS' BEHAVIORS

The following examples are derived from a wide corpus of students' individual written productions and transcripts of *a posteriori* interviews, collected for other research purposes in the last fifteen years by the Genoa research team in Mathematics Education. In particular, we will consider three categories of students: (a) 9^{th} grade students who are approaching the use of algebraic language in proving; (b) students who are attending university courses to become primary school teachers; (c) students who are attending the third year of the university course in Mathematics.

A common feature for all the considered cases is that the individual tasks require not only the solution, but also the explanation of the strategies followed to solve the problem. Each individual task was followed by *a posteriori* interviews.

EXAMPLE 1: 9^{th} grade class

The class (22 students) was following the traditional teaching of algebraic language in Italy: transformation of progressively more complex algebraic expressions aimed at "simplification". In order to prepare students to the task proposed by the researcher, two examples of "proof with letters" had been presented by the teacher; one of them included the algebraic representation of even and odd numbers.

THE TASK: "Prove with letters that the sum of two consecutive odd numbers is divisible by 4".

Here we report some recurrent solutions (in parentheses the number of students who performed such a solution; note that "**d**ispari" means "odd" in Italian)

E1 (4 students): $d+d=2d$

In this case, we can observe how the *systemic requirements of epistemic rationality* are satisfied (algebraic transformation works well), while the *modelling requirements* fail to be satisfied (the same letter is used for two different numbers).

E2 (8 students): $d+d+2=2d+2$

In this case, both the *systemic and the modelling requirements of epistemic rationality* are satisfied, but the requirements inherent in *teleological rationality* are not satisfied: students do not realize that the chosen representation does not allow to move towards the goal to achieve (because the letter *d* does not represent in a transparent way the fact that *d* is an odd number) and do not change it.

E3 (5 students): $d=2n+1+dc=2n+1+2n+1+2=4n+4$ (or similar sequences)

We can infer from the context (and also from some a-posteriori comments by the students) that "dc" means "**d**ispari **c**onsecutivi" (consecutive odd numbers).

In this case *epistemic rationality* fails in the first and in the second equality, but *teleological rationality* works well: the flow of thought is intentionally aimed at the solution of the problem; algebraic transformations are used as a calculation device to produce the conclusion (divisibility by 4).

EXAMPLE 2: University entrance, primary school teachers' preparation The following task had been preceded by the same task of the Example 1, performed under the guide of the teacher. 58 students performed the activity.

THE TASK: Prove in general that the product of two consecutive even numbers is divisible by 8

Very frequently (about 55% of cases) students performed (without comments) a long chain of transformations, with no outcome, like in the following example:

E4: $2n(2n+2)=4n^2+4n=4(n^2+n)=4n(n+1)=4n^2+4n=n(4n+4)$

In this case, we see how both requirements of *epistemic rationality* are satisfied: *modelling requirements* (concerning the algebraic modelling of odd numbers and even numbers); and *systemic requirements* (correct algebraic transformations). The difficulty is inherent in the lack of an interpretation of formulas leaded by the goal to achieve, thus in *teleological rationality*. The student gets lost, even if the interpretation of the fourth expression would have provided the divisibility of n(n+1) by 2 because one of the two consecutive numbers n and n+1 must be even.

In the following case, both *modelling* and *systemic requirements* are not satisfied: the same letter is used for two consecutive even numbers (note that "**p**ari"means "even" in Italian) and the algebraic transformation is affected by a mistake.

E5: $p*p=2p^2$, divisible by 8 because p is divisible by 2 and thus p^2 is divisible by 4.

The student seems to work under the pressure of the aim to achieve: having foreseen that the multiplication by 2 may be a tool to solve the problem, she tries to justify it by considering the juxtaposition of two copies of p that generates "2". Indeed in the interview the student said that she had made the reasoning "p is divisible by 2 and thus p^2 is divisible by 4" before completing the expression. In this case we can see how *teleological rationality* prevailed on *epistemic rationality* and hindered it.

We have also found cases like the following one:

E6: p*(p+2)=p2+2p=8k because p2+2p=8 if p=2

Also in this case, from the *a posteriori* interview we infer that probably the lacks in *epistemic rationality* depend on the dominance of *teleological rationality* without sufficient epistemic control:

> I have seen that in the case p=2 things worked well, so I have thought that putting a multiple 8k of 8 in the general formula would have arranged the situation.

EXAMPLE 3: The bomb problem - Third year mathematics students

TASK: A helicopter is standing upon a target. A bomb is left to fall. Twenty seconds after, the sound of the explosion reaches the helicopter. What is the relative height of the helicopter over the ground?

The problem was proposed to groups of third year mathematics students in seven consecutive years. Some reminds were provided about the fact that the falling of the bomb happens according to the laws of the uniformly accelerated motion, while the sound moves at the constant speed of 340 m/s. However no formula was suggested.

The problem is a typical applied mathematical problem, whose solution needs an *external modelling* process. In terms of *teleological rationality*, the goal to achieve should result in the choice of an appropriate algebraic model of the situation, in solving the second degree equation derived from the algebraic model, and in choosing the good solution (the positive one).

The difficulties that students meet consist: in the time coordination of the two movements (it is necessary to enter somewhere in the model the information that the whole time for the bomb to reach the ground and for the sound of the explosion to reach the helicopter is 20 seconds); and in their space coordination (the space covered by the falling bomb is the same covered by the sound when it moves from the ground to the helicopter). Let us consider some students' behaviours.

Most students are able to write the two formulas:

E7: $s=0,5\ gt^2$, $s=340\ t$

They are standard formulas learnt in Italian high school in grades 10^{th} or 11^{th}, in physics courses. About 20% of the students stick to those formulas without moving further. From their comments we infer that in some cases the use of the same letters for space and time in the two algebraic expressions generates a conflict that they are not able to overcome. We can see how general expressions that are correct for each of the two movements (if considered separately) result in a bad model for the whole phenomenon. *Teleological rationality* should have driven formalization under the control of *epistemic rationality*; such control should have put into evidence the lack of the *modelling requirements* of *epistemic rationality*, thus suggesting a change in the formalization. In the reality for those students such an interplay between *epistemic rationality* and *teleological rationality* did not work.

In other cases (about 10% of the sample) the coordination of the two times was lacking, and the idea of coordinating the spaces (together with the formalization of both movements with the same letters) brought to the equation:

E8: $0,5\ gt^2 = 340\ t$

with two solutions t=0, t=68 that some students were unable to interpret and use (because 68 is out of the range given by the text of the problem). But other students found the height of the helicopter by multiplying 340x68, with no critical reaction or re-thinking, probably because it is normal that school problems are unrealistic!

Less than 60% of students wrote a good model for the whole phenomenon:

$t_b+t_s= 20$ $\qquad\qquad$ $h=0,5gt_b^2=340t_s$

and moved to a second degree equation by substituting $t_s=20-t_b$ or $t_b=20-t_s$ in the equation: $0,5gt_b^2=340t_s$

Many mistakes occurred during the solution of the equation (mainly due to the management of big numbers). Once two solutions were got (one positive and the other negative), in most cases the choice of the positive solution was declared but not motivated. *A posteriori* comments reveal that the fact that a negative solution is unacceptable (given that the other solution is positive!) was assumed as an evidence, without any physical motivation.

In terms of *epistemic rationality*, three kinds of difficulties arose; they were inherent: first, in the control that the chosen algebraic model was a good model for the physical situation; second, in the control of the solving process of an equation with unusual complexity of calculations (big numbers); third (once the valid equation - a second degree equation - was written and solved), in the motivation of the chosen solution.

In terms of *communicative rationality*, we can observe how (in spite of the request of explaining the steps of reasoning) very few students of both samples were able to justify the crucial steps of the solving process. *A posteriori* interviews revealed that most students who had been unable to justify their choices were sure about their method only afterwards, when checking the positive solution and finding that it was “realistic”, thus putting into evidence a lack in *teleological rationality* (lack of consciousness about the performed modelling choices). However a number of solutions was quite realistic, even if got through a bad system. Many authors of the correct solutions were not able to explain why the other solutions were mistaken. This suggests that lacks in *communicative rationality* (as concerns verbal justification of the validity of the performed modelisation) can reveal lacks in *teleological rationality* (motivation of choices with reference to the aim to achieve) and even in *epistemic rationality* (control of the validity of the steps of reasoning).

DISCUSSION

As remarked in the second section, the usefulness of a new analytical tool in mathematics education must be proved through the *actual* and the *potential research advances* and the *educational implications* that it allows to get.

Research advances

In the frame of our adaptation of Habermas' construct, the distinction between *epistemic rationality* and *teleological rationality* allows to describe, analyse and interpret some difficulties (already pointed out in Arcavi's work), which depend on the students' prevailing concern for rote algebraic transformations performed according to *systemic requirements* of *epistemic rationality* against the needs inherent in *teleological rationality* (see E4). Moreover, the distinction in our model between *modelling requirements* and *systemic requirements* of *epistemic rationality* offers the opportunity of studying the interplay between the *modelling requirements* and the requirements of *teleological rationality* (see E7); we have also seen that formalization and/or interpretations may be correct but not goal-oriented (like in E2 and E4), or incorrect but goal-oriented (like in E5, E6 and E8). Together with the other dimensions of rationality, *communicative rationality* allows to describe and interpret possible conflicts between the private and the standard rules of use of algebraic language, and the ways student try to integrate them in a goal-oriented activity (see E3).

Further research work should be addressed to establish what mechanisms (meta-cognitive and meta-mathematical reflections based on the use of verbal language? See Morselli, 2007) can ensure the control of *epistemic rationality* and the intentional, full development of *teleological rationality* in a well integrated way. With reference to this problem, taking into account *communicative rationality* (in its intra-personal dimension, possibly revealed through suitable explanation tasks and/or interviews) suggests a research development concerning the role of verbal language (in its mathematical register: see Boero, Douek & Ferrari, 2008, p.265) in the complex relationships between *epistemic, teleological* and *communicative rationality*.

Educational implications

We think that the analyses performed in the previous section can provide teachers as well as teachers' educators with a set of indications on how to perform educational choices and classroom actions to teach algebraic language as an important tool for modelling and proving. Some of those indications are not new in mathematics education; we think that the novelty brought by the Habernas' perspective consists in the coherent and systematic character of the whole set of indications.

First of all, the performed analyses suggest to balance (at the students' eyes, according to the didactical contract in the classroom) the relative importance (in relationship with the goal to achieve) of: production and interpretation of algebraic expressions, vs algebraic transformations; and flexible, goal-oriented direction of algebraic transformations, vs rote algebraic transformations aimed at "simplification" of algebraic expressions. These indications are in contrast with the present situation in Italy and in many other countries: teachers' classroom work is mainly focused on algebraic transformations aimed at "simplification" of algebraic expressions. The fact that algebraic expressions are given as objects to "simplify" (and not as objects to build, to transform according to the aim to achieve, and to interpret during and after the transformation process in order to understand if the chosen path is effective and correct or not) has bad consequences on students' *epistemic rationality* and *teleological rationality*. As we have seen, many mistakes occur in the phase of formalization (against the *modelling requirements*), and even when the produced expressions are correct, frequently students are not able to use intentionally them to achieve the goal of the activity (against the *teleological rationality requirements)*.

A promising indication coming from our analyses concerns the need of a constant meta-mathematical reflection (performed through the use of verbal language) on the nature of the actions to perform and on the solving process during its evolution. At present, the only reflective activity in school concerns checking the correct application of the rules of syntactic transformation of algebraic expressions (thus only one component of rational behaviour - namely, the *systemic requirements* of *epistemic rationality* - is partly engaged).

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ARTIFACTS: INFLUENCING PRACTICE AND SUPPORTING PROBLEM POSING IN THE MATHEMATICS CLASSROOMS

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In this report we present a teaching experiment on the relationship between everyday mathematics, in particular the numerical culture children acquired outside the school, and classroom mathematics, and the ways each can inform the other in the development of abstract mathematical knowledge. The teaching/learning environment designed and implemented in this study is characterized by an extensive use of suitable artifacts, whose introduction into the classroom setting brings from the outside world potential norms and ways of reflection, interactive teaching methods and introduction of new socio-mathematical norms. It is focused on a mindful approach toward mathematical modelling and a problem posing attitude.

INTRODUCTION

Tools, artifacts, and cultural representational systems are important components of mathematical learning but do not directly determine ways of reasoning (Schliemann, 2002). Mathematical structures embodied by the tools and symbolic systems do not transfer directly to the user's mind and [‘*cognitive activity is not limited to the use of tools or signs’* (Vygotsky, 1978)]. The mathematical goals emerge for children not only in relation to artifacts but even in relation to structure of activity, social interaction and children's prior understanding (Saxe, 2002).

> Given the complex interaction between the use of the tools and the development of reasoning and learning, the question that should concern educators is not how powerful or effective cultural tools are in promoting learning, but rather what teaching practices and classroom interactions can promote meaningful learning and understanding of the mathematical principles and relations embedded in cultural tools and representations. (Schliemann, 2002, p. 302)

The teaching experiment presented in this report is part of an ongoing research project on the relationship between everyday mathematics, in particular the numerical culture children acquired outside the school, and classroom mathematics, and the ways each can inform the other in the development of abstract mathematical knowledge. The project aimed at showing how an extensive use of suitable artifacts, with their incorporated mathematics, a variety of complementary, integrated, and interactive teaching methods, and the introduction of new socio-mathematical norms (Yackel & Cobb, 1996), can play a role in order to create a substantially modified teaching/learning environment. This environment is focused on fostering a mindful approach toward realistic mathematical modelling and a problem posing attitude.

These socio-mathematical norms are constructed and continually modified through the interaction between teacher and pupils, as well as by the artifacts, whose introduction into the classroom setting brings from the outside world potential norms

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and ways of reflection that open lines of cultural conceptual development to the children.

THEORETICAL AND EMPIRICAL BACKGROUND

About mathematical modelling

The term mathematical modelling is not only used to refer to a process whereby a situation has to be problematized and understood, translated into mathematics, worked out mathematically, translated back into the original (real-world) situation, evaluated and communicated. Besides this type of modelling, which requires that the student has already at his disposal at least some mathematical models and tools to mathematize, there is another kind of modelling, wherein model-eliciting activities are used as a vehicle for *the development* (rather than the application) of mathematical concepts. The '*emergent modelling'* approach (Gravemeijer, 2007) taps into the second type of modelling, and its focus is on long-term learning processes, in which a model develops from an informal, situated model ("a model of"), into a generalizable mathematical structure ("a model for").

Although it is very difficult, if not impossible, to make a sharp distinction between the two aspects of mathematical modelling, it is clear that they are associated with different phases in the teaching/learning process and with different kinds of instructional activities (Greer, et al. 2007). However, in this contribution the focus will be more addressed to the second aspect of mathematical modelling.

An early introduction in schools of fundamental ideas about modelling is not only possible but also indeed desirable even at the primary school level. Further we will argue for modelling can be seen as a means of recognizing the potential of mathematics as a critical tool to interpret and understand reality, the communities children live in, or society in general. Teaching students to interpret critically the reality they live in and to understand its codes and messages so as not to be excluded or misled, should be an important goal for compulsory education (Bonotto, 2007).

About problem posing

It is well recognized that problem posing is an important component of the mathematical curriculum and, indeed, lies at the heart of mathematical activity (English, 1998). Not surprisingly, reports such as those produced by the National Council of Teachers of Mathematics (1989, 1991, 2000) have called for an increased emphasis on problem-posing activities in the mathematics classroom.

Problem posing and problem solving are closely related. As Silver (1994) suggested, problem posing could occur prior to problem solving when problems were being generated from a particular situation or after solving a problem when experiences from the problem-solving context are modified or applied to new situations. In addition, problem posing could occur during problem solving when the individual intentionally changes goals while in the process of solving the problem.

Despite its significance in the curriculum, problem posing has not received the attention it warrants from the mathematics education community. Little is known about the nature of the underlying thinking processes that constitute problem posing, and the schemes through which students' mathematical problem posing can be analyzed and assessed (Christou et al., 2005). We know comparatively little about children's ability to create their own problems in both numerical and non-numerical contexts or about the extent to which these abilities are linked to competence in problem solving. We also have insufficient information on how children respond to programs designed to develop their problem-posing skills (Silver, 1994). Research on these issues is particularly warranted, given the well-documented evidence that young children's creativity and open-mindedness in generating and solving problems dissipate as they progress toward the higher school grades (English, 1998).

Problem posing has been defined by researchers from different perspectives (see Silver & Cai, 1996). In this contribution we consider mathematical problem posing as the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. It, therefore, becomes an opportunity for interpretation and analysis of reality in different ways: i) they have to distinguish significant data from irrelevant data; ii) they must discover the relations between the facts; iii) they must decide whether the information in their possession is sufficient to solve the problem; and iv) they must investigate if numerical data involved is numerically and/or contextually coherent. These activities, quite absent from today's school context, are typical of the modelling process and can help students to prepare to cope with natural situations they will have to face out of school.

About artifacts

For some years now our research has been concerned with the following problematic: i) how can we benefit from the numerical culture children acquired outside the school while simultaneously avoiding the strengths and limitations that are typical of the usual everyday mathematics, and ii) how can we design better opportunities for children to develop new understandings about underlying mathematical concepts and structures and their potential generalizability, in a way that preserves the focus on meaning found in everyday situations.

The connection between students' everyday mathematics and classroom mathematics is not easy, because the two contexts differ is some significant ways. Just as mathematics practice in and out of school differs, so does mathematics learning (see e.g. Masingila et al., 1996). Although the specificity of both contexts is recognized, we deem that those conditions that often make extra-school learning more effective can and must be re-created, at least partially, in classroom activities. Indeed while some differences between the two contexts may be inherent, many differences can be narrowed by creating classroom situations that promote learning processes closer to those arising from out-of-school mathematics practices (Bonotto, 2005).

That can be implemented in a classroom by encouraging the children to analyze some mathematical *'facts'*, which are embedded in opportune '*cultural'* or *'social'* artifacts; these mathematical *'facts'* can be seen as concrete extensions [in the logical sense] of the mathematical concept, which have instead intentional nature [in the logical sense]. The use of artifacts in our classroom activities has been articulated in various stages, with different educational and content objectives (for a description see e.g. Bonotto 2005 and 2006).

Several educators (e.g. Vygotsky, 1978, Schliemann, 2002, Saxe, 2002) have noted that is not the artifact (or tool) in isolation that offers support to the teacher – rather the student use of the tool and the meanings they have developed as a result of the activity. Artifacts take on mathematical meaning only in activity, as individuals organize them as a means to accomplish particular mathematical goals. In his *emerging mathematical goals approach* Saxe (2002) analyze the complex role that artifacts play in processes of teaching and learning in collective practices. He identifies four dimensions of children's activities, considering the way in which each is implicated in children's emerging goals; the dimensions include activity structures, social interactions, valued artifacts, and the prior mathematical understandings that children bring to collective practices.

THE STUDY

There is considerable evidence from studies involving both school students and adults that the system of decimal numbers is neither simple to learn nor generally understood (e.g. Hiebert, 1985; Stacey & Steinle, 1998). A central problem seems to be that few connections are made between the form students learn in the classroom and understandings they already have (or could acquire quickly). Thus it is important that teachers recognize the numerical culture acquired outside the school in order to offer children the opportunity to develop new mathematical knowledge preserving the focus on meaning found in everyday situations (Bonotto, 2005).

In this study we decided to use as artifact some advertising leaflets containing discount coupons for supermarkets and stores in order to strengthen, in particular, the percentage concept. It was found that all students were already familiar with this kind of artifacts and had experience in supermarket or stores shopping.

The official logic-mathematics teacher of the class involved in this study had previously worked together with the students on this mathematical content, focusing in particular her activity on the application of percentages within traditional word problems. Nevertheless the children understanding of the subject was rather poor and superficial; they showed a certain amount of difficult especially in solving word problems with more data and more questions.

Participants

The study was carried out in a fifth-grade class (children 10 years of age), consisted of 18 pupils, in Desenzano del Garda, a lakeside resort in the north of Italy, by the official logic-mathematics teacher, in the presence of a research-teacher. The official logic-mathematics teacher of the classes involved in the explorative study use a traditional teaching method.

Procedure

It was decided to subdivide the teaching experiment into two sessions, at weekly interval; each session, lasted approximately two hours, involved, as artifact, different advertising leaflets. Each session was divided in the following phases: i) through whole-class discussion the information and the numerical data present in the artifact were investigated and interpreted; ii) each pair (chosen by the teacher so as to stimulate reciprocal support and assistance) selected the problem's data by extracting it from the artifact provided; iii) each pair created a problem containing the previously selected data: each problem had to contain a percentage calculation and two questions; iv) each pair had to resolve a problem written by another group; v) the results obtained were collectively discussed; eventual errors or incongruences will be discussed; vi) the entire group was called upon to write collectively the final version of some texts of some more complex problems, which were created by the group itself after problem critiquing activity and suggestions made by some of the children.

Data

Data from the teaching experiment include students' written work, fields' notes of classroom observations and audio recordings for all collective discussions. We used qualitative methods of analysis to examine these data.

Research questions and hypotheses

The overall aim of the teaching experiment was to examine the relationship between the mathematics incorporated in real-life situations and school mathematics, and the ways each can inform the other in the development of abstract mathematical knowledge; in this case the focus was on the understanding of rational numbers considered as percentages, in order to foster what Hiebert (1985) calls site 1 ("*symbols and their referents*"), in a way that was meaningful and consistent with a disposition towards making sense of numbers.

Furthermore, we hypothesized that, contrary to the practice of word-problem solving documented in the literature (see e.g. Verschaffel, Greer & De Corte, 2000), children in this teaching experiment would not exclude real-world knowledge from their observations and reasoning (hypothesis II).

Finally we wanted to evaluate the impact of the problem posing activity it itself, process that was unusual for the students, and to begin to investigate the relationship between *problem solving* and *problem posing* activities. The idea is that there is a connection between *problem solving* and *problem posing*: it is impossible to solve any new problem without first having completely understood, in a way that was meaningful and consistent with a sense-making disposition, the assignment, also by raising new problems in *a critique way* during the solution phase, just as it is impossible to write the text of a problem without first having understood, in a way that was meaningful and consistent with a sense-making disposition, the mathematical area that is its foundation.

SOME RESULTS

The children had no difficulty translating typical everyday data into problems suitable for mathematical treatment and all of the pairs succeeded in solving the problems created by their classmates, except for one pair, which encountered difficulties because the data selected by their classmates required somewhat more elaborate calculations. After the group discussion, it was decided to modify some of the data in order to render the resolution of the problem more straightforward.

The *problem posing* activity was experienced as a new and positive procedure for the children; this process, in addition to having created interest and motivation, encouraged the children to create problematic situations that were both original and sometimes complex, but nevertheless much more realistic than those present in traditional word problems.

An example: the text created by Sofia and Giorgio was

> A mother sees that in a drawer there that are several socks with holes and that all the pyjamas are too small. The following morning she looks at a calendar and realised that it is the start of the sales. She goes to a shopping centre and after looking long and hard she decides to buy three pairs of pyjamas at a full price of € 11,90 but with a 20% discount. How much will she spend on pyjamas? Subsequently she buys five pairs of socks that cost € 4,90 a pair and that have a 50% discount. How much will she spend on socks? If she has € 100,00 how much money will she have left*?*

When the classmates were invited to give their contribution by adding, enhancing or changing the problem, the following questions emerged:

> If the mother has run out of credit on her mobile phone and needs to call home urgently because she cannot remember her sons' sizes, will she be able to recharge her phone? By how much?
> If the supermarket's car park costs € 1,50 per hour and the mother enters the supermarket at 10.30 and leaves at 12.00, how much will she spend? Will she be able to pay with the coins given to her as change from the supermarket or will she have to change banknotes?
> When entering the shopping centre the mother sees that the following week all clothing will be discounted by 50%. How many pyjamas and socks can she buy?

In these contributions children showed remarkable originality due to their wealth of experience outside school, which involves different and complex aspects.

Furthermore a subsequent process of *problem critiquing* was set up whereby the children attempted to criticize and make suggestions or correct the problems created by their classmates or the results obtained.

When a pair of children crated the following problem

> To celebrate his son's birthday Mr. Gianni will go a supermarket during its sale period and buy 9 boxes of ice cream treats that cost 2,99€ each, but with a 50% discount. How much does he spend for the ice cream treats? In addition, he also buys 7 packages of "mini-smarties" that cost 2,49€ each but with a discount of 20%. How much does Mr. Gianni spend for the "mini smarties"? To conclude his shopping, he also buys 3 cartons of apple juice that cost 0,99€ each, with a discount of 20%. How much does he spend in all?

a part of the class discussion regarded the result obtained, which was 29,775€:

> Davide: You can't pay that amount since there is no such coin as a 5 mil piece.
> Teacher: What happen when the supermarket bill turns out that way?
> Giovanni: You have to ay a little bit more, for example 29,80€.

Filiberto: Yes, that's true, but if you include the money you pay for the shopping bag to carry the goods home (which one also pays for), it seems to me OK if you give them, let's say, … 30€

and when a pair of children created the following problem related to the second advertising leaflet

It's sale week and Mr. Mario would like to renew his wardrobe. So he goes to the store called "Always more". He buys 8 pullovers of various colours at 42€ each. There is a discount of 5% on the merchandise. How much does Mr. Mario spend in all? If he starts with 350€ to renew his wardrobe, how much does he have after this purchase?

a part of the class discussion regarded the result obtained, which was 319,2€, and show a *problem critiquing* attitude by children based on other realistic considerations:

Michele: OK, but if I give 320€ it comes out the same, since I have to pay for the shopping bag.

Filiberto: But if you go to clothing stores, you don't have to pay for the shopping bag, because you've already spent so much to buy the pullovers that they give you the shopping bag for free.

These examples as many other of written works and discussions demonstrated that the children have by no means ignored the relevant, plausible and familiar aspects of reality in their observations and reasoning (hypothesis II confirmed).

By presenting the students activities that are meaningful because they involve the use of material familiar to them, increased their motivation to learn even among the less able ones. For this reason, even children with learning difficulties related chiefly to linguistic problems are helped.

Yuri: This is not a problem. Problems are full of words and I can never do them because I do not understand very much. I can do these though because anyone can read prices on a flyer!

This confirms what "*Roughly speaking using a receipt, which is poor in words but rich in implicit meanings, overturns the usual buying and selling problem situation, which is often rich in words but poor in meaningful references*" (Bonotto, 2005).

CONCLUSION AND OPEN PROBLEMS

From the results it appears that the teaching experiment had a significant positive effect on achieving learning goals, in particular enhancing the understanding of rational numbers as percentages, in a way that is meaningful and consistent with a sense-making disposition. In our view, the positive results can be attributed to a combination of closely linked factors: (a) an extensive use of suitable cultural artefacts; b) the application of a variety of complementary, integrated, and interactive instructional techniques; c) the introduction of particular socio-mathematical norms; d) an adequate balance between problem-posing and problem-solving activities.

In future research, we would like to look more deeply at the nature of the underlying thinking processes that constitute problem posing, at the relationship between problem posing and problem solving, and between problem posing and creativity, and at how children respond to programs designed to developed their problem-posing skills on long-term.

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ANTICIPATING TEACHERS' LEARNING WITHIN THE INSTITUTIONAL SETTING OF THEIR WORK

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In our recent effort to design a teacher development program, we decided to take into account the institutional setting in which collaborating teachers work. This decision has been necessitated by the particularities of our school system. The analytic approach we adopted is based upon the notion of the 'community of practice'. By analyzing the interconnections between the 'communities of practice' we thus identified, we delineate a hypothetical learning trajectory for the initial emergence of a professional teaching community.

INTRODUCTION

The goal in our teacher development workshops used to be an effort to help teachers revise their textbook-based teaching of mathematics. Teachers had to question the effects of their traditional practices on students' meaningful understanding. However, many of them were reluctant to consider activities that were not included in the textbooks. This situation did not fit well with the fact that in our country there is not high-stakes testing. The lack of accountability measures should have facilitated the reorganization of their ways of teaching mathematics. Thus the need to take account of the institutional setting of their work became apparent. The same need was apparent, when we collaborated with teachers to conduct teaching experiments in their classrooms. Shortly after leaving their classrooms we noticed that they returned to their traditional instructional practices. The change of these teachers' way of teaching mathematics was not as lasting as we assumed. In both examples, the institutional setting was neglected when we first attempted to explain why teachers defied efforts to support the reorganization of their practices. By that time, our assumption was that teachers are autonomous agents in their classrooms.

Two years ago new textbooks in mathematics have been introduced to all the primary schools in our country. The school system is centralized. The new textbooks were meant to be the means to reform instruction. Teachers disparaged them and strongly challenged the change of the old textbooks. Surprisingly, in a very short time, things have settled down. Once more, the institutional setting comes to the front. Now it seems to explain the teachers' change of behaviour. They were constrained to adopt the new textbooks. Though we had a hope for it, the weaknesses of textbooks did not play out in this confrontation.

The explanatory power of the institutional setting, in the above examples, has come to our attention, while we tried to organize a teacher development program. In reviewing the relevant literature we came across several papers that point to the

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importance of the institutional setting in teacher development (Ball, 1996; Stein & Brown, 1997). According to Cobb, McClain, Lamberg, & Dean (2003) the immediate institutional setting within which teachers develop and refine their instructional practices is constituted by the efforts of members of different communities of practice, and whose enterprises are concerned with the teaching and learning of mathematics. Research in mathematics education has started to view teachers' instructional practices as situated in the institutional settings of the schools and districts in which they work. The improvement of teachers' classroom practices is investigated in terms of the school and district structures.

By taking into account the relevant research and our prior experiences, in our current teacher development program, we do not overlook the power of the institutional setting. The goal of our program is focussed on sensitising the teachers in students' reasoning and enabling them to support its development in their classrooms. Analysing teachers' learning as they develop suitable practices is necessary for the success of our program. The design experiment methodology will be an appropriate means for the eventual study of how teachers will improve their instructional practices (Gravemeijer, 1994).

During the first phase of our research program, we attempted to document the institutional setting within which the collaborating teachers work. Given the particularities of the policy environment in our country (lack of standardized tests and accountability measures, centralized school system), we viewed our attempt to be indispensable for the design of a program adapted to the needs of the participating teachers. This design will be effective to the extent that we conduct analyses of the teachers' ongoing learning. So, in this paper, we first present the results of analysing the institutional setting within which teachers will develop and refine their practices. On the basis of these results, we then delineate a conjectured learning trajectory related to the means of supporting the development of a professional teaching community.

THEORETICAL FRAMEWORK

Spillane, Halverson and Diamond (2001) in advancing a distributed perspective on leadership claim that it is not simply a function of individual leaders. As they argue leadership practice is distributed across leaders and their social and material situations. In a similar way, we view teaching as a distributed activity across several communities of practice whose members are concerned with the teaching and learning of mathematics. The institutional setting is constituted by the interaction of these communities. So, a distributed perspective on teaching makes the study of the institutional setting plausible.

We note that even if teachers do not apparently collaborate with others apart from their students, the nature of their work proves that the institutional setting is ubiquitous. For example, one of the functions of teaching is the organizing for mathematics teaching and learning (Cobb and McClain, 2006). The use of a variety of tools is necessary for organizing their lessons (e.g. textbooks, teachers' guides, curriculum manuals, manipulatives). The intended use of the textbooks is based upon lots of instructions on content and pacing of coverage. It is obvious that a long chain of people has contributed to the production of these instructions. It is of course not expected that teachers will use them in the ways intended. This example hints to our assumptions concerning teachers' learning in the context of the institutional setting.

We assume that teachers' activity in their classroom is not determined by the institutional setting. Instead their classroom practices are enabled or constrained by the institutional structures and at the same time they may contribute to the reorganization of these structures. In other words, teachers' participation in several communities of practice is reflexively related to their participation in their classroom community.

METHODOLOGY

In the analysis of the institutional setting we used the analytic approach developed by Cobb and his colleagues (Cobb et al., 2003; Cobb and McClain, 2006). This approach is based upon the work of Wenger (1998) and identifies the communities of practice by using three dimensions: 1) a joint enterprise, 2) mutual relationships, and 3) a well honed repertoire of ways of reasoning with tools and artifacts. These three interrelated constructs will allow us to document the communities of practice whose enterprises are about the teaching and learning of mathematics. However, a more precise picture of the institutional setting emerges from analyzing the interconnections between the identified communities. Three types of interconnections will be distinguished: 1) boundary encounters, 2) brokers, and 3) boundary objects.

Our data for documenting the institutional setting come from a school district in the area of Athens with 14 primary schools. We initially interviewed four teachers of the first three grades, from the same school. These teachers were also observed in their classrooms, so that we would understand their classroom practices. In their interviews teachers have spontaneously referred to their connection to other persons that influence their work. These other persons were the two principals, the district school counsellor, and the parents. Interviewing them has led us to the identification of more people being related to the way mathematics is taught in school. These people were the representatives in the teachers' union and the county leaders. The interviews were semi-structured and were audiotaped. They included questions on the following subjects: 1) professional development activities in which they have participated, 2) people to whom they feel accountable to, 3) collaboration with other people on their work, 4) their beliefs about mathematics and mathematics teaching and learning, 5) the role of students' reasoning in their instructional decision making, 6) the use of the textbooks. The data we gathered include also our notes from observing a one-day seminar organized by the school counsellor on issues related to the teaching of mathematics in the first grades.

RESULTS

Documenting the characteristics of the communities of practice

From the interviews we carried out, the following five communities of practice were identified:

Teachers

Their joint enterprise is to ensure that students would understand all the content contained in their textbooks. From our classroom observations we noticed that their instructional practices are in agreement to the 'school mathematics' tradition (Cobb, Wood, Yackel, and McNeal; 1992). The meaning of understanding is confined to the replication of rules and procedures taught in class to obtain correct answers. Drill and practice are the primary means to secure students' understanding. Adherence to the textbook and its sequencing of mathematical topics is their main concern as they try to teach on the basis of the textbook. We would like to mention that this concern is also expressed in the teacher development seminar we observed. The rationale for covering all the contents is based upon their fear to challenge the expectations of students' parents and of their colleagues. As one teacher said: "I feel better when I finish the textbook because next year the teacher will not be displeased with the job I did."

The division of labour in which the two first-grade teachers are engaged is mainly the sharing of worksheets for students' drill and practice. In addition, teachers report that when having difficulties with the solution of a mathematical problem they might draw on other colleagues for assistance. However, only one teacher mentioned that she had once visited another teacher's classroom to observe a lesson on science. We may therefore assume that teachers are rather isolated from each other.

The tools used for organizing their instruction are the textbooks and the teachers' guides. The curricular objectives are included in the teachers' guides. Their main concern in using these tools is managing a correct sequencing of topics and the pacing of covering the textbook. These issues emerged as topics of discussion also in the teacher development seminar. At this seminar we observed that teachers compared the number of chapters they had covered. Teachers gave special attention to the review lessons included in the textbook. Students' assessment is mainly based on these lessons. As we noticed teachers assess students in terms of their deficits.

District school counsellors

The members of the school counselling community consist of the county coordinator and the 14 district school counsellors of the county. Primary school counsellors do not specialize exclusively in mathematics. Their joint enterprise, as related to mathematics, is to improve the quality of mathematics education by helping teachers with the use of the reform textbooks.

From their interviews we were informed that they organize one-day professional development seminars for teachers and principals on the use of the reform textbooks four times annually. They usually conduct these seminars in collaboration with each other and with teachers. Sometimes, counsellors cooperate with teachers to organize and present exemplary mathematics lessons. In the seminar we observed, the district counsellor had invited a counsellor from another district to make a presentation on the first grade textbook. The whole seminar was structured around issues related to the daily practices of teaching early number arithmetic. Through generalities like using real-world problems, manipulatives, or small group work, the counsellor thought that reformed instruction of mathematics could be realized. However, there was no reference to students' interpretations and solutions of instructional activities. Only traditional instruction goals were encouraged and there was no reaction from the audience. In this sense, interviewed teachers were justified when they expressed their reluctance to accept counsellors' visits in their classrooms. As they commented, counsellors did not have something really new to offer.

The tools which members of this community reason with when organizing for mathematics instruction include the textbooks, the teachers' guides and the curricular objectives. In their interviews the ways they appeared to reason with these tools showed us that covering the curriculum was their main concern. Students' ways of reasoning are not within their scope.

Principals

The community of principals in the school we selected to work with consists of the principal and the assistant principal. They both have 70% release time from teaching. The joint enterprise of each school's community of principals is to ensure the good function of the school unit by solving administrative tasks. However their agendas may extend to pedagogical issues related to the teaching and learning of mathematics. For example, parents objected to the homework assignments being inappropriate in terms of content and quantity. The principals decided to draw teachers' attention on this issue by scheduling some meetings with the teachers. In these meetings, the goals of instruction, as presented in the teachers' guides, were taken as points of reference. Students' needs in terms of their ways of reasoning were neglected. In his interview the principal argued: "We must take into account the curricular objectives. If we do not, then we cannot select effective assignments. In fact assignments should test the content and avoid overlapping…"

Parents

The community of parents was brought to our attention by the principal of the school informing us on a special meeting organized by the parent association. In this meeting counsellors had been invited to address parents' concerns related to the introduction of the new textbooks. Thus we scheduled an interview with the leader of the parent association. Their joint enterprise is to avoid gaps in their children's learning. This is achieved by the assistance they provide themselves or by hiring private teachers. In the regularly scheduled meetings of the parent-teacher association they primarily engage in finding resources to cover the school needs. Occasionally, in these meetings they exchange views on issues related to mathematics instruction. Their experiences from their own participation as students in school constitute a motto for their views about the teaching and learning of mathematics. Their tools for reasoning with teaching and assessment include the textbooks and the homework assignments. The ways of reasoning with these tools are based rather on their children weaknesses than on their capabilities.

Teachers' union representatives

Most of the members of this community receive some release time from teaching. Their joint enterprise is to fight for teachers' rights and to promote their demands. When teachers feel that someone is interfering with their work, they have their union representatives function as their advisers. Within their community, union representatives arrange contacts to people who influence educational policy issues. When a bill concerning education is under discussion, the positions they stand for depend on their political party affiliations.

Documenting the interconnections between communities of practice

By focusing on the interconnections between the above communities of practice the delineation of a conjectured learning trajectory for the community of teachers will be facilitated.

Boundary encounters

School counsellors organize professional development seminars for teachers and principals. These seminars constitute boundary encounters between the members of three communities. Teachers' comments indicate that their instructional practices could not evolve by attending these seminars. Though these seminars did not present something new to the teachers, we could say that they serve as sites to sustain their traditional orientation to mathematics instruction. Their only positive comments referred to the exemplary teaching of lessons prepared by teacher colleagues. As a consequence, we might assume that teachers will not oppose the prospect of observing each other. For the professionalization of their community this might be promising for the evolution of alternative instructional practices.

Other boundary encounters like those between parents, teachers, and principals, or between teacher union representatives and teachers, served to conceal any sort of differences in viewpoints.

Brokers

Brokers are peripheral members of at least two communities of practice. They may act as a bridge for the activities of different communities. The principal of the school we observed, in the case of excessive homework assignments, seemed to persuade teachers by using the curricular objectives. In this way, he functioned as a broker. The counsellor's emphasis on curricular objectives was made clear to teachers. Even though the principal has acted as a broker between parents, teachers, and the counsellor, his activity was aligned to the traditional ideas of mathematics instruction. If the focussing of his activity had been different, we might have capitalized his role to promote a different orientation in the teachers' ways of teaching.

Boundary objects

The boundary objects are common tools used by members of two or more communities in a routine way. Such a boundary object is the set of textbooks. As an example, the counsellors' meaning of textbooks was faithfully attached to the curricular objectives contained in the Analytic Program of Instruction. Similarly, teachers considered textbooks as a set of topics with narrow goals. Since there is no difference in viewpoints, the initiation of a change in the meaning of textbooks would be difficult.

From a teachers' community of practice to a professional teaching community: A hypothetical learning trajectory

On the basis of the above results we realize that teachers' instructional practices are enabled by the institutional setting in which they are situated. On the other hand, these instructional practices contribute to the sustenance of the institutional setting. Therefore, an effort to change teachers' traditional instructional practices necessitates the search for the starting points of teachers' change within the institutional setting and at the same time in the instructional practices themselves. In this context, we make the following conjectures for an initial learning trajectory: *1) Teachers making aspects of their practice problematic*. For these teachers the difference between covering the textbook and attending to students' reasoning might lead to a new meaning of understanding. Supporting activities may include: clarifying the intent of activities in the textbook, analyzing students' work, etc. *2) Teachers starting to collaborate with each other on issues related to their practices*. Jointly planning instructional activities that they would use in their classrooms and then reflecting on the effects of using them might be a supportive means. *3) Teachers making aspects of the institutional setting within which they work problematic.* For example, reflecting on the compulsory professional development seminars organized by the counsellors could lead them to a new meaning of participating in the practices of different communities. *4) Teachers starting to collaborate with members of other communities for a change of the institutional setting.* Jointly planning activities that they would use in their new boundary encounters and then reflecting on the effects of using them might be a supportive means.

DISCUSSION

The trajectory we conjectured reflects a bottom-up process for the transformation of the institutional setting within which teachers work. However, as the institutional setting changes we might expect a top-down process to be initiated. For example, union representatives might fight for some release time so that teachers collaborate with each other under better conditions. Counsellors could also promote a change in the textbooks. Students' reasoning would then take a central position. In this way, starting from the top a new cycle of bottom-up changes could be engendered. This cyclical process of innovation reminds Freudenthal's (1978) idea that innovation is a big social process.

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DELINEATIONS OF CULTURE IN MATHEMATICS EDUCATION RESEARCH

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ABSTRACT: *This paper interrogates the notions of culture and subjectivity in Radford's cultural theory of objectification, and how this theory treats the production and sharing mathematical entities. It follows his analysis of some children in school working on mathematical sequences and considers how roles are understood for both teachers and students. It argues that Radford's research perspective delineates a cropped conception of culture that takes insufficient account of how that perspective presupposes conceptions of curriculum reform.*

INTRODUCTION

Radford's *Cultural Theory of Objectification* is centered on the cultural and historical dimensions of mathematical objects. His theory perhaps provides the most sustained and substantial excursion into this area, supported by extensive empirical research. Through examining how mathematical objects are apprehended by students he endeavors to situate these objects in relation to the cultural historical formation of mathematics more generally. In this theory "teaching and learning involve a particular conception of the milieu, the social interaction, the student and the teacher" (Radford 2008). This work has resulted in a questioning of a prevalent tradition in mathematics education research in which mathematical objectivity has sometimes been seen as transcending cultural specificities. Radford, Bardini & Sebena (2007, p. 2) also question conceptions of learning where the learning is understood as "something mental, as something intrinsically subjective, taking place in the head". Rather, for Radford "thinking is considered to be a *mediated reflection on the world in accordance with the form or mode of the activity of individuals"* (Radford, 2006, his emphasis). Radford's encapsulation of mathematical phenomena as cultural historical objects is to be welcomed as it furthers discussion in a highly productive area. But what conception of culture is envisaged and how does this conception shape analytic possibilities?

MAKING GENERALIZATIONS (OR COUNTING AS ONE)

I shall begin with a crucial question and consider how this is dealt with in Radford's formulation: How does one create or apprehend a mathematical object, relationship or generalization? Radford et al (2007) describe some classroom events in which

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students are examining numeric sequences with view to establishing how generalizations might be understood in a "progressive manner" (p. 3). The acquisition of such generalization is depicted as a developmental process through which students encounter awarenesses (cf. Gattegno, 1971) progressively as they inspect mathematical phenomena through a range of perspectives and filters over time. In this hermeneutic process successive frames are introduced to eventually encapsulate an emerging sense of generalization through identifying features common to each perspective. Here the introduction of each successive filter requires further work as it is no mere neutral exercise to see the generalization in another way. Each new filter is productive in terms of intellectual engagement and in terms of the constitution or frame of that generalization. The authors utilize an unusually broad range of research instruments in examining the many dimensions of the learning processes that they seek to identify and depict. The emphasis in their discussion is on how the acquisition of mathematical understanding is mediated through a variety of bodily experiences that supplement mere mental manipulation. "The resulting qualitative analysis shows how discourse, gestures, actions, and rhythms orchestrate one another and how, through a complex and subtle coordination of them the students objectify different aspects of their spatial-temporal mathematical experience" (p. 1). The instruments include, for example, video analysis of gesture and "prosodic" analysis student speech to check the intensity of word pronunciation. This analysis seeks to pinpoint how emerging generalizations come into being through a variety of mutually supportive constructs. So, for example, when a student utters the word "another" in connection with a sequence and starts making a rhythmic sequence of six parallel gestures he is seen as expressing the "idea of something general, something that continues further and further, in space and time" (p. 2). Meanwhile, student perceptions of specific arrangements change through time as new awarenesses are achieved. The analysis concerns a pattern comprising two horizontal lines of circles, one line above the other, where the lower line has one more circle than the upper. The two additional circles that are initially pointed to as added to each successive term as an iterative step are later seen separately as part of a more holistic view as respective members of the upper and lower layers, as indicated by being physically pointed to with a finger to emphasize a pattern (p. 12, p. 15), or written, e.g. *23 cercles; 11 en haut, 12 en bas*, *203 cercles; 101 en haut, 102 en bas* (p. 10). At another stage the students' use of brackets is "unnecessary" in conventional mathematical terms but are seemingly used as a prop to a particular organization of the story of their mathematical experience ($(n+1) + (n+2)$ is used as an algebraic formula for totaling circles in the pattern, pp. 11-12). These are clearly student observations yet ones that might be predicted to arise within the teacher's prescribed frame. The activity synecdochially suggests a particular way of approaching and hence constructing the field of mathematics more broadly. The student engagement comprises a discovery evaluated against the teacher's expectations, suggesting a reproductive conception of education (Bourdieu & Passeron, 1977) rather than a construction of a new perspective. In this respect the child's construction seems rather unlike a piece of art or poetry seen "as a way of expressing ourselves and making sense of the world" (Radford et al, p. 2), since the constraints to movement within the prescribed mathematical domain are rather more directive to those commonly understood within art or poetry. (Although Atkinson (2002) counters this romantic characterization of art in discussing how children's drawings are often evaluated by British teachers with respect to culturally specific artistic codes that disenfranchise some children from specific cultural groups, such as Pakistani children following artistic styles in which Western conceptions of perspective are not followed).

In Radford's analysis the student's activity is understood as transitive. There is a defined target where there is some issue as to who has ownership of defining that target. There is a right and a wrong way of continuing the sequence. Generalization is sought yet it is the teacher who has decided that it is to be taught and has set the terms of the generalization being sought through shaping activities towards identifying the generalization in these terms. The activity described clearly leaves some space for student construction, more than would be allowed in some teaching schemes, but rather less than might be permissible, for example, in investigational tasks as pursued within progressive models of education deployed in the UK some years ago (e.g. ATM, 1977). Whilst the student may have some input in terms of providing her or his own constructs, these constructs will to some extent be evaluated against the teacher's pedagogical framework and the curriculum constraints that define her practice. "The goal is clear for the teacher, but, generally speaking, not for the students" (p. 6). In this process "the subjective and the cultural" are seen as becoming "entangled" where the object goes beyond that which is seen perceptually and hence it becomes possible to "become aware of general properties that are not visible in the realm of the concrete and the particular" (p. 4).

Yet at this stage it is perhaps necessary to unfold the meanings that are being brought by the authors to the words "subjective" and "cultural". The terms are being used in a rather restrictive fashion that results in particular perspectives being seen as natural rather than ideological. Specifically, the teacher's pedagogical frame and the conceptions of generality that suggests are a function of her own history and of the curriculum constraints she is working to now. So insofar as learning objectives are referenced within that frame there is an inevitable reproductive aspect to learning, an implicit aspiration for the student to understand the mathematics in the teacher's own terms.

"Subjective" might be understood in terms of how teacher and student are *subject* to specific discursive frames, where actions are evaluated with respect to that discursive register. So here a mathematical generalization would not be seen as a "thing" in itself but something understood with respect to a particular discursive frame, that is, to a specifically ideological way of making sense, defined at the level of the pedagogical layer and the materials that support that, such as curriculum specification or favored ways of setting algorithms. For example, Cooper & Dunne (1999) report how the formulation of mathematics in to word problems can sometimes penalize children from poorer backgrounds less able to engage with the particular form of linguistic subtlety.

Meanwhile, "culture" is a function of the perspective one has of its scope and of how one is positioned within it. Radford et al (2007) are looking at the mathematical exchanges defined in conventional mathematical symbols as if from an individual teacher's perspective on how learning takes place with respect to the activities posed, the curriculums that they serve and according to particular ideological conceptions of how situations can be mathematized. Yet the students are not only recipients of culture but also creators of it insofar as their fresh perspectives on mathematical situations can be voiced, rather than being merely evaluated with respect to an existing register. (Although there is also a pedagogical job to be done of enabling students to recognize how mathematical conceptualization can be linked to cultural forms so that they can engage with and share culturally preferred approaches to tackling everyday problems.) Further, the perspective of Radford et al precludes mention of a mechanism that might enable the dissemination of such a perspective and an operational adjustment to practices across a population of teachers. Teachers are not things in themselves as the term teacher is constituted with respect to a particular social construction of that term and the expectations that go with it, expectations that differ markedly across schools and countries. As an individual teacher I may have all sorts of personal optimistic aspirations but if I want a government job I have to fit in with the regulative structures, and understand (or subjectivise) myself through the terms of that regulation. The point of contention here perhaps is whether mathematical generalization can be understood as a thing in itself and hence be universal. We cannot decide this absolutely, however, since the terms of such a decision would be culturally specific, since as Radford correctly asserts, generalization, or objectification, (or any multiple entity "counted as one", Badiou, 2007, p. 23) is a function of the cultural or subjective entities that produce it.

> The students' dialogue suggests the presence of an intense dialectics between the concrete examples and the incipient general. In this dialectics, the concrete accomplishes a twofold function: First, it endows the students with a means to 'craft' the general; second, it provides the students with a way to validate their general statements. (Radford et al, p. 18)

Such dialectics may or may not be convergent to an agreed end. It depends on how the task of teaching is conceptualized.

Radford's fine tuned analysis seeks to pinpoint a range of dimensions to the process of reaching different levels of generalization. Yet how is this generalization conceptualized respectively by the teacher and by students? And how are learning objectives understood in relation to such conceptualizations? How much is the students' quest for a mathematical object previously conceptualized by the teacher, or framed within the pedagogical apparatus? Is the generalization they seek defined in the assessment regime or is it in any sense left to the student's own formulation? That is, whose generalization is it? And how do the students experience the demand to reach this generalization? How is that demand distributed across the specific teacher formulation, the curriculum constraints, school ethos, societal expectations on education or mathematics generally, etc? And if the teacher is culture's custodian how do we understand the cultural formulation of mathematical objects since now the teacher is implicated in that cultural construction, an agent of contemporary cultural dimensions as well as historic aspects?

These questions can be considered against a framework offered by Gallagher (1992) who has highlighted alternative conceptions of hermeneutics in educational contexts; *conservative* where the student's task is to reproductively understand the teacher of the teacher's terms, *moderate* where shared understanding evolves through the teacher-student encounter, *critical* where the student questions the teacher's motivation within the specific pedagogical frame, and *radical* where the teacher's input is mere stimulus to the student's own construction, yet within prevalent discursive orders.

Radford's agenda is on the teacher bringing out the students' meaning but ownership of such meaning is difficult to delineate in such situations given the deterministic aspects of the educational context, and where the students and teachers subjectivity derives from the discursive terrain. Whilst Radford (2006, p. 42) argues that, "ideas and objects … are conceptual forms of historically, social, and culturally embodied reflective, mediated activity", they may still appear as found objects for students and teachers. The students are seen as having a relatively minor role in this hermeneutic mediation, as regards the constitution of the ideas and objects in to objects of consciousness. Undoubtedly Radford is correct in promoting "an essentially social conception of learning" (2008, p. 3), yet there is some issue to be decided as to the status teachers and students hold in such social arrangements. The mutual constitution of "learner", "teacher" and "mathematics" in this instance seems to drag behind an understanding of mathematical ideas and objects largely fixed in advanced by the supposition that they might have been offered from within an objective framework, if it were possible to defend the status of such a framework. The Vygotskian premises that underlie Radford's account possibly suggest an easy entry for individuals into this "culturally embodied reflective, mediated activity". Key figures in this model are teachers equipped to shape mathematics in line with some socially approved structure, with children embracing those expectations.

Radford's model is predicated on the existence of experienced teachers able to administer the classroom in the terms he outlines. This assumes assent from the teachers and also their technical capacity to carry out lessons on this format, or a training course that might be able to produce that capacity in teachers more widely. It similarly presupposes that children will be compliant participants appreciative of the teacher's benevolence and in agreement with this external specification of learning objectives. There is an apparent assumption here that there is some notional model of good teaching that children will recognize and support. We might object, however, that many teachers are alienated from such conceptions at least insofar as their capacity to conceptualize in those terms will be limited, and children may react negatively to externally defined plans for them no matter how ideal their conception. That is, Radford's conceptualization of teachers and their task are somewhat incommensurate with the actual supply of teachers in many instances. Children, perhaps humans in general assert their understandings of who they are through a personal exploration of the boundaries they encounter or perceive. For example, Lacan (2006) sees subjective formation resulting through an engagement with the all encompassing symbolic circuit that shapes thinking, where one's identity is formed through exploring and testing these limits. Teachers and students are often alienated from cultural or pedagogical tools and that compliance with them, or accommodation even, is not the only educational choice. Radford echoes Vygotsky's developmental framework in which students appropriate cultural voices, yet Wegerif (2008, p. 355) argues that it is possible to read this alternatively as the cultural voices appropriating the students. "Vygotsky interprets differences as 'contradictions' that need to be overcome" (ibid, p. 347) yet this is not the only interpretation of an educational interaction; students or teachers may wish, consciously or otherwise, to counter the educational agenda. Radford paints a sympathetic version of cultural accommodation that disregards "symbolic violence" (Žižek, 2008), such as: student performance being understood within a pass/fail categorization; compulsory education fixing choices; differential access to different social groups; insecure teachers reducing the power of student mathematical engagement, perhaps through overly didactic approaches and the closing down of exploration; international curriculum criteria being applied in specific local contexts; the resistance of adolescents to adult guidance; or, the external imposition of perceptual schema (e.g. privileging teacher constructs of social objects; etc). Such symbolic violence cannot be resolved since its existence is consequential to multiple ideologies coexisting. Yet, Radford Miranda, & Guzman's (2008, p. 8) suggestion of locating a "common conceptual ground for the evolution of the students' meanings" points to a more moderate perspective and the authors argue for a transformative process that they describe, after Bakhtin, as *heteroglossic* that suggests a more powerful voice for the students in a situation where "differing views and forces collide … awaiting nonetheless new forms of divergence and resistance" (p. 8). Such an encapsulation seems altogether more critical or even radical in Gallagher's taxonomy. Yet at the same the perspective seemingly offers little engagement with the wider forces that may govern curriculum policy and the specifications such policies demand of teachers activating students' mathematical productions.

CONCLUSION

Radford appears ultimately to be addressing his conception of a model of teaching to a sympathetic audience who could be persuaded by his culturalist aspirations. But what is the scope of that audience? In this aspect of his work his audience is seemingly constructed as teachers and teacher educators who might be seen as seeking to adjust their individual practices, or researchers seeking understanding. Less obviously does it address policy makers who set their task in terms of what can be afforded within attainable educational frameworks and infrastructures. Such infrastructure is an obligatory dimension of contemporary culture that dictates how mathematics can be understood and limits the possibilities that are achievable within the given frames. Such outcomes require strategic manoeuvres perhaps not through direct address to individual teachers and teacher educators. Whilst it is often customary to direct findings to teachers and teacher educators within mathematics education research this is not necessarily the best assessment of the audience reading research journals, nor of the best point of leverage. The instrument of influencing teachers and teacher educators to adjust their individual practices is a very specific conception of how practices may be encouraged to change more generally. It is not the only conception and there are grounds for questioning this orientation to mathematics education research. Radford's theory is centered on controlling events in the classroom albeit with liberal/ progressive intent. In his perspective research questions are predicated on:

- How children could learn better
- How the teacher could assist them
- How the researcher can alert the teacher to possible strategies

But such perspectives rest on certain assumptions:

- That the child is displaying some deficit in relation to a particular ideological perspective (e.g. "raising standards", "reform math", problem solving, performance on SATs, motivation, etc)
- That the researcher focuses primarily on the local classroom interactive level (rather than on socio-economic factors, policy instruments, structural adjustment, etc)
- That the teacher could understand the research provided and could/ would change their practice (rather than this being done by the school board, local authority, government, etc).

Such assumptions result in a partial perspective on the classroom environment, a clipped account of the cultural context and hence a reductive account of the parameters governing the formation of mathematical practices in the classroom. They promote an idealist attitude in which consensus could be readily achieved and where the resources would be available to bring this about. Yet such an acquisition of the necessary resources would entail a radical shake-up of social arrangements beyond the reach and influence of mathematics education researchers. The charge of overemphasizing teacher input could be leveled at many examples of mathematics education research yet Radford invites this response as a result of his specific attention to cultural dimensions of learning. The cultural parameters governing learning are complex and the common assumption that the best point of access in the teacher seems optimistic to say the least. Meanwhile, the cultural parameters seen as governing the research perspective can favor certain models of change.

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MULTIPLICATIVE REASONING AND MATHEMATICS ACHIEVEMENT

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Multiplicative reasoning is involved in young children's mathematics learning in two ways: it forms a basis for learning about composite units in place value and for learning about ratio and proportion. We analysed young children's multiplicative reasoning in two studies. In Study 1, we investigated the conditions under which this reasoning is best elicited in young children. In Study 2 we analysed whether individual differences in young children's multiplicative knowledge contribute to the explanation of individual differences in mathematics achievement in school. When children's multiplicative reasoning is measured at school entry with appropriate techniques, it predicts their achievement better than general cognitive skills or number skills one year later.

INTRODUCTION

Different researchers have suggested that multiplicative reasoning is crucial to the development of children's mathematical thinking beyond constituting an arithmetic operation. Two major roles are attributed to early multiplicative reasoning in the development of children's mathematical thinking.

First, multiplicative reasoning is implicit in understanding place value: e.g. 52 is interpreted as "five tens and two ones". Children do not need to know multiplication facts to understand this, but they need to understand one-to-many correspondence: i.e. they need to understand that each value in the tens place corresponds to 10 units (Carraher, 1985; Kamii, 1981; Resnick, 1983; Steffe, 1994). Behr, Harel, Post, and Lesh (1994), Kaput (1985) and Steffe (1994), among others, use the term composite units to refer to units composed of several objects and suggest that this is a key idea in understanding proportions.

A second important role for multiplicative reasoning has been examined by other researchers (Confrey, 1994; Kaput & West, 1994; Vergnaud, 1983). They argue that additive and multiplicative reasoning are fundamentally different: additive reasoning is used in one-variable problems, where quantities of the same kind are put together, separated or compared, whereas multiplicative reasoning involves two variables linked by a fixed-ratio. Thus multiplicative reasoning forms the foundation for children's understanding of proportional relations and linear functions.

These two aspects of multiplicative reasoning, i.e. the formation of composite units and two variables connected by a fixed ratio, are not radically different from the

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psychological perspective. Both ideas can be attained by children through the scheme of one-to-many correspondences. When teaching place value, for example, teachers often ask children to exchange ten ones by one ten, when the ones and tens are represented by physically distinct tokens. This procedure draws on children's understanding of one-to-many correspondence: although there is only one digit in the tens column, each token in the tens place corresponds to 10 single units. In situations where the correspondences are between different measures in a fixed ratio, researchers like Kaput and West (1994) and Steffe (1994) have developed materials to make the correspondences between measures explicit through the use of computer diagrams in their proportions instruction programs. Thus primary school students' knowledge of correspondence may have a profound impact on their mathematics achievement in the classroom both in understanding place value and proportions.

We report two studies that analyse early multiplicative understanding in young children. Study 1 investigates the conditions that affect children's performance when solving multiplicative reasoning problems. Study 2 reports on the impact of individual differences in children's knowledge of correspondence at school start on their later mathematics achievement.

BACKGROUND

Previous studies of children in kindergarten and in their first two years of school report that many children are able to use one-to-many correspondence to solve multiplication and division problems. Kouba (1989) asked children in 1st, 2nd and 3rd grade to solve some multiplication and division problems. She reports the use of appropriate strategies in 43% of the problems. Among the 1st and 2nd graders, the overwhelming majority of strategies used were based on correspondences, either using direct representation or partial representation (i.e. tallies for one variable and counting or adding for the other); very few answers were obtained by recall of multiplication facts. The recall of number facts was significantly higher after the children had received instruction, when they were in third grade.

Becker (1993) asked kindergarten children, aged 4 to 5 years, to solve multiplication and division problems in which the ratios were easier than those in Kouba's study. The children were more successful with 2:1 than 3:1 correspondences, and the level of success improved with age. The overall level of correct responses by the 5-year-olds was 81%.

Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) also gave multiplicative reasoning problems to kindergarten children involving correspondences of 2:1, 3:1 and 4:1. They observed 71% correct responses to these problems.

Although the success rates in these studies leave no doubt that many young children start school with some understanding of one-to-many correspondence, Sophian (2007) suggested that success in one-to-many correspondence problems is a later achievement. However, her tasks differed from those used in the studies reviewed above. We hypothesise that a critical difference was in the symbolic representations available to the children during problem solving: in the previous work, children were able to represent both variables in the correspondence problems, but this was not the case in Sophian's study.

Two questions arise from these studies. First, what is the best way to describe young children's competence in multiplicative reasoning? Second, what is the significance of individual differences in multiplicative reasoning at school entry for mathematics achievement in school?

We report two studies that analyse young children's multiplicative reasoning. Study 1 investigates the conditions under which children are relatively more successful in solving multiplicative reasoning problems. Study 2 analyses whether individual differences between children in multiplicative reasoning problems predict school achievement in mathematics.

STUDY 1

Methods

The study was carried out in England with 81 children from two state schools in the first two months of their third year in school (mean age 7 years 2 months). The children had been taught about multiplication as the repeated addition of equal objects (dots on a number line, coins of the same value) at the end of their second year in school. They are also taught to represent multiplication as equal jumps on the number line. This teaching is limited to small number combinations and does not involve problems where there is a fixed ratio between measures. We selected this group for study because of their experience with the number line.

The children were presented with 12 multiplication, 12 partitive division and 12 quotitive division problems; 6 addition and subtraction problems were also used to vary the type of actions required. They were assessed by an experienced experimenter in three individual sessions, and solved 14 problems in each session.

Our hypothesis was that children's performance in multiplication and division problems is affected by the material resources available to them when they solve the problems. So we randomly assigned the children to one of four testing conditions defined by the materials that would be available to them during problem solving.

The children in the first three groups had manipulatives at their disposal but their access to manipulatives was varied during the sessions in order to explore how representation affected their performance. For 1/3 of the problems, they were given cut out shapes (circles, rectangles) and blocks so that they could represent the variables in the problems with different materials. For example, for the problem "There are 7 kennels in the dog home; each kennel can fit 3 dogs; how many dogs can this dog home receive?" the children could use the rectangles to represent the kennels and the blocks to represent the dogs. They could represent the problem fully and find the solution. For 1/3 of the problems, the children were only given bricks; if they were solving the dog-home problem, they could form 7 groups of 3 bricks in order to solve the problem. For 1/3 of the problems the children were given only the cut out figures; if they were solving the dog-home problem, they could use the rectangles to represent the kennels and would need to count three times as they pointed to each rectangle in order to solve the problem. The problems were divided into three blocks and the children were randomly assigned to solve a group under each of these conditions, with the restriction that each block was solved by the same number of children. This controls for problem difficulty in the analysis of differences between conditions of access to materials. The order of conditions of access to materials was varied in a Latin Square to control for order effects.

The fourth group was assigned to solve problems with a number line. The children were provided with a page on which four number lines had been drawn, displaying the values 0 to 32 at equal intervals (no answer was larger than 32). This was used for each set of four problems and then a new page was provided.

Results

Two analysis were conducted. The first was an analysis of variance in which the independent variables were access to manipulatives (3: material for both variables vs shapes vs blocks) and problem type (3: multiplication vs partitive vs quotitive division). The mean correct responses (out of 12) for the different problem types were 8.4 for multiplication, 9.8 for partitive division and 8.1 for quotitive division. The main effect of problem type was not significant and the interaction between problem type and access to materials was not significant. However, the main effect of access to materials was significant. The mean correct (out of 12) for children using representations for both variables was 10.50, for children who only had blocks was 9.60 and for children who only had the cut-out shapes was 7.25 ($F_{2,58} = 29.14$; $p<.001$). Post-hoc tests showed that the children with access to representations for both variables and those who had blocks did not differ significantly from each other but both of these groups performed significantly better than the group that had only received the cut-out shapes. An analysis of their strategies showed that the children who had blocks were able to form groups of blocks and use number in the group to represent one variable and number of groups to represent the other variable. So even though they only had one set of objects they could use this to represent both variables. In contrast, the children who had only the cut-out shapes did not have enough shapes to represent the number of objects in the group: these would have to be represented by counting gestures, requiring double counting (e.g. 1,2,3 for the number of dogs in the first kennel, 4,5,6 for the number of dogs in the second kennel etc.). Not all children discovered this strategy and among those who did discover it some counting mistakes were observed.

The second analysis contrasted children who had materials at their disposal with those who had the number line. This analysis cannot take into account problem type and type of access to materials because different children solved the same problem under different conditions of access. The mean correct (out of 36) for the children using the number line was 18.14 and for the children using manipulative was 27.35. This difference was significant according to a t-test for independent groups (t=5.40; df=80; p<.001).

Conclusions

Although the children had received minimal instruction on multiplication and none on division, they were able to solve multiplicative reasoning problems with relatively high levels of success. The best performance was observed when the children had access to materials that facilitated the representation of both variables. Performance deteriorated significantly when they had materials to the represent one variable but had to rely on gestures to represent the other.

The children found it difficult to use the number line to represent two variables. Although they could use equal jumps of 3 to represent the dogs in the dog home problem and 7 jumps to represent the kennels, this was much more difficult than establishing explicit correspondences between both variables.

We conclude that children can show a great grasp of one-to-many correspondence if tested under conditions enable acting out correspondences. A reduction in the representational resources is associated with lower levels of success.

STUDY 2

This study investigated whether children's multiplicative reasoning makes a significant contribution to the prediction of their mathematics achievement in school. The study was conducted in England; the teaching context for multiplication was described in Study 1.

Methods

A longitudinal design was used to examine whether children's knowledge of correspondences when at school start predicts their mathematics achievement about one year later.

The children (N= 52) were tested on their understanding of multiplicative reasoning in their first year in school when they were between 5 and 6 years old. They were asked to solve 18 problems: 6 involving composite units, 6 multiplication and 6 division story problems. The problems about composite units involved counting the total sum of money when coins of different values are used (for example, one 5p and four 1p coins; see Carraher & Schliemann, 1990, for a description of this task). Typically, children who do not understand composite units count the 5p coin as "one", disregarding its value when they count, even if they correctly identify the coin as 5p. The multiplication problems were of the type described by Vergnaud (1983) as isomorphism of measures (e.g., the children were shown a row of 4 houses and told: in each house in this street live 3 rabbits; how many rabbits live in this street? and given blocks to use if they wished). The division problems involved sharing; the unknown in some problems was the size of the share (partitive problems) and in others the number of shares (quotitive problems). The children's access to resources was varied across these problems: sometimes they had access to materials to represent both variables, sometimes not.

About 14 months later, the children were assessed by their teachers in mathematics achievement according to the government guidelines. This assessment is totally independent of the researchers in design and procedure.

Our hypothesis was that the children's performance in these correspondence problems would be a specific predictor of their mathematics achievement. So we needed to control for general cognitive ability and knowledge of number in the regression analyses. The measures used for the factors to be controlled were: for cognitive ability, the British Abilities Scale (BAS-II; Elliott, Smith, McCulloch, 1997) and a Working Memory Test, Counting Recall (Pickering & Gathercole, 2001); for knowledge of number, a sub-test of the BAS-II, Number Skills, entered in the analyses separately and independently of the estimate of general cognitive ability. These measures were then used in a fixed-order multiple regression analysis, which tested whether multiplicative reasoning at school start predicts their mathematics achievement one year later after controlling for each of the general predictors.

Results

This analysis produced two important results. First, the children's performance in the multiplicative reasoning task added significantly to the prediction of their mathematics achievement 14 months later even after these stringent controls. This result supports the view that knowledge of correspondences does offer a basis for learning mathematics in school.

Second, inspection of the βs, which result from model fitting and do not depend on the fixed order test, showed that the multiplicative reasoning items contributed more to mathematics achievement than any of the other factors. The only significant β value in this analysis was that for the prediction based on the multiplicative reasoning task: $\beta = .31$; t=2.50; p=.02. The BAS without the Number Skills subtest showed the same β value but it was not significant. For the Number Skills subtest, $\beta = .21$, which was also not significant.

CONCLUSIONS

Study 1 showed that children do develop some competence in multiplicative reasoning before they are taught about multiplication and division in school. Their performance in the same problems varies when the representational resources at their disposal vary: high levels of performance can be observed when the representational resources available enable the implementation of the scheme of one-to-many correspondence. Study 2 showed that individual differences in this measure are important predictors of children's mathematics achievement. The measure makes a significant contribution to the prediction of children's achievement after controlling for general cognitive factors and children's number skills at school entry; its impact is larger than that of general cognitive factors and number skills.

The educational implications of these findings are quite clear. Presently in schools (at least in England) children are taught about multiplication rather late and using repeated addition as the starting point. Children's ability to solve multiplicative reasoning problems is largely ignored. This means that their potential for learning about multiplicative thinking could be used more effectively in school.

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A FRAMEWORK FOR UNDERSTANDING THE STATUS OF EXAMPLES IN ESTABLISHING THE VALIDITY OF MATHEMATICAL STATEMENTS

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We offer a mathematical framework for dealing with the question "What does it mean to understand the status of examples in determining the validity of mathematical statements?" Our framework presents the interplay between four types of examples (confirming, non-confirming, contradicting, and irrelevant) with respect to the validity of two types of mathematical statements (universal and existential). This framework provides a basis for constructing tasks that assess and support students' understanding of the logical connections between examples and statements. We illustrate the strength of this framework by examining one student's evolving understanding of the status of examples in proving or refuting a mathematical claim, in the course of dealing with one of the tasks that were designed in accordance with the framework.

BACKGROUND

A tension between empirical and formal aspects of mathematics has been broadly recognised as one of the major sources of students' difficulties with proving (Epp, 2003; Fischbein, 1987). This tension can be seen in students' tendency to rely on specific examples as sufficient for determining that a general claim is true. Harel & Sowder (1998) term this type of reasoning *empirical proof scheme*, while Rissland (1991) and Zaslavsky & Shir (2005) refer to it as *example based reasoning*. At the same time, there is evidence that students have difficulties with counterexamples, thus often regard them as exceptions, insufficient to refute a (false) claim (Reid, 2002). This tension suggests that understanding of the logical relations between examples and statements is a non-trivial accomplishment, which is critical for proving. However, this kind of understanding is not usually addressed explicitly in school curriculum.

Our study stems from this unfortunate state of affairs and focuses on the question "What does it mean to understand the status of examples in determining the validity of mathematical statements, and how may this understanding be captured?" In order to answer this question be began with a mathematical analysis that partly builds on the works of Zaslavsky & Ron (1998), Buchbinder & Zaslavsky (2007), Barkai et al (2002). In particular, similar to the work of Barkai and her colleagues, we examine two types of mathematical statements (universal and existential). While they deal

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mainly with methods of proving, we specifically examine the status of examples in proving and disproving. We offer a general framework that can be used to guide investigation of this kind of mathematical understanding.

MATHEMATICAL FRAMEWORK

Generally, mathematical statements can be classified into two main categories: *universal* and *existential*, according to the type of quantifier that appears (sometimes implicitly) in the statement. A universal statement asserts that a proposition *P(x)* is true for all values of the variable *x* in a particular domain *D* ($\forall x \in D, P(x)$); while an existential statement asserts that there is an element in a domain *D* for which a proposition *P(x)* is true ($\exists x \in D, P(x)$).

We propose a framework and illustrate it through the domain *D* - *the set of all parallelograms*, and the proposition *P(x)* - indicating the property of *having two diagonals of equal length*. Thus, the corresponding universal statement is: *"All parallelograms have two diagonals of equal length"* while the corresponding existential statement is: *"There exists a parallelogram which has two diagonals of equal length"*. Depending on whether *x* is an element of the above domain *D* (that is, whether it is a parallelogram or not), and whether the proposition *P(x)* holds for it (that is, whether it has two diagonals of equal lengths or not), it can be classified as one of four types of examples (Figure 1).

Thus, we illustrate the first type of example by a rectangle, which is a parallelogram with equal-length diagonals ($x \in D, P(x)$); we term it a *confirming example*, for both the universal and existential statements. However, its status with respect to determining the truth value of these two types of statements is different: while, this example is sufficient for proving the existential statement, it is insufficient for proving the universal one.

