What is your family name, Ms. Function? Exploring families of functions with a non-conventional representation

Citation for published version:
Arcavi, A & Nachmias, R 1993, 'What is your family name, Ms. Function? Exploring families of functions with a non-conventional representation', *The Journal of computers in mathematics and science teaching.*, vol. 12, no. 3/4, pp. 315-329.

(article begins on next page)

Jl. of Computers in Mathematics and Science Teaching (1993) 12(3/4), 315-329

What Is Your Family Name, Ms. Function? Exploring Families of Functions With a Non-Conventional Representation

ABRAHAM ARCAVI
Department of Science Teaching
Weizmann Institute of Science, Rehovot, Israel

RAFI NACHMIAS
ORT Israel and School of Education
Tel-Aviv University, Ramat Aviv 69978, Tel Aviv, Israel

The Parallel Axes Representation (PAR) and a computerized environment within which the representation can be explored was described in this Journal (Vol. 9(4), Summer, 1990, p. 79-88). In this paper we present exploration activities and problems related to the concept of linear function which can be performed using the PAR. The goal of these activities is to consider and explore families of linear functions both in the PAR and in the other representations in order to support the construction of the notion of function as an entity which can be manipulated and operated on.

INTRODUCTION

There is a wide consensus in the mathematics education community about the centrality and importance of the concept of function. It is also agreed that, at a certain point of their mathematical experience, students should start to regard a function as an entity. This "entification" of the concept, implies, among other things, grasping a function as a "single object" upon which one can define relations, operations, and which can be grouped into families.

The study of functions is closely related to the use of multiple representations (tables, Cartesian graphs, algebraic symbolism). By understanding and learning to manipulate several representations within themselves, and by translating from one representation to another, different relationships and/or processes may become explicit, facilitating reflection and perhaps leading to further cycles of mathematics building (Kaput, 1987).

In this paper, we illustrate how the use of a non-conventional graphical representation, and the translation from and to it, may support the process towards "entification" of the function concept. We start by introducing the Parallel Axes Representation (PAR), and continue by showing how PAR lends itself well to the representation of families of linear functions. We conclude by suggesting exploration and game activities to support the inquiry of family properties. We invite readers to engage in "doing" while reading (using paper and pencil and/or the computer programs we provide), and we hope the work will be thought provoking and also fun.

WELCOME TO THE *PARTY*

The *Parallel Axes Representation (PAR)* consists of two vertical parallel axes (number lines). The axis on the left is used to represent the domain of the function and the one on the right is used to represent its co-domain. The mapping of a number (pre-image) of the domain to its image in the co-domain is represented by the segment that joins them, called the *mapping segment*. The representation of the function takes the form of a bundle of mapping lines. Figure 1 shows the representation of $f(x)=2x+1$ and $f(x) =-x+3$.

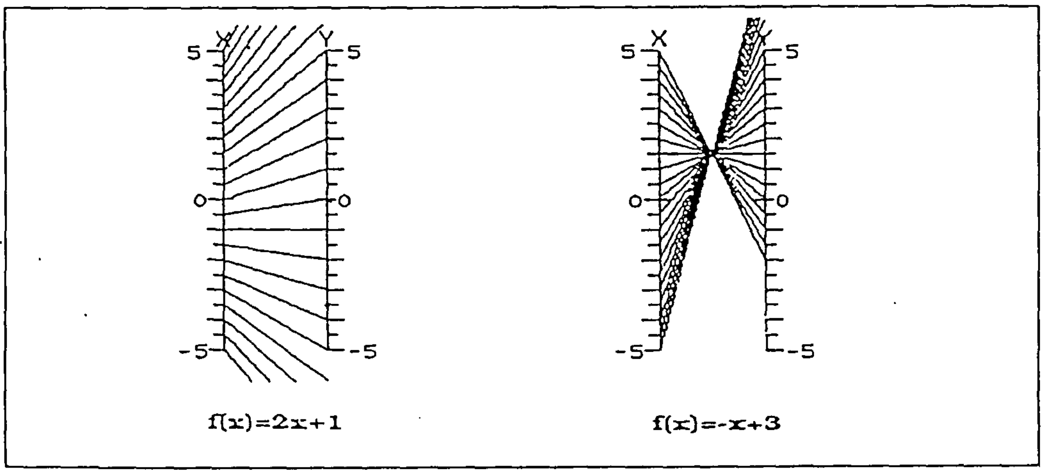


Figure 1.

In the case of f(x)=-x+3 the mapping lines intersect in a single point. If we extend the mapping lines, this is also true for **f(x)=2x+1**, as shown in Figure 2.

