The Languages of Mathematics

4. The Languages of Mathematics[1]

Abraham Arcavi

ABSTRACT

Students of mathematics have to juggle with at least three mathematical languages: rhetoric, symbolic and graphical. Each of these languages have distinct characteristics and can be used in different ways to support, or to alienate, sense-making. How can insights into the nature and characteristics of these languages enlighten mathematics education in all its branches – curriculum development, the practice of teaching, research on learning, teacher education? This talk will provide thoughts, proposals and open questions on these matters.

FOREWORD

An Argentinean comedian used to say "before I talk, let me say some words". Likewise, let me preface the presentation with the following introductory words.

- Although this presentation bears the title of "review talk", it will not consist of a comprehensive recapitulation of the issues related to language. Rather, I will attempt to re-view, that is to look again and to unpack, certain issues related to mathematics education and to reflect upon them.
- Much can be said about mathematics as a language. However, let me state outright: I believe that mathematics is much more than a language, it is a whole culture with dispositions, viewpoints, routines, specific tools, norms and values, and it has a language. I will address this issue: the language**s** of mathematics, stressing the plural: - since I am interested in the different ways in which mathematics is expressed. I will not address deep ontological questions such as "what are symbols?", "what are mathematical objects?" and the like. I will rather refer to the characteristics of the language. According to Anna Sfard's characterization, I may be switching from dualistic to non-dualistic ways of looking at mathematics and back, sometimes with a "Platonistic" approach and sometimes with a

[1] I thank Professor Ravi Subramaniam for inviting me to this conference. It is a pleasure and an honor to be one of the speakers.

discursive approach. I want to think that, in a certain sense, this distinction may not be terribly relevant for the purposes of what I want to discuss.

- I want to refer to the languages of mathematics in relation to thought, to problem solving, to understanding and to communication – but I will not address the questions whether thought, language and communication are indistinguishable or whether one precedes another, I rather concentrate on the main characteristics of the languages of mathematics.
- The discussion of the issues will be mostly carried on the shoulders of examples (taken from junior high school and high school mathematics) and the morals I propose to draw from them.
- Professor Chitra Natarajan in her talk presented a matrix contrasting the goals and tools of the sciences, the arts/humanities and the field of design/technology. As it happened to me before, the comparison triggered my thoughts about the identity of our field of science education: is it a science? Does it belong to the field of arts and humanities? Is it a design area? Probably, science education has elements from all the three. However, I am personally interested in instruction and in the ways in which our academic research results and the products of our reflection can guide and inspire the design of better practices. Therefore a focus of the discussion will lean towards instructional implications of these reflections.

THE LANGUAGES OF MATHEMATICS

Issues and challenges

Let me start with an example, which may be familiar to some of you. If you are not, take a few seconds to let your first intuitions emerge.

> “Consider any rectangle. What would happen to its area if one of its dimensions were increased by 10% and the other decreased by 10%?”

If one requests from students to respond without making any calculation, one encounters many kinds of different intuitive answers. Some say that the area increases, others think it decreases, and some suggest that the increase and the decrease compensate each other leaving the area unchanged. We also may encounter responses like: “the change depends on which dimension is increased and which is decreased”. If one tries numerical examples, calculations show an apparent decrease, however it is only when we resort to symbols that the result becomes obvious and conclusive (Arcavi, 1994). If a and b are the dimensions of the rectangle, its area is ab. After we apply the conditions of the problem, the area will become either $1.1a$ x $0.9b$ or $0.9a$ x $1.1b$. In both cases, the new area becomes $0.99ab$. From the symbols we can read the following: firstly, the area decreases;

secondly, it decreases by 1%. Moreover, the symbols provide a reply to a question which was not explicitly asked (yet raised by some students): it is irrelevant to which of the dimensions we apply the decrease or the increase, the result will still hold. I would like to claim that this solution to the problem is an example of the communicability of the symbolic language of mathematics and it shows how much information can be compressed by it. As characteristics of communicability, I include:

- *Clarity and precision* – the result is unambiguous.
- *Generality* – it applies to all possible dimensions of rectangles, and it can be generalized further to other percentages.
- *Conciseness* – in at least two senses: (a) the symbolic calculation itself is very brief and (b) from the result obtained one can read out a large amount of information, including an answer to a question which has not been asked (the 'direction' of the area change, its exact value, and the irrelevance of the question of which dimension was increased and which was decreased by 10%).
- *Aesthetics* – although beauty is in the eyes of the beholder, for those of you who have a "Hardian" (à la Hardy) soul, I would like to claim that this expression is beautiful.

