Using self-video-based conversation in training mathematics teacher instructors. Accepted to Research Report

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Proceedings

Of the 42nd Conference of the International Group for the Psychology of Mathematics Education

Editors: Ewa Bergqvist, Magnus Österholm, Carina Granberg, and Lovisa Sumpter

Volume 4

Research Reports Pr – Z

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EXPLORING PERSPECTIVES ON MATHEMATICAL MODELLING: A LITERATURE SURVEY

Armando Paulino Preciado Babb[1], Armando Solares Rojas[2], Fredy Peña[2], Andrea Ortiz[3], Marisol Sandoval Rosas[3], Remedios Soriano Velasco[3], Vicente Carrión Vázquez[3], and Mauricio Farrugia Fuentes[4]

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Mathematical modelling has a long tradition in mathematics education and has been gaining international attention, not only in research and practice, but also in official perspectives reflected explicitly in programs of studies around the world. Despite extensive publications on diverse aspects of mathematical modelling, systematic literature surveys on this topic are scarce. We highlight some qualitative results from a systematic survey of 452 publications related to different perspectives on mathematical modelling, extending and complementing previous reviews of the state of the art. In particular, we elaborate on the notion of 'authenticity' and the purposes of mathematical modelling in education. Finally, we identify two trends in Latin American countries.

NEED FOR A SURVEY

The roots of mathematical modelling, as per Niss, Blum and Galbright (2007), date from the late 1950s when mathematical modelling advocates attempted to restore focus on the utility and applications of mathematics in schools and universities. By the 1970s, several countries incorporated mathematical modelling their curriculum. A key moment for the international move toward mathematical modelling in education was the inauguration of the biennial *Conference on the Teaching of Mathematical Modelling and Applications* in 1983, organized by the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA). Another key event was the publication of the 14th *International Commission on Mathematical Instruction* (ICMI) study (Blum, Galbraith, Henn, & Niss, 2007). Since then, international research has increased significantly, and research methods and focuses have extended beyond traditional approaches (Stillman, Blum, & Kaiser, 2017). Current international focuses on science, technology, engineering and mathematics (STEM) education have also stressed the importance of mathematical modeling (English, 2015). Despite the large number of publications on mathematical modelling in education, systematic reviews in the literature are scarce.

Different perspectives have influenced the integration of mathematical modelling in educational contexts. In particular, Kaiser & Sriraman (2006) proposed a classification

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consisting of six perspectives, as described in this paragraph. The *realistic* perspective aims to solve real-life problems beyond mathematics: authentic problems from industry and science are particularly relevant here. In contrast, the *epistemological* perspective focuses on the development of mathematical theories, and includes intra-mathematical models that are used to advance theory in mathematics. The *educational* perspective, based on an integrative approach (Blum & Niss, 1991), considers different aims for modelling that serve scientific, mathematical and pragmatic purposes harmoniously. The *contextual* perspective, also called the model-eliciting approach, focuses on problem solving activities constructed using specific instructional design principles. According to Kaiser and Sriraman, with this approach, "students make sense of meaningful situations, and invent, extend, and refine their own mathematical constructs" (p. 306). The *socio-critical* perspective emphasises the need to develop a critical stance towards the role and nature of mathematical models, as well as their impact on social issues. The *cognitive* perspective on modelling is transversal to the previous five and focuses on cognitive aspects of the mathematical modelling process.

Due to this diversity of perspectives, it is difficult to provide a single definition for mathematical modelling. In this paper, we highlight two common elements of a *mathematical model* consistent across diverse perspectives, namely: a situation or phenomena of interest, commonly but not exclusively, from the world beyond mathematics; and a collection of mathematical entities and relationships that correspond to certain aspects of the situation or phenomena of interest. The collection of entities and relationships is often represented visually and can be manipulated and studied with mathematical tools to make predictions or inferences about the situation or phenomena of interest. Modelling can therefore be understood as the creation or the application of a model to solve a problem, make predictions or estimations, study certain phenomena, inform decisions, or even create policy (Blum & Niss, 1991).

Because there are diverse perspectives and purposes related to mathematical modelling, it is important for researchers, teachers, administrators and policy makers to understand and be explicit about the differences among such perspectives. In this paper, we synthetize some results from a literature survey that includes key publiccations in journals and books. We focus on aspects that complement some of the key reviews of the state of the art in this field, including Latin American trends.

SURVEYING THE LITERATURE

This survey is the result of a seminar consisting of graduate and undergraduate students and educators from the National Pedagogical University and the Mathematics Education Department of the Center for Research and Advanced Studies in Mexico, and the University of Calgary in Canada. We identified multiple perspectives on modelling and found the literature on this topic to be vast. We also noticed that systematic reviews, such as the one conducted by Frejd (2013), were scarce; this influenced our decision to conduct our own review. This paper presents insights from an exploration in the literature focused on the different perspectives on mathematical modeling. Kaiser and Sriraman's (2006) widely cited classification served as a point of contrast to identify salient themes in this exploration.

We initiated our survey by searching peer-reviewed articles with 'modeling' or 'modelling' in the title through the SpringerLink database. Then, we refined the search using 'Education' as discipline and 'Mathematics Education' as subdiscipline for each of the two words. Book reviews and other articles that did not relate to mathematical modelling were excluded, resulting in a list with 73 articles. This list can be considered as representative of the literature because: (a) Springer publishes many of the most influential journals in mathematics education identified by Toerner and Arzarello (2012), and (b) searching the key words in the titles suggests that mathematical modelling is a main focus for the selected articles. The list served as a starting point for the survey, and the initial analysis not only helped to clarify and refine the categories that guided the review, but also allowed for the identification of key publications in books and articles in special journal issues.

In a second stage of the survey, we included: (a) articles from the special issues on mathematical modelling, and articles published in 2017 not included previously; (b) articles from the *Journal for Research in Mathematics Education* (*JRME*); (c) five books related to ICTMA and the 14th ICMI Study (Blum, Galbraith, Henn, & Niss, 2007); (d) articles from journals on mathematics education published in Spanish; and (e) a recent Latin-American book addressing research on mathematical modelling (Arrieta Vera & Díaz Moreno, 2016).

We chose *JRME* because it is at the top of the list of journals identified by Toerner and Arzarello (2012). The same title criterion as in the first stage was followed to search articles in this journal. The five books related to ICTMA, extracted from its biannual conference, correspond to the series *International Perspectives on the Teaching and Learning of Mathematical Modelling* published by Springer. We included the Spanish journals and the Latin-American book to extend the scope of the review beyond publications in English. The selected journals were *Revista Educación Matemática* and *Revista Latinoamericana de Investigación en Matemática Educativa*, because they are specialized in mathematics education and are the most relevant among the Spanish journals. Similar to our search of the journals in English, we searched for articles with words in the title related to modelling.

A total of 452 documents were included for this paper: 111 journal articles, and 341 book chapters. Here, we report results from a thematic analysis on the perspectives on mathematical modelling as presented in these documents.

EMERGENT THEMES AND PERSPECTIVES ON MODELLING

We identified two themes with strong connection to the perspectives on mathematical modelling: authenticity and purpose. We also identified two common trends in mathematical modelling from Latin American countries from the Spanish literature.

Within the recent literature, there is clearly a debate on the notions of 'authenticity' and 'real world,' commonly invoked by several authors from different perspectives on modelling. In one sense or another, most (and perhaps all) perspectives allude to something 'authentic.' Regarding the realistic perspective, Kaiser and Sriraman (2006) claimed that "modelling processes are carried out as a whole and not as partial processes, like applied mathematicians would do in practice" (p. 305). Something similar could be said for mathematicians and scientists who have developed mathematical theories based on phenomena from other fields, such as financing, chemistry, astronomy or biology. Such theories are often extended to models that are applied to subjects beyond mathematics. In this sense, the process of theory generation can be considered as authentic to the work of mathematical modelling.

With respect to this debate, Jablonka (2007) suggested that authentic mathematical modelling in the classroom can take place "when students and teachers are *bona fide* engaging in a modelling or application activity about an issue relevant to them or to their community" (p. 196). This framing could be related to any of the perspectives in Kaiser and Sriraman's (2006) classification.

A proposal in this debate is to consider elements of authentic modelling within a task. Vos (2011) suggested a definition for authenticity in which components of a task, instead of the task itself, include objects that are "clearly not created for educational purposes" (p. 721). In this sense, many tasks within different perspectives have authentic elements of mathematical modelling. Indeed, many reports in the consulted literature do not include the whole modelling cycle or process due to limitations in implementation. In other cases, the instructional approach does not include the whole process of modelling, or does not start from the 'real world.' For instance, Silva Soares (2015) suggested *model analysis* as a teaching approach in which students analyse an already existing model instead of creating a model from real data.

Most recently, Carreira and Baioa (2017) addressed the concept of 'credibility' rather than 'authenticity' in mathematical tasks. While authors have argued that real life situations have the potential to make the learning experience more attractive, this focus on credibility places the relevance at a personal level for students.

Finally, many publications focused on simulations using computer systems. Simulations are used as a part of the modelling cycle involving real data (e.g. Niss, 2015). However, simulations are also used as models to teach specific content within and beyond mathematics (e.g. Gomes Neves, Carvalho Silva, & Duarte Teodoro, 2011). While students may not engage with real data when using a simulator, they can experiment within the model and learn both mathematical and extra-mathematical content.

Regarding the purposes for mathematical modelling, we identified a list, summarized in Table 1, that extends the purposes considered in Kaiser and Sriraman (2006). Many of these might be included in one or more of the perspectives in this classification. In particular, *awareness of social and global issues*, *participatory attitude*, and *culture of innovation* could be considered purposes within both the realistic and the socio-critical approaches, if we extend their description. For the realistic approach, for instance, the purpose can involve an "ultimate goal," as suggested by Carreira and Baioa (2017).

Purposes To:	Examples
Learn mathematics content	Algebra, Geometry, Calculus, Statistics
Apply mathematics	Problem solving
Learn other disciplines	Chemistry, Biology, Finances, Heath Care
Conduct research	Research on learning in virtual environments
Design learning environments	Design simulators and virtual environments for learning purposes
Develop modelling competencies	Elements of modelling; criteria for quality in mathematical modelling
Develop learning skills	Generalize the solution of a problem to other similar problems
Generate mathematical theory	Conceptual understanding; mathematical proof
Develop critical thinking	Judge models used in daily life; question purpose and assumptions of different models
Understand mathematics as a discipline	Historical, social and political aspects of mathematics as a discipline
Develop awareness of social and global issues	Create and critique models used to predict: economic growth; global warming; tax revenue
Promote a participatory attitude	Engage in addressing real problems and decision-making within the community
Promote a culture of innovation	Create something for a customer; program software for an audience
Engage in emancipation strategies	Decolonization; cultural practices in mathematics and mathematical modelling

Table 1: Purposes for mathematical modelling in education, with examples.

While developing mathematical modelling competencies is a common purpose within the literature, other aspects of modelling are barely considered, such as the quality criteria proposed by Perrenet, Zwaneveld, Overveld and Borghuis (2017), comprising: *genericity*, *scalability*, *specialization*, *audience*, *convincingness*, *distinctiveness*, *surprise*, and *impact*. In fact, most of the modelling tasks do not include constructing or experimenting with physical objects, creating something for a customer, writing computer code, or making a decision that will affect the local community. Niss (2015) proposed the term *prescriptive modelling* that involves designing, prescribing, organizing or structuring certain aspects of the real world. Papers that included these activities could be considered as promoters of a culture of innovation and a participatory attitude. For example, Orey and Rosa (2017) reported a task addressing the issue of tariffs in public transportation.

Finally, a few papers addressed mathematical modelling for research or for developing learning environments, which are not considered as purposes in Kaiser and Sriraman's (2006) classification. Campbell (2011), for example, addressed the use of virtual reality, which requires mathematical modelling to create virtual spaces and objects. In contrast to the other purposes for modelling, students may not engage in elements of modelling in the corresponding learning, or research, environments.

Latin American trends

As we reviewed articles written in Spanish, we identified two main trends on mathematical modeling for Latin America, namely: the number of publications, and the innovative aspects in their approaches. These contributions strongly emphasise the social and cultural influences of modelling education.

Mathematical modelling research from Spanish speaking countries produced a modest number of papers in the publications from Springer. However, the review revealed activity in mathematical modelling in Latin America since the 1990s, and in the case of Brazil, since the 1970s (Salett Biembengut, 2016). This brings into contention Blum and Niss's (1991) claims that mathematical modelling was initially developed in regions such as Germany and the UK.

While the number of publications from Spanish speaking countries has increased modestly in the last few years, publications from Brazil are conspicuous in documents published by Springer. Particularly, the 16 ICTMA Conference, held in Brazil in 2013, resulted in an increased number of authors from the host country.

Regarding innovative aspects in Latin American approaches, Stillman, Blum, & Biembengut (2015) identified elements of "a unique Latin American perspective to modelling" in the work of Brazilian author, Ubiratan D'Ambrosio, who discusses knowledge generation (cognition), its individual and social organization (epistemology) and the way it is confiscated, institutionalised and given back to the people who generated it (politics). His perspective on mathematical modeling extends the socio-critical perspective and is a strategy for building up systems of knowledge in different cultural environments.

Another Latin American modelling trend corresponds to research reported as socio-epistemological (see for instance Arrieta Vera and Díaz Moreno, 2016; Quiroz Rivera & Rodríguez Gallegos, 2015). This approach understands mathematical modelling in terms of social practices, both in school and in formal mathematics.

FINAL REMARKS

This report complements other reviews of the state of the art regarding the debate on authenticity, the purposes of modelling, and Latin American trends. The discussion on authenticity and the identification of purposes for mathematical modelling problematize Kaiser and Sriraman's (2006) classification of perspectives on mathematical modelling. Perhaps, while this classification has been useful in the past, it may be appropriate to pay closer attention to the purposes of mathematical modelling on an individual basis, and consider the elements of authenticity in tasks, as suggested by Vos (2011). These elements of authenticity may vary based on the mathematical content addressed in each task. For instance, modelling with statistics may involve elements of authenticity (e.g. using real data) that differ from the elements of authenticity for modelling with calculus (e.g. using simulations).

References

Arrieta Vera, J. & Díaz Moreno, L. (Eds.) (2016). *Investigaciones latinoamericanas en modelación: Matemática educativa*. Barcelona, Spain: Gedisa.

Blum, W., Galbraith, P. L., Henn, H., & Niss, M. (Eds.) (2007). *Modelling and applications in mathematics education: The 14th ICMI study*. New York, NY: Springer-Verlag.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects: State, trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*(1), 37-68.

Campbell, S. R. (2013). Mathematical modeling and virtual environments. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies: ICTMA 13* (pp. 583-583). New York, NY: Springer.

Carreira, S., & Baioa, A. M. (2017). Mathematical modelling with hands-on experimental tasks: On the student's sense of credibility. *ZDM Mathematics Education*. Retrieved from https://doi.org/10.1007/s11858-017-0905-1

English, L. (2015). STEM: Challenges and opportunities foe mathematics education. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference,* (Vol. 1, pp. 1-16). Hobart, Australia: PME.

Frejd, P. (2013). Modes of modelling assessment—A literature review. *Educational Studies in Mathematics, 84*(3), 413-438.

Gomes Neves, R., Carvalho Silva, J., and Duarte Teodoro, V. (2011). Improving Learning in science and mathematics with exploratory and interactive computational modelling. In Kaiser, G., Blum, W., Borromeo Ferri, R., & Stillman, G. A. (Eds.), *Trends in teaching and learning of mathematical modelling: ICTMA 14* (pp. 331-339). New York, NY: Springer.

Jablonka, E. (2007). The relevance of modelling and applications: Relevant to whom and for what purposes? In W. Blum, P. L. Galbraith, H.W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 193-200). New York: Springer.

Kaiser, G., & Sriraman, H. (2006). Modelling in mathematics classrooms: Reflections on past developments and the future. *ZDM Mathematics Education*, 38(3), 302-310.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. Galbraith, H.W. Henn & M. Niss. (Eds.) (2007), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 3-32). New York, NY: Springer.

Niss, (2015). Prescriptive modelling—Challenges and opportunities. In G. Kaiser & G. A. Stillman (Eds.), *International Perspectives on the teaching and learning of mathematical modelling*, (pp. 67-79). New York, NY: Springer.

Orey D.C. & Rosa, M. (2017). Mathematical modelling in a long-distance teacher education in Brazil: Democratizing mathematics. In G. A. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical modelling and applications: Crossing and researching boundaries in mathematics education* (pp. 577-586). New York, NY: Springer.

Perrenet, J., Zwaneveld, B., Overveld, K., & Borghuis, T. (2017). Quality criteria for mathematical models in relation to models' purposes: Their usefulness in engineering education. In G. A. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical modelling and applications: Crossing and researching boundaries in mathematics education*, (pp. 141-152). New York, NY: Springer.

Quiroz Rivera, S., & Rodríguez Gallegos, R. (2015). Análisis de praxeologías de modelación matemática en libros de texto de educación primaria. *Educación Matemática*, 27(3), 45-79.

Salett Biembengut, M. (2016). Modelaje matemático en la educación brasileña: Historia de la ideas e ideas de las historias. In J. Arrieta Vera & L. Díaz Moreno, (Eds.), *Investigaciones latinoamericanas en modelación: Matemática educativa* (pp. 89-108). Barcelona, Spain: Gedisa.

Silva Soares, D. (2015). Model analysis with digital technology: A "hybrid approach." In G. Kaiser & G. A. Stillman (Eds.), *International perspectives on the teaching and learning of mathematical modelling* (pp. 453-463). New York, NY: Springer.

Stillman, G. A., Blum, W., & Biembengut, M. S. (2015). Cultural, social, cognitive and research influences on mathematical modelling education. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 1-32). New York, NY: Springer.

Stillman, G. A., Blum, W., & Kaiser, G. (Eds.). (2017). Mathematical modelling and applications: Crossing and researching boundaries in mathematics education. In G. Kaiser & G. A. Stillman (Eds.), *International perspectives on the teaching and learning of mathematical modelling*. New York, NY: Springer.

Toerner, G., & Arzarello, A. (2012, December). Grading mathematics education research journals. *European Mathematical Society Newsletter* (86), 52-54. Zürich, Switzerland: European Mathematical Society.

Vos, P. (2011). Practical knowledge of research mathematicians, scientists, and engineers about the teaching of modelling. In Kaiser, G., Blum, W., Borromeo Ferri, R., Stillman, G. A. (Eds.), *Trends in teaching and learning of mathematical modelling: ICTMA 14* (pp. 712-722). New York, NY: Springer.

ARE TEACHERS' LANGUAGE VIEWS CONNECTED TO THEIR DIAGNOSTIC JUDGMENTS ON STUDENTS' EXPLANATIONS?

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When teachers analyze students' explanations in a language-responsive mathematics classroom, they explicitly and implicitly activate various categories and hold different views on language and mathematics learning. This study investigates how typical views on language in mathematics classrooms are related to what teachers consider relevant in their diagnostic judgments, in this case on students' explanations of the slope formula for linear functions. Seventy-eight teachers' personal constructs were elicited using a diagnostic activity and related to their self-reported views on language and mathematics learning. The group of language reducers can be shown to focus significantly more on the surface level of language whereas the language pushers *group focuses more on the discourse level. In contrast, worries about language responsiveness being time consuming do not seem to influence diagnostic judgments.*

Due to the increasing language diversity in mathematics classrooms, fostering language learners' mathematics learning has become a major task for mathematical teachers all over the world. Although the teachers' crucial role in language-responsive classrooms has often been acknowledged in classroom research, research surveys show that few studies have been conducted on *teachers'* resources and the obstacles they face in developing language-responsive mathematics classrooms (Radford & Barwell, 2017; Barwell et al., 2016)

In this paper, mathematics teachers' resources and their obstacles when developing their language-responsive classrooms are conceptualized by typical *views* (on language and mathematics learning) and mathematics- and language-related *categories* they apply when noticing students' products (following Sherin, Jacobs, & Philipps, 2001). We report on an inquiry with mathematics teachers (n = 78) that investigates their personal categories elicited in comparative diagnostic judgments which pursues the following research questions:

Which categories do teachers activate when conducting diagnostic judgments on students' written explanations of slope?

How are these categories connected to the teachers' views on language?

THEORETICAL BACKGROUND: TEACHERS' PERSONAL VIEWS AND CATEGORIES IN LANGUAGE-RESPONSIVE CLASSROOMS

As Sherin et al. (2001) have argued, teachers' classroom practices rely heavily on what they notice in classroom complexity. Noticing in language-responsive mathematics

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classrooms starts with diagnostic judgments on students' utterances and products, especially on students' written explanations of mathematical concepts, because explaining mathematics concepts is the most important discourse practice, especially for language learners (Moschkovich, 2010).

Personal views and personal categories in teachers' diagnostic judgments

Sherin et al. (2011) describe noticing as a complex process "through which teachers manage the blooming, buzzing confusion of sensory data with which they are faced" (p. 5). Hence, noticing selected aspects involves filters of perception and consists of two processes: "attending to particular events in an instructional setting" and "making sense of events in an instructional setting" (ibid., p. 5). For the second process of making sense, they hint at the relevance of categories: Interpreting means "relating observed events to abstract categories" and characterizing what they see in terms of familiar phenomena (ibid., p. 5). As Prediger and Zindel (2017) have shown that extrapolating the personal categories in teachers' diagnostic judgments can be an interesting research approach for unpacking their thinking and noticing. As the personal categories that teachers activate in diagnostic judgments can be widespread and very individual, they cannot be captured by predefined items but should be elicited with open-ended diagnostic activities.

Beyond the specific categories, teachers' diagnostic judgments can be influenced by their general views on the topic in view, in our case views on the role of language in mathematics classrooms and the individual interpretation of what language-responsiveness might mean (Short, 2017).

Categories and language views relevant in language-responsive classrooms

Although Moschkovich (2010, p. 160) recommended investigation of teachers' judgements in language-responsive classrooms, little so far is known on teachers' personal views and categories about language and mathematics learning. This can be partly explained by the fact that most research studies on language are conducted as classroom observation studies (Radford & Barwell, 2016).

Although teachers' diagnostic judgments have not been investigated directly, these classroom observations provide important hints on potentially crucial categories and views. As the research on language in classroom observation studies has shown, the role of language in mathematics classrooms cannot be adequately grasped on the *word level* alone, as it is tightly connected to the *sentence level* and especially the *discourse level*: Classroom studies in linguistics and mathematics education research have shown that the epistemic function of language (i.e., the role of language for higher order thinking practices) is mainly reflected by participation in classroom *discourse practices*: Language learners in their early stages of language proficiency can have difficulties participating in classroom practices such as explaining meanings or describing general patterns, whereas they can participate in reporting procedures or describing by examples (Moschkovich, 2010). Thus, categories on the discourse level are crucial for focusing relevant phenomena, as they allow for integrating language and mathematics learning in much deeper way than simple categories on the language surface level (such as orthography) or the word level (such as identifying relevant technical terms).

A typical view that might hinder language learning concerns the interpretation of language-responsiveness as language simplification: In order not to exclude language learners from mathematics learning, many teachers tend to reduce the language demands by simplifying all texts and reducing the production expectations to keywords and half sentences. In contrast, language education research has emphasized the requirement of comprehensible but demanding language input and pushing much language output in order to enhance language learners' learning opportunities (e.g., Short, 2017). Pushing language in the zone of proximal development thus seems to be an important overall view on language in classrooms.

In general, each classroom innovation can be hindered by teachers' view that this innovation is too time consuming. Understanding the backgrounds of time worries might therefore help to overcome them and increase the chance that teachers adopt approaches of language responsiveness. Existing case studies have led to the hypothesis that fewer time worries may be held by teachers who see how language and mathematics are deeply connected, which means that they are already addressing the discourse level on which content- and language-integration mainly occurs.

In order to investigate how categories and views are connected, the following two hypotheses are tested in this study:

H1 Those teachers who worry that language responsiveness is time consuming focus less on the discourse level and more on surface levels than those who do not worry.

H2 Those teachers who try to reduce language in their classroom focus less on the discourse level and more on the surface levels than those who push language.

METHODS

Methods of data gathering

Sample. The sample consisted of German middle and high school mathematics teachers (n = 78) in their first session of a volunteer professional development series on language-responsive mathematics classrooms. The teachers had 2-30 years of experience in math teaching (with a median of 6-10 years) and between 0 hours and several days of previous encounters with ideas of language-responsive classrooms (with a median of 6-8 hours).

Questionnaire for general views on language in math classrooms. General views were captured using e.g. the following items:

- *Language reducer vs. language pusher*: *"*For language learners, I try to reduce the language."
- *Time worrier vs non-worrier*: *"*Language responsiveness is an additional task for math classrooms which steal us much time from mathematics learning."

Teachers' views captured on the six-point Likert-type scales allowed the formation of sub-samples of *language reducer* vs. *pusher* and *time worrier* vs. *non-worrier*, containing those teachers who selected strongly disagree/quite disagree or quite agree/ strongly agree, without those who chose partially dis-/agree.

Diagnostic activity. Eliciting the teachers' implicit personal categories on language and mathematics followed the variation principle: Diagnostic judgments were requested for three contrasting students' explanations (Fig. 1). Teachers were asked to name their criteria, evaluate the three texts according to them, and justify their evaluation.

Diagnostic Activity

The Grade 8 class has introduced the meaning and the formula for the slope of linear functions.
The homework task (formulated as on the right) requested to write a summary.

> Explain what $m = \frac{y_2-y_1}{x_2-x_1}$ means and what you can describe with it.

Analyze the three students' texts using four self-defined categories, two for mathematical aspects and two for language aspects. Evaluate by 0,1 or 2 points and justify your decisions.

Ali	Suleika	Tom
You want to calculate the slope. First you chose two points, e.g. x=3, y=1 and x=5, y=8. Second, you evaluate: $m = \frac{8-1}{7-3} = \frac{7}{2}$. Ready	The slope saying, how much growing the function per x-step. Thus, how much get y more per how much get x more. The PER makes the DIVIDED. 	Iff yu for exempel have y=0.2+10 of the mobile phone tarif. For exempel at 10 minutes it is 12 €, at 30 minutes it is 16 €. Thus, distance 20, price more 4 € and than $\frac{4\,€}{20\,min} = \frac{0.2\,€}{1\,min}$. Thenn, costs per minute my price 0.20 € more costly.

Fig. 1: Diagnostic activity for teachers (translated from German with errors preserved)

The slope of a linear function provides a mathematically rich exemplary topic for students' explanations. This topic demands not only procedural knowledge when evaluating the slope formula for specific values but also conceptual knowledge explaining its meaning as a whole (the slope captures how much a function grows) as well as the components of the quotient: The ratio of two distances is an interpretation that requires conceptual understanding of different arithmetic models (Usiskin, 2008). Four discourse practices can be distinguished here, reporting procedures, explaining meanings, general phrasings, and concrete phrasings (Suleika explains meanings in a general way, whereas Ali reports procedures concretely). The three students' texts were chosen to show a wide spectrum of language features on the surface level (e.g., orthography), word level (technical terms), sentence level (grammatical structures), and discourse level (with the four discourse practices mentioned).

Data analysis procedures

The manifold personal criteria that teachers stated for the diagnostic activity in Fig. 1 were analyzed by a specifically developed categorial scheme. The first version of the categorial scheme was derived from the current state of research and then adapted to the data in order to capture all personal criteria. As the teachers used the same words for criteria with different individual meanings, the verbatim criteria, their assessment scores for each student text and their justifications also had to be taken into account for the categorization. Table 1 shows examples for the categorizations, and Table 2 shows the complete categorial scheme. Within the categorized data, frequencies of category use were determined for the whole sample and compared for the sub-samples. In order to test Hypotheses H1 and H2 in terms of the differences of the sub-samples, *t*-tests were administered.

RESULTS

Insights into two cases

The cases in Table 1 show that the diagnostic activity can elicit very different personal criteria, as intended: The two teachers (here called Peter Tremnitz and Anne Schäfers) assess the students' explanations differently (the bold numbers indicate the evaluations they assigned to Ali, Suleika, and Tom), these assessments scores are based on different personal constructs underlying their diagnostic judgments. Some teachers' criteria are categorized under more than one category, mostly because their justification address several aspects. These personal criteria vary between very vague aspects such as mode of expression and core categories on discourse level. It is typical that criteria on discourse level (general/concrete phrasing, explaining meanings/reporting procedures) appear sometimes as mathematical criteria, sometimes as language criteria.

	Mathematical Criterion A	**Mathematical Criterion B**	**Language Criterion C**	**Language Criterion D**
Peter Tremnitz language pusher time non-worrier	**202** "Which meaning (calculating slope): finding 2 points and using formula" *(→ procedural knowledge)*	**111** "What to describe (difference quotient)": Word slope. meaning of quotient and using (the word) slope / difference quotient is developed for an example" *(→ conceptual knowledge, technical terms, concrete phrasing explaining meaning)*	**021** "Explanation of slope (Why-question): does not explain, only describe, row-per-step / more per…" *(→ conceptual knowledge, explaining meaning, technical terms)*	**101** "How to use formula? (How-question): abstraction is missing" *(→ procedural knowledge, describing procedure, general phrasing)*
Anne Schäfers language reducer partial time worrier	**121** "mathematical correctness of formulations" *(→ mathematical correctness)*	**222** "application" *(→ procedural knowledge)*	**110** „technical language: slope, difference" *(→ technical terms, surface level)*	**210** "understandability" *(→ understandability)*

Table 1: Examples of evaluations and elicited criteria with justification of two teachers (**in bold- faced type**: assessment scores for Ali, Suleika, Tom; *in italics*: categories assigned by researcher)

Peter has a strong focus on the connection between language and mathematics in three of his four criteria. When he mentions technical terms, they serve as indicators for deeper aspects on the discourse level, as his main distinctions concern the different discourse practices. In contrast, Anne presents disconnected criteria without addressing the discourse level at all. Interestingly, these findings correspond to their views expressed in the self-report scale in the hypothesized ways: Peter favors views as language pusher and does not really worry about time, whereas Anne is a language reducer and tends to partially worry about language responsiveness being time consuming.