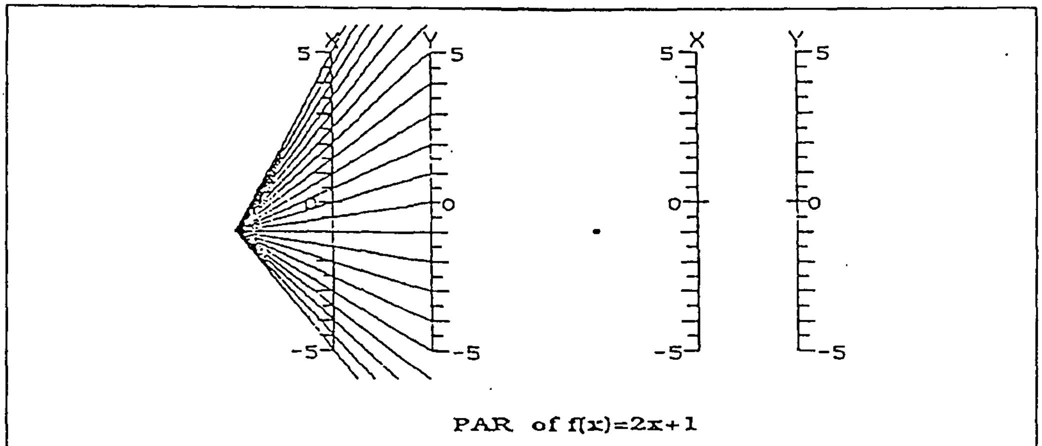


Figure 2.

It can be proven that all the mapping segments in the PAR of a linear function—extended where necessary—always intersect in one point, which we call *focus*. Always?, well..., almost always—when the coefficient **m**, in f(x)=mx+b is 1, the mapping lines are parallel. The converse can also be proven: every bundle of intersecting lines in PAR represents one linear function. Therefore, every point in the PAR plane can be regarded as the focus of a certain linear function (also here we should watch out for exceptions!).

For those readers who, at this point, would like to stop and explore PAR of linear functions by themselves, we suggest the following simple computer program[1] as an enquiry tool. A description of a "fancier" microworld for exploring multiple representations of functions, including PAR, is provided in Nachmias & Arcavi (1990).

```
100 '*Entering the function parameters*
150 PRINT "Enter m and b in f(x)=mx+b";
160 INPUT m,b
200 '*Drawing the mapping lines*
210 LINE (200,10)-(200,210)
220 LINE (250,10)-(250,210)
230 LINE (195,110)-(205,110)
240 LINE (245,110)-(255,110)
300 '*Drawing PAR (unit=50 pixels)*
310 FOR x=-2 to 2 STEP 0.2
320 Y=m*X+b
330 LINE (200, -X*50+110)-(250,-Y*50+110)
340 NEXT X
```

FUNCTIONAL PUNCTUALITY

In this section we will briefly investigate the "punctual" property of linear functions in PAR. A first question may be the following: if a function can be represented by a single point in the PAR plane, how is all the information related to that function, and not to any other, contained in that specific and unique point?

Many times in our PAR investigations, we may gain some insights by contrasting to and comparing with representations we already know and with which we feel comfortable. For example, in the Cartesian plane one way of characterizing the information we need in order to identify the graph of a linear function is to determine its inclination with respect to the positive x-axis—the slope, and its intersection with the y-axis. In other words, we need to determine the **m** and the **b** in **f(x)=mx+b**.

Coming back to the PAR plane, the focus which represents a function, can be uniquely determined, for example, by the horizontal and vertical components of its position. How does this information on the focus position relate to the (Cartesian) slope/y-intercept and to the (algebraic) **m, b** duos?

Let's concentrate in the parameter **m** first. In the Cartesian plane it is the "rise over run", which is the slope of the line. In PAR, one perceptually salient aspect of **m** is the representation of its multiplicative effect in f(x)=mx+b. The image of a domain interval will expand (without reversing) the original interval (whenever m>0). Figure 3 highlights this effect on two domain segments.