However, there may be some "dark clouds" in the horizon: all the above depends on how familiar one is with this language and its characteristics. This familiarity includes both manipulating and reading:

- the capability to "model" symbolically the information of an increase of 10% of a certain number as $1.1a$ and the decrease of 10% as $0.9b$. This is not trivial for students, even for those who have some competence with algebraic symbols
- the capability to manipulate symbols, knowing how to apply syntactic rules and
- the capability of reading information out of the symbols in order to conclude that $0.99ab$ means 1% less than ab, and noticing and interpreting commutativity.

Let me share with you an example in which manipulating and reading symbols can become interrelated for some students. This example is based on an event to which I paid attention when I started to become interested in how people make sense of symbols. While simplifying an equation to obtain a solution, a student arrived at the semi-final stage $3x + 5 = 4x$, just one step away from the desired goal of $x = \ldots$. Instead of proceeding mechanically (as she did until reaching that step), namely "subtracting-$3x$-from-both-sides", she stopped and switched to a different mode: symbol reading. She observed that in order to obtain $4x$ on the right from the $3x$ on the left, one would have to add an x. Thus the actual addend, 5, must be the value of that x. Even though the mechanical solution and her reading of the value of x (as performed by this student) may be considered as mathematically indistinguishable, psychologically there is a subtle but important difference. In this case, the student was displaying some familiarity with symbols by flexibly moving from manipulation to reading, an instance of what I call "symbol sense" (Arcavi, 1994).

Let's consider another example, this time from the realm of elementary number theory, which was proposed in this conference as a good candidate for student learning about proofs and proving.

> "Take any odd number, square it and then subtract 1. What can be said about the resulting numbers?"

Again, we will see manipulating and reading but with a twist. The problem can be represented as follows: $(2n–1)^2 – 1$. Then we may proceed to obtain the equivalent expression $4n^2 – 4n$ in order to reach a conclusion. At first sight, the conclusion is that the resulting number is a multiple of 4. However, by rearranging the symbols, we can obtain $4n^2 – 4n = 4n(n–1)$, and if we stop manipulating and start reading, we may notice that the result is always a multiple of 8 (since n and $n–1$ are consecutive integers and one of them must be even). Can we engineer an expression, from which it can be read directly that it represents a multiple of 8? If we go back to manipulations, we can rearrange the symbols in such a way that instead of $4n(n–1)$ we can write it as $8\frac{n(n-1)}{2}$ which is in fact $8k$ ($k \in \mathrm{N}$). Now the reading becomes straightforward: one can read directly from the expression that the resulting numbers are multiples of 8. A closer reading may show more: these are not all possible multiples of 8, but only very special ones - those in which the factor multiplying the 8 is a triangular number. This is yet another example of the power and communicability of the symbolic language, its clarity, its generality and its conciseness.

I would like to stress another aspect of the language which is very often overlooked: freedom with language use. One of the characteristics of a language is that we can use synonyms and paraphrasing to express the same thing in different ways. Is that the case with this language? Namely, in the example above, do we have a different symbolic way to represent it? Although it is not very common, an odd number can also be represented by n. There are no restrictions for doing so, as long as we remember that throughout the problem. The expression now becomes $n^2–1$ and from $n^2–1 = (n–1)(n+1)$, we can read that the resulting number is always the product of two consecutive even numbers. One of these numbers must be a multiple of 4, thus we can conclude that the result is a multiple of 8. This time the expression is more concise, the manipulation required less effort, but we have to read more into the symbols in order to obtain the information sought. Moreover, this time is not so clear, as it was previously, what kinds of multiples of 8 are obtained. The choice of $2n–1$ to represent an odd number, as opposed to n, displays more of the given information built into the choice of symbol, and thus the final outcome lends itself better to exhibit more aspects of the "structure" of the final product. I claim that these aspects of the language are very rarely discussed in instruction.

Issues and Challenges

It can be argued that the characteristics previously described are subtleties of the language that are well beyond the reach of many students. The following figure (see Figure 1) may be an appropriate illustration of the "down to earth" difficulties that the symbolic language poses for many students.

Even if this may be more of a caricature than a true depiction of how students handle the symbolic language, it reflects how symbols may be perceived and handled by many students. These kinds of "mistakes" are appalling, nevertheless, they deserve some serious thought. We can regard these symbolic products as genuine attempts to engage oneself with a practice still nebulous and alien. We can even discern some attempts at making sense of symbolic manipulation, by striving towards consistency of rules within a language that is not fully understood, and by handling "discursive objects" in a spirit believed to be the one expected by the teacher.