The second type of example is an element of *D*, which does not satisfy the property *P(x)* ($x \in D, \neg P(x)$); for example, a (non-rectangle) rhombus, which is a parallelogram with unequal diagonals. With respect to the (false) universal statement it constitutes a counterexample that *contradicts* it, thus is sufficient for disproving it. However, with respect to the corresponding existential statement, this example violates the property *P*, does not confirm it, and yet does not contradict it either. Thus, the above rhombus is *non-applicable* for proving, and insufficient for disproving the existential statement. We term such example a *non-confirming example* for an existential statement, opposed to contradicting its corresponding universal statement.

The other two types of examples can be viewed as members of the same category since they both represent elements that do not belong to the domain *D*. As such, they are *irrelevant* for evaluating the validity of both universal and existential statements. Nonetheless, there is an important distinction between these two types of irrelevant examples. An isosceles trapezoid is not a parallelogram[1] but it has two diagonals of equal length ($x \notin D, P(x)$), while a right-angle trapezoid, which in addition to not being a parallelogram has diagonals of unequal length ($x \notin D, \neg P(x)$), is another type of irrelevant example.

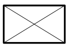
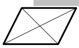
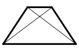
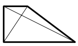
The Domain: The Proposition:	*D*: The set of all parallelograms *P(x)*: Equal-length diagonals			
Type of Statement	Universal statement $\forall x \in D, P(x)$ (All parallelograms have equal-length diagonals)		Existential Statement $\exists x \in D, P(x)$ (There exists a parallelogram with equal-length diagonals)	
Goal Type of Example	To prove	To disprove	To prove	To disprove
Confirming $x \in D, P(x)$  (a rectangle)	**Insufficient**	**Non applicable**	**Sufficient**	**Non applicable**
Contradicting (the universal statement) **Non-confirming** (the existential statement.) $x \in D, \neg P(x)$  (a rhombus)	**Non applicable**	**Sufficient**	**Non applicable**	**Insufficient**
Irrelevant $x \notin D, P(x)$  (an isosceles trapezoid)	**Non applicable**	**Non applicable**	**Non applicable**	**Non applicable**
Irrelevant $x \notin D, \neg P(x)$  (a right-angled trapezoid)	**Non applicable**	**Non applicable**	**Non applicable**	**Non applicable**

Figure 1: A framework for examining the status of examples in determining the validity of mathematical statements (illustrated with respect to a specific proposition)

[1] According to the definition accepted in many countries, a trapezoid has exactly two parallel sides.

As suggested by their name, irrelevant examples are non-applicable for neither proving nor disproving any type of mathematical statement. In spite of their logical resemblance, the two types of irrelevant examples may affect students' decision process quite differently.

We consider "understanding of the status of examples in determining the validity of mathematical statements" as becoming fluent with the logical inferences that can and cannot be made based on the different types of examples with respect to the validity of the two kinds of statements that appear in Figure 1.

THE STUDY

Goals

We report on one part of a larger study, the goals of which were:

1. To learn about students' existing conceptions associated with the roles and status of examples in determining the validity of mathematical statements;
2. To examine students' tendency to generate and use examples in the context of determining the validity of mathematical statements;
3. To trace the development of students' understanding of the roles and status of examples in proving and disproving.

Data Collection and Analysis

The data for this paper was collected in the course of an individual interview with Debby, a 10^{th} grade, average level student, while she was dealing with one of the tasks, specially designed for the study (see below). No time constrains were set. She was also allowed to go back and check her previous answers. The interview (conducted by the first author) lasted for one hour. It was audio taped and transcribed.

Following our framework (Figure 1), the student's work was analysed in terms of the manifestations of her understanding of the status of examples in determining the validity of mathematical statements. In particular, four types of cognitive activity in which she was engaged were examined: example generation, example recognition (i.e., in terms of Figure 1), logical inference, justification and explanation supporting her inferences.

Instruments

The research instruments consist of a collection of tasks that were constructed according to our above framework. One of the tasks includes three false universal and three true existential statements, which deal with basic arithmetic and algebraic facts taken from the content of school mathematics (Figure 2). The task is to determine, for each statement, whether it is true or false, and to explain why. For most statements, several confirming, non-confirming and counterexamples can be found.

For each of the following statements determine whether it is true or false. Explain (prove) your answer.

1. If n is an even number, then $a^n > 0$ for any integer a.
2. If $(a+b)^2$ is an even number, then both a and b are even.
3. For any prime number, if we change the order of its digits, the resulting number is also a prime.
4. There exist two numbers a and b the sum of which is greater than one of them and smaller than the other.
5. There exists a two-digit integer for which the sum of its digits is equal to their product.
6. There exists a natural number n, the square of which is smaller than the number n itself.

Figure 2: The task used for the pilot study.

FINDINGS

Debby began with the first three (universal) statements and used both confirming and contradicting examples in her attempts to evaluate their validity. She was unpleasantly surprised by the fact that she had found these two types of examples that seemed to her that could not co-exist (she explicitly stated this later during the interview), nonetheless, she correctly concluded that these three statements were false. Debby claimed that she "…found an example that the statement is true and also an example that it is false…. Conclusion - the statement is false…it's not true in all cases".

Using our framework at this point to asses Debby's understanding of the logical connections between examples and statements, we could say that Debby was able to *construct* several relevant examples and *recognise* their type. She *inferred* and correctly *explained* that confirming examples were insufficient for proving the statements, but a single contradicting example was sufficient to disprove them. Inconsistencies in Debby's understanding only became apparent when she tried to apply the same techniques to evaluate existential statements.

For statement #4 (Figure 2) Debby found two non-confirming examples and one confirming example: $-1+1=0$. She concluded: "Here the statement is true. 0 is smaller than 1 and greater than (-1). This means that the statement can be true and [at the same time] can be false." When pressured by the interviewer to choose a single answer: "true" or "false", after an initial state of uncertainty, Debby decided that the statement is true: "It's true. […] I can't choose "false" if I give you a proof that it's true".

From Debby's correct answers to all the items in the task (up to this point), it seemed that she was fully aware of the distinction between universal and existential statements. However, her remark about the statement being "both true and false" stood in contrast with her actions, so we decided to ask her to further explain.

Int: What is the difference between statements #3 & #4?

Debby: The difference is ... that here it's true and here it's false. Here [# 3] we provided a proof that it is false and here [# 4] I gave you a proof that it is true.

This excerpt shows that Debby was either unaware that one statement is universal and the other is existential or that she just couldn't express it. Her response to statement #5 revealed that she was indeed unaware of the difference between universal and existential statements and of the status of confirming examples in determining the validity of the latter.

For statement #5 (Figure 2) Debby had found two non-confirming examples, and was just about to infer that there is no two-digit number for which the sum of its digits is equal to their product, when she discovered a confirming example.

Debby: I think no such number exists. Ah, sorry, it does. 22. The sum of the digits is 4, their product is also 4. So such statement exists. It means that the statement is….both true and false! So?! [...] I would choose "false". I can't determine that it's true when it's not true for all cases.

Note, that for both existential statements (#4 & #5) Debby had found two non-confirming and one confirming example, but inferred that statement #4 is true and statement #5 is false, because "it's not true for all cases". This inconsistency further indicated that Debby did not distinguish between universal and existential statements. In addition, she stated (repeatedly) that a statement can be true and false at the same time. However, Debby found this "conclusion" problematic, as indicated in the next excerpt:

"In all my previous answers it could be both true and false. And also here: it doesn't matter which answer I select! That's the problem. [...] For any answer I choose, even if it is true or false, I've *proved* that it is both true and false. So, any answer will be incorrect. What I could write is, that the statement is true but not for all cases. […] I see that it keeps coming up, so I think that something is wrong here. …I have to say this."

In order to help Debby resolve the uncertainty she was experiencing, we asked her to reflect again on her responses to statements #3, #4 & #5 and to try to reach consistency among them. After careful examination Debby replied:

Debby: Here [#3] they are talking about all numbers, and here [#4 & #5] they don't refer to all numbers… Here [#3] it can be both true and false, but they say "all numbers", so it has to be only one correct answer.

Int. : What do you mean?

Debby: A single answer. If I choose "false" it has to be "false". […] For these two [#4 & #5] they say that there are numbers that satisfy these statements. […] but they don't say that this is so in all cases.

Int. So is it true or false?

Debby: Both! Just like I said before. […] They say that there exist such numbers, but not all of them.

Int.: Have "they" asked for that?

Debby went back to compare her answers once again. After a while she said:

> I told you before that here [#4] it can be both true and false? I was wrong. It's only true.[...] I didn't understand the statement properly. I was sure that they are talking about all numbers, as I wrote here [#3]. Now I'm sure that #4 is true, and this answer [false] is out of the question. Beforehand, I thought that they mean all numbers, so I said that it can be both true and false. I understand now that they only say that such numbers exist. Not all of them, but there are such.

This was a moment of insight for Debby. After that she was able to determine and correctly justify that statement #5 is also true, and to complete the rest of the task.

We apply our framework to analyse Debby's initial and growing understanding of the logical connections between examples and statements in course of dealing with the task. All the examples that Debby *constructed* were relevant, and she was able to *recognize* their type (without using our terms): confirming, non-confirming or contradicting. For universal statements Debby was able to correctly *infer* their validity and logically *justify* her conclusion. The gaps in her understanding surfaced when she tried to use a rule of thumb "a theorem must be true in all cases" with existential statements (#4 & #5). Despite her ability to *construct* and *recognise* correctly the type of examples with respect to the statements, Debby had trouble to *infer* their truth value correctly or consistently (i.e., statements #4 & #5). She was especially confused by the existence of both confirming and non-confirming examples.

During the interview, we tried to minimize our intervention, allowing Debby to reason freely and reach the conclusions by herself. At the same time we kept insisting that she commits to a single response for each statement (true or false), and that she responds consistently. It seems that these two demands contributed to the development of Debby's understanding, by allowing her to become aware of the distinction between universal and existential statements, as well of the status of different types of examples in determining their validity. As she stated explicitly, this awareness helped her realise that a statement can be either true or false, but not both. Another indication of the development in Debby's understanding can be seen in her increasing ability to correctly *justify* her responses to the task.

CONCLUDING REMARKS

In this article we presented a framework for dealing with the status of examples in determining the validity of mathematical statements. As shown above, this framework proved useful in constructing tasks that elicit the logical connections between examples and statements, as well as in assessing this kind of understanding.

The specific features of the task, which included both universal and existential statements that have confirming and non-confirming or contradicting examples, seemed to contribute to the development of Debby's understanding. These features created for her a sense of uncertainty and doubt, which have been broadly recognized as a powerful vehicle for learning mathematics (Hadas, Hershkowitz, & Schwarz, 2000; Zaslavsky, 2005). Debby's resolution of the state of uncertainty resulted in her better understanding of logical issues associated with the status of examples in determining the validity of mathematical statements.

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SLOPE AS A PROCEDURE: THE IMPACT OF AXIS SCALE

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This paper examines an episode in which a group of students represented a linear function using more than one graph. The episode shows the students' ideas about slope as a procedure to be enacted versus the understanding of slope as a property of the function. This is illustrated through a disagreement about the value of the slope for a particular function. Those students using only a procedural understanding of slope faced challenges when confronted with a different graph of the same function on which the scale of the coordinate axes had been changed. This highlights how the procedural understanding of slope places great importance on choice of scale, and shows the value in examining graphs of the same function at different scales.

INTRODUCTION

In this paper we provide an example of Grade 7 students discussing the meaning of slope in the course of graphing and discussing a linear function that was presented in an extra-mathematical context. The "Stairs and Floors" problem was stated to the students as follows:

> Suppose that each flight of stairs in a building has 20 steps. Show how many steps you have to climb depending on what floor you live on.

The case described here shows that: (a) students' procedural understandings of slope are insufficient to determine an accurate interpretation of slope for the Stairs and Floors problem; (b) it is possible for a particular graphical representation to validate incorrect procedures, and (c) the outcome of instruction is strongly connected to the seemingly minor choice of scale of axes.

Consistent with research on the development of algebraic reasoning (e.g., Carraher, Schliemann & Schwartz, 2007; Schliemann et al., 2003), the Stairs and Floors problem does not specify a particular number of floors. Such a problem design enables consideration of multiple values of x and the corresponding values of y. Those proficient in mathematics may discern that since there are 20 additional steps for each additional floor, the linear function may be expressed as $y=20x$ and the slope that describes the pattern of change between one variable and another is 20. As our examples will illustrate, students' applications of previously learned procedures result in a variety of values for slope.

Note that in this episode and this paper, the discussion corresponds to a real-life situation in which the ground floor is considered simply to be the ground floor, and the first floor is up one flight of stairs. As a result, we arrive at the function $y=20x$.

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Slope is a property of functions that students often learn to calculate through a process or procedure, without necessarily having a reason to consider the underlying mathematical meaning. This treatment of finding slope for linear functions is commonly connected with: (a) examination of a formula; for example, identifying the slope as m in $y=mx+b$, (b) executing a calculation given two points, as in the use of $m=(y_1-y_2)/(x_1-x_2)$, or (c) determining slope from a graph by finding "rise over run", the increase in the vertical direction over the increase in the horizontal direction. As such, the meaning of slope as a description of change between two variables remains unexplored. To those proficient in mathematics, this distinction between slope *as a procedure* and slope *as a property* of a functional relationship may seem trivial; however, to learners of algebra, we cannot assume that this is obvious. By remaining focused on procedure rather than property, students may lack the tools to connect slope to real-world and scientific phenomena. Following existing definitions of algebraic reasoning (e.g., Arcavi, 1994; Schoenfeld, 2007), such a procedural application of slope is insufficient. We present this case in order to consider the role of multiple graphs, with differently scaled axes, of a single linear function in understanding slope as a property of the functional relationship.

Other work has explicitly considered the use of physical representations of slope as a tool to foster an understanding of slope as a property (e.g. Nemirovsky and Noble, 1997). Issues of axis scale in graphical representation have been illuminated previously by Zaslavsky, Sela, and Leron (2002). Their task-based study used two different graphs of the same linear function; in one graph, the scale on both axes was the same, whereas in the second, the scale is changed on the y-axis only. They refer to this use of different scales as creating a "non-homogeneous coordinate system" (p. 121), and this lack of homogeneity will be seen in the graphical representations described in this paper. Zaslavsky et al. (2002) did not study algebra learners, so the majority of their participants deemed the slope to be the same in both graphs. However, most of them did display some conflict when discussing the graphs and when considering the geometric properties of these. Interestingly, the motivation for Zaslavsky's work was concern about the prevalence of mathematical software that allows easy change of scale. As we see in the example presented in this paper, though, the choice of hand drawn graphs at different scales has as much potential for creating conflicting ideas about slope.

METHOD

The problem we focus on here was presented on day two of a seven-day algebra summer camp program. The program sought to follow up on previous work implementing algebra lessons in elementary school, and a subset of the camp participants had received this early algebra instruction. The research team designed and taught all lessons. All mathematical sessions were videotaped and student work collected. At the program, students worked on mathematics for three hours each morning and then participated in recreational activities during the afternoon.

Participants did not have to have high grades or test scores in mathematics or meet any other performance-based criteria in order to attend.

Participants had just completed the school year in either 5th grade, 6th grade, or 7th grade, so were generally between 10 and 14 years of age. The students were divided into three groups for the mathematics sessions, based on their grade levels. Group 1 was composed of 12 students who had completed 7th grade, the highest grade level at the camp; this group is the focus of this paper. Many of the younger students had not been exposed to graphs and slope in school. As a result, their work on this problem did not touch on the issues connected to axis scale and slope, which are the focus here. In Group 1, all students lived in or around Boston, Massachusetts, USA, and 10 out of 12 students attended school in the Boston public school system. Of the two remaining students, one was home schooled and one attended parochial school.

The mathematics sessions during the camp included some time for individual student reflection, but focused primarily on group activities and whole-class discussions. Instruction attempted to avoid categorizing student ideas as right or wrong, but instead to attempt to understand the meaning of the responses and use them as a window into student thought. The authors of this paper were instructors in Group 1 and Group 2. On day two, when students arrived, each individual had the Stairs and Floors problem taped into their notebooks. They began by working independently or discussing with the other students at their tables. The problem then served as a platform for whole-class discussion.

RESULTS

The Group 1 students began by volunteering specific values for floors and the corresponding values for number of stairs, while the instructor recorded them in a table on the board. The instructor quickly moved to graphing by drawing the coordinate axes on a large piece of chart paper with pre-marked grid lines. Having volunteered to create the graph, Kara, a student in Group 1, began to label the y-axis in which she skipped two ‘blocks’, using the grid lines, for every value of 20, so that each block corresponded to increments of 10 but only multiples of 20 were labelled (see leftmost graph in Figure 1).

The word slope first arose when another student, Olivia, came up to plot the points (1, 20), (5, 100), and (10, 200). She plotted the first two points without incident, but the x-axis did not extend to the point for x=10. She did plot a third point, but one that corresponded with x=8. The instructor asked her what she had done:

Instructor (Darrell): What points did you plot?

Olivia: 1, 20. 5, 100. And 8, um. 8, something, I don’t know.

Instructor: How did you know where to put it then?

Olivia: Because I just did the slope and followed the graph.

Olivia demonstrated how, starting at the point (1, 20), she counted 4 blocks to the right and 8 blocks up to get to the point for (5, 100). She then used this counting over 4 and counting up 8 to reason about the value for slope.

Olivia: So, that means that the slope is, I think 1/2 or 2. I don't know. Wait, 8 over 4. So it's 2.

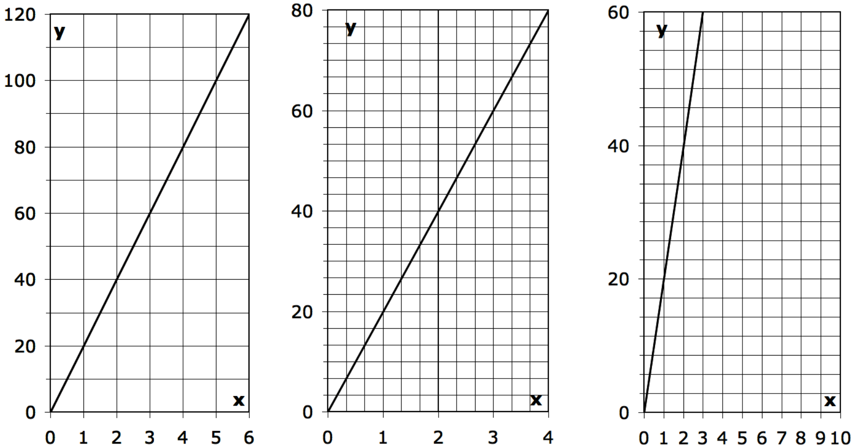

Figure 1: Graphs of *y=20x* used in the discussion

Bianca then questioned Olivia's assertion that the slope was 2, still using the procedure of 'counting over' and 'counting up' to assert that the slope was 1/2. Thus far, for both of these students, slope was treated as a procedure (or even a trick) used to extend points of a linear function without having to find the precise ordered pair. Olivia supported her claim that the slope was 2 by referring to a previously learned formula that places change of y over the change in x.

Bianca: I think it's 1/2 because didn't she say you go across 4 and then 8 up? Wouldn't that be 1/2 instead of 2 if you simplified it?

Olivia: Delta y over delta x is the change in y over the change in x. The x is the across and the y is the up and down.

At this point, the instructor attempted to refine the concept of slope by using the original problem. He first restated Olivia's claim and then asked about the 20 stairs.

Instructor: So you go over 1 and count up 2 blocks… What about the 20?

Olivia: It doesn't really matter. Right?

Other students agreed that the 20 stairs for each floor did not pertain to slope. However, the question prompted Tariq, and later Hannah and Kara, to reconsider both the slope of the function and the role of the grid lines on the chart paper.

Tariq: 20 is actually the slope. Because when you move up 2, the 20, it's actually a value of 20 because it's increasing by 20 all the time. So as you increase by 2, it's actually 20.

Unconvinced, Olivia re-stated her understanding of how to find the slope of a line.

Olivia: In my math class, my teacher said that when you go up, that's the slope. Like, change in y over change in x. So you just basically move it. Right one, up two to get to the same point. … I think the slope is 2.

The instructor created a new graphical representation that changed the scale of both x and y axes (shown in the middle of Figure 1). He marked integers on the x axis every third block, while the y axis showed each multiple of 20 every 6 blocks. Without realizing it, he had created a graph that still adhered to the 'over 1, up 2' model since both scales were increased by a factor of 3.

Instructor: What is different about these two graphs? The reason I'm asking is because we have a debate right now. And I think Olivia has a really good argument and Tariq has a really good argument.

Bianca: You know how Olivia did over 1 up 2 [*on the first graph*]? You can still count over 1 and up 2, and then over 1 and up 2 [*on the second graph*]. It's still the same thing, right? I don't think that really matters, what you use for the floor and the stairs.

Despite the fact that Olivia's procedural application worked on the new graph, Hannah reflected on the similarities and differences between the two graphs.

Hannah: I think Tariq is right, that [*the slope*] can't change. … So then the slope is 20 because no matter what the number is [*on the axes*] it's always going to be 20 stairs. So if that's constant then that should be the slope.

In an effort to challenge the idea that the slope might be 2, the instructor created one more graph in which 7 grid blocks corresponded to an increment of 20 on the y-axis, but one grid block corresponded to an increment of 1 on the x-axis (seen as the rightmost graph in Figure 1). In this case, it was not possible to count 'over 1 and up 2' and still remain on the line that had been graphed for the function. Upon seeing this new graph of the same problem, Bianca became convinced that the slope was not 2, but rather 20.

Bianca: Now I think it's 20. Because if you do 1 by 2 you won't get what you just got.

Bianca's last statement drew upon Hannah's statement that the slope of the same problem could not change despite the scale of the graphical representation. Olivia, however, was not convinced, and seemed to be grounding her explanation in the first two graphs without considering the third. She again explained that what she called 'slope' referred to counting over and counting up, and not what she perceived that the other students were referring to, which was one point on the line, (1, 20). As a counterexample, she said that the line might cross the point (2, 5), but this did not mean that the slope was suddenly 5 over 2.

Olivia: I said that with slope you're not supposed to go up the line with slope. Just because it's 1, 20, doesn't mean that's the slope. The slope can be 2, 5, and that doesn't mean that it's not the slope. … The slope is never on the line. The slope is not a point.

Bianca, who agreed with Olivia earlier, now challenged her. She based her claims n the third graphical representation.

Bianca: You can't go 1, 2 [*referring to over 1, up 2*] because it wouldn't be on the line!

More students supported their claims through the use of the available graphical representations. Hannah tried to convince Olivia by applying Olivia's procedure to the third graph which, since it showed the same function, must have the same slope.

Hannah: Can I show Olivia something? If the slope was 2 [*on the first graph*], as you say, then, okay, wait. That's just for this graph, with different increments. If that's the case [*on the first graph*], then [the slope] is 7 over 1 over here [*on the third graph*]. And 7 doesn't equal 2. He's making a point that the increments don't matter.

While by the end of class Olivia remained unconvinced, several other students not only believed that the slope was 20, but grounded their explanations in the information across the three graphs. In an eloquent final point, Kara summarized why the slope was 20 while at the same time providing the greater mathematical purpose for talking about slope:

Kara: I think the slope is 20 because really and truly we're talking about floors and stairs. And '2' has nothing to do with it because it's just a way to show the data on the graph.

DISCUSSION

In this episode, we can see that some students understood slope to be a procedure to be carried out on a graph. We also see that executing purely procedural understanding can lead to incorrect conclusions about slope, particularly in a case when the scale differs between axes. This tension between procedure and property led to spirited debate over whether the slope was 2 or 20. In applying their existing knowledge of slope, students like Olivia and Bianca made reasonable interpretations of the features of the initial graph to support their assertions that the slope was 2. To help resolve the disagreement, the instructor created a different graph of the same function as a way to provoke students to look at the slope anew. Through these extensions, changing the scale of the x and y axes, students began to examine their ideas about slope. However, this depended on a prudent choice of scale by the instructor. When the second graph was introduced in this example, students were still able to execute their procedure of moving over one block and up two blocks and remain on the line. It was only when the third graph was introduced that this procedure failed. One of the students, Hannah, then drew upon the need for consistency of the value of slope across the three graphs, and tried to disprove Olivia's procedurally based claim by applying it to more than one graph. Since this would yield a slope of 2 on the first graph and a slope of 7 on the third graph, this caused the procedure to fail the implied test of returning the same slope for the same function.

This shows the importance of instructional choices, as well as the luxury of time to work with more than one graph, something that is not always available during the school year. As a result of the extended discussion, students moved from grounding their assertions of slope in one particular graph, independent of floors and stairs, to grounding their assertions in the rates of change as related to the function.

While the younger students in the camp did not progress to such in depth discussions of slope and scale, we still can see the treatment of slope as a procedure in Group 2. When the instructor introduced the term “slope,”, new to many, students offered several meanings drawn from extra-mathematical situations, including referencing a ski slope and mentioning steepness. Then one student, Kayla, suggested that slope had something to do with a triangle, and a second student agreed. Kayla came to the board and tried to explain what slope was by referring to the process of drawing right triangles under the graphed linear function, using the line itself as the hypotenuse. She was executing the same process of moving “over” along the grid lines and then “up” along the grid lines that was seen in Group 1, with the difference being that she wanted to mark on paper the legs of the triangle formed by this path. In this excerpt, she told the instructor how to draw the triangles.

Kayla: You have the line, and you put a triangle under it. […] *(pointing)* You go right there, and right there, and right there, and right there.

Instructor (Mary): So I start here? And where do I go? Across like this?

Kayla: Yeah. Then you go up.

Instructor: Okay, now what’s that, tell me?

Kayla: You keep doing it.

Instructor: You keep doing it? Like this? *(making additional triangles)*

Kayla: Yeah. That’s how my teacher told me to do it.

The triangle method that Kayla explained suggests an understanding of slope as a procedure, rather than a property of the linear function, just as we saw in Group 1. When using this procedure, the expressed conception of slope depends on the specific graph, and therefore on the scale of the axes, with no reference to the numeric intervals that are actually being drawn as legs of the triangle. We may conjecture that this would lead to conflicts similar to those we saw in the episode from Group 1 and that this conflict was not unique to that group of students.

Note also that the seemingly innocuous and helpful pre-marked grid lines were interpreted by the students as relevant information independent of the labelled values on the x and y axes. This is not to suggest grid lines be banished from algebra class forever, but rather that when used as assistive devices in creating and reading graphs, we should be certain to address the possibilities of different scale values for the horizontal and vertical intervals on the grid.

Students brought prior understandings both of slope as a procedure and of slope as a property of the function. The example provides a window into the role of small choices, such as scale on an axis, in classroom discussion and student thought. This corroborates the idea that instructional decisions have great bearing on how students interpret and continuously refine their understanding of mathematical concepts. In the words of Stevens and Hall (1998), instructional decisions 'discipline' the perception of learners to, in this case, evaluate the meaning of slope independent of a graph. While procedural understandings of slope allowed students to reach incorrect conclusions about the Stairs and Floors problem, and these were supported by the students' examination of both the first and second graphs, careful choice of the scale on a third graph provided structure for students to debate the validity of the procedural approach as well as the value, and meaning, of slope.

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CURRICULAR IMPACT ON THE DEVELOPMENT OF ALGEBRAIC THINKING: A LONGITUDINAL STUDY

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We examined longitudinally the impact of using two different types of mathematics curricula on the development of students' algebraic thinking over three years (grades 6-8). Students who used a Standards-based curriculum like Connected Mathematics Program (CMP) showed significantly higher growth rates than Non-CMP students on both Representing Situations tasks and Making Generalization task. On the Equation Solving tasks, the growth rates for CMP and Non-CMP students were similar. These findings not only showed the positive impact of the use of CMP curriculum, but also showed that the development of students' higher-order thinking skills does not necessarily come at the expense of the development of basic mathematical skills when using Standards-based curricula like CMP.

Given the gatekeeper role of algebra as well as growing concern about students' inadequate understanding and preparation in algebra, algebra curricula and instruction have become focal points of mathematics education research (Bednarz et al. 1996; Carpenter et al., 2003; Katz, 2007). In particular, researchers have tried to understand the nature of students' algebraic thinking and how they react to differing types of curriculum and instruction (Katz, 2007; Kieran, 2007; NRC, 2001). The purposes of this study are to examine how middle school students' (grades 6-8) algebraic thinking develops over the three years, and how instruction using two different types of curricula influences the development of their algebraic thinking.

BACKGROUND AND THEORETICAL CONSIDERATIONS

Recent reviews of research on the teaching and learning of algebra clearly show the need to study the development of students' algebraic thinking over time (Jones et al., 2002; Kieran, 2007). It is widely accepted that algebraic thinking should be developed across all grade levels (NCTM, 2000), but we know very little about how we should characterize the growth of students' algebraic thinking. For example, in their review, Jones et al. (2002) explicitly called for future research to build a general model that characterizes the growth of students' algebraic thinking over time. In this paper, we attempt to empirically examine how middle school students develop their algebraic thinking over time.

Classroom instruction is considered a central component for understanding the development and the organization of students' thinking and learning (Rogoff & Chavajay, 1995). Bruner (1990/1998) proposed that it is culture and education, not

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 241-248. Thessaloniki, Greece: PME.

biology, that shapes human life and the human mind. Similarly, Gardner (1991) argued that once a child reaches age six or seven, the influence of culture and schooling "has become so pervasive that one has difficulty envisioning what development could be like in the absence of such cultural supports and constraints" (p. 195). Because of the central role of instruction in the development of students' thinking, we situated our investigation of the development of middle school students' algebraic thinking in the context of instruction using two different types of curricula.

In the late 1980s and early 1990s, the National Council of Teachers of Mathematics (NCTM) published its *Standards* documents, which provided recommendations for reforming and improving K-12 school mathematics. In the *Standards* and related documents, the discussions of goals for mathematics education emphasize the importance of thinking, understanding, reasoning, and problem solving, with an emphasis on connections, applications, and communication (e.g., NCTM, 1989, 2000). This view stands in contrast to a more conventional view of the goals for mathematics education, which emphasizes the memorization and recitation of decontextualized facts, rules, and procedures, with the subsequent application of well-rehearsed procedures to solve routine problems.

With extensive support from the National Science Foundation, a number of *Standards*-based school mathematics curricula were developed in the United States and implemented to align with the recommendations in the *Standards* (see Senk and Thompson, 2003 for details). The Connected Mathematics Program (CMP) is one of the *Standards*-based school mathematics curricula developed with the support of the U.S. National Science Foundation. The CMP curriculum is a complete middle-school mathematics program. The intent of CMP is to build students' understanding in the four mathematical strands of number and operation, geometry and measurement, data analysis and probability, and algebra through explorations of real-world situations and problems (Lappan et al., 2002). Because NSF-funded curricula like CMP claim to have different learning goals and also look very different from commercially developed mathematics curricula, it provides an interesting context from which to examine the impact of curriculum on the development of students' algebraic thinking.

Middle school algebra lays the foundation for the acquisition of tools for representing and analysing quantitative relationships, for solving problems, and for stating and proving generalizations (Bednarz et al., 1996; Carpenter, et al., 2003; Nathan, & Kim, 2007). While there are many important aspects of algebraic thinking, in this paper we focused on the following three important features: representing situations, solving equations, and making generalizations.

METHODS

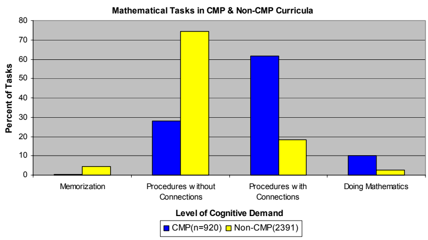
Sample

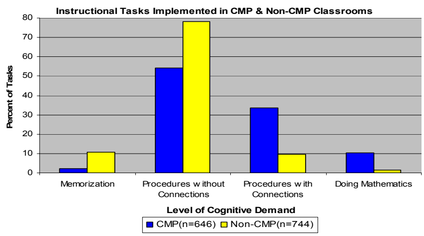
The study reported in this paper is part of a large project, Longitudinal Investigation of the Effect of Curriculum on Algebra Learning (LieCal Project). The LieCal

Project is designed to longitudinally compare the effects of the Connected Mathematics Program (CMP) to the effects of more traditional middle school curricula (hereafter called non-CMP curricula) on students' learning of algebra. The first three years of the LieCal Project were conducted in 16 middle schools of an urban school district serving a diverse student population. In the school district, 27 of the 51 middle schools had adopted the CMP curriculum while the remaining 24 middle schools used other curricula. Eight CMP schools were randomly selected from the 27 schools, which had adopted the CMP curriculum. After the eight CMP schools were selected, eight non-CMP schools were chosen based on the comparable ethnicity, family incomes, accessibility of resources, and state and district test results.

Curriculum and Instruction

We thoroughly analysed the CMP and Non-CMP curricula, as well as the instruction they received. The analyses showed that CMP and non-CMP students had very different mathematical experiences. The extent of the differences is illustrated in the two graphs below. Using a scheme developed by Stein et al. (1996), we classified the mathematics tasks in the CMP curriculum and one of the non-CMP curricula into four increasingly demanding categories of cognition: memorization, procedures without connections, procedures with connections, and doing mathematics. As the left graph illustrates, significantly more tasks in the CMP curriculum than in the non-CMP curriculum are higher level tasks (procedures with connections and doing mathematics) ($p < 0.01$). Accordingly, our analysis of classroom instruction showed that significantly many more high-level tasks were *implemented* in CMP classrooms than in non-CMP classrooms ($p < 0.01$) (see the right graph below).





Tasks and Data Analysis

Six representing situation tasks, six solving equation tasks, and one making generalization tasks were used (see appendix A for sample tasks). These tasks were administered to the selected middle school students for four times (fall 2005, spring 2006, spring 2007, and spring 2008), along with other 22 tasks in the LieCal Project. The six representing situation tasks and the six solving equation tasks are in multiple-choice format. From one testing administration to another, only two of the six (representing situation tasks or solving equation tasks) were identical, and the other four were parallel. Parallel items were determined through piloting and expert judgment. The making generalization task is in open-ended format, in the sense that students need to provide answers and explain or justify how they got them.

For the multiple-choice tasks, we used scaled scores in our analysis. A scaled score is a generic term for a mathematically transformed student raw score on an assessment. Even though we used parallel items, it was still possible that the students responded to the parallel items differently. Using scaled scores, assessment results can be placed on the same scale even though students responded to different, but parallel, tasks and at different times. In particular, we used the two identical items from representing situations to scale students' performance on the representing situations, and used the two identical items from equation solving to scale students' performance on equation solving. For the open-ended task involving making generalizations, we conducted qualitative analyses to capture the correctness of their answers and the kinds of strategies they employed.

RESULTS

Representing Situations

Table 1. Mean Scaled Scores and Standard Deviations for CMP and non-CMP Students on Tasks Involving Representing Situations

	Fall 2005	Spring 06	Spring 07	Spring 08
CMP Students	449	501	536	563
(n=312)*	*(92)*	*(94)*	*(96)*	*(91)*
non-CMP Students	461	502	544	554
(n=309)	*(90)*	*(92)*	*(94)*	*(89)*

*The results were reported based on the cohort of students who took all four assessments.

Table 1 shows the mean scaled scores on the representing situations items for both CMP and non-CMP students across the four testing administrations. Both CMP and non-CMP students showed significant growth from the fall of 2005 (6^{th} grade) to the spring of 2008 (8^{th} grade) ($F = 275.73$, $p < .001$). CMP students started lower than non-CMP students in the fall of 2005, but by the spring of 2008, CMP students performed better than the non-CMP students. CMP and non-CMP students not only exhibited different patterns of growth, but also different rates of growth. Specifically, a repeated measure ANOVA analysis indicated that CMP students showed a significantly higher growth rate than the non-CMP students on the representing situations items ($F=2.61$, $p < .05$). In particular, from the fall of 2005, CMP students increased 25%, but non-CMP students increased 20%.

SOLVING EQUATIONS

Table 2 shows the mean scores on solving equations items for both CMP and non-CMP students across the four testing administrations. Both CMP and non-CMP students showed significant growth from the fall of 2005 (6th grade) to the spring of 2008 (8th grade). CMP students started lower than non-CMP students in the fall of 2005, and the dominance of the non-CMP students continued, although not significantly so. A repeated measure ANOVA analysis indicated that there is no significant difference between the growth rates of CMP and non-CMP students on solving equations items ($F = .451$, $p = .717$).

Table 2. Mean Scaled Scores and Standard Deviations for CMP and non-CMP Students on Tasks Involving Equation Solving

	Fall 2005	Spring 06	Spring 07	Spring 08
CMP Students	459	466	505	505
(n=309)	*(80)*	*(76)*	*(93)*	*(91)*
non-CMP Students	498	504	543	543
(n=312)	*(90)*	*(92)*	*(94)*	*(89)*

MAKING GENERALIZATIONS

For convenience, we call the Making Generalization Task as MG Tasks. The MG Task includes four questions (see Appendix A), with the later questions requiring more generalization skills. To examine how CMP and non-CMP students grew over the three years in making generalizations, we focused only on the data from the fall of 2005 and the spring of 2008. This decision was based on the consideration that the incremental success rates over successive administrations of the later questions were relatively low. By focusing on the fall 2005 and spring 2008 data, it is easier for us to demonstrate the changes over the years.

Table 3. Percentages of Students Having Correct Answers for Questions A, B, and C

	Fall 2005			Spring 2008		
	QA	QB	QC	QA	QB	QC
CMP Students (n=306)	54	4	4	76	18	14
Non-CMP Students (n=322)	61	5	3	72	21	9

Success Rates for Answering Questions A, B, and C. Table 3 shows the success rates for CMP and non-CMP students on Questions A, B, and C. The success rate on Question A for the CMP students increased from 54% in the fall of 2005 (6th grade) to 76% in the spring of 2008. Non-CMP students' success rate also increased significantly (from 61% to 72%). However, the increase rate for CMP students was significantly greater than non-CMP students ($z = 3.99$, $p < .005$). For both CMP and non-CMP students, the success rates were lower on the later questions. Only about 20% of the CMP and non-CMP students got correct answers in the spring of 2008. In answering Question C, only 4% of the CMP and 3% of the non-CMP students in the fall 2005 got the correct answer. In spring 2008, the percentage increased to 14% for CMP students and 9% for non-CMP students. Even though only a small proportion of the CMP and non-CMP students were able to answer Question C correctly, the increase for CMP students was significantly greater than that for the non-CMP students ($z = 2.06$, $p < .05$).

Solution Strategies. We coded the solution strategies used to answer each of these questions into two categories: abstract and concrete. Students who chose an abstract strategy generally followed one of two paths. Some noticed that the number of guests who entered on a particular ring of the doorbell equalled two times that ring number minus one (i.e., $y = 2n - 1$, where y represents the number of guests and n represents the ring number). Others recognized that the number of guests who entered on a particular ring is the ring number plus the ring number minus one (i.e., $y = n + (n - 1)$). Using their generalized rule, these students determined the ring number at which 299 guests entered. Those who used a concrete strategy made a table or a list or noticed that each time the doorbell rang two more guests entered than on the previous ring and so added 2's sequentially to find an answer. In fall 2005, only one CMP student and none of the non-CMP students used an abstract strategy to answer Question A, but in spring 2008, nearly 9% of the CMP students and 9% of the non-CMP students used abstract strategies to answer Question A.

Table 4. Percentages of Students Using Abstract and Concrete Strategies to Answer Questions B and C of the Doorbell Problem

	Fall 2005			Spring 08		
	Concrete Strategy	Abstract Strategy	No Strategy	Concrete Strategy	Abstract Strategy	No Strategy
QUESTION B						
CMP Students (n=306)	26	1	73	13	19	68
Non-CMP Students (n=322)	21	2	77	15	19	66
QUESTION C						
CMP students (n=306)	9	0	91	10	9	81
Non-CMP Students (n=322)	10	2	88	14	5	81

Table 4 shows the percentages of CMP and non-CMP students who used concrete and abstract strategies in answering Questions B and C. In answering Question B, nearly 20% of the CMP and non-CMP students used abstract strategies in the spring of 2008. In answering Question C, even though only a small proportion of the CMP and non-CMP students used abstract strategies in the spring of 2008, but the increase rate in the number of CMP students who used abstract strategies in the fall of 2005 to the spring of 2008 was greater than that of non-CMP students ($z = 2.70$, $p < .01$). Thus, these results confirmed that both CMP and non-CMP students increased their generalization skills over the middle school years. However, the CMP students developed their generalization skills further than non-CMP students over the years.

CONCLUSION

In this paper, we examined the impact of instruction using two different types of curricula on the development of students' algebraic thinking over three years (grades 6-8). Students who used CMP curriculum showed significantly higher growth rates than Non-CMP students on both Representing Situations tasks and Making Generalization task. On the Equation Solving tasks, the growth rates for CMP and Non-CMP students were similar. These findings suggest that the use of CMP curriculum has a positive impact on students' development of algebraic thinking, as measured by Representing Situations tasks and Making Generalization task.

In *Standards*-based curricula like CMP, one important focus is on developing students' conceptual understanding and higher-order thinking skills. However many parents, teachers, and professional mathematicians worry that the Standards-based curricula's attention to the development of students' higher-order thinking skills will come at the expense of the development of basic mathematical skills. The findings presented in this paper showed that the development of students' higher-order thinking skills does not necessarily come at the expense of the development of basic mathematical skills when using *Standards*-based curricula like CMP.

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Appendix A. Sample Tasks

Representing Situations

Which number sentence is correct?

One pound of shrimp costs $3.50 more than one pound of fish.

a. shrimp's cost per pound = fish's cost per pound + $3.50
b. shrimp's cost per pound + $3.50 = fish's cost per pound
c. shrimp's cost per pound + fish's cost per pound = $3.50
d. shrimp's cost per pound = fish's cost per pound – $3.50

Solving Equations

Find the value of x so that $x-5=5$

(a). 0 (b). 1 (c). 10 (d). 25

Making Generalizations

Sally is having a party.

The first time the doorbell rings, 1 guest enters.

The second time the doorbell rings, 3 guests enter.

The third time the doorbell rings, 5 guests enter.

The fourth time the doorbell rings, 7 guests enter.

Keep going in the same way. On the next ring a group enters that has 2 more persons than the group that entered on the previous ring.

A. How many guests will enter on the 10th ring? Explain or show how you found your answer.
B. How many guests will enter on the 100th ring? Explain or show how you found your answer.
C. 299 guests entered on one of the rings. What ring was it? Explain or show how you found your answer.
D. Write a rule or describe in words how to find the number of guests that entered on each ring.

The research reported in this paper is part of a large project, Longitudinal Investigation of the Effect of Curriculum on Algebra Learning (LieCal Project). LieCal Project is supported by a grant from the National Science Foundation (ESI-0454739). Any opinions expressed herein are those of the authors and do not necessarily represent the views of the National Science Foundation.

VISUAL TENSIONS WHEN MATHEMATICAL TASKS ARE ENCOUNTERED IN A DIGITAL LEARNING ENVIRONMENT

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This paper reports on elements of an ongoing study examining how understanding is influenced by engaging mathematical phenomena through digital pedagogical media. As the students sought interpretations and explanations, they frequently negotiated investigative pathways based on the visual feedback they received. Unexpected visual output, encountered when students engaged mathematical investigative tasks using spreadsheets, evoked tensions that influenced their actual learning trajectory. These visual perturbations often stimulated dialogue, which in conjunction with associated activity, fostered mathematical thinking, the formation of informal conjectures, and hence a reshaping of their understanding.

INTRODUCTION

Several studies have investigated how the formation of informal conjectures, and the dialogue they evoke, might influence children's learning trajectories, and enhance their mathematical thinking (e.g., Carpenter, Franke, & Levi, 2003). In a digital environment, the visual output and its distinctive qualities can lead to interpretation and response of a particular nature. In this paper, the notion of visual perturbations is explored, and situated within the data obtained, when the students engaged with number investigative tasks in a spreadsheet environment.

An interpretive theoretical perspective was used to frame the research study. Hermeneutics is understood as a theory of interpretation of meaning. There is a range of historical and philosophical positions that help situate hermeneutics, but a common strand is that in the process of interpretation no one facet can be seen as existing in isolation. Each, whether author, text, listener, meaning, etcetera has its own cultural, sociological, and historical elements that fashion the interpretive process. Gallagher (1992), who explores hermeneutics from an educational perspective, contends that interpretation is always a balance of constraint (with tradition) and transformation (of tradition). There is a play between familiar and unfamiliar horizons. Everything is framed and contextualized by the preconceptions and intentions of the user, yet interpreted through the lens of the receiver's prevailing discourse.

Seeing learning as a process of interpretation, with understanding and 'concepts' being states that are in ongoing formation, rather than fixed realities, recognises that our understandings, and who we are, evolve by cyclical engagements with phenomena through the constant drawing forward of prior experiences and understandings that are consequently influenced by that engagement. This appears consistent with the contention that understanding evolves from the negotiation of

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meaning, and that the situating of learning is within the context of the experience. Vygotsky's articulation of the perception of tools as mediators and the semiotic mediation of language provide an historically situated, socio-cultural version of the process of understanding (Lerman, 2006).

When learners engage in mathematical investigation, they interpret the task, their responses to it, and the output of their deliberations through the lens of their preconceptions; of the mathematics, and the pedagogical medium through which it is encountered. Their understandings are filtered by means of a variety of cultural forms, with particular pedagogical media seen as cultural forms that model different ways of knowing (Povey, 1997). The engagement with the task likewise alters the learner's conceptualisation, which then allows the learner to re-engage with the task from a fresh perspective. This cyclical process of interpretation, engagement, reflection, and re-interpretation echoes Borba and Villarreal's (2005) notion of humans-with-media, and continues until some resolution occurs.

While investigating in a digital environment, learners enter some input, borne of the students' engagement with the task. The subsequent output is produced visually, almost instantaneously (Calder, 2004) and can initiate dialogue and reflection. This leads to a repositioning of their perspective, even if only slight, and they re-engage with the task. They either temporarily reconcile their interpretation of the task with their present understanding (i.e. find a solution) or they engage in an iterative process, oscillating between the task and their emerging understanding. When the students' preconceptions suggest an output that is different from that produced, a tension arises. There is a gap between the expected and the actual visual output. It is this visual perturbation that can either evoke, or alternatively scaffold, further reflection that might lead to the reshaping of the learner's perspective, their emerging understanding. It shifts their conceptual position from the space they occupied prior to that engagement.

As learners re-engage with the task, informal mathematical conjectures often have their speculative beginnings (Calder, Brown, Hanley & Darby, 2006). Other researchers have noted that the development of mathematical conjecture and reasoning can be derived from intuitive beginnings (e.g., Bergqvist, 2005). This intuitive, emerging mathematical reasoning can be of a visual nature. Meanwhile, Lin (2005) claims that generating and refuting conjectures is an effective learning strategy. Visual perturbations, and the dialogue they evoke, can generate informal conjectures and mathematical reasoning as the learners negotiate their interpretation of the unexpected situation. It is reasonable to assume that if the learning pathway induced by investigating in a digital medium meant the learner framed their informal conjectures in a particular way, their understanding will likewise emerge from a different perspective.

The participants in the data used in this paper were drawn from year six (10-year-old) students. There were twelve boys and nine girls from a range of socio-economic backgrounds. The participants were located in a classroom situation with a computer available for each pair of students, as they worked on a programme of activities using spreadsheets to investigate mathematical problems. They were observed, their conversations were recorded, and their investigations were printed out or recorded. There were school group interviews, and interviews with working pairs.

The following episodes from the data illustrated different types of visual perturbation, and the ways in which these influenced the learners' interpretation and learning trajectories. Interestingly, these various forms of visual perturbation don't necessarily emerge discretely; an episode can illustrate several types of visual perturbation in an interrelated manner.

RESULTS

First Episode: Alternative conceptual and technical understandings

The following episode arose from a group's engagement with *Rice Mate,* a traditional problem involving the doubling of grains of rice for each consecutive square of the chessboard, and estimating how long the total amount of rice would feed the world. The initial engagement was constrained by their memory of technical aspects, but the unexpected output that was generated from engaging with the task, permitted alternative approaches to be considered and explored. The tension that arose when there was a gap between their expected output and the actual output promoted the restructuring of their perspective and they approached the task in a slightly modified manner. The recursion of their attending to the task, and interpretation through modified perspectives, allowed the evolution of understanding of technical and conceptual elements of their activity. They began by considering the first square of the chessboard and negotiating a way to double the number of grains of rice in subsequent squares:

Tony: A1 times 2. Where is the times button.

Fran: Times is the star button.

Tony: OK, A1 * 2.

The following output was generated:

A1*2
A1*2
…etc.

The output was unexpected and related to a technical or formatting aspect. Their mathematical preconceptions probably enabled them to envisage a sequence of numbers doubling from one in some form, but the screen output being different and unexpected led them to re-evaluate the manner in which they engaged the exploration of the task. Their alternating engagements with the task, then reflection on the output through their mathematical and spreadsheet preconceptions was facilitating the evolution of their approach to the task, and the emergence of the technical aspects. Tony continued, articulating his perception of their desired approach.

Tony: In A1 we want 1 and then you go something like =A1*2 then you go fill down and it times everything by 2. So 1 by 2, then 2 by 2, then 4 by 2, then 8 by 2, 16 by 2.

Fran: To double it? Times 2 more than the one before.

Tony: The amount of rice for each year will be in each cell.

Fran: What's the first thing we need to start off with? =A1*2.

Tony: We have 1 in cell 1 [for one grain of rice], and then we add the formula in cell A2 now.

Fran: And then fill down.

They entered =A1*2 and filled down to produce the sequence of numbers they anticipated would give them the number of grains of rice for each square of the chessboard. They encountered something unexpected with the output generated:

	A
1	1
2	2
3	4
…	…
26	33554432
27	67108864
…	…
64	9.22E+18

Fran: Ok, that isn't supposed to happen!

Tony: 9.22E +18, that makes a lot of sense.

The output was unforeseen and in a form they weren't familiar with (scientific form). There was a tension between the expected and actual output causing them to reflect, adjust their position, and re-interpret. These pupils initially sought a technical solution to resolve their visual perturbation. They looked for a way to reformat the spreadsheet to alleviate their uncertain perspective.

Tony: You can make the cell bigger. Pick it up and move it over.

Fran: That should be enough.

Tony: It still doesn't work.

Still perturbed by what the spreadsheet displayed, they sought intervention regarding the notion of scientific form. They indicated that they better understood the idea and proceeded with the task. Tony considered the output 2.25E+15:

Tony: When you get past the 5 you will need a lot of zeros. We'll need 13 more.

Fran: You can still just do it from here where it is.

They continued with the task, maintaining the numbers in scientific form as they negotiated a way to sum the column of numerical values. This they managed, drawing on their prior understanding of the technical process required. This generated:

1.84467E+19

Tony: Yeah!!!! It worked.

Fran: We got it!

Tony: Wow. It's a really, really big number.

Drawing on their freshly modified perspective, they considered how it might appear in decimal notation. Their shared understanding required further negotiation.

Tony: How many zeros.

Fran: 19.

Tony: Did you count these numbers here?

Fran: No.

Tony: You need to count from the decimal point to the end and then add the zeros.

Fran: Those numbers count as well.

Tony: How do you think we say that number?

Fran: A bagilliganzillan!

They continued with the task, but carried forward their modified perspective; a perspective moderated through iterations of engagement and interpretation, but initiated by the visual perturbation. The sum of the sixty-four numbers that corresponded to the amount of rice on each of the squares was entered into cell A66, and the size and form of the numbers appeared to lead them towards another way of engaging with the task that involved a formatting aspect.

Fran: So 1.84467E+19.

Tony: It's a lot to type in.

Fran: Go down to the bottom, we could use the cell.

Tony: Divide A66 by…

They still had some mathematical thinking and interpretation to undertake related to how long the rice would feed the world, but their learning trajectory was shaped, via interpretation and engagement, by the various associated socio-cultural filters, including the spreadsheet environment. Their preconceptions, and their understanding and explanations, were mediated by the pedagogical medium as evidenced by their subsequent interactions. It appeared that the visual perturbations had instigated, and then influenced the nature of those interactions.

Second Episode: Reshaping of predictions and emerging, informal conjectures

The next scenario relates to the **101 X** activity, a task in which the pupils investigated the pattern formed by the 101 times table. The two pupils had entered the counting numbers into column A and were exploring the pattern formed when multiplying by 101 in column B:

1
2
3
4 etc

Awhi: =A2 * 101. Enter.

Ben: 202.

They considered the output produced before predicting, using an informal, evolving conjecture.

Awhi: Now let us try this again with three. Ok, what number do you think that will equal? 302?

Ben: No, 3003, (they copy the formula down to produce the output below).

1	101
2	202
3	303
4	404 etc.

Ben [continues]: 303.

The actual output was different from the output the pupils expected. This had created a visual perturbation, which in this case was easily reconciled with their prevailing mathematical discourse. The visual perturbation had caused a reshaping of their prediction that allowed them to reposition their conceptualisation. It also initiated the beginnings of an informal conjecture.

Awhi: If you go by 3, it goes 3 times 100 and zero and 3 times 1; 303.

They then explored a range of two and three digit numbers, before extending the investigation beyond the constraints of the task.

Awhi: Oh, try 1919.

The following output was produced:

193819

They seemed to disregard this output and predicted, using their preconceptions.

Awhi: Now make that 1818, and see if it's 1818 [the output].

Ben: Oh look, eighteen 3, 6, eighteen.

There was a visual perturbation, which made them re-engage in the activity, reflect on the output, and attempt to reconcile it with their current perspective. It caused them to reshape their emerging conjecture.

Awhi: Before it was 193619; write that number down somewhere (183618) and then we'll try 1919 again.

Ben: Yeah see, nineteen, 3, 8, nineteen. Oh that's an eight.

Awhi: What's the pattern for two digits? It puts the number down first then doubles the number. This is four digits. It puts the number down first then doubles, then repeats the number.

The visual perturbation made them reflect on their original conjecture and reposition their perspective on the initial, intuitive generalisation. It stimulated mathematical thinking, as they reconciled the difference between what they expected and the actual output, and rationalised it as a new, informal generalisation. This new generalisation was couched in visual terms, referring to the type and position of the digits.

DISCUSSION

The data in this study illustrated the notion of visual perturbation. When the output generated differed to the expected output that the pupils' preconceptions had suggested, a sense of unease was evident. This tension disturbed the pupils' perceptions of the situation leading them re-engage with the task from a modified position. It influenced their interpretations and decision-making and consequently transformed their learning trajectory. The output, in visual form, influenced the pupils' reactions, interpretations, articulated accounts, and their subsequent re-engagement with the task. They posed and tested informal conjectures, incorporating their interactions from within the visual tabular form. The engagement with the task, and with the medium, often evoked dialogue. This was an inherent part of the negotiation of understanding. The conceptual perceptions to which they subscribed prior to that engagement were revised, and they re-engaged with the tasks. The visual perturbations invoked at various junctures through the engagement with mathematical phenomena in the spreadsheet environment shaped the learning trajectories in particular ways. This facilitated the ongoing evolution of their mathematical thinking. Within the notion of visual perturbation, there seemed to be several manifestations or variations. When the visual perturbation:

- Led to a change in prediction.
- Caused a reshaping of the conjecture or generalisation. This was similar to that above, but the re-engagement was more reflective in nature.
- Made the students re-negotiate their sense making of the task itself. This process was often interwoven with the investigative trajectory, with each influencing the other.
- Was associated with an idea or area new to the students.

- Led the students to further investigate and reconcile their understanding of a technical or formatting aspect associated with their exploration. This was often also symbiotically linked to conceptual exploration.

The visual perturbations shifted the students from the conceptual space they occupied prior to that engagement. This facilitated mathematical thinking. Discussing mathematical thinking when using digital images, Mason (2005) also suggested that when response to a particular action does not meet the expectation, the tension arising might provoke further reflective engagement. The data also supported the contention that engaging with the mathematical phenomena through the spreadsheet fashioned the learners' approach due to the distinctive characteristics of the digital medium, its associated affordances, and their interplay with other influences. As the learners alternatively attended to the task, and their prevailing discourses, the learning pathway was influenced by unexpected visual output. Hence, their perspective was repositioned and consequently their understanding appeared to evolve in distinctive ways.

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USE OF *DRAGGING* AS ORGANIZER FOR CONJECTURE VALIDATION

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In this article, we report on a study centred on the teaching and learning of proof in which there is evidence that dragging becomes a source for significant student participation in the validation of conjectures. The findings highlight the teacher's use of dragging as an organizer of the activity, in cases when there are conjectures that students consider acceptable but for which they do not have the theoretical elements to validate them.

INTRODUCTION

The teaching and learning experience of proof reported here took place in a university plane geometry course, during a problem solving activity, which required a construction carried out with a dynamic geometry program. The problem is one of a set of tasks proposed throughout the academic term to favor student participation in the collective construction of part of an axiomatic system. Students get involved in the exploration of geometric figures, formulation or interpretation of conjectures and their proof. Our general premise is that genuine student participation in the production of ideas with which mathematics knowledge is constructed —thanks to the dynamic geometry context — leads to a significant approach to proof.

We think that this experience contributes to the request formulated by Herbst (2002) which expresses the need to devise class organizations that favor student participation in proof formulating activities, for the different levels of education. Particularly, we want to communicate a novel use of the *dragging* function — specific to dynamic geometry software— employed by the teacher to treat some of the conjectures that the students consider acceptable but cannot validate since the required theoretical elements are not yet part of their axiomatic system, since it is constructed throughout the semester. A review of the literature shows that studies carried out about the *dragging* function in teaching and learning to prove have centered mainly on how students use it to solve problems (e.g. Olivero, 1999; Arzarello et al., 2002; Stylianides and Stylianides, 2005) but its potential use as a class activity organizer has not been explored enough.

THEORETICAL REMARKS

The context in which the activity that we report took place is based on the following ideas about proof and learning to prove. For us, *proving activity* includes two

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processes, idea that coincides with the proving process described by Olivero (2002). The first process consists of actions that lead to the production of a conjecture; these actions generally begin with the exploration of a situation to seek regularities, followed by the formulation of conjectures and their validation. The actions of the second process are concentrated on the search and organization of ideas that will become a proof; this is considered as an argument of deductive nature based on an axiomatic system in which the proven statement can be included. In this sense, we coincide with other researchers (Hanna, 2000; Mariotti, 2006) who consider proof as the fundamental activity mathematicians carry out to remove any doubts about a statement's truth and to organize ideas in a deductive discourse, with the purpose of validating it within a theoretical system. Since the principles and deduction rules that govern the production of the discourse are established by a specific human group, we recognize the sociocultural character of the proving activity, conditioned by the context and the specific domain within which it takes place (Alibert & Thomas, 1991; Hoyles, 1997; Radford, 1994; Godino & Recio, 2001; Mariotti, 2006).

We view the mathematics class as a community of practice (Wenger, 1998) in which students have the opportunity to learn to prove as they commit themselves with a repertoire of practices suitable for proving activity. With these practices, they gain competency and develop ideas of what it means to prove and how they can participate legitimately in the production of proofs. Undoubtedly, the repertoire of practices is conditioned by the class community resources available to carry out the proposed enterprise and the norms that are negotiated for their use. Particularly, when a dynamic geometry program is available, the use of its functions becomes a characteristic aspect of the community's practices. Sometimes, in midst of proving activity actions in class, an unplanned use of a software function can appear, that is later evaluated as favorable for the practice that is taking place and considered as a useful form of taking advantage of the dynamic geometry program. It is the case of the use of the *dragging* function that we discuss in this article.

RESEARCH CONTEXT

The paper focuses on a series of two 2-hour sessions in which pairs of students taking a university level geometry course were asked to solve the following problem, using dynamic geometry software (no figure was included):

> Given line m and two points P and Q in the same half-plane determined by m, determine point R on m for which the sum of the distances PR and RQ is least. a) Describe the geometric construction used to find R. b) Formulate a conjecture. c) Write the main steps of the proof of the conjecture.

The sessions took place at the end of the semester. Throughout the semester, the students had participated in the collective construction of a portion of the Euclidean geometry axiomatic system including properties of angles and triangles. They were used to solving open problems and could skillfully use dynamic geometry software to explore figures and verify conjectures. The data collected consists of transcriptions of class audio and video recordings complemented by video recordings of private conversations of the teacher with each group of students and field notes of one the members of the research group, who acted as a non-participant observer.

In the next sections, we analyze the way students and teacher used dragging during the solution of the problem and the process of validating conjectures.

USUAL USE OF DRAGGING

Ten pairs of students were formed. All groups constructed the required elements, measured the segments and found the sum *PR* + *QR* (Figure 1).

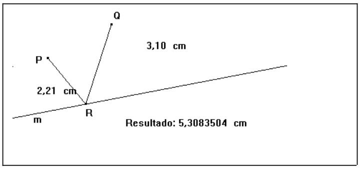


Figure 1: Initial construction

Each group began the exploration process using “linked” *dragging* (Arzarello et al., 2002), that is, moving point *R* on line *m* to determine the position of the solicited point *R*. When they were sure that such a point existed, they searched for the geometric properties that characterize the position, which led them to make auxiliary constructions, find measurements, and move points *P* and *Q,* to determine special configurations or regularities in the figure. Each group wrote their result of the exploration as a conjecture.

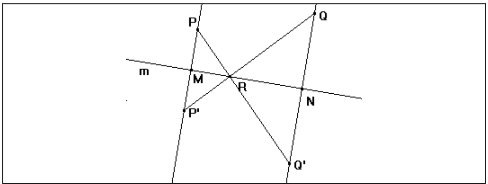
Due to her conversations with each group, the teacher obtained information about the exploration process carried out, the conjecture formulated and the ideas brought up for the corresponding proof. In the explanations that three of the groups gave to the teacher, their use of “linked” *dragging* to verify whether the conjecture was plausible is mentioned.

Having found what the different student conjectures were, the teacher organized them according to their degree of complexity and then moved onto the discussion of results. Some groups presented their conjectures, showing their Cabri representation. Using *dragging*, the class decided whether they were acceptable or not. Seven different conjectures were proposed, one of which was refuted by a student, at the end of the second session, by showing a counterexample by *dragging*. With the presentation of the conjectures, the first session terminated. The teacher asked the students to work on a proof of their conjecture. She suggested using another point on line *m* and comparing the sum of their distances to *P* and *Q*.

In sum, in the course of solving the problem and verifying the formulated conjectures, the students used *dragging* as a means to explore and verify the properties of the figure they constructed. This use of *dragging* has been widely documented (e.g. Olivero, 2002, Arzarello et al., 2002).

NOVEL USE OF DRAGGING IN THE ORGANIZATION OF THE CONJECTURE VALIDATING ACTIVITY

The teacher started the second session asking students for their proofs. Darío offered to prove Leopoldo's and his conjecture (Figure 2). To do his proof, Darío explained that he only needed segment *P'Q* because its intersection with segment *PQ'* is on line *m*. (Figure 3(a)).



P' and *Q'* are points on the lines, perpendicular to *m,* through *P* and *Q* such that *PM* = *MP'* and *QN* = *NQ'*. If *R* is the intersection point of *PQ'* and *P'Q* then *PR* + *RQ* is the minimum sum.

Figure 2: Darío's and Leopoldo's conjecture

Darío used triangle congruency criteria to show that segment *PR* is congruent to segment *P'R* and, therefore, that the sum of *PR* and *RQ* is equal to the sum of *P'R* and *RQ*. (Figure 3(b)):

Darío: [...] since I have a point [*M*] on the line and a perpendicular [to *m* through *M*] and I draw $\overline{PR}$, then [the triangles] *PRM* and *P'RM* are going to be congruent using side-angle-side.

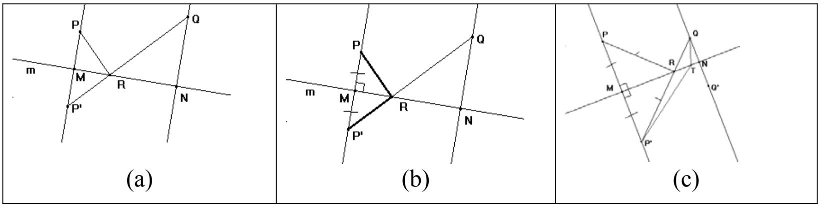


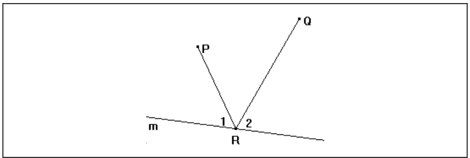
Figure 3: Figures that support Darío's proof

He then used the Triangle Inequality Theorem to show that for any other point *T* on line *m*, the sum of *P'T* and *TQ* is greater than *P'R* plus *RQ* (Figure 3(c)):

Darío: [Draws segments *P'T* and *TQ*, Figure 3(c).] Yes… then we do not have *T* between *Q* and *P';* then we have a triangle. By Triangle Inequality, I have that *P'T* plus *TQ* is greater than *P'Q*; that is, *P'T* plus *TQ* is greater than *PR* plus *RQ*. And this happens with any point that I use.

The student uses in his proof theoretical statements of the axiomatic system that the students have at their disposition, linking statements starting from the properties they fixed for point *R* in their construction. *Dragging* does not play any role in the validation of his conjecture.

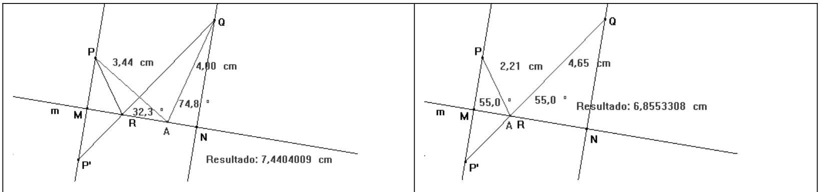
Afterwards, the teacher invites the students to look at Henry’s and Antonio’s conjecture (Figure 4) and makes them notice that it refers to congruent angles, a geometric property that can be checked, but that they did not give a geometrical construction proposal for point *R*, a marked difference with Darío’s and Leopoldo’s conjectures.



If angles 1 and 2 are congruent then *PR* + *RQ* is the minimum sum.

Figure 4: Henry’s and Antonio’s conjecture

At this point, the teacher could have opted for explaining to the students that there was no way to validate Henry’s and Antonio’s conjecture, using the available axiomatic system. According to the norms established in the class, since the conjecture could not be validated, it had to be discarded and could not become an element of the axiomatic system being constructed. The student’s effort would not be valued as relevant mathematical production. Instead, the teacher decided to take advantage of *dragging* to find a way to validate the conjecture. She projects Darío’s Cabri construction on the wall, locates point *A*, different from *R*, on line *m,* constructs $\overline{PA}$ and $\overline{AQ}$, finds the sum of the distances and, additionally, measures $\angle PAM$ and $\angle QAN$ (Figure 5 (a)). Then, she *drags* point *A*, until the sum of the distances became a minimum; at that moment the angle measurements were equal and *R* and *A* coincided; this meant that the constructions proposed by both groups produced the same point (Figure 5 (b)).



(a) (b)

Figure 5: Comparing conjectures

Due to the comparison carried out by *dragging* of the conjectures, the teacher's idea is to use syllogisms to prove that point *A,* as proposed in Henry's and Antonio's conjecture, has the same geometric properties of point *R*, as suggested in Darío's and Leopoldo's conjecture. Since Darío had already proven that *R* satisfied the condition established in the problem, they could conclude that *A* also satisfied the condition. The teacher suggests using as the only "given" condition that $\angle 1$ and $\angle 2$ are congruent (Figure 6 (a)) – as the conjecture states – and to prove that *A* is collinear with *Q* and a point on the perpendicular line *PM* the same distance from *m* as *P*, as indicated in Darío's conjecture (*P'* in Figure 2).

María, another student, suggests drawing the ray opposite to $\overrightarrow{AQ}$ and finding point *S,* the intersection of that ray with $\overleftrightarrow{PM}$. As María explains, this guarantees that *S, A* and *Q* are collinear, due to the definition of "opposite rays" (Figure 6 (b)). Showing that the distance from *S* to *M* is equal to the distance from *M* to *P* remains. Melisa shows this is true because triangles *PMA* and *SMA* are congruent since $\angle 1 \cong \angle 3$ ($\angle 3$ and $\angle 2$ are vertical angles and the latter is congruent to $\angle 1$, Figure 6 (c)), angles *PMA* and *SMA* are right angles and the triangles share $\overline{MA}$. Therefore, *PM* is equal to *MS* and point *S* corresponds to point *P'* of Darío's and Leopoldo's conjecture.

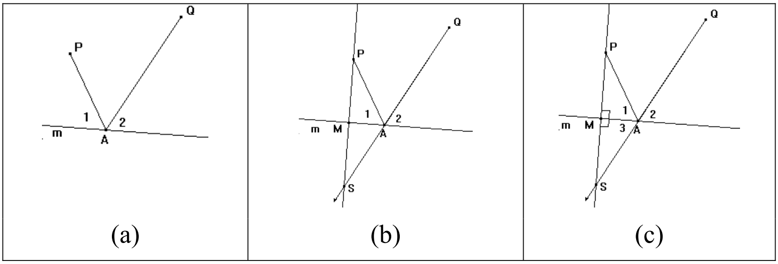


Figure 6: Figures that support theoretical validation of Henry's and Antonio's conjecture

This way, they prove that if angles 1 and 2 are congruent, then *A* is collinear with points *S* and *Q* and *PM* is equal to *MS*. Therefore, point *A* is R. They have already proven (Darío's and Leopoldo's conjecture) that if *R*, *S* and *Q* are collinear points, and *PM* is equal to *MS* then *PR* + *RQ* is the minimum sum. Therefore, they concluded that if $\angle 1 \cong \angle 2$ then *PR* + *QR* is the minimum sum.

To summarize, since Henry's and Antonio's conjecture did not provide geometric properties that were useful for a proof that is within the available axiomatic system, the teacher suggests using *dragging* to verify the coincidence of point *A* and point *R*. The latter point was obtained through a geometric construction that does provide the necessary elements to construct a proof. This allowed validating the conjecture in a way in which, instead of trying to show directly that the sum *PA* + *AQ* is a minimum, the teacher, together with the students, constructs a deductive argument to show the coincidence of points *A* and *R*. Through this ingenious resource, the teacher organizes the validating activity of Henry's and Antonio's conjecture.

CONCLUSIONS

The analysis of the events that we carried out permits us to state, as other researchers have mentioned (Olivero 1999, Olivero 2002, Arzarello et al, 2002; Stylianides and Stylianides, 2005), that the *dragging* function has an important role in the generation of a favorable environment for learning proof, not only during the exploring, discovering and conjecture verifying moments, but also as a means to generate ideas that are a source for the construction of a proof.

In this article, we emphasize the teacher's creative use of *dragging* to organize student activity during the proving activity. Undoubtedly, since the validation done does not ascribe to the type of proofs constructed in class, it is quite improbable that a student would have thought it up. This is why we think that *dragging* becomes a teacher resource with which student proving activity can be fostered.

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MIDDLE SCHOOL TEACHERS' INFORMAL INFERENCES ABOUT DATA DISTRIBUTIONS

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This paper describes the informal inferences made by middle school teachers in response to a task involving a comparison of two data sets. Of specific interest were not only the decisions the subjects made about the scenario they were presented with, but also the nature of their distributional reasoning behind their decisions. Results showed that teachers used both centres and spread in their distributional reasoning as they made probabilistic generalizations that often included evidence of contextualization. As teachers begin to examine the aspects of distributional reasoning which they themselves bring to bear in making informal inferences, they can better promote the informal inferences made by engaging their own students in authentic statistical inquiry.

INTRODUCTION

The purpose of this paper is to report on research describing the informal inferences made by middle school teachers in response to a task involving a comparison of two data sets. Informal inference can be seen as including the decisions or judgments people make in the face of information they receive. Whereas more formal inference relies on tools such as those involved in hypothesis testing, informal inference may involve simpler notions such as the centre, shape, and spread of a distribution. Research over the years has described how people reason about averages and variation and shapes of distributions (e.g. Mokros & Russell, 1995; Ben-Zvi & Garfield, 2004; Bakker, 2004), and coordinating these elements of statistical literacy contributes to distributional reasoning about data (Konold & Higgens, 2002).

While precollege curricula still leans toward a rather thin treatment of descriptive statistics (Shaughnessy, 2006), such as creating and reading graphs or computing averages or other statistical measures, "using this information to make decisions when comparing distributions of different data sets has not received very much attention until recently" (Watson, 2007, p. 139).

Moreover, although precollege students have been the focus for many researchers interested in statistical reasoning, this paper focuses on inservice teachers in addressing the following research question: How can the informal inferences of middle school teachers be described? Of specific interest was not only the judgments the subjects made about the scenario with which they were presented (*What* decisions did they make?), but the nature of the distributional reasoning behind their decisions (*Why* did they make those decisions?). After describing some research relating the

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aspects of distributional reasoning and inference, both of which informed our conceptual framework, the methodology for the study will be explained. Then, preliminary results to the research question will be summarized along with a discussion and implications for future research and teacher training programs.