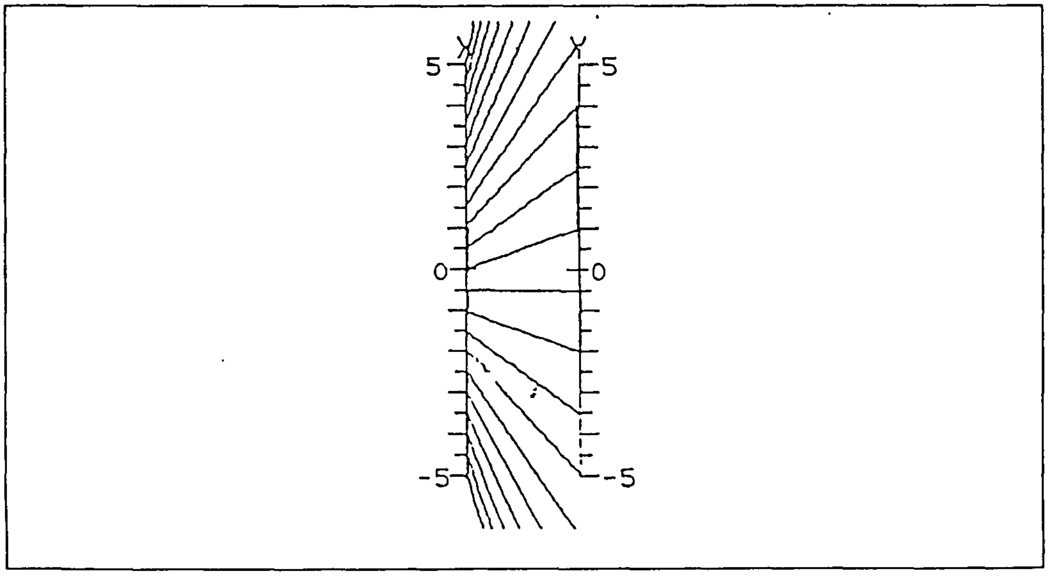
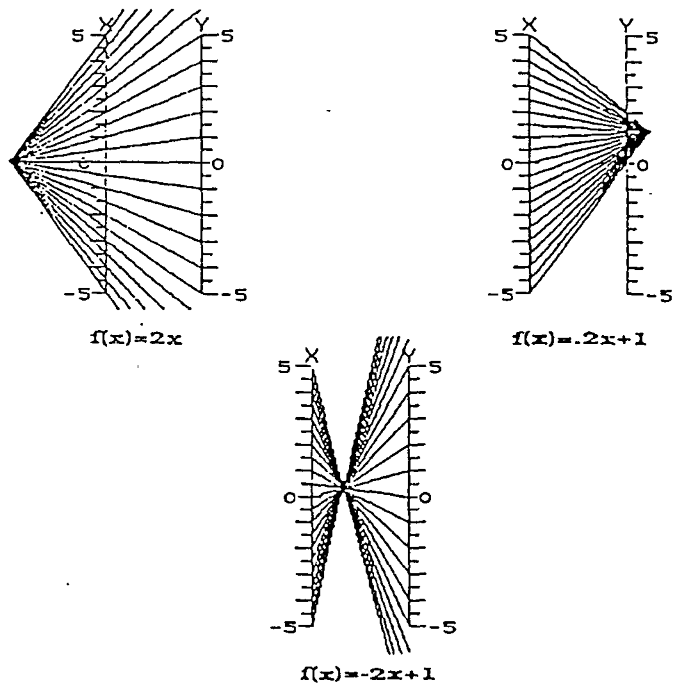


Figure 3.

Thus, the multiplicative effect of **m** is manifested in the direction in which the bundle "converges" (or "diverges"). Different domains of values of **m** determine the location of the focus with respect to the axes. When **m** is greater than 1, the image of an x-axis interval consists of its enlargement (by the factor m). Thus, the focus is located to the left of the x-axis, and the larger the **m** value, the closer to the x-axis the focus will lie. When **m** is between 0 and 1, the image will shrink the x-axis interval, and the focus will lie to the right of the y-axis. When **m** is negative the focus is located between the axes (the absolute value of **m** will determine its proximity to either axis). Finally, as we already implied, when **m=1** the mapping lines are parallel and there is no focus. It is not hard to see that when **m=0** the focus is somewhere along the y-axis, and that no function can be represented by a focus on the x-axis. Figure 4 illustrates different relative positions of foci.



Different relative positions of foci

Figure 4.

So far, we determined qualitatively the position of the focus in relation to the parallel axes. It remains to calculate the distance of the focus to the axes. Let's consider, without loss of generality, the case when **f(x)=mx+b** when **m>1**. We can draw the following diagram.