Challenge 1**.** At this point, I would like to state a first instructional challenge: How to introduce the symbolic algebraic language in order to support the development of familiarity?

On the one hand, we would like students to appreciate the wonderful characteristics of the language (e.g. clarity, generality, conciseness, aesthetics) but on the other hand, we

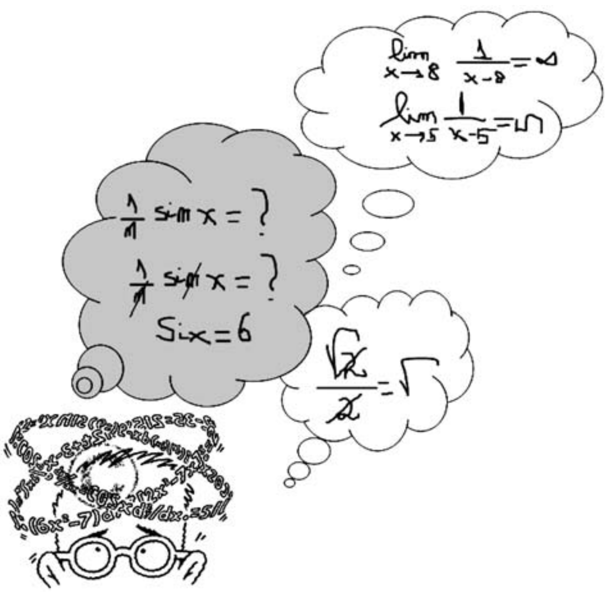


Fig. 1: Illustration of students' difficulties with algebraic symbols

have to face the difficulties students have with something they perceive as an esoteric and meaningless rule oriented endeavor. How can this gap be bridged? I maintain that a main resource for instruction is students' informal wisdom and common sense. Thus, the question becomes: Can we build bridges between informal wisdom / common sense on the one hand, and formal languages on the other? Or, in other words, is it possible to establish a certain contiguity between common sense and formal languages? If so, how? Or, alternatively is there an intrinsic rupture between formal languages and common sense? If we take literally Bertand Russell's famous saying "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true", we may conclude that common sense and formal languages are incompatible. However, I will dare to say that I don't fully believe that Russell, although he was playing the formal game, had no inner sense of what he was talking about. I propose that there is no irremediable rupture between common sense and formal languages, and that an interesting theoretical and empirical research project would consist of "mapping" the extent and the nature of the needed bridge building efforts in different areas within mathematics.

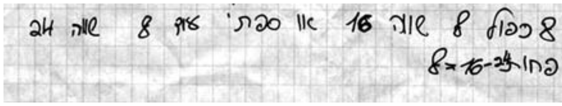
Let us examine some nuances of this issue. Suppose you want to introduce linear equations trying to bridge between common sense and the formal symbolic language of algebra by playing the well known "guess my number" game. For example,

> "I thought of a number, I multiplied it by 3. I subtracted 10 and I got 5.What is my number?".

Most students have no difficulty in finding the number, by thinking backwards from the result: one adds up 10 to the resulting 5, and then one divides by 3. This mental reversal of the operations is common wisdom for most students, including those who are not strongly inclined to mathematics, and who usually become very proficient at this guessing game. What students perform mentally is not far away from what one actually does with the symbols when solving an equation formally. An analysis of the symbolic solution of $3x–10=5$ reveals a strong parallelism to the mental solution (adding 10 and then dividing by 3). So it would seem that, in this case, it is not that difficult to bring the powerful resources of common wisdom and mental solutions close to the symbolic statement of the problem and its formal solution. However, this may be a double edged sword, since students may question the need of an esoteric formal language for purposes which can be achieved so easily by other means. Many teachers reply to such questioning by posing problems in which the mental solution is much harder. Consider, for example, the following.

> "I thought of a number, I multiplied it by 3. I subtracted 16 and I got the number I had at the beginning. What is the number?"