CONCEPTUAL FRAMEWORK

Shaughnessy, Ciancetta, Best, and Canada (2004) investigated how middle and secondary school students compared distributions using the same task as that used in the current study reported by this paper. Given two data sets with identical means and medians, subjects showed how they reasoned about averages and variation, with higher levels of distributional reasoning attributed to those responses that integrated both components of centres *and* spread. Their findings and recommendations echoed that of Mellissinos (1999), who stressed that although many educators promote the mean as representative of a distribution "the concept of distribution relies heavily on the notion of variability, or spread" (p. 1). Thus, two key elements comprising the conceptual framework for looking at distributional reasoning are a consideration both aspects of centre (average) and variation. This approach implies taking an *aggregate* view of data as opposed to considering individual data elements (Konold & Higgens, 2002). Coordinating these two aspects is what enables a richer picture of a distribution to emerge (Shaughnessy, 2006; Makar & Canada, 2005). Underscoring the importance of variation in statistical reasoning (Wild & Pfannkuch, 1999), Canada (2006) also posited that a person's interpretations of variation would have an effect on decisions made about data. For example, research with preservice teachers showed that many subjects expressed a lack of confidence in making decisions, "perceiving that 'anything can happen' in situations involving variation" (p. 11).

Additionally, Curcio (1987) was able to identify three levels of difficulty in making sense of graphs. The first level is simply *reading the data*, in which the subjects could attend to the basic facts stated in the graph, including numerical values shown, titles, and axis labels. The second level of comprehension is *reading between the data*, and the third level is *reading beyond the data*, which requires "extension, prediction, or inference" (p. 384). Consistent with Curcio's description of that third level, Makar and Confrey (2007) articulate principles of informal statistical inference in terms of making "probabilistic generalizations about data" (2007, p. 3). In unpacking this phrase, the authors describe how informal inference would include explicit attention to the uncertainty inherent to inference making, and also about the generalizations that can be made by reading "beyond the data" (p. 4). As a part of *reading beyond the data*, researchers have also focused subjects' regard for the context of the data as an important aspect in making informal inferences (e.g. Makar & Canada, 2005; Shaughnessy et. al., 2004; Canada, 2006).

Thus, in our framework for describing the informal inferences made by our subjects, we were looking for their "probabilistic generalizations", their attention to context, and their use of the distributional reasoning aspects of averages and variation.

METHODOLOGY

The task chosen to look at middle school teachers' informal inferences when comparing data sets was called the Movie Wait-Time task, which was also used along with similar tasks in previous research (e.g. Shaughnessy et. al., 2004; Canada & Ciancetta, 2007). The task introduces the notion of *movie wait-time* as the difference between the advertised start time of a movie (such as might be listed in a newspaper) and the movie's actual start time (after previews and advertisements, for example). Data for the wait-times for 21 different movies are shown in stacked dot plots for each of two theatres, the *Maximum* and the *Royal*. The task was deliberately constructed so that the Maximum and Royal wait-times have the same means and medians, yet different amounts of variation are apparent in the graphs (Figure 1).

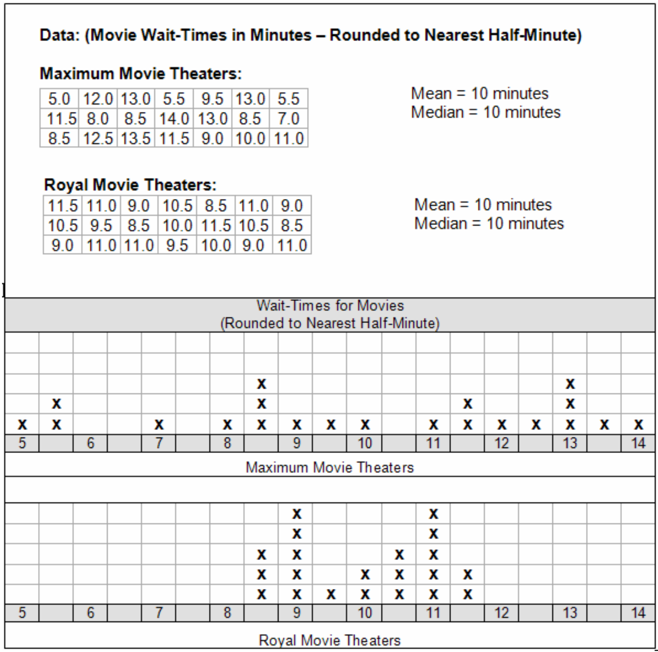

Figure 1: The Movie Wait Times Task

In a sequence of written prompts, subjects were first asked the open-ended question of "What can you conclude about the wait-times for the two theatres?" Then, subjects were asked to react to a hypothetical argument that "there is really no difference in wait-times for movies in both theatres, since the averages are the same," and to provide their reasoning. Finally, subjects were asked which of the theatre chains they would choose to see a movie in (based solely on wait-time), again giving the rationale for their choice.

Nineteen middle school teachers participated in this study as part of a professional development project that included a week of full-day instructional interventions, with half of the time spent in sessions on probability and statistics. Prior to beginning the interventions in probability and statistics, subjects were given the Movie Wait-Times task as a written-response item for completion as part of a pre-survey. The pre-survey was not given as part of the formal evaluation for the professional development project, but rather as a way of having the subjects show their informal sense of how they were initially thinking on a range of concepts in probability and statistics.

The interventions during the professional development project comprised a series of small-group and whole-class activities and simulations which were related to the concepts on the pre-survey. The emphasis was on providing teachers with multiple hands-on experiences, investigations, and opportunities to interpret data in an applied context. Group discussions were videotaped so that comments could be recorded and transcribed, and exit cards were also gathered to glean participants' reflections on the mathematical ideas for each day of the project. Post-survey written instruments were also given to participants to further probe their thinking. The corpus of data for the entire project comprised of the transcriptions of the class discussions, project leader debriefing notes, and written responses to pre- and post-survey instruments as well as exit cards. The excerpt of data included in this paper focuses on the teachers' initial responses on the pre-survey, which were then qualitatively coded according to the components of conceptual framework which related to informal inference as well as the distributional aspects of centres and spread.

RESULTS

All of the teachers disagreed with the hypothetical argument (that "there is really no difference in wait-times for movies in both theatres, since the averages are the same"), with all but two teachers referring to both centres and spreads in their response. The two exceptions were responses that only referred to variability, such as MC who wrote "There is greater variability in the wait times of Maximum Theatres, [a] 9 minute range versus a 3.5 minute range at Royal". Aside from the incorrect range stated by MC (Royal's range is actually 3 minutes), the point is that all the teachers were attentive to the differences in the two data sets, and not anchored to means and medians as the definitive way to compare data sets.

Since the main goal of this paper was to look for the inferences made, we paid particular care to the responses whereby subjects explained their preference for which movie theatre they would chose to visit (based solely on wait-time). One teacher said it didn't matter and chose not to express a preference, while only three teachers chose the Maximum Theatres – the rest chose the Royal Theatres. Here are sample responses from those choosing Royal Theatres:

AG: I would choose Royal…because the wait-time is more predictable as it would range from 8.5 to 11.5 minutes.

FH: The range of Maximum is greater – one could wait anywhere from 5 to 14 minutes with little consistency.

SC: While both average 10 (median and mean), Maximum has a range of 5 - 14 or 9 while Royal's range is 8.5 - 11.5 or 3. Royal seems more consistent in its wait time.

The above responses are representative of the informal inference that the tighter range for Royal Theatres implied more predictability and consistency, and the wider range for Maximum Theatres implied unpredictability and inconsistency.

All three teachers who chose Maximum Theatres have an excerpt of their response listed below. Surprisingly, the last two of these three teachers pointed out that Royal was more consistent, but still said they preferred Maximum:

KA: I would choose the Maximum Theatre and hope I get the low end of wait time rather than the average of 10 minutes.

MK: Royal is more predictable, but I would choose Maximum because I could get lucky and have a very short wait-time.

JP: Although averages are the same, there is a bigger range and therefore more unpredictability at the Maximum. The Royal Movie Theatre wait time is more consistent, with wait times closer to the average. [At] Maximum, there is a better chance of a wait time as short as 5-8 minutes.

JP's response appeals to idea of data being "closer to the average," and this offers thinking that makes it easier to build the notion of data not merely being spread out, but being spread relative to a centre value. A commonality to all three subjects who chose Maximum is that they appeal to the probabilistic notions such as "hope", "luck", or "chance" in attaining a short wait time. That is, even acknowledging the wider range in Maximum Theatres, they based their decision on the probability of attending a movie in the lower end of the distribution. Other teachers (who had chosen Royal Theatres) also used probabilistic language in their generalizations:

RG: I would be more likely to miss less of the beginning of a movie at Royal.

WD: There seems to be an even chance one could wait less than 10 min. and more than 10 min.

ST: The other theatre [Maximum] has a chance I could wait for 14 minutes.

VB: If I were running late, I know I would not likely miss anything up to 8.5 min. after "show time" [at Royal] whereas Maximum is more up in the air!

Note how ST focuses on the maximal wait-time of 14 minutes (at Maximum Theatre), while VB focuses on the minimal wait-time of 8.5 minutes (at Royal Theatre). Also, VB brings up context in the sense of picturing how the scenario of the task might play out in real-life (by conjecturing "If I were running late…").

The issue of context was brought up by seven teachers, including VB. To be coded for context, a response needed to have some personal element that suggested the subject was envisioning the scenario as it related to his or her own lives. The one teacher who declined to make a preference gives a good example:

GG: Because I don't mind watching previews and advertisements, this data would not impact my decision of theatres.

Choosing a preference based solely on wait time doesn't make sense for GG, since this attribute seems irrelevant for him personally. For others, the context of wait time was very relevant:

HA: I would choose Royal because I would be very irritated to wait through lengthy advertisements.

FL: I would choose to see a movie in Royal Movie Theatres because my family tends to make me 10 minutes late to things.

In contrast to GG, HA makes it clear that she is "irritated" by wait times, and FL anticipates getting to the theatre late. It seems clear for those who brought up context in their responses that their inferences were influenced by a personal reflection on going to a theatre and experiencing wait times in real life.

DISCUSSION

We were surprised that all of the middle school teachers in this study disagreed with the hypothetical argument (that "there is really no difference…"), since some subjects in other studies with the Movie Wait-Time task (or similar tasks) have indeed agreed, pointing to the identical means (or medians) as their justification. For example, while none of eight middle school students interviewed about the Movie Wait-Time task agreed, three of eighteen high school students *did* agree, with one eleventh-grader saying simply that "I agree because, well the averages are the same" (Shaughnessy et. al., 2004, p. 16). Also, the results of Canada and Ciancetta (2007) with preservice teachers on a similar "Train Times" task reported that almost 35% of their preservice teachers initially agreed with their hypothetical claim of "no real differences" (p. 965). While only the means were identical in the "Train Times" task (the medians were slightly different), the subjects in Canada and Ciancetta's study showed a far stronger preference for reasoning via centres alone than did our middle school teachers, most of whom considered both centres and spread in their reasoning. We wonder if the experience of classroom teaching allowed our subjects to be more attuned to both centres and spread than the preservice teachers.

Nine of the subjects drew an inference about the consistency or predictability of both theatres, and this echoes the idea of *reading beyond the data* in the sense that these teachers were making a generalization to what they could expect as far as future wait-times. Moreover, ten subjects included probabilistic language in their responses, which showed sensitivity to the uncertainty inherent in making inferences. By speaking in terms of what was "likely" to happen, or in terms of the "chances", teachers actually lay the foundation for authentic statistical inquiry that cannot be answered in simple right-or-wrong terms. The nature of statistical inference hinges upon ideas of well-reasoned argumentation and confidence, which can represent a paradigm shift in those who view mathematics more deterministically.

CONCLUSION

This study set out to describe the informal inferences made by middle school teachers, and in addition to seeing how they used centres and spreads in their distributional reasoning and probabilistic language in their generalizations, we also saw seven teachers pay explicit attention to the context of the data. Interestingly, middle and high school students who were interviewed about the Movie Wait-Times task often expressed a preference for advertisements and previews (Shaughnessy et. al., 2004). The big idea, however, is that since data is often described as numbers with a context, it is natural for context to influence the inferences made.

The changes needed to improve student achievement require a great deal of learning on the part of teachers and are difficult to make without support and guidance (Ball and Cohen, 1999; Borko, 2004; Wilson and Berne, 1999). Unfortunately, "the professional development currently available to teachers is woefully inadequate" (Borko, 2004). As an exploratory study, it was useful to examine the informal inferences made by middle school teachers because we wanted to see them empowered to move their own students from descriptive statistics to more of a focus on the purpose and utility of working with data. Watson noted that "informal inference is a phrase that represents a continuum of experience from when students first begin to ask questions about data sets to the point where they are about to meet formal statistical inference" (2007, p. 142). We found with our middle school teachers that tasks such as Movie Wait-Times afforded useful opportunities to discuss not only *what* decisions they made, but also *why*. As teachers begin to examine the aspects of distributional reasoning which they themselves bring to bear in making informal inferences, they can better promote the informal inferences made by their own students as they engage in authentic statistical inquiry.

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THE REFLECTIVE THINKING OF THREE PRE-SERVICE SECONDARY MATHEMATICS TEACHERS

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This paper reports on the reflective thinking of three pre-service teachers during a one-year teacher education program. We interviewed the participants three times during their practicum and once more in their first year of teaching to investigate the nature and depth of their self-reflections. We developed a three-stage, hierarchical model of reflective practice to interpret the interview responses. Results show slight improvement in the participants' practicum reflections and a greater capacity for reflection in their first year of teaching, but even then their responses were generally descriptive in nature. We suggest some reasons for this situation.

REFLECTIVE PRACTICE AND BEGINNING TEACHERS

The term 'reflective practice' is commonly used to describe a disposition to reflect critically on one's classroom practice and is regarded as an important part of teacher development (Jaworski, 2006). For pre-service teachers, the school-based fieldwork or practicum can become an important "site for learning" but only "if the learner possesses a disposition to be reflective" (Smith, 2003). However, research suggests that pre-service teachers find it difficult to develop a reflective stance on their classroom practice (Alger, 2006) and their reflections are cursory or superficial (Bean & Stephens, 2002). Yet, if pre-service teachers do not cast a critical eye on their practice they are likely to reinforce their existing beliefs and attitudes about teaching (Grootenboer, 2005/2006). They may focus too much on the practical concerns of teaching, such as lesson planning and classroom management (Moore, 2003), rather than becoming more aware of student learning (Jaworski & Gellert, 2003).

Practicum classrooms may not reflect the reformist vision of university courses (Goos, 1999) and many pre-service teachers have practicum experiences that simply reinforce outmoded pedagogies from their own school days (Grootenboer, 2005/2006). Pre-service teachers seek approbation and acceptance from supervising teachers (Roberts & Graham, 2008) but have often not developed a personal philosophy of teaching so simply copy the teaching techniques they observe, leading to a simplistic and technically-based view of teaching (Putnam & Borko, 2000). Passive imitation of supervising teachers does little to encourage a stance of critical reflection or promote the beginning teachers' capacity to learn from their field experiences in any meaningful way (Zeichner, 1992).

Yost, Senter and Forlenza-Bailey (2000) recommend investigating the role of the school-based practicum in assisting pre-service teachers become more reflective. Lerman and Scott-Hodgetts (1991) propose discussion of critical classroom incidents

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as a useful means of identifying and developing reflective practice. This paper focuses on three members of a teacher education program as they recount significant classroom incidents from their practicum experiences. We examine what their reports indicate about the development of their reflective practice.

THEORETICAL PERSPECTIVES ON REFLECTIVE PRACTICE

We developed a three-stage, hierarchical model for analyzing the reflective practice of our participants based on the work of Lee (2005) and Muir and Beswick (2007). Lee investigated pre-service teachers' reflective thinking and described *Recall* (descriptions of experiences without looking for alternative explanations), *Rationalisation* (searching for causes to help explain experiences), and *Reflectivity* (analyzing experiences from various perspectives with the intention of future changes in action). Muir and Beswick examined experienced teachers and identified *Technical Description* (general accounts of classroom experiences often focusing on the technical aspects of teaching), *Deliberate Reflection* (identifying and explaining critical incidents), and *Critical Reflection* (considering the perspectives of others and contemplating alternative actions). To highlight the particular concerns of beginning teachers, we developed the following three levels of reflective practice, with examples for each one taken from our interview data:

- *Descriptive Recall*. General descriptions of classroom practice; evaluating the success or failure of actions; focusing on the technical aspects of teaching (e.g., "I ran out of material at the end of the lesson").
- *Practical Rationalization*. Accounts of critical incidents; explaining the actions; searching for causes (e.g., "I want to do more group work but my supervising teacher says I'm not giving the class enough practice exercises. I want her to write me a good report").
- *Critical Reflection*. Analysis of experiences; considering various perspectives; offering alternatives (e.g., "I tried to teach them multiplication of fractions but they didn't know their tables. A lot of students got confused and I had class management problems. I should have revised some tables questions and kept the numbers smaller").

CONTEXT AND METHOD

This study is part of a larger study (see Prescott & Cavanagh, 2008) and took place at a university in Sydney, Australia, during a one-year professional program for secondary teaching. The 50-day practicum was completed in a single school under the direction of one supervising teacher, predominantly in one teaching day per week over the course of an entire school year. The mathematics methodology units at university allow students opportunities for reflection and analysis of their practicum experiences through, for example, reflective tasks completed in conjunction with the supervising teacher, action research projects on student learning, and regular 'school experience discussions' similar to the 'Episodes and Issues' framework suggested by

Jaworski and Watson (2001, cited in Jaworski & Gellert, 2003). In these discussions, pre-service teachers "bring significant episodes (or anecdotes) from their teaching to share … [and] are encouraged by tutors to take a positively critical stance towards such issues, relating the questions that arise to teaching situations they have personally experienced" (p. 847).

This study occurred during the final implementation of a new Mathematics Syllabus for Years 7 to 10 (Board of Studies New South Wales [BOS NSW], 2002) that emphasized *Working Mathematically*, an approach to teaching mathematics providing "opportunities for students to engage in genuine mathematical activity and to develop the skills to become flexible and creative users of mathematics" (BOS NSW, 2002, p. 45). Working Mathematically is a focus of our research.

Three semi-structured interviews of 20 minutes' duration occurred with each participant during the practicum year. In the first interview, participants recalled their own school days, discussed their motivation for choosing a mathematics teaching career, and shared their concerns about the impending practicum. In the second and third interviews, they described their own and their supervisors' mathematics lessons, and gave an impromptu lesson outline for a specified topic. In a fourth interview in June of the following year, the participants reviewed their practicum experiences from the stance of a first-year teacher. In all interviews, the participants discussed their teaching practices and the roles played by their supervising teachers so we might gain insights into the quality and depth of their reflective processes. All of the interviews were transcribed and read carefully so that emergent themes could be identified and coded. Our reflective practice model guided the coding of the data as we looked for patterns in the participants' responses. After the initial analysis, we revisited the data to look for illustrative examples for the themes we had identified and formed the three narrative case studies (Miles & Huberman, 1994).

THE CASE STUDIES

Sam's reflective thinking

Sam (pseudonyms used for all participants) described his best teacher as "very caring" and his own motivation for teaching as "a great need to help people and I guess I'm caring, so teaching and helping students is something I enjoy. I care a lot about students". Sam expressed a desire during practicum to "engage students in practical activities; to try to link the concepts to the real world", rather than the way he had been taught with teachers who "came in, wrote on the board, we wrote notes and we did exercises in class and we did exercises for homework". However, in outlining how he might teach a lesson on decimals he said, "I'd write up a heading and I'd write the theory behind it. ... And then run through some examples".

Sam completed his practicum in a difficult school in a low socio-economic area and he quickly became preoccupied with classroom management issues. His supervising teacher spent much of the time "screaming to keep the kids under control. The amount of learning wasn't that great, but she kept them under control". Early in his practicum Sam worried that he was becoming "too focussed on trying to implement the lesson as I planned. Hopefully as I progress I'll be able to focus less on what I'm trying to teach and more on the discipline." He later described one of his best lessons as one where "I didn't get much discipline issues".

Sam continued to report difficulties in the classroom, so his supervising teacher suggested that he try hands-on activities in lessons as a means of improving student behaviour. She provided him with many resources he could use, but as she did not model this approach in her own lessons, Sam was unsure about how to implement them effectively. As a result he found lesson preparation time consuming and saw little benefit in the classroom. He commented that the practicum was "wearing me out and I get de-motivated". Sam could not explain why his classroom management problems continued nor could he offer any alternative strategies for dealing with them other than persisting with activities as suggested by his supervising teacher.

It was only after the practicum, when Sam had been working as a teacher in a school with similar management issues to his practicum school, that he began to recognize some other sources of the difficulties he experienced in trying to manage his classes. In the final interview, he commented on students' poor behaviour saying, "I think it could stem from students not understanding or having learning difficulties". He also spoke about some of the unruly classes he was currently teaching, "I don't like teaching something that's simply just got to be taught for the sake of it; and that confuses them so then I have class management problems. It's sort of a vicious circle". Sam realized the futility of teaching some topics, like fractions, when students "don't know their tables and can't work with whole numbers". Instead, Sam said he would prefer to devote more time to basic numeracy but felt he was under "pressure to cover the program". He commented, "As a first-year teacher you're afraid to tread on toes, you don't want to do anything that's too 'out there' because you're on probation so you don't want to be taking too many risks".

John's reflective thinking

John saw mathematics teaching principally as "clarity in explanations where everyone understands" since this was what he remembered most from his own schooling. John commented that he expected his practicum classes would be "well behaved, motivated, like I was when I was in secondary school".

John's practicum occurred in a school which he described as "good, with minor class management issues". He characterized a successful lesson from his supervisor as "chalk and talk" with "content delivered quickly and clearly" and "key principles well explained". However, as John heard more about alternate pedagogies at university he began both to question some of the practices of his supervising teacher and to recognize the limitations of his own schooling. He described constructivism and the working mathematically approach to teaching as "a breath of fresh air that changed my attitude to teaching dramatically" but he was disillusioned that "the methods we discuss at uni [are] not practised at school". For John, the disconnect became more apparent when he tried unsuccessfully to implement this more student-centred approach in his lessons. He recognized that his difficulties arose because students were not used to Working Mathematically and the "school culture doesn't promote this". Even though John was "allowed to do lessons my way", he resolved to follow his supervisor's more traditional style and continue to "actively discuss [with the supervisor] what we learn at uni". John described how his supervisor was "lending a sympathetic ear" and had challenged John with "if you can prove to me that [Working Mathematically] can work, I will try to adopt it more". However, the challenge proved too great for John and, as his subsequent attempts at Working Mathematically did not succeed as he hoped, he began to question the efficacy of the approach because he did not think he could sustain it and complete all of the syllabus content. He concluded that it was better to "proceed with caution", adopt a transmissive style of teaching, and build his confidence.

Later, as a first-year teacher, John repeated his claim that students were not prepared to engage in higher-order thinking required by the working mathematically approach because they want "memory work, a lot of drill". Hence "it's easier to develop your teaching skills in an instrumental teaching scenario and once you've got the basic skills, then you can start to try new things". John reconfirmed his ideas about clear explanations in teaching, noting, "your ability to explain things must be flawless". He stated, "I'll be focussing on my ability to explain things, try and keep at the right level, use the right words".

Peter's reflective thinking

Peter spoke about his passion for teaching noting that in high school and all through university he had "spent a lot of time teaching other students" and felt he was "good at it". His best mathematics teacher was one whose lessons were "quite unstructured and we could explore things we wanted to explore. So lessons were a lot more discovery-oriented and we did the exercises for homework". Peter wanted "to inspire students' interest in maths" and show students who may not like maths how it can be "applicable and important in their lives". He said he wanted to "stretch" the students.

Peter's practicum took place in a comprehensive high school where achievement levels in mathematics had historically not been high. Peter described his supervisor as a "traditional" teacher whose lessons were always "very structured with a quick quiz, review of homework, examples, and well graded practice exercises". He concluded that "students need structure so lessons are predictable" and described one of his own lessons on angles as successful because "they [students] remembered the rules next lesson". Peter also reported that the lessons he designed to engage students more in their learning had little success as the "class wasn't used to doing group work". He also commented that his supervising teacher had very strong views about how mathematics should be taught and expected Peter to follow a transmissive style. Peter said he became frustrated when his supervisor did not support a working mathematically approach and when he criticized Peter for the lack of practice exercises he provided for students. Peter eventually decided that a "good [practicum] report necessitated following his way", but he looked for opportunities to try out new ideas when his supervisor was absent from the class.

In the fourth interview, Peter described being a full-time teacher as "a real high" because he was "finally able to do what I wanted to do" and had "more freedom to try things, to experiment" rather than having to "please my supervising teacher". He now tried "to begin [lessons] with concrete examples and generalize them to formulate the rules" because that was his own preferred learning style. However, Peter noted that he could not take an investigative approach in every topic due to work pressures such as increased preparation time and the need to keep up with parallel classes for examinations. He recognized that it was "easier just to pick up the textbook" sometimes because "students [are] not interested in taking up the challenge of Working Mathematically". Peter commented that he was "still learning the content" since he had not taught most topics before, so he was relying on a textbook more than he would like but would adopt his preferred Working Mathematically style as he felt "more confident".

DISCUSSION AND CONCLUSION

The participants' initial reflections were at the level of *Descriptive Recall* (e.g., Sam's concern about student behavior, John's desire to provide clear explanations, and Peter's comments that students prefer more structured lessons). The strong focus on technical aspects of teaching is consistent with previous studies, highlighting the dominance of such matters in the practicum (Moore, 2003) as an essential first step in developing one's reflective practice (Shoffner, 2008).

Some development in the participants' accounts of their reflective thinking did occur, albeit in varying degrees, to the level of *Practical Rationalization* (e.g., Sam's acknowledgement of student learning difficulties as the likely source of their poor behaviour, and John's growing realization that adopting a working mathematically approach was made more difficult due to the pedagogical style of his supervisor). Peter made progress towards *Critical Reflection* (by realizing that waiting until the practicum ended would allow him to adopt student-centred activities more freely in his own classes). All participants showed greater capacity for critical reflection on their practicum experiences in the final interview, perhaps confirming Schön's (1987) view that significant reflection may occur "only when a student moves out of the practicum into another setting" (p. 299).

The practicum classrooms of our participants were perhaps not ideal settings for them to experience Working Mathematically in practice, but we do not believe they have to be so in order for critical reflection to occur. However, practicum classrooms do need to "provide enough room for student teachers to envisage alternatives, interact with children, and try out new approaches" (Ebby, 2000, p. 95). The reports of all three participants in our study indicate that there was little space made available for such considerations during the practicum so it is perhaps not surprising that there was little evidence of critical reflection. All three case studies demonstrate the crucial role of the supervising teacher and confirm the supervisors' tendency to give advice rather than help student teachers "problematise what they observe, and to reflect on what might be done to improve things" (Jaworski & Gellert, 2003, p. 836).

Our results indicate that reflective thinking can be risky and daunting, particularly for beginning teachers who lack the insights that can come with greater classroom experience. Despite opportunities for reflection in the university program, the participants made disappointingly little progress in developing their reflective practice. Perhaps the often stressful nature of the practicum makes self-reflection too difficult for novice teachers, particularly if they attach little importance to reflective practice or perceive it to be of little benefit. Instead, it appears that the work of reflection is more likely to be done when beginning teachers make the transition to their own classrooms where they can shape their professional identities more freely.

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INTERACTIVE PATTERNS IN A TAIWAN PRIMARY FOUR MATHEMATICS CLASSROOM

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Class interactive patterns in a Taiwan primary four mathematics classroom are explored in this study. Lessons conducted in a mathematics classroom and interviews after each lesson, are video recorded for two weeks to collect data. The pattern, students write and teacher interprets, is one of the primary class interactive patterns found in this study. In this pattern, each student plays as a problem solver and a recorder, while teacher plays as interpreter. It can be divided further into three sub-types, which are students write and teacher selects one record, students write and teacher supplements, and students write and teacher summarizes, based on the process how teacher elaborates after students solve a problem.

INTRODUCTION

After long-term interaction, some interactive patterns always appear in a classroom (Bauersfeld, 1988; Richards, 1991). These interactive patterns not only guide students in class, but also have a great impact on their behavior outside the class. When students are imbued thoroughly with what they frequently hear and see in a classroom, the way they behave and think in the classroom may eventually become their own habit. The longer the period of interaction is, the deeper students are influenced. In the end, the patterns are more likely to influence their ideas about what mathematics is (Bauersfeld, 1995). On the other hand, these class interactive patterns, to a teacher, are of great help in their teaching. Therefore, the objective of this study is to explore class interactive patterns in a mathematics classroom.

There are two types of study regarding characteristics of class interaction. The first type of study distinguishes class interactive patterns in the features of behavior that teachers and students express during class interactive sequences. For example, researchers using an emergent perspective have identified "funnel pattern", "focusing pattern", "elicitation pattern", and "discussion pattern" (Bauersfeld, 1988; Voigt, 1994; 1995; Wood, 1994). In *funnel* patterns, the teacher pops out a series of questions to narrow down possibilities until students figure out the teacher's presupposed solving procedure. The characteristics of a *focusing* pattern is that the teacher asks the students a question to indicate a key point and then leaves them to solve the problem (Wood, 1994). In this type of study, the nature of teachers' and students' interaction is not depicted specifically enough. Class interactive patterns are formed by a series of interactions amongst students and a teacher. Therefore, different participants develop different class interactive patterns. Even two classes, which are both categorized as

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having the "discussion pattern" mentioned above, have different interactive sequences. This explains why it is crucial to do case studies in individual classrooms.

The second type of study specifies discourse patterns in classrooms with decontextualised behavior sequences. Generally speaking, "initiation-response-evaluation (IRE)" (Mehan, 1979) is the most common type used in classrooms. This may analyze the discourse in classroom exactly, but it is totally decontextualized. Therefore, it is unable to reveal a complete picture of interaction amongst teacher and students in the process of explanation and discussion.

Based on Maturana's (1978, 1987) theory of coupling, this study considers social interaction patterns as repeating behavior sequences of hint-choice occuring among individuals in a community in order to make the social system work. This study uses the ideas of *hint*, which results from an individual's interpretation of others' signs, and *choice*, which is the responding action chosen thereafter by his/her own. When this operational definition is applied in an exploration of class interactive patterns, teacher's and students' behaviors and interpretations are analysed into a complete picture of the interaction. How a teacher explains a concept to students and their interaction during the process are recorded specifically and carefully in this study, therefore, the flaws occuring in previous studies can be avoided.

METHODOLOGY

Research participants

A class taught by Mr. Wu is chosen as the object of this study. Since the study uses four video recorders in the classroom and interviews with teacher and students after class for two weeks, the teacher's cooperation is crucial. The class, Mr. Wu's first primary four, has been taught by him for two years. According to Mr. Wu, most of the students are getting distracted easily, less passionate for knowledge, and not active. In terms of their performance, the students who score high marks listen with interest; those who are mediocre are quiet, and the last one-fifth who have given up do not even know where the teacher is teaching. The sitting arrangement in the classroom is three groups, formed by five to six students per group in the front, and three in the rear.

Data collection and analysis

During two weeks that this study collected data from this classroom, four lessons on angle and four lessons on addition and subtraction of decimals were taught. In this study, class interactive patterns are considered as types of repeating hint-choice behavior occurring amongst teacher and students in a mathematics classroom. Similar choice behavior, which is triggered by the same signal, may be generated by different ideas. Based on the reasons stated above, this study collects data by video recording the class and interviews after class. Four video cameras are positioned in the classroom to collect four types of data. These are Teacher Camera, Group Camera, Teacher-view Camera, and Group-view Camera. Teacher and students are both interviewed. Here is how the interview with teacher is taken. The observer and teacher watch the classroom videotapes together and the observer asks the teacher questions spontaneously whenever any doubt about the teacher's or students' behavior arises. By doing so, the observer is able to understand the teacher's intentions as well as the trigger point of his behavior.

There are three steps in data analysis. The first step is to transcribe and to edit all videotapes to protocols. Eight protocols, each of three types of data, including from the classroom, interview with teacher, and interview with group members are produced. In step two, all protocols are divided and classified. Interview protocols are classified according to the corresponding results of the classroom protocols. Lastly, models are built. To avoid flaws of decontextualization in previous studies and to stress the mutual influence occurring in the interaction amongst teacher and students, one complete behavior sequence of hint-choice in certain situations is presented accordingly with each class interactive pattern. To show the suitability of the explication, protocols of classroom and interviews are listed as instantiations.

CLASS INTERACTIVE PATTERNS

Student writes and teacher interprets is one of the primary interactive patterns in this case study. In this pattern, students act as problem solvers and recorders, while Mr. Wu acts as interpreter. This pattern is divided further into three sub-types, based on how the teacher elaborates students' problem solving records.

Pattern 1: Students write and teacher selects one record

Teacher gives a mathematics problem and assigns one student from each group to come to the front and solve it→Representatives from each group solve problem on black board while others work on their seats→teacher selects the record, which he intends to explain, and teaches in a brief conversation

Since Mr. Wu hopes for students to solve certain problem with specific methods, the records prepared by students in this pattern are only some materials ready for the teacher to choose and meet his own purpose. For example, when representatives from each group write down their working of 2-0.63, the teacher does not invite anyone to explain his/her own record. Instead, he chooses one record to explain and offers a strategy, adding zeros behind the number, to help solving the problem. Protocol One is listed below.

Protocol 1

1	Teacher:	This problem, (teacher points at group 3's working of "2.00-0.63=1.37"), 2 zeros are added after two, right? Let's say, if you don't look at it (teacher covers decimal point of 2.00 with finger), it's 200 minus 63, isn't it? In this case, can you do it?
2	Jun:	Yes.
3	Teacher:	Zhi, group 5, look here. 200 minus 63 can be treated as whole numbers and get the result of 137, right? When you put decimal point back, it becomes one point three seven, doesn't it? So, every time when you see whole number with nothing following and you

are afraid you don't know how to do subtraction, what can you add?

4 A few students: Zero.

5 Teacher: That's right, add zeros in the back.

In line 1, teacher selects the record by group three and explains further. His own interpretation of this record is to think of this question as a subtraction between two whole numbers, that is 200-63 and put decimal point back afterwards. Yet, it remains unknown if the teacher's interpretation is compatible with the original writer's, since the writer has no chance at all to present his/her own idea.

In the interview with the teacher, the teacher mentions that his teaching targets here are actually Bao and Pei while he knows other students are capable of doing such work. Some students do notice the teacher's intention when this kind of situation happens and they not only dislike it but also interpret it as making efforts to help students with low IQ.

Pattern 2: Students write and teacher supplements

Teacher gives a mathematics problem→each group solves the problem on one white board→ teacher requests each group to provide its solving record→one student from each group submits their white board and puts it upright on black board→teacher compares all records and makes notes of their similarities and differences→if teacher discovers certain possible methods listed in textbook are missed from all students' record, he tries to supplement the missing methods by propping questions and giving hint→students answer his questions over and over again→as teacher finds students are unable to give the answer he expects, he stops giving hint but announces the answer; students are requested to make a record of it afterwards

In fact, all solving records prepared by students in this pattern are solely reference for Mr. Wu to judge if he needs to add anything further, since the teacher expects students to know every single method introduced in the textbook. For instance, when the students from each group hand over their records of changing angles as the teacher requests, the teacher does not invite anyone to interpret their records, on the contrary, he compares similarities and differences of all records. The following interactions occur when the teacher finds no signs of the method 'arc', shown as one of the methods children may adopt in textbook, in the records of each group.

Protocol 2

1 Teacher: Look. Now I open a fan slowly (teacher opens a fan and then close it), what is changing (teacher moves in arc edge of a closed fan with finger)? Up (teacher points at the circular edge of a closed fan), what happens to this arc (teacher opens a fan)?

2 Many students: Bigger and bigger.

3 Teacher: Longer and longer, have a look again, who can draw this (teacher opens a fan)? What can show the edge, point, and degrees, what can we draw (teacher draws in arc edge of a closing fan with his finger)?

		What can we draw (teacher opens a fan and then close it up and then moves along circular edge of the fan)?
4	Someone:	Degree.
5	Teacher:	Like running the athletic ground, some people run one round and some run half round, if we pull the trip they've run into a straight line, who run longer?
6	Some students:	One round.
7	Teacher:	One round is longer, isn't it? This is the same (points at a fan). People run on the fan, one runs this long (teacher opens the fan), one runs this long (teacher opens it wider), these two arcs aren't the same, are they? So when we present an angle, we may not need to draw things inside (teacher points at an angle with an arrow inside on a white board), but arc outside (teacher draws arc in the air). Now, you draw. Look (teacher takes a fan out). If you have white board on hand, draw it. Here, group 3, group 3.

In line 1, 3 and 5, Mr. Wu starts to give out some hints and hopes for students to offer the fourth method of recording, 'arc'. These hints appear in his language, "now I open a fan slowly, what is changing," and in motions, "move in arc edge of a fan with your finger". During the process, students continue to give the teacher answers, yet disappoint him over and over again. At last, in line 7, the teacher announces the answer, "when we present an angle, we may not need to draw things inside but the arc outside," and then requests them all to make a record of it.

In the interview with the teacher after class, Mr. Wu gives his own explanation about this interaction, which is quoted below. "They have written down standard ones, however, there is still one, arc, which I never mentioned. It's in the textbook and I think it would be nice if they can figure it out on their own, so I supplement length of arc here." When the teacher is asked if arc is an option in recoding an angle in mathematics, he replies "no" confidently. In another words, the teacher supplements one method listed in the textbook as one of the means children may utilize when that is missing from the students' records, even though he knows clearly that the method is rarely employed.

Pattern 3: students write and teacher summarizes

Teacher gives a mathematics problem→each group solves the problem→teacher invites one student from each group to provide their solving record→a representative goes onto stage to write down their record on black board→teacher summarizes all records on black board and presents to the class

In this pattern, due to Mr. Wu's teaching habit, induction, all records from the students become his teaching materials in the end. For example, after each representative writes down their answer to the problem, the result of decomposition of one, the teacher does not invite anyone to present their work but sums all records up and explains. Their interaction is shown below.

Protocol 3

1	Teacher:	… 0.1+0.9, 0.2+0.8, 0.3+0.7, 4 and 5, and 5 and 5, right? Only these five pairs, right? So, strip by strip, there are only these five pairs, aren't they (teacher shows them hundred square board)? It's ten strips in total and if one strip is 0.1, there are only five pairs when we match them together. The rests are combination of two bits then?
2	One student:	Right.
3	Teacher:	Is there one bit plus two bits?
4	Some students:	No.
5	Teacher:	The rests are two decimal bits plus two decimal bits. Ok. Have a look here. What's this (teacher circles 0.99+0.01 in record of group 6) then?
6	June:	0.99 plus 0.01.
7	Teacher:	0.99+0.01, right? Look at the black board. Is it only 1 (points at 0.01), and another is 99 (points at 0.99)? This one is two decimal bits together, too. The second decimal fraction (points at 0.01) has nothing on tenth decimal bit.
8	Teacher:	(Teacher erases part of answers by group 5) Ok. Look (teacher circles 0.39+0.61 by group 4) 0.39 and 0.61, one is 39 and the other is 61, correct? What's the difference (teacher links 0.39+0.61 by group 4 and 0.99+0.01 by group 6) between these two?
9	(No one replies)	
10	Teacher:	One decimal bit is …(points at tenth decimal bit of 0.01)?
11	Some students:	zero.
12	Teacher:	One decimal bit is zero (points at 0.01) and these two both have numbers on tenth and hundredth decimal bits (points at 0.39+0.61).
13	Teacher:	These are about all. One is one decimal bit plus one decimal bit (points at 0.1+0.9), the other is two decimal bits and two bits. Yet, there are two sub-types in this kind, one is zero on tenth decimal bit (points at 0.99+0.01), isn't it? And the other is both decimal bits with numbers (points at 0.39+0.61). Any problem so far? . …

In line 13 lies the results of decomposition of one from the teacher's induction on students' records- two numbers with one decimal bit (i.e. 0.1+0.9); two numbers with two decimal bits, yet the tenth decimal bit in one of them is zero (i.e. 0.99+0.01); and two numbers with two decimal bits and none of their tenth decimal bit is zero (i.e. 0.39+0.61).

The original problem in the textbook is "please draw on the hundred-squares board and try to decompose into two parts. Explain how you do this and think if there is another way." In the interview with the teacher, the teacher explains why he revises the original problem and says: "I find that I can only ask students to raise hands to see which type of method he/she adopts if I follow the textbook and present the three types of results listed in it. Actually, I wish to discuss this topic and find more answers so I apply this method." From his reply, it shows that the teacher does notice the results in textbook, which are 0.1 and 0.9 sheet, 0.25 and 0.75 sheet and 14/100 and 86/100 sheet, but disagrees with the teaching methods it suggests, which is that students raise their hands to express their own choice. He also feels that it is important to look for more answers to do further classification. Thus, the teacher revises the problem- which group gets the most answers of decomposition of one in two minutes to dig out more answers for further classification and adopts two teaching modes, group discussion and competition, to motivate students for more answers.

DISCUSSION

This study reveals three class interactive patterns of 'students write and teacher interprets' based on the operational definition of hint-choice behavior sequence in a mathematics classroom. The method conducted in this study eliminates the shortcomings of non-specification and decontextualization found in previous studies (Bauersfeld, 1988; Mehan & Wood, 1975; Mehan, 1979; Voigt, 1994; 1995; Wood, 1994) and also provides participants (teacher and students) chance to express their thoughts. It is suggested that exploration of the interactive patterns between different units in a mathematics classroom, such as pattern of teacher and each group, pattern of teacher and an individual, pattern between two students or more in a group etc., could be made in the future.

It must be noted that the class interactive pattern 'students write and teacher interprets' found in this case study is very common in Taiwan. In the three sub-types - teacher selects one record, teacher supplements, and teacher summarizes - it is always the teacher who asks questions and explains, and students do not get any chance to present and to express their own ideas. Although a few teachers do regard the solving records prepared by students as the proof of their participation in class discussion, it cannot be called discussion at all, since students never argue and debate over their own concept, but listen and answer the teacher's questions passively. Thus, it brings up concerns that, in the long run, students may misunderstand the meaning of class participation and think that presentation and argumentation are merely the teacher's duty. As the teacher's habit of lecturing or his misunderstanding of mathematics discussion may be the cause of this problem, it is important to let teachers speak about the difficulties the face in the classroom. Moreover, a model of a mathematics classroom can be developed upon teacher's request, so that the teacher is able to become aware of the patterns of his/her own classroom and so improve his/her way of teaching.

This study finds that the teacher's interpretation of the textbook and his own beliefs have a great influence on his reactions in class interaction. In protocol 1, one problem, which may be caused by teacher's belief, is found, that is, the teacher spends most of the time making a few students understand one mathematics concept during a lesson. Also, there is one common problem, misinterpretation of the textbook, that is, the teacher does not know that methods listed in the textbook are solely examples to show the teacher what children may write. Without reading the teacher's guide beforehand, some confusion may arise and cause misinterpretation. For instance, in protocol 2, it is found that the teacher presents a method to record an angle, arc, while he knows that is rarely employed. In this case, it is arguable that arc is shown as one method that children may utilize in the textbook without further explanation. However, the teacher should notice it before the lesson and refer to the teacher's guide or discuss with other teachers. Furthermore, in protocol 3, the teacher revises the original problem in the textbook, to list possible solutions of decomposition of a hundred-square board, and eventually summarizes three ways of decomposition. Here, he overlooks the combination of three digits (i.e.0.008+0.992) and that of four digits etc., while he over-interprets the problem in the textbook. Last but not the least, this study strongly suggests that all teachers must study the teacher's guide carefully before teaching, but go to class and interpret with common sense.

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A REALISTIC CONTEXT DESIGN ON ALGEBRAIC GRADE SKIPPING LEARNING FOR EIGHTH GRADERS

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The purpose of the study was to explore how the "realistic context" instructional design influenced 8th graders' performance on algebraic grade skipping learning of "linear programming". A quasi-experimental design was employed in this study. Samples were selected purposely from thirty-six 8th graders of a junior high school as the Experimental Group, while seventy-nine 12th graders of a senior high school were chosen as the Control Group. Findings indicated that there was no significant difference between 8th graders (Experiment) and 12th graders (Control) on the performance of the linear programming achievement test. The instructional material with a realistic context design used in this study did help students to learn the abstract algebra effectively.

INTRODUCTION

Background

In the highly civilized world of 21st century, mathematical knowledge and abilities became basic requirements in our daily lives and careers (Lin, 2003; Polya, 1945; Romberg, 2001). In our daily lives, we actually used the most popular mathematic ideas and theories, not even recognizing that we were living with mathematics. The more one knew about mathematics, the broader one may develop in his career life (Stein, 1999). However, mathematic educators reviewed mathematical teaching materials used in the past and found that they were designed with an emphasis on mathematician and adult thinking approaches. In addition, the instruction emphasized heavily on abstract mathematical symbols and the training of calculation skills (Huang, 2003). The de-conceptualized, de-experienced thinking of the materials also ignored the cognitive principles and the processes of discovering the truth. Since mathematic teachers overvalued symbolic calculations and proofs as well as answers for mathematical questions, students may naturally sense that mathematics was disconnected with real-life situations; and then they may form the habit of insisting more on calculation skills for answers instead of on the process of thinking and reasoning. Thus, more and more students thought that "math was a boring symbolic game which concerned nothing about the real-life", or "math was so difficult to master (learn)" (Wu & Ye, 2002; Zheng, 2003).

According to Piaget's (1970) theory of cognitive development, the junior-high students were at the right developmental period of "formal operational period", and they should be able to practice hypothesis and deduction with abstract symbols (Huang, 2001). Therefore, most of countries put the concept of written symbols into the

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mathematical curriculum at 7th or 8th grades. However, there were a large number of students facing problems in algebraic learning (Huang, 2001). A possible reason was because students' mathematical learning experience in the past (in the elementary level) was mainly about concrete practices of numbers and graphs. Consequently, when it came to more abstract algebraic content, if the instruction were not designed properly connected with their life experience and prior knowledge, the degree of learning difficulty would be then notably increased.

In recent years, our government endeavored to work on the educational innovations to enhance the civic quality and national competitive ability (Ministry of Education [MOE], Taiwan, 2000). The implementation of the Grade 1~9 Consecutive Curriculum was the core in this reform, where teachers could design curriculum and instruction based on their instructional belief and goal in order to match local conditions and students' personality and special needs (MOE, 2003). Based on the purpose of this reform, the research team applied the theory of Realistic Mathematics Education (RME) (Freudenthal, 1973; Lange, 1995) to an instructional designing for eighth graders' learning in algebra. Both students' prior knowledge and experience were considered while designing this algebraic learning material, as well as how the content was closely related to their real-life situations.

RME was initiated by Freudenthal and his companies in the early 70'. According to the ideas of "mathematics as a human activity" and "mathematics must be connected to reality", Freudenthal (1973) developed the theory of RME. The theory of RME advocated that mathematics education should be based on students' cognitive development, and focused on the real life context. Thus, students could actively construct their own knowledge gradually with the contextual questions containing mathematical concepts, and then realize the mathematic relations and laws through experiences. Moreover, through interactions with classmates and teachers, students may gradually advance their thinking level, and further internalize the mathematical concepts. Besides, scholars in the United States progressed an experimental innovation on math curriculum (Mathematics in Context, MiC), which was similar to the concept of RME and greatly successful (Romberg, 2001). Except for providing an interesting and comfortable learning environment for learners to discover and learn (Freudenthal, 1973 Lange, 1987), RME also emphasized that students should gradually mathematicalize and upgrade their study level through trials, interactions, and discussions. At the end, students were able to concretely handle the knowledge of formal and structural mathematics for effective implementations (Freudenthal, 1991; Treffers, 1987). Therefore, RME was correspondent to the requirements of modern mathematic education, which valued both processes and outcomes. In this study, the research team believed that, through a well-designed mathematical instruction based on RME, students could learn easily while facing the traditionally difficult algebraic unit. They would also be more confident and motivated in learning math because they are trained with methods and habits of active learning in a positive and interactive learning context. Finally, better learning achievements would be accomplished.

To proof that the mathematical instruction with a realistic context design based on RME was truly helpful for normal students in their algebraic learning, the research team chose an algebraic unit from the curriculum of “senior high schools”, and then experimented it as a grade skipping learning process on 8th graders. Students were guided by the contextual questions within this instructional material, and then might perform self-construction and social-construction in the learning process. Their thinking and reasoning abilities might be cultivated and inspired. Further, their learning achievement on the algebraic unit would be promoted. After analyzing and comparing algebraic units among curricula of junior and senior high schools, “linear programming” in 12th grade was selected to be the experimental unit, which was highly practical while facing decisions in daily lives (Fang, 1993)

Purpose

Accordingly, the main purpose of this study was to explore eighth graders’ achievement on algebraic grade skipping learning by using mathematical instructional material with a realistic context design. Besides, evidences regarding to what problems they faced while learning and their feelings and solutions were also collected as a reference.

RESEARCH DESIGN

Method

A quasi-experimental design was employed in this study. Samples were selected purposely from thirty-six 8th graders of a junior high school in Changhua County, Taiwan, as the Experimental Group (Experiment), while seventy-nine 12th graders of a senior high school were chosen as the Control Group (Control). The Experimental Group was taught by the mathematical instructional material in a realistic context design for exploring its influence on eighth graders’ algebraic grade skipping learning. After the instruction, the post-test, the linear programming achievement test, was administered to both groups.

The mathematical instructional material in a realistic context design used in this study was designed by the research team based on the theory of RME. The content in this instructional design was extracted from the Nan-I High-School Textbook. There were eight units in this material. The first and second units were designed for reviewing what students learned before (their prerequisite knowledge, linear inequalities with two variables and its graphic solution). The third, forth, and fifth units introduced the main concept of “linear programming”. By considering the local conditions around the targeted schools (i.e. it was a rural and agricultural environment; rice was the main product in that area), the content in the textbook was irrelevant to these students’ life experience. Accordingly, a new version of the instructional material with a realistic context design, closely relevant to their daily lives, was designed to cope with the farming, the agricultural product, and its sale business. A series of the contextual questions were also developed in order to guide these students to learn the mathematical concepts. The sixth unit focused on the application of the knowledge on linear programming, which aimed to reinforce what they learned in previous units. Besides, this instructional material was implemented associated with the cooperative learning model. Thirty-six eighth graders in the experimental group were teamed up heterogeneously for group activities. There were six teams totally in the experimental group, six students for each team. Another 79 twelfth graders from a senior high school served as the control group, which implemented a traditional instruction with the original content of the textbook.

Moreover, based on the instructional material used in the experimental group and the original textbook used in the control group, a linear programming achievement test was developed to examine these students' learning achievement. This fifty-minute test was sent to two professors, three high school math teachers, and two junior high school teachers for establishing the construct validity. The revised version was then administered to 52 twelfth graders of another high school in Changhua as the pilot test. According to the analysis of the pilot test, 20 questions were chosen based on the analysis of item difficulty index and item discrimination index, which included 5 multiple-choice, 6 fill-in, and 9 calculation questions. The average difficulty index was .65 and the average discrimination index was .45. Table 1 showed the feature of these 20 items corresponding to Bloom's cognitive levels.

Content	**Knowledge**	**Comprehension**	**Application**	**Analysis**	**Synthesis**	**Evaluation**	# of Items
Linear Inequalities with Two Variables	**A** **B** **C**	**A** **B 1(1)** **C**	**A 2** **B 2** **C**	**A 1** **B** **C 1(3)**	**A** **B 4(1)** **C**	**A** **B** **C**	6
Linear Programming	**A** **B** **C**	**A** **B 1(2)** **C**	**A 4(1)** **B 3** **C 1(2)**	**A 3** **B** **C 2(2)**	**A 4(2)** **B** **C 2(3)**	**A** **B** **C**	8
Application of Linear Programming	**A** **B** **C 1(4)**	**A** **B** **C 1(1)**	**A** **B 4(2)** **C 1(5)**	**A** **B** **C 2(1)**	**A** **B** **C 2(4)**	**A** **B** **C**	6
# of items	**1**	**3**	**7**	**5**	**4**	**0**	20

Note: "A" is "multiple-choice question"; "B" is "fill-in question"; "C" is "calculation question"

Table 1: Items of the linear programming achievement test

Data were mainly gathered by the linear programming achievement test after executing the instruction[1], associated with supplementary qualitative data collected from researchers' journals, students' learning records, and informal after-class interviews. Statistical analyses were performed to answer the research question. Quantitative and qualitative findings will be integrated to provide a thorough understanding and discussions of the algebraic grade skipping learning by using analysis in context strategy.

FINDINGS

Performance comparison in linear programming achievement

In order to explore the difference between the learning achievement of Experiment and Control, a simple comparison was made on the average grade of each group. It showed that Experiment had the average score of 51.11 points, which was a little bit higher than Control, 50.28 points. Table 2 showed the summary of frequency and percentage in four different score levels for both groups. A chi-square significant test was carried out and the result was not significant, $\chi^2_{(3)}$□.285, p□.963□.05. Further, the Shapiro-Wilk normality test was conducted, which was not significant for both groups, (Experiment) p .856 .05 and (Control) p .321 .05. In addition, the result of Levene's Test showed that F (1,113) .005, p .945 .05. These two results indicated that these data conformed with three basic assumptions: interclass independence, normality, and homogeneity of variances.

Group	0-25 pts	26-50 pts	51-75 pts	76-100 pts	Total
Experiment	1(2.8%)	18(50.0%)	15(41.7%)	2(5.6%)	36
Control	2(2.5%)	38(48.1%)	36(45.6%)	3(3.8%)	79
Sub-total	3(2.6%)	56(48.7%)	51(44.3%)	5(4.3%)	115

Table 2: Summary of frequency and percentage in four different score levels

Moreover, a *t*-test was conducted to examine the difference between students in Experiment and Control for their performances on the linear programming achievement test. The result was not significant, t (113) .280, p .05. This result clearly showed that, after receiving the experimental "algebraic grade skipping instruction (i.e. the instruction with a realistic context design)", there was no significant difference between Experiment, 36 eighth graders, and Control, 79 twelfth graders, on the performance of the linear programming achievement test. In fact, students in Experiment scored a little bit higher than those in Control (mean difference was .83). Consequently, t mathematical instructional material in a realistic context design did assist eighth graders to learn the abstract algebra (the content usually taught in 12th grade) effectively.

Group	N	M	SD	t	df	p
Experiment	36	51.11	15.039	.280	113	.780
Control	79	50.28	14.651			

Table 3 Summary of the *t*-test

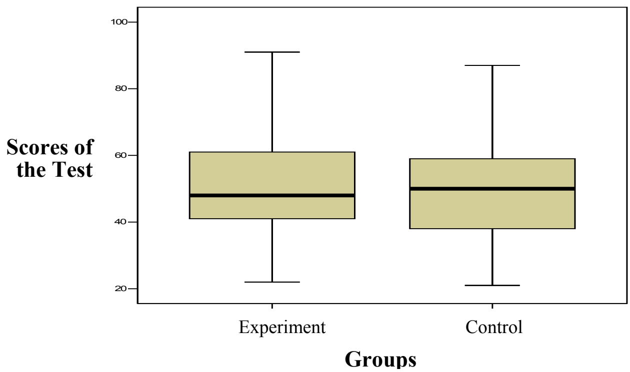

Figure 1 Distribution of the linear programming achievement test

Eighth graders' feelings while facing learning problems and their solutions

First, through the informal after-class interviews, over half of students in the experimental group thought that this instructional material with a realistic context design were more lifelike and easier to comprehend, which did help them better understand what they learned and how to apply what they learned into practice. They also thought it was a friendly, vivid, and interesting design. Besides, for the cooperative learning model, students in Experiment felt more freely to ask questions to classmates, express their own opinions, and listen to others within groups. During the process, they also felt the advantage of positive competitions within or among groups, which promoted them to be more active in learning the linear programming. Finally, they thought this kind of deign (the experimental instruction) was better than traditional ways, especially for those who were low-achievers. Therefore, they were expecting to experience more in their future learning processes.

Secondly, most of Students in Experiment held positive opinions and evaluations toward this instructional material with a realistic context design. According to students' responses (in their learning records), "I do not know how to list the equation" was the most common problem they faced in the learning process. "I do not understand the meaning of some question, so I can not answer the following questions" and "the classmates are not cooperative" appeared to be other problems or complains. Moreover, most students tried to "discuss with classmates" to solve the problem faced. This was probably why "going to the teacher" became not so popular for their problem-solving strategy during the learning process of the last unit.

DISCUSSION AND CONCLUSION

The main purpose of this study was to improve the learning deficiency in algebraic learning and to enhance students' achievement in algebra. BY using this grade skipping experimental design, the research team intended to find out an effective way to benefit these students' leaning in abstract algebraic concepts. The traditional algebraic course usually represented the most simple, formalized symbolic system and the calculation laws. Such a de-conceptualized and de-experienced design of teaching material was not able to connect students' knowledge and experience with real-life situations. Therefore, the degree of learning deficiency naturally increases (Huang, 2001). Besides, we found that eighth graders' algebraic learning abilities can be trained. Weng (2003) thinks that it is an infeasible thought to trade students' happy learning with simplification of teaching materials. Such a thought only weakens students' knowledge level, and even worse, limits the development of students' potential. In Romberg's (2001) study, learning achievement of high school students in U.S.A. and Netherlands were obviously advanced after receiving mathematical curriculum and instruction in realistic context. While in this study, eighth graders' achievement in the algebraic grade skipping learning of linear programming is greatly closed to the normal twelfth graders' achievement. Thus, what junior high students lack in algebraic learning may rather be the appropriate material and instruction than the ability itself. The only meaningful method for students shall be to design a well-planned material and instruction and practically apply them into practice. By doing so, their mathematical knowledge and abilities will be truly promoted, and the national competence will then be raised as well.

In summary, the instructional material with a realistic context design used in this study did help students to learn the abstract algebra effectively. Further, in this algebraic grade skipping learning, evidences indicated that students' potentials in learning could be promoted once an appropriate theory and corresponding instructional strategies were employed in the learning process. In fact, human brain's potentials in forming dispersions and connections are boundless (Wu & Chang, 2006). According to the recent findings of brain-based research, human brain is a seeker of models and meanings, which is characterized with self-invention, and will automatically learn numerous significant and useful knowledge and abilities in a learning context that is full of meaning and context (Wu & Chang, 2006). Consequently, educators must take the responsibility of creating real-life learning environment, applying authentic learning activities, and encouraging interactions among all students in order to reach the ultimate goal of improving their learning process and equipping them proper knowledge and capability for their future lives.

Endnote

1. Actually, for the experimental group, a "mathematical learning attitude scale" was also administered as pre- and post-test for the purpose of understanding whether the instructional material with a realistic context design could influence these eighth graders' learning attitude or not. In addition, it was also to explore how the two factors, "gender and ability", would influence upon their performance in algebraic grade skipping learning achievement. However, because of the page limit, these results were excluded from this report. We will present all results to PME 33 participants in the conference.

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DISCOURSE TO EMPOWER "SELF" IN THE LEARNING OF MATHEMATICS

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This paper discusses what and how specific features of whole-class discourse and the teacher's actions support and promote learner-focusedness in learning mathematics. A learner-focused perspective based on agency, collaboration, and reflection frames this study of classroom discourse. Analysis of data obtained through interviews and classroom observations produced four categories of self-reflection that formed a unique and key basis of learner-focused discourse. These categories are discussed with examples of how they occurred and the teacher's role in facilitating them. Learner-focused discourse and an intersubjective stance of the teacher's role are shown to be important to empower "self" in the learning of mathematics.

This paper is based on a two-year study of discourse that facilitates mathematical thinking as practiced in an elementary mathematics classroom. The focus here is on what and how specific features of whole-class discourse and the teacher's actions supported and promoted "learner-focusedness" in learning mathematics.

RELATED LITERATURE

Recent studies of discourse in mathematics education have sought to explicate the nature of meaningful classroom discourse, the mathematical thinking of students and teachers, and the role of discourse in the learning of mathematics and in mathematics teacher education (e.g., Cobb & Bauersfeld, 1995; Cobb et al., 1997; Cobb, Yackel & Wood, 1993; Peressini & Knuth, 1998; Sfard, 2001; Steinbring, Bussi, & Sierpinska, 1998; Wertsch & Toma, 1995). The increased interest in discourse in recent years can be linked to learning theories and reform recommendations in mathematics education (e.g., NCTM, 1991) that emphasize the importance of a different form of classroom communication from that of traditional classrooms. Given this importance, there is need for researchers to continue to strive to better understand mathematics classroom discourse as a means to improve how students experience the learning of mathematics and the depth of the mathematical knowledge and thinking they develop.

Discourse, as promoted in current reform perspectives of mathematics education, is not about classroom talk intended to convey exact meaning from teacher to student; instead, it is about communication that actively engages students in a way that allows them to construct new meanings and understandings of mathematics for themselves. It attends to reasoning and evidence for sense making and the development of ideas and knowledge collaboratively (NCTM, 1991). It makes thinking public and creates opportunities for the negotiation of meaning and agreement (Bauersfeld, 1995). It provides collective support for developing one's thinking, drawing it out through the

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interest, questions, and probing ideas of others (Cobb, Yackel & Wood, 1993). It enables students to articulate what they know as a way to clarify their own understandings (NCTM, 1991). It is important in helping students develop and sharpen their mathematical thinking (Schoenfeld, 1992; Watson & Mason, 1998).

Discourse with these qualities is dependent on both the nature of the mathematical tasks and questions or prompts used to initiate and sustain it during a lesson (Knuth & Peressini, 2001; NCTM, 1991; Watson & Mason, 1998; Wood, 1999). For example, tasks that would support a more dialogic-type of discourse should be more exploratory in nature, allowing for students' discoveries and reasoning to sustain meaning-making dialogue. Open-ended questions allow students to express their own ideas and employ higher-order thinking skills. In addition, as Hiebert and Wearne (1993) reported, higher-order questions in discourse promote reflection and integration in learning mathematics in a way that recall or rote response cannot.

Both the teacher and students have vital roles in this discourse process in order to initiate and sustain it (NCTM 1991). But these roles and this form of discourse can be difficult for teachers to implement and manage, particularly if their prior experiences with discourse were predominantly in traditional mathematics classrooms. However, classrooms of teachers who are able to implement meaningful discourse can provide opportunities to explore these roles in ways that could inform teaching and teacher education. This paper reports on a study based on such a classroom and provides insights of discourse that is learner focused and the teacher's role in facilitating it.

THEORETICAL PERSPECTIVE

Theoretically, discourse can be considered from different perspectives (Steinbring, Sierpinska, & Bussi, 1998). In this study, a learner-focused perspective of discourse is adopted and conceptualized based on Bruner's (1996) description of four "crucial ideas" of framing learning: agency, collaboration, reflection and culture.

First, Bruner (1996) explains that agency involves one taking more control of one's own mental activity. This assumes that "one can initiate and carry out activities on one's own" (p. 35). "Decisions, strategies, heuristics – these are key notions of the agentive approach to mind" (p. 93). Thus, "the child … [is viewed as] somebody able to reason, to make sense, both on her own and through discourse with others" (p. 57). This view of children as thinkers requires the teacher to give "effort to recognize the child's perspective in the process of learning" (p. 56). Agency, then, is about learner-focusedness and vice versa. This means that in learner-focused, classroom discourse, students are allowed to participate in ways that might include: initiation of a discussion; re-direction of discussions in a relatively teacher-unscripted direction; responses unanticipated by the teacher; responses with an element of creativity and students' intentions, personalization, and expression of students' interests or agendas.

Second, Bruner (1996) explains that agency and collaboration need to be treated together to account for the individual and the collective in learning. He notes:

> Mind is inside the head, but it is also with others. It is the give and take of talk that makes collaboration possible. For the agentive mind is not only active in nature, but it seeks out dialogue and discourse with other active minds. (p. 93)

This implies that agency and collaboration should be integrated in the design of a learner-focused classroom culture. For example, students should not only generate their own hypotheses, but also negotiate them with others – including their teachers.

The third idea, reflection, is "not simply 'learning in the raw' but making what you learn make sense, understanding it … turning around on what one has learned through bare exposure, even thinking about one's thinking" (Bruner, 1996, p. 58). Thus, like agency, reflection is about learner-focusedness and vice versa. This means that learner-focused discourse, for example, should allow or prompt students to notice for themselves and to become aware of their own thought processes, that is, to become more metacognitive. It should also allow students to confront and attempt to make sense of problematic aspects of learning (Dewey, 1933).

The fourth idea, culture, according to Bruner (1996), is the way of life and thought that we construct, negotiate, and use for understanding and managing our world. Thus, the learner-focused, discourse culture is the way of classroom life that must be created to facilitate agency, collaboration and reflection in the learning process.

In general, then, as used in this study, discourse from a learner-focused perspective takes account of students' personal experiences, thoughts, and feelings. It capitalizes on and values students' contributions to their learning. It provides opportunities for students to bring their own backgrounds, personalities, and beliefs into the construction of their mathematical knowledge. Thus, through it, students can come to realize that their ideas are valued and, as a result, have more authority over their learning.

RESEARCH PROCESS

A case study was conducted with an experienced elementary teacher and her Grade 3 class. The teacher's practice embodied social constructivist principles with discourse playing a prominent role. The teacher regularly engaged her students in whole-class and small-group discussions and inquiry-oriented mathematical activities. The classroom culture was oriented towards learner-focusedness. Data sources consisted of interviews with the teacher, weekly classroom observations throughout three consecutive school terms, and classroom artifacts. The open-ended interviews focused on the teacher's thinking about discourse and her discourse behaviors in the classroom. For example, she was prompted to talk about her understanding of, goals for, and role in the discourse; her goals for students in relation to discourse; her approaches to questioning, listening, and task selection; how/why she intervened during discourse; how she established the classroom context; and her understanding of mathematical thinking. The interviews and all whole-class discourses for the lessons observed were audio taped and transcribed. Field notes were made of learning tasks, board work, and non-verbal teacher-student interactions relevant to discourse. The artifacts obtained included relevant students' written work and teacher's notes.

Data analysis for the larger project focused on identifying characteristics of discourse and the relationship to facilitating students' learning and mathematical thinking. Initially, a process of open coding was carried out. Strauss and Corbin (1990) describe this as taking data and segmenting them into categories of information. Two research assistants conducted this open coding independently of the researcher, and independently of each other. Only after initial categories had been identified were the results discussed and compared and revisions made where needed based on disconfirming evidence. Coding included identifying: (1) types of questions/prompts that elicited mathematical thinking (guided by Watson and Mason, 1998) and reflection; (2) what the teacher attended to in students' responses; and (3) different teacher's actions and/or thinking that determined different features of discourse and supported learner-focusedness. Themes emerging from the initial coded information were used to further scrutinize the data and then to draw conclusions. The theme "self-reflection" emerged as a unique and key basis of learner-focused discourse in this classroom. In this context, self-reflection involves students reviewing or exploring their own thinking and personal in- and out-of-class experiences in order to describe, analyze, and evaluate them as a basis to inform their learning of mathematics.

LEARNER-FOCUSED DISCOURSE AS SELF-REFLECTION

The findings of the study showed that the dominant way of achieving learner-focused discourse during whole-class settings in this classroom was by engaging students in self-reflection. Specific features of this learner-focused discourse as self-reflection are highlighted here in terms of the categories of self-reflection and the teacher's actions that were integral to empowering students in their learning of mathematics.

Categories of self-reflection: Four categories of self-reflection were dominant in the whole-class discourse of this classroom: reflection on "real-world" experiences; reflection on conceptions; reflection on preconceptions; and reflection on thinking. This section briefly describes each with examples/cases of how it was initiated by the teacher and discusses implications for classroom discourse.

Reflection on real-world experiences: This category required students to "see the math" (commonly used by teacher) by reflecting on their out-of-school, real-world experiences to identify and decide on what mathematics they embodied, as in these three cases. Case (1): students reflected on their experiences to identify examples of mathematics. Their challenge was to decide on what was an example of mathematics in their real-world experiences. This was initiated by the teacher posing questions, usually at the beginning of lessons, such as: "Where is math in your world?" "Did anything happen in your life that involves math that you want to share?" "Who experienced a math situation since we met in class yesterday?" Case (2): in contrast to case (1) where a mathematics concept was not specified, for this case, students reflected on their experiences to identify a specific mathematics concept. This occurred during the introduction and discussion of a mathematics concept and involved students associating real-world applications or significance of the concept.

For example, in introducing a discussion of the concept of one million, the teacher asked, "Where would you find the number one million used in your world?" Students' responses included: in computers, money in the bank, all the trees together, and all the people. During a discussion of a line graph, students drew on their experiences to respond to the teacher's question: "where have you seen it?" Students' responses and follow-up discussions centered on "in the hospital," "TV shows about hospital (or doctors)," and "patients' heartbeat goes like this." Case (3): students reflected on their experiences to associate real-world meanings or interpretations as in the following situation: "We're going to look at numbers on the calendar and try to think of how many ways to make this number. … Is there anything that you can think of in your life that makes you think of 17?" These three cases of reflecting on experience offer possibilities of how students can be allowed to empower "self" to provide meaningful contexts for discourse in learning mathematics.

Reflection on conceptions: In this category, reflection on self involved students thinking about what they knew about a mathematics concept based on their past experiences, e.g., what they learned in prior Grades. The teacher prompted students to unpack a concept based on the conceptions they had constructed of it. For example, the teacher drew students' attention to a bar graph she had drawn on the flip chart then probed their thinking about the shape making up the graph. She asked, "How do you know it is a rectangle? Make me believe that it is a rectangle." Students were able to recognize the rectangle, but initially encountered a problem explaining why. To prompt their reflection, the teacher asked, "How did you know it wasn't a circle." This led them to talk about what the rectangle was not. The teacher then prompted, "Think about the art project we did," as a way for them to find the language to describe the rectangle, which they were able to do. This discourse allowed the students to reflect on what they knew, i.e., their conceptions, based on making comparisons and connections within and outside of their mathematical experiences.

Reflection on preconceptions: In this category, discourse required reflection that involved students thinking about their preconceptions of a mathematics concept/ procedure, that is, what they thought they knew about it before formally learning it, as in the following three cases: Case (1): students reflected on a mathematics concept they were likely to have preconceptions of, based on their formal or informal learning experiences. For example, at the beginning of a lesson on linear measurement, the teacher initiated the discourse with: "We are going to determine our height today. … How could we do that? … What tools do we need to use for math today to determine our height?" The students' responses led to a discussion of both the tools and units of measurement based on their preconceptions. Case (2): students reflected on a concept of which they were unlikely to have preconceptions, but to which they could relate. For example, during a lesson on representing a number numerically in different ways, the teacher asked: "But just talking about numbers, does anybody really know where numbers came from and why we have numbers? … How many digits do we have?" Case (3): students reflected on a mathematics concept they likely or actually had formed a preconception of, but had not explicitly thought of or articulated, as in the situation when the teacher asked: “What’s the biggest number you can tell me?” One student said “infinity”, which led to a discussion of “What does that mean?”

Reflecting on thinking: In this category, the discourse required students to think about their own thinking, i.e., to reflect on a metacognitive level, as in these four cases. Case (1): students reflected on what they looked for, or thought of, in order to make sense of, or interpret, a mathematics concept. For example, the teacher asked, “We are going to take a look at how numbers are made up … what do you do when you read a large number?” Case (2): students reflected on their problem-solving process or strategy, as when asked, “What did you think of first when you read the riddle?” Case (3): students reflected on their choices of tools to aid learning, as when asked, “Who used the place value mat? … Can you tell us why you chose to use that?” Case (4): students reflected on the affective aspect of their problem-solving experience, as when asked, “How many people had a little bit of difficulty trying to solve it? … How many people found it a challenge?” These four cases deal with reflection as integral in helping students to take control of their own thinking and learning.

Implications for discourse: These four categories of self-reflection provide a prospective of learner-focused discourse that includes both the students’ thinking and real-world experiences in their learning of mathematics. The main feature of this discourse is its focus on the individual and her or his experiences as the source for reflection, i.e., students become the subject in the process of his or her own reflection and learning. Thus, there is a connection between personal experience and conceptual sense making. This discourse goes beyond mere context for the mathematics; appealing to what students already have experienced and from which rich mathematical ideas can emerge. As Bruner (1966) explained, to personalize knowledge one does not simply link it to the familiar but makes the familiar an instance of a more general case and thereby produces awareness of it. Self-reflection facilitates this. Students get to personalize mathematics, to see mathematics in their personal world, and to mathematize their personal world. This discourse also recognizes agency by focusing on what students can do or initiate and think they are doing and basing the learning of mathematics on this.