In the accompanying group discussion, the connection became also apparent: As language for her is only located on the surface level and not really connected to mathematics, it is rational in her view to reduce language demands. In contrast, for Peter, the mathematics and language criteria are tightly connected, so language is to be pushed to foster mathematics learning. Also for other aspects, the insights provided by the questionnaire resonate with richer qualitative video data from group discussions with these two teachers.

However, even if these two cases resonate with the hypotheses, the hypotheses must be tested for a larger sample.

Quantitative results of typical categories and connections

Table 2 presents the frequencies of categories built from the elicited personal criteria as exemplified in Table 1.

Teachers' categories with different focus	Frequency of category use… …in whole sample	…among language reducers – pushers	Effect size *d*	… among time worriers – non-worriers	Effect size *d*
Focus on mathematical criteria					
Mathematical correctness	24%	**53% – 06%****	**1.18**	09% – 14%	0.16
Conceptual knowledge	73%	65% – 72%	0.16	91% – 77%	0.37
Procedural knowledge	54%	**35% – 78%****	**0.92**	55% – 49%	0.12
Focus on discourse level (Language or Math)	*39%*	38% – 60%	0.43	40% – 42%	0.05
Concrete, example-bound phrasing	26%	80% – 57%	0.46	75% – 67%	0.17
General phrasing	26%	40% – 71%	**0.61**	50% – 67%	0.31
Explaining meaning	32%	**12% – 44%***	**0.76**	18% – 37%	0.42
Reporting procedure	16%	18% – 22%	0.11	18% – 17%	0.03
Focus on Language beneath discourse level					
Only surface level or very vague criteria	65%	**80% – 53%***	**0.71**	70% – 70%	0.01
Orthography	12%	13% – 06%	0.24	18% – 18%	0.01
Mode of expression	20%	06% – 22%	0.46	18% – 26%	0.19
Understandability	26%	31% – 17%	0.34	09% – 26%	0.46
Word level: technical terms	42%	50% – 33%	0.33	45% – 47%	0.03
Sentence level: syntactical issues	23%	31% – 17%	0.34	36% – 26%	0.21

Differences between subsamples that are significant in *t*-test are marked in bold with * for $p < .05$ and ** for $p < .01$. Medium effect sizes ($d > 0.5$) and high effect sizes ($d > 0.8$) are marked in bold even if not significant.

Table 2: Frequencies of different categories: Comparison of sub-samples

In the whole sample, 65% of the teachers address only very vague or surface criteria for language, and 42% adopt the often criticized focus on isolated technical terms. Interestingly, 39% of the teachers address categories on the discourse level, distinguishing in some ways between general and concrete phrasing and/or reporting procedures and explaining meanings. Interestingly, half of these teachers mention discourse practices as a mathematical instead of a language criterion.

The comparison of sub-samples shows that the pattern exemplified by the two cases only partly re-appear: For the sub-samples of time worriers and non-worriers, the frequencies of categories are similar without any significant differences (with a maximal difference of 19% for explaining meanings) and all have small effect sizes ($d < 0.47$ for all categories). Thus, Hypothesis H1 must be rejected: Worries about language responsiveness being time consuming does not seem to be systematically connected to the personal constructs applied for diagnostic judgments.

In contrast, the language reducers and language pushers have significantly different priorities in their diagnostic judgments: Whereas the language reducers often address the very general category of mathematical correctness, the language pushers differentiate more thoroughly between procedural and conceptual knowledge. 80% of the language reducers focus exclusively on surface levels (including orthography or technical terms), whereas only 53% of the language pushers do. In contrast, 60% of the language pushers focus on the discourse level while only 38% of the language reducers do; the difference is specifically significant for the most important discourse practice of explaining meanings. So, Hypothesis H2 can be confirmed, with the corresponding null hypothesis being rejected.

DISCUSSION

This study followed Moschkovich's (2010, p. 160) recommendation to investigate teachers' judgements in language-responsive classrooms, in this study, the diagnostic judgments on explanations of a mathematical concept. Similar to other investigations of teachers' diagnostic judgments (Prediger & Zindel, 2017), the thorough exploration of individual categories turned out to provide insightful windows into teachers' thinking.

The empirical identification of teachers' personal categories revealed the problem of surface level categories being dominant for 65% of the teachers. For these teachers, the recommended shift of focus to the discourse level should be a crucial part of professional development programs (Moschkovich, 2010; Short, 2017). This is specifically important as the general view of language responsiveness as language reduction turns out to be significantly connected to the missing focus on the discourse level. As long as professional development programs fail to address the discourse level (as criticized by Moschkovich, 2010), this study provides indications that a crucial precondition for an adequate view on language-responsiveness is missing.

While 39% of the teachers in the study already paid some attention to the discourse level, half of these teachers mention discourse practices as a mathematical instead of a language criterion. So this focus of attention reveals an important resource on which professional development programs must build upon in order to shift teachers' language focus to the discourse level.

Teachers' attitudes have often been described as quite stable orientations that have an impact on teachers' noticing and practices. In this paper, the relation may be turned around: For a new challenge, such as language-responsive mathematics classrooms, the presented findings provide indications that the scope of teachers' categories may also influence their personal views on the role of language in mathematics classrooms. However, this seems to apply more for language reducers (Hypothesis H2 was confirmed) than for those who worry about language responsiveness being time consuming (Hypothesis H1 had to be rejected).

In order to overcome limitations of the study, future research will increase the (so far limited) sample size and study also the development of teachers' views and personal categories during a PD program, with the survey and also qualitative means.

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References

Barwell, R., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J., … Villavicencio, M. (2016). *Mathematics education and language diversity: The 21st ICMI Study*. Dordrecht: Springer.

Moschkovich, J. N. (2010). Recommendations for Research on Language and Math. Ed. In J. Moschkovich (Ed.), *Language and Mathematics education* (pp. 1-28). Charlotte: Infor. Age.

Prediger, S., & Zindel, C. (2017). Deepening prospective mathematics teachers' diagnostic judgments. *European Journal of Science & Mathematics Education, 5*(3), 222-242.

Radford, L., & Barwell, R. (2016). Language in mathematics education research. In A. Gutiérrez, G. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education.* (pp. 275-313). Rotterdam: Sense.

Usiskin, Z. (2008). Arithmetic Curriculum and the Real World. In D. Bock, B. Søndergaard, B. Gómez, & C. Cheng (Eds.), *Proceedings of ICME11, TSG10* (pp. 9-16). Monterrey: ICMI.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). *Mathematics Teacher Noticing*. New York: Routledge.

Short, D. J. (2017). How to Integrate Content and Language Learning Effectively for English Language Learners. *EURASIA Jour. of Math., Science & Techn. Ed*. 13(7b), 4237-4260.

PROBLEM SOLVING: HOW PRESERVICE TEACHERS UNDERSTAND IT DURING THEIR PRESERVICE LEARNING

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Teaching mathematics requires writing, finding, and modifying mathematics problems relevant and appropriate for the students in the course. The preservice program sets the stage for teachers' conceptualizations, appreciations, and understandings of what teaching and learning mathematics means. This mixed methods study explores 44 secondary school preservice mathematics teachers' beliefs about problem solving as they progress through a preservice mathematics education course. Ontologically, problem solving begins closely bound to textbook examples and irrelevant 'real-world' contexts and then shifts to socially relevant experiences. Epistemologically, a gradual release of responsibility reduces structural rigidity in teaching and increases purposeful critical thinking.

INTRODUCTION

Problem solving is ubiquitous with mathematics. There is problem solving in other areas, such as the arts, the humanities, and the sciences, but often, when school learning of mathematics is discussed, problem solving appears to become a synonym for mathematics. The often perceived domain for learning problem solving is the school mathematics course; in fact there might even be an expectation that successful completion of mathematics courses makes successful problem solvers. For the sake of improved learning of mathematics, and functioning as a problem solver in general, perhaps the preservice mathematics teacher learning to employ problem solving is a necessary focus.

Pólya is likely the most familiar person associated with describing the steps of problem solving. In mathematics curricula, for example, the various National Council of Teachers of Mathematics (NCTM) principles and standards publications, and the Ontario Ministry of Education curriculum (OME, 2005), problem solving is described as a mathematics process or performance skill, a way of 'doing' some (but not all) mathematics, where "[i]t is considered an essential process through which students are able to achieve the expectations in mathematics" (OME, 2005, p. 12).

Problem solving is more than just a set of steps to successfully find a solution to a mathematics problem; it is also how one thinks as a problem solver. Some have explored ways one can be a problem solver, and explain what knowing mathematics is all about, for example, "knowing-to" from Mason and Spence (1999), the Harvard verbs of mathematical inquiry (Harvard, 1995), and Cuoco, Goldenberg, and Mark's (1996) mathematical habits of mind.

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It appears problem solving has two meanings, a) a noun: the way some mathematics is done, and b) a verb: the thinking and reasoning mathematically through a problem to come to a successful conclusion or solution. People who are proficient at mathematics may be unaware of how problem solving can exist as a noun and as a verb, and that this distinction can make a difference in their thinking and actions, for example, preservice teachers who are teaching others to learn mathematics and mathematics problem solving. If preservice teachers are not crystal clear that teaching mathematics as problem solving is different from teaching mathematics through problem-solving, what learning success can we expect from students?

Wrapped up in all of this is the understanding of the nature of problems themselves, as well as teachers' beliefs. Teaching mathematics requires writing, finding, and modifying mathematics problems relevant and appropriate for the students in the course. Designing quality problems requires understanding the purpose and intent of the problem. Teachers' beliefs about mathematics, teaching, and learning, also have an influence on what they see as important, what mathematics needs to be emphasised, and how mathematics should be taught. The preservice program sets the stage for teachers' conceptualizations, appreciations, and understandings of what teaching and learning mathematics means. Thus, the purpose of this mixed methods study was to explore secondary school preservice mathematics teachers' beliefs about problem solving as they progress through a preservice mathematics education course in an Eastern Ontario faculty of education. An overarching research question guided this study: What are the ontological and epistemological beliefs about problem solving held by secondary school preservice mathematics teachers?

THEORETICAL FRAMEWORK AND LITERATURE

Three areas of literature support this study, the nature of problems, problem solving, and teachers' beliefs. The nature of problems begins with understanding the characteristics of problems that preservice teachers might consider when creating mathematics problems and assigning problems as a part of their students' learning and course-work. The perspective of problem solving taken in this study includes both the way mathematics can be performed, and a strategy for thinking and reasoning to come to a solution or conclusion for complex multi-step problems. The perspective on teachers' beliefs for this study focuses on those beliefs preservice teachers have about mathematics problem solving.

The nature of problems

There are two key characteristics of problems, routines and structure. These characteristics do not exist as static points of definition, but rather as two continua. The characteristic of routine can be understood as a continuum of routine to non-routine (Mayer & Wittrock, 2006), and the characteristic of structure can be understood as a continuum of well-defined to ill-defined (Hollingworth and McLoughlin, 2005). Routine problems are familiar and require a well-established procedure that can be followed from start to end. Non-routine problems are not necessarily unfamiliar, but they have a complexity to them which precludes using a recent procedure. Creative thought and critical thinking are required to figure out what can be done to solve the problem.

Well-defined problems have a structure that is readily laid out and easily identifiable. The data for the problem is complete and available, there are few rules and procedures to use, a possible solution is anticipated, and the decisions one can make in problem-solving thinking clearly converges to an answer. Conversely, for ill-defined problems, the data is not complete or readily available, any procedures or steps could be taken, a solution or answer is not necessarily evident that it even exists, and the decisions one makes in the problem-solving thinking may not converge towards a solution.

The intent of the problem given to a learner will determine what place on each of these continua of problem characteristics the problem sits. For example, routine and well-defined: becoming familiar with a particular set of facts or relationships necessary to develop speed and facility with numerical calculation (e.g., multiplication tables) or algebraic manipulation (factoring trinomials), or routine and ill-defined: solving trigonometric identities. Problem design must be completed carefully to consider the many aspects of the intent of the course, the problem, and the abilities of the students.

Problem Solving

Problem solving as a way mathematics is done is employing appropriate strategies to arrive at an acceptable solution, given an initial, unclear problem state (Metallidou, 2009). Mathematics problems start with a question that is to be answered, some data, and some resources and tools. Considering and selecting appropriate tools such as a particular piece of technology, or a particular concept or algorithm move the work of problem solving towards the final state of a complete solution and/or an answer.

Problem solving can also be a strategy for thinking and reasoning with a mathematics problem to come to a successful conclusion or solution. Mayer and Wittrock (2006) suggest four aspects for a problem solving framework: problem-solving is a cognitive experience within a problem space in the students' mind, there is an iterative process of mental representation and knowledge manipulation, there is observable directed behaviour, and the qualities of the student's mathematics knowledge and skillset influence the outcome of problem solving. The teacher has the responsibility to create a problem solving learning environment or situation and arranging the context in which meaningful problem-solving experiences increase students' understanding of the mathematics (Hiebert & Wearne, 2003).

Teachers' beliefs

Teachers, being a vital contributor to the creation of the environment for quality problem solving are influenced by their beliefs. Beliefs are "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 2003, p. 2). Preservice teachers' beliefs about the nature of what mathematics is, their ontological beliefs, and the nature of how mathematics is to be learned, their epistemological beliefs, are critical influences to the pedagogical approach employed. Teaching and learning beliefs influence pedagogical decisions (Philipp, 2007).

Methodology and Methods

This study employed a mixed method design, one that provided opportunities for quantitative data for confirmation and identification of important points, qualitative data for explanation and description, and mixing quantitative and qualitative results to unpack the tricky aspects of preservice teachers' ontological and epistemological beliefs as they were learning in a preservice program. Clearly a pragmatic approach (Morgan 2007) was warranted, and in particular a sequential (data collected and analysed in successive phases) and convergent (mixing of results to make conclusions) design (Creswell, 2014) was implemented.

There were a total of seven phases of data collection and analysis: three phases to data collection, and four interspersed phases of data analysis. For phase one and two, the Beliefs about Mathematical Problem Solving (BMPS) (Kloosterman & Stage, 1992) questionnaire and seven interviews were completed, and the quantitative data from the BMPS was analysed. These two phases were performed at the beginning of the preservice mathematics course, and then repeated (a) when the preservice teachers returned from their first teaching practicum at the beginning of December, and (b) when they returned after their second teaching practicum for their last month of the preservice course. The sixth phase involved separately analysing all three sets of qualitative data from the interviews, and the seventh phase involved mixing of all quantitative and qualitative results for inferences and conclusions.

Forty-four secondary school preservice mathematics teachers participated in the study, 34 female, and 10 male, of which 28 had a science as a second teaching subject, and 16 had a non-science subject as a second teaching subject. Seven of those preservice teachers also agreed to be interviewed. All participants were from the same secondary school preservice mathematics education course in the same teacher preparation program in an Eastern Ontario mid-sized university.

The BMPS questionnaire measured beliefs on six scales: difficult problems, steps, understanding, word problems, effort, and usefulness with a total of 36 Likert items using a 5-point scale. The overarching beliefs for each scale are, difficult problems: I can solve time-consuming mathematics problems; steps: There are word problems that cannot be solved with simple, step-by-step procedures; understanding: Understanding concepts is important; word problems: Word problems are important in mathematics; effort: Effort can increase mathematical ability; and usefulness: Mathematics is useful in daily life. The BMPS took approximately 20 minutes of class time to complete. The interview protocol incorporated the Teacher Beliefs Interview from Luft and Roehrig (2007), and also included questions pertaining to each the BMPS scales. Each interview lasted approximately 30 to 45 minutes.

For the quantitative data, reliability for the BMPS was first confirmed with Cronbach's alpha. A process of item removal and participant case removal improved the reliability for the difficult problems, understanding, effort, and usefulness scales. The steps and word problems scales did not reach the reliability threshold of .7; these two scales were not used in subsequent statistical analysis.

Exploratory factor analysis (EFA) was then performed to determine the factor structure that emerged for this data set. Since the scales were related (all measuring beliefs about mathematical problem solving), to allow for correlation amongst the scales a direct oblimin rotation method was decided upon prior to conducting the EFA. Three factors were retained in the phase one data set: effort, understanding, and difficult problems. Four factors were retained on the phase two data set: effort, difficult problems, usefulness, and understanding. Three factors were retained in the phase three data set: understanding, effort, and usefulness.

A two-way mixed ANOVA was conducted for each of the four retained scales to examine the interaction of time in the preservice program against gender, and against second teaching subject. Table 1 shows results across phases 1 and 2, and for phase 3.

Scale	**Significant Two-Way Interactions** (Phase 1 to Phase 2)	**Significant Simple Main Effects** (Phase 1 to Phase 2)	**Significant Two-Way Interactions** (Phase 3 only)
Difficult Problems	Time in program and gender (p = .017, η^2 = .138)	Males' scale scores were greater in phase 2 compared to phase 1 (p = .004, d = 0.486)	
Usefulness		Teachers with science teaching subjects possessed greater scale scores than teachers with non-science teaching subjects (p = .004, d = 0.984)	Time in program[1] (p = 0.05)
Understanding	None	None	Time in program[1] (p = 0.037)
Effort	None	None	None

[1] Simple main effects were not possible to find because of attrition in phase 3

Table 1. Summary of Significant Findings from Two-Way ANOVAs

The qualitative data was analysed inductively in a three-step process of text segments, to categories, to themes (Thomas, 2006). The data was separated into two sub-sets, data relating to ontological questions, and data relating to epistemological questions. The ontological and epistemological themes that emerged are presented in Table 2. The themes are aligned numerically for those that are related across phases.

	Phase One	**Phase Two**	**Phase Three**
Ontological Themes (What is mathematics problem solving?)	1) Problem solving is a collaborative thinking process for solving real-world problems 2) Problem solving is bounded by mathematics education	1) Problem solving is a challenging and collaborative thinking process for solving real-world problems 2) Problem solving is a process for solving academic mathematics problems.	1) Problem solving is a purposeful creative and critical thinking. 2) Problem solving is collective interaction with the world using mathematics. 3) Problem solving in schools is applying strategies to different problem types.
Epistemological Themes (How is mathematics problem-solving acquired?)	1) Establishing a problem-solving learning environment 2) Balanced problem-solving instruction and assessment	1) Exploring problem solving and communicating thinking 2) Demonstrating problem-solving techniques and assessing students' ability with those techniques	1) Collegial building of process-oriented thinking. 2) Developing students' knowledge of problem, solving strategies and resources. 3) Epistemological conflict: thinking critically about problem solving, and perceived tensions

Table 2. Ontological and epistemological themes for the three phases.

Mixing of results

Quantitative and qualitative results were then mixed in a process similar to Li, Marquart & Zercher's (2000) cross-over track analysis. Inferences were formed, shaped, and revised in a fluid drifting over all the results finding connections and relationships amongst particular aspects of qualitative and quantitative data. Two important themes related to changing ontological beliefs and two important themes related to changing epistemological beliefs emerged.

Results of themes and discussion

Ontologically, there is a contextual attempt, and a sense of boundedness to problem solving. The contextual attempt concerns the initial traditional use of 'real-world' examples where problem solving can be observed. This is really a real world façade as these contexts are often not relevant or pertinent to the everyday lived experiences of the students in the classroom, that is, these contexts are unconnected to students' adolescent lives. Such a lack of connectedness emphasises an apparent primary role of problem solving for academic (curriculum) purposes and reduces student motivation and engagement to work on the mathematics problems. Gradually though, preservice teachers begin to understand what it means to engage students as people and agents in their own learning, and problem-solving contexts become a vehicle for engaging with and interacting with the world. Often a social justice flavour begins to appear as problem-solving contexts focus on students' interests and concerns.

Boundedness is about the level of control and ownership preservice teachers feel is important in their classroom practice. Problem solving changes from being confined to the textbook and examples from curriculum documents, to a service aspect—using some concepts to learn other concepts, and finally to contexts that provide an experience of problem solving in meaningful ways.

Epistemologically, how students work as learners and what needs to be taught are important aspects to teaching problem solving. Initially preservice teachers believe students must work individually as they are taught all the steps, strategies, and thinking of problem solving. But then preservice teachers find students working collaboratively amongst themselves can be beneficial; students begin to be more imaginative and creative with thinking about various problem-solving strategies and need less structure and control over their problem solving. Teacher-centred problem solving lessons become collaborative student activities, and then collegial problem-solving experiences. Simultaneously, preservice teachers begin by teaching everything they know about problem solving for every problem, and gradually shift to differentiating between overarching problem-solving approaches (such as Pólya's steps) and particular problem-solving skills for the purposeful critical thinking unique to specific problems.

References

Creswell, J. W. (2014). *Research design: Qualitative, quantitative, and mixed methods approaches* (4th ed.). Thousand Oaks, CA: SAGE.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behaviour*, 15, 375-402.

Harvard. (1995). *Assessing mathematical understanding and skills effectively*. An interim report of the Harvard Group, Balanced Assessment Project. Harvard Graduate School of Education. Retrieved from http://hgse.balancedassessment.org/amuse.html

Hiebert, J., & Wearne, D. (2003). Developing understanding through problem solving. In H. L. Shoen (Ed.), *Teaching mathematics through problem solving*: Grades 6 - 12 (pp. 3-13). Reston, VA.: NCTM.

Hollingworth, R., & McLoughlin, C. (2005). Developing the metacognitive and problem-solving skills of science students in higher education. In C. McLoughlin, & A. Taji (Eds.), *Teaching in the sciences: Learner-centered approaches* (pp. 63-83). New York, NY: The Haworth Press.

Kloosterman, P., & Stage, F. K. (1992). Measuring beliefs about mathematical problem solving. *School Science and Mathematics*, 92, 109–115.

Li, S., Marquart, J. M., & Zercher, C. (2000). Conceptual issues and analytic strategies in mixed-method studies of preschool inclusion. *Journal for Early Intervention*, 23, 116-132.

Luft, J. A., & Roehrig, G. H. (2007). Capturing science teachers' epistemological beliefs: The development of the teacher beliefs interview. *Electronic Journal of Science Education*, 11(2), 38–63. Retrieved from http://ejse.southwestern.edu/

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics*, 38(1/3), 135-161.

Mayer, R. E., & Wittrock, M. C. (2006). Problem solving. In P. A. Alexander & P. H. Winne (Eds.), *Handbook of educational psychology* (pp. 287–303). Mahwah, NJ: Lawrence Erlbaum.

Morgan, D. L. (2007). Paradigms lost and pragmatism regained: Methodological implications of combining qualitative and quantitative methods. *Journal of Mixed Methods Research, 1*, 48-76.

OME. (2005 (revised)). The Ontario Curriculum: Grades 9 and 10 Mathematics. Toronto, Ontario: Queen's Printer for Ontario Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age.

Richardson, V. (2003). Constructivist pedagogy. *Teachers College Record*, 105(9), 1623-1640.

Thomas, D. R. (2006). A general inductive approach for analyzing qualitative evaluation data. *American Journal of Evaluation*, 27, 237-246.

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 57-78). New York, NY: MacMillan.

A STUDY OF MIDDLE SCHOOL STUDENTS' ALGEBRAIC PROOFS IN CHINA

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This study examined how middle school students in China constructed written proofs for six elementary number theory (ENT) statements. The participants were 80 junior middle school students (year 9). It finds that students seemed not to have a good awareness of what mode of justification was needed to verify or refute a particular statement. The common problems in students' algebraic proofs include the lack of necessary steps of reasoning and incorrect explanations.

INTRODUCTION

Mathematics proof is at the heart of mathematics competencies and thereby an essential part of mathematics education (Zaslavsky, 2012). The teaching and learning of proof is challenging because the process of proving involves a sophisticated set of mathematics competencies, such as the identification of assumptions, the isolation of given properties, and the organization of logical arguments (Healy & Hoyles, 2000). The previous studies found that it was difficult for students to construct mathematics proofs and some students were reported to be unaware of the need to provide a proof to verify a general mathematics statement (Carpenter et al. 2003; Chazan 1993; Harel and Sowder 1998; Healy and Hoyles, 2000). Due to its difficulty, proof and proving are hardly covered in most Western nations' school mathematics curriculum until senior secondary level (Healy and Hoyles, 2000). And most of the previous studies mentioned above were also undertaken in senior secondary level. The case in mainland China is different from that in most of the Western nations. Mathematics curriculum for junior middle school in mainland China places a significant emphasis on mathematics proof, especially in geometric proof (Ministry of Education, 2001). Most of the previous studies regarding Chinese teaching and learning of mathematics proofs concerned mainly the geometric proofs. For example, Ding and Jones (2007) analysed Chinese teachers' teaching of geometric proofs in Grade 8 in Shanghai and found the typical teaching model of geometric proofs is "new theorem - simple problems - complicated problems"(P.9). Huang (2005) compared the teaching of Pythagoras' Theorem in Shanghai and Hong Kong, arguing that the teachers in both cities emphasized on developing students' conceptions of geometric proofs. However, very few studies have been done to examine the teaching and learning of proofs in Algebra in mainland China. As a matter of fact, students' development in the conception of algebraic proofs is very significant because it is connected to students' conceptions of algebra and its usefulness in solving mathematics problems (Healy& Hoyles, 2000). Therefore, this study aims to analyse Chinese students' knowledge and conceptions

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regarding algebraic proofs. This study intends to answer two research questions: (1) how Chinese junior middle school students make justifications to support their claims in algebra; (2) what difficulties do Chinese junior middle school students have when constructing algebraic proofs.

METHODOLOGY

Participants and instrument

Participants were 80 middle school students (all in the year 9, the last year of junior middle school) from an urban middle school in the city of Hengshui, Hebei Province in mainland China. The Hengshui city is a middle-size city in northern part of China and it is about 300 miles south to Beijing, the capital of China. The overall student academic achievements in the selected urban middle school have been above the average in the city of Hengshui. The instrument consists of six elementary number theory statements (shown in Table 1). The students were asked to judge whether these statements were true or not and then give their justifications. These statements were adapted from the study of Tsamir, Tirosh, Dreyfus, Barkai and Tabach (2009). Half statements (S1, S4, S5) are true while the other half (S2, S3, S6) are false.

Quantifier Predicate	Always true	Sometimes true	Never true
Universal	S1. *The sum of any five consecutive natural numbers is divisible by 5.*	S2. *The sum of any three consecutive natural numbers is divisible by 6.*	S3. *The sum of any four consecutive natural numbers is divisible by 4.*
Existential	S4. *There exists a sum of five consecutive natural numbers that is divisible by 5.*	S5. *There exists a sum of three consecutive natural numbers that is divisible by 6.*	S6. *There exists a sum of four consecutive natural numbers that is divisible by 4.*

Table 1: The six elementary number theory statements Adapted from Tsamir, Tirosh, Dreyfus, Barkai and Tabach (2009)

The six statements were developed by Tirosh and his colleagues to examine how mathematics teachers to verify a valid mathematics statement and refute an invalid mathematics statement. But as pointed out by these researchers, these statements could be also presented to middle school students in order to investigate what possible justifications could be provided by the middle school students. Therefore, these statements were adopted in this study to examine how middle school students in China verify a valid mathematics statement and refute an invalid mathematics statement. In addition to the above consideration, there are other two reasons why these six statements were chosen in this study. Firstly, these six statements are not covered in mathematics curriculum and textbooks in China. Compared to those common proof tasks in algebra and geometry (e.g., tasks about congruence and similarity), these six statements are non- routine tasks to most of the Chinese students who don't have immediately apparent strategies in mind. Secondly, the year nine students could make sense of the expression of these six statements. Compared to some mathematical tasks which require students to spend a lot of time on reading and understanding the descriptions of tasks, these six statements are all very short and easily accessible to most of the year 9 students in China. The students could easily come up with some ideas on how to prove or refute these six statements by making use of their mathematics knowledge. The statements were translated into Chinese collaboratively by the researcher and some middle school mathematics teachers. In the process of translation the expressions of these statements were adjusted so that the Chinese translations look more naturally to Chinese students and thus make sense to the students more easily. For example, the fourth statement "*There exists a sum of five consecutive natural numbers that is divisible by 5*" was translated into two short sentences in Chinese which literally mean "*There exists five consecutive natural numbers. The sum of them is divisible by 5.* " The translated statements were presented in a worksheet in the order from statement S1 to S6. The participating students were given 30 minutes to respond to the six ENT statements.

Data analysis

In this study, the students were asked to make a judgement on the validity of each statement and then provide a justification for the judgement. In other words, the students' responses to each of the six statements include two parts: the first part is a judgement whether the statement is true or false, and the second part is the justification constructed to support the judgement. The first part was coded as correct judgement, incorrect judgement and null (i.e., the case in which no judgement was made). Please refer to Table 2 for the nature of each statement.

Universal statements			Existential statements		
S1	S2	S3	S4	S5	S6
True	False	False	True	True	False

Table 2: Nature of the statements

The second part was coded as correct mode of justification, incorrect mode of justification and null (i.e., the case in which no justification was made). Here my focus is whether the students were able to provide the right way to justify their judgements, regardless of the correctness of their judgements. More details can be found in Table 3. For example, if a student wanted to verify the validity of a universal statement, the right way of the corresponding justification would be constructing a proof rather than providing an positive example or counter-example. By contrast, if the student wanted to refute the validity of a universal statement, the right way of the corresponding justification might be either constructing a proof or providing a counter-example. Similarly, if a student wanted to verify the validity of an existential statement, then both constructing a proof and providing a positive example could be accepted as the right way of making justifications. While if the student wanted to refute the validity of an existential statement, the right way of making justification would be constructing a proof.