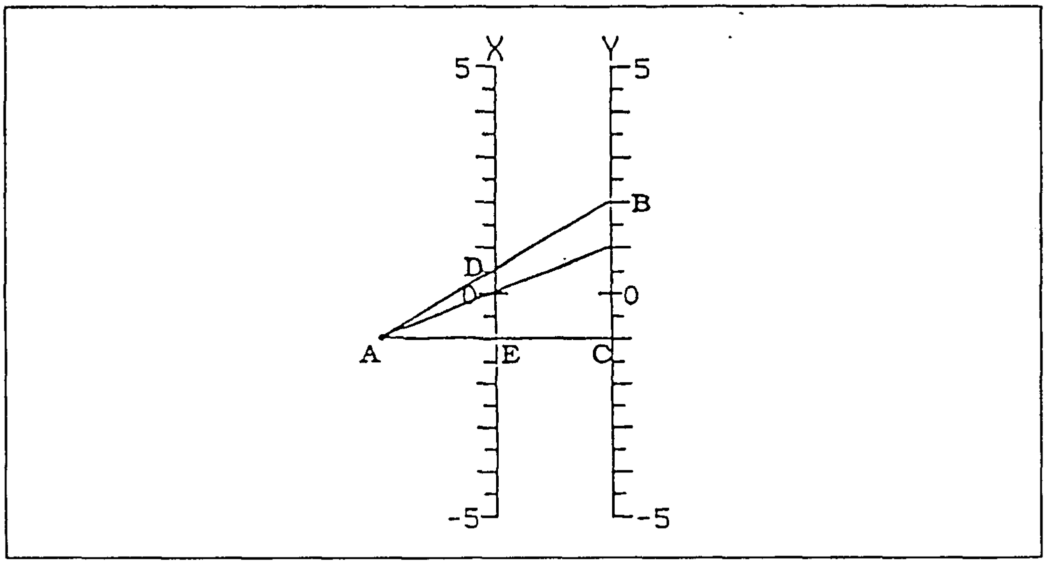


Figure 5.

AE is a very special mapping line: the one which is horizontal (namely, perpendicular to the PAR axes). Obviously, it corresponds to the ordered pair (x,x), which is called the **fixed value** or **fixed point** of a function. In the case of linear functions, this value always exists and is unique[2]. Since the triangles ABC and ADE are similar, then BC/DE=AC/AE. But BC/DE is no other than m, and we can always take EC, the distance between the axes, to be 1.

Therefore $m = \dfrac{1 + AE}{AE}$, whence $AE = \dfrac{1}{m - 1}$ [3]

What about b? The vertical component of the focus can be determined by establishing the location of the fixed value. This we can do by solving the equation x=mx+b. the result is that for a given function f(x)=mx+b, the focus will be at the same "height" (vertical component) as

$$x = \frac{b}{1 - m}$$

It is clear now, that whereas the horizontal component of the focus location depends only on the m value, its vertical component depends on both, m and b.

According to what we found, we can modify our simple computer program in order to assist our explorations of the foci locations. The following program receives the values of m and b of the function f(x)=mx+b as inputs, and draws it focus.[4]

```
150 PRINT "Enter m and b in f(x)=mx+b";
160 INPUT m,b
200 '*Drawing the mapping lines*
210 LINE (200,10)-(200,210)
220 LINE (250,10)-(250,210)
230 LINE (195,110)-(205,110)
240 LINE (245,110)-(255,110)
300 '*Drawing the focus location*
310 IF m=1 THEN GOTO 360
320 Xfocus=(1/(1-m))*50 + 200
330 Yfocus=-(b/(1-m))*50 + 110
350 CIRCLE (Xfocus, Yfocus), 1
360 PRINT "There is no focus when m=1"
```

Further details about the exploration of the parameters m and b in the PAR plane and their rates of change can be found in Arcavi and Nachmias (1990).

PUNC-DUALITY

In the above we came across the PAR property of "condensing" the representation of a linear function to a single point. Now let's examine PAR's "punctuality" in the light of the following "duality". In the Cartesian plane, a point represents an ordered pair, and a collection of points in a line represents the linear function. In PAR, the role of the Cartesian point is played by the mapping line, and the function itself, can be represented by the focus. Thus the duality: a Cartesian point is a PAR line, and a Cartesian line is a PAR point!

In the light of this duality, let's try to understand the meaning of the PAR graph presented in Figure 6.