Learner-focused discourse allows students to talk mathematically from and about their experiences. It thus treats mathematics as connections, not merely in terms of applications within and outside of mathematics independent of students, i.e., having students talk about them as being “out there,” but as related to the personal or the self. This way of empowering the self makes the learning of mathematics personal, real, relevant, important, and meaningful. It allows students to return to their real-world experiences with new eyes framed by mathematical sense-making. As was also evident with the students in the Grade 3 class of this study, it encourages more voluntary participation and allows students to shift from assisted to independent use of reflection on self, indicating learning beyond the content. This discourse can be used to encourage individual expression and revitalizing content, i.e., the lived mathematics curriculum. This means that the teacher's actions are central to whether or how learner-focused discourse is enacted in the classroom.

The Teacher's Actions: These were critical in promoting and supporting the learner-focused discourse and included the following six behaviors: (i) Posing open questions and prompts focused on the self as illustrated in the four categories of self-reflection described earlier. The teacher was not consciously trying to promote self-reflection; instead, she wanted students to "see the math in their lives and their heads" and often posed questions to achieve this. This goal made her aware of when and how to pose questions without, for the most part, preplanning them. (ii) Questioning students' understanding of the mathematics in their examples. In addition to sharing examples of their mathematical experiences, students were asked to think about and further unpack the mathematics. (iii) Extending students' mathematical understanding based on their responses. (iv) Resonating with students' experiences or thinking. The teacher sometimes used the students' responses as a prompt for her to share her own related personal experiences or thinking. For example, after students shared why they chose to use or not use the place-value mat, she shared, "For me it helps me see the number. It helps me keep track of how many hundred, how many tens, and how many ones." (v) Encouraging students' thinking. For example, the teacher would tell students, "Don't put a hand up yet. I want all brains working on this. … Hands down, all brains are working." (vi) Providing affective support. For example, after asking students if they experienced difficulty with the task, the teacher prompted, "Out your hands up nice and high and be proud of it, because that means you've learned something."

These examples provide a perspective of the teacher's role that is necessary to create a learner-focused discourse community with a culture to support self-reflection, agency, and collaboration in a whole-class setting. In this perspective, there is an "intersubjective" (Bruner, 1996) stance, where the teacher applies the same theories to herself as she does to her students, which allows her to create approaches that are as useful for students in organizing their learning as they are for her. For example, this Grade 3 teacher's actions placed her on the same level as the students in enacting the discourse with a focus on self when she resonated in the students' responses with her own personal experience, more on the level of a peer and not as the superior mind. Thus, they supported each other's reflection and learning. In the intersubjective stance, the teacher is also interested in what the student is thinking and thus is concerned with formulating a basis of discourse that she can use to satisfy this and facilitate the efforts of the student, as evidenced in the case of this Grade 3 teacher.

CONCLUSION

The study identified four categories of discourse involving self-reflection and examples of how they occurred in a Grade 3 classroom. These are not intended to be exhaustive, but to provide a view of discourse that can be used to inform our understanding of it beyond this classroom; a view that considers self-reflection as an essential part of learning mathematics. This learner-focused discourse empowers "self" in learning mathematics. It allows students to learn mathematics with meaning and to make sense of: mathematical ideas; mathematics in their lives; their world in a mathematical way; and their ways of thinking or learning. The teacher's role needs an intersubjective stance to establish a leaner-focused classroom culture in which agency, reflection, and collaboration are instrumental to how the discourse is enacted.

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MATHEMATICAL KNOWLEDGE FOR TEACHING AND PROVIDING EXPLANATIONS: AN EXPLORATORY STUDY

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This study investigated the explanations that 20 preservice teachers (PSTs) of different levels of Mathematical Knowledge for Teaching (MKT) offered while working in a simulated environment. Data were collected at the beginning of a teacher education program and after an intervention consisting of a two-term mathematics content/methods sequence. The quantitative analysis of the data showed a moderate correlation between the PSTs' MKT and their performance in providing explanations; the qualitative analysis also showed the PSTs' MKT (and changes thereof) to relate to the quality of the explanations they provided. These results suggest that teachers' performance in providing explanations is contingent not only on their expertise, as previous studies have suggested, but also on their MKT.

INTRODUCTION

Designing and delivering coherent and meaningful instructional explanations is regarded a central task of teaching in which teachers engage at several instances: when presenting ideas to their students, when responding to students' questions, or when scaffolding struggling students. This teaching task is therefore fundamental to the learning process and one that significantly shapes student learning opportunities (Leinhardt, 2001; Leinhardt & Steele, 2005). Even more, teacher explanations clarify subject matter and potentially influence the knowledge that students develop *about* mathematics, since, as Leinhardt (2001) explicates, teacher explanations "demonstrate, convince, structure, and convey, [but also] … suggest the appropriate metacognitive behavior for working in a given discipline" (p. 340).

Despite the pivotal contribution of teachers' explanations to student learning, providing appropriate explanations often constitutes an arduous task for teachers, especially for novices. As suggested by a series of studies conducted by Leinhardt (e.g., Leinhardt, 1987, 1989), notable differences exist between the explanations provided by expert and novice teachers. Expert teachers often build their explanations on representations and examples known to their students; they also present more complete and error-free explanations for the concepts or procedures at hand. Novices, on the other hand, find it difficult to provide explanations (cf. Borko, Eisenhart, Brown, Underhill, Jones, et al., 1992); when they do so, their explanations are often incomplete and error-prone. The exploratory study reported herein aims at extending this line of research by investigating the performance of preservice teachers (PSTs) differing in their level of *Mathematical Knowledge for Teaching* (MKT) in providing explanations.

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THEORETICAL FRAMEWORK

Building on Shulman's (1986) notion of pedagogical content knowledge, during the last decade Ball and colleagues (Ball, Thames, & Phelps, 2008) have coined the notion of MKT to capture "the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (p. 399). Rather than viewing MKT as a single construct, Ball et al. (2008) proposed that it consists of at least four domains: common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching. The first two types of knowledge capture the content knowledge needed for teaching (Ball et al., 2008), which does not necessarily depend on knowledge of students or of teaching. *Common content knowledge* (CCK) is the mathematics knowledge common to all mathematically intensive professions and therefore, not unique to teaching (e.g., computing $2 \div \frac{3}{4}$); this knowledge is necessary, but not sufficient for teaching. A teacher also needs to know why an algorithm works and select and use appropriate representations to help students make sense of it. She also needs to analyze students' errors and size up students' conventional or non conventional solutions. All these require a special type of knowledge known as *specialized content knowledge* (SCK).

According to Ball and colleagues (2008), MKT in general, and its CCK and SCK components in particular, are consequential for several mathematical tasks of teaching, including the task of giving and evaluating mathematical explanations. Two recent studies have somewhat validated this theoretical association between teachers' MKT and their providing of explanations. In the first study, Hill et al. (2008) explored whether teachers who differed in their level of MKT also differed in the quality of their instruction. Analyzing 90 lessons taught by ten teachers with different levels of MKT, these researchers found that, compared to their lower-MKT counterparts, teachers with higher MKT-levels responded more appropriately to their students and chose examples that helped students construct meaning of the targeted concepts and processes. Focusing on two of these teachers, Charalambous (2008) found that the high-MKT teacher's explanations illuminated the concepts under examination; in contrast, those of the low-MKT teacher were largely procedural. Despite providing some empirical validity to the foregoing association, both studies were limited by realities that inhere in examining teachers' performance in real-life settings: the teachers at hand used different curricula and taught dissimilar topics to different student populations. To account for these limitations, the present study – which was part of a larger project focusing on several teaching practices – explored PSTs' performance in providing explanations via a teaching simulation. This simulated environment asked the study participants to consider the same lesson – an introductory lesson on fraction division – and immersed all of them into the same carefully designed teaching episodes. Specifically, the study reported below sought to provide answers to the following questions: (1) Is there an association between PSTs' MKT and their performance in providing explanations? and (2) Are the explanations offered by PSTs with different levels of MKT qualitatively different? If so, how?

METHODS

Participants and Setting

The study was conducted at a large Midwestern U.S. university. Its participants were 20 PSTs who possessed an undergraduate degree and were enrolled in a one-year teaching education program leading to a K-8 teacher certification. The participants (16 White, 2 African-American, and 2 Asian-American) represented a diverse group in terms of the math content/methods courses taken in high school and during their undergraduate studies. During the first half of this program, these PSTs took seven courses, including the math content and methods courses that comprised the intervention this study considers. The content course was designed to help the PSTs unpack and develop flexible understanding of key concepts and processes within the realm of number theory and operations; it also sought to offer them opportunities to practice using representations and providing explanations. Four of the 12 meetings of this course focused on fractions and fraction division. During these meetings, the PSTs discussed different fraction representations and interpretations, were introduced to the partitive and measurement interpretation of division, and considered the meaning of quotients. The methods course afforded the PSTs more opportunities to hone their skill in providing explanations and using representations; this course largely focused on place value and operations with whole numbers.

Data collection and analysis

The PSTs' MKT was measured by a *Learning Mathematics for Teaching* (LMT) multiple-choice test including 41 selected CCK and SCK items pertaining to concepts and processes related to fractions and fraction division; this test was administered to all participants twice: during the first class of the content course (pre-intervention) and the last class of the methods course (post-intervention). The PSTs' performance in providing explanations was tapped through individual semi-structured interviews developed around a teaching simulation that created an arena for exploring (i) if the PSTs noticed without any prompting the problematic features of the virtual teacher's explanations (noticing); (ii) how the PSTs evaluated the virtual teacher's insufficient explanations (evaluating); and (iii) the PSTs' performance in developing and delivering explanations for an imaginary 6th-grade audience; the PSTs were asked to explicate the invert-and-multiply-rule and the quotient in a fraction division whose answer involves a fractional part (i.e., $2 \div \frac{3}{4}$) (performing). This paper focuses only on the performing component of the PSTs' interview performance. Interviews with all participants were conducted at the commencement of the content course; interviews were also conducted with 16 of the participants at the end of the methods course.

A two-phase, mixed-methods approach was utilized to analyze the collected data. In the first phase, *Spearman* r_s non-parametric criterion was employed to explore the association between the PSTs' MKT and their performance in providing explanations. To this end, participants were assigned a pre- and a post-MKT score based on the number of questions they answered correctly on the test. The interview data were "quantified" following Chi's (1997) guidelines: a scoring rubric was developed, validated by three experts, and subsequently refined; the refined rubric was then applied to code the interview data;[1] finally, 4 of the 36 interview transcripts were coded by two independent coders for inter-coder reliability purposes, yielding an acceptable Cohen's κ reliability index (κ=0.73; cf. Landis & Koch, 1977, p. 165). In the second phase, seven PSTs' cases – both convergent and divergent (cf. Yin, 2003) – were scrutinized to develop more nuanced characterizations of the association between the PSTs' MKT and their performance.

FINDINGS

The association between the PSTs' MKT and their performance in providing explanations was examined with respect to one time point – the beginning of the intervention (i.e., static perspective) – and with respect to changes between two time points – before and after the intervention (i.e., dynamic perspective).

First Insights into the Association under Exploration: Quantitative Analysis

The quantitative analysis offered some first insights into the association at hand. On a scale from 0 to 41, the PSTs' pre-intervention performance on the LMT test ranged from 10 to 39; their mean performance was 22.10 with a standard deviation of 7.35; collectively, these findings suggested that the study participants represented a diverse group in terms of their MKT explored in the study. The study participants were also relatively diverse in their pre-intervention performance in providing explanations (on a 0-3 scale, their scores ranged from 0 to 2, with a mean of 0.55 and a standard deviation of 0.69). The use of *Spearman's* r_s non-parametric criterion yielded a moderate correlation between the PSTs' pre-intervention MKT and their pre-intervention performance in providing explanations ($r_s=0.49$, $p<.05$). Using the pre- and post- intervention data from the 16 PSTs for whom complete data were obtained, the association at hand was also explored by considering the changes in the PSTs' MKT performance ($\overline{x}=6.31$, SD=4.59) and the changes in their performance in providing explanations ($\overline{x}=1.13$, SD=0.72); this analysis yielded a marginally significant moderate correlation ($r_s=0.49$, $p<.10$). In conjunction, the results of both the static and the dynamic perspective provided converging evidence corroborating the theoretical assumption about the association between teachers' MKT and their performance in providing explanations. Further insights into this association were gleaned from the scrutiny of the respective performances of the seven selected cases.

Developing Further Insights into the Association at Hand: Qualitative Analysis

Due to space constraints, here I focus only on the findings pertaining to explaining the quotient in $2 \div \frac{3}{4}$. A grounded-theory approach (cf. Strauss & Corbin, 1998) was pursued to develop a classification scheme to classify the quality of the seven PSTs' explanations. The five categories of this scheme are presented in Figure 1.

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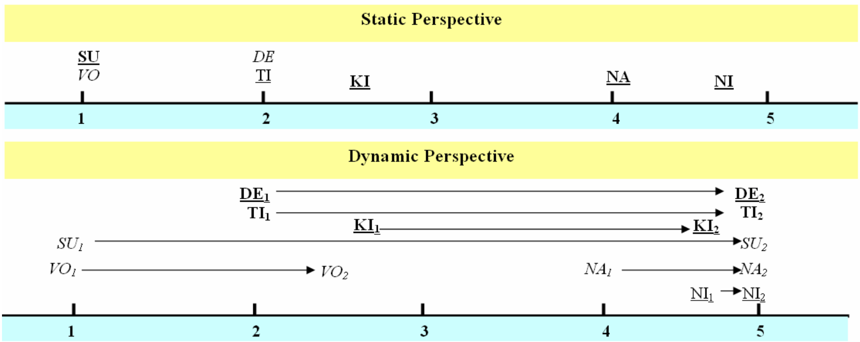
[1] The PSTs' explanations were scored based on criteria such as their correctness, completeness, the extent to which they explicated rather than described the procedure at hand, and their calibration to the intended student population.

1. *No explanation/Description*: No explanation is provided or the "explanation" provided describes rather than explains the procedure under consideration.
2. *Numerically driven explanation*: The explanation is driven by the numbers involved in the procedure and particularly the final answer.
3. *Conceptually driven explanation*: The explanation is grounded in the underlying meaning of the procedure at hand; yet, it is not calibrated to its intended population.
4. *Conceptually driven and calibrated explanation*: The explanation provided is grounded in the underlying meaning of the procedure under consideration and it is calibrated to its intended audience.
5. *Conceptually driven, calibrated, and unpacked explanation*. The explanation provided is grounded in the meaning of the procedure under investigation, calibrated to its intended audience, and adequately unpacked.

Figure 1: Classification scheme for the practice of providing explanations.

Based on their pre-intervention MKT scores, Nathan, Nicole, Suzanne, and Kimberley were clustered in the high MKT category; Tiffany in the medium-low category; and Deborah and Vonda in the low category (all names are pseudonyms). The upper panel of Figure 2 presents how these PSTs' pre-intervention explanations were coded based on the classification scheme of Figure 1.



Notes: (a) Participants: DE: Deborah; KI: Kimberley; NA: Nathan; NI: Nicole; SU: Suzanne; TI: Tiffany; VO: Vonda; (b) Quality of Explanations: The classification scheme is detailed in Figure 1; (c) Pre-intervention MKT: Italics: Low-MKT; Underlined: Medium-low MKT; Bold and underlined: High-MKT; (d) Changes in the PSTs' MKT: Italics: Low changes; Underlined: Moderate-low changes; Bold: Moderate-high changes; Bold and underlined: High changes; (e) Performance in providing explanations: 1. Pre-intervention 2. Post-intervention.

Figure 2: Classification of the PSTs' performance in explaining the quotient in 2 ÷ ¾.

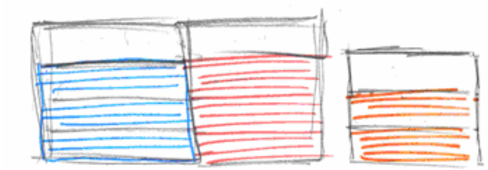
If one excludes Suzanne, the upper panel of Figure 2 shows that the three high-MKT participants were positioned in higher categories relative to the medium-low MKT participant, who, in turn, was presented at the same category than the first low-MKT participant and a higher category than the second low-MKT participant. The lower panel of Figure 2 represents the changes in the PSTs' MKT and the changes in their performance in explaining the quotient in 2 ÷ ¾. As this figure suggests, Deborah and

Kimberley experienced the greatest changes in their MKT scores; Vonda, Suzanne, and Nathan the smallest. The changes in Tiffany's and Nicole's MKT performance were somewhere in between, with Tiffany experiencing more gains in her performance than Nicole. This figure also shows that if, again, one excludes Suzanne, the PSTs of the two upper categories of MKT changes experienced more notable growth in their performance, compared to the PSTs of the two lower MKT-change categories. Given that the scheme of Figure 1 represents increasing levels of performance when moving from its lower to its upper categories, both foregoing patterns corroborate the results of the quantitative analysis reported above.

Deborah was the participant who entered the program with a relatively low MKT score but experienced the greatest changes in her MKT and performance in providing explanations. Presented below are her explanations during the pre- and the post-intervention interviews. The presentation of her explanations seeks to illustrate the coding of the PSTs' performance and to depict her progress in giving explanations, which was consonant with the growth in her MKT, as tapped by the LMT test.

When during the pre-intervention meeting Deborah was asked to outline how she would explain the division 2 ÷ ¾ to 6^{th} graders, she drew two circles to represent the dividend, put a comma, and drew a third circle (see left panel of Figure 3). After shading in ¾ of the last circle to represent the divisor, she was mystified: "I don't know how I'd go about doing divided by three-fourths." Prompted to identify what made this explanation hard for her, she admitted that she was still "at the stage of [considering division] just a calculation." Because the quotient of this division includes a fractional part that warrants an explanation in *divisor*-units, Deborah could not explain this division; she could, nonetheless appropriately explain divisions whose quotient did not involve fractional parts (i.e., ½ ÷1/6). The comparison of Deborah's performance in these divisions showed that her explanations were largely driven by the numbers involved in the quotient; thus, her pre-intervention explanation for the quotient in 2 ÷ ¾ was coded as a numerically driven.

Pre-intervention



Post-intervention

Figure 3: Deborah's pre- and post intervention drawings when explaining 2 ÷ ¾.

Deborah's post-intervention performance in the same task was remarkably different. Although it took her some time to formulate an explanation for the quotient of this division, she ended up drawing two rectangles, one attached to the other, as shown in the right panel of Figure 3. She then divided each rectangle into "four equal pieces" to show the ¾. Next, she colored a ¾-portion of the first rectangle in blue and a ¾-portion of the second rectangle in red. Pointing to the portions she colored, she said:

> This right here [pointing to the blue ¾-portion] is three-fourths of this one [pointing to the whole first rectangle]; so this is three-fourths. And then [pointing to the red ¾-portion:] this is three-fourths. So, right off the bat, how many three-fourths do we get off of two? It would be two, which is these [pointing to the blue and red ¾-portions].

She then compared the two non-shaded portions of her figure to the blue ¾-portion and explained that "we don't even have this". Next, she drew a rectangle equal to the blue portion (see rectangle on the right end of Figure 3) and divided it into three parts. Pretending to be transferring the two non-shaded regions into the ¾-rectangle, she colored two of its parts in orange, and continued:

> So this [pointing to the blue ¾-portion] is three-fourths; this [pointing to the red ¾-portion] is three-fourths; this [pointing to the ¾-rectangle] is the same three-fourths [as the previous two ¾-portions]. And if I transfer these two [pointing to the two ¼ non-shaded pieces] over here [showing the portion colored in orange in the ¾-rectangle], we have how much of three-fourths? I broke it into three pieces and I shaded in two; it's two-thirds of three-fourths. And that's how you get two-thirds. So, in total we get one [pointing to the blue ¾-portion], two [pointing to the red ¾-portion], and two-thirds [pointing to the ⅔ portion of the ¾-rectangle].

Several aspects of Deborah's post-intervention work are noteworthy. For example, she clearly identified the two different units of a division problem (dividend- and divisor-units); after explaining that she could make two whole divisor-units from the available dividend units, she explicated that she could not make another whole ¾-portion; to explicate what part of the ¾-portion she could cover with the remaining dividend units, she explicitly illustrated the size of the divisor-unit and showed that the leftover dividend-part covered only a fraction of this unit; once finished, she showed the correspondences between each part of the quotient and her drawing. Taken together, these features suggest that her explanation was sufficiently unpacked and could be understood by an average sixth grader; hence, her post-intervention explanation reflected a remarkable progress in her respective performance.

DISCUSSION

Previous studies (Leinhardt, 1987, 1989, 2001) have shown teachers' performance in providing explanations to relate to their level of expertise. This study extended this line of inquiry by exploring the association between PSTs' MKT and their performance in providing explanations for fraction division, a particularly difficult topic to understand, let alone to teach (cf. Borko et al., 1992). Both the quantitative and the qualitative analyses undertaken in this study provided converging evidence supporting the association between PSTs' MKT and their giving of explanations.

The quantitative analysis yielded a moderate association between the PSTs' MKT and their providing of explanations. This association is higher than those reported in educational production function studies (e.g., Rice, 2003), which also explored the relationship between teachers' knowledge and their effectiveness. The qualitative analysis largely corroborated this association by showing PSTs' MKT to relate to the quality of their explanations; this was shown in two ways. From a static perspective, PSTs with higher MKT were found to provide explanations that were more conceptually oriented and calibrated to their intended audience than those given by their lower-MKT counterparts, whose explanations were numerically driven, at best. From a dynamic perspective, the changes in the PSTs' MKT were consistent with the changes in the quality of their explanations, as Deborah's case suggested.

To be sure, Suzanne's case, which appears to challenge both foregoing patterns, warrants further examination. Future studies could also explore the degree to which (preservice) teachers' performance in providing explanations in vitro – as explored in the present study via the teaching simulation – transfers to designing and delivering explanations in real classroom settings, which often needs to be done on the spot.

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DEVELOPING EIGHTH GRADERS' CONJECTURING AND CONVINCING POWER ON GENERALISATION OF NUMBER PATTERNS

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This study orchestrates thirty-four eighth graders in a pattern-finding activity of a two week extra curriculum programme to develop conjecturing and convincing power. The activity is designed by applying the binary system to construct six tables with numbers from 1 to 63. A student is asked to select a number between 1~ 63, and check if the number appears in each of the six tables, then the teacher can tell the number accurately. After playing the game for several rounds, the students are divided into six groups to discuss the hidden patterns as well as propose conjectures and make justification. The results clarify that students could not only apply mental representations for generalising but propose impressive justifications to make others convinced during the cognitive process of conjecturing. Besides, their dispositions towards mathematics learning also alter significantly compared with the attitudes they possessed formerly in the regular math classes.

THEORETICAL BACKGROUND

Prominences of features of pattern and generalisation have been outlined by many researchers (Dörfler, 1991; Mason, 1996; Polya, 1954), these features could be a corner stone of numerical, early algebra and algebraic thinking. Mathematics is perceived as the science of pattern and relationship (AAAS, 1990). Consequently, pattern-finding tasks could be a crucial factor for learning of mathematics. Carraher and Schliemann (2007) point out that a pattern is not necessarily a well-defined mathematical object with clear connections to other mathematical objects; rather, it could be considered as a coherent research base in early algebra. Additionally, exploring patterns, relations and functions is an essential focus of algebra (NTCM, 2000).

Researchers acknowledged that pupils' generalising ability could be facilitated efficiently whilst they engage in solving the pattern problems (Becker & Rivera, 2007). Suggestions of some empirical studies also support this viewpoint. Hargreaves, Shorrocks-Taylor and Threlfall (1998) have tested 315 children aged between 7 and 11 years who participated in several number patterning activities, the research results reveal that children could not merely make generalisations with specific strategies, but evolve different cognitive processes.

Actually, generalisations are both objects for individual thinking and means for communication (Dörfler, 1991). Above all, conjecturing is an ongoing process which

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 313-320. Thessaloniki, Greece: PME.

is built on specialising and generalising as an ascent and descent (Polya, 1954). Generalising and specialising are two sides of a coin, in accordance with this view point, Mason (2002) points out two perceptions particularly, which are *seeing the particular in the general* and *seeing the general through the particular*. For that reason, pattern-finding tasks in generalisation can be acknowledged as an important activity for getting students involved in a conjecturing atmosphere. Despite that the significance of conjecturing has been recognised by plentiful researchers (Lakatos, 1976, 1978; Mason, Burton, & Stacey ,1985; Davis, Hersh, & Marchisotto, 1995), it could be recognised that evolving a conjecturing process in patterning approaches is one thing, justifying it for convincing others is quite another. Mason (2002) further states that once a conjecture is made, it needs to be challenged, justified, and possibly reconstructed. As a result, conjecturing accommodates fruitful opportunities for reasoning in behalf of justifying conjectures. Empirical studies have already approved this stance. For instance, Lannin (2005) has held a study with 25 sixth-grade students as they approached patterning tasks in which they were required to develop and justify generalisations. The research result shows that students have demonstrated remarkable abilities to construct generalisations with using different strategies of different justifications to construct and justify the same generalisation.

Lin (2006) stresses that a good lesson must provide opportunities for learners to think and construct actively, in addition, conjecturing is not merely the core of mathematising, but the driving force for mathematical proficiency. According to such standpoint, we organise this study to examine how pupils develop conjecturing and convincing power on reasoning and justifying the generalisation of number patterns.

METHODOLOGY

The study was conducted at a junior high school located in the suburb of central Taiwan. Thirty-four 8^{th} graders participated in a two-week extra curricular programme aimed to foster the power of conjecturing and justifying. According to the students' performances on the previous monthly exam, most of them not only carried out barely below the average, but behaved towards cold end of affect (in the sense of Mandler, 1989).

This pattern-finding activity was purposefully designed to support student development of conjecturing and convincing power. This activity included six tables of numbers (Figure 1) from 1~63 that were constructed by transition of number systems from base-10 to base-2. For instance, decimal number 10 could be represented as the binary number 1010, that is, number 10 is equal to $(1)*2^3+(0)*2^2+(1)*2^1+(0)*2^0$. Among the six tables, table 1 contains the binary numbers with the first digit is 1. The rest may be deduced by analogy, such as table 5 contains the binary numbers with the fifth digit is 1. Hence number 10 can be found in tables 2 and 4 only. The research subjects had not learned the binary system. Hence, randomly placing the numbers in each table but not sequencing them in order might increasing the degree of difficulty of observing the pattern was increased.

The activity started with a game asking a student to select a number from 1~63 and bear it in mind firstly, and then showing the student these six number tables sequentially for she/he to examine whether the selected number is in the table or not. In the end, the teacher notified the student the accurate number she/he had selected. After playing the game for several rounds, students were divided into six heterogeneous groups for working together to find out the hidden patterns, and asked to cooperate together for answering the following questions: (1)What properties can you find from these six tables? (2)Can you induce any generality of the tables from the properties you found? (3)Can you propose any conjecture of how the game works and justify your conjecture? In the end, each group assigned a representative to address their result to the class and answer the students' or teachers questions. After the class, representatives of different groups were interviewed to illustrate their group works based on the worksheets they filled as a record. The whole process of the activity was video-taped, and all these qualitative data were analysed and triangulated.

No. 1

17	19	61	43	25	27	57	31
1	51	37	7	9	11	13	15
33	35	5	39	29	23	45	47
49	3	53	55	41	59	21	63

No. 2

50	51	10	43	58	15	62	63
2	27	6	7	54	22	14	38
18	19	11	23	26	3	30	31
34	35	59	39	42	55	46	47

No. 3

15	5	44	37	12	13	39	4
20	21	61	23	28	29	38	53
52	31	54	47	60	22	62	63
36	7	30	14	6	45	46	55

No. 4

46	13	26	63	28	11	30	31
12	62	42	43	44	45	29	47
56	8	58	59	60	61	41	27
57	9	10	24	40	25	14	15

No. 5

61	17	62	60	20	21	22	23
48	49	50	19	52	53	57	55
24	25	26	27	28	29	58	31
56	54	30	59	51	16	18	63

No. 6

58	33	59	32	36	51	41	39
40	38	42	43	56	48	34	47
63	49	50	37	52	57	54	55
44	53	61	46	60	35	62	45

Figure 1. Number tables for pattern-finding activity

RESULTS AND DISCUSSION

Specialising and generalising, as well as *conjecturing and convincing* are two main subjects of learner's power towards mathematics learning (Mason & Johnston-Wilder). Hence, through the perspectives of these two subjects, students' qualitative data were synthesised and elaborated.

Specialising and Generalising

For the particular cases the students proposed, most of them picked a number from the tables firstly, and then counted the frequency it appeared in these six tables. According to the pattern of the frequencies of different numbers appearing in the tables, students could express the generality from the particular cases. For example, Mike introduced the number pattern which his group found:

Interviewer: How did you find the properties of these tables?

Mike: We will take any number from the number tables in order to find the frequency which it appears in different tables.

Interviewer: Why?

Mike: Because of that, we could outline the pattern of these numbers.

Interviewer: Can you show me how it works?

Mike: Take11 as an example. It appears in tables 1, 2 and 4. If someone tells me the number which appears in tables 1, 2 and 4, since only 11 appears in these tables, I can eliminate the others and say the answer is 11.

Interviewer: Can you tell me the secret?

Mike: We just rearranged the numbers sequentially in each table, and then observed the frequency of a number is the key to find the pattern.

Apart from Mike's group, George's group employed the similar strategy and took the numbers of even and odd into consideration. They marked all the even numbers in each table (as figure 2), and tried to find the patterns of the odd and even numbers through counting the frequencies. The following transcription is quoted from the interview with George:

Interviewer: Can you tell me what your group found from the tables?

George: All the numbers in table 1 are odd numbers and the other tables comprise odd and even numbers half and half.

Interviewer: What do you mean then?

George: That is, except table 1, the frequency of odd numbers is in concert with even numbers.

Figure 2. George's group's patterning strategy
(Numbers circled are originally marked by highlighter.).

Examining the students' different generalising strategies, we also noticed that some groups applied the idea of checklist for generalising the number patterns (as figure 3). It makes an echo to the finding of Filloy, Rojano and Rubio (2001) that checklist serves as a tool for helping students move from focusing on a specific example to describing general relationships. Above all, using checklist for students is a facile approach which is also effective in observing the hidden patterns. As in the case of this study, students could go a step further to see the relationship among the numbers.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	O	X	O	X	O	X	O	X	O	X	O	X	O	X	O	X	O
2	X	O	O	X	X	O	O	X	X	O	O	X	X	O	X	X	X
3	X	X	X	O	X	O	O	O	X	X	X	O	O	O	O	X	X
4	X	X	X	X	X	X	X	X	O	O	O	X	X	O	O	X	X
5	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	O	O
6	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X

Figure 3. Expressing the generality by using the checklist.

Jessica's group was the one who drew a checklist and then detected a crucial pattern and specific regularity of each number table. Based on these critical discoveries, they were able to propose a conjecture which totally fitted the goal of this activity. The following is the transcription from the interview with Jessica:

Interviewer: Can you tell me what your group found from the tables?

Jessica: We found that 1, 2, 4, 8, 16 and 32 are the first numbers in these six tables after rearranging the numbers sequentially.

Interviewer: Your group drew a checklist of the numbers [on their worksheet], can you explain about it?

Jessica: When drawing a checklist, we found some interesting regularity of the tables. For example, table 1 comprises the odd numbers and starts with number 1; table 2 comprises two successive integers and starts with number 2; table 3 comprises four successive integers and starts with number 4; table 4 comprises eight successive integers and starts with number 8; table 5 comprises sixteen successive integers and starts with number 16; table 6 comprises thirty-two successive integers and starts with number 32.

Interviewer: Any further ideas you got?

Jessica: Yes, we noticed that these six numbers [what she meant are 1, 2, 4, 8, 16, and 32] are powers of 2. Additionally, any two of these numbers cannot appear in the same table. That's why there are six tables in this game.

From the students' responses (especially the above examples), it might suggest that this pattern-finding activity could not only provide opportunities for students to think actively, but also create a proper circumstance for students to evolve their cognitive processes. Moreover, all of the cases clarify that students could generate mental objects for thinking through observing and manipulating the number tables and representing for communication within the cooperation with peers in the pattern-finding activity. It coincides with the viewpoint of Dörfler (1991) that generalisation is not only an object for thinking but a tool for communicating.

Conjecturing and Convincing

To make a conjecture in the pattern-finding activity is one thing, however, to make others persuaded is another. Before persuading others, making oneself persuaded is essential. Therefore, together with observing the patterns through specialising and generalising, the students were also encouraged to propose conjectures based on the patterns they found, and to justify their conjectures to convince their peers.

We take Jessica's group as an example first since they proposed a strong conjecture but only applied some particular examples to make their justification. It might because they have not learned formal mathematical proofs yet.

(continued from the episode quoted formerly on this page)

Interviewer: Based on this discovery, did your group make any conjecture?

Jessica: We would guess that all the numbers from 1 to 63 can be expressed as a sum of 1, 2, 4, 8, 16 or 32.

Interviewer: How can you justify the conjecture?

Jessica: For instance, number 5 is equal to 4+1, number 9 is equal to 8+1, number 21 is the sum of 16+4+1, 61 is the sum of 32+16+8+4+1 and so on.

Interviewer: Can you explain to me how the game works?

Jessica: If the number you choose appears in table 1, then 1 is added. If it appears in table 2, then 2 is added. If it appears in table 3, then 4 is added. By the same rule, if it appears in table 6, then 32 is added.

Interviewer: Can you elaborate it more clearly?

Jessica: You just check a number, say, 11. Since 11 appears in tables 1, 2 and 4, so you have to count 1+2+8 which is 11.

Another example should be Tiffany's group. They noticed that the sum of 1, 3, 8, 15 and 28 is 55, so they made a conjecture that any number between 1 and 63 can be a sum of these five numbers [1, 3, 8, 15 and 28]. However, they could not apply it to most of the numbers. Until hearing Jessica's group's report, they seemingly suddenly saw the light, and rectified their answer.

Interviewer: What conjecture did your group make?

Tiffany: We noticed that the sum of 1, 3, 8, 15 and 28 is 55, so we speculated that any number can be made by these five numbers. But it seems not working with other numbers. Until Jessica addressed her group's result, we found we selected wrong numbers.

Interviewer: So, what did you do then?

Tiffany: We checked that 55 can be the sum of numbers 1, 4, 8, 16 and 32 [it should be 1, 2, 4, 16 and 32] as well. We also found that the sum of 1, 2, 4, 8, 16 and 32 is 63, which is the biggest number in every table.

Although all the six groups of students could find some generality of these number tables through observing and manipulating, most of them could not propose appropriate conjectures based the generality they found. However, from the case of Jessica's group, a suitable activity is still able to help the students develop the ability of conjecturing and convincing. Besides, through the cooperation with peers in group and sharing findings with the whole class, the students have more opportunities to construct their own understanding and thinking actively. Moreover, in contrast with the majority of students' former disposition, interestingly, they exhibited rather positive attitudes mathematics learning in this problem-solving activity.

CONCLUSION

The results of this study disclose that, through observing and manipulating, most of the students could develop the ability of specialising and generalising. Based on the results of specialising and generalising, some students might continue to make proper conjectures and justify them on their own. In the whole learning process, the collaboration within peers plays a vital role. Besides, the teacher's guidance is also crucial when students are stuck with a choke point, or miss some important clues. Furthermore, this activity seems to improve the research subjects' attitudes towards learning mathematics as they possessed very low learning motivation. Therefore, we could suggest this kind of activities should be good for the preparation of introducing a new mathematical concept. For instance, the activity designed in this study could be a good pioneering activity for introducing the concept of binary system.

This study offers some empirical evidence of Lin's claim in the plenary speech presented in APEC-TSUKUBA International Conference held in Tokyo, that "*A good lesson must provide opportunities for learners to think and construct actively*", and "*Conjecturing is the centre and pivot of all phases of mathematics learning – including conceptualising, procedural operating, problem solving and proving, and provides the driving force for developing these phases of mathematics learning*" (Lin, 2006). Above all, learning with understanding is essential for enabling students to solve the new kinds of problems they will inevitably face in the future (NCTM, 2000). According to their former learning experiences, most of the students are excessively focused on instrumental understanding without any extra energetic thinking. Consequently, the study is orchestrated for developing pupils' conjecturing and convincing power as well as for facilitating students' learning of mathematics towards relational understanding (in the sense of Skemp, 1987).

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AFFECTIVE, COGNITIVE, AND SOCIAL FACTORS IN REDUCING GENDER DIFFERENCES IN MEASUREMENT AND ALGEBRA ACHIEVEMENTS

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The results of the TIMSS 2003 study indicated that boys had higher measurement achievements than girls and girls had higher algebra achievements than boys. It was predicted in this present study that affective, cognitive, and social factors could reduce these gender differences. The results of a series of regression analyses showed that gender differences in measurement achievements could be reduced by the sub-factors of inductive affect, social backgrounds, and cognitively closed learning experiences, while those in algebra achievements by the sub-factors of deductive affect, cognitively open learning experiences, and social resources, in a descending sequence.

INTRODUCTION

Gender differences in math achievements have long been an issue in math education as there should be equal opportunity, treatment, and outcomes for both boys and girls (Fennema, 1990). There appears a trend that gender differences in math achievements have gradually diminished and remain only in the band of high achievers at the school stage, but more males still enroll in advanced math courses than females (Askew & Wiliam, 1995; Köller, Baumert, & Schnabel, 2001). In addition, qualitative differences between genders in math achievement still remain, as indicated by the result of the Trends in International Math and Science Study (TIMSS) of 2003 that boys were better at measurement than girls and girls were better at algebra than boys (Mullis, Martin, Gonzalez, & Chrostowski, 2004). Similar findings were obtained by Guiso, Monte, Sapienza, & Zingales (2008) using the data from the Program for International Student Assessment (PISA) of 2003. The identification of likely factors in reducing these gender differences can not only aid in creating an equal environment for both boys and girls to learn math but also foster an understanding of the qualitative differences in the relationships between gender and content domains in math.

Affective, cognitive, and social factors have all been found to be related to math achievement (Chiu, 2006, 2007). It is also likely that these factors are effective in reducing gender differences in math achievement. However, it is still necessary to ascertain the specific sub-factors to explain why there are differential gender differences in specific content domains in math.

Two affective sub-factors

There are two kinds of affective sub-factors which are likely to be effective in reducing gender differences in math achievement: inductive and deductive affects. Inductive

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affects are developed based on a long-term interaction with the world or an accumulation of a large amount of data from the world. The most significant affect in an inductive manner is confidence, two major sources of which are external or social comparisons with others' achievements and internal or intra-personal comparisons in achievements between different domains of knowledge (Chiu, 2008). Deductive affects are developed largely from drives or wills, which will help channel resources, focus attention, and overcome obstacles in order to search for some specific goals. One of the most significant affects in a deductive manner is academic aspiration.

Two cognitive sub-factors

Boaler (1998) compared math teaching strategies between two schools in England. In the school taking a content-based approach, students worked alone on a booklet and collected another one when finishing. There was no whole-class teaching and teachers interacted with individual students. On the other hand, students in a school focusing on a process-based approach were given open-ended problems and encouraged to develop ideas, extend problems, and relate math to daily lives. In addition, students discussed the meaning of their math work with peers and negotiated possible solutions. It was found that students in the content-based school had difficulty in solving real-life problems, while students in the process-based school had a deep approach to learning and using math. There were no significant gender differences in math achievement in the process-based school but boys had a higher achievement than girls in the content-based school.

Two social sub-factors

The sub-factors of social support may include (1) social backgrounds or the distal sub-factors, e.g., social and economic status (SES), and (2) social resources or opportunities, e.g., extra tutoring that is not part of regular school courses. It was found that social support was related to math achievements (Byrnes, & Miller, 2007). Guiso et al. (2008) found that gender differences in math achievements diminished in more gender-equal nations, which emphasized education, well-being, and political and economic status for females, but gender differences in geometry and arithmetic still remained in such nations. There appears to be a lack of evidence for the effect of social backgrounds and resources on gender equality in specific content domains in math.

The above three kinds of sub-factors in the affective, cognitive, and social aspects are likely to be related to gender differences in measurement and algebra achievements. The above claim is based on the following rationales. Girls are sensitive to external, social, and contextual messages and are likely to be highly influenced by inductive affects and social backgrounds. In addition, girls' tendency toward active reactions to social messages may imply that girls need cognitively closed learning experiences to concentrate on the pattern of measurement problems, which need cognitively focused thinking. On the other hand, boys' insensitivity towards social messages may decrease their ability to solve algebra problems, which requires dealing with complex messages and relationships. As such, open learning experiences may help foster boys' ability to deal with complex messages and relationships. In addition, boys' focus on one specific goal and adult investment by figures such as parents in channeling their efforts toward that goal is likely to compensate for their weakness in algebra, which requires practice and effort to achieve familiarity with and concentration in dealing with complex messages and relationships. Based on the above rationales, three hypotheses, also as depicted by Figure 1, are posited as follows.

1. Affective, cognitive, and social factors can reduce gender differences in measurement and algebra achievement.
2. Gender differences in measurement achievements can be reduced by the sub-factors of inductive affects (e.g., confidence), cognitively closed learning experiences (e.g., working on problems alone and reviewing homework in class), and social backgrounds (e.g., parental education levels).
3. Gender differences in algebra achievements can be reduced by the sub-factors of deductive affect (e.g., academic aspiration), cognitively open learning experiences (e.g., working in groups and relating math to lives), and social resources (e.g., receiving extra math tutoring).

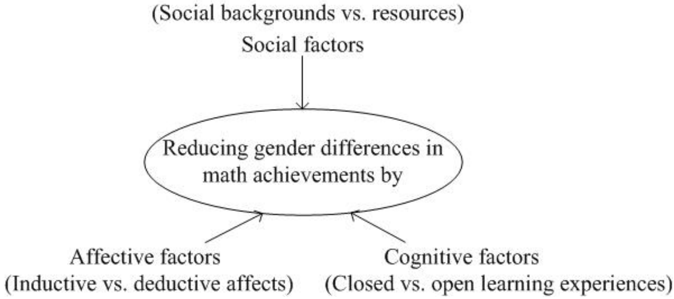

Figure 1: A model of affective, cognitive, and social factors in reducing gender differences in math achievements

METHOD

Participants

The participants were 230,229 Grade-8 students (50.3% girls, 49.2% boys, and 5% missing) from 47 countries participating in the TIMSS study of 2003.

Indicators

Five kinds of indicators (including 11 items) were taken from the database.

(1) Math achievements included students achievement results for *measurement* and *algebra* (TIMSS-variables bsmmea01 and bsmalg01).

(2) *Gender* (girls = 0; boys =1; TIMSS-variable itsex).

(3) Affective factors included students' *confidence* in learning math, e.g., 'I usually do well in math' (TIMSS derived-variable bsdmscl) and students' academic *aspiration* as to how far in school they expect to go (TIMSS-variable bsbghfsg).

(4) Cognitive factors referred to closed and open teaching strategies or learning experiences. Closed learning experiences included *working on problems on their own* and *reviewing their homework* in class (TIMSS-variable bsbmhwpo and bsbmhroh). Open learning experiences consisted of *working in small groups* and *relating math to daily lives* in class (TIMSS-variable bsbmhwsg and bsbmhmdl).

(5) Social factors comprised *parents' highest education levels* (TIMSS derived-variable bsdgedup) and *extra lessons or tutoring in math* that is not part of regular class (TIMSS-variable bsbmexto).

The achievement scores were obtained based on students' answers to a set of math problems in the content domains of measurement and algebra. The scores on the other indicators were derived from students' self-reports on a questionnaire. A higher score on all the indicators, except for gender, represented a higher achievement, degree, or frequency in the present study.

Statistical analysis

The major analysis method used here is linear regression. As suggested by the TIMSS 2003 user guide, student weights had to be used in all analyses in order to generate results representing the populations and SENWGT was used in the present study as it treated each country equally by setting a sample size of 500 for each country. Missing data were dealt with by pairwise exclusion in regression analyses.

RESULTS

Correlations between factors

The results of correlation analyses revealed that there were low correlations between all the items (below .331), except for a high correlation between measurement and algebra achievements (.873) (Table 1). The low correlations indicate a low degree of the problem of multicollinearity in regression analyses. No regression analysis was performed between the measurement and algebra achievements.

Factors in reducing gender differences in measurement achievements

The relation between gender and measurement achievements, or the regression coefficient for the effect of gender on measurement achievements, was small but significant (.022), as can be seen in Table 1 and in Model 1 (M01) in Table 2. The results mean that the .048% variance in measurement achievements could be explained by gender differences and that the positive value could indicate that boys are favored in solving measurement problems.

	M	A	1	2	3	4	5	6	7	8
Measurement achievement (M)										
Algebra achievement (A)	.873									
1. Gender	.022	-.045								
2. Confidence in math	.198	.201	.066							
3. Academic aspiration	.226	.265	-.069	.207						
4. Working on problems alone	.144	.139	.020	.133	.094					
5. Reviewing homework	-.069	-.045	-.029	.075	.050	.140				
6. Working in groups	-.260	-.255	.042	.029	-.026	.048	.117			
7. Relating math to daily lives	-.160	-.153	.051	.126	.031	.095	.201	.280		
8. Parental education levels	.330	.327	.017	.106	.240	.130	<u>.005</u>	-.146	-.071	
9. Extra math tutoring	-.169	-.138	.054	<u>-.003</u>	<u>-.018</u>	<u>-.016</u>	.050	.168	.114	-.073

Table 1: Pearson correlations between the 11 indicators. The correlations underlined are not significant at the .05 level.

The sub-factors that could reduce the regression coefficients for the effect of gender on measurement achievements included confidence (.022 in M01 → .009 in M02), working on problems alone (.022 → .019 in M04), reviewing homework (.022 → .020 in M05), and parental education levels (.022 → .017 in M08). The other sub-factors showed an increase in gender differences in measurement (M03, M06, M07, and M09). In addition, confidence alone could successfully reduce gender differences from significant (M01) to non-significant (M02). The two most effective sub-factors were confidence and parental education levels, which together could reduce the effect of gender differences from .022 to .006 (non-significant) (M10), and the three most effective sub-factors (i.e., confidence, parental education levels, and reviewing homework) all together could reduce gender differences from .022 to .005 (non-significant).

Factors in reducing gender differences in algebra achievements

The regression coefficient for the effect of gender on algebra achievement was -.045, which meant that .203% of the variance in algebra achievements could be explained by gender differences and the negative value revealed that girls were favored in solving algebra problems (M12 in Table 3). The effect of gender differences on algebra achievements was around four times (4.23 = .203% / .048%) larger than that on measurement achievements.

The sub-factors that could reduce the regression coefficient for the effect of gender on algebra achievements were academic aspiration (-.045 in M12 → -.023 in M14), working in groups (-.045 → -.030 in M17), relating math to daily lives (-.045 → -.033 in M18), and extra math tutoring (-.045 → -.033 in M20). None of these sub-factors could successfully reduce the significant gender effect to a non-significant one, perhaps partly because of the large effect of gender on algebra achievements. The two strongest sub-factors (i.e., aspiration and working in groups) together could reduce the regression coefficient for the effect of gender on algebra from -.045 to -.017 (M21), which, however, was still statistically significant. The two strongest sub-factors (i.e., aspiration and working in groups) with extra math tutoring all together could reduce the regression effect of gender on algebra from -.045 to -.013 (M22), which was non-significant. A point to note is that the two open learning experiences, working in groups and relating math to daily lives, and extra math tutoring were negatively related to algebra achievements but that these interventions and investments could effectively reduce gender differences.

Factors \ Models	M01	M02	M03	M04	M05	M06	M07	M08	M09	M10	M11
1. Gender	.022	<u>.009</u>	.038	.019	.020	.033	.030	.017	.031	<u>.006</u>	<u>.005</u>
Affective factors											
2. Confidence in math		.197								.164	.154
3. Academic aspiration			.229								
Cognitive factors (Math in class)											
4. Working on problems alone				.144							.048
5. Reviewing homework					-.068						
6. Working in groups						-.261					
7. Relating math to daily lives							-.162				
Social factors											
8. Parental education levels								.329		.312	.302
9. Extra math tutoring									-.170		

Table 2: Beta estimates obtained by regression analyses for the sub-factors in predicting measurement achievements. The estimates underlined are not significant at the .05 level.

Factors \ Models	M12	M13	M14	M15	M16	M17	M18	M19	M20	M21	M22
1. Gender	-.045	-.054	-.023	-.048	-.047	-.030	-.033	-.046	-.033	-.017	<u>-.013</u>
Affective factors											
2. Confidence in math		.201									
3. Academic aspiration			.261							.257	.256
Cognitive factors (Math in class)											
4. Working on problems alone				.140							
5. Reviewing homework					-.047						
6. Working in groups						-.253				-.247	-.232
7. Relating math to daily lives							-.150				
Social factors											
8. Parental education levels								.329			
9. Extra math tutoring									-.138		-.094

Table 3: Beta estimates obtained by regression analyses for the sub-factors in predicting algebra achievements. The estimates underlined are not significant at the .05 level.

DISCUSSION

The above findings indicate that affective, cognitive, and social factors can be effective in reducing gender differences in math achievements, but that there exist qualitative differences between the sub-factors in reducing gender differences in measurement and those in reducing gender differences in algebra. Gender differences in measurement achievements can be reduced by sub-factors such as confidence (inductive affects), parental education levels (social backgrounds), working on problems alone, and reviewing homework in class (cognitively closed learning experiences), in a descending sequence. On the other hand, gender differences in algebra achievements can be reduced by sub-factors such as academic aspiration (deductive affects), working in groups, relating math to lives (cognitively open learning experiences), and receiving extra math tutoring (social resources), also in a descending sequence. In addition, affective factors are the strongest factors in reducing both the weakness of girls in measurement and weakness of boys in algebra. The second strongest factor, however, is social factors for girls and cognitive factors for boys. This qualitative difference is further depicted in Figure 2.

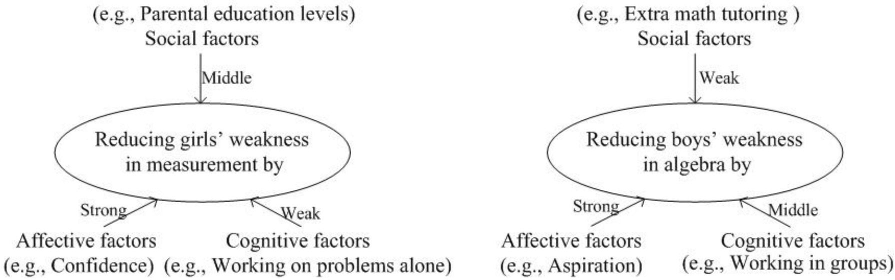

Figure 2: Differential affective, cognitive, and social sub-factors in reducing gender differences in measurement and algebra

The findings are consistent with the results of related studies that indicate that girls' weakness in math problem-solving is at least partly related to their weakness in affective factors, especially girls' low confidence in math (Gallagher & de Lisi, 1994). Academic aspiration is likely to be an important affective factor for boys. Closed and open teaching strategies or learning experiences were found to be related to gender differences in achievements in different math content domains. Past research on social factors in education typically focuses on SES. The recognition of the effect of social resources, which are provided to students in an active way, is a manifestation of the benefit that social investment can bring in improving students' math achievements.

The researcher took an integrated, domain-specific, and context-dependent approach to researching multiple factors in the relationships between gender and math achievements. In other words, it is argued that there is an integrated relationship between gender, math content domains, and cultural tools. Future research can further identify other effective sub-factors in affective, cognitive, and social aspects that may reduce gender differences in math achievements. For example, the frequent use of computers in learning math may be of benefit to boys and an interest-induced teaching program of benefit to girls.

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MISINTERPRETING THE USE OF LITERAL SYMBOLS IN ALGEBRA

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The present study focuses on the ways students interpret the use of literal symbols to stand for numbers in algebra. Students tend to think of literal symbols to stand for natural numbers only and not for any real number. Also students' tendency to think of literal symbols as natural numbers supports the ***phenomenal sign misinterpretation*** *which is the tendency to interpret seemingly negative algebraic expressions to represent negative numbers only.*

INTRODUCTION

Literal symbols are used as variables in algebra to stand for any real number. Because variables refer to numbers, students' prior knowledge of numbers could constrain their understanding of this concept. When students are first introduced to the concept of variable they tend to interpret literal symbols to stand for objects or abbreviated names of objects and not for numbers. For example, they may think that 'D' stands for David, or 'h' for height. Or, they may believe that 'y'-- in the task "add 3 to 5y"-- refers to something that starts with a 'y' like a yacht, a yoghurt, or a yam (Booth, 1984; Kieran, 1992; Stacey & MacGregor, 1997). While the above mentioned mistakes are usually abandoned by 10th grade (Asquith, Stephens, Knuth, & Alibali, 2007; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Stacey & MacGregor, 1997), some of the other difficulties students have with literal symbols are more persistent and not easily corrected. One such difficulty is the tendency to think that literal symbols represent only one, specific, number as opposed to generalized number (Asquith et al., 2007; Booth, 1984; Knuth et al., 2005; Kuchemann, 1981).

It is generally believed that the mathematical concept of variable has been mastered when students come to understand that variables stand for multiple numbers. In previous research we have provided some evidence that there is another possibility for error in the transition from arithmetic to algebra, which centers on students' ideas regarding the kind of numbers that can be substituted for the literal symbols (Christou & Vosniadou, 2005; Christou, Vosniadou, & Vamvakoussi, 2007). Our analysis of the problem, from a conceptual change point of view, leads us to believe that even when students understand that variables stand for many numbers and not a specific one, they could still believe that these numbers are natural numbers only. This error is predicted on the grounds that students' initial concept of number is that of natural number.

Using the conceptual change theoretical framework we argue that by the end of elementary school most children have built a relatively coherent explanatory

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 329-336. Thessaloniki, Greece: PME.

framework for number which is close to the mathematical concept of natural number (Gelman, 2000; Vosniadou, Vamvakoussi, & Skopeliti, 2008). Some researchers refer to this explanatory framework as the "whole number bias" (see Ni & Zhou, 2005). This initial understanding can subsequently constrain the development of more advanced mathematical concepts which differ from natural numbers both in their symbolism and their properties; see for example the case or fractions or rational numbers (Ni & Zhou, 2005).

Previous studies

In previous studies we investigated the hypothesis that students' prior knowledge of numbers and their special commitment to natural numbers would affect them to tend to think of literal symbols in algebra to stand for natural numbers only. It appeared that when students were asked to assign any number they wished to the literal symbols given in algebraic expressions, the majority of them substituted only natural numbers for the literal symbols ending up with responses such as natural numbers in the case of 'a', positive fractions in the case of 'a/b', multiples of 4 in the case of '4g', and negative whole numbers (-1, -2, -3, etc.) in the case of '-b' (Christou & Vosniadou, 2005). Very few students responded with numbers other than natural numbers, such as decimals, fractions, negatives, or real numbers.

It also appeared that students' interpretation of the numbers that can be represented by an algebraic expression was affected by the *phenomenal sign* of the variables and the algebraic expressions. We call *phenomenal sign* the external sign that a variable (or an algebraic expression which contains one or more variables) has, as a superficial characteristic of its form. As we know, a variable does not have a specific sign of its own and the value of an algebraic expression with a negative phenomenal sign, such as '-b', can be either positive or negative depending on the numbers substituted for the literal symbol. Students tended to respond that only positive numbers can be substituted for seemingly positive algebraic expressions such as 'a', '4g', 'k+3', etc., and they excluded negative numbers as possible substitutions. In the same line, the majority of the students excluded positive numbers as possible substitutions for '-b' which they tended to interpret as representing only negative numbers (Christou et al., 2007). Students' tendency to erroneously interpret the phenomenal sign of the algebraic expressions to be the actual sign of the numbers they can represent is called the *phenomenal sign misinterpretation (PSM)*.

Current study

From the conceptual change perspective we do not consider the PSM in an isolated manner but as related to aspects of students' prior knowledge. More specifically, we interpret the PSM to be supported by students' belief that literal symbols stand for natural numbers only. The purpose of the current study was to examine this hypothesis, namely that students' tendency to think of literal symbols as natural numbers supports the PSM. Also, to further investigate the initial hypothesis that students tend to interpret the literal symbols in algebra to stand for natural numbers only when they appear in a more specific mathematical context such as that of functions.

We measured students' phenomenal sign misinterpretation in two mathematical domains: in solving inequalities and in defining the domain of square root functions (SRF); that is functions that contain square roots with algebraic expressions to appear in the radicands. Those two mathematical domains were used because they both are of significant importance in the school mathematical activity and the role of the phenomenal sign in those domains is regulative.

There is a fundamental rule concerning transformations in inequalities mentioning that when multiply or divide the sides of an inequality with a negative quantity then the inequality sign must be changed. One of the most frequent mistakes students do in such tasks is to divide or multiply both side of the inequality with an algebraic expression without changing the inequality sign accordingly (Bazzini & Tsamir, 2004). Based on our previous findings we would expect that even the students who know the above mentioned rule might do this mistake because of the phenomenal sign misinterpretation. In the following study we investigate this hypothesis.

Also, as far as the square root functions are concerned, there is a rule concerning square roots mentioning that they are defined as real only for non-negative radicands. We would predict that the phenomenal sign misinterpretation would lead students to specific mistakes concerning the domain of a square root function.

We would expect that the students would do better in the tasks concerning square root functions than in the inequalities because of the different characteristics of these two domains. In the domain of algebraic inequalities, the sign of the algebraic expression is used in an indirect way during the transformation process. On the other hand, the sign of the algebraic expressions that appear in a SRF is an ontological characteristic of this concept and students have learned right from their introduction to square roots to question about the sign of the radicand.

The last hypothesis we wanted to test in this experiment was students' tendency to be more affected by the phenomenal sign of a seemingly negative expression than from a seemingly positive one. In our previous studies it appeared that students were more affected by the phenomenal negative sign of '-b' which has been interpreted as an expression which stand only for negative numbers than from the phenomenal positive sign of the other expressions that were used.

METHOD

Participants

The participants were 110 10^{th} graders; 60 of them were boys and 40 were girls.

Materials

In order to investigate whether students' performance would be better in the domain of SRF than in the domain of the inequalities independently of their position in the questionnaire we designed two types of questionnaires which contained the same tasks but were presented with different order. The first two tasks in both questionnaires included two functions: f1(x): 2x+1, f2(x)=1/x. Students were asked to fill in a table of values with seven cells and draw the graph in a void frame; Cartesian axes were not provided. In this task the students had the opportunity to assign any number they wished to the variable of the given function in order to fill the table of values and make the graph. These tasks were specifically designed to test with an indirect way the kind of numbers the students tend to assign to the variables. Students who would tend to think of variables as natural numbers only would end up with incomplete graphs.

The tasks that followed included three algebraic inequalities and three square root functions (SRF). In questionnaire A (QR/A) the inequalities appeared before the SRF and in questionnaire B (QR/B) the SRF were appearing before the inequalities. Both the inequalities and the SRF contained algebraic expressions with either positive or negative phenomenal sign. Half of the population completed QR/A and the other half completed QR/B.

As far as the inequalities are concerned, the questionnaire contained three inequalities completely solved by a hypothetical solver through a series or transformations. Each step of the transformation was explained in accompanied comments inside of brackets. One of the inequalities for methodological reasons was presented fully correct and the other two each with a deliberate mistake. The mistake was that both sides of the inequality was multiplied with the algebraic expression which appeared in the denominator of a fraction and the inequality sign was changed or not changed according to the phenomenal sign of this expression. In other words, the hypothetical solver correctly applied the rule which concerns transformations in inequalities but considered the phenomenal sign of the algebraic expression to be its real sign. Such an example is presented in Figure 1.

$3 < \frac{1}{2x}, x \neq 0$

Step 1: 6x <1 (because 2x is positive and we can multiple with a positive without changing the inequality sign)

Step 2: $x < \frac{1}{6}$ (because 6 is positive and we can divide by a positive without changing the inequality sign)

Do you think this inequality was correctly solved? YES☐ NO ☐

If not, what was the mistake and in which step?

Figure 1: An example of a task concerning an inequality with positive phenomenal sign solved with a deliberate mistake

The square root functions were presented to the students with a statement arguing about whether each SRF was defined for any real number that the variable could take. One of those SRF was presented with a correct and the rest with a false statement. The mistake was that the statement was taking under consideration the phenomenal sign of the radicands (which was either positive or negative) and not their real sign. For example, the students were asked:

"$\sqrt{-2x-1}$ is never defined for any number that could be substituted for x, because -2x-1 is always a negative number. Agree or disagree and why?".

We would expect that the students who would tend to think of the phenomenal sign of the given expressions to be their actual sign they would make specific mistakes in the above mentioned tasks.

Results

As it was expected, the majority of the students substituted only natural numbers for the variables. More specifically, 81.8% of students' responses in f1 and 78.2% in f2 were only with natural number substitutions for the variables. Those students drew graphs which were incomplete and were extended only in the first quadrant of the Cartesian axes.

In the remaining tasks that included inequalities and SRF, students' responses were scored in right/wrong basis and students' mean scores were calculated both for the two inequalities and the two square root functions that contained a deliberate mistake (see Table 1). Students' responses in the correctly given tasks were not taken under consideration because those questions were used for methodological reasons and were not contributing to the measurement of the PSM in the same way as the other questions did. Mean scores were also calculated for the tasks that contained algebraic expressions with positive phenomenal sign and for that with negative (see Table 2).

Tasks	**Performance**			
	Mean score	N	St. deviation	maximum
Inequalities	0.36	109	0.7	2.0
SRF	1.18	109	0.86	2.0

Table 1: Students' performance in the incorrectly given inequalities and SRF

As it appear from Table 1, students' tendency to be affected by the phenomenal sign of the algebraic expressions and to interpret it as the sign of the numbers they can only represent resulted in low performance both in the domain of inequalities and SRF. As it was expected, students did better in the SRF than in the inequalities *$F(1,107)=83.197$, $p<.001$* but analysis of variance with repeated measurement indicated no interaction between the questionnaire type and students' performance in the two different domains, *$F(1,107)=0{,}438$, n.s.* This result showed that students did better in the domain of SRF independently of their position in the questionnaire.

Students did significantly better in the tasks that included expressions with positive phenomenal sign than with negative *$F(1,108)=18.616$, $p<.001$.* In order to further examine this later result we compared students' responses in the task that contained inequality with a positive phenomenal sign with the inequality that contained negative phenomenal sign.

Tasks	Performance			
	Mean score	N	St. deviation	maximum
Positive phenomenal sign	0.89	109	0.69	2.0
Negative phenomenal sign	0.66	109	0.68	2.0

Table 2: Students' mean performance in the tasks that contained expressions with positive or with negative phenomenal sign

As it appears in Table 3, only 2.4% of the students who failed in the inequality with a positive phenomenal sign succeeded in the inequality with negative phenomenal sign. On the other hand, 40% of the students that responded correctly in the inequality with positive phenomenal sign failed in the inequality with negative phenomenal sign, *x^2(I, N=110)=49.133, McNemar Test value $p<0.05$.*

Type of the given inequality		Incorrect inequality (negative phenomenal sign)	
		wrong	right
Incorrect inequality (positive phenomenal sign)	wrong	97,6	2,4
	right	40,0	60,0

Table 3: Percent of students' responses in the inequalities that included a deliberate mistake

The same was the case also for students' responses in the tasks with square root functions. More specifically, as it appears in Table 4, only 11.1% of the students who responded successfully in the SRF with a negative phenomenal sign failed the one with positive phenomenal sign. On the other hand, 28.8% of the students who responded successfully in the positive-like SRF failed in the negative-like one, *x^2(I, N=109)=34,886, McNemar Test value $p<0.001$.*

Those results show that, as it was expected, students were less likely to be affected by the positive phenomenal sign than by the negative phenomenal sign of an expression. The students appeared more willing to accept that positive-like expressions could also stand for negative numbers than to accept the reversed, that negative-like expressions could also stand for positive numbers.

Type of the given square root function		Incorrect SRF (negative phenomenal sign)	
		wrong	right
Incorrect SRF (positive phenomenal sign)	wrong	88,9	11,1
	right	28.8	71.2

Table 4: Percent of students' responses in the SRF that included a deliberate mistake

In order to examine the main hypothesis of this study, namely that it is students' tendency to think of literal symbols as standing for natural numbers only that supports the phenomenal sign misinterpretation, we created a new variable that counted the number of the functions (that is zero, one or two) in which each student substituted at least one non-natural number to the variable of the function. Regression analysis showed that the way students responded to the tasks concerning substituting numbers for the variables in order to make the graph of the functions contributed significantly on their performance in the remaining tasks that were measuring the phenomenal sign misinterpretation, $\beta=.318$, $t=3,472$, $p<.001$. In other words, students who tended to substitute at least one non-natural number to any of the given variables they were less likely to show the phenomenal sign misinterpretation.

DISCUSSION

The results of this study supported our main hypothesis as stemming from the conceptual change theoretical framework (Vosniadou et al., 2008) that students' initial knowledge of numbers, which is organized around natural numbers, would constrain students' interpretation of the use of literal symbols in algebra to stand for any real number. Students tended to interpret literal symbols to stand for natural numbers only. The majority of them substituted only natural numbers in the variables of two given functions even though this resulted in incorrect and incomplete graphs.

Also, students tended to interpret the phenomenal sign of the algebraic expressions to be their actual sign and that resulted in specific mistakes and low performance in certain mathematical tasks such as in transformations with inequalities and defining the domain of square root functions. The results supported our hypothesis that students' erroneous assumption that literal symbols are used in algebra to stand for natural numbers strengthen the phenomenal sign misinterpretation. Students who tended to substitute at least one non-natural number to the variable of the given functions were less likely to show the PSM.

It appeared that students were more likely to be affected by the negative phenomenal sign of the given expressions than by the positive phenomenal one. The majority of the students tended to interpret seemingly negative algebraic expressions such as '-2x-1' to represent negative values only (see also Vlassis, 2004). Also, the mathematical domain in which the algebraic expression appear play a role in the PSM; students were less likely to be affected by the phenomenal sign in the context of square root functions than in the transformation with inequalities.

We believe that for students to remedy the PSM they must accept real number substitutions for literal symbols and that requires that they have fully acquired the mathematical concept of variable as a symbol that can stand for any real number. For students to do that they must reorganize their initial knowledge of number which is organized around natural numbers, though the process of conceptual change. We would expect this to be a gradual and long lasting process (see Vosniadou, 2006). The natural numbers bias is deeply ingrained in students' knowledge base (Ni & Zhou, 2005) and for students to expand their conceptual fields beyond natural numbers specific educational approaches who focus on this issue is required. It is important for mathematics teachers to be familiar with the natural number bias and the problem of conceptual change in learning mathematical concepts in order to better understand students' mistakes and the possible reasons why they may appear and reconcile their teaching approach to deal with these mistakes.

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WHAT ARE THE PRACTICES OF MATHEMATICS TEACHER EDUCATORS?

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Stemming from questions raised at the 100th anniversary meeting of the International Commission on Mathematical Instruction (ICMI) held in Rome, Italy, this research examines the practices of three mathematics teacher educators in different international contexts. Using personal narrative and a "researcher as subject" framework, we engage in what we refer to as panoptic and telescopic reflection to contrast and compare our practices, whilst at the same time attending to the evocative nature of the comparisons.

INTRODUCTION

In 2008, the International Commission on Mathematical Instruction (ICMI) held the 100th anniversary meeting in Rome, Italy. An integral aspect of this meeting was collaborative working groups on a variety of topics related to past and future intellectual interests of ICMI. One working group, *The Professional Formation of Teachers*, led by Dr. Deborah Lowenberg Ball and Dr. Barbro Grevholm, focused on the historical evolution of conceptions of the various forms of knowledge and skills for mathematics teaching practice. Each of the authors of this paper was a participant in this working group.

An important question raised by Ball and Grevholm (2008) regarding the deliberations of this working group was related to the practices of mathematics teacher educators; namely, *what are the practices of mathematics teacher educators and how do these practices develop?* We saw this question as one of considerable importance, particularly as related to more prominent discussions about "mathematics for teaching" (Adler & Davis, 2006; Ball, Bass, Sleep, & Thames, 2005). We reflected on our development as mathematics teacher educators in an earlier paper (Clark, Kotsopoulos, & Morselli, 2009). Our intent in this paper is to explore and contrast our practices as mathematics teacher educators across three contexts. Our goal is to move beyond descriptive and/or indoctrinated views of what ought to occur in mathematics teacher education, but rather to examine critically the ontological and historical basis of our practices as a means for opening discussions across other contexts as well about the work of mathematics teacher educators.

THEORETICAL FRAMEWORK

We draw on Feiman-Nemser and Remillard's (1996) *learning to teach* framework to examine our practices as mathematics teacher educators in three contexts. The authors proposed that teacher educators make decisions about the process of learning

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to teach around answers to the following questions: *Who* are the students learning to teach? *What* should students be learning? *How* shall the learning to teach proceed? And, *when* and *where* will the learning to teach take place? The first question examines the beliefs and histories of pre-service teachers. The second and third questions are ones that can be easily isolated to mathematics teacher educators' own teaching philosophy. All the questions, and particularly the final question, may be significantly impacted by individual contexts, both political and institutional views of mathematics education (da Ponte & Chapman, 2008). Furthermore, the answers to these questions are highly subjective.

LITERATURE REVIEW

As Llinares and Krainer (2006) noted, the professional growth of teacher educators may be studied according to two main streams of research: analysis of in-service professional development programs with a focus on the development of teacher educators (e.g., Even, 2005; Zaslavsky & Leikin, 2004) and self-reflection on one's path of development as a teacher educator while teaching teachers (e.g., Onslow & Gadanidis, 1997; Tzur, 2001). In both cases "theoretical references conceived for the reflection on professional development of mathematics teachers are borrowed to describe and discuss teacher educators' professional development" (Llinares & Krainer, 2006, p. 447). Llinares and Krainer proposed that the work of mathematics teacher educators' is a "field where theory and practice of teacher education inevitably melt together and we thus face the challenge of self-applying our demands on teacher education" (p. 429).

Analysis of professional development programs reveals the reality that there is no specific curriculum for mathematics teacher education that needs to be "covered" (Zaslavsky, 2007). Consequently, the degree of subjectivity and autonomy between curricula of mathematics teacher educators may only be aligned to varying degrees (Blömeke, Felbrich, Müller, Kaiser, & Lehmann, 2008; Llinares & Krainer, 2006). The lack of specific curriculum is further challenged by the fact that mathematics teacher educators differ from mathematics teachers in terms of access to suitable tasks for engaging in mathematics learning for the purpose of learning to teach mathematics (Zaslavsky, 2007). Watson and Mason (2007) noted that "although effective mathematical tasks for teachers share many of the features of effective mathematical tasks for learners, tasks for teachers also serve a higher-order purpose" (p. 208). Therefore, although mathematics teacher educators borrow and self-apply research and approaches from mathematics teaching, there are differences nonetheless.

Llinares and Krainer (2006) noted that mathematics teacher educators' growth is "a learning-through-teaching process supported by reflective practice" (p. 447). Similarly, Chapman (2009) discussed the education of teacher educators seen as reflective practitioners and as researchers, arguing that both processes (reflecting on his/her practice, researching on his/her practice) are crucial for professional growth.

METHODOLOGY

In earlier work (Clark, Kotsopoulos, & Morselli, 2009), we explored our development as mathematics teacher educators across our three contexts (i.e., United States, Canada, and Italy) using Ellis and Bochner's (2000) framework of *researchers as subjects*. Here again we engage in the sharing of personal narratives as a means of contrasting, comparing, and reflecting across our own and each other's narratives to answer the research question. We engaged in what we refer to as *panoptic and telescopic reflection*: 360˚ of reflection from all angles with the ability to zoom in and out at particular instances. To facilitate our research across our three contexts, a secure virtual communication space (i.e., WIKI), was created where we could (a) post questions to one another; (b) engage in discussions; (c) post self-reflections; and (d) share literature. As a starting point, we shared the syllabi of our courses (for primary teachers and secondary teachers) and engaged in discussion and comparison in our virtual space. Our analysis moved beyond a mere comparison of syllabi, in that "living" exchanges also formed an additional source of information that facilitated further questions and reflections. Excerpts of the exchanges are set apart (using italics) in the text that follows. We organize the results of our exchange according to the dimensions outlined by Feiman-Nemser and Remillard (1996).

FINDINGS AND DISCUSSION

Who

In Canada, teacher education is predominantly a post baccalaureate degree. Programs vary between one and two years. Some programs in Canada offer concurrent degree programs in education and another degree (i.e., Bachelor of Arts, etc.), and very few offer a terminal baccalaureate degree in education. In comparison, traditionally-prepared teachers in the United States are required to complete an undergraduate degree in teacher education (with an emphasis in their subject area) or an undergraduate degree in their subject concentration (with secondary education course work). In Italy, teacher education for primary and secondary teachers is different. In order to become a primary teacher it is necessary to obtain the university degree in Primary Education. Prospective secondary teachers must enter a two-year training program after a university degree relating to the taught subject. Across all three contexts, there is a great variety of ages, social backgrounds, and motivations of those pursuing teaching as a career. Admission to teacher education in Canada is highly competitive. In Italy, only a limited number of applicants have access to the training program through an admission test. Admission to teacher education is much more accessible in the United States.