In this case, the equation has the unknown in both sides. The expectation is that since a straightforward mental strategy will not work, the necessity of a formal language would be more natural. However, this may not be always the case. A student from a class of low achievers in mathematics solved also this problem mentally, as follows: if from three times the number I subtracted 16 and got the number, 16 must be twice that number, namely the number is 8. This story questions the usefulness of bringing “harder” problems in order to convince students to move to a formal language. In this case, the teacher still wanted to use this mental solution as a leverage towards a more formal solution, and asked this student to write it down. He wrote the following:



Translating from the original (Hebrew), it says: “8 times 8 is 16, I added another 8, equals 24, minus 8 = 16–24”. This answer reveals genuine difficulties in transferring ideas from a verbal to a written language. Not only does the student write ‘times’ when he apparently means ‘plus’, and ‘8 = 16–24’ instead of ‘24–16 = 8’, but also the flowing and straightforward nature of his original verbal argument seems to be lost (Karsenty, Arcavi & Hadas, 2007).

The above examples show that in this case the attempts at bridging common wisdom and formal languages were not particularly successful, firstly, because some students challenged the need of a formal language, and secondly because organizing one’s thoughts towards formalizing (for example, in writing) can be difficult for students, even for those who are capable of producing sophisticated verbal solutions.

Let me share with you another example, which raises similar issues but with more encouraging results. The following problem was posed more than ten years ago in the matriculation exam (exit exam at the end of the 12th grade) in mathematics which is one of the pre-requisites for obtaining a high-school matriculation diploma in Israel. The following is taken from the lowest of the three levels at which the exam can be taken and it is a example of a typical problem in the topic of arithmetic sequences.

> The sum of the first ten terms of an arithmetic sequence is $S_{10}= 65$. The tenth term is $a_{10}= 20$. Find the first term a_1, and the constant difference d.

The intentions of the designers of this problem were to assess the use of the following knowledge resources:

- recall of the relevant formulae (or their identification from a given list), for example, $S_n = na_1 + \frac{n(n-1)}{2}d$ and $a_n = a_1 + (n–1)d$
- substitution to obtain $65 = 10a_1 + 45d$ and $20 = a_1 + 9d$ respectively and
- solution of the system of equations.

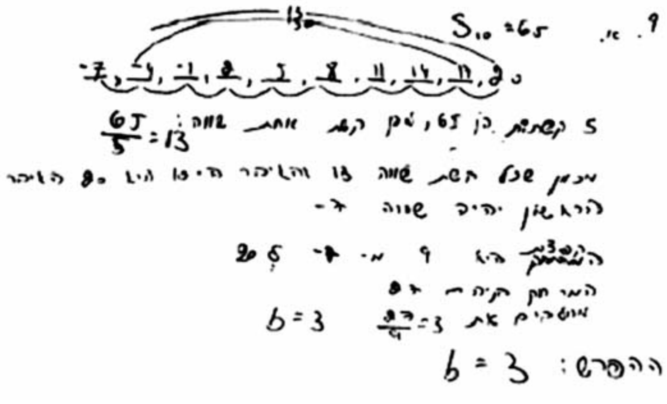


Fig. 2: A non-formal solution

What is the mathematics involved in such solution approach? These resources highlight the absence of conceptual knowledge related to arithmetic sequences (e.g. constant difference, linearity, equal internal sums of pairs of 'symmetrically located' terms). Thus, the problem only assesses knowledge of the algebraic language and the rules thereof. The following (see Figure 2) is a solution produced by a student of the "three units curriculum" (Arcavi, 2000).

Even without knowledge of the Hebrew language, it can be seen that this solution does not make use of any formulae. Instead, there is a predominance of verbal and visual/ diagrammatic elements (Arcavi, 2003). The process of solution is as follows. The student found the first element and the constant difference mostly relying on a visual element: arcs, which he envisioned as depicting the equal sum of two symmetrically situated

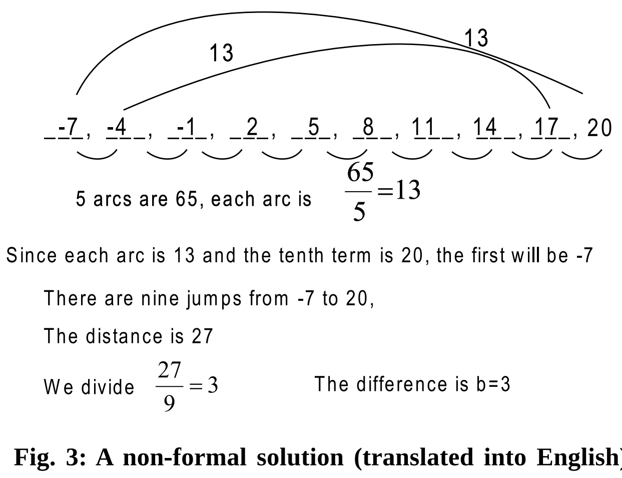


Fig. 3: A non-formal solution (translated into English)

elements in the sequence. Five such sums (arcs) add up to 65, thus one arc amounts to 13. Therefore, the first element must be 13–20 = –7. Then, the student used another visual aid: the ‘jumps’, and wrote that since there are 9 jumps in an arithmetic sequence of 10 numbers, starting at –7 and ending at 20, namely when the “distance” is 27, each jump must be 3. Note that the 27 was the result of calculating a “distance” rather than the result of the formal operation 20–(–7). The following (see Figure 3) shows the solution in free translation into English.