		Correct modes of justification	Incorrect modes of justification
Universal statements	S1 S2 S3	1. Claiming it is true by constructing a proof; 2. Claiming it is false by giving one or more counter examples; 3. Claiming it is false by constructing a proof.	1. Claiming it is true by giving one or more examples; 2. Claiming it is false by giving one or more positive example; 3. Others.
Existential statements	S4 S5 S6	1. Claiming it is true by giving one or more positive examples; 2. Claiming it is true by constructing a proof; 3. Claiming it is false by constructing a proof.	1. Claiming it is false by giving one or more examples; 2. Claiming it is true by giving one or more counter examples; 3. Others.

Table 3: Correct modes of verifying or refuting the statements

FINDINGS

Overview of students' judgements

As was introduced above, a total number of 80 year 9 students participated in this study. Figure 1 provides an overview of all the students' judgements on each of the six statements.

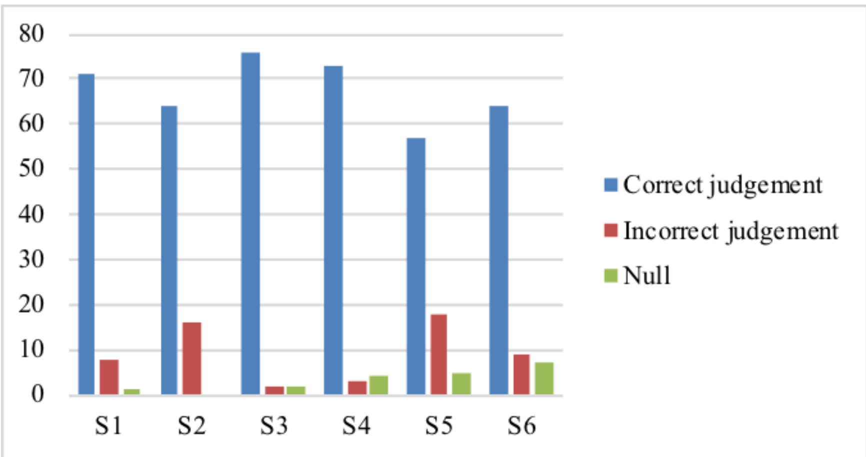


Figure 1: Overview of students' judgements on the six statements

It can be seen that a majority of the participating students provided correct judgements on the six statements. Almost all the participating students made judgements on the first three statements S1-S3, whereas about 5 students left blank for the last three statements S4-S6. Around 70 students made correct judgements on the six statements except S5, on which fewer than 60 students made correct judgements. The possible reason why fewer students failed to make correct judgements on the fifth statements is partial proof and counter examples were provided similar to S2.

Do the students know what is required to justify their judgements?

This section reports whether students were aware of what was required to justify their judgements, regardless of the correctness of their judgements.

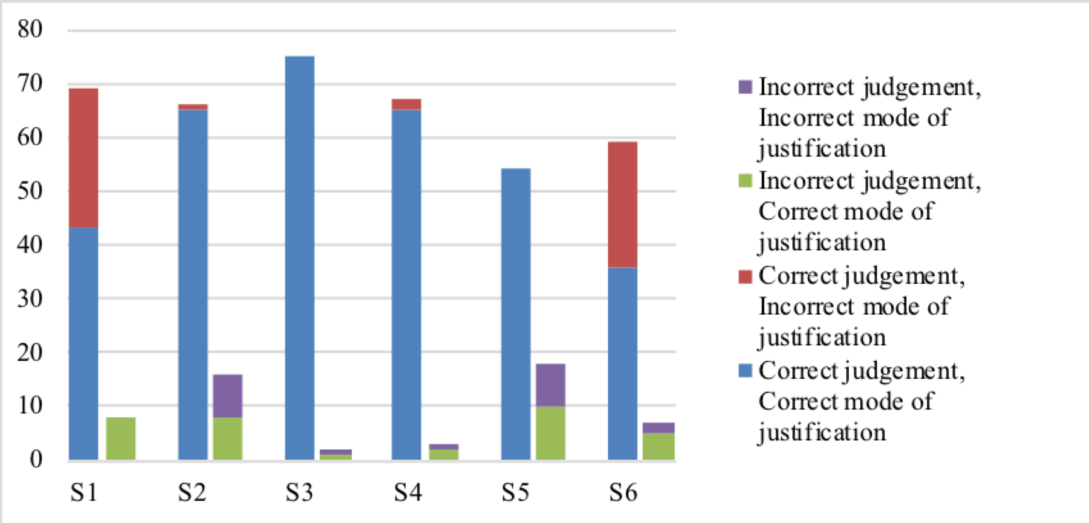
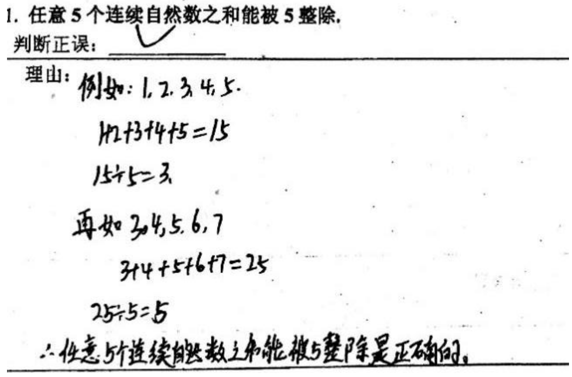


Figure 2: Modes of justifications in the six statements

Figure 2 outlines the students' performances in each of the six statements. Those students who made judgements but did not give justifications were not counted and presented in the Figure 2. Take the statement S1 for example, more than 70(the left column corresponding to S1) out of the 80 participating students made the claim that S1 was true, whereas the remaining students (the right column corresponding to S1) claimed that S1 was false. About 60% of those who made the correct claim (i.e., S1 is true) employed the aligned justifications, which means these students constructed a proof to verify that S1 is true. By contrast, 40% of those who made the correct claim (i.e., S1 is true) provided one or more positive examples to verify that S1 is true. For those (the right column corresponding to S1) who made the claim that S1 is false, the majority was able to employ the aligned justifications. That is, the majority were attempts to provide one or more counter-examples or constructed proof to refute the statement S1. For each statement, most of the students were documented to be able to make the correct claim. The lowest record is documented in the case of S5, for which 57 students (out of 80 in total) made the correct claim. For the statements S2-S5, the majority of those who made correct claims could find the aligned way to justify their claims, for both providing examples and constructing proofs are available for these four statements. By contrast, for the statements S1 and S6, almost half of the students who made correct claims failed to provide aligned justifications. In summary, the majority of the participating year 9 students did not experience significant difficulties in responding to the statements S3 and S4, but more students failed to give aligned justifications for their claims in S2 and S5. While nearly half of the students indeed have difficulties in giving responses to the statements S1 and S6.

Here lists one example (see Figure 3) in which the student failed to realize what are required to his judgements on statements S1. From this student's written responses, it can be seen that this student wanted to justify the statement S1 is true. A general proof is required to make the justification but this student simply gave two examples to support his judgement. In this case, the student was not clear about what was needed to verify the validity of a universal statement.



Statement 1. *The sum of any five consecutive natural numbers is divisible by 5.*

Judgement: √ .

Justification:

An example, 1, 2, 3, 4, 5

1+2+3+4+5=15

15÷5=3

Another example, 3, 4, 5, 6, 7

3+4+5+6+7=25

25÷5=5

∴"The sum of any five consecutive natural numbers is divisible by 5" is true.

Figure 3: One student's responses in S1

Do the students construct complete and correct justifications?

This section focuses on the students who made correct judgements and also successfully demonstrated the awareness of what were required to justify their judgements. The quality of these students' construction of justifications is shown in Figure 4. Here those students who made correct judgements but did not provide justifications were excluded. A total number of 80 students participated in this study. For each of the six statements, the number of students who made correct judgements and also successfully demonstrated the awareness of what were required to justify their judgements is respectively 43, 62, 75, 65, 54, and 36.

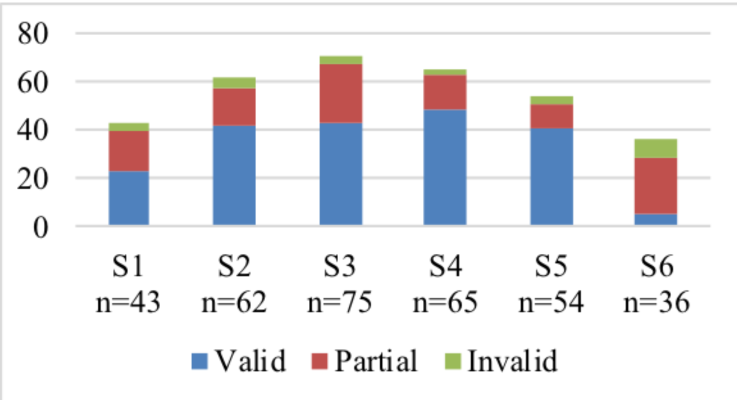
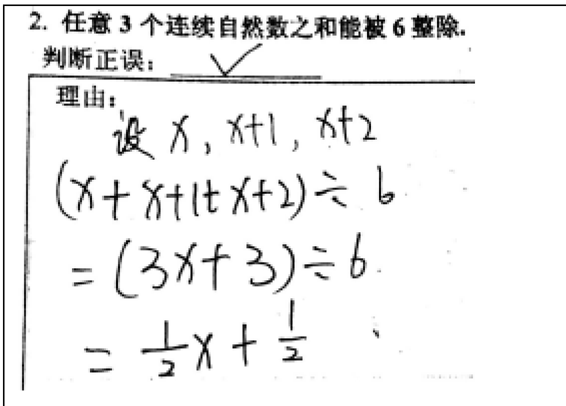


Figure 4: Quality of students' justifications

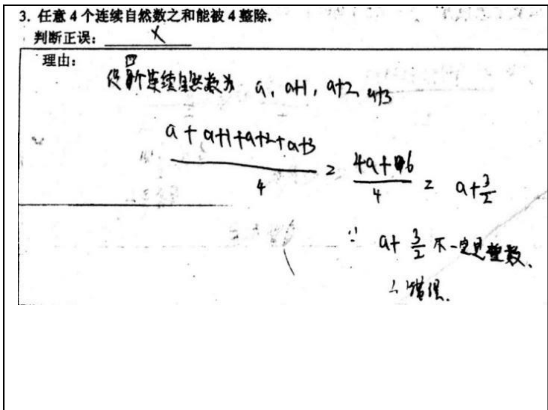
It can be seen that, for each statement, if the students made correct judgements and also successfully demonstrated the awareness of what were required to justify their judgements, most of them could successfully provide valid or partially valid justifications. Some students (about 7 or less in each case) constructed invalid justifications. But it is noteworthy a substantial number of students constructed partially valid justifications. Two examples are presented here to show the partially valid justifications constructed by the participating students.


2. 任意3个连续自然数之和能被6整除.
判断正误：√
理由：
设x, x+1, x+2
(x+x+1+x+2)÷6
=(3x+3)÷6
$=\frac{1}{2}x+\frac{1}{2}$

Statement 2: *The sum of any three consecutive natural numbers is divisible by 6.*
Judgement: √ .
Justification:
Denote x, x+1, x+2
(x+x+1+x+2)÷6
=(3x+3)÷6
$=\frac{1}{2}x+\frac{1}{2}$

Figure 5: One student's responses in S2

In Figure 5, it can be seen that this student successfully wrote down the algebraic expression but did not provide any explanations. This student seemed to know the necessary procedures to make the justifications. But there is still a reasoning gap between the final result ($\frac{1}{2}x+\frac{1}{2}$) and the conclusion that S2 is correct. In Figure 6, it can be seen that this student gave an explanation of his final result ($a+\frac{3}{2}$). But this explanation is incorrect. Because a is a whole number, the final result ($a+\frac{3}{2}$) cannot be a whole number. The student seemed to be unable to distinguish "cannot be" from "not necessarily be" in an expression.


3. 任意4个连续自然数之和能被4整除.
判断正误：×
理由：

Statement 3: *The sum of any four consecutive natural numbers is divisible by 4.*
Judgement: ×
Justification:
Let's denote the four natural numbers are a, a+1, a+2, a+3

$$\frac{a+a+1+a+2+a+3}{4}=\frac{4a+6}{4}=a+\frac{3}{2}$$

$\because a+\frac{3}{2}$ is not necessarily a whole number,

$\therefore$ (The statement) is false.

Figure 6: One student's responses in S3

CONCLUSION

This study shows the majority of participated students could make correct judgements for the given statements. But it identifies two major difficulties Chinese students experienced in making justifications to support their claims of ENT statements. Firstly, Chinese students tended to have difficulties in identifying the correct mode of justification to verify or refute a universal statement. For a large proportion of students attempted to provide positive examples to verify the validity of a universal statement (S1), and give counter examples to refute the validity of a existential statement (S6). Secondly, a lack of necessary reasoning steps was found in the algebraic proofs constructed by the students. It is quite common that students were able to construct proofs with algebraic representations successfully, while they went to conclusions without necessary reasoning or even gave incorrect explanations to the results.

References

Bieda, K., & Lepak, J. (2014). Are you convinced? Middle-grade students' evaluations of mathematical arguments. *School Science And Mathematics, 114*(4), 166–177.

Carpenter, T., Franke, M., & Levi, L. (2003). Thinking mathematically: Integrating arithmetic &algebra in elementary school . *Portsmouth: Heinemann*.

Chazan, D. (1993). High school geometry students’ justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24* , 359–387.

Ding, L., & Jones, K. (2007). Using the van hiele theory to analyse the teaching of geometrical proof at grade 8 in shanghai, china. *European Society for Research in Mathematics Education*, 612-621.

Ministry of Education (2001). *Curriculum Standards for Year 1-9 School Mathematics* (Trail) [in Chinese]. Beijing: Beijing Normal University Publisher.

Healy, L., & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education, 31*(4):396-428.

Harel, G., & Sowder, L. (1998). Students’ proof schemes: Results from exploratory studies. In A.H. Schoenfeld, J. Kaput, & E. Dubisnky (Eds.), *Issues in mathematics education* (Research in collegiate mathematics education III, Vol. 7, pp. 234–283). Providence: American Mathematical Society.

Huang, R. (2005). Verification or proof: justification of Pythagoras' theorem in Chinese mathematics classrooms. *International group for the psychology of mathematics education*, *3*(6), 758–762.

Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R., & Tabach, M. (2009). Should proof be minimal? Ms T's evaluation of secondary school students’ proofs. *Journal of Mathematical Behavior, 28*(1), 58-67.

Zaslavsky, O., Nickerson, S. D., Stylianides, A. J., Kidron, I., & Winicki-Landman, G. (2012). The need for proof and proving: Mathematical and pedagogical perspectives. In Hanna, G., & de Villiers, M. (Eds), *Proof and Proving in Mathematics Education* (pp. 215–229). The Netherlands: Springer.

SECONDARY PROSPECTIVE TEACHERS' INTERPRETATIVE KNOWLEDGE ON A MEASUREMENT SITUATION

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With a focus on teachers' perception about the area formula for a rectangle, we discuss components of the knowledge mobilized by prospective secondary teachers' when answering a particular task aimed at accessing and developing their interpretative knowledge. In particular, we investigate the relationships between the focus of the prospective teachers' content knowledge and their ability to expand their own space of solutions. Our findings reveal that the prospective secondary teachers seem to understand area as a surface measurement, but struggle to give meaning to the area formula of a rectangle when the reasoning involved differs from their own.

INTRODUCTION

There is a growing interest among international scholars to better understand mathematics teachers' professional knowledge and how such knowledge can be developed in teacher education. One often-used approach is to conceptualize and implement tasks that are carefully designed for the development of such knowledge in teacher education. The context of the tasks is practice-based, aimed at developing prospective teachers' mathematical ideas, their understanding, and awareness when considering students' productions—whether correct, incorrect, or non-standard (Ribeiro, Mellone, & Jakobsen, 2016).

When considering the specialized nature of teachers' knowledge and the particularities of knowledge involved in giving meaning to productions and comments (as part of a solution process) given by others, a new kind of teacher knowledge and awareness has been identified (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013). We have named this particular and specialized knowledge *interpretative knowledge* (Jakobsen, Ribeiro, & Mellone, 2014); it corresponds to the knowledge involved in interpreting and giving meaning to students' productions and comments as part of their work on mathematical problems. Interpretive knowledge includes the ability to expand one's own space of solutions, looking at situations from a wide range of different points of view—as an outsider (Jakobsen et al., 2014).

In order to explore, understand, and develop interpretative knowledge, the nature, aims, and focus of tasks included in teachers' education are crucial. Obviously, a mathematical task should be initiated from a mathematically fruitful activity (Mason & Johnston-Wilder, 2006). We also argue that in order to develop teachers' professional knowledge, it is more efficient and meaningful to work through practice-based situa-

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tions (Smith, 2001). One must take into consideration that teachers' mathematical knowledge (and beliefs) influence their practice. All these arguments influenced our own development of tasks within teacher education (e.g., Ribeiro, Mellone, & Jakobsen, 2016).

In previous works, we have addressed the interpretative knowledge grounded on the analysis of productions and comments of prospective primary teachers in arithmetical contexts (Jakobsen et al., 2014) and prospective secondary teachers working on a problem related to the powers of ten (Jakobsen, Mellone, Ribeiro, & Tortora, 2016). The implemented tasks addressed topics that prospective teachers would have to teach in their future practice. In this paper, we shift the foci of attention to include prospective secondary teachers' (PSTs) interpretative knowledge and revealed beliefs when solving and giving meaning to students' answers to a question involving the use of the measurement area unit (dm^2). The question involves a topic the PSTs do not necessarily have to teach but will be used in class. In particular, we address the question: What mathematical knowledge is revealed by PSTs when they are asked to interpret students' productions to a problem to use and make sense of the area formula of a rectangle.

THEORETICAL FRAMEWORK

Teachers' practices are molded by their knowledge and beliefs. Teachers' beliefs are organized in systems (Liljedahl & Oesterle, 2014) that include, among others, beliefs about mathematics and the teaching of mathematics, beliefs related to the learning of mathematics, and beliefs about the students' and teachers' roles. Such beliefs are rooted in teachers' previous experiences, both as students and as teachers. Moreover, these beliefs affect teachers' attitudes and actions and have a direct and crucial link to their mathematical knowledge, hence having an impact on the mathematics education process(es) (pedagogical dimensions).

Students' reasoning when solving mathematical problems, even when the solution is wrong, can provide precious and meaningful learning opportunities if suitably used by the teacher; in this perspective, interpretative knowledge is perceived as a core element of teachers' knowledge. Having students' productions, argumentations, and associated reasoning as a starting point for engaging students in the process of doing mathematics (instead of teaching the procedures), demands a shift from what one has (probably) experienced previously—aligned with the experiences that shaped teachers' belief systems. For such a shift to be effective, the interpretative knowledge needs to be developed, which research indicates does not happen merely over time with practice (Jakobsen et al., 2014). Thus, in order to allow such development to occur—creating the genesis of interpretative practices—one approach that can be used in teacher education is to engage (prospective) teachers in tasks focusing on contingency moments of real school practice (Rowland, Huckstep, & Thwaites, 2005) for example, to interpret students' solutions to a particular mathematical activity.

With the aim at deepening our understanding of the nature and content of interpretative knowledge and beliefs, we built on some aspects of the Mathematics Teachers' Specialized Knowledge (MTSK) conceptualization (Carrillo et al., 2013). This conceptualization assumes that teachers' knowledge consists of six subdomains (three are part of the subject matter knowledge and three are part of the pedagogical content knowledge). Teachers' beliefs are at the core of the MTSK representation in order to enhance the fact that beliefs influence and are influenced by teachers' knowledge. Due to the focus of our work here, we address only two of those subdomains—Knowledge of Topics and Knowledge of the Mathematical Structure.

In this paper, we deal with the formula for the area of a rectangle. Knowledge of Topics (KoT) includes teachers' knowledge of: definitions (what is area), properties of the mathematical content, phenomenology (the measurement process), applications, meanings (e.g., the meaning of the superscript 2 in the area measurement unit) and examples, different forms of representation, as well as procedures. In particular, teachers are required to be in possession of knowledge on the different key concepts that underlie measuring (e.g., Clements & Stephan, 2004) which demands, for example, considering whether the unit of measurement is of the same nature as the "element" to be measured. Indeed, the case of measuring the area of a rectangle demands defining the unit of measurement of the area (two dimensions) and subsequently determining the number of times such unit of area is needed to cover the rectangle.

We can recognize the above measurement process in the rectangle area formula by interpreting one of the factors representing the number of units of measure of the surface to cover the length (or the width)—a stripe—and the other factor as the number of times in which the stripe has to be repeated. The formula of area of a rectangle is commonly interpreted in many textbooks as Area = length x width, which can be seen as a sort of mathematical packing of the previous measurement process. In the MTSK perspective, teachers should be able to unpack and explain the rationale behind the formula.

Knowledge of the Structure of Mathematics (KSM) concerns teachers' knowledge of an integrated system of connections (allowing them to understand and develop advanced concepts from an elemental standpoint), and elemental concepts from an advanced mathematical standpoint. Examples of the content of such teachers' knowledge domain concerns the connections on the reasons grounding the possibility for doing, and meaning attributed to *dm* x *dm* in the context of dimensional analysis problems, and on the different meaning given to *dm* as a unit of length measurement and dm^2 as a unit of area measurement. Area as a magnitude to be measured can be seen as the sum of equal quantities (the amount of area units considered) that can be represented in the form of product—in a more elemental standpoint—or as the value of an integral, in Riemann perspective—from a more advanced mathematical standpoint. Although, teachers do not necessarily need to be able to provide a mathematical justification for how to determine the area of a certain surface using the integral. Teachers need to possess the knowledge that allows them to connect the notion of area with the reasoning of "how to determine" such area—explicating the rationale behind the procedure.

Thus, teachers who allow their students to understand what they are doing and why they are doing it open the door to future learning. They need to be in possession of knowledge on attributing meaning (in the school context) to the area unit (dm^2) as something different from the product of the units of two lengths. Nevertheless, it is important to underline that in other contexts (e.g., Physics), we can perform the calculation of *dm* x *dm*, obtaining dm^2. Moreover, the International System of Units (SI) has only a standard unit for metrology and not for the surface, assuming in this way the unit of the surface as directly obtainable from the one of length.

Hence, in the context of the rectangle area formula, what at first glance can appear trivial and elementary, actually offers rich and powerful issues that, if suitably exploited, can develop into teachers' new awareness and beliefs about mathematics.

THE CONTEXT OF THE STUDY

In this study, we focus on PSTs who were supposed to have strong knowledge of mathematics. Indeed, the PSTs of our sample were in their last year of education (fifth year), and the majority of them already had some teaching experience. Data were collected during their last course in mathematics education (one out of a total of five mathematics courses) in Brazil, and the 12 PSTs answered a task developed by the authors. The course was audio and video recorded.

The task consisted of three parts. The first and second parts were solved individually by the PSTs, while they worked in pairs in the third part. In part one, the participants were asked to answer two similar mathematical problems and to provide a mathematical argument that would justify the correctness of the provided answer. Both problems involved finding the area of a rectangle, first with sides measuring 3 and 4 units of length, and then sides measuring 3 *cm* and 4 *cm.* In the second part, the PSTs were asked to imagine how second-grade pupils would answer the first problem, and fourth-grade pupils would answer the second problem. One of the aims of the task is to discuss the meaning of the area formula for the rectangle, not only as a product of two units of length. This was done by mobilizing PSTs' knowledge on the notion of area in a practice-based context where they cannot use the interpretation of the area formula as a direct reading of length x width, as students in grade 2 do not know this formula.

In part three, the participants were given four fifth-grade pupils' productions to the second problem—presented in Figure 1. The PSTs were asked to make sense of these four students' solutions and provide a constructive feedback to each student in order to support their mathematical learning (Ribeiro, Mellone, & Jakobsen, 2013).

Caio: Multiplying the length by the width, we get 4 *cm* x 3 *cm*=12 cm^2.

Douglas: The area is a surface measurement and thus it has two dimensions (length and width) so we need to put the 2 in the exponent and we get 3 x 4=12 cm^2.

Camila: We just need to count the number of square centimeters needed to cover the square, and thus we get 3 cm^2 x 4 cm^2=12 cm^2 or, similarly, 4 cm^2 x 3 cm^2=12 cm^2.

Fernanda: I think the area is 12 cm^2 as we have to do 4 x 3 cm^2=12 cm^2 or 3 x 4 cm^2=12 cm^2.

Figure 1: Students productions to be interpreted

Although all the students provided a correct answer to the area of the rectangle (12 cm^2), their reasoning and argumentation differ and are associated with different interpretations of area, area units, and the meaning associated with the formula (A=length x width). Thus, all of the students' productions included in part three are aimed at discussing different interpretations when giving meaning to the area and how to determine it.

Caio's answer is the most commonly given answer. She reads the formula linearly, considering the area measurement unit (cm^2) as a result of the product of two variables (a x $a = a^2$, for any a). Although Douglas in the first part expresses the notion of area as a surface measurement (two-dimensions), and the main final mathematical argumentation follows Caio's idea, he writes the numbers without reference to its meaning and at the end he "adds" the 2 associated to the second power because "area has two dimensions." Camila's production is grounded in the notion of area as the number of square units needed to cover the surface (number of cm^2), but she does not attribute a meaning to the product. Fernanda perceives area as a measurement surface and she is counting (repeated addition) the number of area units (cm^2) required to cover the rectangle – repeating the considered stripe.

RESULTS

When answering part one of the task, all the PSTs provided a final numerically correct answer to both of the problems. Independently of the problems, different arguments were provided as to how to find the area of the rectangles. The three following answers represent the categories of answers that emerged: (i) counting squares; (ii) calculating squares using the product; (iii) area as the product of two lengths; (iv) the argumentation in which one of the factors representing the number of units of measure of the surface to cover the length (or the width)—a stripe—the other factor as the number of times in which the stripe has to be repeated. We have to note that only (iii) did not appear in part two of the task, imagining grade 2 pupils' answers. Below are some examples of the PSTs' answers to part two of the task:

PST1: We divide the side of the rectangle into 3 and 4 parts and we get 12 small squares with side 1cm. Each square has, by definition, 1cm^2 of area. As we have 12 squares (4 columns and 3 rows), the area is 12cm^2. [All the squares were numbered.]

PST2: One square of side 1cm has $1cm^2$ of area. We have 12 squares (3x4) so we can use the formula *length* x *width* and we get $3x4=12cm^2$. [Also, the answer 3cmx4cm was provided here.]

PST3: We have 3x4 squares. Each square of side 1 has $1cm^2$ of area. We have 12 squares so the area of the rectangle is $12x1cm^2=12cm^2$.

Concerning part three (interpret students' productions and provide meaningful feedback), we will separately present the PST pairs' answers to the four students' productions included in the task. Caio's solution was considered adequate by all pairs:

Pair1: Congratulations, your application was perfect, so you understood the formula.

Pair2: Caio correctly used the area measurement unit through the multiplication of cm x $cm = cm^2$.

Pair3: Caio's solution is correct as he correctly applied the formula for determining the area of a rectangle (measure of length x measure of height). He did the same thing as I did.

These comments reflect the PSTs' own answer and approach to the posed situation. The fact that Pair3 mentioned that Caio's answer is correct as it coincided with his own answer is consistent with results from previous work when participants' space of solutions consisted of only one element, their own (Jakobsen et al., 2014).

All pairs considered Douglas's answer as correct. One provided comment was:

Pair1: Your answer is correct. You carefully mention that the exponent goes to the unit, leading to the cm^2 in the answer.

The provided interpretation focused on the final part of the answer (the 2 in the exponent) or on showing the PT's "correct way" for using the formula of the area of the rectangle.

When commenting on Camila's production:

Pair1: The answer is correct but the calculation is incorrect; has $cm^2xcm^2=cm^4$, and thus, the correct calculation would be $3cmx4cm=12cm^2$.

Pair4: Your result is correct but your operation is incorrect. You made 3 cm^2 x 4 cm^2 and the result should be 12 cm^4. The formula for determining the rectangle area (size of the basis times size of the height) says that the area is 4 cm x 3 cm = 12 cm^2. Your mistake was that you assumed the measurement of the basis and height, respectively, 4 cm^2 and 3 cm^2, which is not true has cm^2 is not a length measurement unit but of area (the basis and height are lengths).

In these comments given to Camila's solution, from one side, we can appreciate the dimensional check of the terms of the multiplication and the consistency of the dimension of the result. From the other side, we can see that the PSTs are not able to go "out of the box" and see the opportunity to use Camila's solution to interpret the two factors of the product as different things: one representing the stripe (number of units of measure of the surface to cover the length or the width) and the other factor corresponding to the number of times in which the stripe has to be repeated. In other words, when confronted with a different reasoning from what they expected, the PTS were unable to go out of their own space of solution and provide feedback explaining the student's mistake. For Fernanda's production, we provide one comment:

> Pair4: Although the numerical result for the area is correct (12 cm^2), using the area formula, when you write 4 x 3 cm^2 you are considering that the measurement of the basis is 4 (without dimensional unit) and the height is 3cm^2 (with two-dimensional unit). You should note that both measures are uni-dimensional. Thus, the basis and the height have, respectively, 4 cm and 3 cm of length, and the area is 4 cm x 3 cm=12 cm^2 or 3 cm x 4 cm=12 cm^2.