Unlike Italy, each state/province (and conceivably each college and university with teacher preparation programs within a state) in Canada and the United States establishes its own standards and requirements for teacher education; there are also national standards and state standards that govern programming and education mandates. In the United States and in Italy, there may be pre-service teachers who already have temporary teaching positions in school prior to completing their teacher education. In particular in the United States, this most often occurs in alternative certification programs. This is relatively rare or only in specific circumstances (e.g., private schools) in Canada.

Common across all three contexts is the reality that elementary pre-service teachers' mathematics is limited to their own experiences in elementary and secondary schooling. As a consequence, many of our pre-service teachers report an ill-ease when dealing with mathematics. Francesca explained, "*I feel a real challenge to conceive a course that is suitable for the different histories and backgrounds of my teachers. I have to take into consideration that some of them already have strong beliefs about teaching and also have a negative relationship with mathematics*." It is also evident that a negative relationship with mathematics is not just characteristics of pre-service primary teachers. Kathy noted that, *"At least half of the middle grades pre-service teachers at my institution admitted to changing from the secondary program because of their inability to master secondary topics or their lack of desire to teach such 'advanced' mathematics."*

What

We all agreed on the importance of our experiences as producers and consumers of research in structuring our work as mathematics teacher educators. In each of our courses, pre-service teachers engaged in some form of critical examination of research (e.g., reading, personal reflection, discussing articles, or lectures based upon research articles). In Donna's course, pre-service teachers engage in classroom-based research projects (e.g., Jaworski's (2006), communities of inquiry) where pre-service teachers conduct research based upon authentic questions about their own learning.

Our virtual discussions revealed some important tensions in terms of accessibility to mathematics education research which is published predominantly in English. Francesca pointed out, that although many of her pre-service teachers "*could understand and even read English, the level of proficiency often is not at a level that is conducive to understanding mathematics education research that is predominantly written in English*." This led us to reflect deeply on the issue of language transparency in mathematics education research. As Francesca explained, her options to adopt research articles were limited to four: abandon providing research articles in English and limit the choice to articles in Italian; translate the articles for or with the teacher candidates; provide extended time for a translation in small in groups; present a summary of the research articles that results in a "filtering process" carried out by the mathematics teacher educator.

We also found some commonality in the role of mathematical tasks in learning to teach mathematics. Mathematical tasks may have different pedagogical purposes: "to consider implications for teaching, including designing and trying out related mathematics-related tasks with learners, as well as extending and varying task structure and task presentation (teaching experiments) with the aim of developing the habit of ongoing innovation, observation and reflection" (Watson and Mason, 2007, p. 207). In Donna's course, pre-service teachers engage in mathematical tasks via mock teaching experiments to one another, where the selection and didactic decisions about the mathematical task were relatively autonomous. Kathy's pre-service teachers engage in mathematical tasks in four different ways: individually, in groups, or whole class as part of the weekly class sessions, in groups to investigate and present a sample mathematical task from one content standard (NCTM, 2000), in groups to investigate and present mathematical problems found in practitioner journals, and in groups to plan and implement a micro-teaching project.

Francesca distinguishes in her teaching between tasks at pupil level and tasks at adult level, meaning that the second ones are useful for teachers even if they are not directly transferable to the classroom. Pre-service teachers are not asked to create tasks for pupils. At the pupil level, pre-service teachers are asked to analyze and discuss in groups exemplary tasks created by a national commission of researchers, mathematicians, and mathematics educators according to the most recent research findings ("*Mathematics for the citizen*" project). At the adult level, pre-service teachers are involved in doing mathematical tasks created by her in order to work on mathematical concepts, experience real mathematical activity, and possibly question their beliefs about mathematics.

We agreed, that collectively a natural consequence of task-based mathematics education teaching was, as Kathy put it, "*to challenge prospective teachers' beliefs about what it means to do mathematics and the skills required to construct meaningful tasks.*"

How

A key difference was noted between the extents to which the mathematics teacher education curriculum is made visible to the pre-service teachers. In Donna and Kathy's courses, pre-service teachers have access from the onset of the course to all course readings, assignments, class topics, assessment tools, and so forth, which is a common format for university-level courses in Canada and the United States. Although, as Kathy pointed out, a hybrid scenario may exist, where most course materials and readings available from the onset and others coming available during the course when ready for dissemination. In Francesca's course pre-service teachers only have a general overview of the content of the course at the beginning, but the topics and readings are presented week by week.

Although each of us provided many opportunities for reflective practice in our courses, the differences in the transparency of our pedagogy raised discussion amongst us about the ways in which *a priori* reflection occurred – the ability for pre-service teachers to think about course content in advance. As Francesca pointed out, "*My students receive in my course some hints for reflection, as well as theoretical reference in order to frame their reflection. The real reflection is 'in itinere' (i.e., 'on journey') and a posteriori. During the course, they often intervene linking what I say* *to what they see during the practicum, or what they learnt during other courses. Usually, they appreciate that the lectures are not completely established a priori, rather negotiated*." This is very different from Donna's position: "*As far as reflection and course progression - my own course has a script, as you have seen. This is not to say that there is no organic or emergent component - topics do often veer to address the on-going reflective stance of the pre-service teachers. My view is that if we are looking for true reflective practice - one that teachers will carry forward into their practice as teachers, I would want to develop that skill of a priori reflection, and place value on it*."

We found consensus in the kinds of reflections pre-service teachers shared and our challenges as a result of those reflections. As Kathy explained, "*It seems that pre-service teachers skip a critical step of reflective practice: that they need to experience teaching in some way before they can know what works for them as a teacher and more importantly, for the pupils they will teach*."

Where and when

Significant restrictions on each of our practices existed – particularly in relation to how pre-service teacher education was delivered. For example, each of us had different national mandates in terms of the nature and duration of teacher education. Additionally, each of us had institutional demands that cannot be obliterated by our shared adherence of some fundamental approaches (i.e., reflection, tasks, etc.). For example, Francesca's program separated the theoretical portion from the applied portion (i.e., developing lesson plans, etc.), whereas Donna and Kathy's courses were integrated. Additionally, the relationship of our courses to other aspects of mathematics teacher education and the practica were inconsistent. Donna did not have practica supervision but had access to the practica. Kathy did have practica supervision. Francesca did not have practica supervision or access, despite that the practica occurred parallel to her course.

CONCLUSIONS AND EDUCATIONAL IMPLICATIONS

The act of engaging in panoptic and telescopic reflection across our three contexts provided a unique opportunity to raise questions and engage in discussion. Our research shows, that although in principle we held similar beliefs and values for some of the meta-narratives in education and mathematics education, our practices nevertheless significantly differed. Numerous conclusions emerged from our virtual explorations of our practices. These are: (1) practices across context vary despite common beliefs in what matters in mathematics teacher education, (2) the autonomy of mathematics teacher educators' practice may be over estimated, (3) differences in course structures may provide different opportunities for reflective practice, (4) research transparency may be limited for pre-service teachers who do not have English language proficiency, (5) mathematics teacher educators may take on the role of research filter for pre-service teachers, (6) mathematical tasks are emphasized and used differently, and (7) pedagogical transparency varies between mathematics teacher educators.

A limitation of our research is that the conclusions we have outlined may be isolated to those contexts in which each of us is practicing. We feel, nevertheless, that our work raises some important questions for the field of mathematics education: (1) *What are the effects of language transparency of English dominant research publications on mathematics teacher education in areas where English is not the lingua franca?* (2) *What are the implications of a priori, in itinere, and a posteriori reflection in mathematics teacher education and in what ways can reflection be meaningfully occasioned?* (3) *What are the sorts of essential mathematical tasks pre-service teacher must engage in whilst learning to teach?* (4) *What are the implications of significant variability between the practices of mathematics teacher educators to "mathematics for teaching" frameworks?* And, (5) *What impact does variation in pre-service teacher education have on classroom learning?*

Significantly more and varied research is needed to examine these questions. Our examination across three contexts is the strength of this research. Our findings demonstrate that research across contexts is necessary to interrogate individual contexts and to explore taken-as-shared assumptions about the practices of mathematics teacher educators and the effectiveness of mathematics teacher education.

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EQUIPARTITIONING/SPLITTING AS A FOUNDATION OF RATIONAL NUMBER REASONING USING LEARNING TRAJECTORIES

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Rational number reasoning is synthesized with a map and learning trajectories towards three key meanings of a/b: ratio, fraction-as-number, and operator. Primary distinctions of many-to-one, many-as-one, and times-as-many from fair sharing and equipartitioning/splitting form the foundation of rational number reasoning.

INTRODUCTION

Rational number reasoning and multiplicative conceptual fields are perhaps the two most intensively researched areas in mathematics education. These topics are regularly given a place in handbooks (Lamon, 2007; Thompson & Saldanha, 2003) and are the subjects of entire books (Hart, 1984; Streefland, 1991) and edited volumes (Carpenter, et al., 1993; Harel & Confrey, 1994). Areas of agreement include the recognition of multiple interpretations of what is symbolically represented as a/b, which vary according to contexts based on students' thinking and representational expression. But there remain needs for wide consensus in the field on key concepts and for major improvements in instruction. An enormous number of distinctions also remain, which have swelled the research to such an unwieldy size that it cannot readily serve as a guide to instruction. Teasing this richness into a coherent resource is the challenge of synthesis.

THE RNR SYNTHESIS: GOALS AND SCOPE.

We identified seven major areas of research in rational number reasoning (RNR): 1) equipartitioning/splitting; 2) multiplication and division, 3) ratio, proportion, and rate; 4) fractions, 5) area and volume; 6) similarity and scaling; and 7) decimals and percents. For each, we gathered research by reviewing major (English-language) journals, books, and refereed conference proceedings, and solicited contributions over the web. Our database summaries of 600+ articles includes: title, author, source type, theoretical/empirical nature of each study, topic, grade level, assessment items, the study demographics, analysis, and abstract (Confrey, J., et al., 2008). Articles are listed once under their primary area.

Once we have reviewed a topic area, we write a synthesis it by examining the theoretical claims and the empirical evidence. Syntheses are not simply literature reviews. We subscribe to the approach of Cooper (1998), that "the investigator must propose overarching schemes that help make sense of many related but not identical studies." (p. 12). Our aim in these syntheses is to identify areas of consensus about

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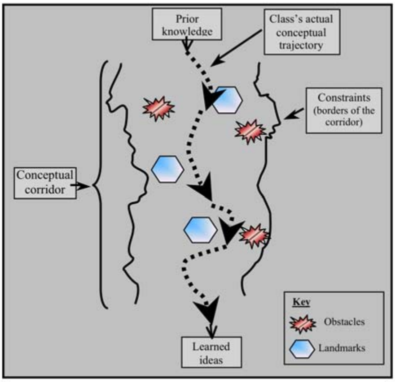
children's thinking, with the anticipated result that "The cumulative results are more complex than any single study, because they have to explain higher-order relations." (p. 13). "Perhaps the most challenging circumstance in the social sciences occurs when a new concept is introduced to explain old findings." (p. 17) Our syntheses have four goals:

- Identify common significant themes and findings;
- Introduce new distinctions to resolve differences or to integrate results;
- Discern the most compelling results among controversies;
- Prepare to implement robust results into practice.

We selected the construct of learning trajectories as the basis of our synthesis work, because of its potential to unpack complexity by revealing characteristics of gradual student learning over time. Building on the work of (Brown, 1992; Cobb, et al., 2003; Lehrer, et al., 2007; Simon & Tzur, 2004) and others we have constructed a working definition for Learning Trajectory:

> A researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey, Jere, et al., 2008)



We see learning trajectories as embedded within conceptual corridors (Figure 1). This implies multiple possible paths, but also includes various landmarks and obstacles that students typically must encounter. We see local (district), state, and national standards as appropriate components of the boundaries of the conceptual corridor. We are focusing on the design of better assessments to gauge student progress in ways that provide instructional guidance to teachers.

Figure 1. Conceptual Corridor

A number of features of this view of learning trajectories distinguishes our approach from others:

1. "researcher-conjectured" refers to the fact that learning trajectories are models, by researchers, of students' likely paths;
2. "empirically-supported" refers to a three-step process: reviewing the literature, asking outside experts to review our syntheses, and conducting further studies;

3. "through instruction" denotes the recognition that students will only progress if provided appropriate opportunities and technology/tools to learn the material and that the sequence of those activities must be designed intentionally to support the trajectory;
4. "through successive refinements" indicates the need for students' active involvement in the learning process and the need to engage in cycles of problem-solving behavior.

EQUIPARTITIONING/SPLITTING AND FRACTION, RATIO AND DIVISION/MULTIPLICATION

We initially identified six content strands for synthesis, but existing literature had a number of articles that explored the construct of 'partitioning.' These defined partitioning in a variety of ways, applied to various situations, but all had in common the generation of equal-sized groups or parts. We identified four distinct cases: sharing a collection (A), sharing a single whole (B), sharing multiple wholes with more persons than objects (C), and more objects than persons (D). Eventually, we determined that "equipartitioning" is a more mathematically satisfactory descriptor for the overall construct, because, unlike breaking, fracturing, segmenting, etc., equipartitioning pertains to behaviors that create *equal-sized groups*. In addition, we would assert that division is most directly derived from equipartitioning, with multiplication following as its inverse, rather than the traditional view that multiplication precedes division.

We posited that the generation of fair shares underlies, as a foundation, the most essential elements of rational number reasoning, and developed a map as a means of displaying the interconnections among the various rational number learning trajectories (Figure 2). This paper unpacks elements of the map and how they are related to learning trajectories (shown in different colors).

EQUIPARTITIONING AS AN OPERATION PRIMITIVE.

Equipartitioning/splitting (pink) has been studied in a variety of research projects as the fair sharing of collections or wholes. Its earliest expression in children is an early primitive operation of halving and doubling, or 2-splits, of either collections or wholes (Confrey, 1988; Pothier & Sawada, 1983). Equipartitioning/splitting as an operation leads to partitive division as well as to multiplication.

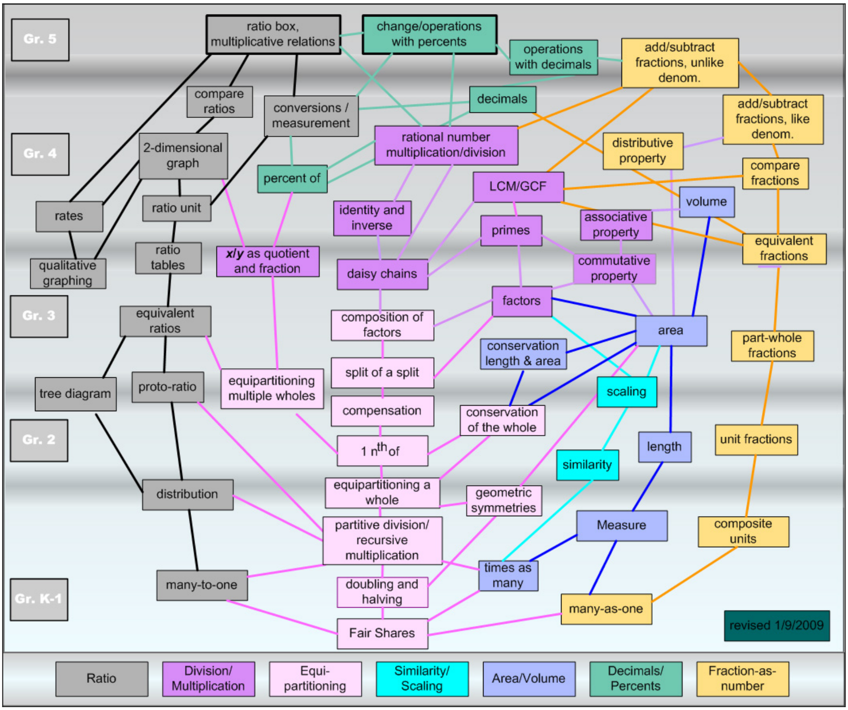


Figure 2. Learning Trajectories Map for Rational Number Reasoning.

We claim that equipartitioning is inherently recursive, not iterative, a fact not evident until one repeats splits two or more times. In the case of paper folding, for example, two parallel folds in one direction (a 3-split) and one fold in the orthogonal direction (a 2-split), the 2-split applies to all the parts resulting from the 3-split.

Reassembly of fair shares into the whole (for collections as in case A or shares of a whole, as in case B) is a basis of multiplication. In reassembling, the recursive character of the operation can be described by replacing a single fair share or part or size m, by n times as many of that part, to produce the original collection or whole (later, the product) mn. Based on our synthesis, we posit that helping children understand the meaning of "n times as many" is of fundamental importance, but often neglected or underemphasized in early grades. This is distinct from the iterative or incrementing approach of repeating the initial part n-1 times, the repeated-addition model for multiplication most commonly taught. The necessity of a recursive approach to multiplication becomes even more evident later, as students engage in repeated multiplication such as with exponential functions (Confrey & Smith, 1994).

In equipartitioning a whole (case B), one observes the emergence of another important element. When asked to find ways to share a rectangular cake among 6 people, for example, children will make perpendicular cuts (by 2 in one direction, and by 3 in the other). Although children readily use this approach to generate 4 fair shares, they seem to need to re-invent it for higher numbers of shares such as 6, in order for it becomes a reliable strategy or property for them. Pothier and Sawada (1983) named this strategy "composition of factors." This generalized strategy of a split of a split, with one of the splits in one dimension, and the other in a second dimension, introduces geometric dimensionality into equipartitioning. However, from the standpoint of equipartitioning as a learning trajectory, the term "composition of factors" now seems somewhat misleading: because the child is partitioning rather than reassembling, the "factors" are actually divisors, and the children use the strategy well before either factors or divisors are formalized, while the operations are yet implicit.

Fischbein et al. (1985) recognized the importance of implicit primitives of operations, and distinguished two—partitive and quotative—for division. Their remained an unsettling asymmetry, however, because they proposed only one primitive (repeated addition) for multiplication. This asymmetry spawned the splitting conjecture (Confrey, 1988) in the first place, partly because two different primitives for division inferred two different inverses for an operation that would evolve into multiplication. Our analysis of equipartitioning/splitting now allows us to reconcile the asymmetry by matching the two primitives for division with two primitives for multiplication, namely, recursive and iterative.

MANY-TO-ONE AND MANY-AS-ONE: RATIO AND FRACTION-AS-NUMBER

Further study of case A, sharing collections evenly, led us to employ a novel distinction between "many-to-one" and "many-as-one" as related to ratio and fraction. We conjecture that if in reporting the result of equipartitioning a collection into fair shares, one maintains both dimensions (coins and persons), a many-to-one grouping (coins per person) is produced, which gravitates towards the ratio unit (n:1). In these situations, to reverse the process of equipartitioning, the number of objects, formed by the equal sharing, *covaries* with the number of people.

In contrast, we conjecture that if, in reporting the result of equipartitioning of collections (mn items), one reduces the result to one dimension (coins), a many-as-one grouping (m coins as a group) is produced, which gravitates towards the idea of composite unit (one [group of] m as the fair share, rather than m ones per person). Reversing this process tends to proceed to a definition of multiplication as an (iterative) operation in which one factor describes the size of the group (m), and the other acts as a count of those groups (n). This iterative approach supports multiplication as repeated addition.

In case B, multiple children share one cake. The two-dimensional version of case B (though one-to-many in this context instead of many-to-one) is that 1 cake shared among n children results in a fair share of $1/n^{th}$ of the cake per child. (It follows from this that the whole cake is "*n* times as much" as a single piece.) In this case, maintaining the two dimensions introduces $1/n^{th}$ as a description of a single person's share of a single cake in relation to *n* people sharing a whole cake as seen in this ratio table. It entails an equivalence, namely that the relationship of 1 cake to *n* people is the same as the relationship of $1/n$ of a cake to 1 person.

Cake	Persons
1	n
$1/n$	1

n people:1 cake = 1 person: $1/n$ of a cake.

The one-dimensional version of case B is that the fraction $1/n$ is defined as a "unit fraction" which describes the size of the part that results from equipartitioning the cake into *n* parts. However, even if one works in one dimension in describing the relationship of the whole and the part, the ratio of *n* people to 1 person lurks in the background, as seen in the representation 1 cake:$1/n$ cake = *n* people:1 person. The *n* becomes the "splitter," which we subsequently view as a scalar or a divisor. We emphasize that while this approach is viable as a means to invent the "fraction," dropping the role of the second dimension (which enforces covariation explicitly) burdens the single dimension with a new restriction, namely that the whole and the part must share a common unit of one. From this we claim that a fraction is a subset of ratio, further distinguished by the requirement of a common unit of one for comparisons, addition, or subtraction. This explains why the many-as-one approach leads one to readily represent fractions on the number line, where a single unit underlies the structure.

Based on this synthesis, fair shares are located at the base of map. Progression through the many-to-one construct of equipartitioning leads to the learning trajectory for ratio, proportion, and rate (grey) (Streefland, 1991). While ratio, proportion and rate are also expressed as a/b, the values a and b imply quantities of two different dimensions. For equipartitioning resulting in many-as-one fair shares, we claim that the progression is toward fraction-as-number (gold) or drawing on "times as many" through measure to length, area, and volume (blue).

ONE-n^{th} OF AND *n* TIMES AS MANY: AN *a*/*b* OPERATOR CONSTRUCT

Cake	persons
m	n
$1/n$ *(m)*	1

n people: m cake = 1 person: $1/n$ of m cakes.

The final piece is the role of $1/n$ (and more generally, of a/b) as an operator, an idea that cannot be fully developed in this paper. Fundamentally the argument is that through equipartitioning across cases A and B, children develop an understanding of "1/nth of" as one share of any sized collection.

From very young, children learn that if they share a collection with one other person, they each receive "half," where half is half of the collection. Likewise, if they share a single cookie, they are as likely to call their share, "half of a cookie" as they are to call it "one half." Now, if one asks children whether all halves are equal, they draw the distinction that two halves of the same object are equal, but that a half of a small object or collection is smaller than a half of a larger group or collection. In our analysis, we build on this elementary operator idea in relation to equipartitioning and then multiplication/division to develop "*a*/*b* of" as a composition of equipartitioning (shrinking) and times as many (stretching). This third key idea of *a*/*b* as operator takes its place along side of *a*/*b* as ratio and *a*/*b* as fraction-as-number.

IMPLICATIONS AND CONCLUSION

The map of the RNR learning trajectories is two-dimensional, with the equipartitioning and multiplication/division trajectories forming a central spine of the net. The constructs of ratio, fraction, and *a*/*b* as operator are all interrelated. A 3-dimensional illustration—wrapping the map in Figure 2 around a cylindrical axis—would show ratio and fraction closer to each other. The map overall is constructed to depict the foundational role of equipartitioning in both constructs, and the central role of equipartitioning and multiplication and division, in students' ability to understand and interconnect all the constructs of *a*/*b*. What remains is to explain in detail how each of the trajectories develops.

Overall, we believe that the Learning Trajectories map provides a visual way of reinforcing the distinctions and the linkages among three main meanings for the expression *a*/*b* that students must master to be well-prepared for algebra and higher mathematics, and that envelopes most of the school-level meaning of rational number. These uses of *a*/*b* include: *a*/*b* as a fraction (number), *a*/*b* as a ratio—a relation between two numbers in various contexts, and "*a*/*b* of" as an operator. The foundational equipartitioning/splitting learning trajectory, which should be strengthened in schools beginning at the kindergarten level, is shown in research to build the reasoning that underpins all three major meanings for *a*/*b*. It is becoming clear that understanding and reasoning in rational number constructs is complex and develops gradually across years of education. Most curricula in the U.S. and many other countries current neglect equipartitioning constructs or allowing children's early proficiencies in them to languish in elementary school, focus primarily on fraction-as-number for several years before returning to retrieve the operator and ratio versions of *a*/*b*. We believe that curricula should instead build on the flexibility that young children can demonstrate, and developing their reasoning in all three meanings for *a*/*b* in parallel. In this way, children will gradually build their understanding of all three meanings for *a*/*b* and simultaneously build their intuitive and explicit understanding of the links between these them, and that as a result, they will be far more prepared for and successful in algebra and the higher mathematical courses so essential for their careers and for citizenship in the 21st century.

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MATHEMATICAL INTEREST SPHERES AND THEIR EPISTEMIC FUNCTION

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This paper presents the re-analysis of data taken from a case study that investigated individual construction of knowledge as a cognitive process. This data set is re-analysed from the perspective of social interactions including the social construction of knowledge and the respective emergence of interest-based actions. The results show that, depending on the task and the given tools, the students create individual interest spheres that shape the frames for the emergence of situational interest, the specific use of epistemological resources and the kind of knowledge that is constructed.

INTRODUCTION

This paper presents the first step of the project *effective knowledge construction in interest-dense situations*. This project is a joint study between two research teams from Israel and one team from Germany. Its aim is to shed light on processes of in-depth knowledge construction and its background conditions. Based on the theory *abstraction in context* (Hershkowitz, Schwarz & Dreyfus, 2001; Dreyfus, Hershkowitz & Schwarz, 2001), the Israeli teams want to investigate the need to construct knowledge, the process of individually constructing knowledge and its consolidation including epistemological issues. The German team investigates constructing knowledge from the perspective of social interaction, illuminating the role of interest in mathematics and so-called interest-dense situations which are situations in maths classes that foster learning mathematics with interest (Bikner-Ahsbahs, 2003; 2005).

Based on empirical data the aim of the joint project is networking the two theories (Prediger, Bikner-Ahsbahs & Arzarello, 2008) in order to link individual processes of coming to know with the social processes of constructing knowledge; to gain insight into the way of generating the need to construct knowledge and to investigate the epistemological issues accompanying these processes.

The first step of this project is an exchange of data sets that are analysed from the other perspective. In this paper we present the results of the analysis of an Israeli data set from the perspective of social interaction concerning the emergence of interest, interest-dense situations and the students' use of epistemological resources. This approach provides the opportunity to illuminate how the need for a new construction is constituted. It will be investigated in detail when the three teams meet to share the results.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 353-360. Thessaloniki, Greece: PME.

Leading questions for the re-analysis of the Israeli data set are:

- Do the students show some specific interest activities?
- How are they linked to the construction of knowledge and its epistemological background?
- How is this related to the creation of the need for the construction of knowledge?

THEORETICAL BACKGROUND

The theory about the emergence of interest-dense situations (Bikner-Ahsbahs, 2005) describes discursive situations in maths classes that lead to in-depth knowledge construction. It regards knowledge construction as an epistemic process constituted by social interactions. The specificity of social interactions shapes the quality of constructing mathematical meanings for example whether this is carried out in an interest-based way. Interest in this sense is a relation between a subject and a mathematical thing (Krapp, 2007). It may occur situationally or it may be a stable. Situational interest is shown when a student is involved in a mathematical topic and shows that the activity and the topic are meaningful to him or her (Mitchell 1993). Personal interest is a stable form of interest that the students bring with them into the class independently from the situation. It is shown by epistemic actions that enlarge knowledge about a topic; that are accompanied by positive emotions and acknowledgement of the topic or the activity. Interest-dense situations are situations of working on a mathematical task in which students get deeply involved, constructing progressively deepened mathematical meaning and valuing highly the activity or the content (see also Arzarello, Bikner-Ahsbahs & Sabena, 2009). In these situations students most likely become – at least situationally - interested in the mathematical content or activity. This view on learning situations was developed for class discussions in school in which the teacher takes part. It could be shown that the emergence of interest-dense situations is supported if the students and the teacher focus on the students' train of thought. It is disturbed if the students are forced to follow the teacher's train of thought. However, the given data represents a learning situation shaped by an interview without the teacher. In what way the emergence of an interest-dense situation is fostered in such a dialogical learning situations is not fully understood yet.

The epistemic action model consisting of gathering, connecting, structure-seeing was developed to analyse the epistemic process in interest-dense situations compared to other learning situations. According to this model, the epistemic process is shaped by the three epistemic actions named above; these are gathering mathematical meanings, connecting mathematical meanings and structure-seeing. Gathering means assembling similar mathematical entities; connecting means linking a limited amount of these and other entities; structure-seeing means constructing or reconstructing a mathematical structure that refers to an unlimited number of mathematical entities. Research results show that all interest-dense situations lead to structure-seeing.

METHODOLOGICAL CONSIDERATIONS

The data-set of the Israeli teams consists of a transcript of an interview: A pair of students works on a sequence of tasks that leads to constructing the double distributive law. Some of the questions in the task refer to working with software based on Excel. There is no teacher; however, in some instances the role of the teacher is played by an interviewer.

Our analysis is based on the assumption that according to the contextual setting, the two students generate an interest sphere in which they approach the given tasks. A mathematical interest sphere is an area in which students prefer to act mathematically within the situation. This preference is shaped by the students' biographies, by their social experiences, by their experiences of competence and of autonomy (see also Deci, 1998), by the circumstances of the given situation, norms, etc. The assumption of such an interest sphere provides the possibility to explain the emergence of distinctive preferences; not necessarily interests; activated by the same task.

The analysis of the transcript is done through reconstructing the social interactions according to the three questions above. The kind of student involvement and the epistemic process is reconstructed through interpretation on three levels: the locutionary level consists of what is said or shown, the illocutionary level consists of what is told through saying or showing something and the perlocutionary level consists of the people's intentions and the impact of what is said (Davis, 1980). On the locutionary level the construction of mathematical knowledge is re-constructed. On the illocutionary level the object of research is the situation generated through the negotiation of the interlocutors. The perlocutionary level provides indicators of the students' intentions, preferences and interests. The illocutionary and the perlocutionary levels answer the question how mathematical knowledge is socially constructed during the current setting (see also Arzarello, Bikner-Ahsbahs & Sabena, 2009).

SOME RESULTS

An electronic learning environment was prepared as a tool for a task given to the two students R and Y. Two seals (rectangle 1 and 2) generate numbers in their fields concerning an implemented pattern. The learning situation is organized as an interview in which the students are supposed to learn the double distributive law. A worksheet consisting of the following two rectangles and six related tasks is given to the students.

7	13
9	15

Rectangle 1

3	9
5	11

Rectangle 2

Table 1: The seals in the worksheet

Here you find the first two questions of the worksheet.

1. In the spreadsheet, build a 'seal' of expressions for such rectangles. For this purpose, enter any number in the left upper cell and write appropriate expressions for the other three numbers in the seal.

   Verify your 'seal': Enter the numbers **7** and **3** in the upper left cell and check whether the above rectangles result.

2. Try to find as many properties as possible common to all rectangles of this type.

Right from the beginning the two students show different spheres of interest. R is interested to work on the computer and Y prefers to work algebraically. These preferences are situational in the sense that the tasks provide the opportunities to work with the computer and to work algebraically. If the students become involved and their constructions of mathematical meanings are meaningful to them they generate situational interest in the current mathematical thing (Mitchell 1993).

The data show that the two interest spheres provide specific epistemological resources that the students use to construct knowledge. They prefer arguing and creating mathematical objects within their interest spheres.

61	R:	We don't need the sum. Write down, the difference between X1 and X2 will always be 2.
62	Y:	But why? I won't write it without understanding it! What is X1 and what is X2?
63	R:	Here we put 1 [changes the number in the left upper cell], the difference is 2.
64	Y:	But you have to define what is X. You can't just write that the difference between X1 and X2 equals 2.
65	R:	X is instead of A or B or C.
66		(empty line)
67	Y	Yes?
68	R	Yes.
69	Y:	OK. Let's do: I one you one.
70	R:	Whoever finds, says.
71	Y	[Continues writing;] The difference between X what?
72	R	X1 and X2
73	Y	You know that if I write X1 it's like X times 1, and if I write X2 it's like X times 2.

R has found out that the difference of one number in the seals and the number below is always 2. Y looks for more features and tries to find a pattern using the sums of the numbers. Both are asked to write down what they found. R dictates Y to write down R's findings, but Y refuses to do this stating locutionarily that he does not understand the meaning of X1 and X2 (62). This locutionary meaning is enriched by complementary meanings on the illocutionary level. Illocutionarily, Y tells R that his view is different and that he follows the norm only to write things down that he has understood. For R the signs X1 and X2 are not problematical. His interpretation of Y`s question "what is X1 and what is X2?" shows that R refers to the software assuming that Y does not know the pattern "the difference of the numbers in the two cells is 2". Therefore R changes the number in the left upper cell to show how the pattern can be checked (63). Illocutionarily, R tells Y that his sphere of interest is working with the computer. His way of coming to know is checking and his epistemological tool is the computer.

In line 64 the difference between the students' interest spheres becomes obvious. Y follows another norm: whenever X is used it has to be defined. The way X1 and X2 is used in this case does not make sense within Y's interest sphere. Defining X as an unknown belongs to the area of algebra. Illocutionarily, Y tells R that in his view it is even forbidden to write "the difference between X1 and X2 equals 2" (64).

The software shapes the epistemological frame as a filter for R`s interpretation: X has the same function like A, B or C as cell links. Illocutionarily, R informs Y that the current work has to be done with the computer. In 67 and 68 both agree. Then they decide that both of them have the same right to express findings. However, in line 71 Y shows that he still has the problem to write down the sentence using X1 and X2. In line 73 Y expresses what kind of obstacle he experiences: X2 looks as if it means "X times 2" just how this sign would be interpreted algebraically. Saying this, Y shows that he uses the signs algebraically und his preferred epistemological resources stem from algebra.

During the following scene the students decide to use words instead of terms if they want to write down their discoveries.

This episode shows that each student situationally establishes a specific sphere of interest including specific epistemological tools, languages, and signs. We can find three different instances of working together:

- They decide to use a language that is able to include both interest spheres.
- Y accepts the use of the computer as a calculator or when he is not able to express the current pattern by an algebraic term.
- The interviewer forces both to work algebraically.

As soon as it is possible to continue working in their own sphere of interest the students give up to take over tools from the other perspective and continue to argue within their interest spheres again.

The interest spheres also predetermine the kinds of epistemological resource the students select and the kinds of mathematical structures the students observe. Y is interested in algebraic structures that are not obvious:

89 R: Between the left upper one and its diagonal a difference of 8, that's clear because here the formula is +8.

90 Y: OK, that's clear but that's what they want us to write. Never mind, let's find something more beautiful.

For R the software is the epistemological resource. An algorithm says what can be found out. Y is not interested in this kind of findings. He wants to find "something more beautiful" in the sense of an algebraic structure.

In line 92 Y shows what he means: "the sum of the diagonals will always be even" is a law of the seals.

R prefers finding mathematical patterns through exploring examples: R puts numbers into the left upper cell and realises that "two numbers that complete to 10" always lead to the same difference between the product of the right column and twice the product of the left column. This feature seems to be too complex for Y to express it algebraically. Therefore he accepts checking examples with the software to validate the feature (163).

163 Y: Yes, the result after all the computations will be the same!

164 I: Just a moment. Wait just a moment.

165 R: Ah! I got it! I think that ... [types into the upper left cell the number 2] one second. I'll check and then, I'll tell you whether what I think is really correct [goes on typing in the upper cell the number 8]. Eight, that's it! I know the formula. You take two numbers that complete to 10 and then, it's the same.

In line 163 Y identifies the computer as a calculator without any attempt to become involved deepening insight. In contrast, R gets involved deepening insight through the use of the computer as an epistemological tool (164, 165): R becomes situationally interested in the construction of knowledge with the computer. Line 165 describes the kind of action R selects to find a pattern: Ideas are generated and checked with the computer. A formula in this case is a pattern that is verified with the computer. In contrast, Y regards a formula as an algebraic expression that is understood through using laws. The discovery of the double distributive law progresses as Y recognises the simple distributive law (263) during the students` attempts to find out the difference between the product of the right column and twice the product of the left column.

Although, the sphere of interest is understood as a situational concept describing a preferred area to work in, the student`s spheres of interest are stable during the current interview. They continue working in *their* sphere of interest. Only when their preferred tool reaches a limit they take over an epistemological tool of the other perspective; but as soon as it is possible, they return to the epistemological tools of their own sphere of interest.

Three different kinds of interest-dense situations emerge during the interview. In two situations the emergence of an interest-dense situation is shaped by one of the students' structure-seeing within their specific interest spheres. In these cases, the role of the other student is to question the ideas and to become convinced. The emergence of the third interest-dense situation is supported by the struggle between the two students about which explanation is the right or the more logical one (197-201).

197 R: Did you understand, we have X and in each diagonal we actually have 2X+8.

198 Y: No I think it's more logical that "X divided by 2 equals diagonal". Each diagonal is X divided by 2.

199 R: No

200. Y: Let's write the two ways. Write yours! [Hands the worksheet to R.]

201 R: No, you write it and I'll explain it to you, OK?

[R writes his justification then gives the worksheet back to Y and Y writes his justification. Writes:

In each diagonal there are 2X + 8:

X	X+6
X+2	+8

2x+8= =2X+8

SUMMARY AND REFLECTIONS

The students' mathematical interest spheres are situationally created and determine their interpretations of the task, the way they work, act, become involved and construct mathematical meanings. The students show situational interest within their spheres and less interest in the other. An interest sphere provides the tools that are used to construct knowledge. These tools shape the epistemological resources and they predetermine the kind of knowledge that is constructed. The differences between interest spheres in a working group create a frame difference (Krummheuer, 1992) that is an obstacle for establishing a common ground for working together. At the same time, this difference is a resource for deepening understanding because it provokes a debate in which the students are forced to explain their own view. The participating students become involved in a deep negotiation of mathematical meanings, interest-dense situations emerge together with situational interest in the mathematical objects. The difference between mathematical interest spheres provides the opportunity to broaden one's own interest sphere through the experience of competence and autonomy which may further develop interest in a mathematical thing (Deci, 1998).

In this case study, the individual need to construct mathematical knowledge is rooted within the respective interest sphere. It occurs

- as a deep personal interest in finding a mathematical pattern (see Y in line 90),

- as a situational interest when the student is aware that a mathematical pattern will answer a question or solve a problem (the single distributive law, in 262),
- as a situational interest when the student is aware that he is about to grasp a pattern. This happens intrinsically, as R realises the emerging pattern in his explorations (see R in Line 165) and, extrinsically, as the interviewer tells the students “you almost got it” (262-263).

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THE ROLE OF THE TEACHER IN DEVELOPING PROOF ACTIVITIES BY MEANS OF ALGEBRAIC LANGUAGE

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This work is part of a wider project of didactical innovation aimed at fostering students' conscious use of algebraic language through teaching experiments on proof in elementary number theory (ENT). In this paper we will analyze the role played by the teacher as a guide to the enactment of fundamental skills for the construction of reasoning by means of algebraic language, pointing out the specific characteristics of the teacher's effective action in serving as a role model to his/her students.

THEORETICAL FRAMEWORK

The work presented herein is part of a wide study (Cusi 2009a), within the framework of the Italian research for innovation (Arzarello & Bartolini Bussi 1998). It was conceived to promote, in the actual school environment, activities aimed at helping students in developing symbol sense (Arcavi 1994) towards a conscious use of algebraic language as a thinking tool (Arzarello *et Al.* 2001), possibly re-converting their previous pseudo-structural conception of algebra (Sfard & Linchevski 1994). This led to designing and implementing an innovative introductory path to proof in ENT (grades 9-10), in the frame of an approach to the teaching and learning of algebra focussed on the control of meanings (Cusi 2009b).

Our research takes place in a Vygotskyan perspective of approach to the teaching and learning processes, specifically with reference to: (a) the central role played by social interaction in thought-development processes (Vygotsky, 1978); (b) the importance of experts' contribution in helping students bridging the gap between their potential and their actual mental development (Vygotsky, 1978); (c) the role played by activities performed in a social context in the development of both the personal meaning that an individual attributes to them and the motives that determine his/her awareness of learning processes (Leont'ev, 1977). In this framework, the role played by the teacher is fundamental. This role (has been and) is the topic of several studies, especially as far as collective discussions are concerned. If we look at the social aspects of interaction, it turns out that the teacher should be able to create a good context for classroom discussions, by stimulating and regulating argumentative processes (Schwarz et Al., 2004) and, most of all, focusing students' attention onto the need to listen to others carefully, so that they might decide whether what they say makes sense and, possibly, criticise or give suggestions (Wood 1999). If we look at the mathematical content of the discussion, the teacher is in charge of determining the direction this content should take in the different phases of the discussion, "filtering" students' ideas, so that they can focus on those contents the teacher considers more relevant and meaningful (Sherin, 2002). Besides this, the teacher is in charge of the quality of the discussion because it is through his/her reactions to students'

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 361-368. Thessaloniki, Greece: PME.

interventions that he/she implicitly evaluates the solutions they propose (Yackel & Cobb, 1996), leading them to become aware of the finest forms of reasoning (Anghileri, 2006). Many researches have highlighted methodologies and behaviours to approach mathematical discussion, that the teacher should be able to manage and develop in a flexible and dynamic way (Anghileri 2006; Leikin & Dinur 2007).

The role played by the teacher becomes even more complex and delicate when mathematical proof is the object of discussion. Balacheff (1991) underlines how, sometimes, interaction may be an obstacle to learning within proving processes, because it favours the development of those "argumentative behaviours" which contrast with the achievement of an awareness of both the specificity of mathematical proof supporting one's conjectures and the role played by logical deduction. More recently, Martin *et Al*. (2005) have drawn attention to the importance of whole-class discussions and the role they play in the learning of mathematical proof, highlighting the impact of teachers' choices and actions both on individual and collective understanding. The authors stress that the teacher is in charge of favouring the inclusion of students in the process of class-based negotiation of a conjecture, as well as in the process of collective construction of the proving reasoning referring to precise rules. Moreover, Blanton *et Al.* (2003), exploring the role of instructional scaffolding in the development of undergraduate students' understanding of mathematical proof, highlight that students who engage in whole-class discussions that include metacognitive acts make gains in their ability to construct proof. These aspects also concern the construction of proofs by means of algebraic language, since it requires that students employ skills that may be attained only when they participate in contexts, such as those advocated by Schoenfeld (1992), wherein mathematical sense-making is practiced and developed. In previous works (see, for instance, Cusi 2009b) we highlighted those skills that students must develop in order to become able to succeed in this kind of activity and to acquire an awareness of the role played by algebraic language within these processes. In order to analyse the development of thought processes carried out by means of algebraic language we referred to some theoretical constructs, in tune with the view of teaching algebra that we are promoting, drawn from the works of Arzarello *et Al*. (2001), Boero (2001) and Duval (2006). The first authors highlight the use of conceptual frames (defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem") and changes from a frame to another as fundamental steps in the activation of the interpretative processes of the expressions progressively constructed. Boero focus on the concept of *anticipating thought* as a key-element in producing thought through processes of transformation (he defines anticipating as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process"). Duval, who defines representation registers as those semiotic systems "that permit a transformation of representations", asserts that a critical aspect in the development of learning in mathematics is represented by changes of representation, be they either from one *representation register* to another (conversions) or within a single register (treatments). The analysis of students' discussions enabled us to highlight the role played, in the development of proofs in ENT, by three essential components (a. the appropriate activation of conceptual frames and coordination between different frames; b. the correct application of appropriate anticipating thoughts; c. a good flexibility in the coordination between algebraic and verbal registers) and the mutual relationships existing between them (Cusi 2009b). Difficulties met in making students develop the essential skills we have outlined, as well as in helping them become aware of the meanings that algebraic language can transmit, if used appropriately, enabled us to highlight the crucial role played by the teacher as a model students should refer to. This led us to analyze teacher's actions during those class discussions aimed at introducing the construction of proofs in ENT. This paper propose the main results of this part of our research.

RESEARCH HYPOTHESES AND AIMS

The idea this work is based on is that, during classroom interaction, students acquire, through a process of cognitive apprenticeship (Collins *et Al.* 1989), teachers' attitudes and behaviours. This led us to the following two hypotheses: **(a)** The teacher must orient his/her actions to foster students' construction of the three skills (mentioned above) which are fundamental in the development of reasoning by means of algebraic language; **(b)** The teacher has to serve as a *role model* in leading his/her students to a gradual and conscious acquisition of these skills. The research aims related to these hypotheses are: **(1)** To study teachers' attitudes and behaviours in leading their students to the construction of proofs by means of algebraic language and to highlight **(a)** the productivity or negativity of their interventions, pointing out the typical behaviours of an *aware and effective* teacher, and **(b)** the effects of their approach on students, from the point of view of both awareness shown and competencies acquired. **(2)** To identify the specific characteristics of a teacher which effectively act as a role model for his/her students and to propose a first characterisation of the theoretical construct of *teacher as a role model*.

METHODOLOGY

The class-based work was articulated through small-groups activities, collective discussions and individual tests. The data being analysed were students' written productions and the transcripts of the audio-recordings of both small-groups and whole class activities. Each transcript was analysed from different points of view, depending on the typology of activity being analyzed. The analysis of transcripts of collective discussions has been performed by highlighting: (1) weaknesses and strengths of the discussion, with reference to the three key components in the development of proofs in ENT; and (2) the role played by the teacher as a "stimulus" to foster an approach to algebra as a tool for thinking, and, at the same time, as a "model" and "guide" in the construction of reasoning. Transcripts concerning small groups activities were analysed with reference to: (1) weaknesses and strengths of the discussion, with reference to the three key components in the development of proofs in ENT; and (2) the link between types of approach proposed by the teacher in previous activities and types of approach chosen by students, with particular reference to the meta-reflections they propose. Space limitations do not allow us to dwell on the analysis of small-group activities; therefore, we will only report the analysis of a discussion led by an effective teacher.

AN EXAMPLE OF TEACHER ACTING AS AN EFFECTIVE ROLE MODEL

We report here an excerpt from a collective discussion, referring to the construction of the proof of the following statement: *If b is an odd number, the expression 3b represents an odd number*. It is important to stress that, aiming at fostering students' transition from verbal argumentation to algebraic proof, we chose to propose, since the very beginning of the path, an algebraic approach to the justification of statements, even when a verbal one could appear more simple to be adopted. We chose to carry out an in-depth analysis of this particular discussion because it reveals both how the teacher (T) tries to lead pupils to develop those skills that play an essential role in the development of reasoning by means of algebraic language and the positive effects of her approach on students.

During the first part of the discussion, all students agree to formalize the hypothesis through the equality $b=2x+1$ and spontaneously suggest to substitute $2x+1$ instead of b in the expression $3b$. Therefore T writes $3b=3(2x+1)$ on the blackboard. The following excerpt refers to the subsequent part of the discussion.

1 T: What can I do then? (*Silence*) How can you highlight that a number is odd?

2 M: You take a multiple of 2 and add it to 1…

3 T: Does this [*T points* $3(2x+1)$] look like a multiple of 2 added to 1?

4 A: Yes, and then it is multiplied by 3!

5 T: Well... all I have now is 3 times something… you said that my aim is to highlight a multiple of 2 added to 1! Does this look like a multiple of 2 added to 1?

6 A: You can write 3 times $2x$ in brackets, plus 1! [T *writes* $3(2x+1)=3\cdot(2x)+1$]

7 T: Are these two expressions equivalent? [T *points to* $3(2x+1)$ *and* $3\cdot(2x)+1$]

8 Chorus: No!

9 T: Watch out, then! I must carry out transformations which lead me to an expression equivalent to the one I started from!

10 S: You can write 3 times, open bracket, $2x$ in brackets, plus 1, closed bracket [T *writes* $3(2x+1)= 3[(2x)+1]$]

11 M: Why would you do that? It's the same thing!

12 T: Let's keep our aim in mind... What I want to do is to highlight a +1, but in the main expression. Here [T *points to* $3[(2x)+1]$] we have highlighted a +1, but it is inside this factor…

13 A_N: Since we can perform calculations, we could obtain $6x+3$… [T *writes* $3(2x+1)=6x+3$]… Then it becomes $6x+2$, in bracket, then +1 [T *writes* $3(2x+1)=6x+3=(6x+2)+1$]

14 T: She said, "If I perform calculations, I obtain $6x$+3". What is her aim, then? To show that it is possible to highlight a +1. Can we see, now, that it represents an even number added to 1? [T *points to* ($6x+2$)+1]

15 Chorus: Yes!

16 T: Couldn't we do something to highlight this property even more?…

17 M: We can factorize the expression in brackets! Yes! We can take out 2!

18 T: Let's do that… So we have 2 times $3x$+1, plus 1.
[T *writes* $3b=3(2x+1)=6x+3=(6x+2)+1=2(3x+1)+1$]

19 T: Look, now. We've started from $3b$ and we obtained 2 times something plus 1. As M said, an even number plus 1 always gives…

20 S: An odd number.

21 T: So we have accurately proved the statement! Can you follow this reasoning?

22 Chorus: Yes!

Analysis of the excerpt

At the beginning of the discussion T asks how to proceed (line 1), playing the role of an *investigating subject* and acting as a part of the class in the "research" work being activated. In this way, she lets the class guide the activity, although she remains the point of reference for the discussion. Since she gets only silence back, T asks her students whether the constructed expression [$3(2x+1)$] explicitly highlight the property of representing an odd number (line 1). Thus, she becomes an *activator of anticipating thoughts*. Afterward, since her aim is to lead students to syntactically transform the expression they constructed in a way that it could highlight the property of representing an odd number, without relying on any verbal argument, T *stimulates* them through a more explicit question (line 3). Because of A's difficulties (line 4) in following the line of reasoning proposed by T, she decides to *provoke* both a correct interpretation of the examined expression and an effective approach to the manipulations to be carried out, making, once again, the aim of the activity explicit (line 5). Students seem to have followed T's indications, but they have difficulties in identifying what kind of treatments should be applied on the expression in order to better highlight that it represents an odd number. The *anticipating thought* she tried to activate in her students ("I need to highlight the sum between an even number and 1") *prevails on their ability to control the manipulations* they operate, *leading them to perform erroneous treatments* [$3(2x+1)=3\cdot(2x)+1$, line 6]. When A propose an erroneous treatment, T writes on the blackboard the equality he suggests and points out that the two expressions $3(2x+1)$ and $3\cdot(2x)+1$ are not equivalent. Then she puts herself *on a meta level* proposing a comment upon the meaning of the whole activity (the syntactical transformations to be carried out must be allowed, i.e. they must lead to expressions actually equivalent to the starting one, lines 7 and 9). In this way T acts as a *stimulator of reflective attitudes* on the meaning of the activity. She then follow the suggestions proposed by S (line 10), who seems to have lost sight of the aim of the activity: his *anticipating thought* is inhibited *by his need to verify the equivalence between the two expressions* $3(2x+1)$ and $3[(2x)+1]$. Therefore, T's responsibility becomes that of *fostering* in students *an harmonic balance between semantic aspects* (interpreting the expressions in the activated conceptual frames and activating an anticipating thought in relation to the aim of the activity) *and syntactical aspects* (controlling the correctness of the suggested treatments). The meta-comment on the meaning of the activity proposed by M (line 11) highlights that the student has seized the stimuli given by T: his exclamation, “It’s the same thing!”, does not refer to the equivalence between the two expressions, but to the fact that they both activate the same conceptual frame (‘being multiple of’), wherein they have the same interpretation. In order to guide her class toward the right direction, T puts herself once again on a meta-level to draw students’ attention to the objective of the syntactical manipulations being performed (line 12), assuming the role of an *activator of anticipating thoughts*. An “indicator” of this typical role played by T is her frequent use of the term “aim”. At this point, A_N, showing to have understood T’s suggestion, proposes to transform the expression $3(2x+1)$ into an *additive form* (line 13), analyzes the obtained expression [6x+3] into a new conceptual frame (‘even-odd’) and suggests to perform the treatment which allows to highlight the addend +1. In this way she displays a good semantic control in carrying out syntactical manipulations under the guidance of a correct anticipating thought. T (line 14) goes back to A_N’s line of reasoning (*re-phrasing*, in the words of Anghileri, 2006) to stimulate a moment of reflection on the effectiveness of the approach adopted by the student. This highlights another important role played by the teacher: she acts *as a guide who leads students to the identification of effective models to refer to*. When T asks (line 16) to suggest how to transform the expression $(6x+2)+1$ in a way that it could better highlight the property of representing an odd number, M (line 17) interprets his teacher’s question within the frame ‘even-odd’ and suggests the correct treatment to be performed. T closes the discussion (lines 19 and 21), putting herself once more on a meta-level and re-examining the whole process. Her reflections on both the meaning of the performed syntactical transformations and the effectiveness of the obtained expression, together with the continuing stimuli she has given throughout the discussion, are aimed at *inducing a meta-level attitude* also in students. The analysis of the transcripts of small-groups discussions, not documented herein, enabled us to point out how students have successfully adopted the approach proposed by their teacher as a reference model to face the subsequent activities of the didactical path. Such transcripts, in fact, show an actual maturation of both that awareness and those competencies T has tried to develop in her students.

CONCLUSIONS

The analysis of this transcript, together with the analysis of the transcripts of both collective discussions guided by the observed teachers and their students’ discussions while working in small-groups, enabled us to single out the *specific characteristics of an effective action of a teacher in serving as a role model for his/her students*. Moreover, it helped us in sketching the profile of an ‘effective teacher’ and made us develop the idea of introducing a characterization of the theoretical construct of '*teacher as role model'*. The defining elements of this construct are those characteristics that teachers must fulfil. Summarizing, they must: (a) be able to play the role of an "*investigating subject*", stimulating in his/her students an attitude of research on the problem being studied, and acting as an *integral part* of the class in the research work being activated; (b) be able to play the role of a *practical/strategic guide*, *sharing* (rather than transmitting) knowledge with his/her students, and of a *reflective guide in identifying effective practical/strategic models* during class activities; (c) be aware of their responsibility in maintaining a *harmonized balance between semantic and syntactic aspects* during the collective construction of thought processes by means of algebraic language; (d) try to *stimulate and provoke* the enactment of fundamental skills for the development of thought processes by means of algebraic language (being able to translate, to interpret, to anticipate and to manipulate), playing the role of both an *"activator" of interpretative processes* and an *"activator" of anticipating thoughts*; (e) stimulate and provoke *meta-level attitudes*, acting both as an *"activator" of reflective attitudes* and as an *"activator" of meta-cognitive acts*.

We tested this model in the analysis of other discussions guided by teachers during other activities of the planned path: this enabled us to highlight a clear gap between those teachers who are able to *act as role models* in their classes and those who are not; the latter, in the most extreme cases, produce the opposite effect of stimulating a kind of pseudo-structural approach to the use algebraic language as a thinking tool.

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TRANSITION PROBLEMS IN MATHEMATICS THAT FACE STUDENTS MOVING FROM COMPULSORY THROUGH TO TERTIARY LEVEL EDUCATION IN DENMARK: MISMATCH OF COMPETENCIES AND PROGRESSION

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This paper investigates the transition points from compulsory elementary through to tertiary level in Denmark in terms of the mathematics competencies stated as intended learning outcomes in the curricula. It describes the competence levels using the so-called SOLO Taxonomy and investigates if there is progression across these levels. The paper concludes that after each transition point it appears that a process of competence progression starts "from the bottom". It also appears that the more difficult the content, i.e. the higher education level, the lower is the ambition level of mathematics competencies in the curricula.

INTRODUCTION

The study is a part of a research project (Matthiasen et al., 2008) about transition problems that students experience moving from elementary through to tertiary education in Denmark in the subjects of English, French, German, IT, Mathematics, and Science. This paper reports some of the results for mathematics. A purpose of the project is to enhance connections between the education levels.

The transition between for instance secondary and tertiary education "known in anthropology as rites of passage – are events that, in a major way, influence one's decisions about the future" (Kajander & Lovric, 2005, p. 150). Furthermore, students' high school experience seems to be correlated with their success in university mathematics. In Denmark transition problems are frequently discussed, particularly in relation to students dropping out, and the Danish government aims at having 95% of a year's youth passing an upper secondary education and 50% a higher education by 2015. Research in Denmark has so far mainly focused on vocational educations and the emphasis has been on the importance of advising the youth. However, this study focuses on the transition points from an educational perspective. The study is unique as it focuses on the whole education sector from elementary through to tertiary, which has not been done before in Denmark. A central research question for this paper is therefore what mathematics competencies the students are suppose to have learnt by the exits of each education levels and if there is progression and connections across these levels.

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MATHEMATICS COMPETENCIES

The concept of ‘competencies’ is used in all curricula at all levels. Most of the times they use the terminology of the Danish KOM project (Niss, 2002).’KOM’ stands for ‘Competencies and the Learning of Mathematics’. The project was initiated by the Ministry of Education and contains a new competence based system to describe curricula, development, and progression in mathematics teaching at all levels. This is to replace the more traditional syllabi which list contents and topics to “go through”.

The report describes eight mathematical competencies: (1) Thinking mathematically, (2) posing and solving mathematical problems, (3) modelling mathematically, (4) reasoning mathematically, (5) representing mathematical entities, (6) handing mathematical symbols and formalisms, (7) communicating in, with, and about mathematics, and (8) making use of aids and tools. To possess a competence means to be able to perform certain actions with various contents.

Methodology

The mathematics competencies the students are expected to learn during the different school levels are investigated through an analysis of official curricula and guidelines. Thus, this paper focuses on the ’matter meant’, rather than the ‘matter taught’ or ‘matter learnt’ (Bauersfeld, 1979, pp. 204-206). One might argue that the former is not necessarily the same as the two latter. This is true, however owing to a new Danish grade scale that was implemented in the whole education sector during 2006-2007, grades are now to be given based on how well the students meet the curricula formulated intended learning outcomes which are stated in the terminology of competencies. This means that the ‘matter meant’ strongly effects the examination and grading. This in combination with the usual constitutional effect of examinations on learning, means that I anticipate that the ‘matter meant’ has a strong impact on both the teaching and the learning wherefore it is relevant to study the curricula.

Measuring the level of competence possession and competence progression

How do we measure the level of competence possession in a student? Since we cannot simply X-ray their brains, we need to observe them *do* or *say* something, and this something is what convinces examiners on the appropriate grade. Therefore, in order to measure the level of competencies that each curriculum expects the students to have gained, I use the SOLO Taxonomy (Biggs, 2003, pp. 34-53). SOLO (*Structure of the Observed Learning Outcome*) distinguishes five different levels according to the cognitive processes required to obtain them. In Brabrand and Dahl (2008, in press) we define *competence progression* as moving up through still higher SOLO levels; i.e. SOLO progression. When a student moves up the SOLO levels, he first experiences *quantitative* improvements when he becomes able to deal with still more aspects (SOLO 2-3). Later he experiences *qualitative* improvements when details integrate and form a structure and is generalized (SOLO 4-5). A short description of the five levels is below:

SOLO 1: Pre-Structural Level

The student does not understand but might have acquired scattered pieces of information.

SOLO 2: Uni-Structural Level

The student deals with a single aspect and makes obvious connections. The student can use terminology, identify (remember things), perform simple instructions/algorithms, etc.

SOLO 3: Multi-Structural Level

The student deals with several aspects but does not consider these in connection. He sees the trees but not the wood. He is able to describe, classify, combine, apply methods, etc.

SOLO 4: Relational Level

The student understands relations between several aspects and how they form a structure; he sees the wood. He can compare, apply theory, explain in terms of cause/effect, etc.

SOLO 5: Extended Abstract Level

The student generalizes structure beyond what was given, perceives structure from various perspectives, and transfers ideas to new areas. He hypothesizes, criticizes, etc.

Mathematics competencies at tertiary education

The Danish university sector is structured with 3-years Bachelor programmes followed by 2-years Master's programmes. We (Brabrand & Dahl; 2008, in press) investigated SOLO progression at *inter alia* mathematics programmes at University of Aarhus (AU) and the University of Southern Denmark (SDU). As with the present paper, this was a study of curricula descriptions. For each course we calculated a SOLO average levels based on curricula statements of learning outcomes and competences, and subsequently calculated averages etc. for various programmes. These learning outcomes had been added to the traditional content-based syllabi during 2007 in order to fulfil the requirements of the new grading scale of having explicit stated intended learning outcomes. An example of such addition is seen in *Calculus II* that states that at the end of the course, the student should be able to:

Apply (3) basic techniques and results from calculus to solve prescribed exercises within: differentiation and integration of functions in one and several variables, linear algebra, and infinite series.

Give arguments (4) for the steps in the solution of exercises.

Formulate (3) correct mathematical arguments.

Use (3) mathematical terminology and symbols.

The SOLO average is here (3+4+3+3) / 4 = 3.25. The classification of verbs to a SOLO levels builds partly on Biggs (2003, p. 48) partly on our analysis of 632 course curricula at AU and SDU. A list of some if these verbs is seen in Table 1.

SOLO 2	SOLO 3	SOLO 4	SOLO 5
identify	describe	explain	discuss

calculate	account for	analyze	assess
reproduce	apply/use method	compare	evaluate
arrange	formulate	apply/use theory	interpret**
define	solve	argue	reflect
recognize	conduct	construct	predict
find	prove	interpret*	criticize
translate	classify	model	judge
choose	express	derive	reason

Table 1: SOLO Classification of verbs in Brabrand and Dahl (2008, in press).

About the asterisks, Danish has two different verbs that translate into ‘interpret’. ‘*Tolke*’ denotes the SOLO 4 ability to (just) transfer/intepret something but ‘*fortolke*’ incorporates an assessment, hence SOLO 5. ‘Prove’ was in the university curricula used as reproducing proofs “on the blackboard”, not as inventing new proofs.

There appeared not to be much SOLO progression moving from Bachelor to Master’s level in the mathematics programmes, even though we found SOLO progression at the other departments at the Faculties of Science. In fact, at AU the SOLO-level was higher at the Bachelor level (3.2) than Master’s level (2.9) whereas at SDU the same numbers were 2.8 and 2.9; hence a small progression. Progression in the mathematics programmes was instead in the level of abstraction and content.

This calculation method rests on several assumptions such as if the SOLO classification in Table 1 is credible, if the SOLO Taxonomy is an appropriate measure for progression, if each statement of intended learning outcome “weigh” the same, and if the equal numeric steps between adjacent SOLO levels reflects equal steps in progression. These assumptions are addressed in details in Brabrand and Dahl (in press). Even if the levels of competencies may not solely be reduced to a number, it does give an approximate guide to its placements in the SOLO hierarchy.

Mathematics competencies in compulsory education

Formal schooling begins at the age of seven. Then follows 9 years of compulsory comprehensive schooling (the *Folkeskole*) in one unit. There is an optional grade 10. The analysis uses the ministerial curriculum (Danish Ministry of Education, 2009) since most schools use these. Key Stage Goals are formulated for grades 1-3, 4-6, 7-9, and 10 separately and Final Goals are described for grades 9 and 10. The subject mathematics is described through four Central Knowledge and Skills areas throughout grades 1-10. These are: (1) mathematical competencies, (2) mathematical topics (algebra, geometry, statistics/probability), (3) application of mathematics, and (4) mathematical work methods. Since this paper is about transitions to other education levels, the focus is on the Final Goals for grades 9 since this grade give access to upper secondary. The mathematical competencies are formulated based on Niss (2002). Below is the author's translation with SOLO numbers in brackets:

> Formulate (3) questions that are characteristics for mathematics and be able to derive (4) what type of answers can be expected (*Thinking*).
>
> Recognize (2), formulate (3), classify (3), and solve (3) mathematical problems and assess (5) the solutions (*Problem solving*).
>
> Do mathematical modelling (4) and explain (4), interpret (4), analyse (4), and assess (5) mathematical models (*Modelling*).
>
> Construct (4) and explain (4) using own reasoning mathematical conjectures, account for (3) and assess (5) other's mathematical reasoning (*Reasoning*).
>
> Construct (4), account for (3), and apply (3) various representations of mathematical objects, concepts, situations or problems (*Representation*).
>
> Account for (3) and explain (4) symbolic language and formula and translate (2) between daily language and symbolic language (*Symbol*).
>
> Formulate (3) mathematical questions and activities in various ways, express to others (3) and interpret (5) other people's mathematical communication (*Communication*).
>
> Identify, (2), choose (2), and apply (3) aids tools when working with mathematics, including IT, and be able to derive (4) their possibilities and limitations (*Aids tools*).

Calculating a SOLO-average using a double weigh average (as in Brabrand & Dahl, 2008, in press) yields to the following result:

[(3+4)/2 + (2+3+3+3+5)/5 + (4+4+4+4+5)/5 + (4+4+3+5)/4 + (4+3+3)/3 + (3+4+2)/3 + (3+3+5)/3 + (2+2+3+4)/4] / 8 =

[3.5 + 3.2 + 4.2 + 4.0 + 3.3 + 3.0 + 3.7 + 2.8] / 8 = 27.65 / 8 = 3.46

Although this is an approximation, we nevertheless see a very ambitious curriculum in terms of mathematics competences; also in relation to university curricula.

Mathematics competencies in upper secondary education

The common route to university is via the three-year general academic high school (*Gymnasium, stx*). The teaching at upper secondary level is divided into three levels of difficulty: A, B, C, with A as the most advanced. The curricula (Danish Ministry of Education, 2008) describes the identity of the subject mathematics, overall purpose, learning objectives, mathematics content, teaching principles, working methods, IT, and connection with other topics. The learning objectives state a number of competencies the students are to be able to do with the content by the end of the course. I will focus on Levels A and C, since they represent both ends of the spectrum and in some sense are the exit and entrance courses, respectively.

Level C	Level A

Handle formulas (3) including being able to translate (2) between symbolic and natural language and be able to use (3) the symbolic language to solve (3) simple problems with a mathematical content. SOLO: 11/4 = 2.75	Handle formulas (3) including being able to translate (2) between symbolic and natural language and independently use (4) the symbolic language to describe (3) relations between variables and solve (3) problems with a mathematical content. SOLO: 15/5 = 3.0
Use (3) simple statistical models to describe (3) a given data set, formulate (3) questions based on the model and be able to derive (4) what type of answers can be expected, and be able to formulate (3) conclusions in a clear language. SOLO 16/5 = 3.2	Use (3) simple statistical or probability theoretical models to describe (3) a given data set or phenomena, formulate (3) questions based on the models and be able to derive (4) what type of answers can be expected, and be able to formulate (3) conclusions in a clear language. SOLO 16/5 = 3.2
Use (3) the relationship between variable to create models (4) of given data, perform predictions (5) and reflect on these (5). SOLO 17/4 = 4.25	Use (3) function and derivative to create models (4) based on knowledge from other subjects, reflect (5) on the models and analyze (4) given models and perform simulations (4) and predictions (5). SOLO 25/6 = 4.17
	Use (4) various interpretations of antiderivative and various methods to solve (3) differential equations. SOLO 7/2 = 3.5
Use (3) simple geometrical models and solve (3) simple geometry problems. SOLO 6/2 = 3	Model (4) in geometry and solve (3) geometry problems based on triangle calculations, give analytical (4) description of geometry figures in coordinate systems, use (4) this to solve (3) theory/practice questions. SOLO 18/5 = 3.6
Reproduce (2) simple mathematical reasoning. SOLO = 2	Account for (3) mathematics reasoning and proofs and deductive mathematics theory. SOLO = 3
Account for (3) how mathematics is used. SOLO = 3	Account for (3) how mathematics is used in various topics, including how it is used (4) in solving complex problems. SOLO 7/2 = 3.5
Give examples (4) of mathematics in interplay with the scientific and cultural development. SOLO = 4	Give examples (4) to how mathematics developed in interplay between the history, natural science, and culture. SOLO = 4
Use (3) IT aids tools to solve given mathematical problems. SOLO = 3	Use (3) IT aids tools to solve given mathematical problems. SOLO = 3

Table 2: Learning objectives in Level A & C in upper secondary. Author translation.

Based on this, a calculation of double weigh averages of Levels A and C shows a progression in mathematics competencies as follows:

Level C: [2.75+3.2+4.25+3+2+3+4+3] / 8 = 25.2 / 8 = 3.15

Level A: [3+3.2+4.17+3.5+3.6+3+3.5+4+3] / 9 = 30.97 / 9 = 3.44

PROGRESSION?

Below we see a table giving an overview of the SOLO averages for all the levels.

Compulsory, Grade 9	Upper secondary, C	Upper secondary, A	University, Bachelor	University, Master
3.5	3.2	3.4	3.2/2.8	2.9/2.9

Table 3: Overview of SOLO averages in curricula competence descriptions.

It seems that when students shift to a new education level, they "start from the bottom" and then "work up" in terms of SOLO levels. Thus, it seems that each education level has each own "cycle". Furthermore it seems that the overall SOLO competence level of mathematics competencies decreases from compulsory through to tertiary level (3.5 to 2.9). Is this then a problem? One could argue that parallel to this is a progression in content and level of abstraction, and different education levels have different purposes in terms of preparing students for different things. Hence it might be natural that each level has its own cycle. True, however teaching students higher SOLO level competencies may require another teaching style and emphasis than lower SOLO level competencies. In short, lower SOLO levels may fit better with a teaching emphasising procedural knowledge while higher SOLO levels may fit best with a conceptual teaching. This is also noted by Brandell et al. (2008, p. 42):

> There are different views on what it means to learn mathematics at secondary and tertiary levels. … it is clear that at the university level, *routine skills in arithmetic and algebraic computations* are considered as an absolutely necessary ingredient when learning mathematics, and new entrant students are suddenly expected to handle much more complicated expressions and computations than they have met before.

Brandell et al. (2008, pp. 43-44) also states that the tasks in upper secondary national tests do not require heavy computational skills. Instead these problems focus on the students' understanding of numbers and arithmetic operations (conceptual understanding) and this lack of skills becomes a problem when they enter the university where "routine skills and knowledge of formulas and theorems (procedural knowledge) are considered necessary for the understanding of concepts and theory, as well as important tools in problem solving" (Brandell et al., 2008, p. 44).

CONCLUSION

Transition problems occur since the three education levels each has a cycle in terms of starting from more "basic" level of mathematics competencies moving to higher order levels of competencies, described in the SOLO terminology. When students transit into the next education level, they, so to speak, start "all over" in terms of competencies taught. This does not deny that parallel to this is a progression in content and abstraction, but the teaching styles supporting higher level competencies do not necessarily fit the learning of lower level competencies. It also seems that the more difficult the content, the lower is the level of mathematics competencies. This is seen in the decrease of SOLO level competencies from compulsory to tertiary.

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RESEACHING CLASSROOMS: HISTORICITY AS A PERSPECTIVE TO ANALYZE A GEOMETRY CLASS

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Our aim is to present a methodological approach to research in classroom practices, seen as social worlds made of people relating with each other. Taking a cultural-historical perspective we stress the historicity of those practices and we use the notion of expansive cycles to help in identifying the non-linear trajectory of activity systems out of their equilibrium conditions, like in the class involving the calculation of areas of geometric figures used as an exemplification. In it, we could identify two cycles of actions and a marked influence of the teacher in the development an expansive cycle of the activity, through the perturbations he presents. We also show evidence of the non-linearity of the classroom activity time since, simultaneously, different students are taking part into different cycles of the classroom activity.

In this paper we present a methodological approach to research in classrooms that highlights the historicity of the students and teacher activity, exemplifying its potentiality of analysis with a task involving areas of geometric figures.

Comparative studies of classroom practices in different countries alert for the inadequacy of making typifications and comparisons of such practices considering the influence of cultural aspects and the context where they take place (Clarke, 2008; Jablonka, 2008). Clarke (2008) also argues that since a classroom takes a different aspect according to how you are positioned within it or in relation to it, the research methodology should be sufficiently sophisticated to represent the multiple perspectives of the many participants in complex social settings such as classrooms.

Taking different perspectives and levels of analysis, other researchers also make a detailed study of classroom practices (Boaler, 2002; Tomaz & David, 2008; Watson, 2008). All of them consider that the classroom imposes many challenges for the researcher. Our challenge is to contribute with the analysis of practices occurring in classrooms, seen as social worlds made of people relating with each other. Using a cultural-historical perspective we will stress the historicity of these practices regarding classrooms as activity systems (Engeström, 1987).

FOCUS ON THE ACTIVITY HISTORICITY

The concept of activity (Leont'ev, 1978) is a specific form of social existence that includes crucial changes of social reality. According to Leont'ev, the activity emerges from a necessity, which drives motives towards an object-related. To satisfy the motives, actions are needed. These, on their turn, are accomplished in accordance with the conditions of the activity that determine the operations related to each action.

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In using Activity Theory to analyze classrooms we have considered, as Engeström (1996) does, that: the unity of analysis is the activity of the students and the teacher in the classroom; the internal contradictions inherent in human activity may result in perturbations, innovations and changes in the activity; and the analysis of the activity system and its constituent components and actions should be done historically.

The key point of this historical analysis, according to Engeström (1996), is the periodization of actions. To deal with this issue, he proposes a time structure based on the idea of *expansive cycles*, regarded as "phases of any cycle of genuine learning activity" (Engeström, 1987). According to him, for the historical understanding of an activity, the expansive cycles are crucial, because they help us in identifying the trajectory of an activity system out of its equilibrium conditions. Thus, those cycles can be used to show ways of participation on the ongoing activity and how the new social structures in the classroom result from the reproduction/production of actions.

By adopting the idea of the *expansive cycles*, Engeström (1999) seeks to solve two problems he has identified regarding the understanding of the structure of the activity. The first one is related with the differentiation between the structures of time action and time activity, which can be better explained with the *expansive cycles* because, while the action time in these cycles is linear and finite, the activity time is cyclic and recurrent. The action time corresponds to a "time's arrow", the activity time is more like a "time's circle" (Engeström, 1999).