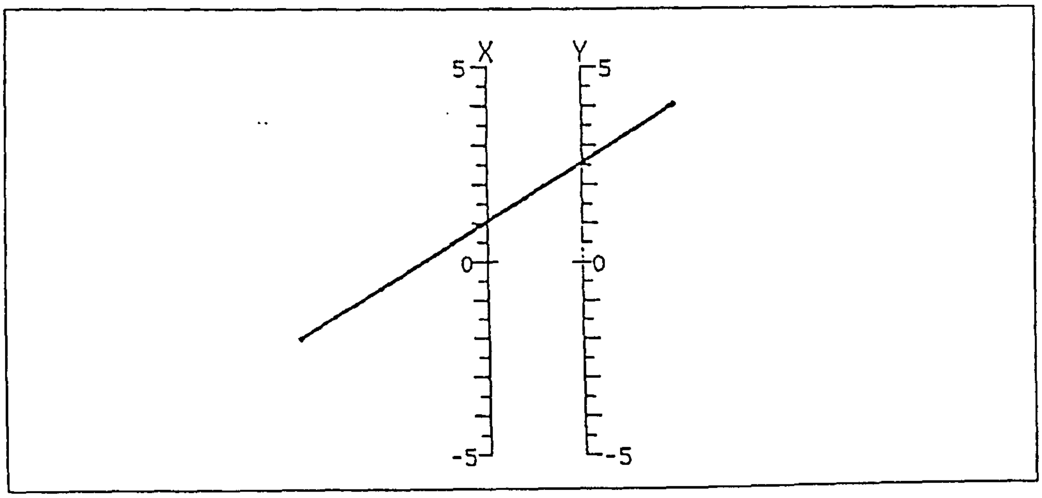


Figure 6.

The graph determines one mapping segment. Its extremes can be regarded as two foci, which correspond to two linear functions. The mapping segment can be regarded as the only one which belongs to the two bundles simultaneously. In other words, it represents the only pair (pre-image, image) shared by the two functions. If we put this in algebraic terms, it represents, in PAR, the solution to the system:

$$y = 1.5x + 1$$
$$y = .5x + 2$$

The line-point "duality" between PAR and the Cartesian representation holds nicely: in the Cartesian plane two lines meeting in a point represent the same idea as two PAR points joined by a (mapping) line.

A second, and subtler, look at Figure 6 above may induce another interpretation. The segment joining the two foci can be regarded as a set of points, each of which (except the one on the x-axis) can be the focus of a linear function. Therefore, the mapping line is now also a set of foci, or in other words, it represents a family of linear functions **f(x)=mx+b** for a collection of **m**'s and **b**'s. In this case, the mapping line represents the family of linear functions sharing (1,2.5) when $.5 \leq m \leq 1.5$ (and $m \neq 1$).[5]

We may stretch our dual thoughts further: if what initially was the representation of a single ordered pair in PAR (a mapping line) becomes the representation of a whole family, the same may hold in the Cartesian plane. Namely, the point (1, 2.5) can also be regarded as representing a family: all the linear functions which contain it.

Now we can look for the algebraic counterpart of the above statement. If we substitute for the variables x and f(x) in the expression **f(x)=mx+b** (which represents all possible linear functions) we will obtain the algebraic representation for a family of functions through a point. In our example, the algebraic family name will be **2.5=1m+b**.

FAMILY TIES

Following what we saw, the name of a certain family of functions can take three different forms: a line in the PAR plane (when considered as a set of foci), a point in the Cartesian plane and an algebraic constraint on the parameters **m** and **b**.

In the following we bring what we believe are thought provoking exercises of translation among the three forms. Consider, for example, the translation into PAR and into Cartesian graphs of the family **2m+b=3**. One way to approach this exercise is to identify individual members of that family, e.g. f(x)=x+1, f(x)=2x-1, etc, and then look for their intersection point, which will turn out to be no other than (2,3). Another way is to just read the intersection point (2,3) out of **2m+b=3**. It is interesting to note that the latter approach enables us to handle a family of functions without identifying any specific member of it.

We leave to the reader the following exercises as practice. Solutions are provided at the end of the paper.

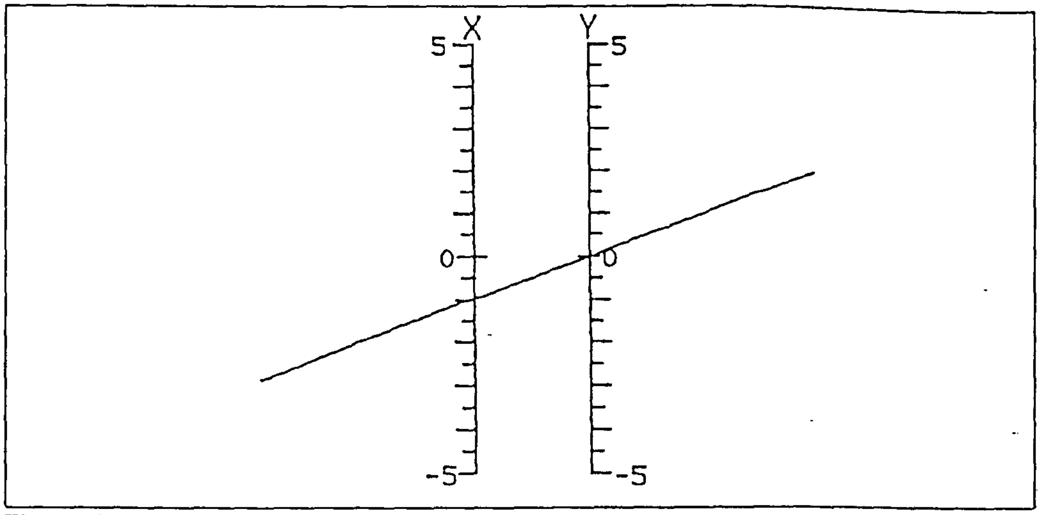


Figure 7.