Pair 4, even if they were able to express explicitly that the unit of measurement of area is different from the unit of measurement of length (when mentioning they need units after the quantities), they do not transition between their own reasoning in two different contexts. Such difficulties in amplifying their space of solutions is grounded on their own previous experiences and knowledge, and on the beliefs associated with the way they use the formula of the area of a rectangle to be "the correct one as we always have used it at school and we use it in some Calculus courses at the university."

DISCUSSION AND FINAL COMMENTS

All the PSTs seem to know what area is (as a surface measurement). However, when asked to provide a mathematical argument to their answers, different levels of justifications were provided. When asked to solve the problem as someone who does not "know the formula of the area of a rectangle," they are aware, at least intuitively, that one of the key concepts that underlie measuring is comparing (Clements & Stephan, 2004), and thus they determine the quantity of area units (here a square) needed to cover the rectangle. They were not able to mobilize such knowledge when giving meaning to that same process when associated with the use of the formula (in students' productions). Broadening their own space of solution (Jakobsen et al., 2014) demands they "move out of the box," which requires a possession and mobilization of knowledge on the connections (KSM) sustaining the mathematical validity to be kept between and within school levels, topics, and contexts.

PST's answers to both parts of the task (specialized and interpretative knowledge) are grounded in a "linear interpretation" of the formula. This fact is also intertwined with their beliefs of the role of formulas in mathematics and related to the nature of their previous experiences both as students and in teacher education. This reinforces the fact that when teachers' beliefs about mathematics (Liljedahl & Oesterle, 2014) are exclusively linked with a procedural/instrumental approach of mathematics, such beliefs implicitly shape the content of their knowledge and of the interpretations they deem to perform. Hence, as the mathematical knowledge and beliefs teachers elaborate and develop are grounded in their own experiences in "doing" mathematics, it highlights the need for teacher education to expose prospective teachers' experiences that generate "questioning momentums" on the validity of their knowledge and their interpretations of students' productions.

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References

Carrillo, J., Climent, N., Contreras, L. C., & Muñoz-Catalán, M. C. (2013). Determining specialized knowledge for mathematics teaching. In B. Ubuz, Ç. Haser, M. A. Mariotti (Eds.), *Proceedings of CERME* 8 (pp. 2985-2994). Antalya, Turkey: ERME.

Clements, D.H. & Stephan, M. (2004). Measurement in pre-K to grade 2 mathematics. In D.H. Clements, J. Sarama & A.M. DiBiase (Eds.) *Engaging young children in mathematics: Standards for early childhood mathematics education*, (pp. 299–317). New Jersey: LEA.

Jakobsen, A., Mellone, M., Ribeiro, C. M., & Tortora, R. (2016). Discussing secondary prospective teachers' interpretative knowledge: A case study. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 3, pp. 35-42). Szeged, Hungary: PME.

Jakobsen, A., Ribeiro, M., & Mellone, M. (2014). Norwegian prospective teachers' MKT when interpreting pupils' productions on a fraction task. *Nordic Studies in Mathematics Education*, 19(3-4), 135-150.

Liljedahl, P. & Oesterle, S. (2014). Teacher beliefs, attitudes, and self-efficacy in mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 583-586). Netherlands: Springer.

Mason, J., & Johnston-Wilder, S. (2006). *Designing and using mathematical tasks*. St Albans: Tarquin.

Ribeiro, C., M., Mellone, M., & Jakobsen, A. (2013). Give sense to students' productions: A particular task in teacher education. In J. Novotná, & H. Moraová (Eds.), *Proceedings to International Symposium Elementary Mathematics Teaching (SEMT 12) – Tasks and tools in elementary mathematics*, (pp. 273-281). Prague, Czech Republic: Charles University, Faculty of Education.

Ribeiro, M., Mellone, M., & Jakobsen, A. (2016). Interpreting students' non-standard reasoning: Insights for mathematics teacher education practices. *For the Learning of Mathematics*, 36(2), 8-13.

Rowland, T., Huckstep, P. & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8, 255-281.

Smith, M.S. (2001). *Practice-based professional development for teachers of mathematics*. Reston, VA: The National Council of Teachers of Mathematics

GEOMETRY IN KINDERGARTEN: FIRST STEPS TOWARDS THE DEFINITION OF CIRCUMFERENCE

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This paper deal with a teaching experiment concerning geometry in kindergarten for children aged 4-5. Through a classroom-based intervention, designed according to the Semiotic Mediation theoretical framework and developed on a multimodal approach, children produce a "pseudo-definition of circumference", which still refers to perceptual elements linked to the shape, but where it is possible identifying the dynamic nature of the curve as a trace generated by the movement of a point. The analysis of the teaching experiment highlights the specific roles of the teacher and of artifacts in supporting the process of semiotic mediation through which the children and teacher transform the signs linked to artifacts into mathematical signs.

INTRODUCTION

The concepts of space and systems of spatial reference are very important for the children's psychological and cultural development. Bartolini Bussi (2008) considers different kinds of experiences referring to space (the space of body, the external spaces and abstract space), which are related to different cognitive activities and different abilities in specific contexts. To move from the space of the body to representative spaces, where objects are represented as shape, up to the space of the geometry, where geometrical figures are identified through their geometrical characteristics, is one of the central ideas of teaching and learning geometry. This doesn't refer to a hierarchical geometrical learning model. As matter of fact, recent researches are developed against the Piaget and Van Heile's views that geometrical thinking can be described through a hierarchical model formed by levels where abstract space is placed at the end of the child's evolutionary process (Owens, 2015). Even if the definition of geometrical figure do not appear explicitly in curricula until primary school (in Italy in 4th grade), recent research is showing how geometry has an intuitive experiential basis well before school (Bryant 2008). Moreover, reviewing some literature available in the field and addressed to children education, we identify different research concerning geometrical teaching experiences in kindergarten (Sinclair & Moss, 2012, Bartolini Bussi & Baccaglini-Frank, 2015). According to theses researches, in children's development of geometrical and spatial competences, the ability to orient itself is one of the first crucial nodes in space conceptualization. Moreover, body and instrument experiences act as a bridge between the physical modeling of space and the conceptualization of a geometric space. With these premises in mind, I argue it makes sense that geometry could be explored much earlier and more robustly than it is at the moment (at least in Italy), already during the kindergarten years. To this aim, I will present an education sequence

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concerning a teaching experiment where children in kindergarten experienced circle close, but not completely isomorph, to the definition of circumference. From a methodological point of view, my aim is showing that a multimodal approach in kindergarten activities can promote the exploration of figures in geometrical sense in order to approach a "pseudo-definition" of circumference.

THEORETICAL PERSPECTIVES

The theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), developed to design and analyze educational activities, is the theoretical framework of reference in this research. It provides a powerful way for teachers and researchers to study the process by which activity with artifacts can be turned into mathematical activity. Moreover, in present research, the teaching-learning process is considered as multimodal activity where, exploiting perceptual-motor components, the body becomes essential in the learning processes (Nemirovsky, 2003). Thus, Multimodality (Gallese & Lakoff, 2005), involving in particular drawing, gesture, manipulation of physical artifacts and various kinds of bodily motion, is regarded as a driving force for the formation of geometrical understanding.

Summarizing the main elements of the Theory of Semiotic Mediation (TSM) (for more details, see Bartolini Bussi & Mariotti 2008), the teacher takes in charge two main processes: the design of activities and the functioning of activities. In the former the teacher makes appropriate choices about the artifacts to be used, the tasks to be proposed and the mathematics knowledge at stake. In the latter, the teacher monitors and manages the students' observable processes (semiotic traces), to decide how to interact with the students in order to transform signs linked to the use of artifact in math signs. The students are in charge of the resolution of the task through the use of the artifact proposed by the teacher. Making this, they produce signs (objects, drawings, words, gestures, bodily movements, and so on), which are linked to the artifact but that aren't yet explicitly math signs. The teacher collects all these signs, in order to analyze them and to organize a path for their evolution towards mathematical signs that can be put in relationship with the chosen mathematical knowledge. The TSM identifies three main categories of signs: artifact signs, which refer to the context of the use of the artifact, mathematical signs, which refer to mathematics context, and pivot signs, which act as bridges between the artifact signs and the mathematical signs. Moreover, according to Gallese and Lakoff (2005) mathematics teaching-learning processes are multimodal activities. Nemirovsky (2003) states that, understanding and thinking, included mathematical thinking, are perceptuo-motor activities, which become more or less active depending of the context. This means that, exploiting perceptual-motor components, the body becomes essential in the learning processes. In this perspective, the term multimodality is used here to underline the importance and mutual coexistence of a variety of cognitive, material and perceptive modalities or resources in teaching-learning processes and, more generally, in constructing of mathematical meanings

(Radford, Edwards & Arzarello, 2009) and of "abstract thought" (Lakoff & Nùñez, 2000).

METHODOLOGY: THE TEACHING EXPERIMENT

The class is composed by 21 children aged 4–5. Three teachers were involved during the classroom activities. Several sessions (13) were carried out during the school year, for 5 weeks (more or less twice or three times a week) either in the classroom or in the gym or outdoor, with a careful alternation of whole-class (with adult's guidance) and some individual activity. Each session was carefully observed by one of the teacher involved, with the collection of photos, graphical productions, videos and transcripts. The teachers and a researcher (the author) designed the educational activities. Due to space constraints it is not possible to report in this paper on all the activities, so I'll focus on certain sessions in which the production of signs was been particularly meaningful and rich.

First sessions: why the choice of the "circumference"

In these sessions, teachers work on the spatial orientation, on different viewpoints and on topological notions of "inside" and "outside". The children physically constructed (in the gym) a path with different materials: circles in plastic, wood blocks, and nine-pins. Then, they are asked to drawing on a sheet of paper the path from different viewpoints. Starting from this, some drawings aren't accepted from all the children because of the circle drawing. Thus, the teacher asked: "what is a circle?". First of all, children categorized drawings. As expected, children's attention was focused on the narrative aspects such as the possible similarities/differences among shapes: the drawings look like eggs, strawberries and so on. Then, they generalized by distinguishing circles as closed curves and "not circles" as opened curves. Then, in order to support the notion of position in the space and spatial reference system, as well as topological notions, children are asked to put themself inside a plastic circle located on the gym floor or outside it, inside a carton box or outside it, inside a circle of children or outside of it. This support also the categorization of "closed" and "opened" curves. We observe that different kinds of space where children have to work, affects the children's cognitive activities during exploration: in micro-space, where they draw circle and they have a global vision from a single viewpoint, the exploration is possible through the sight; instead in macro-space, being each child included in space, the exploration is possible moving in it, taking different viewpoints. In the exploration of these spaces, the reference system is egocentric and, at this point, children identify a circle as a "closed round".

Session 3: construction of a circle

In this session, the teachers choose wood blocks as artifact. Children are asked to build circles by wood blocks (Fig 1a); then, teachers asked to produce a circle of children holding hands (Fig. 1b). In this last case, the artifact is the class of children themselves.

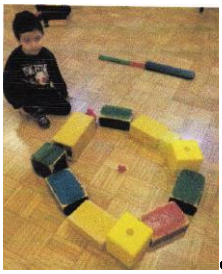
a) b)

Figure 1: a) circles of wood blocks; b) circle of children holding hands.

Teachers set these tasks in order to consolidate the characterization of "closed round shape" through different viewpoints. In this activity, some children make a rotation of their index finger to show their schoolmates how they have to place in order to have a circle of children holding hands. Once the circle of children is realized, spontaneously a child takes the central position inside the circle saying: "I'm the center of the circle". Various signs were produced in verbal and gestural forms: the rotation of the index finger is a gesture linked to the concept of “trace”. This simulates the trace of the circumference (a “closed round shape”) and, for this, is an iconic gesture. The circle has still purely meaning of shape in static sense, as a set of points (children or wood blocks). The word “center” refers here to a privileged position inside the circle, approximately at the middle of it. It is a verbal sign that denotes a “special position” inside the circle, whose children don’t refer in term of "point" and "measure" (length). This word plays the role of pivot sign that teacher will have to transform in geometrical sign ("center of a circumference”). In this two tasks, a change of viewpoint can be observed: using the wood blocks to construct a circle, children take an external viewpoint with respect to the circle and they have a global view of it; instead, taking his/her own place in the circle of children holding hands, each child takes an internal viewpoint and, from this position, he/she can't describes circle through a global view. This is true for all children except for Viola who are asked to describe circle of children from an external position (out of the circle). Since it is not possible to take a single viewpoint for the exploration, she takes a self-centered point of view always moving inside and outside of the circle, but when she tries to read the regularity of the form in terms of distance of the children from the center, the reference becomes allocentric and she considers the center as privileged point. She says: "Pietro is in the middle of the circle". This confirms that the chose of the viewpoint isn’t linked to the child's cognitive development, as considered for a long time according to Piaget's theory, rather to the tasks he/she has to do.

Sessions 10 and 11: pivot signs “sticks” and “center”

The teachers’ aim was to transform the pivot sign "center", linked to the meaning of "privileged point", into math sign linked to the meaning of "center of a circumference". Teachers ask children: “how we can be sure that this point is really at the center?". Children have at disposal different artifacts: wooden sticks of the same length and different length, skittles, paint, brushes, rope. Below we show an excerpt from an important exchange between a child, V, and the teacher, T, from which stemmed a *pivot sign* ("stick") that was eventually embraced by the class and that would eventually be related to the *radius* considered and named as the "equal distance from the center". We observe that, in previous sessions, V introduced the term "center" as Pivot sign linked to the meaning of "privileged point".

V: we have to measure with the feet that is to say, we have to put a foot consecutively to the other [She tries but the shape of circle isn't satisfactory because schoolmates were always moving and they were changing their position in the circle].

V. proposes to use wood blocks and children construct the distance between center and their own position placing consecutively wood blocks. But, since the blocks have different lengths, the distances from the center aren't equal, as V. wanted. The children realized it; someone makes a sign with his finger to show that the shape of the circle is not regular, that is, not "round" even if it is closed.

V: we can try by the sticks! [Each child takes a stick and they put each stick from the center to edge.]

T: ok guys, what you have obtained?

V: a circle, and we all are distant equal! [Fig. 2a]

In this excerpt, a combination of movement of body (to get the equal distance from the center by feet), gestures (with fingers to show the round and closed shape), words ("center", "equal distance") and drawings appear. The word "sticks" associated to meaning of "equal distance" plays the role of pivot sign. We observe that it is the embryonic definition of "radius" through which, together with the pivot sign "center", children will construct the circumference. The class embraces the pivot sign "sticks" when teacher asks children to represent, on a round sheet, what they have obtained (Fig 2b)


a)


b)

Figure 2: a) sticks from center to children; b) figural representation of the circle of children handing hands

Obviously, this isn't a construction in geometrical sense, but we can recognize an embryonic geometrical construction of circumference because of the invariants taken into consideration (center point and sticks). Moreover, we observe that the representation obtained (Fig. 2a and b) still has the meaning of a curve in a static sense, as a set of points generated by the positioning in sequence of points or, if you want, a repetition of action (a child for each stick). As result, some children correctly put into relation the sticks and the extreme points to the center (10 children, Fig 3a), but others don't (3 children, Fig. 3b). Some children consider only the circle and the center (as preferred position) without put them into relation by sticks. (8 children, Fig. 3c)

a) b) c)

Figure 3: a) correct relation among center, sticks and extreme points; b) wrong relationship; c) circle and the center as preferred position

Sessions 12

Teachers ask children "what is a circle" in order to make explicit the geometrical relations among center (central position), radius (sticks), and points on the circumference (children)

T: if you were to explain to other children what a circle is, what would you say?

R: we take a point, the center, and we put down the sticks, all equal sticks, starting from the center. We take the same **measure**, the same **distance**, starting from the center

T: so, what is a circle?

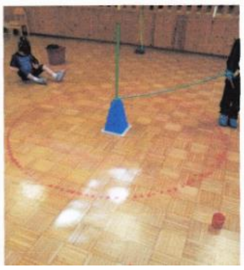
F: it's a "round thing" and there are **equal lines** from the point, the center.

We observe that children, in order to identify "equal length" from the center, embrace the pivot sign "sticks". Moreover, to answer the teacher's request "what is a circle", R describes the circle construction procedure. The action of construction as element identifying and characterizing a shape (not yet a geometrical figure) in perceptual terms (round thing), is still dominating with respect to the generalization. Nevertheless, we observe that some of the words used by children to describe the construction procedure have changed during these last sessions: from words closely related to material experience and perception (sticks, equal sticks, point in the middle) into words related more, but not still completely, to geometry (equal line, measure, distance, center). However, we can observe that the pivot sign "sticks" is now explicitly related not only to "equal length" but also to the "distance from the center", which relays to the meaning of radius. The pivot sign "sticks" and the associated math sign "radius" are put into relation through "measure". We still detect words referring to perception ("round thing") to identify the trace of the circumference. This shows that the trace is not yet identified as a curve generated by the movement of a point, dependent on a fixed point (center).

Session 13: the pseudo-definition of circumference

Teachers ask children to draw a circle on the floor and they provide them with different materials: a cone, a rope, a brush, a stick and some paint. The action of greatest importance for the group seems to be measuring equal distances starting from the center.

To this aim, children use different units of measure, more or less effective: the can of paint, the stick, the brush and the rope. The latter seems to be the most effective if anchored to the cone placed at the privileged position assumed as the center of the circumference. To define the trace of the curve, the children fixed the brush to the other end of the rope and then, dipping brush in the paint and twisting around the cone (center), they traced a set of points on the floor (trace of the circumference). These points, together, define the curve (Fig. 4a, b). Here, the curve is conceived in dynamic sense as a movement of the point around a center. The characterization of circumference produced by children at the end of the teaching experiment is the following: *A circle is a "round" made of many joined dots that have the same distance (the same measure) from the center point.* I label this sentence as "pseudo-geometric definition" because it still refers to perceptual elements linked to the shape ("round") even if it is possible to identify the dynamic nature of the curve as a trace generated by the movement of a point (the tip of the brush) that maintains the same distance from the center. The choice of artifacts and tasks seems to have helped teacher to transform artifact signs (dots on the floor, children in the circle of children holding hands...) into math signs (trajectory, understood as "the set of successive positions assumed by a moving point").


a)


b)

Figure 4: a) set of points on the floor; b) curve

DISCUSSION AND CONCLUSION

Different artifacts, different signs related to these artifacts and the transformation of them into mathematical signs through the teacher's mediation, allowed children to concept the pseudo-definition of circumference. In order to reach this aim, some cognitive and educational aspects seem to play a key role in this teaching experiment: perception and bodily experiences (construction and description of circle of woods blocks, or circle of children being inside or outside it…) are fundamental for the development of conceptualization, for this, the math activities were contextualized into different kinds of spatiality (internal and external space). In particular, children work into Macro-space and Micro-space and, in order to explore and describe them, they need of different systems of reference and different viewpoints. Moving from egocentric to allocentric references seems to allow children to identify the invariants of the geometrical figure (center and distance from the center). What favor this transition seems to be both the variety of artifacts (blocks, drawings, children) and the possibility to discuss about the signs produced (gestures, words, body movements, drawings...). Even if no experience has been developed in geometric model of space, children seem to be able to visually treat information (Bishop, 1988) by moving from figural language ("round", "closed round", "sticks"...) to a "pseudo-abstract" language (points having equal distance from center, equal length, measure, distance from center), what Radford names contextual generalization. For this, we named the reached definition a "pseudo-definition" of circumference.

References

Bartolini Bussi, M. G., & Baccaglini-Frank, A. (2015) Geometry in early years: sowing seeds for a mathematical definition of squares and rectangles. *ZDM Mathematics Education*, 47(3) (this issue). doi:10.1007/s11858-014-0636-5.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: artefacts and signs after a Vygotskian perspective. In: L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (pp. 720–749, 2nd éd.). Mahwah: Erlbaum.

Bartolini Bussi, M. (2008). *Matematica. I numeri e lo spazio*. Azzano San Paolo: Junior.

Bryant, P. (2008). Paper 5: understanding spaces and its representation in mathematics. In: T. Nunez, P. Bryant, & A. Watson.

Bishop, A. J. (1988), *Mathematical eculturation: a cultural perspective on mathematics education*, Dordrecht, Kluwer Academic Publishers.

Gallese, V. & Lakoff, G. (2005). The brain’s concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology* 22, 455-479.

Lakoff, G. & Nùñez, R. (2000*). Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. New York: Basic Books.

Nemirovsky, R. (2003). Three Conjectures Concerning the Relationship between body activity and Understanding Mathematics. *Proceedings of PME 27*, Hawaii (US), Vol. 4, pp.113-120.

Owens, K. (2015). *Visuospatial reasoning: An ecocultural perspective for space, geometry and measurement education*. New York: Springer.

Radford, L., Edwards, L. & Arzarello, F. (2009). Beyond words. *Educational Studies in Mathematics*, 70(3), 91 - 95.

Sinclair, N., & Moss, J. (2012). The more it changes, the more it becomes the same: the development of the routine of shape identification in dynamic geometry environments. *International Journal of Education Research*, 51–52, 28–44.

USING CRITICAL EVENTS IN PRE-SERVICE TRAINING: EXAMINING THE COHERENCE LEVEL BETWEEN INTERPRETATIONS OF STUDENTS' MATHEMATICAL THINKING AND INTERPRETATIONS OF TEACHER'S RESPONSES

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The participants of this study were asked to report on critical events that hold the opportunity for teachers' decision-making to build on students' mathematical thinking. The goal of this study is to analyze the coherence between pre-service mathematics teachers' interpretations of students' contributions and their interpretation of teachers' responses in these critical events. An analysis of 37 reports indicates that while PTs show the ability to identify critical events and to interpret students' thinking within the event, there is low coherence between their interpretations and their analyses of the teacher's responses. One possible implication is that teacher educators and expert teachers who train pre-service teachers should be more explicit when discussing the reasoning behind teachers' responses to students' mathematical thinking.

INTRODUCTION

Classrooms are a complex environment in which many things often happen simultaneously, and the teacher is required to exercise constant decision-making. Some moments may be more critical than others in terms of the opportunity they provide for teachers to build on students' mathematical thinking (Stockero & Van Zoest, 2013). Considering those critical moments' potential to foster students' learning, Stockero and Van Zoest (2013) suggested that training for pre-service mathematics teachers (PTs) should focus on both identifying critical moments and thinking about how the teacher might best respond in order to optimize students' learning. To date, researchers have examined each of these goals separately, and there is limited research regarding how PTs construct a coherent view of teaching practice (e.g., Barnhart & van Es, 2015). In this study we asked PTs to identify critical events they witnessed during their clinical training observations and to use a structured critical event report (1) to interpret the mathematical thinking of the students (2) to interpret teachers' actions and (3) to offer an alternative response as if they were the teacher responding to the critical event. The work described in this paper is a part of a longitude research project that examines the use of critical events in teacher training over the course of three years, from teaching certificate clinical training to the second year of in-service teaching. This paper focuses on the first stage of PTs' training. We examine the level of coherence in

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PTs identifying, interpreting, and responding to student mathematical thinking as reflected in their critical events reports.

Literature review

The proposal to use Pedagogy of Cases was raised in 1986 by Shulman (L. Shulman, 1986), who referred to a case as a description of an event that represents a broad pedagogical phenomenon or a dilemma with theoretical aspects. In PTs' training, cases serve as a focus for teachers' guided discussions, offering an opportunity for reflection on teaching and connecting teaching practice with teaching theories (Putnam & Borko, 2000). Unexpected moments often exhibit valuable opportunities for students' mathematical learning (Rowland & Zazkis, 2013). Stockero and Van Zoest (2013) define these moments of opportunity for the teacher to build on students' mathematical thinking as *pivotal teaching moments* (PTMs). They underline the importance of teachers' training programs promoting awareness toward those PTMs, as "this preparation would improve beginning teachers' abilities to act in ways that would increase their students' mathematical understanding" (p. 125). In these moments teachers' ability to "build capital" depends on their knowledge for teaching and awareness of the mathematical potential for learning (Rowland & Zazkis, 2013). In the words of Mason: "It is almost too obvious to say that what you do not notice, you cannot act upon" (p. 7). Mason defines *noticing* as being awake to possibilities, being sensitive to the situation, and responding appropriately. More recent researchers refer to professional noticing as (1) attending to student thinking within student-teacher interactions, (2) interpreting student understanding based on these interactions, and (3) offering a response based on this analysis (e.g., Jacobs, Lamb, & Philipp, 2010). The premise of teacher noticing is that the power of teachers to regularly elicit, and build their instruction on, children's thinking is linked to their ability to interpret children's mathematical thinking. Based on these noticing skills teachers formulate their immediate responses in the classroom; for example, deciding to ask students for more clarification or to follow up on a student's comment. Connecting instructional methods to the mathematics and to student mathematical thinking is considered a part of the *Mathematical Knowledge for Teaching* (MKT: Loewenberg Ball, Thames, & Phelps, 2008). So far, researchers have investigated PTs' different noticing-related skills - attending, analysis and response - separately. Some researchers focused on PTs' ability to attend to student thinking in a lesson; others focused on PTs' abilities to articulate learning goals. There is limited research regarding how one skill relates to the others (e.g., Barnhart & van Es, 2015). Barnhart and van Es (2015) characterized the relationships science PTs drew between attending to student thinking, analysing it and responding. They indicated that being attentive to students' thinking does not promise in-depth analysis or response. In the current research, PTs were asked not only to analyze students' thinking but also to analyze the observed teacher's responses, in order for us to investigate the extent to which PTs have students' mathematical thinking in mind when interpreting teachers' responses. Our research question is: what is the level of coherence between pre-service mathematics teachers' interpretations of students' mathematical thinking and their interpretations of teachers' responses, when describing a critical event they witnessed during clinical training?

Research context: ACLIM-5 clinical training program

The study took place in the context of ACLIM-5, a large university's clinical training program for unique 5-unit mathematics teaching (high track mathematics program in secondary school). ACLIM-5 is a three-year training program designed to support the development of high track mathematics teachers through the use of critical events. Due to the limited space of this paper, and that the program is not at the focus of this research, we will not elaborate regarding the program details. For clarifications reasons we will note that this study focused on the first year of the program, while PTs study for their teaching certificate. PTs participated in critical events course in which they were required to submit reports, based on lessons' observations. The reports consist of three main parts: (a) prompts for describing the critical event, the mathematical context, and the student-teacher interaction. (b) prompts for interpreting the students' contributions. E.g., "offer interpretations for the student utterances: 'what was going through his mind?' on what is it grounded?", (c) prompts for interpreting the teacher's responses "offer interpretations for the teacher's actions: 'what was going through his mind?' on what is it grounded?", "offer different ways of responding, other than the teacher's response. explain.". These reports served as the data source for this study.

DATA COLLECTION

This data for this paper was taken from reports submitted in the first year of the first ACLIM- 5 course. Course participants were one male and 12 females. Five studied toward a dual degree in Mathematics and Education, and eight graduated with a B.Sc. from the university's department of Mathematics. The data consist of 37 critical events reports describing real-life classroom events from PTs' observations, submitted in three cycles during the academic year.

DATA ANALYSIS

Data analysis was conducted in two phases. First, we categorized the critical events chosen by the PTs by using Stockero and Van Zoest's (2013) framework for PTMs, while being open to the possibility of new categories coming up. The framework describes four types of critical events: (a) when students ask a question or make a comment beyond the mathematics the teacher plans to discuss; (b) when students try making sense of the mathematics in the lesson or express mathematical confusion; (c) when students make a mistake; and (d) when a mathematical contradiction occurs. A new category emerged in our bottom-up analysis: (e) when a student suggests an original solution.

In the second phase of analysis, we examined the PTs' interpretations of (a) the students' contributions in the event and (b) the teachers' responses, and (c) their suggestions for possible alternative responses. The analysis was adapted from Barnhart and van Es' (2015) category scheme, with some modifications made to allow us to focus on the coherence between interpretations of students' expressions and interpretations of teachers' responses (table 1 describes the three-level coding scheme). "High coherence" means that the PT interprets the students' contributions while taking onto account the students' mathematical thinking and the mathematical context of the critical event as described in the report. While interpreting teacher's responses, "high coherence" indicates that the PT kept the student's mathematical thinking in mind when interpreting the teacher's statements and response. "Medium coherence" level indicates that in the interpretation of the teacher's response the PT attended to the student's thinking in a general way, without considering the particular mathematical thinking. "Low coherence" level indicates that the PT did not attend to the student's thinking when interpreting the teacher's statements and response.

	Low coherence	Medium coherence	High coherence
Interpreting student's contributions	Highlights classroom events, teacher pedagogy, student behaviour and/or classroom climate.	Highlights student's thinking without respect to the mathematics of the critical events.	Highlights student's thinking with respect to the mathematics of the critical events.
Interpreting teacher's responses	Highlights teacher's pedagogy, student behaviour, and/or classroom climate. No attention to student thinking.	Makes sense of teacher's statements while relating to student's thinking with no respect to the particular mathematics of the event.	Consistently makes sense of teacher's statements while keeping student's thinking in mind, with respect to the mathematics of the specific situation.
Responding as the teacher in the critical event	Offers vague general ideas of what to do differently next time. The idea does not build on a specific student idea.	Offers a general idea of what to do differently next time. The idea does not build on the student's mathematical thinking and relates to general mathematics instruction.	Offers a specific idea of what to do differently next time. The idea is built on the student's mathematical thinking and takes the mathematics of the critical events into account.

Table 1: Three-level coding scheme for interpreting student's statements, teacher's statements and responses to the critical event.

PRELIMINARY FINDINGS

PTs' interpretation and response

The first phase of data analysis indicated that all PTs were able to identify PTMs during which teachers had an opportunity to build on students' mathematical thinking and promote learning. PTs were more attentive to moments when a student suggested an original solution to a given problem (41%) and when trying to make sense of the mathematics in the described critical event (30%). The results of the second phase are noted in table 2.