This solution is very different from the previous. The following are its main characteristics:

- It is conceptually immersed in the topic of arithmetic sequences and their properties,
- the tools are informal,
- visualization plays an important role, and
- the conceptual proximity at all the steps helps to naturally monitor the progress made during the solution and its correctness.

As opposed to the first examples on equation solving, this student was able to fully explicate in writing the solution. However, in this case, the gap between the informal knowledge and the formal way to solve this problem seems much larger, providing even further grounds for a potential questioning of the need for formal methods.

***Challenge 2*.** Do mathematically proficient students who reach a good command of the symbolic language fully trust that language? What do they “see” or fail to see in it? And what can be done about it? Consider the following problem requesting the sum of an infinite geometric series.

Find the sum of $\frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + (\frac{1}{4})^4 + (\frac{1}{4})^5 + (\frac{1}{4})^6 + \ldots =$

The usual solution is to apply the formula for the sum of convergent series, namely $\sum_0^\infty aq^n = \frac{a}{1-q}$ $(0 < q < 1)$, where $a = q = \frac{1}{4}$, to obtain $\frac{1/4}{3/4} = \frac{1}{3}$. Usually, this is the end of the solution process. For some students, success in solving such a problem may imply more than just reaching the result – they would want to “see” where the 3 comes from. The answer that it comes from applying the formula is insufficient, it does not fully “speak” to their inner sense of understanding. In such cases, it is our pedagogical responsibility not to leave such seeds of dissatisfaction unaddressed and to go beyond a flawless symbolic solution. The demand of further explanations should be heard and we need to provide (or at least attempt to provide) an answer in a spirit similar to that of Papert’s (1980, p.147) masterful visualization of the concise and clear albeit cryptic symbolic solution to the “string around the earth” problem. The following visualization (Mabry, 1999) can address this request (see Figure 4).

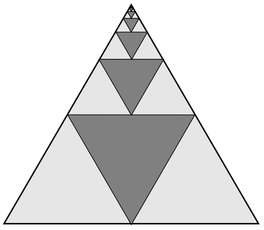


Fig. 4: A visual solution to $\frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + (\frac{1}{4})^4 + (\frac{1}{4})^5 + (\frac{1}{4})^6 + \ldots =$

The largest inverted colored triangle represents one fourth (of the whole), the second largest is one fourth of one fourth, and so on. The colored stream of inverted triangles represents the series itself, and it is 'surrounded' by two other identical series. Thus we can "see" the numerical result of one third emerging in front of our eyes.

Is this geometrical presentation a proof that the result is one third? This is not the important point here. The point is that this is a different language which we propose to present in instruction at least for the sake of those students who, regardless of their proficiency with symbols, want to "see" mathematical phenomena beyond symbolisms.

However, the potential visual power of the graphical language may have some limitations. Consider, for example, what happens if we slightly modify the series and pose the following problem instead.

Find the sum of $\frac{1}{9} + (\frac{1}{9})^2 + (\frac{1}{9})^3 + (\frac{1}{9})^4 + (\frac{1}{9})^5 + (\frac{1}{9})^6 + \ldots =$

The power and beauty of the algebraic language to generate general solutions is indisputable. In this case, we only need to resort to the same formula $\sum_{0}^{\infty} aq^n = \frac{a}{1-q}$ $(0 < q < 1)$ for $a = q = \frac{1}{9}$, to obtain $\frac{1/9}{8/9} = \frac{1}{8}$. Graphically, we need to