The second problem relates with the opposition between internalization and creative externalization. According to Engeström (1999), an *expansive cycle* is a process of development that contains both the processes of internalization and of creative externalization. Those basics processes work continuously in every level of human activity. On a side, represented by internalization, we have the objective reality and its internalized subjective form; on the other, we have creative externalization, the subject activity that includes both external and internal processes. Internalization is related to the cultural reproduction and creative externalization with the explanation of how the creation of new artifacts makes the transformation of the activity possible.

In the model of the development of the *expansive cycles*, presented by Engeström (1999), they start with the internalization, socialization and training of new participants to make them able to participate in the activity. As these cycles progress, the design and the implementation of new models for the activity will be set, with the predominance of creative externalization. This one occurs, first, in the form of individual innovation. Through the growing contradictions and ruptures of the activity, the internalization takes the connotation of self-critical reflection and externalization is identified by creative solutions, reaching its peak when a new model for the activity is designed and implemented.

Engeström (1999) considers the *expansive cycles* to be the equivalent to the zone of proximal development of Vygotsky. From his perspective, the key process of the *expansive cycles* is that they do not develop in a pre-determined unidirectional sense.

Therefore, to identify *expansive cycles* is to look at the historicity of the activity gathering different viewpoints and approaches from the various participants.

Thus, the activity that structures the geometry practice which we are going to analyze is not simply a sum or a series of individual discrete actions, although the students' and the teacher's agency is necessarily perceived in the form of their actions (Leont'ev, 1978). Our focus on the classroom practice leads us to consider how the activity can be explained as it develops, not ignoring its local character. Therefore, we use the notion of historicity to situate it in time and space, pointing out how some changes on the form of participation of the students and teacher, seen as expansive transitions from the individual to the collective, can promote fluctuations of the object of the activity.[1]

CONTEXT

Roberto[2] has a large experience (over 30 years) as a mathematics teacher and we have been observing his school practice for some years, both in private and public schools in Belo Horizonte[3]. We have developed a particular interest for this teacher because he is considered, in the communities he participates in, as a mathematics teacher with a differential practice. We thought he was an interesting case to study, since we were concerned to present examples and to stress 'real' practices in the teaching of mathematics which could be considered successful in terms of the participation of the students in the class. Our interest increased when, after many years teaching in private schools[4], he decided to apply for a teaching post in the public system[5] and was hired by a school in the periphery of Belo Horizonte with students from a very low income origin, to teach 5th grade classes (ages 11-12).

[1] This research was developed with partial support from CNPq. The opinions expressed herein are those of the authors and do not necessarily represent the position or policies of CNPq.

[2] All names mentioned in this paper are fictitious names.

[3] Belo Horizonte is the capital of Minas Gerais, a state in the southeast area of Brazil, with about 2.5 million inhabitants.

[4] Since they are paid, usually only middle class and upper middle class families can afford them, and this has a number of predictable implications in these schools' practices, at all levels.

[5] Public schools in Brazil tend to be poorly evaluated with respect to several aspects, including the performance of the students in all school subjects, and especially in mathematics.

METHODOLOGY AND ANALYSIS

In this school, we have observed and tape recorded some classes of the same group of 25 students, along one academic year: 12 classes in the beginning of the year, when the subject matter was problems and operations in **N**, and 11classes towards the end of the year, when the subject was area, followed by fractions and decimals. Eight classes were videotaped. Since we wanted to understand how Roberto managed to develop in his students a disposition to participate actively in the class, we have made observations over a longer period of time trying to interfere as little as possible, to be able to analyze his practice in a situation as close as possible to its 'natural' context.

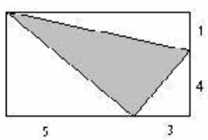
To analyze his practice we took the Activity Theory principles, considering the activity of the students and the teacher in the classroom as the unity of analysis. Alternating with selected segments of the transcription of the class selected for this paper, we insert some observations and punctuate the perturbations perceived, aiming to identify *expansive cycles* that can lead to innovations on the activity.

The episode we are going to analyze belongs to a videotaped class of the second period of observations, following a series of lessons in which the students were calculating areas: first, the area of rectangles and squares, then the area of a right angled triangle, in which case they were taught to "complete" a rectangle and then "halve" its area. We have selected this class because it is quite representative of the classroom practice of this teacher and, as we will see, the fact that the subject matter was geometry has facilitated the identification of *expansive cycles*.

In the previous lesson the teacher had handed out a work sheet with several composite figures for the students to exercise the calculation of the shaded areas at home. After looking over the students' homework, table after table, Roberto initiates the discussion of one of the exercises.

001 Teacher: In this work sheet there are some exercises we have already talked about…Now, this one here…it looks as if it is too difficult, it has a different style.

The teacher draws on the blackboard:



002 Teacher: Here, this is a rectangle, ok? With these measures here…was anyone able to make this one here? (…) Nobody succeeded? With respect to this triangle, I do not have any information yet about this triangle…I (only) have information about these pieces of the sides here…isn't it?

003 George: You can remove that white rectangle.

004 Teacher: Which white rectangle?

005 George: That big one there, then you … take that triangle and lay it down.

006 Teacher: Can you explain this better?

007 Ramon: You place that triangle upright so it becomes a vertical triangle.

008 Teacher: You place this rectangle here upright.

009 Several voices: Yes.

010 George: There, it is a triangle, isn't it? Then you go there and you make it a rectangle.

George seems to be recalling the completion procedure they were using before to calculate the area of a right angled triangle, apparently already internalized. So far, the participation of the students is centered on the visualization and manipulation of the figure, which characterizes the first cycle of actions in this activity.

011 Ramon: You place two triangles on its side pointing down.

012 Teacher: But I do not have the measures of the sides this triangle, I do not have any information of any data about it. (…)

First attempt of perturbation of the activity introduced by the teacher, directing the students' attention to the unknown sides of the triangle. Following this, on the right side of the blackboard Roberto draws a right angled triangle, calling the students attention to the right angle on it, and recalls the procedure to calculate its area.

014 Teacher: Not every triangle has a side perpendicular to the other, when it has a side perpendicular we say the following…right angled triangle… they are triangles (…) If I want to calculate their area what do I do?

015 Voices: Transform it in a rectangle.

016 Teacher: I transform it into a rectangle, ok. And then?

017 Voices: Then, mister, basis times height divided by two.

018 Edna: You make basis times height and then you divide by two.

The teacher recalls this procedure and adds to the drawing on the blackboard a mark on the three right angles of the three right angled triangles completing the rectangle.

023 Teacher: So this angle is a right angle and this side is perpendicular to the other, which is perpendicular to the other, which is also perpendicular… See if it gets any better; see if anyone can discover something here.

Second attempt of perturbation of the activity introduced by the teacher, with some influence in at least one student (line 25). This can be seen as the beginning of a new cycle centred on actions of algebraic calculation of areas.

025 Debora: There, look… you just make four times three and divide by two.

026 Teacher: Sorry?

027 Debora: Four times three and divide by two…

028 Voices: (…) four times eight and five times five…

029 Teacher: Wait a little bit. To calculate the area of the rectangle is no problem, but I have asked to calculate the area of the triangle, calculate this area here in the middle…

030 Voices: ahhh…. ahhh…

After Debora (line 25), some students start calculating the areas of the 'external' triangles but the teacher deviates them from this, focusing again on the figure as a whole. Follows a discussion (lines 31-43) leaded by Edna, with the participation of several students, all of them still working with the idea, typical of the first cycle of actions, that the "middle triangle" is a right angled one and that if you rotate and slide it, it will adjust to the sides of the rectangle, as the figure visually suggests.

The second cycle of actions is retaken when the teacher makes a third attempt of perturbation in the activity, questioning the procedures they are using (line 44). However, it takes some time before the class as a whole understands that the visualization and manipulation of the figure is insufficient to determine the area of the "middle triangle".

044 Teacher: Look, which warranty do you have that this thing here is going to fit there (meaning the sides of the rectangle)?

While some students, like Priscila, George and Edna, are still trying to convince the teacher that the middle triangle is going to adjust to the sides of the rectangle, showing by gestures how this could be done, typical actions of the first cycle, Debora seems to initiate the process of creative externalization of the second cycle.

048 Debora: But you have no warranty, there is no measure…

049 Teacher: Just one question, do you see this opening here (pointing to the angle in the central triangle that 'looks like' a right angle)? It is going to fit here (the angle of the rectangle) only if I have an opening that is perpendicular. But the problem does not inform about that, the problem does not say if I have two perpendicular sides. So, the thing… Is it going to fit? Would you be able to find the area of this triangle here (the middle one)? Would it be of any help to find the area of these triangles here (the right angled triangles surrounding it)?

In line 49 there is a fourth attempt of perturbation in the activity instigated by the teacher. In the meantime, George and Priscila are still trying to find the area of the 'middle' triangle, now using a different procedure inspired in another previous exercise. Suddenly, all of them got attuned to the procedure suggested by the teacher. Finally they all join the second cycle of actions focused on the calculation of the areas of the three right angled triangles to find the area of the 'middle triangle'.

056 George: You multiply three times four and five times five and one times eight and it will give this here…

057 Teacher: Wait…Bernardo. Speak out Bernardo (he was raising his hand)!

058 Bernardo: I have done it the same way as Giovan, but afterwards I have done eight times five and I have added the whole figure he said, and I have calculated each area. Five times five, one times eight and three times four. I have

added the result and I have made minus forty, which gave the area of that figure there.

Bernardo seems to have already figured out how to solve the problem and the teacher asks him to go to the blackboard to show his work. Bernardo organizes all his calculations on the board, ending up with 45. He gets confused and erases everything. He goes back to his seat without solving the problem and the teacher questions him.

061 Bernardo: It's because the result of the 'empty' there (meaning not shaded) was forty five and the result of the rectangle was forty.

Follows a recapitulation of Bernardo's procedure with the participation of several students while the teacher reminds them that they should not forget to divide the areas of the triangles by two and resumes the calculations on the blackboard, ending up with twenty two and a half for the sum of the areas of the three right angled triangles. Some students seem reluctant to accept these 'broken numbers' but the teacher tries to make them accept that it is perfectly all right to have them as a result. Finally, the teacher focuses again on the visualization of the whole figure - rectangle with a triangle inside - for the calculation of the area of the central triangle. He goes back to the actions of the first cycle, to proceed with the actions of the second, by a cyclic and recurrent procedure that we see as a characteristic of the activity time.

084 Teacher: Just a minute. Sorry to have interrupted you. Twenty two and a half is the area that I want to calculate?

085 Voices: Yes… No…

The discussion goes on with the students looking attentively to the blackboard, where the teacher is writing all the calculations. Then Edna resumes the calculation.

094 Edna: Then you make forty minus twenty two and a half, and what remains is from the triangle.

At this point, the teacher asks Edna to go to the blackboard to explain it all over again to everybody. He gives the pointing stick to her and she repeats the whole explanation while the colleagues look attentively to her 'acting' as a teacher, in a process of creative externalization. When she is finished he checks once again with the class if they have understood the whole procedure.

CONCLUSION

In this geometry class we could identify two cycles of actions. The first cycle can be characterized as one of Visual Manipulation of the figure and the second as one of Algebraic Calculation of the area of the figure. In the first cycle we can notice a predominance of the process of internalization, as the students reproduce procedures of visual manipulation of figures. In the second we could identify a process of creative externalization characterized by the introduction of the algebraic calculation, as a new artifact. Together, they compose an *expansive cycle* of the activity.

Our analysis shows the marked influence of the teacher in the development of an expansive cycle of this activity. The perturbations he presents result in disequilibrium of the conditions of the activity system and in fluctuations of the activity object, which we could identify because we were analyzing the historicity of the activity.

This kind of analysis also allowed us to verify that there are students which are taking part, simultaneously, into different cycles of the classroom activity. Thus, showing evidence that the classroom activity time is not linear.

Future further research exploring the expansive cycles of classroom activity should be developed discussing the process of learning by expanding (Engeström, 1987) in mathematics classrooms.

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SAYING CORRECTLY, WRITING INCORRECTLY: REPRESENTATION OF SETS

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Sets are one of the important domains in mathematics curricula and may influence students' performance in other topics such as functions. The aim of the study is to determine difficulties that year 9 Turkish students experience in the "sets", trying to reveal their thoughts about sets and their representations. Multi method approaches were used in this study. The data were collected by a diagnostic test of the sets and semi constructed interview.59 high school students participated in this study. Semi constructed interview was conducted by 10 students. The findings showed that students had different type of difficulties in representation of sets; they speak sets correctly but they write sets incorrectly. It is also find that they know what they know but they cannot use their knowledge in different mathematics context.

INTRODUCTION

This paper reports on one aspect of a wider study on freshmen secondary students' performance and understanding of aspects of representing sets in contextual and context-free questions. This paper merely concentrates on difficulties and 'representation' aspects of describing a set.

Set is one of the fundamental concepts of mathematics. Set Theory constitutes a basis for many subjects such as number theory, functions, linear algebra, etc. in mathematics. German mathematician Georg Cantor (1845-1918) introduced the concept of set in the nineteen century. According to Eves and Newsom (1958), the modern set theory is one of the most remarkable creations of the human kind. Many domains of mathematics were enriched, clarified, extended and generalized by the set theory, and the influence of set theory is deep on the study of foundations of mathematics (citied in Pinker, 1981). Cantor expressed the meaning of 'Set' as a collection of definite and separate objects which can be considered by the mind and to which one decide whether or not a given element belongs (citied in Bagni, 2006). Though there are many verbal descriptions of sets, the definition of sets does not exist (Bagni, 2006; Brenton and Edwards, 2003; Fischbein and Baltsan, 1999). Mathematical set concept possesses its own features and perspective. At that point, students have a tendency to explain the concept of set with its characteristics and aspects. For instance, one of the intended goals in "Sets" topic is "students show sets by using listing method, Venn diagrams and common property method" (Ministry of National Education [MEB], 2007) in secondary school mathematics program in Turkey.

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Anyone mentions about sets as a general concept, some pictures take shape in our mind that is called "concept image", it is not really set itself (Tall and Vinner, 1981; Vinner, 1983). Recently, multiple representations are getting more interest in the mathematics curriculum (Piez & Voxman, 1997; cited in Patterson and Norwood, 2004). The National Council of Mathematics Teachers (NCTM, 2000) suggested the followings:

- enable all students to create and use representations to organize, record, and communicate mathematical ideas;
- enable all students to select, apply, and translate among mathematical representations to solve problems;
- enable all students to use representations to model and interpret physical, social, and mathematical phenomena.

Representations play an important role in understanding mathematical concepts and ideas and set is such a topic in mathematics to use representations in either describing a set or solving problems. Three methods have been widely used for representing a set. First method was introduced by Euler, which is called "Euler Diagram" and resumed by John Venn, namely "Venn Diagram". By using this method, it is quite possible to demonstrate the relationships between the sets and placed the elements of a set visually in a closed curve. The second method is listing elements. By this way, sets are described by using parenthesis, which includes elements of set with commas to distinguish distinct elements. Third way is to represent a set is common property method. Each of these methods is a way to describe a set and a set may be described by more than one representations but the important aspect of these representations are being equivalent of each other. Tsamir (2001) says:

> "The ability to recognize the underlying mathematical identity of given processes, notions, or tasks, which are presented in different forms is an important aspect of mathematical knowledge. However, it has been widely documented that different representations of essentially the same mathematical tasks often trigger responses that differ and sometimes even clash".

Despite different representations of a set, which are equivalent, enrich the concept image and help student to approach the sets from different perspective, it may also let students to make mistakes. In three different representation methods of a set, different symbols are used to describe a set. The experiences of the researchers with students showed that students have difficulties in perceiving sets that are given by common property method and translating verbal text into mathematical symbols. Therefore, in addition to converting one representation of a set to another representation writing what is said in mathematical symbols seemed to be important for representing a set in equivalent forms. According to Rosnick (1981), translation from verbal relational statements to symbolic statements brings about students a great deal of confusion (cited in Capraro and Joffrion, 2006). This diagnosis underlines the significance of understanding mathematical texts and representations such as symbolic expressions, equations, formulas, graphs. Symbols and representations are widely utilized in almost every field of mathematics to support thinking and to exchange ideas in terms of understanding and constructing the knowledge (Delice and Aydin, 2006). Pimm (1987) stated that mathematical language, which includes symbols and representation, is constructed in classroom environments to communicate with using mathematical argumentation, mathematical vocabulary and mathematical writing (cited in Delice and Aydin, 2006). Therefore, students also need to notice the meaning of symbols and representations that are used in mathematical contexts.

Some students, especially science and mathematics students, encounter set theory throughout their education since the set theory is the fundamental domain, which is included in other domains such as relations, functions, matrix theory, etc. For example, in Turkey, students are introduced to mathematics by starting subject sets in first year in the primary school. They faced with set theory gradually in details at the grade 6 and grade 9 again. New secondary school mathematics program involves the concept of sets as a second topic after logic (MEB, 2007). There is a strong relationship and order between the topics in the mathematics program. Hence, students need to follow courses more carefully in order not to miss any important point during the instruction. If students have difficulties in understanding set theory, this situation may influence their attitudes against other topics of mathematics in negative way. Since the topics are ordered and related, students may have difficulties while learning new subjects. It is observed that some students fail in exams because of not understanding the set given in the question although they participated in most of activities during the lessons. Students also have sufficient knowledge about former subjects such as numbers, inequalities. The role of language in mathematics is critical issue while learning mathematics. Ferrari (2004) declared that language is important in the building of meanings as developing mathematical thinking. Consequently, In this research, the obstacles that influence students' performance and understanding the representation of sets are investigated.

METHODOLOGY

This research is carried out with multi method approach (Cohen et al., 2007) which is also called mixed-methods approach (Lodico et al., 2006) in order to gather wealthy data to answer research questions. The research instruments are a test of the sets and a semi-constructed interview. So, the research is qualitative in terms of the data.

A diagnostic test is developed to reveal particular weakness or difficulties that students are experiencing (Cohen et al., 2007) in sets, more specifically in understanding representation of sets. When test items were constructed, ultimate goal was to discover what kind of difficulties students are experiencing and possible causes that influence their learning. Therefore, the test was also criterion-referenced test (Cohen et al., 2007). Content validity of the data collection instruments was obtained by a detailed consideration of the scope of research by three tutors in Department of Mathematics Education and fifteen mathematics teachers. Test was also piloted with 20 students to see whether it is working or not. In the test there are six questions asking students to write equivalent form of the given set. In order to ensure reliability qualitative data were categorised and coded (Miles and Huberman, 1984:23). Compatibility rates among these categories were then calculated. The coding revealed a compatibility rate of %88.

Purposeful sampling technique (Patton, 1990) of non-probability sampling methods, which accept individuals or events as they are, was used for the selection of the sample for this qualitative research (Cohen *et al.*, 2000). The sample of this study was selected from nine-grade freshmen secondary school of the Ministry of National Education. A diagnostic test was administered with 59 students and then semi-structured interviews were also conducted with ten students to get insight about their thoughts about sets and representation of sets.

RESULTS

The analysis of the diagnostic test and interview are presented in this section.

Equivalent forms of representations of a set

The answers which are equivalent forms to the sets given in the question column of Table 1 were categorized in two ways. First, the answers were categorized if the equivalent forms were correct (C), incorrect (IC) partially answered (P) and non-attempted (NA). Independent from whether the equivalent forms are C, IC and PA the representations used in equivalent forms were also categorized in the same way as C, IC, PA and NA. NA means both non-attempted and non-applicable in second categorization. The percentage table of both categorizations is presented in Table 1.

	Answers (%)				Representations (%)			
Questions	C	IC	P	NA	C	IC	P	NA
$A=\{2,4,6,8,...\}$	61	20	10	9	26	5	49	20
$B=\{x \mid 3\le x<7,\ x\in N\}$	68	12	9	11	18	5	41	36
$C=(-1,4)$	4	50	17	29	2	37	10	51
$D=\{105,110,115,...,155\}$	9	48	31	13	15	3	32	50
$E=\{y \mid -4<y\le 7,\ y\in R\}$	2	68	7	23	0	20	28	52
$F=\{a,b,c\}$	20	19	31	30	13	3	36	48
Total	27	36	18	19	20	20	53	7

Table 1: Percentages of students' answers and representations in diagnostic test.

There are interesting results in the Table 1; Overall, highest percentage of students gave the wrong answer to the question and more than half of the students either left answer partial.

Answers column shows that more than half of the students given the correct equivalent forms of the sets A and B whereas less than 10% of students wrote incorrect representation form of C, D and E. Interestingly 68% and 50% of students wrote the incorrect equivalent form for the sets E and C which are both about Real numbers.

Representations column shows that in all representations less than quarter of representations was written correctly by students. Most of the students written some expressions that is not exactly correct or incorrect, namely, partial in almost all of them. In NA categorizations, students mostly preferred verbal forms rather than mathematical symbols or forms. Almost half of the students did not write an equivalent form for the sets C, D, E and F or written verbal forms (without any mathematical form).

Another categorization also made for the representation methods used by students in writing the equivalent form of the questions in Table 1. More than half of the representations were written by Common property method and then 34% of the most used method was listing method. 10% of students however used Venn diagram and other representation methods.

Interviews by students

The questions in interviews were also categorized and presented question by question.

The concept of set; More than half of the students said a set is a closed curve with something inside. Almost quarter of them said a group of something and there were few students who were not able to say anything.

Types of representation of a set; Interestingly all students said there are three types of representation methods which are Venn diagrams, common property and listing methods.

Representation of set encountered the most; More than half of the students said listing methods is the most encountered method whereas 25% said Venn diagram. Common property and Universal set was the representation methods 6% students encounter the most in total.

Representation which is difficult to comprehend; Interestingly 32% of students said they have no difficulty with representation of the sets. However, 32%, 17% and 10% of students have difficulties to understand common property, listing and Venn diagram methods respectively.

Subjects which are influenced by sets; Students thinks functions, relations, operations, numbers, problems, logic and some other are mostly influenced by the sets in that order. 15% of students did not think of any topic influenced by sets.

When students were asked to show the set of real numbers, they had real difficulties with representations and with the set itself. Some of the students said since there is infinite numbers they cannot show it. Some of them have difficulties with representations (Figure 1)

Figure 1: Two examples of representation of Real numbers.

It seemed that even though they have difficulties to show students thinks of empty set as interesting set. Most of the students gave impossible examples and situations to get an interesting set such as flying cow, flying elephants and four leg human beings.

DISCUSSION

Since "Sets" is one of the basic subject in mathematics and form a basis for many topics such as numbers, function, etc., it has been attributed more importance among mathematicians and mathematics educators (Bagni, 2006; Fischbein and Baltsan, 1999; Delice and Aydin, 2006). It is important that students understand sets sufficiently due to some of following topics are related to sets and also important topics in mathematics. In this research, it was dealt with representations of sets and investigated students' perception about sets and representation of sets.

Research findings demonstrated that students aware of the significance of sets and their representation. Nevertheless, in the application, especially in diagnostic test, they could not write down what they really know. Between the process of understanding representation of sets and expressing them by using mathematical language, they had difficulties. The difficulty appears in writing rather than speaking. They say they can express the set but it is hard to write with symbols or mathematically. This implies that there is a cut point in transferring knowledge from mind to text. So, it may be said that students' minds confuse in translating verbal statements to symbolic statements (Rosnick, 1981; cited in Capraro and Joffrion, 2006).

The results indicated that students also had difficulties in real numbers which included in the common property representation of sets. They did not know how to show infinite elements of the real numbers. The symbols are widely used in common property method in representing sets. Students may overcome some difficulties by having symbol sense which is a similar kind of familiarity with algebraic symbols as "number sense" to arithmetic that increasing of students' ability to deal with symbols give them symbol sense and helps to understand algebra. Symbol sense can be seen as the power of symbolic thinking, an understanding of when and why to apply it, and a feel for mathematical structure (Bergsten, 1999). As algebra can be viewed both as a symbol system and as a way of thinking (Sierpinska, 1995, p. 157; cited in Bergsten, 1999) the representations of the sets depending on the symbols may be seen as a way of expression and thinking as well. Moreover, representation of a set and symbols used are kind of learning another language such as symbol sense as a level of mathematical literacy beyond number sense (Picciotto and Wah, 1993, p. 42). Students said "like learning to speak..write English but very hard and forgettable".

Results also showed that over half of the students tried to write down equivalent forms by using common property method. However, it is observed from their writings that they could not understand and comprehend the common property method rather that other methods adequately. That might be because of the meaning they would probably not constructed (Delice and Aydin, 2006) or the signs and symbols students are not able to observe advantages of using it.

> "In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labour is wonderfully diminished." (Leibniz quoted in Bergsten, 1990)

Although students meet the set with Venn diagram first in their school life and then repeat it later in some years, they rarely used Venn diagram and they wanted to use algebraic way listing and common property methods. This might be because of the sets (mathematical context) or concept image of the sets in their mind. They perceive sets as a closed curve, namely, they are conceptualized metaphorically as containers (Lakoff and N´u˜nez, 2000, p. 45). However Euler–Venn diagrams cannot be identified with the concept of sets (Freudenthal, 1983). Therefore, though there is not clear definition of the sets and the Venn diagram is the one students use the most in the past, they preferred to use symbolic ways, signs to represent sets. Having the concepts of sets and concepts of representations is embedded in long processes of sign and meaning production (Radford, 2003).

CONCLUSION

In this study, students' understanding of sets and representations of sets was investigated. The research findings displayed that even though students had sufficient knowledge about sets and their representations, they did not transfer their knowledge into action. Due to the fact that the concept of set is domain of some important topics in mathematics such as functions, number theory, linear algebra, etc., it ought to be given more importance in teaching sets.

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A STRUCTURAL MODEL FOR THE UNDERSTANDING OF DECIMAL NUMBERS IN PRIMARY AND SECONDARY EDUCATION[1]

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This study aims to propose and validate a model which describes the structure of decimal number addition understanding related to multiple-representation flexibility and problem-solving ability. Data were collected from 1701 primary and secondary school students. Confirmatory Factor Analyses (CFA) affirmed the existence of seven first-order factors indicating the differential effects of the modes of representation, the transformations of representation and the place-value concept, two second-order factors representing multiple-representation flexibility and problem-solving ability and a third-order factor that corresponds to the understanding of decimal number addition. Results revealed the invariance of this structure across the two educational levels and provided suggestions for a smoother transition to secondary school.

INTRODUCTION

A representation is any configuration of signs, characters or objects that stand for something else (Goldin & Shteingold, 2001). Nowadays the centrality of different types of representation in teaching and learning mathematics seems to become widely acknowledged by the mathematics education community. The NCTM's Principles and Standards for School Mathematics (2000) document includes a new process standard that addresses representations and stresses the importance of the use of multiple representations in mathematical learning. In fact, the abilities of recognizing the same concept in multiple systems of representations, manipulating the concept within these representations and converting flexibly the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh, Post, & Behr, 1987).

Furthermore, problem solving is an integral part of all mathematics learning (NCTM, 2000; Reys, Lindquist, Lambdin, Smith, & Suydam, 2001). The role of multiple representations in problem solving is widely acknowledged (Elia, Gagatsis, & Demetriou, 2007). Hitt (1998) underlines that the ability to solve problems entails the ability to convert with the preservation of meaning from one system of representation to another in functions. Recently, Deliyianni, Panaoura, Elia and Gagatsis (2008) have proposed a three-level hierarchy across students of two different grades in primary school, indicating the important role of multiple-representation flexibility and problem-solving ability in the understanding of fraction addition.

The concept of decimal numbers is included in mathematics curricula and it is considered to be of great significance due to its application and use in everyday life.

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The specific concept is closely linked with the concept of fractions, since decimals can be considered as the parts of the whole that has been divided into a number of parts that are powers of 10 (Thomson & Walker, 1996). The necessity of using a variety of representations in supporting and assessing students' constructions of decimal numbers is stressed by a number of studies (e.g. Thomson & Walker, 1996; Martinie & Bay-Williams, 2003). Michaelidou, Gagatsis and Pitta-Pantazi (2004) reveal 6th graders' difficulties in recognition and conversion tasks in decimals. However, the cognitive processes underlying the understanding of decimal numbers have not investigated systematically yet especially when the students are at a point of transition from primary to secondary education.

In this study we take into consideration a synthesis of the ideas articulated in previous studies on learning with multiple representations to capture primary and secondary students' processes in decimal number addition multiple-representation tasks. On the theoretical level, this knowledge will contribute to gain a more comprehensive picture of the decimal number addition understanding related to multiple-representation flexibility and problem-solving ability and second, to find out more meaningful similarities between primary and secondary students' representational thinking and problem-solving ability. Keeping in mind the difficulties students face during their transition from one educational level to another (Mullins & Irvin, 2000), this study could be practically useful, as well, as it may offer suggestions for a smoother transition of students to secondary school in mathematics learning.

Our hypotheses are concerned with the structure of the processes underlying decimal number addition understanding, First, we expect that multiple-representation flexibility and problem-solving ability influence the understanding of decimal number addition since using multiple representations flexibly (NCTM, 2000) and problem-solving ability (Reys et al., 2001) are considered as basic components of mathematics performance. Second, the transformations of representation (recognition, treatment and conversion according to Lesh et al., 1987), the modes of representation and the place-value concept are expected to interact with each other and affect multiple-representation flexibility in decimal number addition. Specifically, it is well documented that different types of representation activate different mental processes even when dealing with tasks of the same concept (Duval, 2006). The decimal numbers are also related closely with the place-value concept (Thomson & Walker, 1996). In fact, Michaelidou et al. (2004) revealed that the performance of students varies according to the number of the summands' digits. Third, based on the differential impact the representations exert on additive problem-solving performance (Elia et al., 2007), we expect that problem-solving ability would be influenced by the verbal and the diagrammatic modes of representation. Fourth, drawing on the findings of Deliyianni's et al. (2008) study which provided evidence for the invariance of the structure underlying the understanding of fractions across different grades of primary school, we expect that the structure proposed here for decimal numbers would generally be independent of age, as well.

METHOD

The study was conducted among 1701 students, aged 10 to 14, of primary (Grade 5 and 6) and secondary (Grade 7 and 8) schools in Cyprus (414 in Grade 5, 415 in Grade 6, 406 in Grade 7, 466 in Grade 8).

The test that was constructed in order to examine the hypotheses of this study consists of 19 multiple-representation and 4 problem-solving tasks. Multiple-representation flexibility tasks differed in terms of the following three dimensions: (a) the types of representation transformation, b) the modes of representation and c) the value of the digits. With respect to the transformations, there were three types of tasks: recognition (Type Re tasks), treatment (Type Tr tasks) and conversion (Type Co tasks) tasks. Concerning the modes of representation, there were four types of tasks: number line tasks (Type L tasks), tasks involving rectangular area surface (Type R tasks), tasks consisting of circular area surface (Type C tasks) and tasks in a symbolic representation (Type S tasks). In conversion tasks the first uppercase corresponded to the initial representation and the second one to the target representation. As for the value of digits, distinction was made between the tasks in which both summands consist of tenths (Type t tasks) or hundreds (Type h tasks) and the tasks in which the summands included tenths and hundreds respectively (Type th tasks). Problem-solving tasks (Type Pr tasks) differed in terms of the modes of representation. Thus, there were two types of problems: the verbal (Type V tasks) and the diagrammatic (Type D tasks) ones. In the codification of all task types the task numbering was indicated at the end. Representative tasks used in the test appear in the Appendix. Particularly the test included:

1. Recognition tasks in which the students are asked to identify addition of one and/or two digit numbers in number line (ReLt1, RELh4, ReLth7), rectangular (ReRt2, ReRh5, ReRth8) and circular (ReCh3, ReCth6) area diagrams.
2. Symbolic addition treatment tasks in which the summands consist of tenths and/ or hundreds (TrSt9, TrSh10, TrSth11, TrSh12, TrSt13).
3. Conversion tasks from a symbolic to a diagrammatic representation (CoSLth14, CoSRh15, CoSCt16), and the reverse (CoRSh17, CoLSth18, CoCSt19), in which the summands consist of tenths and/ or hundreds.
4. Diagrammatic decimal number addition problem (PrD20).
5. Verbal decimal number addition problem that is accompanied by an auxiliary diagrammatic representation (PrD21).
6. Verbal decimal number addition problem (PrV22).
7. Justification problem-solving task that is presented verbally and is related to the addition of decimal numbers (PrV23).

RESULTS

In order to explore the structure of the various decimal number addition understanding dimensions a third-order CFA model for the total sample was designed and verified. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI (Comparative Fit Index) and RMSEA (Root Mean Square Error of Approximation). The following values of the three indices are needed to hold true for supporting an adequate fit of the model: x^2/df < 2, CFI > .9, RMSEA < .06.

Figure 1 presents the results of the elaborated model, which fits the data reasonably well [x^2(201) =380.61, CFI=0.98, RMSEA=0.02]. The third-order model which is considered appropriate for interpreting decimal number addition understanding, involves seven first-order factors, two second-order factors and a third-order factor. The two second-order factors that correspond to the multiple-representation flexibility (MRF) and to the problem-solving ability (PSA) are regressed on a third-order factor that stands for the understanding of the decimal number addition concept (DAU). The values of their loadings are both high revealing that students' decimal number understanding is predicted from both multiple-representation flexibility and problem-solving ability. Thus, the findings confirm our first hypothesis suggesting that multiple-representation flexibility and problem-solving ability influence the understanding of decimal number addition.

The first-order factors F1 to F5 are regressed on the second-order factor that stands for the multiple-representation flexibility. The first-order factor F1 refers to the recognition tasks in which the summands have the same number of digits, while the first-order factor F2 refers to recognition tasks in which the summands have different number of digits. The first-order factor F3 consists of decimal number addition treatment tasks in which the summands have the same or different number of digits. Conversion tasks in which the initial and the target representation is decimal number equation and diagrammatic representation, respectively, constitute the first-order factor F4, while the first-order factor F5 refers to the decimal number addition conversion tasks from a diagrammatic to a symbolic representation. Therefore, the existence of the first-order factors F1 to F5 verifies the second hypothesis. Particularly, it reveals that the flexibility in multiple decimal number addition representations constitutes a multi-faceted construct which involves an interaction between representation transformations (recognition, treatment, conversion), the modes of representation (symbolic, diagrammatic) and the place-value concept.

The other two first-order factors F6 and F7 are regressed on a second-order factor that represents problem-solving ability. The first-order factor F6 consists of problems accompanied with a diagram while the factor F7 is comprised of the verbal problems. The existence of these two first-order factors confirms the third hypothesis as it explains the differential effects of the modes of representation on problem-solving ability.

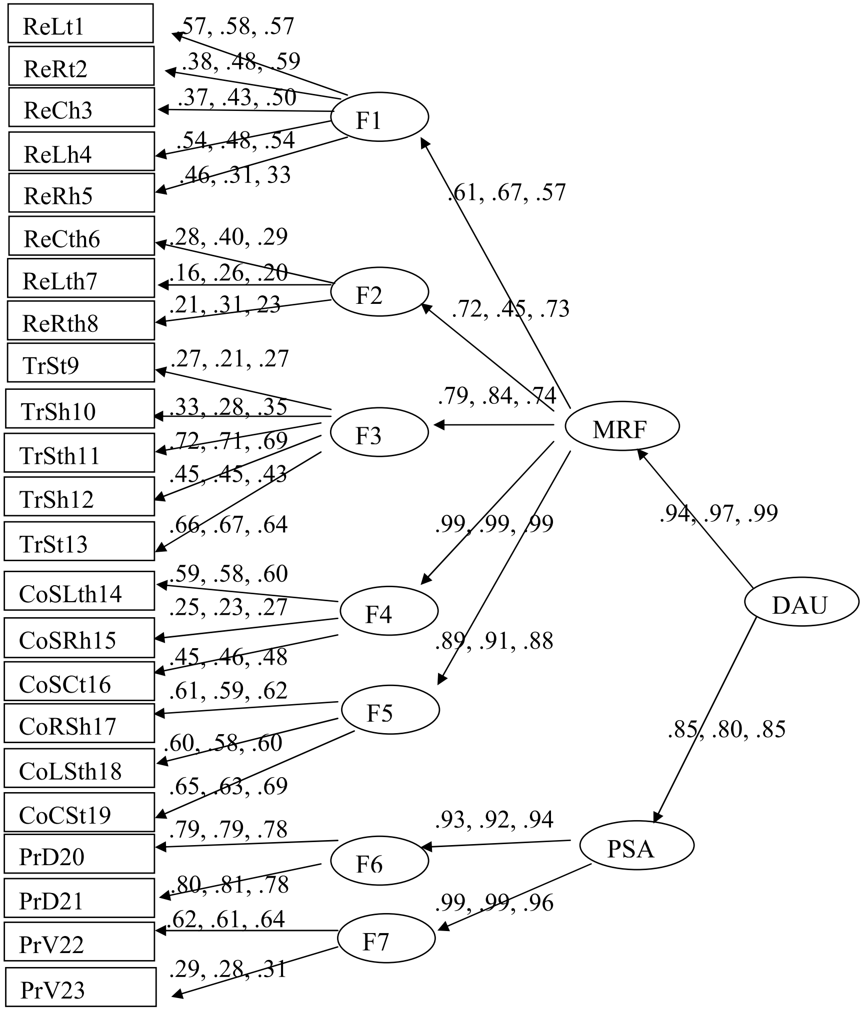

Figure 1: The CFA model of the fraction addition understanding

Note: 1. The first, second and third coefficients of each factor stand for the application of the model in the whole sample, primary and secondary school students, respectively. 2. Errors of the variables are omitted.

To test the invariance of this structure between the two educational level groups' decimal number addition understanding, multiple group analysis was applied, where the proposed three-order factor model was validated for primary (Grade 5 and 6) and secondary (Grade 7 and 8) school students separately. The model was tested under the assumption that the relations of the observed variables to the first-order factors, of the seven first-order factors to the two second-order factors and of the two second-order factors to the third-order factor would be equal across the two level groups. The fit indices of the model tested are acceptable [x^2 (426) =798.24, CFI= 0.96, RMSEA= 0.03]. Thus, the results are in line with the fourth hypothesis that the same structure holds for both the primary and the secondary school students. However, some factor loadings are stronger in the group of the secondary school students, revealing that the strength of the relations between these abilities and, thus, the specific structural organisation potency increased across the educational levels.

CONCLUSIONS

The main purpose of this study was twofold, to test whether flexibility in multiple representations and problem-solving ability have an effect on decimal number addition understanding and to investigate its factorial structure within the framework of CFA, across students of primary and secondary school. The results provided a strong case for the important role of multiple-representation flexibility and problem-solving ability in primary and secondary school students' decimal number addition understanding. CFA showed that two second-order factors are needed to account for the flexibility in multiple representations and the problem-solving ability. Both of these second-order factors are highly associated with a third-order factor representing the decimal number addition understanding.

CFA also verified that five first-order factors are required to account for the second-order factor that stands for the flexibility in multiple representations, while two first-order factors are needed to explain the second-order factor that represents the problem-solving ability. In particular, the ability to recognize decimal number addition, to symbolically manipulate decimal number addition and to convert flexibly from one decimal number addition representation to another differentially affected multiple-representation flexibility. Besides, the value of digits was found to differentially affect the ability to solve recognition tasks of decimal number addition. However, the place-value concept do not affect students' ability to calculate symbolically the sums of decimals since the specific processes are automated by the age group of the students involved here. Furthermore, the ability to convert from diagrammatic to symbolic equation came out as a dimension of performance distinct from the ability to convert from a decimal number addition equation to a diagrammatic representation. This suggests that the different types of representation differentially affect the solution process, because students activate different mental processes when solving these tasks. In fact, the findings revealed that multiple-representation flexibility constitutes a multi-faceted construct in which the representation transformations interact with the modes of representation and the place-value concept. The results indicated also the differential effect of the modes of representation on problem-solving ability as far as the addition of decimal numbers is concerned. Therefore, developing the abilities to recognize the addition of one and/or two digit numbers in a variety of diagrammatic representations, to manipulate symbolically the addition of one and/or two digit numbers, to converse one and/or two digit numbers from a diagrammatic to a symbolic representation and the reverse and to solve decimal number addition verbal and diagrammatic problems may contribute to the development of multiple-representation flexibility and problem-solving ability, which are of primary importance in the understanding of decimal number addition concept.

Besides, the findings of the present study are generally in line with the three-level hierarchy validated for the fraction-addition understanding by Deliyianni et al. (2008). This reveals the potential for developing and validating an integrated cognitive framework of rational numbers, which could be the subject of a future study.

Concerning the educational level, it is to be stressed that the structure of the processes underlying the fraction addition understanding was invariant across primary and secondary grades. This finding suggests that it is important to develop the corresponding cognitive processes in both primary and secondary education so as to enable primary school students to have a smoother transition to secondary school. Certainly, there is still need for further investigation into the teaching implications of the subject. In particular, it would be interesting and useful to examine the effects of intervention programs aiming to develop multiple-representation flexibility and problem-solving ability on primary and secondary students' performance concerning decimal numbers.

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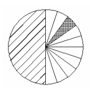
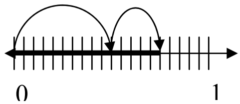
APPENDIX

1. Circle the diagram or the diagrams whose shaded part corresponds to the equation 0.5 + 0.05.



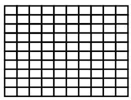


(ReCth6) (ReLth7) (ReRth8) [recognition tasks]

2. Solve the equation: 0.5 + 0.4 = ….... (TrSt9) [treatment task]

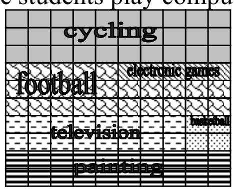
3. Present the following equation on the diagram:

0.05 + 0.04=… (CoSRh15) [conversion task]



4. A research was conducted in a school to find out which activities the students choose to do in their free time. The results are given below:

- 0.25 of the students play football.
- 0.04 of the students play basketball.
- 0.3 of the students do cycling.
- 0.16 of the students watch television.
- 0.2 of the students paint.
- 0.05 of the students play computer games.



Which activities 0.6 of the students do? (PrD21)

5. In the addition of two decimal numbers that do not have zero as a digit of tenths and hundreds the sum may be a whole number. Do you agree with this view? Explain. (PrV23) [problem solving tasks]

[1] This study is a part of the medium research project MED19, funded by the University of Cyprus

THE SECONDARY-TERTIARY TRANSITION: BEYOND THE PURELY COGNITIVE

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Transition from secondary to tertiary level is an important issue in mathematics education, indeed this transition is often problematic for many students. University spontaneously reacts through running bridging courses, lowering the level of the mathematics taught, or reducing the examination standards in order to avoid massive failure. For different reasons these reactions rarely proved to be effective. Instead, research in mathematics education highlights that students' difficulties are related to a multiplicity of factors – cognitive, meta-cognitive, affective and linguistic – the incidence of which is amplified when accessing tertiary education. Starting from these findings, the authors designed and developed two parallel mathematical "university preparatory" courses for grade 12-13 students.

INTRODUCTION

The transition from secondary to tertiary mathematics education is often very problematic for many students, so much that it is considered "*a major stumbling block in the teaching of mathematics*" (De Guzmán et al., 1998; p.756). The existence of specific difficulties in mathematics in the move from secondary to tertiary education is widely acknowledged by both mathematicians and educators (De Guzmán et al., 1998; Gueudet 2008; Wood, 2001).

In addition, the phenomenon is exacerbated by the growing democratization of tertiary teaching and the increasing numbers of those who study some mathematics as part of a degree; this urges the mathematics education community to pay more attention to the issue. In fact, notwithstanding the increasing interest of mathematics educators towards tertiary education issues, so far the amount of research in the field is modest and limited to some specific topics (Selden A. & Selden J., 2001).

While some authors underline the need to make mathematics education findings at the tertiary level more user-friendly and accessible (Artigue, 2001; Selden, 2005), on their part universities and university teachers are spontaneously reacting to students' massive failures in mathematics in different ways:

- adapting curricula to the actual students' knowledge, in particular lowering the level of the mathematics taught, and getting the university teaching style closer to the secondary one; thus the development of students as autonomous learners might be hindered (De Guzmán et al., 1998);

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 401-408. Thessaloniki, Greece: PME.

- reducing the examination standards while formally keeping the usual curricula, thus introducing strong discrepancies between what is taught and what is actually assessed. As Artigue (2001, p.209) observes *"[T]his situation had dramatic effects on their beliefs about mathematics and mathematical activity. This, in turn, did not help them [students] to cope with the complexity of advanced mathematical thinking*";
- running bridging courses aimed at filling gaps in prerequisite knowledge. But, as research suggests, students' difficulties at university entrance cannot be reduced to lack in content knowledge or to purely cognitive factors. In order to develop effective teaching actions to overcome these difficulties one needs to interpret them by analysing students' behaviour, and according to McLeod (1992, p.268) "*researchers who are interested in human performance need to go beyond the purely cognitive*".

Assuming this perspective, we designed and enacted a mathematical "university preparatory" course for grade 12-13 students, which we present in the current report.

CONCEPTUAL FRAMEWORK

Linking research and practice, through the development of teaching design grounded in research findings, is obviously one of the main challenges for research concerning students' difficulties at any level. In that respect the "*clarification of the causes* [of students' difficulties] *plays a fundamental role in the building of appropriate didactical actions*" (Gueudet, 2008; p.251).

Research highlights that several different factors can be related to the difficulties experienced by novice undergraduates. De Guzmàn et al. (1998) provide a wide overview on the possible sources of these difficulties and highlight that lack in the prerequisite knowledge is only one of these causes. The portrait emerging from a recent literature survey (Gueudet, 2008) is rather composite: various research studies, focused on different phenomena (individual, social and institutional) and drawn from different perspectives, contribute to sort out different kinds of difficulties.

Many authors indicate the necessity of developing new thinking modes (Lithner 2000; Sierpinska, 2000). In particular, the necessary shift from *elementary* to *advanced mathematical thinking* is seen as a main source of difficulties: "*The move from elementary to AMT involves a significant transition: that from describing to defining, from convincing to proving in a logical manner based on definitions*" (Tall, 1991; p. 20).

De Guzmàn et al. (1998) also stress the importance and the difficulty for students to become autonomous learners: "*acquisition of a certain level of autonomy in learning is often seen by university teachers as the main stumbling block in the secondary-tertiary passage*" (ibidem, p.751). Lack in autonomy is revealed by the fact that "*many students arriving at university do not know how to take notes during a lecture, how to read a textbook, how to plan for the study of a topics, which questions to ask themselves*" (ibidem, p.756-757).

Research has also stressed the role that metacognitive (Schoenfeld, 1983), affective (Adams & McLeod, 1989) and linguistic/semiotic (Ferrari, 2004) factors play in students' mathematical behaviour also at the tertiary level and then the need to develop abilities related to these features.

Summarizing, it emerges that a variety of factors can be related to students' difficulties in mathematics at the tertiary level; in addition, these factors appear to be deeply intertwined. Hence, we think that in the design teaching actions aimed at alleviating students' difficulties, one should start from the consideration of all these factors.

THE EXPERIMENTAL "UNIVERSITY PREPARATORY" COURSE

Origin and aims.

In the academic year 2006/07, the University of Pisa carried out a project (named PORTA.[1]) which addressed the issue of students' difficulties in the secondary-tertiary transition. As for mathematics, the local mathematics education research team (of which the authors were members) was involved in the project.

The setting up strategy consisted in promoting teaching actions at the secondary level for working in advance on the causes underlying students' difficulties in mathematics, in order to prevent or limit their occurrence. An experimental mathematics course (hereafter PORTA course) for grade 12-13 students was designed and carried out by the two authors.

Drawing on the research findings mentioned in the previous section, we formulated the hypothesis that students' difficulties mathematics in the secondary-tertiary transition are linked to four main aspects:

- lack in the prerequisite knowledge;
- difficulties to move from elementary to advanced mathematical thinking;
- negative attitude toward mathematics. According to Zan & Di Martino (2007) a negative attitude can emerge, for example, as: negative emotions toward maths, an instrumental view of mathematics (Skemp, 1976), or low self-competence (Bandura, 1986; Pajares & Miller, 1994);
- lack in metacognitive and linguistic abilities (to interpret, understand, produce a mathematical text, to communicate knowledge and ideas and to manage different representational systems).

In addition we think that the above factors are strictly interrelated. This hypothesis was also substantiated by the authors' personal experience as assistant lecturers in mathematics service courses.

[1] PORTA is an acronym (Progetto Orientamento e Riduzione Tasso Abbandoni) which stands for Project for orienting students and reducing the university abandoning rate.

Therefore for a didactical intervention to be really effective at the tertiary level, there is the need of going beyond the purely cognitive dimension and intervene also on the other dimensions. Our conviction is that acting on the affective sphere, and promoting the development of cognitive, but also meta-cognitive and linguistic abilities, we can provide students with tools for autonomously recognizing, facing and overcoming possible future difficulties. Zan (2000) described a successful remedial action at the tertiary level inspired by the analysis of students' failures in terms of meta-cognitive and affective factors, supports our conviction.

Consistently with the above hypothesis, the aim of the PORTA course was initiate a path to develop the mentioned multiple abilities, to promote students' awareness of the importance of such abilities in mathematics. We think that in order to pursue this aim it is necessary to make students work on specific basic mathematical contents; a suitable choice of the mathematical contents allows intervening also on prerequisite knowledge for tertiary mathematics.

Population.

Two parallel teaching sequences were provided in two different cities in Tuscany (Pisa and Lucca) open to at most seventy students (35 for each course). Seventy volunteer students from 9 different secondary schools attended the two teaching sequences. The two groups were formed only on the basis of students' geographical provenience. The resulting groups were made of both high and low achievers in mathematics.

Methodology.

The main ideas for the design of the course were based on the previous experience of the research team which had been involved in organizing bridging courses in mathematics for the whole Faculty of Science since 2003.

PORTA courses consisted of seven 3-hours long meetings (held once a week, in extra-school time).

Mathematical topics were selected out of the common prerequisites of first year mathematics courses of the Faculty of Science in Italy: numerical sets, elementary algebra, Cartesian plane and trigonometry.

The table below depicts the contents and abilities dealt with in the meetings: bold-face is used to point out the main focus of the meeting.

	Contents	Abilities
1	**Numerical sets**	Defining, proving, communicating, problem-solving skills
2	**Equations and inequalities**	Defining, proving, communicating, problem-solving skills
3	Trigonometry	**Attending a frontal lesson, taking notes**

4	Trigonometry	**Systematizing own notes, studying a written text**
5	Cartesian plane	**Changing representations and frames**
6	Arithmetic, functions	**Defining**, proving, communicating
7	Arithmetic	Defining, **proving**, communicating

Table 1: Contents and abilities dealt with each meetings.

In the third meeting a frontal lesson (using traditional tertiary teaching style) about basic notions of trigonometry was given; students were asked to take notes.

Each of the other six meetings was structured into three phases:

Students worked in small groups for accomplishing given tasks. The tasks were designed to make students challenge and question their knowledge. Little or no room was given to routine exercises, tasks included open-ended problems, requests of commenting on or producing definitions and argumentations, requests of communicating.

Classroom discussion on the work carried out in the previous phase. The tutor had the delicate role to foster sharing and discussion, drawing on the tasks accomplished in the previous phase. The participation of all the students was important, it helped to develop communication skills and tolerance; and the tutor posed specific attention to that trying to maintain a relaxed atmosphere and a non-judgmental, non-prescriptive style. As far as possible all the contributions were valued, in particular incorrect answers or inadequate ways of thinking were not ignored.

Synthesis. Finally the tutor summed up the discussion trying to synthesize and make clear the main aspects emerged concerning the mathematical topics addressed. Also issues concerning transversal competencies were highlighted.

In this methodology the choice of the tasks assigned to the students has a crucial importance: the actual outcome of the first phase sustains and structures the enactment of the subsequent phases. We report, in table 2, the first activity of the first meeting: it had a great relevance to introduce the students to the course methodology.

Read the list of mathematical terms below. Do you know the meaning of some of these terms? Try to explain the meaning of the terms you know. i) *natural numbers* ii) *integer numbers* iii) *rational numbers* iv) *irrational numbers* (v) *real numbers*

Table 2: Meeting 1 – Activity 1.

The task involve some concepts (as for example natural numbers) that all secondary students have seen many times in their school life; moreover, the request is purposefully vague (a mathematical definition of the terms is not requested), the question is not aimed at getting a *correct* answer and it makes these activities different from those that characterise school practice. The authors' experience in both bridging courses and PORTA course confirms that this task work very well: the weakest students felt that they could say something and participated in the classroom discussion. Vice versa in the discussion some aspects always emerged that surprised also the high-achievers: for example the difficulty to avoid circularity. In particular the students can realise that the single answer is less important than the consistency of the whole set of answers: this seems to be crucial in that it favours a relational view of mathematics.

Results.

It is always difficult to evaluate the effectiveness of bridging courses (Wood, 2001). The evaluation of the PORTA corse relies on direct observations of students' work and involvement in the proposed activities, and on the answers to two purposefully designed questionnaires administered to students (at the end of the course and four months later).

The activities carried out by the students in the different meetings as well as the evolution of their interventions over time (after initial embarrassment, all students continually intervened with observations and well-posed questions), support authors' impression that, during the course, the path toward the development of students' autonomy seems to have been undertaken.

This positive feelings is also supported by the students' answers to the questionnaire: "*I will remember the need to explore the reasons of any odd thing in my next studies*", "*I suggest this experience to other students because it can, in some way, it can be of help to overcome some secondary prejudices that describe mathematics as 'a subject that teaches how to make calculation'*".

New interesting impressions emerged by the answers to the second questionnaire, especially in the comparison between meetings of the PORTA course and regular lessons in the classroom. The major differences that students highlighted concern methodology and aims: "[PORTA course and classroom lessons] *are different because in PORTA we worked in groups, we faced real justification of the concepts taught and above all because there was always a debate and a continuous exchange of ideas*", "[In the PORTA course] *we explained what we knew and we learned from the other students*" "[in the PORTA course] *concepts were not explained, rather I tried to understand them by myself*".

CONCLUSIONS

Linking research and practice, through the development of teaching interventions grounded in research findings, is obviously one of the main challenges for research concerning students' difficulties at the tertiary level; the course described in this report was an attempt to face such a challenge.

Besides authors' positive impression and students' positive reactions, the methodology was also appreciated by many of the secondary school teachers amongst whom the material, the methodology and the results of the course were disseminated.

Some of these teachers began to experiment the methodology in their classroom: they autonomously selected a segment of the curriculum, designed the tasks, carried out few lessons following the described methodology, and collected students' reactions through questionnaires. Even in this case, both teacher's feelings and students' reactions were positive (with few exceptions).

Although these results seem to encourage the adoption of the described methodology in regular lessons, one has to be aware that the presence of a formal assessment of students' performances might hinder the functioning of the methodology. The absence of any assessment was an important condition for students to feel free of participating in collective discussions, as clearly emerges from the answers to the questionnaires we collected: "*I was not anxious of understanding in view of the test*", "*Because there was not a real teacher to assess pupils*".

Moreover, the dissemination of these findings to secondary teachers is difficult, partly due to the conditions that change when activities are carried out in the classroom and partly because any implementation of a design requires many decisions that go beyond the design itself "*This occurs because no design can specify all the details, and because the actions of participants in the implementation […] require constant decisions about how to proceed at every level*" (Collins et al., 2004; p.17).

In any case, our early intervention at secondary level through PORTA project, allowed us to cooperate with secondary teachers sharing material and ideas as well as going toward the fundamental direction marked by Wood (2001, p.98): "*it is time to stop reacting and being more proactive in the transition to mathematics curriculum at tertiary level*".

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FEEDBACK IN DIFFERENT MATHEMATICS TASKS

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The present article aims to show a research carried out in the context of Project AREA[i]*, on the influence that written feedback has in pupil's learning, when given to diversified tasks in the type and in the method of work. The design research was the case study, the participants in the study were four pupils from the 8th year (aged thirteen). The feedback given to tasks of problematic, exploratory or investigative nature seems to be more efficient than in tasks which appeal to the strict knowledge of mathematical concepts. The feedback given to tasks solved in small groups seems to enhance learning in a more significant way when compared to tasks solved individually.*

INTRODUCTION

At this time a new program of Mathematics for the Elementary School (DGIDC, 2007) is going to start in Portugal. This curricular document comes to reinforce the cross capacities to all the mathematics learning – the problem solving, the mathematical reasoning and the mathematical communication – focusing on the diversification of learning experiences that are offered to pupils over their work in Mathematics, as through tasks of different nature, whether in the form of work between students.

With this research, we intend to study how the written feedbacks, given to pupil's written productions, contribute to their learning process, specifically, taking into account the experiences of learning offered. To do so, we state the following issues:

- What kind of tasks allows feedback as a way to regulate learning?
- Which work methods allow really significant learning, given the teacher's feedback?

THEORETICAL ARGUMENTATION

Learning results essentially from the activities developed by the pupils and the reflection they do on them (Bishop & Goffree, 1986). To answer to the new standards for mathematics learning (NCTM, 2000) high-level tasks are necessary (Stein, Remillard & Smith, 2007). To achieve learning, teachers must develop processes in order to support students' engagement, in other words, they must attach special importance to the formative assessment. A new approach to learning and its objectives have led to give particular importance to the assessment that aims and contributes for learning. The concept of formative assessment integrated definitely, by legal imposition, the lexicon of the teachers (DGEBS, 1991; Morgan, 2008; DGIDC, 2007). It seems obvious that it is necessary to diversify the type and the

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form of assessment instruments, but such is not sufficient for the change of attitudes that a process of assessment for learning imposes (APM, 1995; NCTM, 1995; DGIDC, 2007). Good communication between teachers and pupils is a necessary condition so that the assessment is effectively conducive to learning (NCTM, 1995; 2000; Wiliam, 2007). Therefore, the feedback that teachers give to pupils' productions is a central concept in the formative assessment (Black & Wiliam, 1998; Sadler, 1989), since it allows to students to self-correct their mistakes, improving learning, and to teacher to know better about the pupils difficulties, adapting their practices (Shepard, 2000).

Feedback must happen in the development of the process and in specific aspects of the task (Black & Wiliam, 1998; Schunk & Swartz, 1993; Tunstall & Gipps, 1996), assuming a descriptive and open to dialogue nature (Jorro, 2000; Gipps, 1999). If the feedback focuses in the process rather than the final result and it is complemented with information on the pupils' progress regarding the learning objectives, not only the performance of the pupils but also their beliefs on their competences will improve (Black, Harrison, Lee, Marshall & Wiliam, 2003; Pinto & Santos, 2006). A feedback that differs from the task and focuses on the student tends to have more negative effects, since it approaches aspects that can affect the self-esteem of students (Cameron & Pierce, 1994). The feedback must gives clear guidelines on how the students can improve their production, stimulating reanalysis of the given answers and recognizing what is already well done (Santos, 2003, 2008).

METHODOLOGY

In this research a qualitative/interpretative methodology was followed, using the case study as design research (Bogdan & Biklen, 1994; Merriam, 1991). Four pupils from the 8th year were selected (aged thirteen) whom had already been pupils of the teacher/researcher in the previous academic year. The pupils should show different levels of performance in mathematics, though registering some type of evolution at the level of their commitment and performance in mathematics. Albertina was a student with a low achievement in mathematics and not always committed in overcome her difficulties. Tiago was a student with a low achievement in mathematics, but committed in overcome his difficulties. Manuel was a student with a medium achievement, but just committed with the school tasks. Ricardo was a student with a high achievement in mathematics and very committed.

To gathered data we used the observation, the interview and the analysis of documents, where a log book carried out by the teacher/researcher was included. During the classes in which the pupils developed activities in the context of this research, the teacher was a participant observer. The teacher prepared the tasks to be proposed to the pupils and wrote the feedback to their productions. During the research we carried out three interviews to each one of the pupils, not only with the purpose of gathering their opinions and perspectives about mathematics, mathematics as a school subject and assessment, but also to gather information that might not be observable during the lessons, namely how the pupils understand the feedback and the reasons that support their options at the time of the second phase. The observations and the interviews were audio recorded and totally transcript. The first and second students' written productions concerning each one of the tasks proposed, as well as the feedback given to the first phases were gathered as documentation.

Data were analyzed considering each type of task (bibliographical inquiry, tests, investigation task and problems) and the way they were worked by the students (individual or small group). In each one, we analyzed the evolution of the pupils' performance from the first phase to the second of each task and of the first task to the second one of each type.

In the context of this research, the eight tasks that were prepared gave origin to twenty productions in two phases (with the exception of the second bibliographical inquiry that was carried out in an unique phase): eight tests, eight problems, two bibliographical inquiries and two investigation tasks, which meets the recommended by NCTM (1995) on the diversity of the type of tasks which must be given feedback.

The type of assignments and the method of work are set out in chart 1:

Tasks	Work method
Bibliographical inquiry on Pitágoras	Group
Test in two phases	Individual
Investigation task: study of the influence of k and b in the graphic representation of functions of the type y=kx+b	Group
Problem solving	Individual
Bibliographical inquiry on Diofanto	Group
Test in two phases	Individual
Investigation task: divisions by 11	Group
Problem solving	Individual

Chart 1: Types of tasks and methods of work

In all the tasks the pupils had a first moment in classroom to begin its development. After this time, the productions were analyzed and commented. The productions were then delivered to the pupils so that they continued them (Santos, 2008). The second phases of all the tasks were always developed in the next lesson after the end of the first phase, with the exception of the second test, whose second phase was developed a week later. In general, feedback with incidence in the task in analysis was supplied (Black & Wiliam, 1998), in order that each pupil improved in each one of the tasks. Therefore, we tried to use a descriptive feedback (Tunstall & Gipps, 1996) as dialogue (Jorro, 2000) between the teacher and students, trying to lead them to reflect about each aspect of each task.

DATA ANALYSIS

Bibliographical inquiries.

The bibliographical inquiries constituted, along with the tests, the most closed tasks given because the scripts supplied to the pupils were very directive.

The feedback provided to the bibliographical inquiries consisted on short comments. It was mentioned the unfinished of some aspects, the existence of not too explicit information, the need of developing and of clarifying some information. The decisions taken by the pupils were negotiated between them, sharing the meanings that each one was giving to the written comments by the teacher, so that they could define how to proceed:

Tiago: What do we have?

Ricardo: So, are we going to take the Pythagoras Theorem?

Tiago: Do we take it away?

Manuel: Why?

Ricardo: I am asking.

Tiago: What the teacher wanted to tell is that in order to have here the Pythagoras Theorem we should be putting it. Here like work, as we did with the tree. Do you get it?

When the feedback was not sufficient in order that the pupils improve any aspects, they developed strategies to reach this objective, resorting, for example, to the search of different sources of information.

When they developed the second bibliographical inquiry, the pupils used, by themselves, both phases of the first bibliographical inquiry that they had developed. They shown to have learnt to structure a work of this type, what each part had and which type of information they should gather. Specifically, they improved the conclusion of the work, developing a small summary of what they had done. From the first to the second bibliographical inquiry only some orthographic mistakes persisted. The quality of the second bibliographical inquiry was such that there was not a second phase. Thus, we think that there was a quite positive evolution between the first and the second moment of producing a bibliographical inquiry.

Tests.

The given feedback to the pupils productions were as short comments, clues were given on mathematical contents which the students could use, it was asked in order that they improved justifications and aspects of the tests were set out – this explanation could assume the interrogative form with straight or indirect questions, or the affirmative form. This feedback worked like a progress of the oral questions done in the lessons in which the students improved routine tasks. It was the “push” that they need to overthrown some trouble felt in this resolution and that was preventing from showing other knowledge. All the students improved their productions from the first to the second phase. Nevertheless, there was not an evolution on the performance of the students from the first to the second test. For example, the fact that the teacher has given a feedback in the first test with the intention of setting out information on the test, so that the student managed to correct his/her answer, it was not enough for the student to understand the whole test in the second test. The two following examples illustrate feedback given to the two tests from the same student. In both, the teacher makes reference to the context of the statement, explaining it:

1 What can you do if you want to call the police, for example?

2 How can you write the number of matches, using the number of the picture?

It was as if everything started again.

Investigation tasks.

The supplied feedback to the investigation tasks assumed nearly always the form of short comments, though in some cases several instructions were given in the same text. In the comments, was indicated the incompleteness of some aspects and questions placed near incorrect aspects. Showing what was already well done, using examples of the pupils to question on aspects of their answer, a few times on a more straight form, other times less, and asked for a deeply explanation of some aspects.

In the first task, the students faced the multiple questions as feedback like being to answer. They did it in the straight questions, but not in the questions that had as objective to lead them to re-analyze their answer to detect mistakes. So, some aspects of the written report produced maintain wrong.

The most obvious improvement in the performance of the group from the first investigation task to the second one was the fact that the students created more examples to help them to reach a generalization. In the first report, when it was asked the students to replace k by five different values so that they would analyze its influence in the respective graphic representation of the function y=kx+b, the students did it. However, they did not try any more values besides the five presented, even with difficulties in understanding this influence. In the second task, in spite that in the text were given six fractions with denominator 11, the pupils wrote an orderly sequence of 20 fractions with denominator 11. Grouping them in accordance with criteria chosen by themselves, as illustrates the image 1.

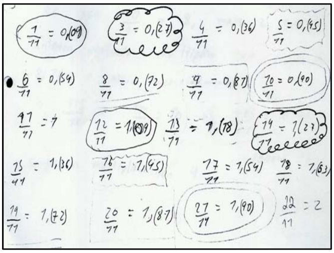

Image 1: Sample of the report

Problems.

The supplied feedback to the pupils productions were long comments, especially in the first problem. Was highlighted what was already well done, set out conditions that the pupils were not respecting, given clues on mathematical contents were needed to study. These were supplied in the affirmative or interrogative form, using pupils and teacher examples in the direction of setting information out. Students were encouraged to improve their written productions.

When the pupils solved the second problem, a very big evolution was noticed regarding the first problem. This evolution was well-known in the justifications of the reasoning, in the sequential presentation of the attempts done to solve the problem, as well as on the inclusion of some attempts, which came to reveal false ones, accompanied by the explanation for their abandon. The following image is illustrative in this respect:

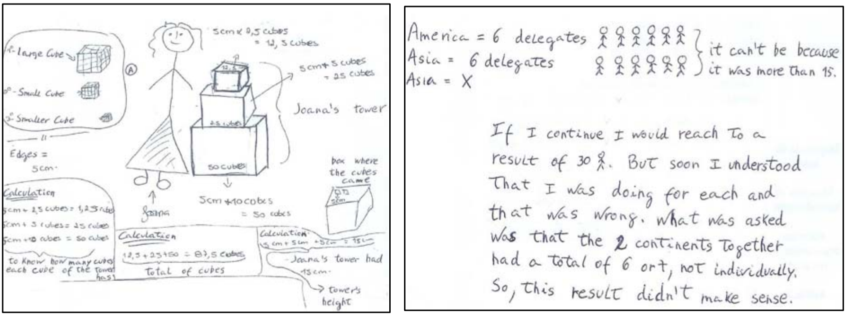

Image 2: Sample of the 2 problems solved by the same student

CONCLUSIONS

This study point out that the feedback that always tried to be descriptive (Gipps, 1999; Tunstall & Gipps, 1996) and with incidence in the task in analysis (Black & Wiliam, 1998) seems to be more efficient in evolutive terms when it is given to task that appeals to mathematics' capacities as problem solving, mathematical reasoning and communication. When the task appeals more to the mathematical knowledge, the feedback can be useful for the improvement of the same task. But in a subsequently task of the same type, if the pupils do not dominate the new contents, it is as if everything started again. Students revealed to learn to structure and to understand what was expected that they did in a bibliographical inquiry, started to organize their attempts and to increase the studied cases, to present strategies, even those who came to be revealed unsuccessful and improved the justifications of their strategies and reasoning. The tests, although all of them have improved from the first to the second phase, there are no known improvements of performance from the first one to the second test. The feedback supplied to the tests helped to realize that the aspects of the routine application of not dominated concepts by the pupils were surpassed momentarily, but not in the constant form (Veslin & Veslin, 1992).

The productions carried out in group and that subsequently to the first phase received feedback, were always improved in the second phase and in the second moment they always revealed important improvements in the group performance. The tasks carried out individually which received feedback, not always gave significant improvements in the next phase. Thus, the productions of individual resolution seem not to favor significant learning in comparison with the resolution tasks in group, given the feedback from the teacher. The possibility of pupils confront the interpretations that each one gives to a supplied feedback and to discuss between themselves its meanings seems to constitute a positive factor for the efficiency of the feedback.

Although the feedback given to different tasks provided learning at different levels, we believe that this diversity of tasks contributes to a more solid learning. Therefore, any type of task is to exclude from the learning process. It is necessary to continue to invest in the practice of giving feedback to all kind of students' productions in order to realize what kind of feedback is most appropriate for each type of task.

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PRIMARY STUDENTS' SPATIAL VISUALIZATION AND SPATIAL ORIENTATION: AN EVIDENCE BASE FOR INSTRUCTION

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This paper reports on the performance of 58 11- to 12-year-olds on a spatial visualization task and a spatial orientation task. The students completed these tasks and explained their thinking during individual interviews. The interview data were analysed to inform pedagogical content knowledge for spatial activities. The study revealed that "matching" or "matching and eliminating" were the typical strategies that students employed on these spatial tasks. However, errors in making associations between parts of the same or different shapes were noted. Students also experienced general difficulties with visual memory and language use to explain their thinking. The students' specific difficulties in spatial visualization related to obscured spatial elements, the perspective used, and the placement and orientation of shapes.

INTRODUCTION

In 2006, the (US) National Academies (National Academy of Sciences, National Academy of Engineering, Institute of Medicine, National Research Council) published a landmark report titled "Learning to Think Spatially" in which they proposed the importance of embedding spatial thinking in the contemporary curriculum. They described thinking spatially as knowing about *space*, *representation* and *reasoning.* Although the National Academies' report was directed towards achieving spatial literacy across the curriculum, it is particularly applicable in mathematics. For example, there is a relationship between adolescents' performance on spatial ability tasks and their performance in and preference for mathematics (Stavridou & Kakana, 2008). Thus, the purpose of this paper is to explore students' thinking on mathematics tasks that incorporate visual representations, in particular, those that place a heavy demand on spatial ability. Henceforth, the term "graphics" will be used to refer to visual representations because the term "representations" has multiple meanings in mathematics education.

INTERPRETING GRAPHICS

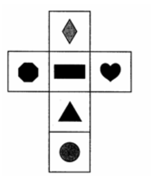
Routinely, in mathematics, students are required to interpret graphics (e.g., maps. number lines, graphs) as well as text and mathematical symbols. However, our previous research has revealed that for many students the interpretation of graphics is problematic rather than routine (e.g., Lowrie & Diezmann, 2007). The interpretation of graphics is problematic for students for at least three reasons.

First, the interpretation of graphics is complex and requires particular knowledge and skills. Specifically, it involves the interaction between a visual symbol system and perceptual and cognitive (i.e., conceptualisation) processes (Winn, 1994). The

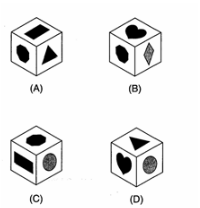
2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 417-424. Thessaloniki, Greece: PME.

symbol system is composed of *visual elements* (e.g., shapes) that represent objects or ideas and the *spatial relationships* among the elements within the graphic (e.g., one shape inside another). Mackinlay (1999) argues that there are six visual symbol systems comprised of particular combinations of perceptual elements and spatial relationships which he terms “graphical languages”. These are *Map Languages* (e.g., topographic map), *Axis Languages* (e.g., number line), *Opposed Position Languages* (e.g., bar chart), *Connection Languages* (e.g., network), *Miscellaneous Languages* (e.g., calendar), and *Retinal List Languages* (e.g., mental rotation task). This latter language presents mathematical information through a combination of perceptual elements (e.g., colour, shape, size, saturation, texture, orientation) and capitalizes on these retinal properties to encode this information (e.g., Figures 1 and 2).

This is the net of a cube.



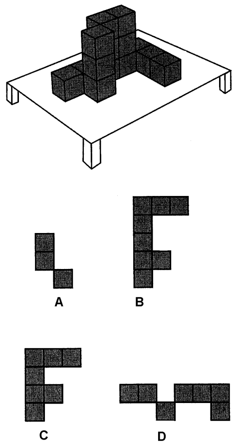
Which one of these cubes could be made by folding the net?



Educational Testing Centre, 2002, p. 9.

Figure 1: Net task (spatial visualization).

What does this model look like from above?



Queensland Studies Authority, 2002, p. 10.

Figure 2: Model task (spatial orientation).

The interpretation of graphics has dual perceptual and cognitive foci. Students need to make sense of various perceptual elements (e.g., shape, size, saturation, orientation, texture) and the spatial relationships among these elements. They also need to employ various spatial perception skills, such as eye-motor co-ordination; figure-ground perception; perceptual constancy; position-in-space perception; perception of spatial relationships; visual discrimination or visual memory (Del Grande, 1990). In tandem, students need to employ the appropriate cognitive processes for the particular graphical language. For example, in *Retinal List*

Languages, they might be required to employ particular spatial abilities such as spatial visualization (Figure 1) or spatial orientation (Figure 2). McGee (1979) argues that visualization and orientation are two distinct factors in spatial ability. Visualization and orientation are of particular importance in enabling the interpreter of a graphic to translate between different representations of the same object. In Figure 1, the translation is between the (2D) net of a cube and the drawn (3D) graphic of the cube. In Figure 2, the translation is between the (3D) model of a set of cubes and the (2D) bird's eye view of the model. Henceforth, these two tasks will be referred to as "cube tasks" because both tasks involve the interpretation of cubes.