	Low coherence	Medium coherence	High coherence	No evidence
Interpreting student's contributions	14%	41%	43%	3%
Interpreting teacher's responses	62%	30%	0%	8%
Alternative responses	62%	14%	3%	22%

Table 2: Results of the three-level coding scheme: interpreting student's statements; interpreting teacher's statements; and responding as the teacher in the critical event.

As shown in table 2, the majority of PTs' interpretations of students' statements were of medium coherence (41%) or high coherence (43%). While interpreting teachers' statements, the level of coherence was low (62%) or medium (32%). When asked to respond in lieu of the teacher during the critical event, the majority of PTs gave a response with low coherence (62%) or did not offer any alternative (22%). Meaning that when PTs interpreted students' statements they usually considered student thinking with respect to the mathematics of the critical event, but when interpreting teachers' statements or offering alternative responses, PTs focused on the teacher's general pedagogy, student behaviour, and/or the classroom climate without attention to student thinking.

Adel's example

We will demonstrate PTs' common interpretation of student's contributions and teacher's response using Adel's critical event report. Further examples will be presented at this paper's presentation. Adel described a critical event when 'Mr. Jones' plot' problem was presented in a 10th grade calculus class (figure 1). Adel's critical event revolved around one student's original solution. The student stated that: "when the width and the length are equal, meaning when it's a square, the width is 350m and the length is 350m." In her report, she described the critical event:

> At the beginning the teacher didn't say anything. Then, when the student repeated his answer, more than once, the teacher told him to wait a few minutes and that they will

> discuss it later. After the teacher solved the problem the way he wanted, meaning after explaining about building a target function, finding extremum and determining their type, the teacher went back to the student's answer. Together they analyzed why his answer was correct. The teacher asked a couple of questions: "Why does the square give us the maximum area?" The student said: "it is always true, because 25 times 25 is bigger than 24 times 26." The teacher asked the students: "$a \times b$, why is it maximum when $a=b$?". After posing questions, the teacher summed up: "a and b are variables but their sum is constant so $b = c - a \rightarrow a(c-a)$ and therefore $a \times b$ is maximum when $a(c-a)$ is maximum. And how do we know when it is maximum?"

Mr. Jones has a rectangle shaped plot, its length is 400m and width is 300m. Mrs. Jones wants to shorten its length by x meters and to lengthen its width by x meters, in order to maximize its area. Find x.

Figure 1: 'Mr. Jones' plot' problem

Adel interpreted the student's statements as follows:

> The student thought immediately… according to his previous knowledge (you can tell by the example he gave, he had those numbers in his mind…). I think that he looked differently on this problem, he said: "length times width, so when is it maximum? When it is the average of both numbers (width and length)" … He disconnected from the possibility that the question is linked with "derivatives, extremum points and so on."

Adel's interpretation reflects the student strategy as relying on previous knowledge. Although she was not explicit regarding the specific previous knowledge, it can be implied from her saying: "you can tell by the example he gave, he had those numbers in his mind". This can be assumed to refer to the student's example $25 \times 25 > 24 \times 26$, which is a commonly used numeric example in middle school. By writing "when it is the average of both numbers (width and length)" she also implicitly addresses $a^2 > (a-1)(a+1)$. Her last sentence offers a possible reasoning for how this path of thought happened: "he disconnected from the possibility that the question is linked with…" Therefore, Adel's interpretation highlights the student-thinking process with respect to particular mathematics, so her interpretation level is high coherence with respect to the mathematics of the critical event.

In her interpretation of the teacher's response to the student's answer, Adel wrote:

> That was the first lesson on extremum problems, and this ('Mr. Jones' plot' problem) was the first question in class. So, in the beginning, the teacher had to explain to them how this topic relates to the previous topic (derivatives)… So, he solved it in the way he should teach, and then addressed the student's answer so he will understand the method he used…

Most of Adel's interpretation highlights the teacher's general pedagogy. She interpreted the teacher actions as pursuing his lesson's plan, teaching extremum problems using the previous topic, derivatives. By the end of her interpretation she relates to student, but she does not relate to the student's thinking but to the teacher's answer. Adel suggested that the teacher was trying to gain better understanding but she did not articulate the understanding the teacher was trying to achieve. Was the teacher trying to understand the student? Or was he figuring out whether the student understood his own answer? Therefore, Adel's teacher's interpretation level was of low coherence with the student mathematical thinking. One might perceive the teacher's question "why does the square…?" as a bridge between the student's utterance and the teacher's lesson goal, teaching extremum problems using derivatives. Exploring the potential benefits of 'Mr. Jones' plot' problem - simple numbers and known properties of the area of a rectangle and a square - may cast the teacher's response in a different light. The teacher might have wanted to approach extremum problems using simple problems, so that the students would not have to deal, cognitively, with complicated calculations. In addition, the teacher may have wanted the students to draw a connection between the maximum area of a rectangle and the area of a square, leading to a general discussion later on regarding the use of derivatives as a general method to prove the maximum area of any type of polygons.

When offering a response as if she were the teacher, Adel also presented a low level of coherence with the student's mathematical thinking:

> …There is a need to tell the student that his answer is correct but we need to think in a different way, to draw connection on previous topics (derivatives)… To show them the correctness of things in general through proof.

Adel's response is vague and relates to the correctness of the student's answer, not to his thinking. Adel offered to "show them the correctness of things in general through proof" but she did not elicit what kind of generality she referred. Did she mean a more general proof for the fact that, while limiting convex polygon's perimeter, it's maximum area is when the polygon is an equilateral? Or did she refer to the teacher's proof?

In conclusion, Adel identified an original solution proposed by a student as a critical event in the lesson, but it was hard for her to address its potential as a starting point for the teacher to build students' mathematical thinking.

DISCUSSION

The PTs' critical event reports in this study reflect their ability to identify moments with valuable opportunities for student mathematical learning (stockero & Van Zoest, 2013) as well as to provide interpretations for students' statements that are anchored within the students' mathematical thinking. However, a low level of coherence was found between those and their interpretations of the teachers' responses. Leatham, Peterson, Stockero, and Van Zoest (2015) indicate that while skilled teachers may notice when critical events occur during a lesson, novice teachers fail to notice or fail to act upon opportunities to use students' thinking to further mathematical understanding. Our analysis partly challenges Leatham and et al. (2015) finding in that the PTs were able to identify critical moments and to interpret students' mathematical thinking. However, in accordance with Leatham et al., (2015) the PTs encountered difficulty "acting upon" the students' thinking when interpreting teacher's strategies. The findings indicate a 'disconnect' from the meaning given to the student's contribution when analysing the teacher's response. Instead, the interpretations of the teacher's response focused on general teaching strategies or other aspects of classroom management, without taking into account the particular student's mathematical thinking - thinking that they had highlighted just minutes earlier. A possible explanation could be PTs' lack of MKT; that is, not connecting students' thinking with teachers' decision-making and instruction (Morris, Hiebert & Spitzer, 2009). We argue that teacher educators and expert teachers who work with PTs in clinical training programs should be more explicit regarding how particular mathematics together with student mathematical thinking may shape instruction, and how teachers' actions may be built on student mathematical thinking to promote learning. The next step within our project is to devise possible strategies to support this goal.

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References

Barnhart, T., & van Es, E. (2015). Studying teacher noticing: Examining the relationship among pre-service science teachers' ability to attend, analyze and respond to student thinking. *Teaching and Teacher Education, 45*, 83-93.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for research in mathematics education, 41(2),* 169-202.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46(1),* 88-124.

Loewenberg Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of teacher education, 59(5),* 389-407.

Mason, J. (2002). Researching your own practice: The discipline of noticing. London: RoutledgeFalmer.

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn. *Journal for research in mathematics education, 40(5),* 491-529.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning?. *Educational researcher, 29(1),* 4-15.

Rowland, T., & Zazkis, R. (2013). Contingency in the mathematics classroom: Opportunities taken and opportunities missed. *Canadian Journal of Science, Mathematics and Technology Education, 13(2),* 137-153.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher, 15(2),* 4-14.

Stockero, S. L., & Van Zoest, L. R. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education, 16(2),* 125-147.

CREATIVITY OR IMAGINATION: CHALLENGES WITH MEASURING CREATIVITY

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In this article, we look closely at the relationship between creativity and imagination. Using a combination of theoretical and empirical analysis, we call into question the validity of measuring creativity by examining products and coding them for flexibility. The framework of imagination proves to be a useful lens for analysing the data.

INTRODUCTION

Since Guilford's (1967) conceptualization and Torrance's (1974) development of according tests, the factors of originality, fluency, and flexibility have been used by many researchers to measure creativity. At PME 41 we looked closely at the characteristic of originality as a metric for measuring creativity and concluded that "the originality of a solution is not a reliable indicator of the creativity of a solution" (Liljedahl & Rott, 2017) and that one needs to be careful about using products as proxies for processes when measuring creativity. In this paper, we look closely at flexibility and the feasibility of using it as a proxy for measuring creativity.

CREATIVITY

Sternberg and Sternberg (2011, p. 479) broadly define creativity as "the process of producing something that is both original and worthwhile"; while Torrance (1966) described creativity as

> a *process* of becoming sensitive to problems, deficiencies, gaps in knowledge, missing elements, disharmonies, and so on; identifying the difficulty; searching for solutions, making guesses, or formulating hypotheses about the deficiencies: testing and retesting these hypotheses and possibly modifying and retesting them; and finally communicating the results. (Torrance, 1966, p. 6)

Nearly all definitions have in common that being creative is regarded as a process and that a creative idea or product is something new, unique, or original. Many approaches to conceptualize (and measure) creativity can be traced back to the ideas of Guilford who differentiated between convergent and divergent thinking. With convergent thinking, he describes logical deductions and using analogies as well as relying on previous knowledge and finding the answers to routine tasks. On the other hand, divergent thinking is described as generating many ideas in a 'non-linear' manner, combining ideas, drawing unexpected conclusions, and, thereby, coming up with original ideas (cf. Guilford, 1967).

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 59-66). Umeå, Sweden: PME.

Measuring Creativity

Because of its importance (e.g., for problem solving) and it being part of intelligence models, there have been several attempts at measuring creativity. Besides trying to measure it by using problem solving and/or problem posing tasks, the construction of creativity tests mainly goes back to Guilford and – building on his groundwork – Torrance. In the 1940s and 50s, Guilford had identified four factors of divergent thinking, namely fluency, flexibility, originality, and elaboration. The basic idea of the tests by Guilford and Torrance is to code test subjects' solutions, assigning points for those four factors, and use the points to generate a total test score. This basic idea is still used in current tests and will be illustrated by the example of the Multiple Solutions Task (MST) test for mathematical creativity by Leikin and Lev (2013).

The participants are asked to find as many solutions as possible to a given problem (MST). For each solution, it is decided whether it is appropriate (measuring *elaboration*). To identify the score for *fluency*, i.e. having many ideas, the number *n* of appropriate solutions is counted. The score for *flexibility*, i.e. having different ideas, is determined by addressing 0.1, 1, or 10 points to each appropriate solution (see below for details). And, finally, the score for *originality*, i.e. having unusual or unique ideas, is similarly determined by rating each appropriate solution with 0.1, 1, or 10 points; this rating is done by comparing the solutions to those within a reference group (it cannot be checked whether a solution is new for the participant).

To evaluate flexibility (the focus of this paper), a solution space is established by grouping experts' solutions to the MST: solutions belong to separate groups if different strategies, representations, and/or properties are used to obtain them. The participants' solutions – in the order in which they were found – are sorted into those groups. If a solution stems from a group that has not been used before, it gets 10 points. A solution gets 1 point, if its group is not unique but it is clearly distinct from previous solutions. If a solution is almost identical to a previous solution, it gets 0.1 point.

The creativity score is calculated by multiplying the flexibility and originality scores for each solution and summing them up. A solution that gets 10 points for each originality and flexibility gets $10 \cdot 10 = 100$ points. This results in a "decimal" scoring system; for example, a total score of 231 suggests two solutions that were recognized as highly flexible and original. Overall, flexibility has a big influence on the total sum.

As can be seen, this scoring system for creativity focuses on the number of different solutions without being able to ensure that those solutions are original sensu Sternberg or Torrance (see above). Therefore, we look into the framework of imagination.

IMAGINATION

Imagination is the ability to see things other than as they are (Greene, 2000) – as reaching out from where you are (Greene, 2000) along lines of conceivable trajectories as determined by your own experiences (Dewey, 1933; Whitehead, 1959). It is the capacity to transcend the actual and to construct the possible, and the impossible.

To understand the imagination we need to experience the imagination. Thus, we begin with a thought experiment. Imagine, for an instant, an animal that lives on a distant planet – a planet with an atmosphere, a day and a night, water, and vegetation (Egan, 1992). What does it look like? Is it unique? Is it unusual? Is it conceivable? Most likely, the creature of your imagination is rooted in some experience you've had (a real animal or a movie creature) along with some standard modifications (fangs, extra limbs, etc.). Perhaps it is a giant winged lizard with horns and colorful stripes or a horse-like creature with a lion's mane and three tails. Regardless of your animal, however, some things are likely true. Although it may be unique and unusual, it will likely have some even number of limbs, or wings, or both. It will propel itself by walking or flying or swimming. In essence, it will be recognizable as an animal. You cannot avoid it. This is because when we imagine we reach out from where we are, not blindly or randomly, but along conceivable trajectories. That is, we build our animal from a repertoire of features and characteristics of things that are animal-like. The combinations and permutations of such features allow for endless possibilities of animals that we can conceive, but they will all be animal-like. Thus, while our imagination may be limitless, it is not unbounded; it is constrained by the conceivable.

This is not to say that the imagination can be reduced to a variation on a theme – a twisting of some recalled experience. What is explored above is meant to be a description of the imagination in action not a prescription for action. When we imagine we are constrained by what we can conceive. This constraint is real and undeniable. It limits and guides our imagination, but the imagination is still free to seek out unique and unusual possibilities within these bounds. To reduce this process to a prescription of intentionally making a slight variation to an old idea – a blue cow, a stripped giraffe, etc. – is an oversimplification of the imagination, at best, and a perversion of the imagination, at worst. The imagination is constrained by the conceivable, not controlled by it. To be otherwise, reduces the imagination to the mundane and the predictable. Although there are constraints and intentionality exercised over it, the imagination still possesses a quality of autonomy to it.

CREATIVITY OR IMAGINATION

Both the imagination and creativity lay outside of the logical forms – they are extra-logical processes (Dewey, 1938) capable of producing ideas and solutions that lie beyond what could normally be produced by reason alone. Creativity and imagination leap ahead of reason, the slow moving and ponderous master, to scope out possible new realities (Kasner & Newman, 1940). This has already been discussed in regards to the imagination's ability to generate unique, yet conceivable, ideas. Unlike the imagination, however, creativity is not constrained by the conceivable. It is capable of producing ideas and solutions that go beyond, not only reason, but also the conceivable. Through creativity, ideas can be generated that are qualitatively different and not linearly attributable to any one prior notion. So, whereas the imagination will produce a horse-like creature with a lion's mane and three tails, creativity can produce a creature that resembles nothing we have seen before, perhaps in a gaseous form existing in time rather than moving through time.

The imagination is not to be viewed as the poor second cousin to creativity, however. The imagination is the source of creativity. It is from the limits of the imagination that creativity takes off; leaping over boundaries of conceivability to explore what lies beyond. Once there, however, the imagination is once again free to explore the bounds of the newly constructed possibility. It is as if creativity carries the imagination across these boundaries, these barriers of conceivability. This is why a creative experience is often referred to as a leap of the imagination (Greene, 2000).

Given the close relationship between imagination and creativity in general, and the distinction between the conceivable the inconceivable in particular, we wonder what the lens of imagination might help us see in creativity data. In particular, we wonder if the imagination may not be a better descriptor of data previously coded as flexibility. This is our *research intention*.

METHODOLOGY

The data for this study was originally collected and analyzed in a German project on mathematical giftedness in upper secondary school (MBF_2, led by Maike Schindler and Benjamin Rott). Twenty students from grades 11 and 12 (age 16 – 18) participated voluntarily in this project, coming to the university every second week (cf. Schindler, Joklitschke, & Rott, in press). In this project, among other activities, mathematical creativity was measured using the previously explained rating scheme by Leikin and Lev (2013). Two of the used MSTs are presented in Figure 1.

For the results presented in this report, the authors consensually discussed and re-analysed selected solutions using the framework of imagination in general and the notion of the conceivable in particular (see above).

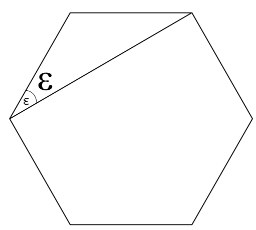
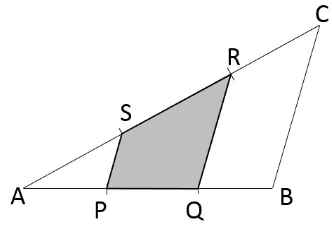
Solve the following problem. Can you find different ways to solve the problem? Find as many ways as possible. This figure is an equilateral hexagon. How big is the angle ε? Remember, in an equilateral hexagon, all sides have the same length and all angles have the same size, which is 120°.	
Solve the following problem. Can you find different ways to solve the problem? Find as many ways as possible. This figure is a triangle ABC. The points P, Q resp. R, S divide sides AB resp. AC each in three equal parts. What is the area of the quadrangle in comparison to the area of the triangle?	

Figure 1: The *hexagon* problem (source: D. Brockmann-Behnsen) and the *triangle* problem (source: Novotná, 2017)

RESULTS AND ANALYSIS

In what follows, we look at four cases from this analysis, focusing on the flexibility scores and the re-interpretations using the framework of imagination.

Case 1: Thomas – *hexagon* problem

Thomas found $n = 6$ different ways to solve the hexagon problem, four of which are given in Fig. 2 (left). (1) His first approach was to measure the angle ε, which was not recognized as an appropriate solution. (2) His second approach was to subtract a right angle from the 120° degree interior angle, what leads to ε = 30°. (3) After that, he drew a right-angled triangle and calculated ε by using the sum of the angles of a triangle (180° – 90° = 30°). (4) He used an isosceles triangle to calculate 2ε = 180° – 120°.

Analysis: Thomas' second and third approach were so different that they each gained 10 points for the flexibility score in the creativity measurement (the fourth approach gained only 1 point for flexibility as it uses mostly the same idea as the third approach). Using the framework of imagination, however, reveals that all approaches are rather imaginative than creative. Each of his solutions is a conceivable extension of something he has experienced before as there is no doubt that in his school career, Thomas had previously divided geometric figures into smaller figures (like triangles) by drawing auxiliary lines. For example, the formula for calculating the sum of the interior angles of polygons is derived by dividing a polygon into triangles.

This is not to say that all instances of drawing auxiliary lines are conceivable extensions of past experiences. But in the case of a hexagon, there are only two interior auxiliary lines that can be drawn and neither requires a great leap to conceive of.

Case 2: Christa – *hexagon* problem

Christa solved the hexagon MST with $n = 5$ approaches, all of which were assessed as appropriate. Four of her approaches are shown in Fig. 2 (right) and two of those approaches are presented here. (1) In her first approach, she halved the 120° interior angle of the hexagon by drawing a diagonal to the opposite vertex; she then argued that the given diagonal halves the remaining angle, resulting in ε = 120°:4 [the second step could be argued more rigorously, though]. In her second and third approaches, Christa mostly used the same argument of dividing the given angles. (4) Her fourth approach was similar to the third approach by Thomas (right-angled triangle).

Analysis: Like Thomas, Christa had two solutions that each scored 10 points for flexibility. However, both her first and fourth approach were not creative in the sense that something subjectively new had been discovered. Like Thomas, Christa experienced division of figures and symmetry arguments in the context of arguing for angle sizes in her school education; therefore, here ideas are conceivable extensions of previous experiences – they are a product of her imagination.

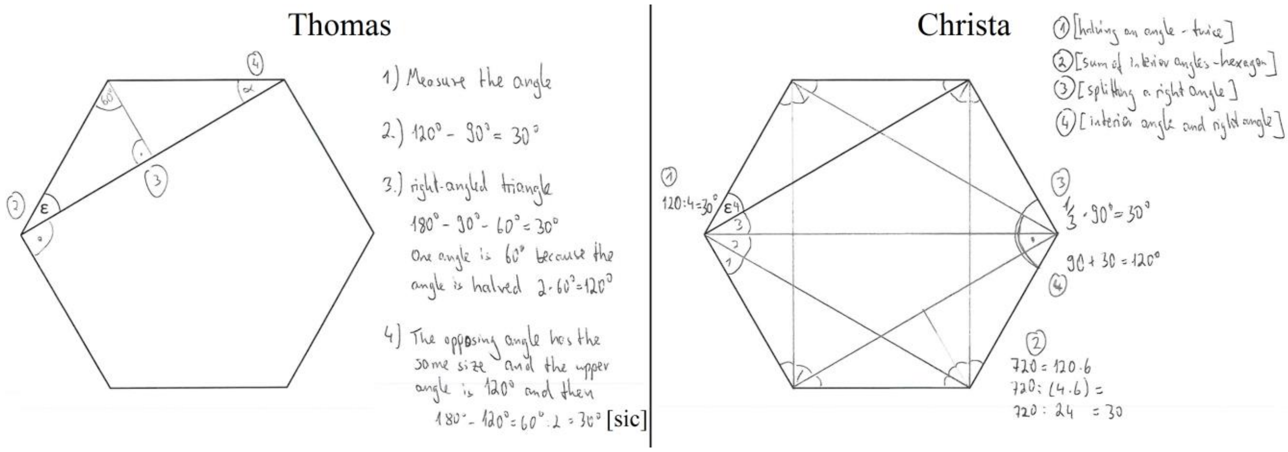


Figure 2: Excerpt of the solutions by Thomas and Christa to the *hexagon* problem

Case 3: Thomas – *triangle* problem

Thomas completed $n = 2$ approaches to the triangle MST (see Fig. 3), all of which were assessed as appropriate. (1) In his first approach, he divided the triangle into three congruent parallelograms and three small triangles with the area of the triangles being half the area of the parallelograms. By stating that each parallelogram has an area of 1 and each small triangle has an area of 0.5, he was able to show that the ratio of the white and grey areas is 3 : 1.5; or, differently, that the grey area makes up one third of the area of the given triangle. (2) In his second approach, Thomas used a point reflection to complete the given triangle to a parallelogram in which the grey area can easily be identified as one third of its total area.

Analysis: Both of these approaches were sorted into different groups of the solution space and, therefore, each scored 10 points for the flexibility score. With the lens of imagination, both approaches do not look creative as they are easily conceivable for him. In his school education, Thomas had previously divided geometric figures into squares or rectangles to estimate their areas. For example, deriving the area formula of a rectangle by dividing it into squares is mandatory content in German mathematics curricula. The same is true for combining figures (e.g., two right-angled triangles to form a rectangle or two trapezoids to form a parallelogram) which is also mandatory in Germany and, therefore, also known to and conceivable for Thomas.

Case 4: Christa – *triangle* problem

Christa solved the triangle MST with $n = 3$ approaches (all assessed as appropriate). (1) Like Thomas, via point reflection, she completed the triangle to a parallelogram. (2) Her second approach (see Fig. 3) was comparable to the first approach by Thomas, she divided the given triangle into nine smaller (congruent) triangles and numbered them. Three of those triangles are grey and six are white, therefore, the grey area is 3/9 or one third of the given triangle. (3) Her third approach was similar to her second approach, she divided the given triangle into parallelograms and smaller triangles.

Analysis. Christa's first and second approach each gained her 10 points for flexibility (her third was quite similar to the second and scored only 1 point). Again, with the framework of imagination, her approaches are not interpreted as creative, but as imaginative as the idea of diving a polygon is easily conceivable for her.

The implications of the re-interpretations are discussed in the next section.

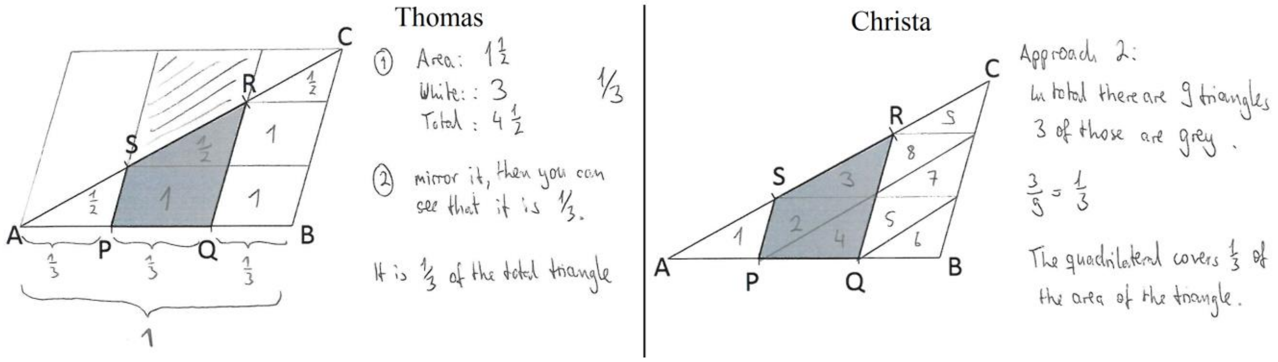


Figure 3: Excerpt of the solutions by Thomas and Christa to the *triangle* problem

DISCUSSION

In this article, we used the differentiation between imagination and creativity to attack the problem of measuring creativity from a different angle. From a theoretical perspective, imagination helps us to grasp the conceivable, whereas creativity helps us to reach the inconceivable. To make this clear, this differentiation is not the same as the interplay of convergent and divergent thinking in the terms of Guilford (1967). Both imagination and creativity are extra-logical processes that are capable of producing ideas beyond standard situations (cf. Dewey, 1938). Therefore, both belong to the category of divergent thinking.

In addition to the theoretical consideration, we used an empirical case study to support our argument. The case study has shown that imagination is a useful lens for analysing data from creativity tests. For example, the case of Thomas (hexagon problem) shows that all of his solutions were variations on problem solving techniques he had experienced before. More generally, approaches that scored maximum points for fluency were identified as imaginative, instead of creative.

Nonetheless, the implication of this is not insignificant, because – at their core – imagination and creativity are very different (cf. Greene, 2000). Therefore, this is a question of validity. If data that has been used to indicate creativity is clearly an example of imagination, this calls into question the utility of flexibility as a metric for measuring creativity. We do not say that flexibility cannot be used to identify creativity; we only say that the identification of flexibility does not imply creativity.

Why do we need an additional theory, one might wonder. "If all you have is a hammer, everything looks like a nail." A different theoretical approach might help us to focus on important aspects of creativity without being distracted by aspects that might not be of particular relevance. Taken together, the results of the research presented here argue that flexibility is not a good indicator of creative products. In a more general view, it is, however, a good indicator of divergent thinking (in the sense of Guilford, 1967), if imagination is included in it.

References

Dewey, J. (1933). *How we think*. Boston, MA: D.C. Heath and Company.

Dewey, J. (1938). *Logic: The theory of inquiry*. New York, NY: Henry Holt and Company.

Egan, K. (1992). *Imagination in teaching and learning: The middle school years*. Chicago, IL: Chicago University Press.

Greene, M. (2000). *Releasing the imagination: Essays on education, the arts, and social change*. San Fransisco, CA: Jossey-Bass Publishers.

Guilford, J. P. (1967). *The nature of human intelligence*. New York: McGraw-Hill.

Kasner, E. & Newman, J. (1940). *Mathematics and the imagination*. New York, NY: Simon and Schuster.

Leikin, R., & Lev, M. (2013). Mathematical creativity in generally gifted and mathematically excelling adolescents: what makes the difference? *ZDM – Mathematics Education, 45*, 183–197 (2013).

Liljedahl, P. & Rott, B. (2017). Creative process vs. creative product: challenges with measuring creativity. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education, Vol. 3* (pp. 161 – 168). Singapore: PME.

Novotná, J. (2017). Problem solving through heuristic strategies as a way to make all pupils engaged. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 29–44). Singapore: PME.

Schindler, M., Joklitschke, J. & Rott, B. (in press). Mathematical creativity and its subdomain-specificity. Investigating the appropriateness of solutions in Multiple Solution Tasks. In M. F. Singer (Eds.), *Mathematical creativity and mathematical giftedness. Enhancing creative capacities in mathematically promising students* (pp. x–y). Springer.

Sternberg, R. J., & Sternberg, K. (2011). *Cognitive Psychology*, Sixth Edition. Belmont, CA: Wadsworth, Cengage Learning.

Torrance, E. P. (1966).The Torrance tests of creative thinking: Norms-technical manual. Research edition. Verbal tests, forms A and B. Figural tests, forms A and B. Princeton, NJ: Personnel Press.

Torrance, E. P. (1974). *Torrance Tests of Creative Thinking*. Bensenville, IL: Scholastic Testing Service.

Whitehead, A. (1959). *The aims of education and other essays*. New York, NY: The Macmillan Company.

TENSIONS IN IMPLEMENTING MATHEMATICS JOURNALING

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Research suggests there is a strong connection between mathematical writing and mathematical learning. As a result, many educators are implementing journaling in their mathematics classroom, which can be a challenging process. This paper identifies the tensions faced by an individual teacher implementing journal writing for the first time and interprets those tensions through the lens of activity theory. The results suggest that pinpointing the areas of tension within an activity system may provide a means of mitigating the challenges.