1. What are the algebraic and Cartesian representations of the family given in the following PAR graph?
2. Translate into Cartesian and PAR terms, the families
   i) m+b=3 ii) b=2 iii) m+2b=4
3. What are the algebraic and PAR versions of a family of functions represented by a point on the Cartesian bisector of the first and third quadrant, say (2,2)?
4. What will be the PAR property shared by the lines representing the following families: m+b=0, 2m+b=0, 3m+2b=0?

So far all the PAR lines we dealt with as representations of families, could be translated into the Cartesian plane as points. What about the PAR line, in Figure 8, as representing a family?

The PAR graph shows that all the foci have the same horizontal component for their location, namely that these are functions which share the same m coefficient. Another way to see the same characteristic is to observe that the line does not intersect the PAR axes, namely there is no

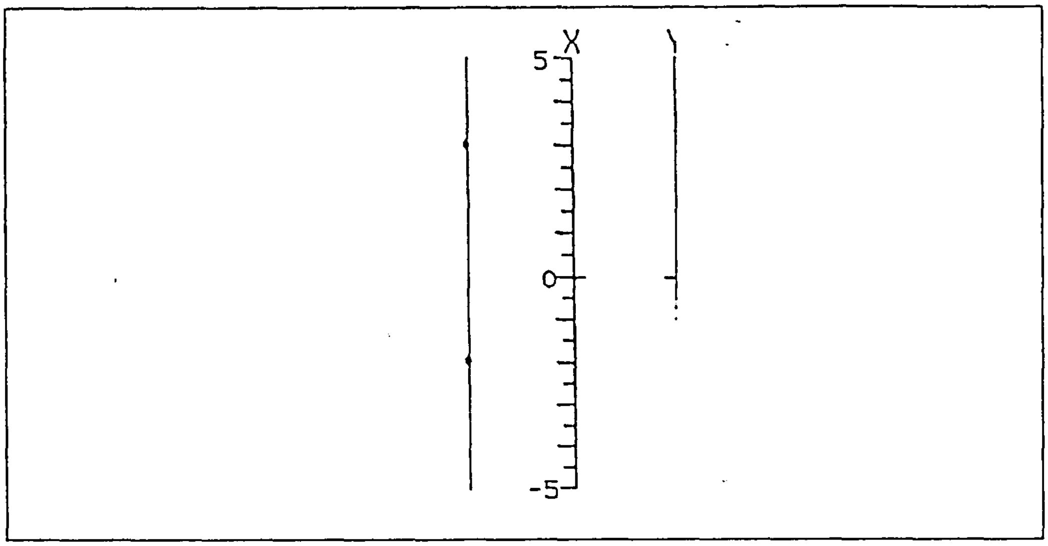


Figure 8.

f(x1)=y in common to this family. In Cartesian terms, we are in the presence of a family of lines parallel to each other. Thus, in this case, there is no Cartesian "dual" to the PAR line, and the algebraic family name is m=2.

REGIONAL FAMILIES

After considering the family m=2, what about m>2? According to what we already know about foci locations, m>2 will correspond to the region of the PAR plane to the right of the vertical PAR line m=2, and to the left of the x-axis (see Figure 8a).

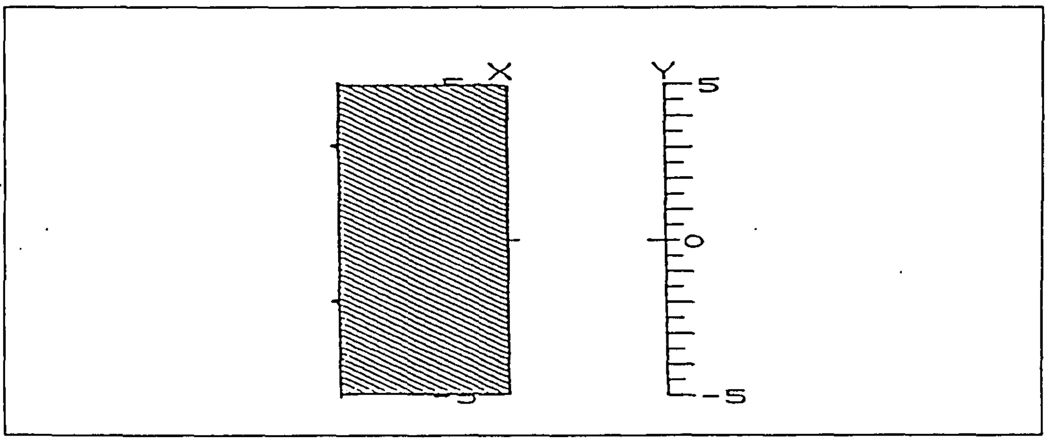


Figure 8a.