Second, students experience particular difficulties in each of the graphical languages. For example, on structured *number line* items (Axis languages), students' difficulties included overlooking the relative position of an unnumbered mark to identify its numerical value (Diezmann & Lowrie, 2006). Whereas on a *map* (Map languages), students experienced difficulty identifying which landmarks they should use in the solution process (Diezmann & Lowrie, 2008). Thus, we anticipate that students will experience unique difficulties interpreting Retinal List languages because it is a distinct graphical language.

Third, there is scant guidance for teachers to support students' interpretation of graphics in mathematics. Thus, it should be worthwhile to explore students' interpretation of graphics in relation to five aspects of pedagogical content knowledge (PCK) proposed by Carpenter, Fenema and Franke (1996): (1) what tasks students can typically solve and how they solve them; (2) an understanding of individual students' thinking; (3) how students connect new ideas to existing ideas; (4) common errors made by students; and (5) what is difficult and what is easy for students.

METHOD

This investigation is part of a 3 year longitudinal study which sought to describe and monitor primary students' capacity to interpret information graphics in mathematical test items. The aims of this study were:

1. To describe students' knowledge and thinking about cube tasks;
2. To document the errors students made on cube tasks; and
3. To identify the difficulties that students experienced on cube tasks.

The Participants

The participants were 58 primary students aged 11 to 12 years drawn from two schools in moderate socio-economic areas. Fewer than 5% of students had English as a second language.

The Interviews

The interview tasks were a pair of Retinal List items (Figures 1 and 2) drawn from the 36-item Graphical Languages in Mathematics test (Diezmann & Lowrie, in press). This test comprises six sets of graphic items corresponding to each of the six graphical languages. These two items are similar in that they each included the interpretation of 3D cubes. The items are dissimilar in that the Net task required students to identify the correct net for a cube whereas the Model task requires students to identify the bird's eye view of a set of cubes. The students completed the two items during an individual interview and then explained their thinking. They also explained which of these tasks was more difficult for them. The analyses of data were guided by Carpenter et al.'s (1996) five aspects of PCK. It involved the thematic coding of students' responses and frequency counts.

RESULTS AND DISCUSSION

The results focus on three research questions. The first question addresses three facets of PCK (Carpenter et al., 1996): (1) which tasks students can typically solve and how they solve them, (2) students' thinking, and (3) connections students made between new ideas to existing ideas. The subsequent two questions focus on the other two aspects of PCK, namely (4) students' errors and (5) difficulties respectively.

1. What did students know about these cube tasks?

These tasks were not particularly difficult for Grade 6 students (N=58) with 76% and 66% of students successful on the Net and Model tasks respectively. Thus, the Net task was relatively easy for students and the Model task was of moderate difficulty. Higher results had been anticipated because these tasks were designed for students one to two years younger than this cohort.

Across the two tasks, successful students used a variety of strategies. However there was one predominant strategy for each task. On the Net task (Figure 1), 68% of successful students (n=30) used a *matching strategy*. Paul's response, in which he identified the same parts of the shape on two different graphics, was typical.

> I chose A (answer) because to make a cube, the one (shape) that's in the middle of the cross (net) is the one that's going to be on the top and A is the one on the top (*matching*).

Matching was also part of the typical strategy employed for the Model task (Figure 2) with 61% of successful students (n=23) using a *matching and eliminating strategy*. Heather's response of *matching* aspects of one representation to another and *eliminating* multiple choice answers was typical.

> First I had a look at the model and I had a quick look at the A B C and D and then I counted how many blocks were along that side and I saw that it was 3 and on this side it was 4 and one down so I had a look on here and I thought cause that one (Answer A) was too small, so 1 2 3 (counting cubes) and then I saw 1 2 3 4 (*matching*) and I got that so I thought it was probably B and then I just checked C and D and I didn't think it was D cause you would be seeing all the blocks and there's not a space there (*eliminating*), I can see it there and C was too short going this way so I thought it must be B.

The exploration of students' thinking revealed two unanticipated results. First, notable in the successful (and unsuccessful) students' responses was a difficulty using language to describe their thinking. There were considerable pauses and reference to vague language such as “it”, “that” and “there” as in Megan’s response:

> **It** folds down ... **that** would fold down to **there** and **that** would be on top like **that** and then **it** would be like **that** (emphases added).

Second, only one student made a link between one of the tasks and prior knowledge. Colin’s comment provides evidence of a connection between the Net task and a previous task in an earlier year albeit using concrete materials.

> Well we did this in Grade 5 folding the net of a cube and so and we did colour it in before so I learnt a bit about shapes and possible configurations.

The paucity of student explanations linking the tasks to prior knowledge is surprising given that a constructivist philosophy underpins the mathematics syllabus in this state and building on prior knowledge is a central tenet of constructivism. However, there are three plausible reasons why relevant prior experience might not have been described by the students. The students might have had no previous experience with similar tasks; they might have had previous experience with similar tasks but did not think to refer to these experiences in their explanations; or they may have failed to make a connection between prior knowledge and these tasks.

2. What errors do students make on the cube tasks?

Students made four types of errors across these two tasks. On the Net task (Figure 1), the dominant error was *incorrect association* with 93% of the 14 unsuccessful students using this strategy. This code was assigned when students made an incorrect association between two parts of the same shape or between one part of a shape and the corresponding part on its alternative graphic representation. For example, Sue made the correct assumption that the heart <u>could</u> be on the top but then made an incorrect spatial association between the location of the hexagon and the heart.

> I picked B and the love heart could be on the top (correct) and then the hexagon would be on the side (incorrect it would be on the bottom) so that means that the diamond would be on the other side.

Similar to the Net task, *incorrect association* was also the dominant error on the Model task (Figure 2). Seventy percent of the 20 unsuccessful students made this error. The second most frequent error on this task was *incorrect elimination* which was made by 15% of unsuccessful students. Paul’s response was typical of an incorrect elimination error:

> I chose C – it couldn’t be that one (A) because there’s more (cubes). I can see three blocks there and I can see another block there (pointing to the model). It couldn’t be that one (B) because three down and two (incorrect elimination) there so I chose that one.

Across the two tasks, one to two students also made errors because they assumed the diagram *looked correct* (without checking) or because they *misread the graphic*.

3. What difficulties do students report experiencing on the cube tasks?

After the students had completed the two tasks, they were asked to identify which task was harder for them and why. The results indicate that approximately 10% more students identified the Net task as more difficult than the Model task (53% : 41%) (Table 1). However, students' perceptions do not mirror their performance because approximately 10% more students were successful on the Net task than the Model task (76% : 66%).

Net task was harder	Model task was harder	Both tasks were similar	Students were asked the question
53% (n=31)	41% (n=24)	2% (n=1)	3% (n=2)

Table 1: Relative difficulty of the tasks.

A thematic analysis of students' explanations of why particular tasks were difficult highlights the complexity of graphic interpretation and the specificity of difficulties with particular tasks. On the Net task, in which students were working from a 2D graphic to a drawn 3D graphic (Figure 1), students reported four difficulties: a lack of prior experiences; limited visual memory; difficulty imagining an obscured view; and difficulty imagining the placement and orientation of shapes when a net was folded (Table 2). On the Model task, in which students were working from a 3D graphic to a 2D graphic (Figure 2), students reported two difficulties: imagining an obscured view and a bird's eye view (Table 3).

Lack of prior concrete experiences: "You have to have an (concrete) example or something because sometimes you're not sure" (Bridget)
Visual memory: "Hard to imagine them being folded and forgot which way each one went" (Rachel)
Imagining an obscured view: "Didn't see the whole cube, you could only see three sides" (Molly)
Imagining the placement and orientation of shapes on a folded net: "Hard to work out which shapes were next to each other" (Alan) (placement); "You had to work out which way to fold them and whether you could turn them around" (Ned) (orientation)

Table 2: Type of difficulties and examples for the Net task.

Imagining an obscured view: "Because you can't get the exact photo because you've got like blocks there and then you can't see the blocks behind and you've got to sort of guess like those blocks or how many blocks there are." (Isobel)
Imagining a bird's eye view: "It was just hard imagining what it would look like from above." (Colin)

Table 3: Type of difficulties and examples for the Model task.

The students' difficulties across both tasks highlight the importance of concrete experiences and strong visual perception skills particularly visual memory. Students' difficulties with various aspects of imagining (obscured view, placement, orientation, bird's eye view) suggest the importance of both spatial visualization and spatial orientation in these types of tasks. On the Model task, students' difficulty imagining what blocks were hidden might have been exacerbated by the perspectives shown (See Parzysz, 1991 for a discussion of optimal perspectives of shapes). On this task not only did students need to translate from a 3D graphic representation to a 2D graphic representation but they also had to coordinate an orthogonal projection (model) with oblique projections (answers) (Figure 2).

CONCLUSION

The importance of spatial skills in our technological world is increasing with new devices becoming commonplace (e.g., Global Positioning Systems [GPS], new virtual worlds to traverse (e.g., Google earth), and new careers that rely heavily on spatial abilities (e.g., deep sea imaging). Hence, spatial literacy is indisputably a fundamental literacy in the 21st century. Our investigation of students' performance on spatial visualization and spatial orientation tasks indicates six ways that educators can foster students' spatial abilities and work towards spatial literacy for all students. First, ensure spatial skill development and a variety of spatial activities are embedded in the mathematics curriculum. Second, support students to develop their spatial vocabulary and provide opportunities for them to use this language. Third, foster the development of students' visual memory and spatial abilities with particular attention to the visualization of obscured views, the placement and orientation of shapes, and different viewpoints. Fourth, provide concrete examples of tasks prior to expecting students to visualize tasks and encouraging them to make links to these previous experiences. Fifth, follow up on students' difficulties and errors and provide practice tasks on each of the sub components of complex graphics. Finally, capitalize on 21st century technologies to provide opportunities to develop spatial literacy. For example, 3D games that include virtual avatars provide multiple opportunities for students to learn about orientation in an informal environment (Amorim, 2003).

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A TEACHER'S IMPLEMENTATION OF EXAMPLES IN SOLVING NUMBER PROBLEMS

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In this paper, we report a study of a teacher's choice and use of examples in a Year nine fraction teaching experiment. This study is part of a larger study of mathematics classrooms in New Zealand, nested within the national component of the Learner's Perspective Study. During this teaching experiment, the teacher applied teaching strategies and examples advocated in the New Zealand Secondary Numeracy Project (SDP). *Our findings indicate that for instructional examples to be more effective, teachers need more assistance in understanding the relationship between an instructional model and the development of thinking patterns in mathematics.*

RESEARCH CONTEXT

How teachers choose and use examples can make a great difference in developing students' thinking in mathematics. The study by Zaslavsky, Harel, and Manaster (2006) indicated that the effectiveness of a teacher's treatment of examples seemed to be influenced by her mathematical content knowledge (MCK) and pedagogical content knowledge (PCK). Moreover, Chick (2007) found that although professional development may enhance a teacher's PCK and him or her selection of examples, the teacher's implementation of an example may not always reflect the potential affordances identified by the professional development designers. To gain a full appreciation of what this means, we examine a teacher's implementation of instructional examples in a teaching experiment. During this teaching experiment, the teacher applied teaching strategies and implemented examples from a model of teaching advocated in the New Zealand *Secondary Numeracy Project* (SNP) (details are available at http://www.nzmaths.co.nz/Numeracy/SNP/SNP.aspx). Our study follows the direction proposed by Zodik and Zaslavsky (2008) and explores the question, "how [could] the findings of teachers' example-related practice…be used for teacher education programs?" (p. 181). Enhancing our understanding of what happens in the classroom will enable us to enrich professional development opportunities that support teachers' capacity to improve student achievement.

THEORETICAL FRAMEWORK

Since 2000 there has been a systematic implementation of a national *Numeracy Development Project* (NDP) in New Zealand. The central aim of the professional development programme is the development of primary school teachers' "pedagogical and content knowledge … to enable them to meet the learning needs of all their students" (Ministry of Education (MoE), 2006, p. 2). A key focus in the

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 425-432. Thessaloniki, Greece: PME.

classroom is the development of students' flexible strategies for solving numerical problems.

More recently the project has been extended into the secondary school with the objective of building a strong foundation for algebraic thinking (Irwin & Britt, 2005). With regards to the long-term goal of improvement in algebra performance the model advocates the development of students' numerical competency and understanding and generalisation of number properties as a basis for algebraic thinking.

As part of the NDP, the SNP shares many features with the other parts of the project. Most notably is a model for teaching strategic thinking in Number. The core feature of this teaching model is a recursive pattern that illustrates a dynamic relationship between the phases of 'Using Materials', 'Using Imaging', and 'Using Number Properties' (see Figure 1).

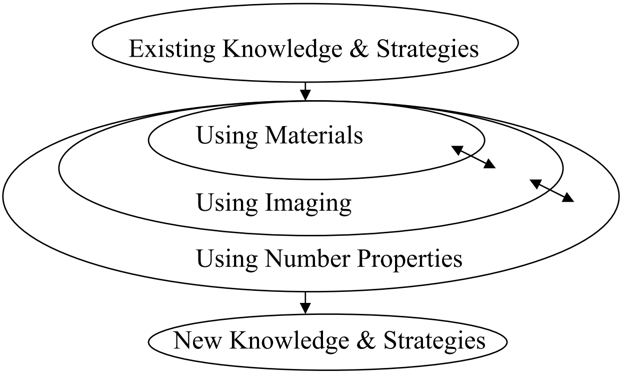

Figure 1. The Strategy Teaching Model (MoE, 2006, p. 5)

Moving through the phases illustrated in Figure 1 demonstrates greater degrees of abstraction in a student's mathematical thinking. In this model, the abstraction means "Using Number Properties when applied to number" (MoE, 2006, p. 7). Thus, in the context of the SNP professional development programme, learning experiences include transitions through physical representations and imaging towards abstract mathematical concepts and algebraic thinking.

In this paper we are interested in seeing how a secondary teacher enacted instructional examples that focused on using material in relation to the final goal of abstraction towards number properties of fractions. In examining classroom specific episodes, we were cognisant of mathematicians' view of examples—it is more important to understand what is done with the examples, how they are probed, generalised, and perceived by students and the teacher than to merely select the examples (Bills et al., 2006). Such understanding will help us identify more clearly links between teacher's instructional strategies and students' thinking in mathematics.

THE STUDY

The data come from a study of a Year nine (Grade 8) fraction classroom within a large urban all-girls school. The class of 13-14 year-old students was a top group of a four class cohort. This study is part of a larger study of three mathematics classrooms from three different schools in New Zealand, which represent the national component of the Learner's Perspective Study (LPS). The teacher Amy (a pseudonym), identified as an effective teacher in the larger study, had over 12 years teaching experience in school mathematics. In the year prior to the study Amy had significant involvement in the SNP professional development programme.

The observation data were collected by three cameras. The first camera focused on the teacher, the second one on the focus students, and the third one on the whole class. We then used video-stimulated recall interview techniques to gain the teacher's and students' interview data. Triangulation of the video and interview data was also enhanced by reference to two researchers' classroom observation notes, photocopies of written work by the focus students, textbook pages, worksheets and teacher questionnaire data. Ethical issues of our research were addressed by strictly following the ethical procedures of the LPS project.

The study focused on understanding the relationship of Amy's implementation of examples and the enactment of her MCK and PCK learned from the NDP teaching model (see figure 1). The data were analysed according to the analytic strategy of case study, "developing case descriptions," suggested by Yin (2003, p. 109). In our analysis we first looked at the teacher's implementation of the kind of 'representations-specific' examples as identified by Rowland's (2008) study of instructional examples—specifically, examples represented by materials and examples represented by numbers. We described three teaching episodes from the first lesson of a sequence of 10 lessons to show in detail how the teacher implemented examples involving the fractional fringes (FF) model—called 'Fraction Strips' in the NDP activities (see Material Master 7-7 at http://www.nzmaths.co.nz/Numeracy/materialmasters.aspx). In addition, to interpret Amy's instructional intention and her students' understanding related to these teaching episodes, relevant teacher's and student interview data are included. Linking the observed data to the teacher's and students' interview data provides a deeper insight into the nexus of teaching events and student learning.

FINDINGS AND DISCUSSION

In this section, we provide teaching episodes to illustrate Amy's implementation of examples in one lesson. Our analysis is intended to highlight the potential affordances and limitations of these instructional examples for developing students' mathematical thinking and understanding. In commenting on the teacher's implementation of examples in these specific teaching episodes we offer possible alternative ways of implementing these examples.

Using materials as the starting point

The teacher's intention in the first lesson (L1) was to review and consolidate students' understanding of equivalent fractions. She started the lesson by providing students with a step-by-step activity for making two Fraction Fringes (FFs). Each FF was based on a set of fractions: 1, 1/2, 1/4, 1/8, 1/16 and 1, 1/3, 1/6, 1/12, 1/24 (see Figure 2 for illustration of part of the FF model the teacher modelled in the class.). In the first activity of making FF of 1, 1/2, 1/4, 1/8, 1/16, the teacher provided students a set of five sheets of coloured paper. In making the FF students were required to cut the various layers of paper to represent successive fraction divisions of the unit 1. The teacher suggested that students choose their own way to determine the successive lengths of 1/2, 1/4, 1/8, 1/16. Some students used a measuring strategy to cut the layers of paper, while others used a folding strategy. On completion of this activity (about 20 minutes), the teacher led a whole class discussion drawing students' attention to the FF pattern (see Figure 2(a)) as follows:

1/4	1/4	1/4	1/4
1/2		1/2	
1			

(a)

1/6	1/6	1/6	1/6	1/6	1/6
1/3		1/3		1/3	
1					

(b)

Figure 2. Making fractional fringes

T (Amy) We started with a whole. OK, and we wanted to halve that and the fraction that we got was a $^1/_2$. Then what we wanted to do was we wanted to work out a $^1/_2$ of $^1/_2$ and what did that turn out to be?

Ss (some students) $^1/_4$.

T $^1/_4$, then, we wanted a $^1/_2$ of $^1/_4$ and what did that turn out to be?

Ss $^1/_8$.

T Can we see a pattern? What's happening? Vicky?

Vicky The number at the bottom is being doubled.

The teacher then drew students' attention to the same doubling relationship of denominators of $^1/_8$ and $^1/_{16}$ and suggested that $^1/_{32}$ would be the next number if they continued halving the layers. Students then went on their FF making activity. Some of them started to make another FF (see part of the model in Figure 2(b)). During this whole class activity the teacher walked around the class offering students individual assistance. The interaction of one student with the teacher is described as follows:

S (a student) What is after 3rds?

T I don't know. What is after 3rds? What was the pattern that we saw happening with 1? (The teacher guided the student to observe the pattern of the set of numbers 1, ½, Ό, $^1/_8$, $^1/_{16}$ they just discussed on the board.) What was happening to the number on the bottom?

S Doubling.

Here, the teacher again guided the student to observe the doubling relationship of the denominators of 1/3 and 1/6.

As has been discussed (see Figure 1), a feature of the strategy teaching model within the NDP is the use of materials (called 'equipment' in Higgins, 2005) as the starting point for teaching students new mathematical idea. Students are encouraged to first describe the features and functions of the equipment, and then to progress to imaging, and then to number properties in their representations of number ideas (Higgins, 2005). Influenced by the NDP teaching model, Amy started the lesson by involving students in the FF making activities (in total about 35 minutes of this lesson). She then guided students to observe the FF patterns for making connections between the ideas of halving and doubling. However, there was no more discussion about other significant features potentially afforded by the FF model, particularly in relation to an understanding of equivalent fractions. For instance, as urged by Irwin and Britt (2005), students could have been engaged in "ongoing algebraic thinking via quasi-variable thinking" (p.183). That is, the teacher could have led students to understand, by the pattern of the FF model, that

$$1={}^{1}/_{1}={}^{2}/_{2}={}^{4}/_{4}={}^{8}/_{8}={}^{16}/_{16,}\ 1={}^{1}/_{1}={}^{3}/_{3}={}^{6}/_{6}={}^{12}/_{12}={}^{24}/_{24}$$

and, the rich algebraic nature of arithmetic represented by the FF model such as:

$$\tfrac{1}{2}={}^{2}/_{4}={}^{4}/_{8}={}^{8}/_{16,}\,\acute{}\mathrm{O}={}^{2}/_{8}={}^{4}/_{16},\ {}^{2}/_{4}={}^{4}/_{8}={}^{8}/_{16},\ \acute{}\mathrm{Y}={}^{6}/_{8}={}^{12}/_{16,}\ \ldots$$

$${}^{1}/_{3}={}^{2}/_{6}={}^{4}/_{12}={}^{8}/_{24,}\ {}^{2}/_{3}={}^{4}/_{6}={}^{8}/_{12}={}^{16}/_{24,}\ {}^{1}/_{6}={}^{2}/_{12}={}^{4}/_{24},\ {}^{2}/_{6}={}^{4}/_{12}={}^{8}/_{24},\ {}^{3}/_{6}={}^{6}/_{12}={}^{12}/_{24,}\ \ldots$$

Such an implementation of the FF model required the teacher to vary the examples afforded by the FF model, from visualising the physical pattern of the material examples to visualising the mathematical structure of the number examples. With explicit attention to the numerical patterns, students would be given the opportunities to develop other ways of thinking such as analogy, induction and generalisation for understanding the law of equivalent fractions. Moreover, given that the class was a high achieving group, this expectation to experience more sophisticated thinking strategies is not unreasonable.

Between 'Using materials' and 'Using Number Properties'

After completing the activity of making FFs, the teacher continued the lesson by encouraging students to use the FF model to solve the number problem (${}^{2}/_{3}={}^{?}/_{6}$) as follows:

T	… How could you use your fractional fringes to solve some problems? … ${}^{2}/_{3}$ is the same as something over 6. … How would you use your fractional fringe to do that? Kate, what would you do?
Kate	You would find ${}^{2}/_{3}$ on your little fringe…
T	OK, so how many would that be?
Kate	1, oh, 4.
T	OK. So ${}^{2}/_{3}$ is the same as ${}^{4}/_{6}$. Everyone agree? (Students all agreed in the class.) So the fractional fringe is actually quite a good way to work that question out.

This teaching episode may reflect Amy's understandings of the NDP strategy teaching model. In this case it appeared that 'Using materials' was intended to provide students an alternative strategy for thinking and solving the number problem. However, rather than moving students' mathematical thinking from 'Using materials' to 'Using Number Properties', the teacher acknowledged that for some students the activity was moving their thinking from Number Properties back to Materials, or simply providing a choice. This was confirmed by the teacher's interview data after a later lesson ('adding and subtracting fraction', L6):

I (Interviewer) So were you expecting them to use their fractional fringes? How helpful were they?

T I think some of them used them but most of them knew what they were doing already, so basically I thought they could have the option of using them as they choose.

T Oh, here, when I'm walking around, some of them couldn't actually do it by hand, and so that's why I was encouraging them to use the fractional fringe to start with.

In contrast to Amy's assumption, some students' post-lesson interview data indicated that the teacher's implementation of the FF model neither challenged their curiosity nor helped their intellectual development:

I What are the main things that you learnt in the lesson?

Tara That you can use a fraction fringe for fractions, I didn't think you could use it for that, like I wasn't aware or anything, it really gave me an idea that this easier than having to work it out in your head. (L1)

Skye I still don't like the fractional fringes, I find it easier to do things in my head rather than on a piece of paper. (L2)

Arguably, the teacher's implementation of the examples based on the FF model provided limited opportunities for students to develop full understanding of equivalent fractions. The consequences of such fragile understanding were evident in one student's interview following the addition and subtraction lesson (L6):

I When you first started (adding fractions), you were using fractions fringes.

June Yeah, and I talked with Leigh and she taught me an easier way. I had seen equivalent fractions before, but I didn't think of using it for adding.

Junita Yeah, I learnt something there ($^3/_4+^1/_8=$), like how it goes times 2, I thought you had to times both of them (denominators 4 and 8), but you only have to do one (denominator 4).

June I was thinking, like times both of them to make it more equivalent, but you only had to do one of them.

I So you might have thought of making it into 32twoths to start with?

June Yeah.

I So why do you think you only needed to do the one?

June I'm not sure. Just to make it so that 4 is the same as 8.

As has been pointed out by the NDP, an important goal of the teaching of Number is to encourage mathematical thinking that will assist in the development of later algebraic thinking. To this end, the NDP encourages students to explore a variety of mental strategies to solve arithmetic problems. For instance, Irwin and Britt (2005) suggest that operations like factorising which are usually taught in algebra class can be applied to numerical problems. Applying this principle in the context of reviewing equivalent fractions, the teacher in question could have led students to observe the arithmetical relations of the two sets of fractions as follows:

$$1={}^{1}/_{1}={}^{2}/_{2}={}^{4}/_{4}={}^{8}/_{8}={}^{16}/_{16} \rightarrow 1={}^{1}/_{1}={}^{1*2}/_{1*2}={}^{1*2*2}/_{1*2*2}={}^{1*2*2*2}/_{1*2*2*2}={}^{1*2*2*2*2}/_{1*2*2*2*2}$$

$$1={}^{1}/_{1}={}^{3}/_{3}={}^{6}/_{6}={}^{12}/_{12}={}^{24}/_{24} \rightarrow 1={}^{1}/_{1}={}^{1*3}/_{1*3}={}^{1*3*2}/_{1*3*2}={}^{1*3*2*2}/_{1*3*2*2}={}^{1*3*2*2*2}/_{1*3*2*2*2}$$

$$\tfrac{1}{2}={}^{2}/_{4}={}^{4}/_{8}={}^{8}/_{16} \rightarrow \tfrac{1}{2}={}^{1*2}/_{2*2}={}^{1*2*2}/_{2*2*2}={}^{1*2*2*2}/_{2*2*2*2}$$

$${}^{1}/_{3}={}^{2}/_{6}={}^{4}/_{12}={}^{8}/_{24} \rightarrow {}^{1}/_{3}={}^{1*2}/_{3*2}={}^{1*2*2}/_{3*2*2}={}^{1*2*2*2}/_{3*2*2*2}$$

If June could perceive number and operational sense from this way of manipulating arithmetical relations, her understanding of flexible numerical structures involving the four arithmetic operations might be developed simultaneously. Given the problem ${}^{3}/_{4}+{}^{1}/_{8}$, June might have more easily understood that ${}^{3}/_{4}+{}^{1}/_{8}= {}^{3}/_{2*2}+{}^{1}/_{2*2*2} ={}^{3*2}/_{2*2*2}+{}^{1}/_{2*2*2} = {}^{6}/_{8}+{}^{1}/_{8} ={}^{7}/_{8}$. However, the implementation of number examples in this way is mediated by a teacher's understanding of what the FF model can afford and what it cannot afford for developing students' mathematical thinking. The pattern of arithmetical relations demonstrated above is not explicit within the FF model, rather it is inherent in the structure of the set of fractional numbers.

CONCLUSION

In this paper, we attempted to understand how a teacher implemented instructional examples for developing students' thinking in mathematics. It was noted that the FF model can be a two-edged instructional tool when developing students' understanding of different concepts and laws in fraction. On the one hand, the teacher successfully used the FF model to make connections to the mathematical ideas of halving and doubling; on the other hand, however, she failed to use the model to move students' thinking forward to the discovery of the more generalisable number pattern of fractions related to equivalent fractions. As a result, her students missed the opportunity to engage in important mathematical thinking processes such as analogy, induction and generalisation. Moreover, when completing number operations such as addition of fractions, the teacher folded students' attention back to the physical pattern of the materials, as an alternative to providing opportunities to develop understanding of the algebraic nature of arithmetic represented by numbers. These findings indicate that it was not merely the understanding of a teaching model that assists teachers to implement instructional examples more effectively, but the understanding of the relationship of the instructional model and thinking patterns in mathematics.

In providing programmatic resources, Chick (2007) urges that it is necessary for professional development programmes to give prominence to an investigation of instructional examples and their pedagogical implications. To enrich teachers' MCK, teacher education and professional development need to assist teachers to be aware of the diverse patterns of thinking in mathematics. To develop teachers' PCK, our study indicates that it is necessary to guide teachers to understand the potential affordances and limitations of an example through learning and comparing its implementation in different teaching and thinking models.

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THE PROCEDURAL AND THE CONCEPTUAL IN MATHEMATICS PEDAGOGY: WHAT TEACHERS LEARN FROM THEIR TEACHING

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Teachers' learning about teaching mathematics is problematic to identify and analyse. Among many other things, teachers need to enact their understandings of how students' ideas about mathematics might develop, what tasks might support that development and how to elicit the expression of students' ideas. This enactment in practice potentially provides opportunities for teachers to learn about teaching mathematics in ways that enable them to improve their teaching over time. By examining data from a substantial longitudinal study by the first author we address the wider issue of what constitutes the body of pedagogical content knowledge in mathematics and address whether it is useful to talk of the procedural and conceptual in mathematics pedagogy.

INTRODUCTION

The nature of the professional knowledge of mathematics teachers has been and continues to be of major interest in research on teaching. Much research (Leikin & Zazkis, 2007) has focused on three major questions: what does the professional knowledge base for teaching mathematics look like; how does it develop; and how do practising teachers acquire it? In this research report we will focus on the third question by examining some of the data from a four-year research project carried out by the first author and her colleagues[i]. The larger focus of the research project was to investigate the role of literacy in mathematics teaching, taken to include speaking, writing and reading. The shared question for the researchers and teachers was how to develop students' skills for communication in the mathematics classroom. An additional focus for the researchers was how experienced teachers can potentially learn from their teaching, in this case about the development of students' abilities to communicate mathematically.

This report is based on an investigation of how teachers' knowledge develops as they interact with their students, other teachers and researchers. We see this knowledge development occurring, in part, as teachers experiment with new ideas that become absorbed into their repertoire of professional knowledge and shared with other teachers. For us, watching what teachers *do* as they engage in practice leads to a focus on pedagogical strategies or actions in the classroom. Going further and unpacking *why* teachers do what they do leads us and them to a focus on interpretation and reasoning about the teaching of mathematics. We want to call this distinction the *procedural* and *conceptual* aspects of mathematics pedagogy. Our

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focus in this paper is on how these aspects of pedagogy develop in practice and to explore the potential usefulness of this distinction.

Several years ago Hiebert and colleagues (Hiebert, Gallimore & Stigler 2002) posed the question of what does the professional knowledge base for teaching mathematics look like? We would add to their question, how does this knowledge base develop? In addressing this, we find it useful to draw on Skemp's (1976) distinction between instrumental and relational thinking and Hiebert and Lefevre's (1986) discussion of procedural and conceptual knowledge in mathematics. We wish to apply their distinction between knowing *how to* and knowing *why* in mathematics to teachers' knowledge of mathematics pedagogy. Just as Hiebert and Lefevre envisage a flowing from one to the other, not simply a progression once and for all from procedural to conceptual, we argue that the interplay between the procedural and conceptual aspects of mathematics pedagogy is potentially useful for gaining insight into teachers' learning in practice. Teachers, such as those in this study, often develop pedagogical strategies (or procedures) to deal with the very specific issues that arise for them in practice. Such strategies are specific responses to problem situations involving particular students, materials, and mathematical goals. But at the same time, some of these strategies can become principles for action (or pedagogical concepts) that can cut across contexts and can be adapted to other problem situations. For example, the teachers in this study developed, over a lengthy period of time, a rich set of strategies (pedagogical procedures) and a sophisticated model for supporting the development of students' mathematical writing (a pedagogical concept) (Doerr & Chandler-Olcott, in press). The teachers subsequently realised that they could use this same approach to support the development of students' mathematical reading skills. In this sense, the pedagogical procedures that the teachers developed became more than 'mere' strategies that were only relevant to a particular time and place and no more. These procedures became generalized principles for action (or pedagogical concepts) that could be shared among teachers and across contexts and problem situations, and thus contributed to the development of the professional knowledge base for teaching mathematics.

We will present some data from the project to serve as a basis for a discussion about the value of developing the notions of the procedural and conceptual aspects of mathematics pedagogy.

BACKGROUND OF THE STUDY

The data reported in this paper are from a four-year research project on mathematics and literacy. The research was carried out by a team of university-based researchers in mathematics education and literacy education, working in concert with mathematics teachers in a mid-sized urban school district in the United States. The school district had recently adopted what are known as Standards-based curriculum materials, namely, *Connected Mathematics Project (CMP)*, (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). These particular Standards-based materials, along with several others, were developed with the support of the National Science Foundation in the 1990's to align with curriculum standards that had just been put forward by the National Council of Teachers of Mathematics (NCTM, 1989). These materials represent a significant shift from traditional textbook materials in that they are structured around a sequence of 'investigations' that require the students to engage with mathematical tasks that need to be interpreted through the stories, pictures, diagrams, and charts in the texts. In addition, the students are expected to provide descriptions, explanations and justifications about their work with the tasks, both orally and in writing. As such, these curricular materials presented new challenges for the teachers who now needed to learn to support the development of students' abilities to communicate mathematically.

The teachers participating in this project were from one high school and its three feeder schools. The results reported here are from one of the teachers, Cassie, in the feeder school where the first author worked with the teachers. This school had approximately 860 students and 45 teachers and support staff and was considered high poverty with over 80% of the students qualifying for free or reduced-fee lunch. The school population was quite diverse with approximately 31% African-American, 21% Asian, 35% Caucasian, and 11% Latino students. Approximately 20% of these students were English language learners, and 25% were identified as having special needs. As we began the project, it was the second year using the *CMP* materials for most of the teachers. Cassie was in her 11th year teaching; she was enthusiastic about the use of the CMP materials. Cassie held multiple teaching certifications as she was certified to teach middle grades mathematics, special education and elementary education. She was teaching students in both grade seven and grade eight (ages 12 and 13 years old) as we began our work together.

Since our primary research questions concerned teachers' learning about mathematical communication in their own practices, we used "lesson cycles" to work jointly on planning, implementing, and debriefing lessons for supporting literacy opportunities for students (Doerr & Chandler-Olcott, in press). The lesson cycles began halfway through the first year of the project and continued through the fourth year of the project. Each teacher participated in a lesson cycle approximately once every three to four weeks with a member of the research team. Each lesson cycle consisted of three elements: (1) A planning session that followed the overall CMP guidelines for the investigations, but asked specifically the question "what are the literacy opportunities in this lesson?" In planning with this focus, the teacher discussed her ideas for reading the text, described opportunities for students to speak with each other, and identified prompts for student writing that would be used in the lesson. (2) The implementation of the lesson, where a member of the research team would observe the lesson, take extensive field notes, and generate questions for discussion that arose during the observation related to the literacy opportunities in the lesson. (3) A de-briefing session with the teacher, where the intent of the session was to collaboratively gain insight into the teachers' thinking about the literacy opportunities of the lesson and to collect shareable artefacts from the lesson, such as insights gained or tools used to support students' learning. The de-briefing session often centred on a discussion of the students' written work and how that might be used to inform subsequent lessons. The planning and debriefing sessions were audio-taped and later transcribed. Brief memos were written based on notes taken and the artefacts of the session.

The observation of the lesson allowed us to focus on the teachers' pedagogic procedures as they were enacted during the lesson. The planning and debriefing sessions gave us a focus on how the teacher interpreted and reasoned about the events that occurred in the classroom, namely the conceptual aspects of her pedagogy. In bringing these two aspects of pedagogy together, these sessions and the observations led to our shared insights into the development of students' abilities to communicate their mathematical thinking. Our complete data sources included field notes from summer work sessions, quarterly project-wide meetings, bi-weekly school-level team meetings, and the field notes and transcripts from the lesson cycles. In addition, each teacher was interviewed five times: once prior to the start of the project and then at the end of each year. At the final interview, each teacher was asked to reflect back on her experiences in the project and to share the stories and events that had been most salient from her perspective and the insights that she had into teaching through participating in the project. We are using our shared analysis of this final interview to examine the procedural and conceptual aspects of the learning that took place for Cassie.

RESULTS

We have chosen these three examples because of the importance that was given to these events by the teacher in her recounting of her learning in practice and because they serve to illuminate our emerging ideas about utility of attending to the procedural and conceptual aspects of mathematics pedagogy. Each example begins with the words of Cassie.

"the whole RAVE format"

Early in the project, the teachers worked together as a group to identify what they considered to be the characteristics of "good math writing." One of the other teachers on the team wanted to develop a rubric that could be used for letting students know what was expected from them when they were producing mathematical writing. The teachers also felt the need to be able to grade students' written work. Another teacher had begun to use a rubric called RAVE with her grade seven students. The acronym stands for R -- restate the question; A -- answer the question; V -- use math vocabulary; and E -- explain your examples. She found that it seemed to convey to the students some of the most important qualities of "good math writing." She also found that it was easy for the students to remember: many students were putting the letters RAVE on the top of their papers as a reminder to themselves. Over the course of the year, as this was shared and discussed during school team meetings, teachers in the other grade levels began to make use of this heuristic.

Cassie felt that the use of RAVE was an important contributing factor to the improvements that the teachers were seeing in their students' writing. Cassie recalled that when their school's students were being graded on the state assessments by teachers from other schools, that these teachers commented to her: "We're grading [your] sixth and seventh grade tests. They're amazing writers. How did they get to be such great writers?" Cassie attributed it to the school-wide use of RAVE: "I think that it's that whole RAVE format. And kids being able to write succinctly and accurately about what they observe. I think that's important. That's a good story and I like[d] hearing that." Using RAVE came about gradually, as Cassie and the other teachers gained experience in using it with their students. We see the use of this heuristic largely as a pedagogical procedure. Beyond meeting the needs to convey expectations to students and to grade students' written work, we found no evidence that this particular heuristic was conceptualized by the teachers in a more principled way. Rather, it primarily served as a "how to" strategy, albeit a particularly robust and widely used one.

"writing over time … because I actually saw growth in the students"

This comment by Cassie is a reference to an episode that occurred early in the project where she had given her students the same writing prompt ("what makes two figures similar") at the beginning, middle, and end of a unit of instruction on similar figures. Cassie and the researcher (first author) had an extended conversation about this writing, and Cassie spent some time classifying various examples of students' work as weak, average and strong. These categories represented her assessment of the level of student understanding that was evident in the writing. In this process, Cassie was often tentative with her judgments about what students might understand, especially for those students with special needs (perhaps diagnosed with a learning difficulty related to written expression) or second language learners. There were many students whose growth she could see over time and she found this exciting. Cassie said: "I actually saw the growth in students … It's just like you really get a picture of what they're thinking and what they know about math." These insights into students' thinking were useful for her instruction. However, what was most striking about her analyses of this student work was her realization that there were some students who started out weak and stayed weak. This greatly concerned Cassie and provided her with compelling evidence that she needed to change her instructional strategies to address the needs of these students.

This example of writing over time began as a pedagogical procedure for Cassie, but then moved in directions for pedagogy which are more conceptually grounded. Her analysis of the students' writing furthered her more general commitment to using writing for insight into student thinking and student growth. Later in the project, Cassie selected an example of this "writing over time" and shared it with her colleagues and at a national conference, articulating its potential for providing teachers with the kinds of insight that will allow them to support the development of students' abilities to communicate their ideas through mathematical writing.

"reading, writing and speaking really are central to math."

As Cassie analyzed her experiences in the math and literacy project, she identified a major shift in her thinking that had taken place, namely that "reading, writing and speaking really are central to math." Cassie reflected that she had "always wanted to think, oh, that's just the math that's the important part." By the end of the four year project, she saw that students' skills and abilities to read, write and speak were central to their mathematical learning. We see this pedagogical knowledge as conceptual in nature; it frames *why* a teacher would want to engage students in communicative practices. But at the same time, Cassie also was quick to acknowledge that students will have difficulty with these practices. She saw that she had a role to play in supporting students' development as they acquire these practices: "You'll have to support that [overcoming difficulties]." Her continuing work with her colleagues and her developing repertoire of instructional strategies related to communicative practices had begun to shape this role for her. We see this as the continuing development of and acquisition of pedagogical procedures, grounded in the needs of her own practice. Cassie acknowledged her difficulties in forging this new role. She commented that "organized chaos isn't my thing" as she wanted her classroom to be well-organized and structured and that this worked against her giving up some control when the consequence of that might be more "chaos" in the room -- often reflected in off-task behaviours and conversations by students. Yet, as the project progressed, Cassie continued to experiment with forms of group work and paired work that became much more common events in her classroom, as students engaged in various communicative practices. We see this as an example of the interplay between the development of Cassie's procedural pedagogical content knowledge (specific strategies for how to engage students in communicating) and the development of her conceptual pedagogical knowledge. This conceptual pedagogical content knowledge includes both a rationale for engaging students in communicative practices and the recognition that students' skills will develop over time with appropriate supports.

DISCUSSION

Our examination of Cassie's perspectives on her practice shows her recognition of how specific responses (pedagogical procedures) to problem situations had wider application and were of more general significance to her and the other teachers in her school and hence became principles for action (or pedagogical concepts). Our main focus in this paper is to propose that charting the flow from procedural pedagogical concepts to conceptual pedagogical concepts (and vice versa) provides educational researchers with a productive distinction that enhances our understanding of the nature of the learning of mathematics teachers from their practice.

In the first example, the use of the RAVE rubric to focus students' attention on what they needed to produce in their mathematical writing, we see an instance of a pedagogical strategy that is very useful to the teachers but that remains a pedagogical procedure for Cassie and the others. We might imagine that Cassie could have commented on the more general issue of the need for teachers to be explicit about their expectations for writing and therefore raising students' consciousness about desired features of mathematical writing. Other moves towards pedagogical concepts (principles for action) could also have taken place, but we did not see evidence of that in Cassie's reflections on the use of the RAVE rubric or in other data sources.

In the second example, we can see Cassie's learning in the interplay between the procedural and conceptual aspects of her pedagogy. Cassie recognized that students' writing gives an insight into the mathematical understanding of the students and of the potential growth of their understanding; writing is therefore a powerful tool for students and for teachers. She had a second recognition in seeing, through examining their writing, that some students were not developing; this led her to review and revise her teaching. We consider, from Cassie's explanations of the significance of these recognitions, that they have become more than local strategies, *knowing how*, and have become principles: for developing students' attention; for playing a role in the development of their learning; and for providing insights for teachers into their teaching, *knowing why*. The teacher is getting "a picture of what they're thinking and what they know about math," leading to her continued experimentation with various communicative practices. We identify this as an example of a flow from procedural (developing writing is needed) to conceptual (powerful tool for students and teachers), returning to procedural (seeing some were not developing) to conceptual again (systematic experimentation).

Finally, in the third example, Cassie saw the necessity of directly addressing students' skills and abilities to read, write, and speak, seeing them as central to students' mathematical learning where previously she saw them as peripheral, that the mathematics was the most important part. Communication became understood as crucial, since it is what drives all learning. Collectively, the use of a rich and shared set of strategies for addressing students' literacy skills led to the development of a new rationale for engaging students in communicative practices. This new conceptual pedagogical knowledge then led Cassie to seek ways to develop a new role for herself in supporting the development of students' skills over time. As in the second example, we identify the teachers' learning as a recurring flow between the procedural and the conceptual.

Our reflections, as researcher (first author) and co-author, on the teacher's learning lead us to recognise the vital role played by the interactions amongst the teachers and between the teachers and the researchers. Even when widely used among a group of teachers, some pedagogical strategies (such as the use of a particular rubric) appear to remain as procedures. Other strategies (such as writing over time) appear to have the potential for becoming conceptually grounded in a teachers' practice as rationales for the strategy are developed and shared with colleagues. Finally, repertoires of strategies, refined through experimentation in practice and by the interactions among teachers as the strategies are shared across contexts and problem situations, facilitate a shift from being merely local strategies (or pedagogical procedures) to becoming principles for action (or pedagogical concepts) through raising their awareness of the meaning of the mathematical teaching activity.

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THE DEVELOPMENT OF ALGEBRAIC REASONING IN A WHOLE-CLASS SETTING

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Generalization is a powerful vehicle for the development of mathematical insight by primary school pupils. However the implementation of this in classrooms is challenging for teachers as pupils often focus on superficial aspects of patterns. In this paper there is a description of a whole class discussion that shows that generalization and justification are closely aligned. In this regard, there is a need for teachers to engage in pedagogical press and for students to attend to the functional relationship between variables rather than pattern finding in single variable data.

INTRODUCTION

Over the past decade the teaching and learning of algebraic reasoning has been a focus of research and reform efforts. Among the arguments posited by Kaput and Blanton (2001) for the inclusion of algebra in primary grades are access to powerful ideas and the addition of a new level of depth and coherence to elementary mathematics. Algebra now appears in the curricular documents of many countries with the recommendation that it be taught throughout the primary school (D.E.S./NCCA, 1999; DfES, 2001; NCTM, 2000). The extent to which students are given the opportunity to access the powerful ideas of algebra is questionable, however. In Ireland, reviews of the implementation of the 1999 Mathematics Curriculum suggest that emphasis remains on the strand of number and on lower-order thinking skills (Shiel, Surgenor, Close, & Millar, 2006). From an examination of text-books which are used prolifically for the teaching of mathematics (ibid), the focus in algebra remains on symbol manipulation and pattern generation with little opportunity for generalization and justification. As whole-class discussion can be fertile ground for the development of higher-order mathematical thinking (O'Connor, 2001) it may be a means of addressing this issue.

LITERATURE REVIEW

In early years' classrooms, the emphasis in algebra is usually on the exploration of simple repeating and growing patterns. As any variation usually occurs within the pattern itself, the focus is on single variational thinking. As pupils move through the primary school system greater emphasis is placed on the formation of functional relationships and the generalization of patterns. Functional thinking is described by Warren, Cooper and Lamb (2006) as a focus on the relationship between two or more varying quantities. Although young children are capable of thinking functionally (Blanton & Kaput, 2004; Warren, 2005; Warren et al., 2006), there is evidence that

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they focus on pattern spotting in one data set rather than on the relationship between the pattern and its position throughout the primary years. Warren (2005) suggests that that this is the case either because single variational thinking is cognitively easier for children or is so engrained from school experience that there is a tendency to revert to it. Lannin, Barker and Townsend (2006) outlined a continuum of generalization strategies:

- Recursive: The students describes a relationship that occurs in the situation between consecutive values of the independent variable
- Chunking: the student builds on a recursive pattern by building a unit onto known values of the desired attribute
- Whole-object: the student uses a portion as a unit to construct a larger unit using multiples of the unit.
- Explicit: a rule is constructed that allows for immediate calculation of any output value given a particular input value.

The framework developed by Lannin et al (2006) was based on a teaching experiment with pairs of fifth-grade students who engaged in the solution of linear generalization problems. They found that students' choice of strategy was influenced by social, cognitive and task factors. In particular, student desire for efficiency played an important role in the method they used. The use of close input values (e.g. consecutive input values) seemed to invoke the recursive strategy while input values that were multiples of prior known values were conducive to the use of the whole-object strategy. In general it was found the students who had poor visual imagery of the problem often used the recursive or whole-object strategies incorrectly. Indeed, the tendency to use the whole object strategy erroneously (to assume, for example, that $f(100) = 5f(20)$) is commonplace among primary and secondary students (Stacey, 1989).

Anthony and Hunter (2008) drew on the framework above in their research with a class of students aged 9 – 11 years. Similar to Lannin et al., they found that social factors impinged on choice of method. The sharing of strategies by students in small groups appeared to encourage some but not all students to consider more efficient strategies. Generally speaking, students seemed to opt for the recursive strategy before risking alternative strategies. Teacher pedagogical press was another factor that appeared to support the development of more flexible, efficient strategies. The need for generalization and justification to be strongly linked is well recognised (Lannin, 2005; Lannin et al., 2006; Rowland, 1999) and is the subject of recent research (Ellis, 2007). The way this link might be forged through whole-class discussion is the focus of this paper.

BACKGROUND

The aim of my research is to investigate the factors that contribute to the development of mathematical insight by primary school pupils. The theoretical framework of the study is derived from the sociocultural perspective. The methodology is that of 'teaching experiment' which was developed by Cobb (2000) in the context of the emergent perspective and in which students' mathematical development is analysed in the social context of the classroom. I taught mathematics to a class of thirty-one pupils (seven girls and twenty-four boys) aged 9 - 10 years on a total of twenty-seven occasions over a six-month period. Many lessons took place over two or three consecutive days, each period lasting forty to fifty minutes. The school is situated in Ireland in an area of middle socio-economic status. All phases of the lesson were audiotaped and detailed field notes were written. When children were working in pairs, audio tape recorders were distributed around the room. Each pupil maintained a reflective diary. Follow-up interviews were held with students who had shown some evidence of reaching new understandings over the course of a lesson.

As it had emerged from data collected at the pilot stage of the project that generalization was conducive to 'insight', I sought to include an element of it in each lesson taught. In keeping with grounded theory methods, data collection and analysis occurred simultaneously. The data were analysed using a data reduction approach. Initially a descriptive account of the events for each session was constructed. I used these descriptive accounts to identify emerging themes and to create codes and categories (Charmaz, 1995; Goetz & LeCompte, 1984). A comparative analysis of lessons was also undertaken (Glaser & Strauss, 1967). Connections to previous literature and the new ideas that arose were documented.

The problem that is the focus of this paper was entitled. 'Friendship Notes' and involved finding the number of sheets of paper required if notes were to be sent by each member of a group to all other members, i.e., if there were ten pupils in a class the number of notes required would be ninety as each child would send notes to nine of his or her class mates. The solution generalizes to n(n-1) and as such is a quadratic[1] rather than a linear problem. The lesson took place over three consecutive days but transcripts from the plenary sessions that took place on the first day only will be considered here.

FINDINGS

In introducing the problem I clarified the notion of a friendship note (a friendly note written by each pupil in a group to all others) and the fact that the teacher had to order a set of blank notes for this purpose. I first asked the pupils to consider a situation where there was only one child in the group and then two. The focus at this stage was on clarification of the problem conditions, i.e., pupil would send one and only note to each member of the group and not to him or herself. When three members of the class were being considered in general a count strategy was used, although erroneously as was the case below with Aidan and Catherine[2], e.g.,

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[1] In the generalized form of a linear function, 1 is the highest power. In a quadratic function, 2 is the highest power.

[2] The following transcript conventions are used: T.D.: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings; Chn: A group of children speaking in unison;…: a hesitation or short pause; […]: a

77 Aidan: Three.

78 T.D.: Right you think three. Why do you think three?

79 Aidan: Because they have to get two, and then there's another one so like they get one each, so three.

80 T.D.: Three. Ok

81 Ch: Ah Miss!

82 Catherine: It would be nine.

85 T.D.: Why do you think nine, Catherine?

86 Catherine: Cos Chris has to send one to Charlie and Sinead and then Charlie will send one to Chris and Sinead and …

87 T.D.: And would that, would that be nine?

88 Catherine: Yeah.

89 T.D.: You think it would be nine, ok. Anybody else think different?

92 Myles: Six.

93 T.D.: Why do you think six?

94 Myles: Because each person has to send two notes, Chris sends one to Charlie and Sinead and then Sinead will have to send one to Charlie and Chris and Charlie will have to send one to Chris and Sinead.

Aidan, it seems, interpreted that the problem concerned receipt of one note each. It is possible that Catherine supposed nine because she had not given any thought to the non-reflexive nature of the activity. The problem was then enacted by the three pupils in much the same way as suggested by Myles (turn 94). A list was made on the blackboard showing number of children alongside number of notes. When a fourth pupil joined the group, Desmond used a recursive strategy incorrectly:

141 Desmond: Ten.

142 T.D.: Ten, you think ten. Why do you think ten?

143 Desmond: Because it is going like two - two, six, so it is going up four.

144 T.D.: Ok so you are looking here like this – two, six, you are looking at the …the list on the blackboard, so you think it is going up four each time. So that could make complete sense…

He is assuming, possibly from prior experience, that the difference between output values is constant. Brenda restated this idea:

145 Brenda: Eh, ten, cos on that side it has like a pattern.

Other students began to use the multiplicative relationship between both variables:

151 Myles: Em twelve.

152 T.D.: Why do you think twelve?

153 Myles: Because it would kind of be like four threes and four threes equals twelve.

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pause longer than three seconds; ( ): inaudible speech; [ ]: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

154 T.D.: And why do you think it's four threes?

155 Myles: Cos eh one person would have to give three notes to the other three people.

156 T.D.: You think twelve. What do you think?

157 Catherine: Twelve.

158 T.D.: Why do you think twelve, Catherine?

159 Catherine: Because each person has to send three notes.

Catherine it seems has moved from a counting strategy (see turn 86) to consideration of the relationship between the variables. When the group size was extended to five, a few pupils used the multiplicative relationship but Dermot thought the answer might be fifteen 'cos four would need twelve and add another three' ('chunking' strategy). Finbar then made the following contribution:

268 Finbar: It's like eh 1, em it actually kind of starts on two, it goes two then like count up in twos but it skips four, goes on to six and then that it skips eight and then it goes on to twelve but then on the fives it actually just em skips em eh fourteen, sixteen and eighteen, just goes on to twenty.

269 Chn: Huah[3]>huah>huah>please

270 T.D.: Right … yes.

271 Liam: It goes two, four, six, eight, ten.

272 T.D.: Does it go two, four, six, eight, ten?

273 Liam: No it doesn't go up, it's cos it's, cos zero plus two, two plus four equals six, six plus six equals twelve, twelve plus eight equals twenty.

Finbar has seen an additive pattern on the right hand side of the table and Liam expanded on his idea. Alex then saw the functional relationship between the numbers on the right-hand side and the left-hand side of the table. However his attention remained on the pattern of that relationship:

279 Alex: Em well on the board cos em 1 goes into zero one and then two goes into two once and then three goes into three (sic) twice, four goes into twelve three times and then five goes into twenty four times.

None of the three, Finbar, Liam or Alex, was able to use the pattern they found to predict answers for higher numbers, possibly because they had not given consideration to the structure of the problem. I distributed the worksheets in which pupils were asked to consider the number required for six to ten children. They worked in self-selecting pairs or triads on the worksheets. During the plenary session that followed the completion of these sheets, most pupils seemed to be using a multiplicative strategy to find solutions between one and ten and were able to justify their reasoning, e.g., in response to the number of notes required by six children the following interchange occurred:

[3] 'Huah' is used to indicate the sound made when there is a sharp intake of breath; it suggests that a child is keen to make a contribution to discussion.

298 Alex: Thirty.

299 T.D.: Right, now you have got to say why, why do you think it's thirty?

300 Alex: Because fi, cos you don't write one to yourself so five by six. [ ]

305 T.D.: …What would it be for seven? Hugh?

306 Hugh: Eh forty-two.

307 T.D.: Forty-two, can you explain why?

308 Hugh: Cos seven sixes is forty two.

309 T.D.: And where are you getting seven sixes from?

310 Hugh: Em cos there's seven pupils, then they each have to send six.

Alan made reference to the multiplicative relationship and justified his reasoning as a result of teacher press:

321 T.D.: Where are you getting the seventy-two from?

322 Alan: Em nine multiplied by eight is seventy two.

323 T.D.: And why is it nine eights?

324 Alan: Em, I don't know. I just found a pattern like that.

325 T.D.: You found … what kind of a pattern did you find?

326 Alan: Em if it was six pupils it would be six multiplied by five.

327 T.D.: And so why is it six multiplied by five if there were six pupils … Why? []

330 Alan: Cos they can't give one to themselves.

331 T.D.: Ok, so if there were ten how many would there be?

332 Alan: Em, ninety.

When discussion moved on to consideration of the number of blank notes needed for twenty pupils, Rory used a whole object strategy but undercounted. He decided on one hundred and eighty on the basis that ninety notes would be required for ten pupils:

334 Rory: One hundred and eighty.

335 T.D.: Where are you getting one hundred and eighty from?

336 Rory: Eh, well I just eh added ten and ninety and then I got a hundred and I added another eighty and I got eh one hundred and eighty.

In considering the solutions for inputs of twenty and later thirty, many pupils did apply the multiplicative relationship but became overly concerned with calculation, e.g.,

389 Colin: Eh you can take away the zero from the thirty …

390 T.D.: Hm, hm.

391 Colin: and take away one from the other thirty and then you have twenty-nine so you take away, you'll have to put up twenty-nine then you'll add three and two and you get five and then you add, you … wait, it's …

392 T.D.: What do you think on the calculator? If you had a calculator now, you are getting into funny numbers now, hard numbers, what would you multiply on the calculator? Colin?

393 Colin: Thirty by twenty-nine.

394 T.D.: Thirty by twenty-nine. Do you agree with that, David?

395 David: Yes.

396 T.D.: Why?

397 David: Eh because em if there are thirty in the class you wouldn't be giving themselves notes ()

The lesson ended soon after and was revisited on the next day when there was further focus on the generalized pattern.

CONCLUDING REMARKS

Initially most of this group of students used a count strategy which was verified by pupil modelling of the situation. As they became accustomed to the problem conditions more of them gave consideration to the multiplicative relationship between the variables. As results were listed on the blackboard, a few pupils began to use a recursive strategy to find the solution for an input value of four. As suggested by Lannin at al (2006) and Anthony et al (2008), this may have been due to the fact that input values were consecutive - it is possible that the use of random rather than consecutive input values would help pupils to focus more attention on the functional relationship. At the conclusion of the lesson some class members seemed to have developed an explicit rule (thirty by twenty-nine) - whole class discussion and teacher press for justification played important roles in the development of this rule. Similar to the findings of Warren et al. (2006), attention shifted from pattern identification to computation when large numbers were introduced. Also the erroneous use of the whole object strategy (see turn 336) was a prevalent feature in many lessons in this teaching experiment and warrants further investigation. Finally although the recursive strategy is not necessarily helpful for the development of a generalization, the pupils appeared very excited when they found a pattern (see turn 269) – how to harness this excitement into a deeper appreciation of algebraic structures remains a challenge for mathematics education researchers and teachers.

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TOOL USE AND CONCEPTUAL DEVELOPMENT: AN EXAMPLE OF A FORM-FUNCTION-SHIFT

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Tool use is indispensable both in daily life as well as in doing and learning mathematics. Research suggests a close relationship between tool use and conceptual development. This relationship is investigated for grade 8 students' acquisition of the mathematical concept of function. The study includes two teaching experiments and qualitative as well as quantitative data analyses. The results show that a form-function-shift can be recognized in the process of tool acquisition and learning about functions. We conclude that this process can be fostered by an instructional sequence in a multimedia environment that supports the co-emergence of external representations and mathematical concepts.

THEORETICAL FRAMEWORK

It is generally acknowledged that tools influence the process of students' mathematical sense making. Cobb (1999) describes the dynamic interplay between students' ways of symbolizing and mathematical reasoning in relation to tool use. In his study, computer minitools supported students' reasoning in the domain of statistics. He noted that this support is not a property of a tool that exists independently of users, but rather is an achievement of users. In this view, learning is characterized as a process in which students construct mental representations that mirror the mathematical features of external representations. This is related to the distinction between artefacts – material objects – and tools as instruments constructed by users in activity. The instrument comprises the cognitive schemes containing related techniques and conceptual aspects. The constructive and dynamic process of tool appropriation in mathematics education can be conceived as a process of instrumental genesis (Artigue, 2002) and progressive mathematization (Gravemeijer, 1999). Research has shown that the concept of emergent modeling can function as a powerful design heuristic for creating such processes in mathematics education (Gravemeijer, 1999). Open questions cast in realistic problem situations offer students opportunities to develop tentative situation-specific external representations. These representations give rise to (informal) models and new mathematical goals emerge. This is guided by the teacher with questions posed in situations that shift focus from real life problem solving to generalizing and mathematical reasoning (e.g. Doorman & Gravemeijer, 2009).

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We investigated the relationship between tool use and conceptual development in the domain of functions. With respect to the concept of function, Sfard pointed to the dual nature of mathematical conceptions (Sfard, 1991). She distinguishes operational conceptions (related to processes) from structural conceptions (related to objects). In the process of concept formation, the operational precedes the structural, but can not completely be separated as a result of this dual nature. Sfard uses the history of the function concept to argue that it evolved from calculations with input-output relationships to studies of sets of ordered number pairs. The dual nature of functions appears difficult for students and most students do not reach that level of understanding.

We expect to be able to foster the operational and structural aspects of functions with the availability of computer tools. The main research question of our project[i] is:

How can computer tools be integrated in an instructional sequence on the function concept, so that their use fosters learning?

The principle of a form-function-shift of notations in use (Saxe, 2002) is particularly suitable for analysing the interplay between tool use and conceptual development. This shift describes the interplay between cultural forms (external representations) and cognitive functions (purposes for structuring and accomplishing practice-linked goals). In the domain of functions, students can be supposed to use representations in a computer tool for investigating input-output relationships. As these investigations initially build on the imagery of repeated calculations, a form-function-shift might take place when the representations start to signify structure and dynamics of the relationships' co-variation.

This paper deals with the role of a form-function-shift in analysing the interplay between tool use and conceptual development.

METHODS

The research method followed is characterized as design research and consists of creating and analysing educational settings. In its cyclical process three phases are distinguished: instructional (re)design, teaching experiments and data analyses (Cobb et al, 2001). We created an educational setting in which we could analyse and come to understand the interaction between the instructional activities, tool use and students' learning.

Instructional design

The instructional goal was to facilitate students' ability to understand and investigate dynamic co-variation of input-output relationships. We aimed at a process of concept formation, within which formal models are built upon situation specific reasoning. Consequently, the instructional design had to provide opportunities for the learner to develop self-generated calculations that can progressively be mathematised into representations of functions.

The instructional sequence starts with three open-ended activities to reveal the students' current thinking, to evoke the need for organising series of calculations and to provide for opportunities for the teacher to introduce the computer tool AlgebraArrows (AA). This tool primarily supports the construction of input-output chains of operations. The chains can be applied to single numerical values as well as to variables. Tables, formulae and (dot) graphs can be connected to the chains (Boon, 2008; Drijvers et al, 2007).

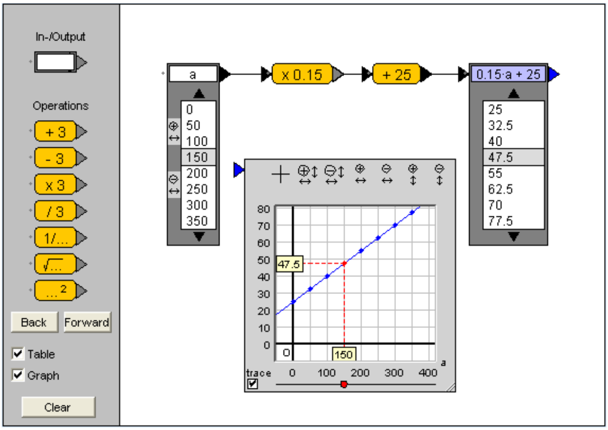


Figure 1: The computer tool AlgebraArrows

The initial computer activities are meant to show that AA is a useful tool for repeated calculations. The arrow chain resembles these calculations and input-output results obtained in prior activities. To structure inquiry of dependency relationships, the next step in the computer sequence is the use of tables and graphs and scroll and zoom tools (see Figure 1).

After the computer lessons the students carried out a matching activity. In this activity, students sort cards showing various representations into sets which represent one single function (c.f. Swan, 2008). This activity was designed to offer opportunities to reflect upon the various representations which they had encountered in the computer environment.

Teaching experiments

The first teaching experiment was conducted with three grade 8 (13-14 year olds) classes at three different schools and the second with 23 classes from 8 schools. In both experiments teaching sessions and group work in two classes were videotaped, and screen-audio-videos of three pairs of students working with the computer tool were collected. Students' answers to the computer activities were saved on a central server. In addition, the researchers collected students' written work and the results of a written assessment at the end of the instruction experiment.

Analyses of the first experiment showed that the principle of a form-function-shift (Saxe, 2002) was helpful in describing what happened in the learning process of two students. We revised the instructional sequence to be able to investigate this shift quantitatively by strengthening the goals of the sequence and redesigning two activities for a second analysis. In addition, we questioned to what extent the final performance of the students was dependent on the use of the available computer tool. Therefore, we added a computer test as well as a written test at the end of the instructional sequence. For the analyses of the second teaching experiment we used data of 155 students from 3 schools. These schools concluded the experiment with both the computer test and the written test.

Data analyses

The analyses started with organization, annotation and description of the data in a multimedia data analysis tool. Initially, the tasks in the instructional sequence served as the unit of analysis for clipping the videos. Codes were used to organise and document the data, and to produce conjectures about patterns in the teaching and learning process following the principles of grounded theory (Strauss & Corbin, 1998).

In analysing the first teaching experiment, we reconstructed the learning process of two students. This storyline motivated the use of a form-function-shift to describe the instrumental genesis of the computer tool. We identified characteristics of two computer tasks that represented different solution strategies. These characteristics were used to construct strategy codes with respect to the use of representations in the computer tool.

In the second teaching experiment, the screenshots of 155 students' answers to the specific two tasks were coded with the strategy codes. Good inter-rater-reliability was achieved (with a Cohen's Kappa of 0.79) on 55 out of 306 items (18%). For an analysis of the relation between results on the written test and the computer test we used a paired t-test.

RESULTS

The results of one of the open-ended activities (comparing two cell-phone offers) are illustrative of attempts by the students to organize the situations mathematically, i.e. organize repeated calculations, construct variables and use various representations (e.g. see Figure 2).

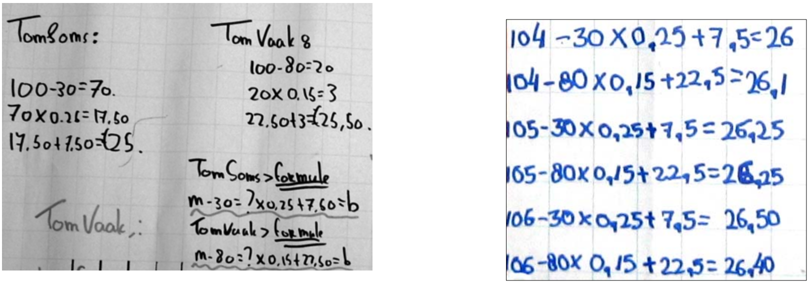
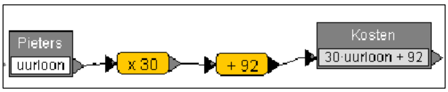


Figure 2 Posters of students' calculations

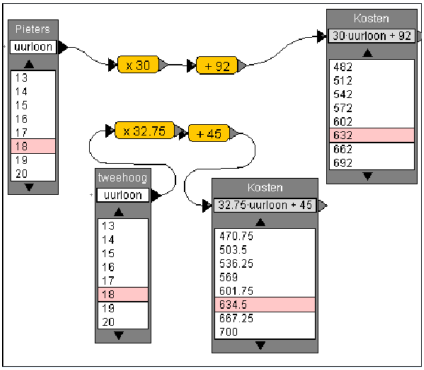
The teachers used various ways to introduce and guide the computer activities (Drijvers et al, in press). With respect to the interplay between tool use and students' conceptual development we analysed the computer work of a pair of students (Lina and Rosy). This resulted in a storyline of their learning process with respect to the computer tool. One of the first activities is similar to the cell phone activity and concerns determining when it is wise to change from one phone company offer to the other. Lina and Rosy built a calculation chain from the input box by adding operations and finally connecting an output box. They started to reason by initially focusing on specific cases. They entered numbers like 100 and 50 successively for comparing the differences between the two offers.

During the third computer lesson we observed quite another strategy by these students with AA while solving a similar kind of activity about two offers by contractors. The use of the representations in the tool as well as the focus of their reasoning seemed to have changed from case-by-case calculations to modeling with input-output chains and using the table tool to investigate the dynamics of both relationships. Lina starts the discussion and operates the mouse and the keyboard.

L: Company Pieters charges start costs and an hour rate … that is
R: plus 92, times 30 …
[L drags an input and an output box into the drawing area and creates a label for the in-box]
L: First company Pieters … I write only Pieters here, eh?
R: euro … plus costs per hour Pieters … and the total costs will come in here. [The mouse moves from in-box to out-box and she creates a label and types the name of the out-box. The variables are identified.]
L: Yes, here are the costs.
[The chain of operations is constructed and connected to the input- and output-box]



(…)
L: And … the other is TweeHoog …



[The inbox and outbox are positioned in the drawing area. Labels are created and names of the corresponding variables are typed in the labels.]
R: Strange costs per hour 32,75.
[The arrow chain of operations is constructed.]
R: Now you have to connect it to a table.
[They start to scroll in the table of the input values and quickly see that the break-even point is at 18 hours.]
(…)
L&R: From 18 hours.
[They type their answer 'from 18 hours'
in the answer box of question b.]

This vignette illustrates how the operation and function of the arrow chain representation changed in comparison to the previous computer activity. Lina and Rosy organized dependency relationships by immediately positioning and labeling boxes for input and output variables and then filling the gap with operations. For investigating the dynamics, they successfully operated the table tool in the software. This way of reasoning is precisely what we aimed for: an immediate identification of the variables, a clear construction of a dependency relationship with the computer tool and finally a purposeful and productive use of the table tool to investigate the dynamics of both relationships.

The second teaching experiment was on an increased scale, and one of the research goals was to obtain quantitative support for the conjectured form-function-shift during the computer activities. The solutions of 155 students of three schools to two specific computer activities were coded with the strategy codes. These results offered significant evidence to support our conjecture. We found that initially 130 (out of 155) students used the tool only for calculating successive input-output values, while in the end 89 (out of 152) students used the tool for structuring and investigating the dynamics of the relationships.

In a paired t-test we found no significant difference (p=0.200) between the final scores on the written test and the computer test. The correlation of 0.38 between these scores was moderate (p=0.001). These results indicate that the students were able to transfer their functional thinking in the computer environment to the paper-and-pencil environment.

CONCLUSIONS

We investigated how computer tools can be integrated in an instructional sequence on the function concept. We conclude that a form-function-shift can be used to describe the process of tool use and conceptual development. The shift took place when students started using representations of functions as tools for investigating relationships. As the investigations with arrow chains initially built on the imagery of repeated calculations, final operations on representations resembled an object conceptualization of functions. The performance of students appeared to be independent of the computer tool.

The emerging and shifting mathematical goals that fostered the form-function-shift mirrored the instructional sequence. During the introductory activities, students invented tentative representations that prepared for the successive computer activities. The representations in the tool linked up with these prior activities and offered opportunities to shift goals from solving situation specific problems to operational and structural aspects of functions. These shifts accomplished mathematical development as a step in the process of progressive mathematization and instrumental genesis.

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THE NUMBER LINE AND ITS DEMONSTRATION FOR ARITHMETIC OPERATIONS

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Teachers' interpretation of the guidance within the Primary Framework for teaching mathematics in England (DfES, 2006) appears to invoke a demonstrational rather than developmental presentation of the number line. Carrying out appears to be more important than developing relational understanding — the number line is mainly seen as a tool. Such emphasis can cause ambiguity in the way teachers refer to it and restrict understanding amongst the children that focus upon the ordering of numbers and the actions that could be associated with this ordering. Reporting on evidence from the use of the NNS (1999) this paper suggests that children's focus on procedures appears to be the reason they fail to use the representation in a 'relational' way.

Introduction

Recognition that the number line possesses certain underlying features has led to a variety of recommended models (Freudenthal, 1973; Anghileri, 2000), suggested modifications (Klein, Beishuizen & Treffers, 1998) and applied uses in, for example, the teaching of whole number operations (Beishuizen, 1997). Curriculum guidance for teachers within England has indicated that the number line is a "key classroom resource" (DfEE, 1998, p. 23) that has been found to help children master numeracy skills. It is therefore no surprise that as a representation it occurs frequently within the Primary Framework (DfES, 2006) as a mechanism to support children's calculation strategies in addition and subtraction. There are recurring references to either its use or the use of a number track within each year group of the primary school. Within the reception year and all other years, this use is associated with ordering and positioning numbers.

Though the notion of the number line may be seen as a representation of the more abstract notion of the number system, within classrooms it usually takes on a concrete form as an instructional representation. Lesh, Post & Behr (1987) identify it as a manipulable model that possesses in-built relationships and operations that may fit every day situations and form the basis for internalised images. Such a model can act as a mediator of mathematical ideas between the teacher and the student. However, Foster (2001) has indicated that representation within the classroom may be used in two ways, either in 'demonstration mode' or in 'developmental mode'. He sees that in the latter children use the representation freely and creatively according to their personal ideas and understanding in order to make sense of it. It is interesting to note that guidance for teaching mathematics within England (DfEE, 1999; DfES, 2006) appears to emphasise the former but does not mention the latter although there is a need for learners to know and understand the nature of the model they are using.