INTRODUCTION AND THEORETICAL FRAMEWORK

It is easy to understand why teachers are interested in implementing journaling in their mathematics classrooms. One of the primary benefits is its strong connection to mathematical learning (Morgan, 1998). There is the suggestion that the act of writing can help students synthesize new ideas and make meaning between old and new concepts (Hamdan, 2005), and that it can also foster positive mathematical beliefs (Sanders, 2009). For teachers, the written reflections can assist in assessing students' mathematical understanding and "give the teacher insights into areas of confusion or misunderstanding" (Sanders, p. 437).

However, there are challenges in journaling. The biggest one relates to students' initial responses to journal writing, which is typically a new type of assignment for them. They tend to perceive it as being something outside the norm, declaring that journal writing "should not be part of mathematics class" (Williams & Wynne, 2000, p. 134). Additionally, it can be difficult for students to express mathematical ideas in writing, as Morgan (1998) suggests, "it has largely been assumed that students will learn to write through experience and that they will develop appropriate forms of language 'naturally'" (p. 2).

Tension may result as teachers attempt to maximize the benefits while minimizing the challenges. Endemic to the teaching profession, tension encompasses the inner turmoil teachers experience when faced with contradictory alternatives for which there are no clear answers (Berry, 2007). At any given moment, a teacher may experience tension with students, with the task, with content, with assessment—tension that ebbs and flows throughout the course of a lesson, a day, or a lifetime of practice. This leads to the possibility of viewing tension as a web or series of interconnections. For example, a teacher says she is experiencing tension with assessing mathematical understanding. Is the tension 'what' to assess? How to assess? Is it because parents prefer summative assessment? Is it because students have test anxiety? Is it a combination of these? All of these? If these tensions are thought of as an alarm panel, which are the sections that

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light up and demand attention? Which sections are unlit? Cole and Engeström's (1993) activity theory offers a way of understanding these tensions as interactions in, and on, a dynamic, active system. Using activity theory, I can localize the tensions within an activity while considering their global effect on the activity itself. My goal then is to identify the tensions experienced by a teacher as he implements journal writing in his secondary mathematics classroom.

Activity Theory

Activity theory is built on the assumption that, at the level of the individual, all intentional human actions are goal-directed and tool mediated (Venkat & Adler, 2008). To expand from individual mediated action to the level of collective activity, Cole and Engeström (1993) introduced activity systems, which they define as communities engaged in activities which share common goals. They suggest these systems exist within socio-cultural settings like a classroom or school and can be seen as "natural units of analysis for the study of human behaviour" (p. 9).

As illustrated in Figure 1, an activity system comprises six elements. The subject is the person, or group of people whose perspective is the focus of the analysis and the object is the overall goal of the system. Tools include anything used to mediate the activity, while rules are the explicit and implicit norms that guide and restrict the activity. The community is the person, or people, who comprise the social context in which the subject belongs, and division of labour regards the roles within.

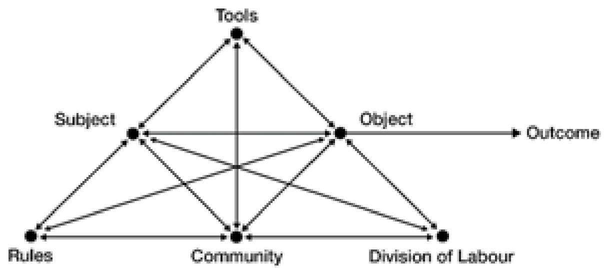

Figure 1: Activity system (Cole & Engeström, 1993)

Gedera (2015) notes that these six elements in an activity system "act as mediators and the relationships between these elements are constantly mediated" (p. 55). This suggests activity systems are dynamic and changeable. Cole and Engeström (1993) add that "activity systems are best viewed as complex formations in which equilibrium is an exception and tensions, disturbances, and local innovation are the rule and the engine of change" (p. 9). This means that the interconnections between the elements, and ways in which they influence and are being influenced, are key areas for exploration in that they "draw attention to those points where contradictions or tensions exists" (Jaworski & Goodchild, 2006, p. 55). These tensions can occur within an element of the activity system, between elements of the activity system or between connected activity systems, and are considered essential for understanding the motivation for particular actions and the overall evolution of a system more generally (Cole & Engeström, 1993).

Tensions also prove useful as a way for teachers to describe their own experiences of practice (Berry, 2007). To develop their classroom practice, it may be helpful for teachers to recognize and define these tensions (Lampert, 1985). “In the process of renaming what they know through their experience, the teachers critically reflect on—and thus begin to renegotiate—their ideas about teaching and learning” (Freeman, 1993, p. 488). To achieve this, the tensions first need to be identified at both the global level of the activity and at the local level of its constituent elements. This leads to my research question: What are the tensions the teacher experienced in the journaling implementation and where are they located within his activity system?

METHODOLOGICAL CONSIDERATIONS

Studies on tensions often rely on semi-structured interviews and self-studies as the primary means for subjects to identify and reflect on the tensions they experience (e.g., Sparrow & Frid, 2001; Berry, 2007). Inherent in revealing and understanding tensions then, is the element of reflection that originates from the subject. This study therefore, utilizes data from the subject's reflection of the activity rather than on the researcher’s direct observation of the activity itself. The rationale for this methodological decision is that it is necessary for tensions to stem from the subject’s reflections rather than their observable actions, and furthermore, to make use of the tension, the subject must come to recognize and name their own tensions. It is in reflecting on the activity that the subject moves closer to understanding the tensions they experience within their activity system.

This paper in particular is a small-scale qualitative study that seeks to prove the existence of a phenomenon rather than its prevalence. It involves one participant, Dan, a secondary mathematics teacher. One of many teachers interviewed as part of a larger study regarding tensions in teaching, Dan was chosen for this study because of a specific experience he shared regarding implementing journal writing. Data used for analysis was obtained during a one-hour semi-structured interview that was recorded and transcribed in its entirety, along with written responses to follow-up questions. Using activity theory as both the theoretical lens and the analytical tool, the data was used in two ways. First, it was used to create a descriptive narrative of the subject, Dan, as he portrays himself as a teacher. This narrative was shared with Dan, to validate its accuracy, and subsequently used to outline his activity system. Secondly, the data was scrutinized to identify tensions. This was done by first searching the transcript for evidence of tensions, that is, for utterances with negative emotional components such as “I was most disappointed with…” or utterances that conveyed doubt such as “I didn’t know what to do.” The list of identified tensions was then narrowed to include only those that stemmed from, or were related to, Dan’s implementation of journal writing. Dan’s own activity system, developed from the descriptive narrative, was then used as the frame for locating which specific elements contributed to, or were impacted by, the tensions he experienced.

Context of the Study

The next section contains two parts. First is a narrative describing Dan, which will be later used to outline his activity system. This is followed by a short description of the journaling Dan implemented in his mathematics classroom. Note that, from here on, all italicized words contained within quotation marks are Dan's.

Dan always planned to teach high school mathematics. He trained as an elementary generalist however, as he found it difficult to fulfill the senior mathematics requirements for secondary concentration. His plan was to use his elementary generalist degree as a stepping stone, thinking *"if I go in this way, maybe I can kind of go in the backdoor and get into secondary math"*. And, after two years of temporary contracts and substitute teaching, Dan was employed as a secondary mathematics teacher. He found this was a dubious success. Quite bluntly he states, *"I hated it. I hated secondary teaching; I did not fit in that culture"*. Calling himself *"philosophically misaligned"*, he spent his first few years figuring out where he fitted in as a teacher. He credits a strong bond with a teacher-mentor and his elementary generalist training with helping him establish his teacher identity. Despite initially viewing his entry into the elementary generalist program as a means to an end, he said that the elementary style of teaching and learning actually appealed to him. Sharing the adage that *"elementary teachers teach kids, secondary teachers teach content"*, his belief is that *"I can't do anything if I don't have a relationship"*.

Dan describes himself as *"kind of an outlier, typically out in front of things"*. He holds a Master's in secondary mathematics education. He also served as president for his province's mathematics teachers' association. His first role with his current district was the district's math consultant, a position he held for five years before going back to teaching secondary mathematics four years ago. Upon returning to the classroom, Dan noted a lack of engagement in his students: Lack of engagement with himself, and lack of engagement with mathematics. He wanted to create a culture in his classroom where his students were thinking mathematically, not simply repeating back words and steps he had provided. To that end, Dan began implementing changes in his practice. He began giving students opportunities to work collaboratively on problem solving at alternative working spaces. He began offering retests and flexible assessments. He also began working on alternative assessment practices that better described qualitatively what his students knew and could do.

Overall, Dan experienced varying success with the changes he implemented. What piqued my interest was his mention of the tensions he experienced regarding one change in particular—his requirement that his students write journals about their problem-solving experiences. Dan had started off the current school year convinced that mathematical writing would be a beneficial experience for the students, helping them create personal connections to their mathematical learning. Instead he found the opposite; the change he wanted to implement was damaging his relationship with his students and their relationship with mathematics.

The journal writing Dan introduced was a weekly activity that his two classes of grade 10 students were expected to complete as homework. The students were to write up their problem-solving processes for a task they had completed collaboratively in class.

In both his classes, the students' reaction to journaling was immediate and overwhelmingly negative. Noting that he had not anticipated the anxiety they would experience, Dan discontinued journal writing after three entries.

ANALYSIS AND DISCUSSION

In the following analysis, activity theory is used to outline the five elements (object, tools, rules, community, division of labour) in Dan's activity system, in which he has the role of ***subject***. His own activity system is then used as a frame for identifying and interpreting the tensions he experienced. In all, six tensions were identified (see Figure 2), but due to space limitations, only three will be presented for analysis, each of which highlight a different arrow in the activity system.

Dan's Activity System

Essential to Dan's activity system are his dual desires to develop meaningful relationships with his students, and for his students to develop meaningful relationships with mathematics. His ***object***, then, is in establishing what he called an *"ethic of care"* with his students, while figuring out how to *"push them forward mathematically"*, in this instance, through writing to learn. The ***tools*** Dan uses are pedagogical in nature. His approach to teaching combines tools of whole group, small group, partner, and individual. He attempts to engage students' mathematical thinking through tools such as journaling and collaborative problem solving. He also uses homework and note-taking, and offers flexible assessments. There are certain ***rules*** within which Dan's practice occurs. The content he teaches is guided by a provincial curriculum and there are school-wide assessment practices such as final exams. There are also well-established school-wide norms that dictate expectations for teachers, students, and classrooms. The ***community*** contained within Dan's immediate activity system comprise his students, parents, teachers, and administration. Lying farther out are the wider educational, professional, and social communities which Dan inhabits. The ***division of labour*** for each establishes expectations of the roles for the community, essentially who is to do what. For example, as the teacher, Dan has the authority to choose and assign homework; his students are expected to do the homework and hand it in.

Tensions between subject and tool.

Dan felt journaling would be an effective way of getting students *"to write for the learning of mathematics, to explain their thinking in written form"*. Initially stating that he likes the idea of journaling, he quickly amended his statement, *"No, that's not true. I don't like the idea of it. I think it's absolutely essential and critical."* His tensions with the tool lay in how to implement it effectively, not its efficacy. He believes he may have rushed the students into the process of journaling too quickly, not allowing them the necessary time or space to adjust and accommodate to this new tool. His students strongly resisted, to the point where he says, *"I was losing relationships with kids over this."* Dan reflects, *"I should have started in a more traditional way and eased into it."* This response is in keeping with Winograd (1996), who suggests that people deal with tensions regarding a tool in two ways; they either find ways to "work around" the tensions or they blame themselves. Dan, it seems, falls in the latter category. He places the blame for the tool failure on his own perceived shortcomings.

Indeed, Dan never mentioned questioning the tool itself. His desire to have students writing to learn in mathematics led him to introduce a standard journaling technique of having the students write to him about their problem-solving processes. Noting that writing in mathematics is "best accomplished in contexts where there is an authentic need to communicate" (p. 15), Phillips and Crespo (1996) suggest that most mathematics writing activities are contrived and have the teacher as the intended audience. This can have a detrimental effect on the writing. It is possible that changing the style of writing and/or changing the intended audience may have managed the tension Dan experienced.

However, for now, this is an ongoing tension that Dan is still facing. Despite having discontinued journaling this year, he continues to reflect on the experience as he wants to try again next year, with a different group of students. *"I'm committed to it"*, he says, *"it's really, really important."*

Tensions between rules and community.

Dan noted that journaling broke the rules regarding what his students believe about mathematics saying, *"I've been challenged by kids around their expectations of what math is and what math class is."* His students adhere very strictly to traditional notions of mathematics classrooms, reinforced by the community in which Dan works. He suggested that his students have figured out a way to *"survive math class, which is, you're going to give notes, you're going to give me homework, I'm going to study, get a tutor or whatever."* Dan said they were looking for *"give me the ten questions I need to know and I'll practice doing them"*, and instead, he was asking them to write. Morgan (1998) suggests disengagement may occur if the students interpret journaling as an incidental extra activity unrelated to the learning of mathematics as they have come to expect.

This is an ongoing source of tension for Dan, who is trying to change how students *"see and do"* mathematics. He's frustrated by the limited notion of math that leads to students questioning *"when are we going to do some math here?"* whenever he tries anything 'untraditional', such as journaling. He goes on to say, *"I can't engage for them so I need them to buy-in to this."*

Tensions between subject and object.

Perhaps one of the most significant tensions Dan faced was maintaining his relationship with his students while trying to engage them in thinking mathematically. Noting that his students were resistant to being pushed to write mathematically, he worried that he was going *"too far outside their comfort level"*. He acknowledges the importance of fostering mathematical thinking and conceptual understanding yet recognizing, *"at the same time I can't be successful unless I have their trust. I can't be* *successful unless I have them with me and they see me as an advocate for them and not a barrier for them."*

In discontinuing the journaling activity, Dan was able to decrease the tension he was experiencing with his students, but it was at the expense of his corresponding object of engaging his students' mathematical thinking. This suggests that, although Dan contends that both are equally desirable, his true object is maintaining his teacher-student relationship. It is possible that this activity caused Dan to rearrange the priority of his goals: he may continue to value mathematical thinking but, in this instance, he gave higher priority to maintaining relationships. The strong emotional response from his students and himself, according to Leont'ev (2009), is necessary for establishing this hierarchy of objects, or what he calls motives. A change in the object hierarchy necessitates changes in the rest of the activity system, and the result may be a different outcome. In this way, tensions fuel the evolution of Dan's activity system.

Although discontinuing journaling managed the immediate tension surrounding relationships with his students, it remains an enduring tension of which Dan is keenly aware, *"and so I have to manage the tension between moving in a pedagogical direction that I think is best for their learning but at the same time that won't cost me the relationships I have with them or then I've lost them entirely."*

CONCLUSION

Imagine Dan's activity system as an alarm panel, with the connections between its components only lighting up between when tensions are experienced—his would have been a flashing array of warning signs (see Figure 2). He experienced tensions in six aspects of his activity system and the ultimate outcome was that he quit journaling. This was unexpected and disappointing for Dan. In reflecting on the experience, he said:

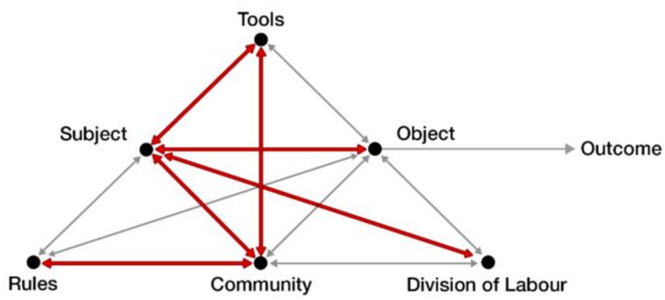

Figure 2: Dan's Activity System.

> "When I think back on this, it didn't occur to me... it wasn't a possibility that this wasn't going to work. So when it started to go sideways, I didn't know what to do. I didn't realize, so I continued to push forward with it. And then the pushback happened with the kids."

Initially, Dan spoke globally of tension in the activity of journaling. However, by focusing on the individual elements it is possible to see where the tensions impacted locally. And, as Dan is determined to try journaling again, identifying the local tensions that arise throughout his activity system could offer a means of reflection as he thinks through his next implementation. This fits with Lampert's (1985) view of a teacher as a "dilemma manager who accepts conflict as endemic and even useful to her work rather than seeing it as a burden that needs to be eliminated" (p. 192).

References

Berry, A. (2007). *Tensions in teaching about teaching: Understanding practice as a teacher educator*. Dordrecht: Springer.

Cole, M., & Engeström, Y. (1993). A cultural-historical approach to distributed cognition. In J. Salomon (Ed.), *Distributed Cognitions: Psychological and Educational Considerations* (pp. 1-46). New York: Cambridge University Press.

Freeman, D. (1993). Renaming experience/reconstructing practice: Developing new understandings of teaching. *Teaching and Teacher Education, 9*(5), 485-497.

Gedera, D. (2015). The application of activity theory in identifying contradictions in a university blended learning course. In D. Gedera and J. Williams (Eds.), *Activity Theory in Education: Research and Practice* (pp. 53-69). Netherlands: Sense Publishers.

Hamdan, M. (2005). Nonlinear learning of linear algebra: Active learning through journal writing. *International Journal of Mathematical Education in Science and Technology, 36*(6), 607-615.

Jaworski, B., & Goodchild, S. (2006). Inquiry community in an activity theory frame. In *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, (3)*, 353-360. Prague, Czech Republic: PME.

Kuutti, K. (1996). Activity theory as a potential framework for human-computer interaction research. In B. Nardi (Ed.), *Context and Consciousness: Activity Theory and Human-Computer interaction* (pp. 17-44). Cambridge: MIT Press.

Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review, 55*(2), 178-195.

Leont'ev, A. (2009). *Activity and consciousness*. CC-SA: Marxists Internet Archive.

Morgan, C. (1998). *Writing mathematically: The discourse of investigation*. London: Falmer Press.

Phillips, E., & Crespo, S. (1996). Developing written communication in mathematics through math penpal letters. *For the Learning of Mathematics, 16*(1), 15-22.

Sanders, C. (2009). Exploring and writing geometry. *Mathematics Teacher, 10*2(6), 432-439.

Sparrow, L., & Frid, S. (2001). Dilemmas of beginning teachers of primary mathematics. In *Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia (2)*, 451-458. Sydney, Australia: MERGA.

Venkat, H., & Adler, J. (2008). Expanding the foci of activity theory: Accessing the broader contexts and experiences of mathematics education reform. *Educational Review, 60*(2), 127-140.

Williams, N., & Wynne, B. (2000). Journal writing in the mathematics classroom: A beginner's approach. *Mathematics Teachers, 93*(2), 132-135.

Winograd, T. (1996). *Bringing design to software*. New York: ACM.

FROM "HOW GOOD I AM!" TO "FORGIVE ME…PLEASE TRUST ME"- MICROAGGRESSIONS AND ANGLES

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The endeavour of this report is to provide findings on how the normative appreciation of preciseness in mathematical concepts evoke micro-aggressions when students in a linguistically and socially diverse classroom reason about angles in a group activity. Results show that Samir, an emergent Swedish speaker, becomes deprived of reliability and hence loses his chances to make claims of knowledge partly due to the rigidity of (Western) mathematics. The analysed interaction begins with Samir confidently saying "How good I am" when solving a task with his peer Darko. However, it ends with Samir´s ways of talking about himself being completely changed from confidence to insecurity and subordinance, begging Darko to rely on his mathematical knowledge saying "Forgive me…please trust me.".

MATHEMATICAL CONCEPTS AND MICROAGRESSIONS – MAKING THE CONNECTIONS

Mathematical concepts used in formal mathematics are rigid, based on axiomatic conceptions. Their use elicits the normative preciseness of mathematics, an aspect of the Romance of (Western) mathematics (Lakoff & Núñez, 2000) also present in school mathematics. Preciseness and rigidity excludes plastic conceptions of mathematics evoking ideas about mathematics, which often play out as a matter of being right or wrong. Putting forward mathematical claims of knowledge is hence risky business while the chance of being "precisely wrong" is at stake. Being wrong exposes the claimer to potential micro-aggressions from which individuals might become traumatized and eventually stop perceiving themselves as members of mathematical communities (Gutiérrez, 2017). Micro-aggressions are "the everyday verbal, nonverbal, and environmental slights, snubs, or insults, whether intentional or unintentional, that communicate hostile, derogatory, or negative messages to target persons based solely upon their marginalized group membership" (Sue, 2010, p. 3). Marginalized group membership is multiple for an emergent second language speaker while s/he might not be a full member of the linguistic, cultural or/and mathematical community of practice situated in the particular mathematics classroom.

Micro-aggressions operate as micro-assaults, micro-insults or as micro-invalidations enacted as exclusion, nullification or disregarding of a person's beliefs, statements or experiences (Sue, 2010). When participating in reasoning activities together with peers, perpetrations of all three types of micro-aggressions may occur. However, I argue that the preciseness of mathematics in particular may elicit micro-invalidations

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while a person´s statements risk being disregarded and nullified if it cannot be justified in a mathematically precise way. Such micro-invalidations might lead to negative talk and feelings about oneself. Drawing on Wittgenstein's ideas on language games to explore students´ reasoning about angles and talk about themselves, I-language games and the game of giving and asking for reasons are theoretically accounted for below.

I-LANGUAGE GAMES AND THE LANGUAGE GAME OF GIVING AND ASKING FOR REASONS

The notion of I-language games does not raise questions about what "I" am nor what it is to be "me", rather they depart from the question "how do I talk about me?", presupposing that my talk about me "is not one and is not universal for it does not refer to any metaphysical or ontological essentiality. Thus, the discourses about myself are countless because they change constantly. Therefore, the language games in which I use the word 'I' or in which I talk about myself are latticed, interwoven in order to form a whole structure" (Beristain, 2011, p. 107). Based on Wittgenstein´s rejection of reference as the fundamental principle for word meaning, the idea of I-language games dissolves the referential outer(object)-inner(subject) dichotomization of "I" while they are not concerned with what "I" refers to. In fact, Wittgenstein claims that the word "I" do not refer to a person or anything at all. I-language games are not connected to any "metaphysical, internal, private, self-conscious or any psychological mental state" (Beristain, 2011, p. 114). Instead, the use of the word "I" should be understood due to the way we learn to draw attention to ourselves in various language games. The many language games in which the word "I" is explicitly or implicitly used allow for us in understandable ways to share talk which states something about ourselves. It is the *function* of the use of "I" in different contexts that should be stressed, not what it refers to. The function of I-language games "enables a sharable language of our mental/psychological states, experiences, feelings, thoughts" (Beristain, 2011, p. 108). Hence, the study of I-language games allows for drawing attention to students´ ways of sharing experiences and feelings when participating in reasoning activities. Moreover, I-language games show students´ ways of sharing feelings and thoughts about themselves for instance in response to micro-aggressions such as micro-invalidations. Feelings and thoughts, which might lead to stigmatization and eventually perceived marginalized membership in mathematical communities (Guitérrez, 2017).

When students reason in mathematics class their I-language games are latticed with other language games for instance, the *game of giving and asking for reasons* (GoGAR). The GoGAR is at heart of the philosophical theory inferentialism (Brandom, 1994, 2000). Following Wittgenstein´s ideas, concepts are not conferred with meaning due to reference but by their inferential roles in reasoning. The inferential relations of concepts are mostly implicit in concept use, but when unpacking conceptual meaning in the GoGAR they are made explicit. Premises, consequences and incompatibilities that follows from using a concept when making claims form the implicit conceptual relations. For instance, from claiming an angle to be 140° follows implicitly that it is an obtuse angle, that it cannot be for instance 40° and so forth. The

GoGAR is built around to two normative statuses; commitments and entitlements, which emerge in a socially articulated structure of authority and responsibility. When claiming that things are such and such what one does is making a judgement which one undertakes a commitment to and can be held responsible for. One makes a normative stand in putting forward a normative belief. Such a commitment entails not only what is explicitly said but also what follows implicitly form it, i.e. by making a claim one also commits oneself to other normative stands or beliefs, which follow from the original claim. For claims to have normative status they must be normatively appropriate in the social practice they are caught up in. This involves assessment of claims, hence "there must be in play also a notion of entitlement to one's commitments: the sort of entitlement that is in question when we ask whether someone has good reasons for her commitments" (Brandom, 2000, p. 43). An entitlement is a social status that a performance or a commitment has within a community, e.g. a mathematical community and/or a classroom community. Normative assessment of conceptual use implies normative appropriateness as well as normative inappropriateness. From the Romance of mathematics follows a normative appreciation of preciseness, which shape the assessment of mathematical claims. Inappropriate conceptual claims violate the norms, which guide the use of the concept. Such violation (i.e. lack of entitlement to a claim) calls for some kind of sanction which "need not consist in external sanctions" (Brandom, 1994, p. 179-180) suggesting that sanctions can be internal as well as external. External sanctions like exclusion, nullification or disregarding of a person's beliefs or statements are micro-invalidations, i.e. as a kind of micro-aggression (Sue, 2010). Furthermore, the sanctions might lead to disqualification from counting as eligible to undertake commitments (Brandom, 1994). Hence, a student who fails to give reasons for a claim involving e.g. a mathematical concept which her/his interlocutors asses as inappropriate risks being exposed to external sanction in the form of micro-invalidation due to the failure. Internal sanctions caused by micro-invalidations brought forward by interlocutors, affecting the student´s feeling and emotions about him/herself, can be made explicit in the student´s use of I-language games.

METHODOLOGY

The empirical material used for analysis in the present paper consists in a transcript from a 44-min recorded interaction part of a regular mathematics lesson in a linguistically, socially and culturally diverse grade 5 Swedish-speaking-only classroom located at a suburban school in the south of Sweden. The interaction, which involves four students (Darko (D), Samir (S), Greta (G), Eva (E)), and occasionally their teacher, was recorded when the author of this paper acted as a participant observer in the classroom. However, the author did not engage with the group referred to in this report, but had simply left a recording device at their table. The students were engaging in a pair task drawing angles that the other pair of students were to either measure using a protractor or to judge (not measure) as right angled, obtuse or pointy. In interviews made prior to the interaction Darko, born in Sweden by immigrant parents, claims to speak both Serbian and Swedish at home. Samir, who shared that he arrived from the

Palestine 2.5 years ago is an emergent Swedish speaker who speaks Arabic and Hebrew on daily bases at home. In interviews, Samir says that he is good at mathematics and that he knows more mathematics than his peers. His teacher refers to him as an above average achiever in mathematics. Greta and Eva claim to speak only Swedish at home. Greta is the one in the group who uses most formal school talk. The study follows Swedish ethical guidelines for studies in the social sciences (Vetenskapsrådet, 2002). All students´ names are pseudonyms.

The full excerpt analysed in this paper begins with Samir saying "How good I am!" when engaging with the task on angles. It ends 310 turns later with him saying "Forgive me…please trust me", begging Darko to rely on his mathematical knowledge. The analysis is done by firstly forming initial analytical tools to view the empirical material through, and then allow the emerging results to influence the initial theorization about the looming intertwining of I-language games and sanctions in the GoGARs played in the analysed material iteratively. The analysis is conducted in two steps. Step 1 aims at locating critical points where Samir´s I-language games turn form mainly positive talk about himself to talk that is more of a negative kind. To do so turns that function as I-positive language games such as self-praising, task responsibility distribution, initiative taking actions, instruction giving and making claims of knowledge were assigned a +. Turns that function as I-negative language games such as self-criticism, task responsibility renouncing, initiative obeying actions and making claims of lacking knowledge were assigned with a -. Of course, some of the above given as example of I-negative language games could in fact under different circumstances function as I-positive language games. For instance, to obey someone´s initiative can be an act of solidarity and as such viewed as an I-positive language game. Moreover, refraining from making utterances could be considered a kind of silent I-negative functioning language game. The step 1 analysis quantifies Samir´s turns as either 0 (neither positive nor negative or not part of an I-language game), + or -. The analysis does not aim at providing a particular number of turns assigned 0, + or -, rather it is performed to unveil critical points where positive I-language games characterised by for example "How good I am!", changes for negative ones. To do so Samir´s turns where divided into sections of 10 turns each. (In the excerpts below only Samir´s turns are numbered.) The relationship between + and – marked turns in each section where calculated in percentages out of the total number of + and – marked turns. The 0 marked turns where not taken into account while they do not provide information about the relation or change between + and – I-language games. Should the foci for instance had been the distribution of I-talk also the 0-turns would have to be taken into account. In Step 2 of the analysis, based on the GoGARs foregoing and/or being part of the critical points unveiled in the step 1-analysis, aims at exploring committing and (un)entitling moves in the GoGAR that appears to evoke Samir´s changed I-language games. The questions used in the step 2 analysis are; a) What commitment(s) are Samir attributed?, b) Does he get/lose entitlement to his claims? How? and c) What kind of in(ex)ternal sanctions do Samir´s lost entitlement evoke?

RESULTS

Analysis step 1

As shown in chart 1 below Samir´s I-language games are usually positive. Out of the 14 sections, including 10 of Samir´s turns each, most of them show an emphasis on positive I-language games. The result coincides with interviews made with both Samir and his teacher also showing that Samir usually talks positively about himself when engaging in mathematics.