Pause and think: What would be the Cartesian counterpart of the PAR region in Figure 8a?

Let's concentrate on family issues in the context of PAR regions. The first regional thoughts which may come to mind refer to what we already know from possible locations of the focus: the family- $0<m<1$ corresponds to the "open" region of the PAR plane to the right of the y-axis; the family $-\infty<m<0$ corresponds to the region between the axes and $1<m<\infty$ to the region located to the left of the x-axis.

If we play with algebraic constraints we can obtain other open regions representing families of functions which are bounded by horizontal and diagonal PAR "border" lines. For example, the region bounded by $2m+b<2$, $b>2$ and the axes (see Figure 9).

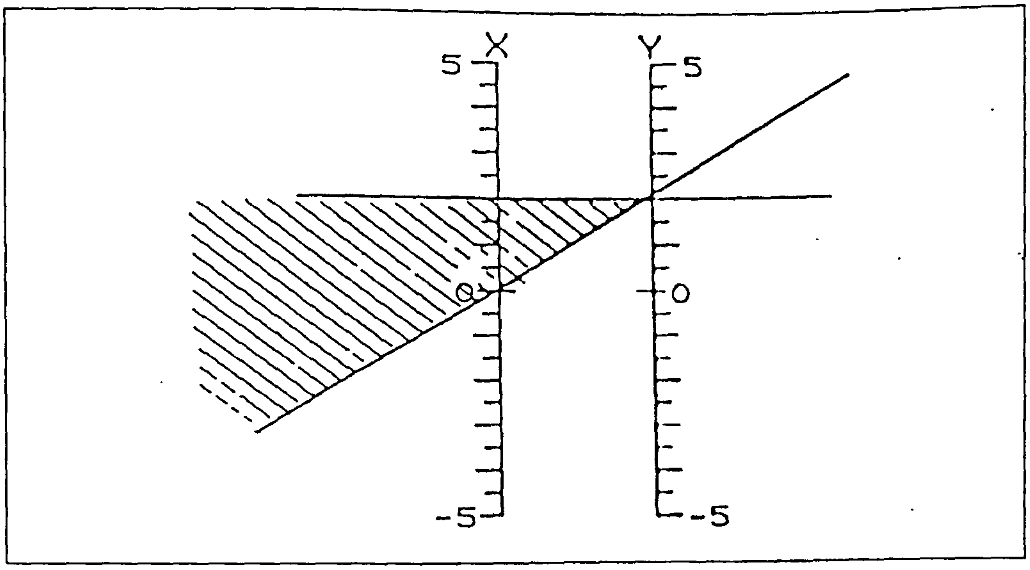


Figure 9.

The possibilities of obtaining PAR regional families of different geometrical shapes seem endless. A new modification of our computer program will assist reader who would like to explore PAR regions effortlessly. The following program is tuned to draw the PAR graph for the family $2 \leq m \leq 3$, $1 \leq b \leq 2$.

```
200 '*Drawing the mapping lines*
210 LINE (200,10)-(200,210)
220 LINE (250,10)-(250,210)
230 LINE (195,110)-(205,110)
240 LINE (245,110)-(255,110)
300 '*Drawing the foci location*
```

```
320 FOR m=2 to 3 step 0.05
321 FOR b=1 to 2 step 0.05
325 IF m=1 THEN GOTO 370
330 Xfocus=(1/(1-m))*50+200
340 Yfocus= - (b/(1-m))*50+110
350 CIRCLE (Xfocus,Yfocus),1
360 NEXT b
370 NEXT m
```

Playing with several constraints will reveal the (expected?) conspicuity of trapezoidal PAR regions. Creating triangular, rectangular, and other regions in the PAR plane in a similar way are not trivial tasks.[6]