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Theoretical Framework

Despite its frequent appearance and plethora of applications the use of the number line tends to emphasise its procedural use as a tool rather than its conceptual structure. Certainly earlier objectives, although less apparent in the more recent would seem to support this view:

> *Make frequent use of a number line*, 100 square, number apparatus, pictures, diagrams,… and games and puzzles where *the rules are picked up quickly by watching a demonstration*. (DfEE, 1998, Introduction; p. 21; Italics not in the original text)

> **Demonstrating** showing, describing and modeling mathematics using appropriate resources and visual displays:… *demonstrating on a number line how to add on by bridging through 10*… (DfEE, 1998, Introduction; p. 12; Italics not in the original text)

Explicit reference to conceptual knowledge associated with the form and use of the number line appears to be omitted whilst the number line is not explicitly defined, but is seen as:

> … a means of showing how the process of counting forward and then back works. It can also be a useful way of getting children to visualise similar examples when working mentally. (QCA, 1999; p. 31)

Thus it could be claimed that the inclusion of the number line within the curriculum is as a means to and end, its use as a tool, but achieving the end appears to ignore what it is the children are using and, beyond the more obvious perceptual differences, also ignores the conceptual differences between number tracks and the number line. The difference between the number track and a number line does not lie simply in the perceptual sense that one has the "spaces numbered" and the other has the "points numbered" but in the conceptual sense identified by Skemp (1989).

> On the number line, numbers are represented by points, not spaces; and operations such as addition and subtraction are represented by movements over intervals on the line, to the right for addition and to the left for subtraction. The concept of a unit interval thus replaces that of a unit object. Also, the number line starts at 0, not at 1.
> (Skemp, 1989; pp. 139-141)

There is an implicit assumption within curriculum documentation, that the differences are recognized by teachers, applied in practice and understood by children. Doritou & Gray (2009) suggest that teachers tend to define the number line in terms of descriptions of particular lines, placing an emphasis on its use in applying procedures rather than on its strength as an abstract representation of the number system. Ambiguous association of the number line with a number track leads to conceptual difficulties in the knowledge reconstruction needed to deal with larger numbers and fractions, limited understanding of its use as a metaphor and its final rejection as procedural support (Gray & Doritou, 2008).

Herbst (1997) identified the number line as a metaphor of the number system and defines it as the consecutive translation of a specified segment U as a unit, from zero. U itself can be partitioned in an infinite number of ways (i.e. fractions of U). This 'density' provides the rationale for Herbst's suggestion that the number line can be seen as a representation of the number system and as a "solving and justifying" tool that can help in obtaining the solution to a problem on the line, as well as providing the justification of the way of thinking and consequently to the way in which an answer was obtained. To use a metaphor to support thinking suggests the need to know and understand the nature of the "helping tool" so that it can be used flexibly.

Method

This paper considers the way that teachers use the number line in their classroom for the teaching and learning of addition and subtraction and provides an indication of children's interpretation of this use. It considers the ways teachers talk about it, represent it and use it and the consequent meanings that are established by the children. The results form part of a case study of a primary school in the English Midlands during 2003 and 2004 (Doritou, 2006) that followed guidance within the National Numeracy Strategy (DfEE, 1999) and was considered to achieve below the national average compared to other schools. It may be seen as an example of the type of school the NNS was designed for in its attempt to encourage a rise in achievement.

The data collection was through lesson observation (Year 1 to Year 6), where the number line featured, and subsequent discussion with a sample of children to establish what they thought they had learned during the lesson. 22 children were interviewed—median ages ranging from 6.5 to 10.5 (four from each of the year groups Y2, Y3 and Y4, and five from each of the years Y5 and Y6) — drawn from a larger sample of 90 children and chosen to represent a cross section of achievement based upon actual and predicted achievement in national Standard Attainment Tasks (SATs), mediated by each class teacher's personal assessment. While lessons within Year 1 were observed, there was no follow-up questioning with the children – due to difficulties in articulating their understanding – but lessons placed a context on the initial development of the number line and therefore are reported. There was no observation concerning addition or subtraction within Year 6.

Results

During Year 1 lessons, the number line was used as a representation to support forward and backward counting associated with ordering of the numbers to 20. The Y1 teacher also referred to a 'chart', which consisted of three rows of numbers. The bottom row had the numbers 1 to 9 written on and the two rows above the first, had the numbers 10 to 90 and 100 to 900 respectively. The teacher indicated:

> [*Pointing to the 9*] And that's the number nine. You've got your number line. (TY1)

The teacher also referred to a sequence of tiles with numbers on them, as a number line and also gave children randomly distributed number symbols, asking them to make their own number lines by placing them in order. At a later stage, she referred to both a number line and a number track as a number line:

> If you look against my board there, you can see a big number line on the top that goes

from zero all the way up to ten and then a smaller number line underneath that goes from one to twenty. (TY1)

The series of lessons gave insight into the emphasis placed on the number line in the children's first year of school. Thus, very early in their mathematical development, the children experience the notion of number line associated with the number track whilst the teacher was frequently ambiguous in her use of terms whilst also referring to the hundred square as an alternative 'tool' that would support the children's development in counting and identifying particular numbers:

If you need help, you can look at our hundred square or number line. (TY1)

During the Year 2 observations, the objective being to emphasise forward and backward counting, the teacher used an empty number line, a ruler, a long plastic card on which the numbers 0 to 20 were ordered, a hundred square and a 0 to 100 number line. The ruler and the row of numbers on the plastic card were referred to as number lines, whilst the other representations were given equivalent status:

You're going to use a ruler as a number line to help you answer the questions. … A ruler is a bit like a number line. (TY2)

Since the ruler has not got all the numbers, you can use the hundred square or the big number line [*0 to 100 under the pin-board*] or use your ruler or try to do it mentally. (TY2)

The ambiguity was further compounded by the statement:

The hundred square is like a number line, but with one bit on top of the other. (TY2)

This ambiguity was later expressed in the interviews with the children. When asked what they had been doing in particular lessons, the children indicated:

Put numbers on number lines and hundred square… counting in tens. The teacher takes out a number of the hundred square and we had to find which is missing. (Child 2.1)

I had to do some work, I had a playtime… I had to put the numbers in the right order in tens…. Used a ruler… all the way up to one hundred. (Child 2.3)

Using a number line (means hundred square), coloured it with pens, got numbers on it and the number line don't go up to one hundred, but the square does. (Child 2.2)

Interestingly, though multiple representations were used within the classroom during the particular lesson, it is the hundred square that dominates the articulations of these children. Child 2.2 was the only child able to make some distinction between the track and a number line. When given a number line to talk about, she indicated:

This is a number line. A real number line. There's nothing. Nothing's between one and two because nothing's there. (Child 2.2)

Observed lessons within Year 3 focused on the use of the number line for addition and subtraction by counting up and counting back respectively. Subtraction was seen to be the opposite of addition. The dominant representations used were a number line and a hundred square and as we have seen earlier within Y1 and Y2, these two representations were regarded as similar, the difference being simply the presence or otherwise of zero:

> Really, they're sort of similar things, but this goes zero to one hundred [*number line*], this goes one to one hundred [*hundred square*], so it's the same really. (TY3)

There being no conceptual reference to the differences in the two forms of representations children reflected the comments of their teacher but gave only part of the difference:

> [*A hundred square*] is a number line coz it's got the one that goes up to a hundred. (Child 3.3)

The children's reflections on these lessons indicated that:

> The teacher done a sum. He had to jump on a number line to get the answers. (Child 3.4)

> Counting forwards and backwards with numbers on a hundred square. (Child 3.3)

Whilst one child could be fairly explicit about a procedure:

> We had to find the difference between the take-away sums. Thirty-nine take-away thirty-six are close together, so you get a number line or a hundred square and count down to the highest number. (Child 3.2)

Others had forgotten it:

> No! [*I can't tell you what the lesson was about*]. I always forget stuff… number line? (Child 3.1)

During the Year 4 lessons on subtraction, the teacher simply used an arrow representing 'jumps' to connect numbers (86 and 95) on the board, but no number line. When the selected children were invited to work out the answers to addition and subtraction problems, only two (Children 4.3 and 4.2) attempted to use a number line and neither of these were successful in obtaining the correct answer although both executed a procedure that emphasised a focus on bridging the tens. For example, Child 4.3 attempted to use the number line to add 84 and 36, but in fact used a procedure to subtract 36 from 84. She drew a line, marked 36 at the left end and 84 at the right, crossed the 36 through to mark 4 and reach 40 (in the middle of the line) and finally marked the 44 to reach 84. Adding 44 and 4 gave her 48. The child applied the exact procedure used by the teacher during the lesson for subtraction, but suspecting she had made an error, casually changed the addition sign from 84 + 36 to 84 – 36 saying — "I was meant to put take-away there".

Child 4.2 on the other hand was asked to obtain the solution to 503 – 103 and whilst the procedure he applied using number line 'jumps' (but as presented by the teacher no representation of the line) was correct, he then used a vertical algorithm to add the components of his 'jumps' but applied this incorrectly:

> I put one four three (143) here and then I jump... I get seven, I get three or fifty and I add it... if we get, that… three hundred. [*503 written to the right and the jumps and their result written between 143 and 503*]. (Child 4.2)

After establishing the sum of the jumps, Child 4.2 obtained a solution of 450 rather than 360. Here there are place value issues but that is a separate issue. It was not unusual for children to become confused when bridging tens and perhaps partly in recognition of this confusion they applied a well rehearsed counting procedure to add or subtract. Child's 4.4 approach emphasised the use of counting unit marks she had placed on the empty number line, but this did not always prove a suitable approach. She had some difficulty attempting to obtain the solution to 18 – 9 using an empty number line and recognising this she concluded:

> I know the answer. Nine… Because double nine is eighteen and eighteen take-away nine is nine. (Child 4.4)

Child 4.1 found the solution to 43 + 37 using partitioning. When she used the number line, she placed 37 on the left and 43 on the right. She initially added 3 to get to the next multiple of ten, 40, but then gave an answer of 46 (43 + 3). She realized something was wrong:

> Coz it ain't the same answer as when I partitioned it… I've done the number line wrong… The partitioning way [*is better*]. (Child 4.1)

It seemed that during the interviews children used the number line with some reluctance. Though they might use an empty number line and follow the indications of the teacher to "put the smallest number at the beginning and the biggest number at the end" (Child 4.1) to repeat "what we done this morning… half jumps like that and we had to add quite to the nearest ten" (Child 4.3), partitioning, but omitting the use of the number line or jumps, seemed better.

The teaching objective for the Year 5 observation included working out addition first by partitioning and then using the number line, first adding an appropriate number of tens onto the larger of the two numbers and subsequently adding the units of the smaller number to the total. After the lesson, the selected children were asked to describe the substance of the lesson. In each instance, the children only referred to their use of the standard algorithm and when asked to give an example wrote the example as in the standard algorithm. Only Child 5.5 indicated that an alternative way of doing the addition was to use a number line. However, when she used the number line to add 462 and 276 she made two unsuccessful attempts: (i) She initially drew a number line near the end of the page and wrote 276 at one end and 462 at the other end. Starting from the 276 she made a left to right jump of 200 and then identified a jump of -14. Unfortunately the jump continued in a left to right direction. (ii) A second line was drawn covering the width of the page in case there was not enough room to put the numbers on. Though not completed, but Child 5.5 indicated that:

> The vertical [*algorithm*] way is the easiest. The number line is the hardest. (Child 5.5)

Child 5.4 on the other hand could use the number line to confirm an estimation or validate answers worked out mentally. When asked to give the answer to 76 + 49 he drew a line with the ends marked 0 and 150 ("It doesn't need to go any further") and then represented successive jumps of 70 then 40, then 6 and finally 9 to give 125. Child 5.1 demonstrated an interesting variant on the use of the number line and on the use of partition to obtain a solution to 24 + 32. She preferred to use the standard algorithm. When requested to do it in another way, she volunteered to use the number line. However, knowing that the answer was 56, did not contribute to the number line's successful use. The child marked the numbers 20, 30, 4 and 2 on the line and made two jumps (one between 2 and 4, and one between 20 and 30) and random marks – which she called "little bits" – between them, applying the partitioning process through jumps on the number line.

Discussion

Within the recommended curricula for schools, we see the use of a variety of representations, each of which are variously used to promote understanding of the sequence and order of the naturals, introduce and develop addition and subtraction, and expand children's knowledge of the number system. However, within the documents, there is no clear indication of the conceptual differences between each of these representations and neither is there explicit treatment of what it may mean to possess a relational understanding of the number system.

The conceptual differences between the number track (and the hundred square) and the number line do not appear to be a cause of concern for the teachers that have been observed — both can be used to illustrate 'jumps'. The most sophisticated child response regarding their difference was given by Child 2.2 but it was a difference articulated from the perception that there is a gap between the positioning of the numbers on the number line "because there is nothing there". The fact that for the number line the jump is from an end point of one interval to the end point of an another interval whilst on the number track the jumps exist between spaces does not appear to be a matter of concern — it is the numeral and the difference between it and another numeral that is focused upon and not upon what the numerals represent. Both can equally be used as perceptual representations that support counting on or counting back in ones or in sequences of ten.

To reach an appropriate level of skill and understanding, children need guidance but this guidance was absent from the documentation that is the key curriculum resource and from the teaching they experienced. Consequently, most of the children that the documentation and teaching was designed to support did not make appropriate conceptual leaps. Their understanding of the number line remained embodied in a whole number world that, it is conjectured, did not reflect teacher assumption. The teachers focused upon what they were encouraged to use to support the development on their children's skill and understanding in numeracy. However, whilst they make the former explicit, the latter requires a process of reflective abstraction.

The interviewed children tended not to use the number line as a representation of choice to deal with addition problems. It seems to have served its purpose as a tool to facilitate the development of partitioning within earlier years, but returning to its use seems to cause problems. Children who appear to see it as a line with a beginning and an end, display little implicit awareness of its underlying characteristics nor can they remember how to use it operationally. The fact that they apply 'rules without reasons' by having learnt 'fixed plans' that tell them what to do at each point, indicates that their understanding of the number line concept is far more 'instrumental' than it is 'relational'. It is a careful reminder of the limitations that may be associated with other forms of mathematical representation used in primary schools.

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A STUDY OF COMPARATIVE PERFORMANCE ON PARTITIVE AND QUOTITIVE DIVISION IN SOLVING DIVISION WORD PROBLEMS

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This paper reports on the results of a one-to-one interview involving 26 Grade 3 students on a range of partitive and quotitive whole number division word problems, following the participation of half the group in a teaching experiment. The focus is on whether students perform better on partitive or quotitive division problems. Results suggest there is little difference between students' performance on partitive and quotitive problems.

THEORETICAL FRAMEWORK

There have been much research on young children (aged 5 to 9) solution strategies to division problems (e.g., Burton, 1992; Heirdsfield, Cooper, Mulligan & Irons 1999; Kouba, 1989; Mulligan & Mitchelmore, 1997; Murray, Oliver & Human, 1992), their understanding of division (e.g., Brown, 1992; Correa, Nunes & Bryant, 1998; Squire & Bryant, 2002) and their performance on division problems (e.g., Fischbein, Deri, Nello & Marino, 1985; Zweng, 1964). This is not surprising given the complexity associated with the learning of division and how distinctly different it is from other operations.

Division may be interpreted in two different ways, namely division by the multiplier (partition division) and division by the multiplicand (quotition division). In partition division (commonly referred to as the sharing aspect) the number of subsets is known and the size of the subset is unknown, whereas in quotition division (otherwise known as measurement division), the size of the subset is known and the number of subsets is unknown (Fischbein, Deri, Nello & Merino 1985; Greer, 1992). The division problem 12÷ 4 could be interpreted as a partitive problem, such as: 'Twelve counters are shared equally among 4 children. How many did they each receive?' Interpreted as a quotitive problem, using the same context, it would be: 'There are 12 counters and each child receives 4. How many children will receive counters?' While the quotient is the same for each, the model is quite different. These examples illustrate the fact that the role of the size of the portion and the number of recipients actually reverses in partitive and quotitive division (Squire & Bryant, 2002).

It is important to have an understanding of both partitive and quotitive division because an over emphasis on division as 'sharing' in the early years can lead to difficulties when dealing with numbers less than one (Anghileri, 1995). For example,

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if the following task, 6 ÷ ½ is interpreted as 6 shared between ½ it does not make sense. Nor is it simple to interpret in a real life context, such as: 'Six cakes shared between ½ of a person giving 12 each'. If interpreted as how many amounts of ½ are in 6 it makes sense, as it does in a real life context, such as: 'Six cakes are each cut in half. How many halves are there?'

Studies on children's performance on partition and quotition tend to indicate that young children find quotition division is more difficult due to their limited experience with it prior to commencing school (Anghilieri, 1995; Correa et al., 1998; Kouba, 1989; Squire & Bryant, 2002). This may explain why partition division has traditionally been taught first, as the sharing aspect was considered to relate much more to a child's everyday life (e.g., Anghileri, 1995; Correa et al., 1998). There is research evidence that suggests that 5 to 8 year-olds are capable of solving quotitive division problems prior to instruction (e.g., Brown, 1992; Burton, 1992; Correa et al., 1998; Murray et al., 1992; Squire & Bryant, 2002; Zweng, 1964).

In fact, Burton's (1992) study of 117 Grade 2 students revealed that there was little difference between their performances on partitive or quotitive division problems, whereas other studies showed that children were less successful with the partitive division than they were with quotitive division problems (Brown, 1992; Zweng, 1964).

Studies on children's solution strategies to division indicate that children generally begin with direct modelling and unitary counting, progress to skip counting, double counting, repeated addition or subtraction, then to the use of known multiplication or division facts, commutativity and derived facts (e.g., Mulligan, 1992; Mulligan & Mitchelmore, 1997). Kouba (1989) found children used two intuitive strategies when solving quotition problems: either repeated subtraction or repeatedly building (double counting and counting in multiples). For partitive division, they drew on three intuitive strategies: sharing by dealing out by ones until the dividend was exhausted; sharing by repeatedly taking away; and sharing by repeatedly building up. However, Brown (1992) reported that children in Grade 2 tended to solve partitive problems using grouping strategies, rather than sharing strategies, but the strategy did not always correctly model the action of the problem. Murray et al., (1992) found the children's solution strategies (Grades 1 to 3) for partitive and quotitive division problems initially modelled the problem structure. As they became more experienced they were more flexible in the strategies they used and ignored whether it was a partitive or quotitive problem.

This study differs from previous studies on division at Grade 3, as the focus was on students' performance on partitive and quotitive division word problems, pertaining to four different semantic (equivalent groups, allocation/rate; rectangular arrays, times as many). The students were also given a choice as to the level of difficulty of the problem for each, to give them some ownership. The level of difficulty refers to the size of the quantities used in the problems. The excerpts of children's solution strategies indicate an intuitive use of multiplication for solving both partitive and quotitive division problems. This was evident in studies of students in Grades 4 to 6, involving numbers beyond the multiplication fact range (Fischbein et al., 1985; Heirdsfield et al., 1999). Unlike earlier studies where the children used physical materials (counters) and drawings to solve the problems, the students in this study were encouraged to solve them mentally.

METHODOLOGY

This paper draws on one of the findings of a larger study conducted from March to November 2007, of young children's development of multiplicative thinking. The study involved students aged eight and nine years from two Grade 3 classes in two primary schools in a middle class suburb of Melbourne. A sample of 13 (case studies) representing a range of performance from each grade was selected using a one-to-one task based interview. Following a teaching experiment on division, over a three-week block, in October, the researcher administered a one-to-one, task-based interview to each case study student in November. The purpose of the interviewing was to gain insights into and probe students' understanding of multiplicative structures and strategies used in division problems.

Instruments

The main sources of data collection were interviews. The researcher developed a one-to-one, task-based interview on division, consisted of two problems for each semantic structure identified by Anghileri (1989) and Greer (1992): equivalent groups, allocation/rate, arrays, and times as many. For each problem there were three levels of difficulty, rated as easy, medium or challenge, from pilot testing.

The division interview consisted of eight division word problems. Each category included both a partitive and quotitive problem to identify whether there was a relationship between the strategies students chose and the division type. Table 1 lists the challenge questions chosen for the division interview, noting the aspect of division (partitive or quotitive). The numbers (dividend and divisor) used in the medium and easy questions are in brackets.

Interview Approach

The case study students in both schools were interviewed 3 weeks after the 15-day classroom intervention. Each interview was audiotaped and took approximately 30 to 45 minutes, depending on the complexity of the student's explanations. Responses were recorded and any written responses retained. Each question was presented orally, and paper and pencils were available for students to use at any time. Generous wait time was allowed and the researcher asked the students to explain their thinking and if they thought they could work the problem out a quicker way. Students had the option of choosing the level of difficulty to allow them to have some control and feel at ease during the interview. If a student chose a challenge problem and found it too difficult, there was an option to choose an easier problem.

Method of analysis

Initially, students' responses were coded as correct, incorrect, or non-attempt as well as on the level of abstractness of solution strategies, drawing on the categories of earlier studies (Anghileri, 2001; Heirdsfield et al., 1999; Kouba, 1989; Mulligan & Mitchelmore, 1997). Giving the students a choice, added to the level of complexity in presenting the data.

Semantic structure	Aspect of division	Problem
Equivalent groups	Partition	I have 48 cherries to share equally onto 3 plates. How many cherries will I put on each plate? (M 18, 3; E 12, 3)
	Quotition	72 children compete in a sports carnival. Four children are in each event. How many events are there? (M 24, 4; E 12, 4)
Allocation/ Rate	Partition	I rode 63 kilometres in 7 hours. If I rode at the same speed the whole way, how far did I ride in one hour? (M 28, 7; E 15, 5)
	Quotition	I have 90 cents to spend on stickers. If one packet of stickers cost 15 cents how many packets of stickers can I buy? (M 60, 5; E 30, 5)
Rectangular Arrays	Partition	One hundred and two pears are packed into the fruit box in 6 equal rows. How many pears are in each row? (M 54, 6; E 24, 4)
	Quotition	I cooked 84 muffins in a giant muffin tray. I put 6 muffins in each row, of the tray. How many rows of muffins on the tray? (M 36, 4; 4; E 20, 4)
Times as many	Partition	Sam read 72 books during the readathon, which was 4 times as many as Jack. How many books did Jack read? (M 36; E20, 4)
	Quotition	The Phoenix scored 48 goals in a netball match. The Kestrels scored 16 goals. How many times as many goals did the Phoenix score? (M 28,7; E 18, 6)

Table 1: Division Interview Word Problems

RESULTS AND DISCUSSION

Table 2 shows the number of correct responses for the combined student cohort on each of the four semantic structures for partitive and quotitive whole number division problems. The correct responses only were recorded. Codes used in the table: Partitive (P), Quotitive (Q), Equivalent groups (EG), Allocation/rate (AR), Rectangular arrays (RA), Times as many (TM). So EG-P means an equivalent groups partitive problem. Blank spaces indicate that no student selected this level of difficulty.

Cohort	Task	EG-P	EG-Q	AR-P	AR-Q	RA-P	RA-Q	TM-P	TM-Q
Grade 3 *(n=26)*	Easy			2	1	5	4	2	
	Medium	14	9	5	8	10	11	4	4
	Challenge	12	16	16	15	11	11	10	11
Total correct		*26*	*25*	*23*	*25*	*26*	*26*	*16*	*15*

Table 2: Number of Correct Responses on Division Problems

Four key findings from these results

There is little difference between the students' responses to the partitive and quotitive type problems. Indeed, when added together the total number of partitive problems correct was 91 and the total number of quotitive problems correct was 90, giving 181 correct responses of the possible 208 (87%). This would indicate that these students were able to perform successfully on both partitive and quotitive problems and can be exposed to both in the early years of schooling. This result is particularly impressive when it can be seen that many students chose problems that involved numbers that would seem to be beyond students at this level, according to the state curriculum.

However, the difficulty factor masks these results, to some degree. Of the 181 correct responses, 66 were for the medium and 14 were for the easy problems. A different result may have occurred, had there only been the challenge problems.

Nevertheless, 101 of the correct responses were by students who chose the challenging problems. Perhaps many students at this level are capable of much more than the curriculum currently indicates they are able to do.

The following excerpts from the interviews illustrate some of the students' correct solution strategies, in solving the following challenge problem:

EG-Q: Seventy-two children compete in a sports carnival. Four children are in each event. How many events are there?

Bindy: I started with 12 x 4 then doubled it to get 24 x 4 and that's 96, but that's too much. So I took away 20 from 96 and that gave me 76, and that's 19 x 4 but I still need to take away another 4 so it would be 18 events or 72 ÷ 4 =18.

Mark: 4 x12 is 48. If I use the doubling strategy it would be 24 x 4 is 96, but that too much. If I take away 4 x 4 it's 80 so that's 20 x 4 but that's still 8 too much. So I need to take 2 x 4 away from 80, so it is 18 events or 72 ÷ 4 =18.

Matt: Four children in each event and 72 children is 18 events, 'cause 4 plus 4 is 8 (that's 2), 8 plus 8 is 16 (that's 4), 16 plus 16 is 32 (that's 8), 32 and 32 is 64 (that's 16), 64 and 2 fours more is 72, so it is 18 fours. The number sentence would be 72 ÷ 4 =18.

Bindy and Mark used known multiplication facts and doubling, while Matt used doubling. Interestingly, Mark used commutativity in his calculation by turning the fact around when he doubled. It appears that these students understand the relationship between the numbers in the context of this problem and indicates their understanding of the inverse relationship of multiplication and division.

A third interesting result is that 100% of the students were able to correctly responded to the rectangular arrays problems. Even though more students chose the easier problems for rectangular arrays than for any other task type, the accuracy and success rate for the challenging problems is much higher than the literature would indicate, highlighting the advantages of exposing students to a variety of semantic structures for division.

Mark and Annie used a multiplicative calculation, and Sharn used wholistic thinking (partitioning the dividend using distributive property or chunking) splitting the 84 into 60 and 24 to solve the following quotitive division rectangular array problem:

RA-Q: I cooked 84 muffins in a giant muffin tray. I put 6 muffins in each row, of the tray. How many rows of muffins on the tray?

Mark: 12 sixes are 72, and another 2 sixes is 84 so that's 14 sixes or 84 ÷ 6 =14 that means 14 rows of muffins.

Sharn: 6 tens are 60 and 6 twos are 12 so six fours are 24 so that's means 84 ÷ 6 =14 or 14 rows of muffins.

Annie: I know 12 x 6 is 72, 13 sixes is 78, 14 sixes is 84, that means 14 rows of muffins.

In the next excerpt we notice that Mark and Annie solved the partitive division rectangular array problem using a multiplicative calculation, and Sharn again used wholistic thinking.

RA-P: One hundred and two pears are packed into the fruit box in 6 equal rows. How many pears are in each row?

Mark: 12 sixes are 72, 24 sixes are 144 but that's too much so I need to come back a bit. 20 sixes are 120, but that's 18 too much, so it is 17 because 3 sixes are 18 so I just needed to minus 3 sixes to get 17 x 6 is 102, or 17 in each row.

Sharn: 20 x 6 is 120 but that's too much, 15 x 6 is 90 and another 12 is 102, so that means 17 x 6 is 102 or 102 ÷ 6 =17 or 17 pears in each row.

Annie: It's something times 6. 12 x 6 is 72, 15 x 6 is 90, then 17 x 6 is 102, that makes 17 pears in each row.

These students used the same strategy to solve both partitive and quotitive division problems. However, they all thought of the partitive problem as a quotitive problem, in that, six was the multiplicand rather than the multiplier. This could indicate their preferred method for solving partitive division problems.

Lastly it is worth noting that the times as many problems were more difficult, with fewer students (15%) being able to correctly answer the question they attempted. One could infer from this that many of the students at this level found this semantic structure difficult. However, the excerpts of students' solution strategies below indicate they are capable of solving such problems, given exposure to them over an extended period of time.

Below are some examples of the children's solution strategies for the partitive times as many challenge problem:

TM-P: Sam read 72 books during the readathon, which was 4 times as many as Jack. How many books did Jack read?

Bindy: I know that 6 x12 is 72 and that 6 x 6 is 36 and half of 72 is 36 and because it's 4 times I need to halve 36 and that's 18, so that means 4 x18 is 72 or 72 ÷ 4 =18 so Sam would read 18 books.

Sandy: I need to think of 4 times something equals 72. 72 take away 40 (which is 4 times 10) is 32, and 32 is 4 x 8. 4 x 20 is 80, but that's too much. I know 6 x12 equals 72, but 12 would be 6 times not 4 so it can't be that. I need to take 8 off 80 to get 72 so it would be 18, because 4 x10 is 40 and 4 x 8 is 32 and together that's 72. So Sam read 18 books, 72 ÷ 4 =18.

Matt: Half of 7 is 3 and a half, so half of 70 would be 35 and half of 72 would be 36 because half of 2 is 1. Half of 3 is one and a half, so half of 30 is 15 and half of 6 is 3 so it is 18 books. 72 ÷ 4 =18 because you halve 72 two times, which is the same as saying 18, four times.

Bindy used multiplication facts and halving, Sandy used a combination of multiplication facts, splitting the product and distributive property, whereas Matt split the product and then used halving. Quite different thinking was employed by the students, which illustrates the benefits of providing students with challenging problems encourages them to develop more efficient strategies (Murray et al., 1992).

CONCLUDING REMARKS

The finding of this study suggest that students at Grade 3 are capable of solving complex partitive and quotitive division problems, relating to a range of semantic structures, involving numbers beyond what is mandated by state curriculum documents. Second, that providing students with experiences in both partitive and quotitive division through a problem solving approach before any formal teaching of algorithms, to allow them to develop a range of efficient mental strategies (Murray et al., 1992). Third, that having an understanding of multiplication supports their development of division and enables students to use the inverse operation to solve division problems, as indicated by the students' preference to use multiplication to solve both partitive and quotitive division problems. This presents a strong case for linking multiplication to the teaching of division.

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EXPLORING MATHEMATICAL KNOWLEDGE FOR TEACHING

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The latest years have provided our research field with several theoretical models describing the mathematical knowledge needed for teaching. All these describe the knowledge through different dimensions. Ball, Thames and Phelps (2007) suggest that common content knowledge and specialized content knowledge are two different dimensions of pure mathematical knowledge within the mathematical knowledge for teaching domain. The research reported in this article re-establishes these two dimensions empirically in a Norwegian context, using a multiple-choice test of 356 teachers in primary and lower-secondary school. The research gives empirical support to the theoretical model as each of the two dimensions forms empirically acceptable constructs.

LITERATURE REVIEW

Research on teachers' knowledge has always been an important part of mathematics education research. But the latest years have seen a considerable growth in the field. This has resulted in several theoretical models. Probably the most cited is from Shulman (1986). He proposes a theory of teacher knowledge that includes three types of content knowledge. One type is *subject matter content knowledge* (or just content knowledge), another is *pedagogical content knowledge* and a third is *curricular knowledge*. The new idea was a description of pedagogical content knowledge as something that 'goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*' (Shulman, 1986, p. 9).

There are several theoretical models that describe what mathematical content knowledge is, with titles focusing on mathematical understanding, proficiency, empowerment and competency (for example Brekke, 1995; Kilpatrick, Swafford, & Findell, 2001; NCTM, 1989, 2000; Niss & Højgaard Jensen, 2002). These models are tools for analyzing what mathematical content knowledge a teacher of mathematics needs. But even if there is an agreement that mathematical content knowledge is a prerequisite to be able to teach mathematics, there is also a growing understanding that this is not sufficient (Ball, 2002; Ball & Bass, 2003; Ball, Hill, & Bass, 2005; Niss, 2007). Consequently, it is also important to look at what the mathematics teachers need in addition to mathematical content knowledge. Shulman (1986) emphasize pedagogical content knowledge. More recent, several models that describe the knowledge a mathematics teacher needs have been developed, as a specification of what pedagogical content knowledge might be within mathematics. Niss and Højgaard Jensen (2002) provides a model of *mathematical teacher competency*. This model contains curriculum competency, teaching competency, uncovering of learning

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competency, assessment competency, collaboration competency and professional development competency. Another model called *proficient teaching of mathematics* is provided by Kilpatrick, Swafford and Findell (2001), and contains conceptual understanding, fluency, adaptive reasoning and productive disposition. Also in The Teacher Education and Development Study in Mathematics (TEDS-M) a model of what 'knowledge of teaching mathematics' is has been developed (Tatto et al., 2008). This model contains mathematics content knowledge and mathematics pedagogical content knowledge. The latter is subdivided into mathematical curricular knowledge, knowledge of planning for mathematics teaching and learning and enacting mathematics for teaching and learning (Tatto et al., 2008, p. 42).

Ball, Thames and Phelps (2007) provides a practice-based theory called *The content knowledge for teaching*, which is elaborated from both *pedagogical content knowledge* and *subject matter knowledge* (Shulman, 1986), where both these aspects are divided into three parts, figure 1.

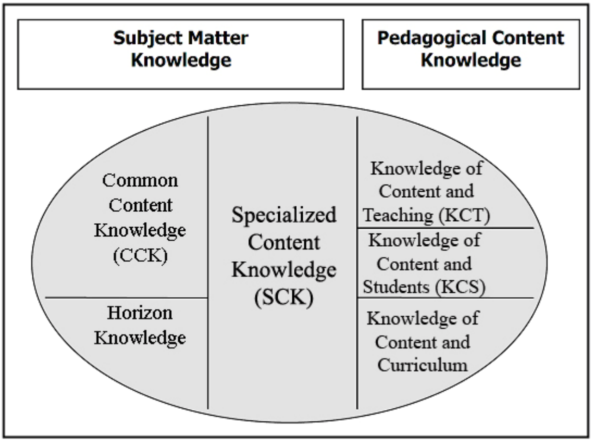

Figure 1: The content knowledge for teaching theory (SCK is bold in original) (ibid, p. 42).

Common content knowledge (CCK) is defined as 'the mathematical knowledge and skill used in settings other than teaching' (ibid, p. 32). It regards questions that are commonly answerable by people who know mathematics, and not knowledge unique to teachers. Specialized content knowledge (SCK) is defined as 'the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work (...) and therefore not commonly needed for purposes other than teaching' (ibid, p. 34). SCK is purely mathematical in the sense that it does not require knowledge of students or teaching. But this is not the mathematics typically learned in the university mathematics courses, it is 'mathematics in its decompressed or unpacked form' (ibid, p. 36). The knowledge of content and students is defined as 'knowledge that combines knowing about students and knowing about mathematics. Teachers need to anticipate what students are likely to think and what they will find confusing' (ibid, p. 36). Knowledge of common student conceptions and misconceptions are a vital part of this domain. Knowledge of content and teaching is defined as 'knowledge that combines knowing about teaching and knowing about mathematics' (ibid, p. 38).

Included in this domain is for example the selection of examples, the sequencing of the content, the advantages and disadvantages of different representations, which student contributions could be used and how, when the whole class needs clarification, when to work with single students or small groups, and the decision of what method or algorithm to introduce. Schulman's curricular knowledge is placed within pedagogical content knowledge and re-named to curriculum content knowledge. Horizon knowledge is 'an awareness of how mathematical topics are related over the span of mathematics included in the curriculum' (ibid, p. 42).

These examples from the manifold of models illustrates that there are several ways to describe the knowledge a teacher needs to teach mathematics. These models all divide the mathematical knowledge for teaching into different dimensions or knowledge types. While TEDS-M does this by grouping teachers' daily tasks to make categories, the *content knowledge for teaching* theory (Ball et al., 2007) describes their different dimensions as knowledge constructs.

RESEARCH QUESTION

The most innovative part of the *content knowledge for teaching* theory is arguably the distinction of specialized content knowledge (SCK) and common content knowledge (CCK) as two different dimensions within the mathematical knowledge needed for teaching. This theory have evolved from empirical studies, and factor analysis suggests that the mathematical knowledge for teaching is multidimensional (Ball et al., 2007; Hill, Schilling, & Ball, 2004). But the existence of SCK and CCK as two separate constructs are only partly documented, and it is necessary to explore this issue further empirically. It is also interesting to see if it is possible to re-establish these categories in a Norwegian context. Consequently, the following research question has been formulated: Are there any evidence for or against the existence of specialized content knowledge (SCK) and common content knowledge (CCK) as two separate constructs among Norwegian primary and lower-secondary teachers of mathematics?

METHOD

As a part of the teacher development project 'Mathematics in Northern Norway', 356 teachers from grade 1 until 10 answered an individual test between October 2007 and February 2008. The teachers came from 28 schools from four regions, schools of different sizes, and from both central and rural parts of the county. All teachers that teach (or normally teaches) mathematics in these schools participated in the projects development courses. As Norway has general teachers' in primary and lower secondary school this means that almost all teachers teaching grade 1 until 7 are mathematics teachers and thus participate in the study. For the grades 8 until 10 this is normally different, as the teachers teach fewer subjects, so for these grades only about one third of the teachers are mathematics teachers and thus participates. The course leader gave standardized information before the test, and did not answer questions that could serve as a help to solve any of the tasks. It was no time limit given, and nobody withdraw from the test. 90% of the participating teachers were tested, as 36 teachers were absent, in most cases due to illness.

The test was primarily designed to describe teachers' mathematical knowledge for teaching, and consisted of 20 tasks with a total of 46 questions all taken from the database developed by the Learning Mathematics for Teaching project, LMT (2009). 18 of the tasks were picked within the topic of numbers, and selected to cover some of the central parts of the number topic from the Norwegian curriculum. As an example the test addresses different number concepts like whole numbers, decimals, fractions and percent, and the operations of addition, subtraction, multiplication and division. To create a wider perspective and not rely entirely on findings within the number topic, one task within basic algebra and one within geometry were also added. Some tasks tested the teachers capability to assess whether a method or a suggestion would be valid for all numbers, other tasks tested if the teachers could assess whether an illustration do explain a given calculation or not, and some tasks were testing the teachers capability to choose the most relevant question to ask a student at a given situation. All the tasks demand some knowledge of mathematics.

ANALYSIS AND EMPIRICAL RESULTS

The test as a whole had a reliability (measured by Cronbachs alpha, see Crocker and Algina (1986)) of 0,84 on 46 items. This reliability can be regarded as high and satisfactory for the test as a whole.

Initially, the test was not designed to check the existence of SCK and CCK, because the *content knowledge for teaching* theory as presented by Ball, Thames and Phelps (2007) was not published when the test was developed. As a result of this, the tasks of the test had to be categorized using the *content knowledge for teaching* theory in retrospect. Two articles from Ball, Hill and colleagues was used as the basis for the categorization (Ball et al., 2007; Hill, Sleep, Lewis, & Ball, 2007).

Ten of the tasks, with a total of 27 questions, were categorized as tasks that are primarily testing SCK. One example is task 26 (figure 2), where it is necessary to consider each method and decide whether the method will work for any two numbers. The considerations needed are purely mathematical. Consequently, this task is not testing knowledge of content and students or knowledge of content and teaching. This is a type of situations that teachers often meet in the classroom and for that reason they need the knowledge to answer these problems. Others will hardly ever have to solve a problem like this. As a result of this, the task demands SCK.

Cronbachs alpha was 0.77 for these 27 questions. By studying the alpha if item was deleted as well as the correlations between the SCK construct and each item, it became clear that two questions could be deleted from the construct without reducing the reliability of the scale (items 4a and 19d). Item 4a is a task where the teachers where asked to inspect the standard algorithm in Norway for multiplication, and consider if it is valid for all numbers. This is probably the reason why this task does not suit well into the construct, where there are many non-standard methods and invented rules to consider. A closer look at item 19d reveals that it is possible to give good arguments for both the answers Yes and No. This is probably the reason why the item does not work well together with the others. After deleting these two questions, the SCK construct has 25 questions and a Cronbachs alpha of 0,78.

Imagine that your students are working with your class on subtracting large numbers. Among your students' papers, you notice that some have displayed their works in the following ways:

932 356 +4 *-356 360 +40* *400 +500* *900 +32* *932 576* *Student A*	*932 932* *-356 - 300* *632* *- 50* *582* *- 6* *576* *Student B*	*932 936 976* *-356 -360 -400* *576* *Student C*

Which of these students is using a method that could be used to subtract any two whole numbers? (Mark ONE answer):

a) A only b) B only c) A and B d) B and C e) A, B, and C

Figure 2: Task 26

The validity question is then whether this construct is a good representation of the theoretical dimension. All 25 questions are testing mathematical knowledge not commonly needed for others than teachers, and at the same time important to know for teachers. Theoretically they are therefore different from CCK. The tasks cover a wide range of central mathematical topics, and they include items where the teacher should assess if a suggestion is valid, if an invented rule works for all numbers, assess rules of thumbs, which answer(s) on a student task can be a good evidence that the student understands and which text is connected to a number calculation. It could be seen as difficult to cover all parts of SCK in only 25 questions. But as the tasks regard important knowledge for a teacher, and test pure mathematics not normally needed by others than teachers, they form an acceptable representation of SCK.

Three of the tasks, with a total of 12 questions, were categorized as tasks that are testing CCK. One example is given in figure 3. To answer this task, the teacher needs to know the equivalent forms of numbers expressed as decimals, fractions and percent. It is possible to answer this correctly even if one does not know anything about students thinking or teaching of mathematics. Consequently, this is neither knowledge of content and students nor knowledge of content and teaching. This task is categorized as CCK, but it is a borderline item between SCK and CCK, and this borderline is often difficult to define. The main argument that this is not SCK is that if the text was reformulated, this task could be found in students textbooks. Consequently, this task belongs to the CCK category, as something all students have to learn at school. But since this is a typical problem in school mathematics, it is also natural to think that teachers need this knowledge more than others, hence a borderline item. This borderline problem is also typical for the other tasks in the CCK category.

Mrs. Bond's students were writing equivalent forms of the same number. However, students did not always find this easy. For each list below, indicate whether the expressions are equivalent forms of the same number. (For each list, circle, EQUIVALENT, NOT EQUIVALENT or I'M NOT SURE.)

	Equivalent	*Not Equivalent*	*I'm not sure*
$\frac{0}{2}$, 0,2 , 2%	*1*	*2*	*3*
$\frac{30}{10}$, 3,0 , 300%	*1*	*2*	*3*
$\frac{20}{5}$, 0,4 , 40%	*1*	*2*	*3*
$\frac{7}{1000}$, 0,007 , 0,7%	*1*	*2*	*3*
$\frac{6}{10}$, 0,6 , 6%	*1*	*2*	*3*

Figure 3: Task 13

Cronbachs alpha was 0.72 for these 12 questions. A study of the alpha if an item was deleted as well as the correlations between the CCK construct and each item showed that a deletion of question 23c would give a slightly higher reliability. A study of the item revealed no theoretical reasons to delete it, so it remained included.

All the tasks in this construct are within the definition of CCK (Ball et al., 2007; Hill et al., 2007). But because the test originally was designed to describe teachers' mathematical knowledge for teaching, all the tasks have a classroom context. It is just after a closer analysis using the *content knowledge for teaching* theory it has become clear that these tasks differ from the ones in the SCK category. None of the tasks in the SCK category are borderline items with the CCK category, as exemplified by task 26. All the tasks in the CCK category are within the definition of CCK, but at the same time borderline items with SCK, as exemplified by task 13, figure 3. As a consequence of this, there are theoretical reasons to look at these two groups as separate constructs.

The CCK construct includes decomposing of whole numbers, equivalent expressions from the example above (fraction, decimal, percent) and assessing the number of solutions to three different number tasks. But even if all the 12 questions are within the definition of CCK they are not representing the whole concept very well, because they are few, and because they all are borderline items with SCK.

The *observed* correlation between the SCK and the CCK construct is 0.60 and the *latent* correlation is 0.80. This indicates that there are two different constructs measured as the constructs turns out to be sufficiently different empirically.

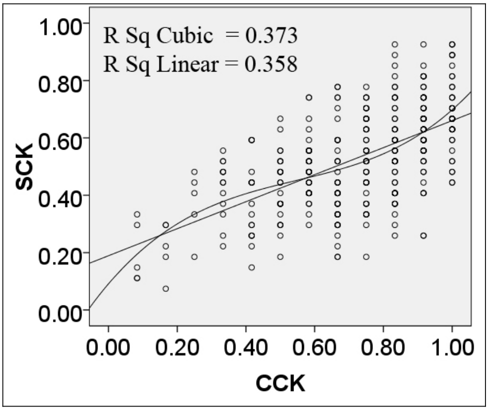

Figure 4: Scatter plot of CCK and SCK

The scatter plot in figure 4 shows that the cubic model offers a slightly better explanation than the linear model does. The first model explains 37 percent of the variance; the second 36 percent. This indicates that the nature of the relationship is not entirely linear, a finding that may be further explored in future analysis. Another interesting aspect of this plot is that the teachers that have full score (1,0) at the CCK tasks have rather different scores at the SCK tasks, varying from 0.40 until 0.95. The implications of this finding will also be looked at more closely in future analysis and research.

CONCLUSION

Specialized content knowledge (SCK) and common content knowledge (CCK) are two different dimensions of pure mathematical knowledge needed to teach. This research has re-established these categories in a Norwegian context, where each of them form empirically acceptable constructs. The SCK also forms an acceptable theoretical construct, while the CCK construct in the test is not representative for the whole dimension as defined originally. The results show that these two constructs are connected, but still sufficient different empirically to indicate that there are two different constructs. Further research is needed to understand more about the relationship between these two dimensions, for example what impact common content knowledge (CCK) has on specialized content knowledge (SCK).

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TEACHERS USING TECHNOLOGY: ORCHESTRATIONS AND PROFILES

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Abstract *Developing ways to use educational technology is not evident for mathematics teachers. With the theory of instrumental orchestration as a framework, we investigate the types of orchestrations three teachers use in a lesson series for grade 8 on the function concept, employing an applet embedded in a digital learning environment. The results show six types of orchestrations that differ in their ICT specificity. Interview data suggest that teachers' preferences for types of orchestrations can be related to their views on mathematics learning and teaching.*

1. INTRODUCTION

It is generally acknowledged that the integration of technological tools into mathematics education is a complex issue. A theoretical framework that acknowledges this complexity is the instrumental approach (Artigue, 2002). According to this approach, the use of a technological tool involves a process of *instrumental genesis*, in which mental schemes and techniques for using the tool co-emerge. Many studies focus on students' instrumental genesis and its possible benefits for learning (e.g., Kieran & Drijvers, 2006).

Students' instrumental geneses need to be guided, monitored and orchestrated by the teacher. To describe this, Trouche (2004) introduced the metaphorical notion of *instrumental orchestration*. Until today, however, the number of elaborated examples of instrumental orchestrations described in the research literature is limited (Drijvers & Trouche, 2008; Drijvers et al., in press; Trouche, 2004). Therefore, we wanted to investigate the different types of orchestrations teachers use, as well as the relationships between these types of orchestrations and teachers' views on mathematical learning and teaching. We address these issues from meta-perspective, i.e. without detailed reference to specific mathematical, didactical and technological aspects. For a comprehensive analysis of these issues, the reader is referred to Doorman et al. (in press).

2. THEORETICAL FRAMEWORK

The notion of instrumental orchestration (Trouche, 2004) addresses the question of how the teacher can fine-tune the students' individual instruments and compose

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coherent sets of instruments, thus enhancing both individual and collective instrumental genesis.

An *instrumental orchestration* is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a – in our case computerised – learning environment in a given mathematical task situation, in order to guide students' instrumental genesis. An instrumental orchestration consists of three elements: a didactic configuration, an exploitation mode and a didactical performance. The first two elements are described by Trouche (2004). As an instrumental orchestration is partially prepared beforehand and partially created 'on the spot' while teaching, we felt the need to add a third component, reflecting the actual performance. As such, the threefold model has a more explicit time dimension.

A didactical configuration is an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it. These artefacts can be technological tools, but the tasks students work on are important artefacts as well.

In the musical metaphor of orchestration, setting up the didactical configuration can be compared with choosing musical instruments to be included in the orchestra, and arranging them in space so that the different sounds result in the most beautiful harmony.

An exploitation mode of a didactical configuration is the way the teacher decides to exploit it for the benefit of her didactical intentions. This includes decisions on the way a task is introduced and worked, on the possible roles of the artefacts to be played, and on the schemes and techniques to be developed and established by the students. Decisions on the exploitation mode can be seen as part of the design of a Hypothetical Learning Trajectory (Simon, 1995).

In the musical metaphor of orchestration, setting up the exploitation mode can be compared with determining the partition for each of the musical instruments involved, bearing in mind the anticipated harmonies to emerge.

A didactical performance involves the ad hoc decisions taken while teaching, on how to actually perform in the chosen didactic configuration and exploitation mode: what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool?

In the musical metaphor of orchestration, the didactical performance can be compared with a musical performance, in which the actual inspiration and the interplay between conductor and musicians reveal the feasibility of the intentions and the success of their realization.

The instrumental orchestration model brings about a double-layered view on instrumental genesis. At the first level, instrumental orchestration aims at enhancing the students' instrumental genesis. At the second level, the orchestration is instrumented by artefacts for the teachers, which may not necessarily be the same artefacts as the students use. As such, the teacher herself is also engaged in a process of instrumental genesis for accomplishing her teaching tasks (Bueno-Ravel & Gueudet, 2007), which include the development of operational invariants: the implicit knowledge contained in the schemes that is believed to be true.

3. METHODS

To investigate the *different types of orchestrations* teachers use, we analysed video tapes of 38 50-minute lessons taught by three experienced female mathematics teachers. Their grade 8 classes participated in an innovative technology-rich learning arrangement[i] for the concept of function (Drijvers et al., 2007). The learning arrangement, which was developed over three research cycles, came with a teacher guide for the teachers including a planning scheme. The main technological artefact was an applet called AlgebraArrows embedded in an electronic learning environment called Digital Mathematics Environment (DME). The applet allows for the construction and use of chains of operations, and options for creating tables, graphs and formulae and for scrolling and tracing. The DME allows the student to access the work in any location, and the teacher to access student work to check progress, and to monitor the learning process.

Data analysis focused in particular on whole-class episodes in which technology was used. The unit of analysis is an episode, which concerns the whole-class treatment of one task. If this treatment consists of different orchestration types, the episode was cut into sub-episodes. This way, a corpus of 83 episodes was identified. These data were organized and analysed with the help of software for qualitative data analysis[ii]. The analysis combined a deductive, theory-driven approach with an inductive, bottom-up analysis. Six orchestration types were identified and the corpus was coded according to this categorization. A second coding of 29% of the episodes led to a good inter-rater-reliability (Cohen's kappa) of .72.

To investigate the relationships between these types of orchestrations and *teachers' views on mathematical learning and teaching*, the researchers drew up a profile for each of the three teachers' orchestrations, possible operational invariants and views on mathematical learning and teaching. These profiles were validated through semi-structured post-experiment interviews with the three teachers concerned.

4. RESULTS

Different types of orchestrations

The data analysis, partially based on a priori, theory-driven codes, led to the definition of six orchestration types: Technical-demo, Explain-the-screen, Link-screen-board, Discuss-the screen, Spot-and-show, and Sherpa-at-work. The first three are predominantly teacher-centred, whereas the last three are more interactive.

The *Technical-demo* orchestration concerns the demonstration of techniques by the teacher. A didactical configuration for this orchestration includes access to the applet and the DME, facilities for projecting the computer screen, and a classroom arrangement that allows the students to follow the demonstration. As exploitation modes, we observed teachers treating a technique in a new situation or task, as well as adding new techniques to students' work, anticipating what will follow.

The *Explain-the-screen* orchestration concerns whole-class explanation by the teacher, guided by what happens on the screen. The explanation goes beyond techniques, and involves mathematical content. Didactical configurations can be similar to the Technical-demo ones. As exploitation modes, teachers sometimes took student work as a point of departure for the explanation, or started with a solution suggested by the teacher herself.

In the *Link-screen-board* orchestration, the teacher stresses the relationship between what happens in the technological environment and how this is represented in conventional mathematics of paper, book and school board. In addition to DME access and projecting facilities, a didactical configuration includes a school board and a classroom setting such that both screen and board are visible. Similarly to the previously mentioned orchestration types, teachers' exploitation modes take student work as a point of departure or start with a task or problem situation set by the teacher herself.

The *Discuss-the-screen* orchestration concerns a whole-class discussion on what happens on the screen. The goal is to enhance collective instrumental genesis. A didactical configuration includes DME access and projecting facilities, preferably access to student work, and a classroom setting favourable for discussion. As exploitation modes, we once more see student work as point of departure, and having students react to it, as well as teachers setting a task, problem or approach as input for discussion.

In the *Spot-and-show* orchestration, student reasoning is brought afore through the identification of interesting digital student work during preparation of the lesson, and its deliberate use in classroom discussion. Besides previously mentioned features, a didactical configuration includes access to the DME during the preparation of the teaching. As exploitation modes, teachers may have the students whose work is shown explain their reasoning, and ask other students for reactions, or herself provide feedback to the student work.

In the *Sherpa-at-work* orchestration the student uses the technology, either to present his/her work, or to carry out operations the teacher requests. Didactical configurations are similar to the Discuss-the-screen orchestration type. The classroom setting should be such that one student can easily manage the technology, with both Sherpa-student and teacher easy to follow by all students. As exploitation modes, teachers may have student work presented or explained by the student using the technology, or may pose questions to the Sherpa student and ask him/her to carry out specific operations in the technological environment.

Orchestration type	**TeacherA cycle 1**	**TeacherA cycle 2**	**TeacherB cycle 2**	**TeacherC cycle 3**	**TeacherA cycle 3**	**Total**
Technical-demo	5	3	2	7	5	**22**
Explain-the-screen	0	0	0	7	1	**8**
Link-screen-board	3	0	6	0	3	**12**
Discuss-the-screen	4	4	3	1	2	**14**
Spot-and-show	0	1	12	2	2	**17**
Sherpa-at-work	2	7	0	0	1	**10**
Totaal	**14**	**15**	**23**	**17**	**14**	**83**

Table 1: Orchestration type frequencies by teachers and research cycles

Table 1 shows the frequencies of each of the orchestrations for each of the teachers and, for teacher A, for each of the three times she taught the lesson series. Technical-demo and Spot-and-show are the most frequently used orchestration types. Concerning the former, the post intervention interviews suggest that the three teachers felt the need to familiarize students with basic techniques, in order to prevent technical obstacles from hindering the mathematical activities. The Spot-and-show orchestration is also quite frequent, but is unevenly spread over the three teachers, even if they all appreciated the opportunity to browse through the digital student work while preparing the next lesson. This brings us to the issue of differences between teachers, and the underlying views they have on the learning and teaching of mathematics.

Orchestrations reflecting teachers' views

Table 1 shows that *Teacher A* has high frequencies for Technical-demo, for Discuss-the-screen and for Sherpa-at-work. She spends most time on the three interactive orchestration types. In the post-intervention interview, she confirmed that she finds interaction in the classroom very important, and sees ICT as a means to stimulate this:

> "…so you could discuss it with the students using the images that you say on the screen, […] it makes it more lively…"

Teacher A mentioned time constraints as the main reason not to use Sherpa-at-work in the final cycle. Technical constraints (e.g. slow internet connections during the first cycle or inappropriate classroom settings in the computer lab) also drive her choices for orchestrations.

Teacher B has high frequencies for Link-screen-board, and particularly for Spot-and-show. She said she used the board to

> "take distance from the specific ICT-environment, otherwise the experience remains too much linked to the ICT"

Establishing the links between the ICT-work and paper-and-pencil mathematics is important to her, as she sees the use of ICT as a means to achieve her mathematical teaching goals. She appreciates Spot-and-show for discussing students' common mistakes or original approaches, as she considers this as fruitful for learning.

Teacher C has high frequencies for Technical-demo and Explain-the-screen, which are more teacher-centered orchestration types. Two reasons seem to explain her preferences. She described herself as a "typical teacher for mid-ability students" who strongly believes that these students benefit from clear demonstrations and explanations. Furthermore, she wanted to be in control of what is happening in the classroom, and believed that more teacher-centered orchestrations would support this.

5. CONCLUSION AND DISCUSSION

One of the goals of this study was to investigate the different types of orchestrations teachers use and to extend the repertoire of exemplary orchestrations. The analysis revealed six orchestration types, which are very different in their specificity. Discuss-the-screen and Explain-the-screen can be seen as ICT-variants of regular teaching practices most teachers are familiar with. More specific for the use of ICT are Technical-demo, Sherpa-at-work and Link-screen-board. Specific for ICT-tools that provide the teacher with online consultancy of digital student work is the Spot-and-show orchestration. We conclude that the repertoire of orchestrations is diverse, and that the orchestrations have different degrees of ICT-specificity. Technological and time constraints may influence the choice and exploitation of the orchestrations.

A second goal was to investigate the relationships between these types of orchestrations and teachers' views on mathematical learning and teaching. The matching of observed orchestrations and interview data reveals clear links between the three teachers' preferences for orchestrations and their ideas on what is important to achieve during the teaching. We conclude that these three teachers' choices for orchestrations and their exploitation are strongly related to their views on mathematics learning and teaching.

While considering these conclusions, we should note that the three teachers, who participated on a voluntary basis, were confronted with a complex and innovative learning arrangement, which they were not familiar with. Dealing with that for the first time is not easy, and it is possible that a lack of familiarity partially guided their orchestration choices. Also, the time schedule for the learning arrangement was very tight, which as well may have affected the didactical performance and the decisions involved.

At the end of section 2, we mentioned the instrumental genesis that teachers themselves are involved in. As developed schemes include operational invariants, the question arises as to what the operational invariants of these teachers are. Although the data are too limited to fully answer this question, it suggests that these operational invariants are at least partially determined by the teachers' views on mathematics learning and teaching. A more fine-grained analysis of the orchestrations, which goes beyond the categorization we presented here, is needed to shed more light on this. A next enterprise to be undertaken!

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[i] For more information on the project see http://www.fi.uu.nl/tooluse/en/ .

[ii] We use Atlas ti software, www.atlasti.com

TEACHERS' TEACHING OF STOCHASTICS AND STUDENTS' LEARNING

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This report focuses on a quantitative research project that combines two aspects of a statistics curriculum related to teachers' classroom practice, and their students' stochastical knowledge. Data were collected with questionnaires. The development of the questionnaires derived from results of a qualitative research project will be sketched. Afterwards, some results of the quantitative research will be discussed.

INTRODUCTION

In recent years, statistical reasoning (SR), statistical literacy (SL) and statistical thinking (ST) have been declared as the three overarching goals of a modern stochastics teaching (Ben-Zvi, & Garfield, 2004). In many countries, e.g. in Germany, these goals involve both a considerable change from teaching probability to teaching statistics, and a considerable change from thinking about "computations, formulas, and one right answer" (Ben-Zvi, & Garfield, 2004, 4) to thinking about data in a realistic context. Hence, Garfield (2002) mentions that teaching stochastics in regard to SR, SL or ST requires a change of stochastics teachers' instructional practice. However, the knowledge about both the classroom practice of "conventional" stochastics teachers and the impact of the "conventional" stochastics teachers' teaching on their students' learning is scarce.

Therefore, in this report, the focus is on a research project involving a quantitative survey concerning the classroom practice of German stochastics teachers and their students´ learning based on the following questions:

- What is the content of the curriculum of conventional stochastics teachers?
- What are the objectives of the curriculum of these stochastics teachers?
- What knowledge and beliefs do students attain in stochastics courses?
- How is the impact of the teachers' stochastics curriculum on the students' knowledge and beliefs?

The research project is based on two results of the research on teachers' beliefs. Firstly, Chapman (1999) states that mathematics teachers' thinking is the key factor in any movement towards changing mathematics teaching. Secondly, there is strong evidence that the knowledge and beliefs students attain are determined by their teachers' beliefs and their teachers' instructional practice (Calderhead, 1996).

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THEORETICAL FRAMEWORK AND METHODOLOGY

The research project discussed in this report is part of a larger research project involving a qualitative designed investigation of the stochastics teachers' instructional planning (teachers' individual curricula), the teachers' classroom practices (teachers' factual curricula) and the influence of the latter on students' learning (students' implemented curricula; Eichler, 2007). The results of this qualitative research frame the design of the quantitative survey.

Teachers

The qualitative research (Eichler, 2008) yields four types of (individual) stochastics curricula that are similar concerning the content, but considerably differ with regard to the teachers' objectives or beliefs. The distinction between the four types is characterised by differences of the teachers with regard to two dimensions. The first dimension can be described with the dichotomous pairs of a static versus a dynamic view of mathematics. The second dimension can be described with the orientation on formal mathematics versus mathematical applications. The four types of stochastics teachers were characterised with reference to their main objectives as follows:

(dimension 2)	static view of mathematics (dimension 1)	dynamic view of mathematics
application	*Application preparers*: their central goal is to have students grasp the interplay between theory and applications. Students firstly must learn statistical theory in order to cope with mathematical applications later.	*Every-day-life preparers*: their central goal is to develop statistical methods in a process, the result of which will be both: the ability to cope with real stochastic problems and the ability to criticise.
formal	*Traditionalists*: their central goal is to establish a theoretical basis for statistics. This involves algorithmic skills and insights into the abstract structure of mathematics, but it does not involve applications.	*Structuralists*: their central goal is to encourage students' understanding of the abstract system of mathematics (or statistics) in a process of abstraction which begins with mathematical applications.

Figure 1: Four types of stochastics teachers

With regard to the characterisation of the four types, a questionnaire was developed including four parts. The first part concerns the instructional contents of stochastics courses. The teachers have to mark those contents (given in a list) that they teach in stochastics courses. The other three parts of the questionnaire concern: teachers' objectives of stochastics and mathematics instruction (part 2), teachers' beliefs about the students' benefit of stochastics or mathematics instruction (part 3), and teachers' beliefs about effective teaching of mathematics (part 4). In each of the latter three parts of the questionnaire (parts 2-4) the teachers were asked to rate typical statements of the teachers who represent one of the four types (from full agreement to no agreement, coded with 1 to 5) . In these three parts respectively two statements of every type have to be rated.

Students

Results of the qualitative research yield that the students differ in their knowledge, especially concerning declarative knowledge and conceptual knowledge (Eichler, 2007). The differences consist between the students of one teacher, and between groups of students of different teachers. In regard to these results the questionnaire for the students involves items concerning *declarative knowledge* and *conceptual knowledge* (Hiebert & Carpenter, 1992). Concerning their declarative knowledge, the students were asked to rate a list of 28 stochastical concepts whether they:

- are not able to remember the stochastical concept (coded with 0),
- are able to remember the stochastical concept (coded with 1),
- are able to approximately explain a stochastical concept (coded with 2),
- are able to exactly explain a stochastical concept (coded with 3)

Concerning the conceptual knowledge, the students were asked to indicate interconnections in the consecutively numbered concepts.

In addition to their stochastical knowledge, the students' have beliefs concerning stochastics or mathematics (Broers, 2006). In view of the qualitative research the students showed beliefs about stochastics and mathematics concerning their relevance for society and their relevance for the own life (Eichler, 2007). For this reason, the questionnaire involves several items (open items and closed items) in which the students were asked to indicate examples

- that give evidence for the relevance of stochastics, and
- of stochastical applications in conjunction with stochastical concepts.

METHODOLOGY

A random sample of 240 German secondary high schools was selected. These schools were asked to conduct the survey. 166 of these high schools agreed. From each of these schools two teachers' and for each of the teachers three students with different stochastical performance were asked to fill out the questionnaire (January to July 2007). The completed questionnaires of 110 teachers and 323 students were analysed. Due to the fact that Germany has a low tradition in stochastics teaching, it must be assumed that the survey includes those teachers who have taught stochastics opposed to teachers who have never taught stochastics.

RESULTS

Teachers

With regard to the contents the teachers' individual curricula show some patterns. For instance, nearly 100 % of the teachers indicate to teach the classical block of probability (see figure 2). In contrast, the teachers teach rarely stochastics concepts, e.g. median, correlation, or confidence intervals that are useful to explore real data sets in a descriptive or an inferential mode (see figure 2).

Inferential statistics
E.g. hypothesis testing (89%), confidence intervals (51%), Bayesian statistics (27%)

Distribution
E.g. normal distribution (79%), hypergeometrical distribution (49%), Poisson distribution (32%)

Descriptive data analysis
E.g. frequencies (98%), mean (87%), spread (87%), median (52%), regression (16%), correlation (16%)

Classical block of probability
(≈ 100%)
E.g. Laplacean probability, statistical probability (only 72%), Bernoulli experiment, binomial distribution, expected value, standard deviation

Conditional probability
E.g. conditional probability (81%), (in)dependence (80%) theorem of Bayes (74%)

Figure 2: Percentage of teachers teaching different instructional content

While the teachers' individual curricula are similar in respect to the contents, they differ considerably with reference to the instructional goals. Factor analysis concerning the objectives of the teachers' stochastics curricula yield three interpretable factors (Table 3) which include 14 of the 24 items referring to the objectives of the stochastics curriculum.

Factor	Factor 1* (5 items, α=0.689)	Factor 2* (5 items, α=0.707)	Factor 3* (4 items, α=0.779)
Interpretation	Traditional curriculum with little reference to real stochastical applications	Curriculum with high reference to real stochastical applications	Curriculum with high reference to process

Table 1: Number of items, Cronbach's alpha and interpretation of factors concerning objectives of the stochastics curriculum

Although these three factors are appropriate to describe differences between the teachers' instructional goals, data analysis yields little impact of the teachers' instructional goals and their students learning (Eichler, 2008). Thus, in the following only two factors concerning the teachers' instructional goals are used. The two factors, examples of the related items, the interpretation of the factors, and the reliability of the factors (Cronbach's alpha) are shown in table 3. In addition to each item the table presents the type of the teacher giving such a statement (traditionalists: T; application-preparers: AP; every-day-life preparers: EP; structuralists: S). Factor 2 includes only the six items of the every-day-life preparers.

About 73% of the teachers predominately agreed with the objectives listed concerning factor 1 (about 32% concerning the objectives listed concerning factor 2). The correlation between the two factors is – 0.13. That means that some of the teachers agree to both the objectives listed concerning factor 1, and objectives listed concerning factor 2.

Factor 1, 8 items, α = 0.678	Factor 2, 6 items, α = 0.706

Curriculum with low reference to real stochastical applications (examples)	**Curriculum with high reference to real stochastical applications** (examples)
The objective of teaching stochastics is to establish a theoretical foundation of stochastics (T)	To teach stochastical methods is only useful, if these methods facilitate to solve real problems (EP)
Students must learn to deal successfully with abstract and formal systems (S)	The main goal of stochastics teaching is the students ability to criticise (EP)
It is possible to include realistic examples into teaching, but the main goal are the mathematical concepts behind these examples (S)	If students understand, that more complex methods help them to cope with more complex real problems they will be motivated (EP)
Algorithmic skills constitute the basis of learning stochastics or mathematics (T)	Students have to learn to cope with real stochastical problems (EP)
Students must be well prepared concerning final exams and studies (T)	For students, math must be a part of an universal problem solving ability (EP)

Table 2: Examples of the items concerning factor 1 and factor 2

Students

In respect to the clusters of contents shown in figure 2, the students estimate their ability to explain stochastical concepts concerning the *classical block of probability* to be very good. In contrast the students did not remember concepts regarding descriptive statistics (e.g. the median) or regarding inferential statistics (e.g. confidence intervals), or, respectively, estimate their ability to explain these concepts to be very low (for details see Eichler, 2008).

Although the students estimated their declarative knowledge by themselves, these estimations give evidence of the students' factual knowledge. The students were asked to give examples concerning connections between stochastical concepts (CON1), stochastical applications (AW), the use of stochastics into reality (NU), and connections between stochastical concepts and stochastical applications (CON2). The variables CON1, AW, NU, and CON2 count the number of the examples the students indicate. In addition the variables AW, NU, and CON2 were divided whether the examples given by the students concern game of chance (…_g; including classical random generators, e.g. urns, cards, dice, etc.) or whether they concern realistic stochastical applications without games of chance (…_a). The Pearsons's correlations between the students' declarative knowledge (d.c.) and the value of the variables described above are shown in Table 6. In this table the *weighted* declarative knowledge, i.e. the quotient of a student's self estimated declarative knowledge and the number of stochastical concepts taught by his teacher, is used. In table 6, the sample is restricted to those 201 students, of which there exist the completed questionnaires of both the teachers *and* the students.

n=201	CON1	AW_a	AW_g	NU_a	NU_g	CON2_a	CON2_g

d.c. (weighted)	0,335**	0,079	-0,144*	0,266**	-0,200**	0,208**	0,103

Table 3: Pearson's correlations between students' knowledge and 7 other variables

The correlations are predominately weak, although some of them are significant different from zero. However, the correlations as a whole give evidence that the students' self estimated declarative knowledge (d.c., weighted) measure in some sense the students' flexibility of dealing with stochastical concepts. Further, there is evidence that the higher the students' flexibility of dealing with stochastical concepts is the higher their reference to realistic stochastical applications is, and the lower the reference to games of chance is. However, all the students indicate few examples for stochastical applications (AW, figure 3) or connections between stochastical concepts and stochastical applications (CON2, figure 3).

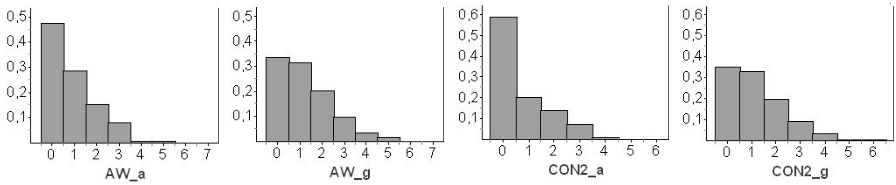

Figure 3: Relative frequencies

Teachers - students

To evaluate the influence of the teachers' teaching on the students learning, the teachers was divided in two groups with high reference to factor 1 (or factor 2) and with low reference to factor 1 (or factor 2) In each case the average of the students' knowledge concerning the variables discussed above concerning the two groups of teachers were compared by a t-test. Table 5 shows only variables with significant differences concerning the two groups of students.

The results of the data analysis give evidence that a teacher's high reference to a traditional curriculum (factor 1) has a negative impact on the students self estimated knowledge. Further, a teacher's high orientation to a curriculum with high reference to stochastical applications (factor 2) has a positive effect

- on the students self estimated knowledge,
- the students' ability to indicate stochastical applications to explain the relevance of stochastics, and
- the students' ability to indicate interconnections between stochastical concepts and real stochastical applications.

n = 201	Factor 1	Factor 2			
Students of teachers	d.c. (weighted)	d.c. (weighted)	NU_a	NU_g	CON2_a

with *high* reference to the factor	$\overline{x} = 1{,}9$	$\overline{x} = 1{,}96$	$\overline{x} = 1{,}25$	$\overline{x} = 0{,}68$	$\overline{x} = 0{,}95$
with *low* reference to the factor	$\overline{x} = 2{,}2$	$\overline{x} = 1{,}66$	$\overline{x} = 0{,}88$	$\overline{x} = 1{,}0$	$\overline{x} = 0{,}6$
	$\alpha = 0{,}43\,*$	$\alpha = 0{,}24\,*$	$\alpha = 0{,}45\,*$	$\alpha = 0{,}28\,*$	$\alpha = 0{,}46\,*$

Table 4: Significant results of the averages concerning different variables

It is, however, noticeable that the differences between the averages are always small. Comparison of the students of the teachers with high reference to a traditional curriculum (factor 1) and the students with high reference to stochastical applications (factor 2) yields predominately no significant differences. The students of the teachers with high reference to factor 1, however, indicate significant lesser real stochastical applications to explain the interconnections between stochastical concepts and realistic applications of these concepts.

DISCUSSION

The results of the quantitative survey concerning the curriculum of stochastics teachers and the learning of students give evidence that:

- "The traditional way of teaching statistics, with its heavy emphasis on formal probability" (Broers, 2006, p.4) is still existent in German secondary high schools;
- the teachers' instructional contents are similar, but the teachers' objectives differ considerably;
- the quality of students' declarative knowledge affects their conceptual knowledge and their beliefs concerning the relevance of stochastics;
- the teachers' orientation towards a traditional curriculum (factor 1) seems to have a (small) negative impact on the students' knowledge;
- the teachers' orientation towards a curriculum with high reference to real stochastical applications (factor 2) seems to have a (small) positive effect on the students' knowledge, and also has a positive impact on the students' beliefs concerning the benefit of stochastics to solve real problems.

To the latter result, it is noticeable, that there is evidence (concerning the results of the qualitative research or the result shown in figure 2), that also the teachers with high reference to real stochastical applications seldom let the students explore real data sets.

The change of a traditional stochastics curriculum towards the teaching of stochastics in regard to SR, SL, ST needs a change of the stochastics teachers' thinking (Garfield, 2002). For this reason, it seems to be self-evident to increase research related to the teachers' thinking. The stochastics teachers' thinking is determined by the stochastics teachers' fundamental orientation, i.e. the teachers' system of objectives concerning stochastics teaching or rather the teachers' central beliefs concerning stochastics teaching. Pajares (1992) stated that it could be difficult to change the teachers' central beliefs. One approach to change these central beliefs may start by the teachers' conviction that a changed curriculum actually will promote students' stochastical knowledge, and students' stochastical beliefs. Thus, it would be desirable to have more research into the stochastics teachers' curricula, the students' stochastical knowledge and beliefs, and, in particular, the relations between stochastics teachers' curricula and the students' stochastical knowledge or beliefs.

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