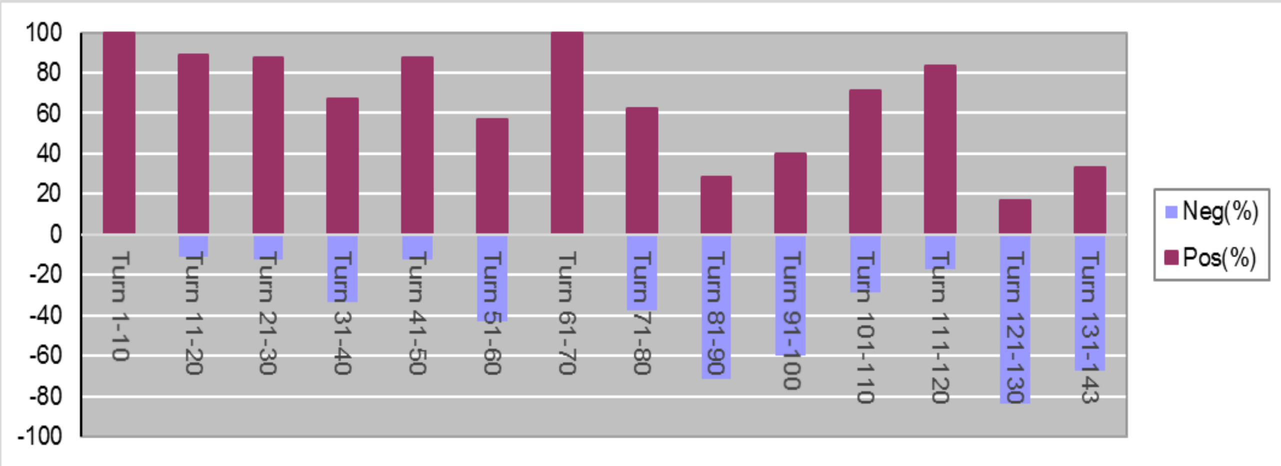

Chart 1: Chart showing the relation between Samir´s positive and negative I-language games. There are 10 of his turns in each bar.

However, at three points in the analysed interaction, turns 51-16, 81-100 and 121-143 Samir´s positive talk about himself is clearly changed, indicating critical events affecting his I-language games.

Analysis step 2

The interaction forgoing the first negative change (in Samir´s turns 51-60) concerns how angles are denoted when using a protractor. Samir taking the lead, he and Darko measured the angles drawn by Greta and Eva and wrote the magnitude on a piece of paper which they handed over to the girls. However, not being fully aware of how to use the scales on a protractor nor finding it normatively inappropriate to denote an angle as 80/100° and another one as 40°/140° (two numbers placed above each other on a protractor scale) that is what they wrote. Greta asks them to give reasons for claiming that the angle is 80/100° and later on also for claiming that the other angle is 40°/140°.

	G:	Yes, but no. It cannot be both [80° AND 100°]
43	S:	Yes, because they are above each other
	G:	Yes, but they…it does not mean that it is the same…it [THE ANGLE] is not 100 slash 80. It must be one of them.
44	S:	Yes, yes…I get it, I made a mistake…where is the protractor…

Samir then uses the protractor to re-measure the angle. He suggests that it is 100° and Darko that it is 80°. Following Darko´s suggestion, Samir writes 80° on the piece of paper. A similar interaction takes place on the behalf of the 40°/140° angle leading to Greta thoroughly giving reasons for claiming that angles are denoted using only one number and explaining how the scales of the protractor works. Though both Samir and Drako initially are committed to claiming the double denunciation, when failing to give normatively appropriate reasons for the claim and thus realizing that they are not entitled to such a claim, Darko tells Samir "Why didn´t you say so.", hence holding Samir responsible for the loss of entitlement to their initial claims. The normative appreciation of mathematical preciseness makes Greta questions the boys´ double denunciation making their entitlement loss explicit and hence exposing them to a micro-invalidation. To avoid in(ex)ternal sanctions Darko places the loss on Samir who alone becomes the victim of the micro-aggression which, in the following turns, changes his I-language games into more negative ones.

In the interaction forgoing and being part of the second negative change in Samir´s I-language games (turns 81-100) the four students are occupied with in pairs drawing one right, one obtuse and one pointy angle that the other pair of students are to judge which is which. Samir takes the lead in deciding on whether he or Darko should draw the right angle ending up with him drawing it. Right after he has finished drawing Greta urges Eva to use the protractor to check that their right angle is exactly 90°. The excerpt below consists in the turns that follow Greta asking Eva to measure their right angle.

	D:	Well done…What degrees is it [THE RIGHT ANGLE THAT SAMIR DREW]
83	S:	It is…eh eh I am good at forgetting.
	D:	Mmm…yes you are good at forgetting.
84	S:	Yes, that is why my name is Forgetty.

Probably inspired by Greta urging Eva to measure their right angle, Darko wants Samir to do the same thing, i.e. he wants Samir to undertake a claim of their angle being exactly 90°. In turn 83 Samir is just about to do that by starting to say "It is…" but appears to change his mind and claims instead to be "good at forgetting". It seems as though Samir tries to avoid being held responsible for a claim about the angle being precisely right which might (like in the first negative change) cause him losing entitlements to that claim and thus expose him to micro-invalidations. To avoid such exposure and its potential following sanctions, he appears to start playing negative I-language games. Hence, Samir rather plays negative I-language games than expose himself to potential micro-invalidations. This I find to be an example of the power that micro-invalidations caused by the normative appreciation of mathematical preciseness holds over students' feelings and thoughts about themselves when engaging in mathematics.

The negative change in Samir´s turns 121-134 occurs when the students are measuring and denoting each other's right, obtuse and pointy angles though that was not the teacher's intention. The boys are faced with using the protractor and measuring angles and again Greta challenges their claims about the magnitude of the measured angles. Playing the GoGAR with Greta on whether to denote an angle as 145° or 155° Samir claims it being 155°, a claim which Darko supports in the first turn. The excerpt below shows the last parts of the interaction.

	D:	Wait [TO GRETA]…look it *is* 155.
	G:	Not 55. 55 is there [SHOWING ON THE PROTRACTOR]…this is 45.
132	S:	I thought it was…
	D:	I am not going to trust you anymore.
133	S:	I thought it was so…
	D:	You cannot just think so.
134	S:	May I…
	D:	I asked you specifically and you just said yes.
135	S:	Forgive me…please trust me.

Initially Darko undertakes Samir´s claim of denoting the angle 155° and uses it when trying to convince Greta to undertake the same claim. However, she finds it normatively inappropriate and hence challenge it, justifying her own claim by showing Samir and Darko where on the scale of the protractor 145° and 155° respectively are located. Samir appears to undertake Selma´s claim and simultaneously states that his initial claim was based on that he "thought it was" a normatively appropriate claim of knowledge. To avoid being exposed to sanctions due to a possible lack of entitlement for claiming the angle to be 155° Darko (who also undertook that claim) dismisses Samir´s claim and seems to argue that "think so" is not enough to justify a claim, hence challenging Samir´s reliability. In Samir´s 134th turn, he asks Darko´s permission for something but he does not complete his saying. In the last turn Samir appears to think that his reliability and thus possibility of making claims that will earn entitlements is lost and he begs Darko to forgive him and to reassign him reliability. As shown in chart 1 above Samir´s last turns of the interaction include the least amount of positive I-language games. This indicates that the micro-aggressions caused by the normative appreciation of preciseness in mathematics which he has been exposed to has caused changes in the way he uses I-language games to share his "mental/psychological states, experiences, feelings, thoughts" (Beristain, 2011, p. 108) that significantly differs from his usual way of thinking and feeling about himself when engaging in mathematics.

CONCLUDING REMARKS

This report shows how the normative appreciation of mathematical preciseness inherent in conceptions of angles and the artefacts used to measure and denote them impact the students´ reasoning evoking micro-aggressions directed towards Samir, an emergent speaker of Swedish. Not only does the exposure to micro-aggressions but also such potential exposure, change Samir´s usually positive ways of talking, thinking and feeling about himself when engaging with mathematics, for negative ones. The students in this report do not to explore alternative ways of denoting angles nor do they explore each other's conceptions of and ways of talking about angles, rather they are in the pursuit of preciseness. A normative space of mathematical reasons that appreciate preciseness, does not allow Samir or the other students to give imprecise and/or in-appropriate reasons that they can elaborate on when playing the GoGAR. Rather, it is hindering their engagement in GoGARs that would allow for more plastic conceptions and ways of talking about angles and their denotations. The normative space of mathematical reasons elicits epistemological invalidations and hence epistemological micro-aggressions, causing Samir´s talk and feelings of being a forgetter, a wrong-doer and a person without reliability. For an emergent speaker of a second language to engage in mathematical reasoning activities in a Swedish-only classroom where pre-ciseness guides the interlocutors´ reasoning appears to be particularly risky business while her/his resources underpinning giving reasons for claims are diminished by mono-lingual, mono-cultural and mono-mathematical normativity. When inviting students to group activities involving reasoning, educators need to be sensitive to the exposed situation of emergent second language speakers and provide student aware-ness on potential micro-aggressions evoked by the normative appreciation of mathe-matical preciseness. At stake are the replacement of positive feelings about oneself when doing mathematics in exchange for feelings and thoughts of sub-ordinance and marginalized membership of mathematical communities.

References

Beristain, C.B. (2011). Language games and the latticed discourses of myself. In Gálvez, J. P., & Gaffal, M. (Eds.), *Forms of Life and Language Games* (pp. 107-119). Walter de Gruyter.

Brandom, R. (1994). *Making it explicit: reasoning, representing, and discursive commitment.* Cambridge, MA, USA: Harvard University Press.

Brandom, R. (2000). *Articulating Reasons.* Cambridge, MA, USA: Harvard University Press.

Gutiérrez, R. (2017). Political conocimiento for teaching mathematics. In Kastberg, S., Tyminski, A. M., Lischka, A., & Sanchez, W. (Eds.), *Building Support for Scholarly Practices in Mathematics Methods* (pp. 11-37): Charlotte: Information Age Publishing.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: how the embodied mind brings mathematics into being.* New York: Basic Books, cop. 2000.

Sue, D. W. (2010). Microaggressions, marginality, and oppression: An introduction. In D. W. Sue (Ed.), *Microaggressions and marginality: Manifestation, dynamics, and impact* (pp. 3–24). New York, NY: Wiley & Sons.

SOME CHANGES OF MATH ANXIETY GROUPS BASED ON TWO MEASUREMENTS, MASS & EEG

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This article investigated how mathematics anxiety (MA) of Korean middle school students could be reduced by comparing analytically their cognitive neuroscience and questionnaire results. We developed a three-hour Complex Treatment Program (CTP) on quadratic functions for the study. In the summer of 2016, we collected data of the pre and post MA questionnaires by Mathematics Anxiety Scale for Students (MASS), the percent of correct answers (PCA) & reaction time (RT) by E-prime program, and also brain-imaging data of the event related potentials (ERP) by Electroencephalograph (EEG) using computer-based functional F-G model. The result indicated the CTP to be effective with the group with higher math anxiety and the group with higher achievement respectively. The MASS result was verified with the better performance of PCA on type G, which was measured by E-prime program. Some interesting patterns were revealed on brain-imaging data by EEG, indicating more mental activities with the MA groups.

INTRODUCTION

Looking back at the Programme for International Student Assessment (PISA) from 1995 to 2012, Korean students ranked in the highest level of achievement in mathematics but consistently marked a very low score in affective domains like self-efficacy and interest (Ko & Yi, 2012). This leads us to conclude that students who are high-performers in mathematics could potentially have low achievement levels in the affective domains of mathematical learning, resulting in mathematics anxiety (MA) which is displayed as the uneasy feeling about the subject of mathematics. While many previous studies about MA were centered on the development of questionnaires to diagnose students' MA, recent studies have been focused on devising a treatment to abate it. For example, researchers applied the Note-taking Model like Divided-page Method (Tobias, 1987; Eun & Lee, 1994), and the Mathematics-friendly Program (Yoon & Jeon, 2010) to students who were feeling mathematics anxiety. However, these studies focused mainly on problem solving or simple algorithms and not on a specific domain of mathematics, e.g. function. In order to reduce MA in the students' functional understanding of mathematics, we developed a complex treatment program that combined a mathematical treatment and a psychological treatment, once used by Yoon et al. (2010). Moreover, we collected data to grasp the effects of the program more closely by measuring the students' brain waves which is the difference of this study from previous ones.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 83-90). Umeå, Sweden: PME.

The purpose of this study was to investigate how to reduce the MA of eighth-grade students after they completed the complex treatment program (CTP), which was a mixture of psychological and mathematical treatments to help students psychologically and mathematically. The research questions were set up as follows:

1. How did the student groups respond to the MASS before and after the complex treatment program?
2. What were the differences of the groups in the percent of correct answers (PCA) and reaction time (RT) gained by E-prime program while getting the brain imaging data through ERP?
3. What were the differences in the brain imaging data of the groups on the functional tasks through ERP?

THEORETICAL BACKGROUND

Mathematics Anxiety (MA)

In the previous research, various researchers developed differing definitions of MA(e.g. Fennema & Sherman, 1976). Since Byrd's (1983) comprehensive definition, several researchers have defined MA differently and mostly used the definition formerly described in the literature. Various definitions, however, were similar in the context of mathematical learning. In this study, we defined MA as "anxiety felt by individuals when they solve mathematical problems".

Brain Research on Students' Understanding of Functions

Akkoc and Tall (2002) suggested the complexity of the function concept arises from its many representations, e.g. graphs, equations, and tables. In our study, we decided to focus on the translation between graph and equation representations—a particularly important concept in understanding functions (Williams, 1998). As such, our study resembles two previous neuroscience studies on students' understanding of functions (Thomas Wilson, Corballis, Lim, & Yoon, 2010; Waisman, Leiken, Shaul, & Leiken, 2014) though neither of them considered the effects of MA.

Thomas et al. (2010) classified function tasks into four formats: graph-to-graph, equation-to-equation, graph-to-equation and equation-to-graph. They used these formats for both linear and quadratic functions, generating a total of eight kinds of tasks for the ten students in their study—ten college students in New Zealand. In addition to measuring RT and PCA, they measured neural activity within the neocortex of the brain through use of functional magnetic resonance imaging (fMRI). Essentially, fMRI measures neural activity by way of blood flow to areas within the brain.

METHOD

Participants

25 students in the 8th grade from middle school in Yongin-city, Korea participated in this study. The students had never participated in a program for the purpose of reducing MA.

Procedure

The data was collected in July, 2016 in which the subjects were available right before the summer vacation started in Korea. In this study, the pre and post-questionnaires by MASS were used to measure students' MA. In the same way as Sheffield & Hunt (2006), the participants were classified as high MA (HAX) group or low MA (LAX) group based on the result of the pre-questionnaire. Also, according to the result of the recent final exam of their school, they were grouped into either the higher achievement (HAC) group or lower achievement (LAC) group. The treatment program of the three-lesson units was applied to these students. The students reviewed the problems on their own, while working with the experimenter. To find the effectiveness of the treatment program, we investigated the degree of MA using the same questionnaire as the pre-questionnaire and measured the students' brain waves to see how they reacted to stimuli of functional tasks while they solved them on the monitor provided.

Instruments

Questionnaire for MA. The Mathematics Anxiety Scale for Students (MASS) revised by Ko & Yi (2012) was used to measure students' MA. This test scale was chosen because it was more appropriate to the middle school students since their MASS was developed for the secondary school students including middle school students with 65 question items which resulted in the internal consistency reliability, the Cronbach's α= .976 as reported in table 3. Each question was answered on one of 1 to 5 point Likert scale.

Complex Treatment Program (CTP). The complex treatment program in three units consisted of the mixture of a mathematical treatment and a psychological treatment. The mathematical treatment that was composed of four kinds of problems provided students a chance to reflect the nature of quadratic function. Firstly, the diagnosis problem was constructed as the basic concepts learned in the previous lesson. Secondly, a lower level (basic) problem was constructed with a basic concept of the quadratic function. Thirdly, the treatment was constructed to help students to understand the type G. Lastly, a higher level (challenging) problem was constructed to challenge advanced level students with a higher level of difficulty. Also, four to six mathematical-hint cards reflecting Vygotsky's scaffolding theory in which they could be assisted by peers or a teacher whenever needed were prepared for each problem.

The psychological treatment which was the other part of CTP was constructed by revising the brain integration in education (BIE) program used by Kim (2010) to reduce the students' anxiety factors shown in the pre-questionnaire. Kim (2010) developed the brain integration in education (BIE) program consisting of three stages: understanding of brain, integration of brain, and application of brain, to improve self-directed learning ability of middle school students.

F-G Model of Functional Translation. Thomas et al. (2010) presented the functional tasks in relation to graphs and equations as stimuli and observed 10 students' brain waves. By actively using it, the tasks developed in this study were composed of 10 kinds of basic quadratic functions with $a = \{-1,\ 1\}$, $p = 0$, $q = \{-2, -1,\ 0, 1, 2\}$ in a algebraic formula, $y = a(x-p)^2 + q$. In other words, the subject was able to grasp the elements such as shape, intercept, vertex, etc. of a quadratic function.

Two hundred items were developed which asked students to decide whether graphs and equations matched. E-prime program randomly decided an item and presented it the subject. Two types of cross-format functional tasks were translating from equation-to-graph (type F) and translating from graph-to-equation (type G) using the ten quadratics functions. The students solved 20 problems as presented randomly for each type.

Brain-Imaging Technique. For this study, Electroencephalograph (EEG) was selected. This decision was based on budget, mobility, accessibility, safety, functionality, and practicality. In detail, we selected Event Related Potential (ERP) among the EEG's techniques. ERP is a method of averaging EEG activity from time domain analysis and multiple-stimulation in an EEG. It usually appears within 50-500ms after stimulation: P50, & N100 which reflect the stimulus detection, and cognitive-related factors, P300, & N400 which appear later and mainly indicate stimulation.

RESULTS

Pre and Post Questionnaires

The mean value of the pretest for 25 students was 2.63. The students who were higher or lower than 2.63, were classified as either HAX (High Mathematics Anxiety) or LAX (Low Mathematics Anxiety) respectively. Also, students who were higher or lower than the final exams' mean value, were grouped as either HAC (High Mathematical Achievement) or LAC (Low Mathematical Achievement) according to the recent final exam administered by their school, to see relationship between cognitive and affective aspects in detail.

The following Table 1 compares the results of the pre-questionnaire by groups with the results of the post-questionnaire after the program was implemented. This comparison in two dependent sample t-tests was to see the effect of the complex treatment program.

Group		Mean	SD	N	t value
HAX	pre	3.41	0.68	11	2.41*
	post	2.94	1.01	11	
LAX	pre	2.02	0.52	14	1.21
	post	1.82	0.75	14	
HAC	pre	2.23	0.72	15	2.57*
	post	1.88	0.72	15	
LAC	pre	3.23	0.86	10	1.08
	post	2.96	1.11	10	
All	pre	2.63	0.91	25	2.51*
	post	2.31	1.03	25	

*p < .05

Table 1: Two dependent samples t-test results for the effect of the CTP in groups

The Percent of Correct Answer (PCA) and Reaction Time (RT)

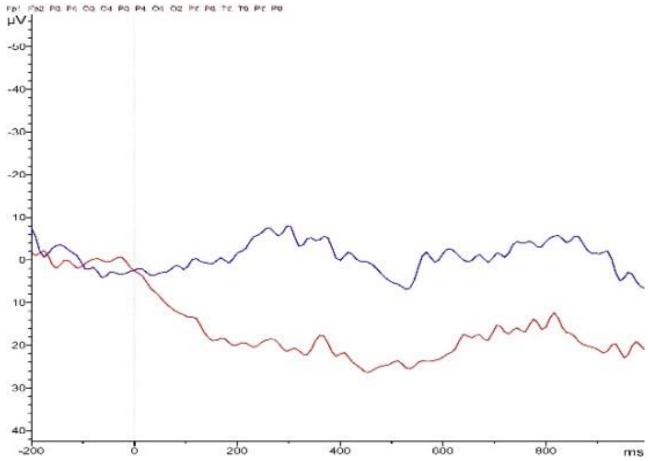
In order to measure the responses of the students who participated in the research, we developed the functional types for the E-prime program along with EEG measurement, which provided us the percent of correct answers (PCA), the reaction time (RT) of the subjects. Before solving the F-G Model, the students solved arithmetical tasks (type A) consisting of simple addition problems to help activate their brain. This is called the dual-task composition in which the students worked on two phases: first, the phase for arithmetical tasks; second, the phase of functional tasks to capture more proper activation of brain. However, the result of type A was not included in this paper. PCAs and RTs of four groups for each type of tasks are as follows.

Group	Type F		Type G	
	PCA	RT	PCA	RT
HAX	72.7%	2.05 sec	86.4%	1.8 sec
LAX	83.6%	1.78 sec	84.2%	1.41 sec
HAC	88.5%	1.62 sec	89.7%	1.26 sec
LAC	66.7%	2.25 sec	79.5%	1.98 sec

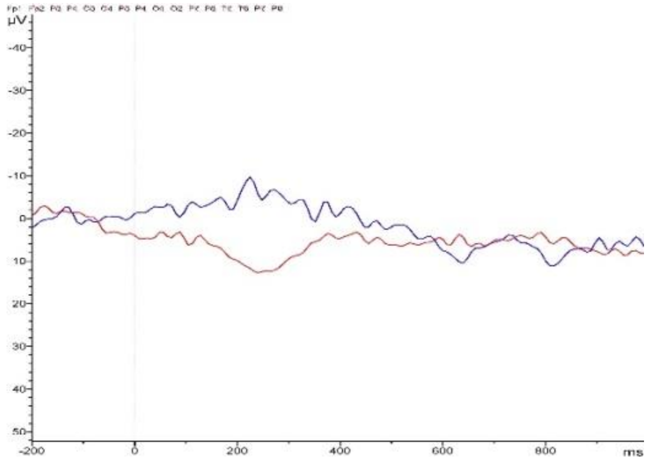
Table 2: PCA and RT by group for the type F and G

EEG

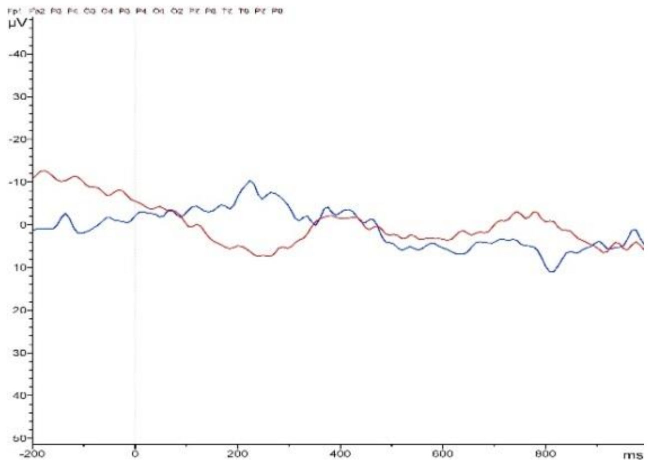
In this study, we analyzed P300 wave in Event-Related Potential (ERP) to measure brain waves, in addition to PCAs & RTs by E-prime program. The P300 wave is a positive wave that reaches the apex about 300 ms after stimulus presentation starts in ERP. The results of measuring brain waves while solving types F and G through EEG are shown in Fig. 1 which compares the groups for the types. In this paper, we just displayed the imaging data for groups because of spatial constraint. In the conference site, we will display all data we gained.



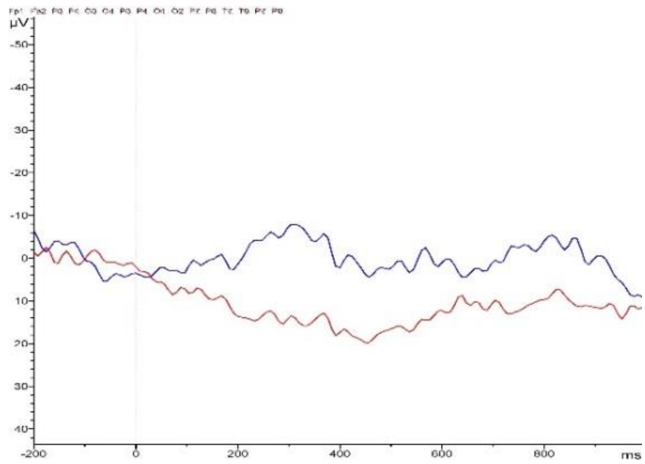
G(Red)&F(Blue) of HAX



G(Red)&F(Blue) of LAX



G(Red)&F(Blue) of HAC



G(Red)&F(Blue) of LAC

Figure 1: EEG by a task for the group

CONCLUSIONS

First, the test results by MASS showed that our program overall was effective for the students to reduce MA. In detail, MA of group HAX and group HAC had significantly been lowered. In fact, because our program was composed of two important parts that help students with psychological and mathematical aspects, this outcome on Tab. 1 made sense to us.

Second, after completing our program, all the groups of students solved type G more accurately and quickly than type F. That is, the CTP we had developed to help the students not only to reduce MA, but also to get over the difficulty on type G for better functional understanding played an important role in achieving the purpose of the study. Interestingly, the result by E-prime that HAX and HAC exceeded the counterpart on type G in Table 2 seemed to verify the result we got by MASS outcomes.

Third, to examine the precise effect of the program, we observed closely at the brain waves by EEG while the students were solving the F-G Model. As mentioned above, although the students solved type G more quickly and accurately, compared to type F, we noticed that the students had more brain wave activity with type G in brain (see Fig. 3). Also, the MA groups presented a bigger gap between two types, which meant more activities on type G in brain. If we had not measured EEG in addition to the questionnaire of MA, we might not have found this kind of difference of type G on which students were asked to decode the graph image first, and then to match it to a corresponding equation of function. Therefore, the MA aspect influences more on students' performance.

Today, the advent of cutting edge-handy devices for measuring brain waves makes it possible to apply them for educational purposes like reducing MA. The importance of mathematics is increasing day by day, but there are many students, who have a sense of discomfort with mathematics. For the sake of national core competence, we should solve the MA issue and develop new methodologies in mathematics education.

References

Akkoç, H. & Tall, D. (2002). The simplicity, complexity and complication of the function concept. In A. Cockburn and E. Nardi(Eds.), *Proceedings 26th Annual Conference of the International Group for the Psychology of Mathematics Education*, *2*(pp. 2-025). Norwich, UK.

Byrd, P.G. (1983). A Descriptive Study of Mathematics Anxiety: Its Nature and Antecedents. *Unpublished Doctoral Dissertation*, India University.

Eun, S.J. & Lee, J.H. (1994). A Study on an Effective Teaching Method for the 7th Grade Students with Math Anxiety. *Mathematics Education*, *33*(2), 285-296.

Fennema, E., & Sherman, J. A. (1976). Fennema-Sherman mathematics attitudes scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *Journal for research in Mathematics Education*, 7(5), 324-326.

Kim, E.J. (2010). A Study on Development of a Brain Education Program to Improve Self-Directed Learning Ability. *Korean Journal of Youth Studies*, *17*(10), 161-188.

Ko, H.K. & Yi, H.S. (2012). Difference in Mathematics Anxiety of Middle and High School Students per Factor according to Background Variables, *Journal of the Korean School Mathematics Society*, *15*(3), 487-509.

Thomas, M.O., Wilson, A.J., Corballis, M.C., Lim, V.K. & Yoon, C. (2010). Evidence from Cognitive Neuroscience for the Role of Graphical and Algebraic Representations in Understanding Function. *ZDM*, *42*(6), 607-619.

Waisman, I., Leikin, M., Shaul, S. & Leikin, R. (2014). Brain activity associated with translation between graphical and symbolic representations of functions in generally gifted and excelling in mathematics adolescents. *International Journal of Science and Mathematics Education*, *12*(3), 669-696.

Yoon, R. & Jeon, I.H. (2010). The Development of a Math Friendly Activity Program for the Alleviation of Mathematics Anxiety. *Journal of Elementary Education in Korea*, *14*(3), 583-613.

TRANSFORMATION OF A GEOMETRIC DIAGRAM TO PRODUCE A CONJECTURE AND ITS PROOF

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Given a set of geometric conditions student-teachers were asked to construct a geometric diagram, explore it, observe it, make a conjecture, and prove the conjecture. The focus of this paper is to analyze one of two proofs, for the same task, offered by a student-teacher to validate his conjecture. This student participated in a constructivist classroom teaching-experiment on the teaching-learning of geometry using the Geometer's Sketch Pad (GSP). The analysis uses Stjernfelt's model for diagrammatic reasoning, rooted in the semiotics of Charles Sanders Peirce, which considers the transformation of diagrams to unveil valid relations among their parts. In the case of geometry, such relations enable the construction of geometric arguments to prove geometric propositions.

THEORETICAL RATIONALE

For centuries it has been recognized that *sense perception* is an essential element of cognition. Kant (1781/2007) asserts that perception without conception is simply blind and that conception without perception is merely empty. Arheim's book on *Visual thinking* (1969) gives specific emphasis to *visual perception* by rephrasing Kant's idea: "vision without abstraction is blind and abstraction without vision is empty" (p. 188). This is to say that the perceptual elements in thought and the thought elements in perception are complementary and their synergy makes human cognition a unitary process that leads the way from the elementary acquisition of sensory information to the most abstract and general ideas. Bishop (1989) emphasizes visualization as a process that translates abstract relations and non-figural information into visual terms (i.e., *visual processing)*; this process allows students to construct geometric diagrams from verbal descriptions of geometric relations. Dörfler (1991) proposes the notion of *image schemata* as the individual's mental construction that connects and integrates related concept images which have been subjectively constructed by the individual. This notion is an open invitation to consider the subjective and evolving interpretations of learners. This notion of interpretation is unfolded as a synergistic process between intra-interpretation and inter-interpretation by Sáenz-Ludlow and Zellweger (2016).

All the above notions interweave the individuals' sense perceptions and thoughts experimented either alone, with the influence of others, or with the influence of technical devices. Such interweaving enables learners to perform transformations on geometric diagrams and to perceptually and conceptually "see" them under a new light with more abstract and general meanings.