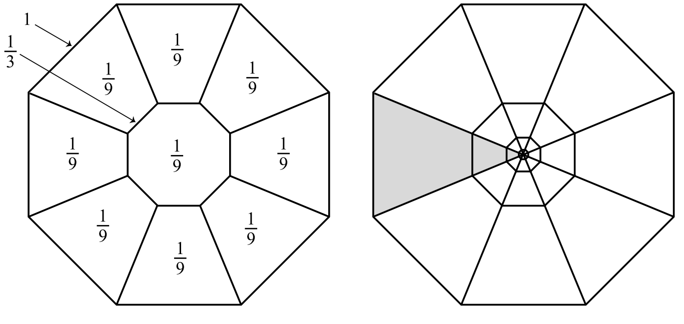


Fig. 5: A visual solution to $\frac{1}{9} + (\frac{1}{9})^2 + (\frac{1}{9})^3 + (\frac{1}{9})^4 + (\frac{1}{9})^5 + (\frac{1}{9})^6 + \ldots =$

work harder in order to find a representation analogous to the one found for the previous case. For example, Tanton (2008) proposed the representation shown in Figure 5.

Thus these two languages of mathematics differ in the different ways in which information can be communicated. Whereas the symbolic language provides us with a general tool, which may be opaque for some, the graphical language may provide with a visual sense of transparency of the phenomena, but it may need to be re-created for different cases. Moreover, it may be impossible to illustrate in one diagram, for example,

$$\tfrac{1}{n} + (\tfrac{1}{n})^2 + (\tfrac{1}{n})^3 + (\tfrac{1}{n})^4 + (\tfrac{1}{n})^5 + (\tfrac{1}{n})^6 + \ldots = \tfrac{1}{n-1},$$

or $$\tfrac{m}{n} + (\tfrac{m}{n})^2 + (\tfrac{m}{n})^3 + (\tfrac{m}{n})^4 + (\tfrac{m}{n})^5 + (\tfrac{m}{n})^6 + \ldots =$$

It would seem that the generality and the transparency of these two languages are at odds with each other: the greater the generality the less their transparency (or conceptual proximity), and vice versa. But is it so, always?

Challenge 3**.** Can generality and conceptual proximity be brought together? Through the process of task analysis (for the purposes of design of a large curriculum project) we worked on instances in which conceptual proximity and generality may go hand in hand, offering students (who have difficulties in handling a fully formal language) useful sense making and general tools. Consider, for example, the slope of a line. Traditionally, the slope of the line which passes through two points (x_1, y_1) and (x_2, y_2) is introduced and applied using the formula $\frac{y_2 - y_1}{x_2 - x_1}$. For many students this general and powerful formula poses numerous difficulties: the use of sub-indices, confusion with the order of the subtraction, identification of which point is which and more. However, this formula may not be necessary for them. All the problems (in the high school syllabus) that involve slope, can be solved by remembering “rise over run”, a visual image corresponding to the formal definition (see Figure 6).

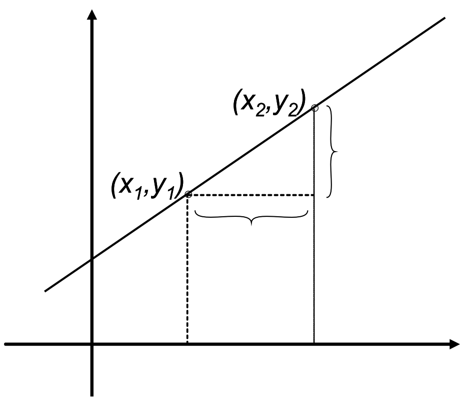


Fig. 6: A visual definition of slope

I claim that, in this case, the visual language has both generality and conceptual proximity. Generality would have to be adjusted for positive and negative slopes, but given any two points (which one does not even need to name formally as (x_1, y_1) and (x_2, y_2)), the figure indicates how to calculate slope in all cases. Moreover, the figure provides a visual reminder of the concept of slope and its entailments (e.g. the larger the rise for a given run the greater the slope, and the steeper the line). This also works for any two pair of points (if the coordinates are large numbers, a schematic sketch can be drawn).

The challenge consisted of finding suitable candidates in which powerful characteristics of two languages (generality and conceptual proximity respectively) can be combined to reduce the "formal" load, taking into account the characteristics of the target population and without harming the scope and depth of mathematical contents to be learned (Arcavi, Hadas & Dreyfus, 1994; Arcavi, 2000).

Challenge 4. Not all graphic and visual representations share the same characteristics. There is a task which became a "classic" in mathematics education task related to the understanding graphic representations. The task (see Janvier, 1981) provides a picture with sketches of several racing tracks and a speed/time Cartesian graph of a certain racing car in its second round in one of the tracks. The task consists of identifying the right track for such car on the basis of the information of the graph. The two graphical representations in this task may share (perception-wise) some visual similarities but they are of completely different nature. Whereas the first is a schematic "photograph" of something out there, the second one is an intellectual construct with very rich conceptual substrates. One of the tracks was chosen as a pictorial distracter in which shapes from both representations can be confused due to perceptual similarities.