A FAMILY GAME

We conclude by presenting a simple computer game which provides an excuse to practice further translations among representations. The computer presents a PAR region. In each turn the player constraints **m** and **b**, in order to choose a family, as a result the corresponding foci are drawn. The objective of the game is "to shoot" as many functions as possible in one turn, towards the given PAR region. The program keeps a positive score for foci falling inside the region and subtract points for each foci outside the target. The following program is tuned to a specific target, but minor changes in line 250 will enable to define other regions as well.

```
100 ' *** A Family game ***
200 ' * Drawing the mapping lines *
210  LINE (200,10)-(200,210)
220  LINE (250,10)-(250,210)
230  LINE (195,110)-(205,110)
240  LINE (245,110)-(255,110)
250 ' *** Drawing the target ***
     Xt=210:    Yt=110:    DX=40:          DY=40
     LINE (Xt,Yt)-(Xt+DX,Yt)
     LINE (Xt+DX,Yt)-(Xt+DX,Yt+DY)
     LINE (Xt+DX,Yt+DY)-(Xt,Yt+DY)
     LINE (Xt,Yt+DY)-(Xt,Yt)
260 ' *** Choice of functions ***
     LOCATE 1,1 :INPUT "m from"; startm
     LOCATE 1,12 :INPUT "to"; endm
     LOCATE 2,1 :INPUT "b from"; startb
     LOCATE 2,12 :INPUT "to"; endb
300 ' ***Drawing foci ***
     FOR m=startm TO endm STEP .1
     FOR b=startb TO endb STEP .1
```

```
            y=m*x+b
            IF M=1 THEN GOTO 100
            mx=(1/(1-m))*50+200
            my=-(b/(1-m))*50+110
            CIRCLE (mx,my),1
            IF mx>Xt AND mx<=Xt+dX AND
my>=Yt      AND my<=Yt+DY THEN
Sco=Sco+1 ELSE          Sco=Sco-1
            LOCATE 1,50: PRINT "SCORE: "; Sco
            NEXT b
            NEXT m
            INPUT w
END
```

Figure 10 shows a typical screen display after one shot

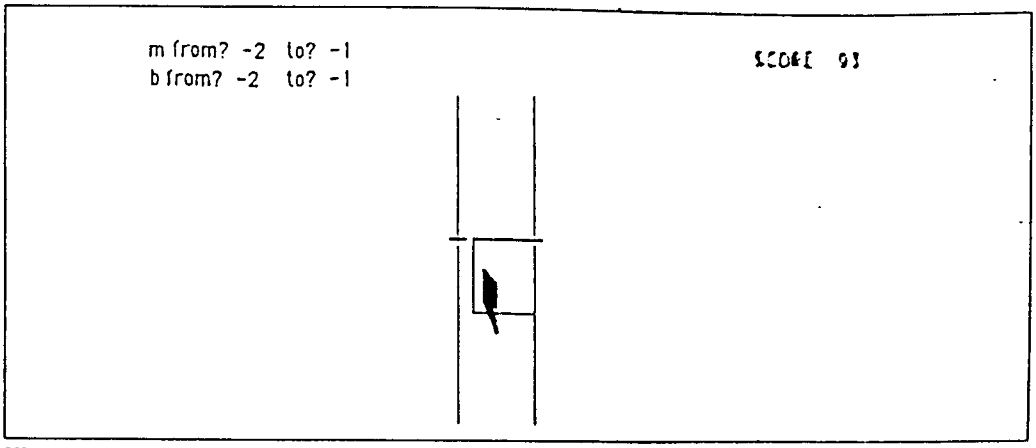


Figure 10.

EPILOGUE

The introduction of computerized tools facilitates the introduction of a new representation, PAR. We suggest that PAR has the potential to further support the consideration of functions as entities, and to focus students attention on families of functions. Whether the computerized introduction of a new representation accompanied by interesting challenges will indeed support the conceptualization of a function as an entity, is still an empirical question. Certainly the learning processes which may take place in such environment will indicate its degree of appropriateness. We can anticipate that PAR in the hands of students will reveal unexpected levels of complexity and surprises as well.

equivalent in PAR, is to have all the mapping lines as horizontal. Or to have no horizontal lines at all, respectively. In both cases there is no focus.

3. Our reasoning was based on M>1, but the formula can be shown to be general. Take for example, M=-1 or m=.5 and according to the formula the distances will be -.5 and -2 respectively. The resulting minus sign has to be interpreted as an indication of "to the right of" the x axis.

4. Since the upper left corner of the screen is taken to be the (0,0) a sign adjustment needs to be made in line 340.

5. If we extend the mapping line beyond the two foci, the family will comprise *all* the linear functions for which f(1)=2.5, except for f(x)=1+1.5.

6. The program as presented takes the **m** and **b** values to be independent. The loop definition needs to be changed if one wants to establish a functional relationship between them.