The following example also shows some of the deep subtleties of the graphical language, which poses challenges even to mathematically sophisticated students. Consider the following speed/time graphs for two cars (see Figure 7).

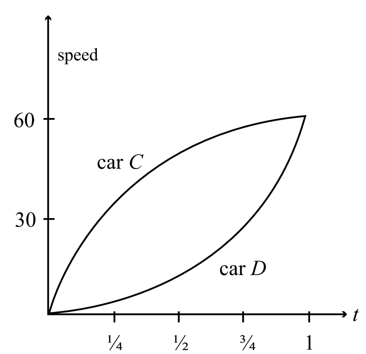


Fig. 7: A speed/time graph

Consider the following question: between three quarters of an hour and one hour are the cars getting closer or farther apart? (Carlson, 1998, p. 129). A first impression would lead us to believe that the cars are getting closer, moving towards a "meeting point". This is certainly correct for distance/time graphs, however in this case the meeting point represents the moment in which the cars reached equal speeds. Thus, contrary to a first and visually luring impression, these cars are getting farther apart during the whole hour, because even when the rate of change of the speed (acceleration) of C decreases and the rate of change of D increases, C still travels faster than D.

Thus, also the visual languages have their own intricacies. In some cases, it can provide an intuitive sense for the "reason" of a certain mathematical result (as in the sum of the geometric series) even if its generality is limited. In other cases, visual languages mask subtle conceptual substrates which may or may not be in conflict with straightforward and direct perception and previous knowledge (see, also, Ben-Zvi & Arcavi, 2001)..

The aspects of the communicability of the symbolic and visual languages discussed so far are: generality, transparency/opacity and conceptual proximity. There are several other aspects, such as changes in perspective (switching within a certain language from one kind of objects to another) and creativity, which will be not addressed here.

WHAT CAN WE DO? - SOME INSTRUCTIONAL HEURISTICS

The characteristics of the languages should deserve much more attention in both the design and the practice of instruction than it is presently the case. The following are some instructional heuristics which I propose in this respect.

Reduce authoritarianism

Language can be a powerful tool for communication. However, language can also be a powerful tool for domination, as illustrated by the following story told by the 19th century mathematician Augustus De Morgan:

> Diderot paid a visit to Russia at the invitation of Catherine the Second. At that time he was an atheist, or at least talked atheism… His lively sallies on this subject much amused the Empress, and all the younger part of her Court. But some of the older courtiers suggested that it was hardly prudent to allow such unreserved exhibitions. The Empress thought so too, but did not like to muzzle her guest by an express prohibition: so a plot was contrived. The scorner was informed that an eminent mathematician had an algebraic proof of the existence of God, which he would communicate before the whole Court, if agreeable. Diderot gladly consented. The mathematician…was Euler. He came to Diderot, with the gravest air, and in a tone of perfect conviction said, "*Monsieur,* $\frac{a+b^n}{n}=x$, *donc Dieu existe; respondez!*" ("Monsieur, $\frac{a+b^n}{n}=x$, whence God exists; answer that!"). Diderot, to

> whom algebra was Hebrew, … and whom we may suppose to have expected some verbal argument of alleged algebraical closeness, was disconcerted; while peals of laughter sounded on all sides. Next day he asked permission to return to France, which was granted. (De Morgan, 1915, p. 339).

This story portrays algebraic language as a pungent source of power "to mystify and intimidate, rather than to enlighten" (Koblitz 1984, p. 254). This can be done in several ways. For example, by means of an unintelligible concatenation of symbols some arguments can be wrapped with highly respected (and feared) formal language in order to paralyze an interlocutor by means of hard to refute "evidence" (for more examples, see Koblitz, 1984). I propose to ask ourselves frequently whether in our practice of mathematics education as teachers or as designers we are using the language of mathematics, or mathematics in general, in an authoritarian way. Moreover, I also urge ourselves to educate to inspecting their own (and certainly other people's) use of the language in order to ascertain understanding and developing a critical appraisal of possible authoritarian misuses. An algebraically literate person, should have defied the symbolic authoritarianism in the story above by questioning the relationship between the formula and the gist of the argument. A central aspect of instruction should not only avoid authoritarian uses of the language, it should also consist of nurturing the ability to question any use, misuse, or abuse of the algebraic language.

It is in this context that I propose to add "education towards democracy" as one of the purposes for mathematics education to the list in Farida Khan's interesting presentation. One of the reasons for learning mathematics is that students should be able to inspect and challenge mathematical language and mathematical arguments that are invoked and used as sources of authority (as it is often the case in the media and elsewhere).

Rethink expertise

One of the aspects of expertise which is seldom discussed and modeled in an authentic way in front of students is the nature of mathematical competence, especially the opportunistic character of expertise. Sometimes experts use one language, sometimes they use another, since expertise allows to flexibly switch from one to the other in the pursuit of efficiency, parsimony or even aesthetics. Examples of flexible uses of language can be found in Arcavi (1994). For a discussion of opportunistic choice of calculation algorithms in arithmetic by mathematicians, see, for example, Dowker (1992). As an example of the extent to which the choice of mathematical language can be tied to the "culture" in which one works, see the section "academic everydayness" in Arcavi (2002).

Legitimate alternative solutions

I have shown above an alternative non-formal yet conceptually rich solution of a traditional problem on arithmetic sequences, and I have contrasted it with the traditional formal solution. I have also shared with you a general visual way to define and apply the concept of slope which may make more sense to students and thus be easier to remember and understand. The legitimization of such alternative ways is very important for improving the learning of low achieving students. It is also important for students who are competent in mathematics. Mathematically proficient students may perceive certain correct solutions as not legitimate and thus not even attempt to pursue them (e.g., Arcavi, 2005, p. 44).

Design appropriately

In the design and practice of instruction, we should dose the level of formalism that we introduce, taking seriously into account the target population. Moreover, it is convenient to precede any formalism with tasks that enable rich conceptual experiences within any language available to the student and only then introduce formal language as it is so carefully done in the Realistic Mathematics Project in the Netherlands (also see, for example, Bruckheimer & Arcavi, 1999 and Arcavi & Hadas, 2000). Similarly, formal solutions (as in the case of the sum of the geometric series presented above) can be supplemented ex post facto with sense-making visual tools, which may enhance students comfort with and trust in the more opaque symbolic language (e.g. Bruckheimer & Arcavi, 1995).

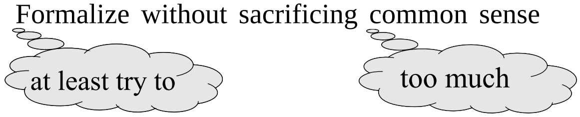
Explicate metamathematical issues

Rarely do we engage in discussions **about** the nature of the mathematical tools we use. It can be informative and productive for students to discuss some characteristics of the languages: what aspects of a certain piece of information are better communicated by symbols or by graphs? How do they compare? Which language is more appealing? When? And why? In Arcavi (2005, p.46), there is an example of such a discussion with low achieving students.

CODA

And finally, a little advice in two versions:

1. Formalize without sacrificing common sense
2. Formalize without sacrificing common sense



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DISCUSSION

Chair: Prof. Parvin Sinclair, IGNOU, New Delhi

Q1. Thank you for a thought provoking talk. You mentioned something in passing that we need to think more about – that mathematics education is possibly design science.... That's another connection between design and technology, mathematics education, science education and so on. My question is about the issue of transparency of symbols, when you bring in the perspective of communication – and this can be done in a very practical sense in the classroom because you can put the pressure, bring in the need to communicate with other students and therefore to make more transparent and therefore to develop and amplify your symbolic resources. Does that add something to the practice of teaching and learning?

AA. Certainly. If you mention the issue of design, I would highly recommend another classic of mathematics education – a paper from 1992 in which a group of researchers and a teacher come to a sixth grade classroom… the design of the task was telling a story about velocity and time, time-space graphing and having the kids inventing representations from that graphing, which the paper tells in great detail. It's an extremely interesting paper, how the kids end up developing Cartesian graphing, it's called re-inventing graphing or inventing graphing. One of the things that the students learn most is not the Cartesian graphing itself but the issue of meta-representations, the issue of how to communicate. Whenever they came up with a certain representation, they were asked by their peers, 'what do you mean?', 'what does it say here?'. So, they have to go back to the drawing and the design and make these transparent to the other. It's a wonderful paper... one of the best in the issue of representation and I recommend it highly as graduate course reading certainly.

Q2. When you talked about the communicability of algebraic language, I'm wondering, are you making the assumption that it is a one-way communication?

AA. No, no, Certainly not. My answer to the previous question was exactly a reply to that one.