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Such a thinking process is not new, in fact, it is historical. Netz (2014) presents insights into the evolutionary and revolutionary nature of the Greek mathematical thinking in which geometric diagrams played a very significant role. These diagrams were considered the metonymy of the proposition and their transformations shaped deductive mathematical reasoning to unveil the mathematical Objects that the diagrams purport to represent. Furthermore, the letters inserted in a diagram were considered *indices* that only indicate clearly certain elements or parts of it but not *symbols* with deeper meanings. In addition, Netz argues the writing of a proof was preceded by oral rehearsal to refine the geometric argument. Euclidean geometry is a classic example of visual-spatial perception accompanied by thought-experimentation and intellectual manipulation performed on geometric diagrams. As Peirce says, "Euclid first announces, in general terms, the proposition he intends to prove, and then proceeds to draw a diagram, usually a figure, to exhibit the antecedent condition thereof" (1976, p. 317).

Nowadays, given the dragging mode of dynamic geometry environments like the GSP, the manipulation of geometric diagrams is expedited and, with it, the perceptual and conceptual transformation of the relations among the parts of the diagram, as well as the possible observation of learners' intentional or unintentional manipulations and their planned or unplanned experimentation. Such manipulation and experimentation guide the observation of variant and invariant relations among the elements of geometric diagrams and facilitate the formation of conjectures and their proofs. That is, the GSP mediates perception and reasoning through the transformations of diagrams. This type of reasoning mediated by diagrams was called by Peirce *diagrammatic reasoning*.

What is Exactly Diagrammatic Reasoning and What is Its Role in Deduction?

The role of diagrams in deductive reasoning has been argued by Peirce as follows:

> By *diagrammatic reasoning*, I mean reasoning which constructs a diagram according to a precept expressed in general terms, *performs experiments* upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses it in a *general form* (CP 2.96, italics added)
>
> *Deduction* is that mode of reasoning which examines the state of things asserted in the premises, *forms a diagram of that state of things*, *perceives in the parts of the diagram relations not explicitly mentioned in the premises, satisfies itself by mental experiments upon the diagram that these relations will always subsist*, or at least would do so in a certain proportion of cases, and concludes the necessary, or probable truth (Peirce CP 1.66, italics added)

Peirce argues that the similarity and analogy between the skeleton structure of a diagram and the abstract structure of its Object bring into being a process of inference of inductive, abductive, and deductive nature. Such a process not only allows perceptual observation of structural relations among the parts of the diagram (the object-as-it-is-perceived) but it also enables thought-experimentation to infer the hidden structural relations among the structural parts of the Object (the object-as-it-is) by means of inferential reasoning. This is for Peirce the essence of *diagrammatic reasoning*.

For Peirce, a diagram is a sign-vehicle that facilitates possible relations. But what is the function of a sign-vehicle besides representing its Object? It is natural to Peirce to give comprehensive definitions.

- "A sign-vehicle is anything which, being determined by an Object, determines an interpretation to determination, through it, by the same Object" (1906, p. 495, italics added).
- "A sign [sign-vehicle] is anything which determines something else, its interpretant, to refer to an Object to which itself refers in the same way…" (CP 2.303, italics added).
- "a sign [sign-vehicle] is not a sign [sign-vehicle] unless it translates into another sign [sign-vehicle]" (CP 5.594).

Peirce (1906) goes even further to say that the *relation* between a sign-vehicle and its Object could be of iconic, indexical, or symbolic nature. These three types of relations are not independent from each other and they, in some way, depend on the learner/Interpreter of the sign-vehicle. Sign-vehicles are symbolic when they allow ways of thinking about thoughts that we could not otherwise think of them, and when they enable us to create abstractions, which are the genuine means of discoveries. *Symbols*, he argues, rest exclusively on already well preformed habits of thinking and therefore they do not furnish any observation of themselves and do not enable the addition of further knowledge. On the other hand, he argues that *indices* only provide positive assurance of the reality and nearness of their Objects and that such assurance does not give any insight into the nature of such Objects. In contrast, he argues that *icons* partake in the covert character of their Objects and therefore they do not stand unequivocally for this or that existing thing; nonetheless, icons afford a skeleton displacement, before the mind's eyes, of possible logical relations among the parts of the Object. He classifies *icons* into images, metaphors, and diagrams. *Images* represent the Object through simple qualities; *metaphors* represent the Object through a similarity found in something else; and *diagrams* represent the Object through skeleton structural similarities that, by analogy, reflect the actual abstract structure of the Object itself.

Stjernfelt's Model of Diagrammatic Reasoning

Stjernfelt (2007) captures the essence of Peirce's diagrammatic reasoning (Figure 1). This process is rooted in perceptual and conceptual activity that produces chains of inductive, abductive, and deductive inferences. His figure is a diagram itself; it synthesizes a manifold of relations which amalgamates a sequence of progressive acts of interpretation (intra-inter) on the part of the individual, namely, the construction of a diagram, the sequential observation of structural relations among its parts, and the physical and mental manipulation to produce a chain of inferences to attain a conclusion.

Stjernfelt describes diagrammatic reasoning as the transformation of a diagram caused by the individual's evolving *interpretants* (i.e., the product of the intra-inter interpreting acts of the individual). In this transformation, the implicit deep structural aspects of the Object-as-it-is, which the diagram purports to represent, can be unveiled by analogy with the structural relations among the parts of the diagram, which represents the Object-as-it-is-perceived.

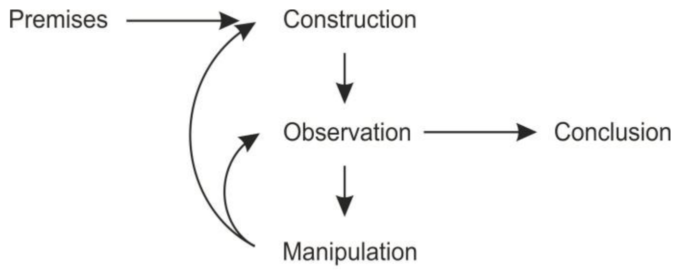

Figure 1: Diagrammatic reasoning process (Stjernfelt model, 2007, p. 104).

That is, the individual/Interpreter mentally refurbishes the initially constructed *interpretants* triggered but the initial diagram (transformand diagram) into more meaningful *interpretants* triggered by the new diagram (transformate diagram). In this process, the individual/Interpreter transforms a diagram-token into a diagram-icon and the latter into a diagram-symbol. It can be said that each time the relations among the parts of the new transformate diagram reveals more and deeper structural relations among the parts of the Object; relations that hinge on mental operations and inferential reasoning product of the acts of intra- and inter-interpretation of the individual. It is in this sense that iconic sign-vehicles are transformed, in the mind of the individual/Interpreter, into symbolic sign-vehicles. It is then not a surprise that Stjernfelt describes the diagrammatic reasoning process as a sequence of emergent *interpretants* (***a, b, c, d, e, f, g)*** growing in structural symbolic meanings:

- ***a***. ***Symbol (1):*** Diagram-symbol, the transformand diagram or symbol-diagram in the mind of the proposer. The diagram the Interpreter has to deal with.
- ***b***. ***Immediate Iconic Interpretant (b < a)***: Produces a diagram-token in the Interpreter's mind; a rule-bound diagram; an initial interpretation of the diagram-symbol ***a***.
- ***c***. ***Initial interpretant (b+c < a)***: Produces a diagram-icon, a skeleton schema of the relations among the parts of the diagram-symbol ***a***; a transformate diagram with possibilities of new relations.
- ***d***. ***Middle Interpretant ((b+c) + d < a)***: Produces a diagram with sources, **a, b,** and **c**; an emergent diagram-icon with possibilities of further conceptual relations and transformations.
- ***e***. ***Eventual, Rational Interpretant****:* Produces a new emergent and more advanced transformate diagram-symbol (closer in meanings to diagram-symbol ***a***) with more general and abstract meanings.
- ***f***. ***Symbol (2)***: Concluding transformate diagram-symbol, in the Interpreter's mind, carrying an advanced schema of the relations among the parts of the diagram-symbol ***a***.

g. ***Post-Diagrammatical Interpretant****:* Produces a more advanced interpretation *of the diagram-symbol* ***a*** *(****Symbol (1)****);* this interpretant of ***a*** is different from ***b*** and more advanced than ***b***.

It can be said that transformate diagrams are somewhat embedded in the transformand diagram with features in a process of potential refinement. That is, diagrammatic reasoning is the mental process of the Interpreter who intentionally endeavors both in the observation and in the manipulation of an initial diagram (transformand diagram/**symbol (1)**), which he only "sees" as a token. This first interpretation is a diagram-token and then, progressively, the Interpreter enriches and transforms it into a diagram-icon and later into a diagram-symbol (transformate diagram/**symbol (2)**). This means that the Interpreter finally unveils, as best as he can, the structural relation of the *Object* that the transformand diagram (**symbol (1))** purports to represent.

METHODOLOGY

Nine pre-service and in-service mathematics student-teachers participated in one-academic-semester classroom teaching-experiment on the teaching-learning of Euclidean geometry using the GSP. The main goal was to improve student-teachers' ability to conjecture and to validate or reject them. An inquiry approach was used in which tasks were posed by giving a set of geometric conditions; students were asked to construct a diagram, to explore it, to make a conjecture and to prove it. Students solved 19 sets of tasks (as homework assignments), each set had from 5 to 10 geometric situations concentrated on certain concepts: angles, triangles and classifications, properties of triangles, quadrilaterals and classifications, properties of quadrilaterals. Student-teachers were free to talk about their exploration of diagrams and about their conjectures and reasoning; however, the written proof was to be done individually. Here we analyse one proof, out of two, that a student-teacher produced for task#3 in homework #7.

DATA ANALYSIS

Task#3

a) Construct a triangle ΔABC.

b) Extend side AB from the point B and take a line segment BE = AB.

c) Extend side AC from the point C and take a line segment CF = AC.

d) Construct the line BC.

e) Construct the perpendicular line segments EH and FG from points E and F to the line BC. (Points H and G lie on the extensions of side BC)

f) Measure the line segments EH and FG and tabulate your measurements dragging any vertex of triangle ΔABC.

g) Write your conjecture about EH and FG.

h) Prove your conjecture.

Proof 1. When the student-teacher explains his proof, he shows on the computer screen the sequence of diagrams in Figure 2. This sequence unveils a sequence of *interpretants*, in the mind of the student/Interpreter, allowing transformations of the initial diagram, constructed under the given conditions (transformand diagram), into more sophisticated diagrams (transformate diagrams) mediating the inference of a conjecture and the construction of an argument for its proof.

He constructs the triangle ΔABC and extends sides AB and AC from points B and C, respectively, creating BE=AB and CF=AC. He then drops lines EH and FG perpendicular to line BC (transformand diagram 2a). Then he tabulates the measurements of EH and FG by dragging a vertex of triangle ΔABC and conjectures that EH=FG (transformate diagram 2b). On his own, he introduces altitude AD on the side BC (transformate diagram 2c) as auxiliary line. Soon after he visualizes two pairs of right triangles as being congruent and which implications allows him to prove the conjecture.

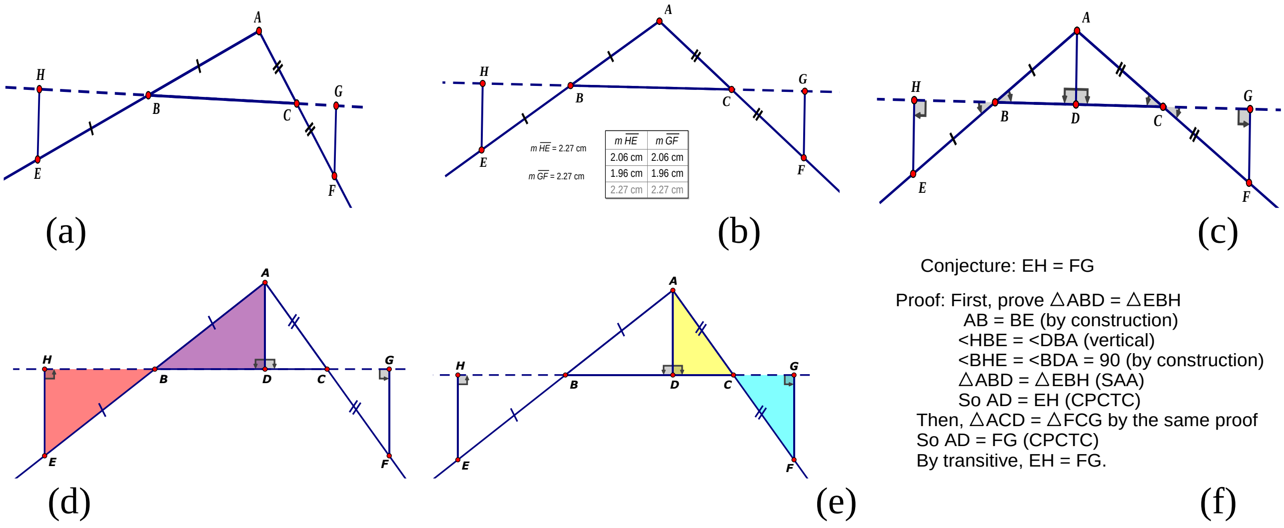

Figure 2: **(**a) initial construction; (b) exploration-conjecture; (c) altitude; (d) ΔADB≅ΔEHB; (e) ΔADC≅ΔFEC; (f) written proof.

Below is the description of the sequence of *interpretants* generated by the student:

- *Inmmediate iconic interpretant:* Visual perception of the given information in the task: triangle ΔABC; points B and C seen as midpoints of segments AE and AF; right triangles ΔBHE and ΔCGF. (transformand diagram, 2a)
- *Initial Interpretant*: Tabulation of measurements of EH and FG and emergence of the conjecture EH=FG. Selection and use of collateral knowledge to construct altitude AD and right triangles ADB and ADC. (transformate diagrams, 2b & 2c)
- *Middle Interpretant*: Triangles ADB and BHE have: vertical angles ∠ABD ≅ ∠EBH, right angles ∠ADB ≅∠EHB; and AB=BE given condition. (transformate diagram, 2d)

- <u>*Rational interpretant*</u>: ΔADB≅ΔBHE are congruent by SAA. Likewise, ΔADC≅ΔFGC are congruent by SAA. Then AD=EH and AD=FG are implications of congruent triangles. (transformate diagram, 2d & 2e)
- <u>*Eventual rational interpretant*</u>: Using the transitive property: EH=AD and AD=FG, then EH=FG.
- <u>*Post-diagrammatical interpretant*</u>: Coordination and integration of interpretants: synthesis of geometric argument and writing of proof (2f). AB=BE (by construction)
  ∠ABD ≅ ∠EBH vertical angles
  ∠ADB ≅ ∠EHB right angles (by construction)
  Therefore, ΔADB ≅ ΔBHE by SAA and AD=EH (CPCTC)
  Similarly, ΔADC ≅ ΔFGC by SAA and AD=FG (CPCTC)
  Therefore, by transitive EH=FG, proving that the conjecture was right

The student constructs the geometric diagram under the given conditions and then performs visual and conceptual experiments on the initial diagram. He observes that regardless of the changes produced in the diagram by dragging the vertices of ΔABC, there is a constancy in the length segments HE and FG. This invariance is the conjecture that he makes after a few trials. This conjecture was an act of inter- interpretation of inductive nature aided by the GSP. He then selects and uses altitude AD on side BC, as auxiliary line, to produce two right triangles. The student-teacher's decision to use altitude AD indicates a creative act intra-interpretation of abductive nature. This altitude introduces two new right triangles ABD and ACD that he visually perceives as being congruent with the right triangles HBE and GCF, respectively. He then validates the congruence of these two pairs of triangles by the SAA criterion. Using this congruence, he infers that the length segments HE and FG is the same as the length of altitude AD. Using the transitive property for equality, he deduces the equality of segments HE and FG.

CONCLUSION

In the process of intra-inter-interpretation, the perceptive elements in thought and the thought elements in perception played out synergistically to enable the emergence of *interpretants* in the student's mind. These interpretants mediated the transformation of the initial diagram into diagrams richer in geometric relations.

The analysis indicates that the student's transformation of the initial diagram (transfromand diagram), constructed under the given conditions, was possible because of his own acts of inferential interpretation of inductive, abductive, and deductive nature. The conjecture of the student was an act of inductive inference mediated and enabled by the dragging modality of the GSP. His introduction of the altitude as an auxiliary line to introduce right triangles was a creative act of intra-interpretation; in other words, this was an act of abduction. Once, new triangles were introduced, his acts of interpretation were deductive in nature and produced by intra-interpretation and inter-interpretation mediated by the GSP as a device to interact with. The writing of the proof, once the student constructed the geometric relations among the parts of the diagram, was an act of intra-interpretation of deductive nature.

The analysis brings to the front a thinking strategy that emerged, in part, due to the inquiry method used during the teaching-experiment, namely, the role of the students' engagement on their own sequential exploration and transformation of an initially constructed diagram to generate a conjecture and to prove it. The exploration and observation of each new diagram (transformate diagram) allowed the student to 'see', perceptually and conceptually, new relations among their parts. Such relations, in turn, triggered a coherent geometric argument that was linearly and deductively encapsulated in the writing of his proof. What is novel in the analysis of the data is not the actual proof of the conjecture but the process through which the proof emerged and the student's own acts of inferential interpretation of inductive, abductive, and deductive nature. Here the student is not re-producing a proof, he is producing a proof on his own.

References

Arheim, R. (1969). *Visual thinking*. Los Angeles, CA: University of California Press.

Bishop, A. (1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics*, *11*(1)7-16.

Dörfler, W. (1991). Meaning, image schemata and protocols. *Proceedings of the 15th Annual Conference for the Psychology of Mathematics Education*, *1*, 17-32.

Kant, I. (1781/2007) *Critique of pure reason*. (Translated by Max Müller). New York, NY: Penguin Books.

Netz, R. (2014). Greek mathematical diagrams: their use and their meanings. *For the Learning of Mathematics*, *18*(3), 33-39.

Peirce, C. S. (1935-1966). *Collected Papers of Charles Sanders Peirce* (CP). Edited by Charles Hartshorne, Paul Weiss, and Arthur W. Burks. Cambridge, MA: Harvard University Press.

Peirce, C. S. (1906). Prolegomena to an apology for pragmaticism (PAP), 16(4), *The Monist*, 492-546.

Peirce, C. S. (1976). *The New Elements of Mathematics* (NEM), vol. IV. Carolyn Eisele, Editor. The Hague, The Netherlands: Mouton Publishers.

Sáenz-Ludlow, A. and Zellweger, S. (2016). Classroom mathematical activity when it is seen as an inter-intra double semiotic process of interpretation. In A. Sáenz-Ludlow and G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics* (pp. 43-66). Rotterdam, The Netherlands: Sense Publishers.

Stjernfelt, F. (2007). *Diagrammatology: an investigation on the borderlines of phenomenology, ontology, and semiotics*. Dordrecht, The Netherlands: Springer.

"GROWTH GOES DOWN, BUT OF WHAT?" A CASE STUDY ON LANGUAGE DEMANDS IN QUALITATIVE CALCULUS

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The instructional approach of qualitative calculus aims at developing conceptual understanding for the relationship between amount and change, e.g., by connecting multiple representations of complex context phenomena. This article presents a design experiment with two Grade 11 students' pathways towards the mathematical concepts of amount, change, and change of change. Qualitative analysis is used to show how deeply concept and language development are intertwined and to unpack the language demands occurring on the students' conceptual pathways.

THEORETICAL BACKGROUND: LANGUAGE DEMANDS IN QUALITATIVE CALCULUS: AMOUNT AND CHANGE OF CHANGE

Amount and change as core concepts in qualitative calculus

Calculus has often been shown to be conceptually challenging for many students. That is why approaches of qualitative calculus have been suggested in order to promote conceptual understanding of the relationship between quantities of amount and change (Thompson & Thompson, 1994) long before change is mathematized as average and instantaneous rate of change and the derivatives and their procedural rules (see Stroup, 2002, for steps even before the rate of change). "Understanding qualitative calculus is cognitively significant and 'structural' in its own right" (Stroup, 2002, p. 170). The "own right" is justified, e.g., by the relevance of qualitative concepts for out-of-school contexts such as newspaper headlines:

> "Fewer child births. In recent years, the population growth has decreased."

Empirical evidence has been provided that many students and adults misinterpret this statement as reporting about declining populations. But it is the *population growth* function f' that decreases, not the *population amount* function f, and f can still grow even if f' decreases, then the growth becomes just slower. As Hahn and Prediger (2008) have shown, this misunderstanding occurs especially in a phenomenon they called *two-directional covariation*, i.e., when the covariation of f and f' have different directions: One increases and the other one decreases. They explained the specific difficulties in understanding two-directional covariation phenomena using their level model (see Fig. 1 from Hahn & Prediger, 2008). It builds upon Confrey's and Smith's (1994) distinction of two approaches for functions, the *correspondence* approach (asking what the value of f at x_1 and x_2 is) and the *covariation* approach (asking how f changes with x).

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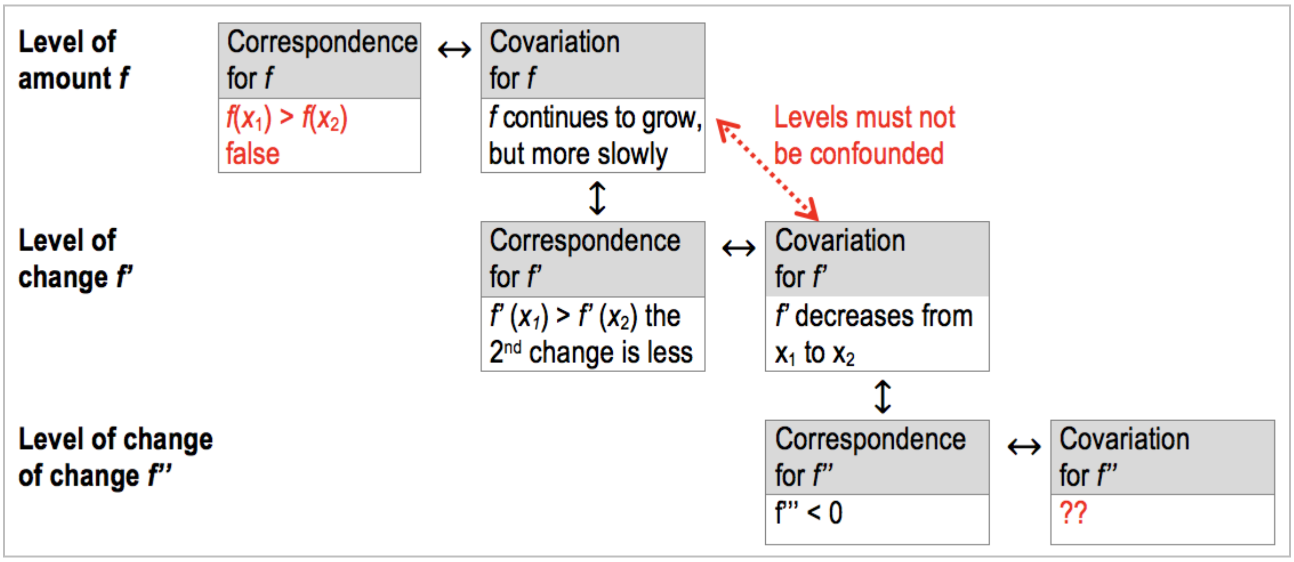


Fig. 1: Complex relationships of amount and change: Shifts of approaches and levels for the example of the headline "population growth decreased"

When shifting the levels (see Fig. 1), the covariation approach for *f* turns into a correspondence approach for *f'*, and on the next level, the covariation approach for *f'* turns into a correspondence for *f''*. In contrast, statements about covariation of *f'* must not be identified with statements about covariation of *f*, as they can have opposite directions.

Research focus on specifying language demands in conceptual development

In general, developing conceptual understanding has been shown to depend much more on language than procedural knowledge, which can be traced back to the epistemic function of language (Moschkovich, 2010; Schleppegrell, 2007): Constructing meanings of abstract mathematical concepts requires the use of concise and powerful language as a thinking tool for disentangling complex thoughts. For these purposes, everyday language must be enriched by the *school academic language register*, as its language features are optimized for expressing reified and abstract relationships in concise and explicit ways (Schleppegrell, 2007; for functions, Prediger & Zindel, 2017).

Although the general affordances and challenges of school academic language have been profoundly analyzed (Moschkovich, 2001; Schleppegrell, 2007), this general knowledge is not yet concrete enough to support language learners in their mathematics learning. Thus, more topic-specific research is required to specify the concrete language demands for specific mathematical topics (Prediger & Zindel, 2017). This paper intends to contribute to this research agenda by unpacking two students' learning pathways in a design experiment towards the distinctions of amount, change, and change of change with the following research questions:

(RQ1) Which language demands occur when students engage in productive struggle with the qualitative concepts of amount, change, and change of change?

(RQ2) How can students' learning be supported?

Design of the investigated teaching learning arrangement

Following Duval (2006) and many other mathematics education researchers, developing conceptual understanding can be fostered by the design principle of *connecting multiple representations*. This design principle has also proved to be powerful for language learning, as relating different registers and representations of the multiple semiotic system supports the language learners' construction of meanings (e.g., Prediger & Zindel, 2017). Furthermore, a key language challenge in mathematics teaching "is to help students move from everyday, informal ways of construing knowledge into the technical and academic ways that are necessary for disciplinary learning [and connect with subject-specific] ways of using language to construct knowledge" (Schleppegrell, 2007, p. 140). In this quotation, the sequencing for language registers is directly combined with the sequencing of epistemic practices; hence it calls for integrating conceptual and language-related learning opportunities.

For sequencing learning opportunities along a concept- and language-related continuum, a good starting point is students' intuitive distinctions of amount and change as documented by Stroup (2002): "Learners are observed to be able to move between rate and amount renderings — most often in graphical forms" (p. 170). Thompson and Thompson (1994) have shown in a case study that "computational language" is not enough to construct meanings that allow for distinguishing amount and change, but they leave open how "conceptual language" should look.

The design is already a first answer to RQ2: With activities as printed in Fig. 2, the learning arrangement combines all three design principles: (1) relating representations

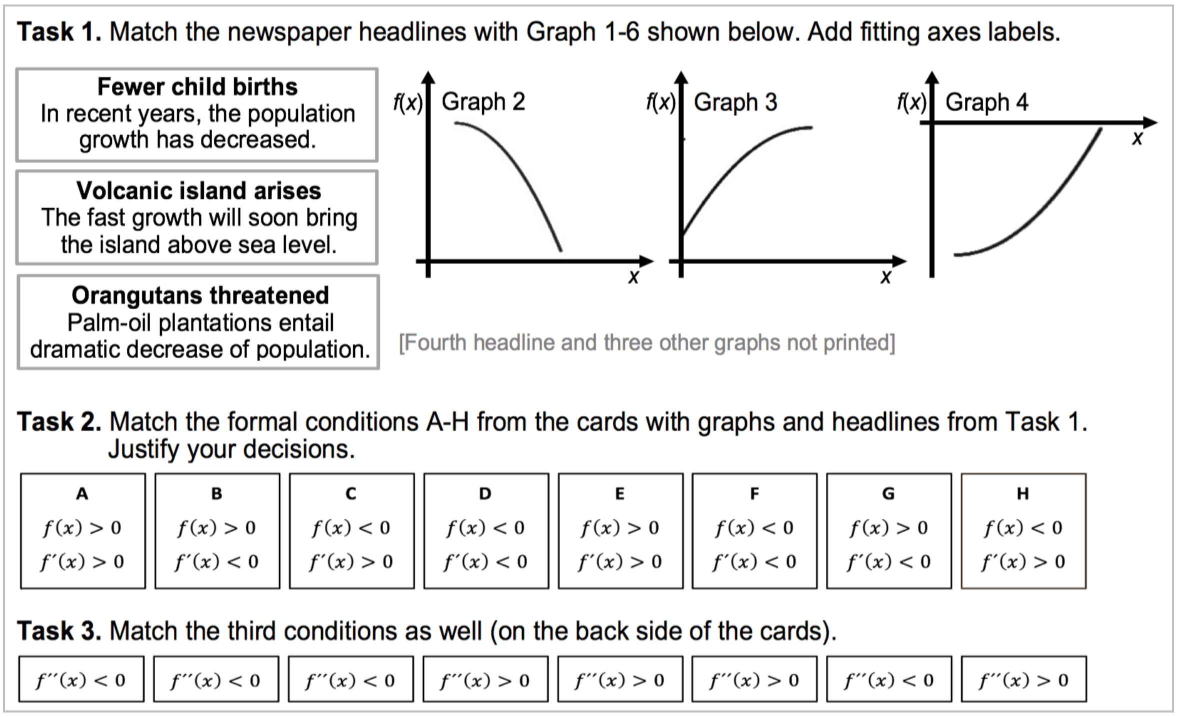


Fig. 2: Activity for connecting multiple representations of two-directional covariation

and registers, (2) eliciting students' intuitive conceptual and language-related resources in order to construct meanings, and (3) providing challenges for productive struggle in order to elaborate students' concepts and language integratively in more concise and formal ways. Thereby, the phenomenon of two-directional covariation is in the core and worked through by relating representations and contrasting similar texts, graphs and formal conditions. Task 1 of the activity was treated in a similar form in the beginning of calculus (as suggested by Stroup, 2002). In order to ensure a pertinent conceptual focus during all of the teaching units of calculus rather than only in the beginning, the activity sequence in Fig. 2 was given after introducing the formalization for the derivative and the second derivative in order to reconstruct meanings for it.

METHODOLOGY OF THE DESIGN RESEARCH STUDY

Design Research as methodological framework. The overarching Design Research project in which the presented case study is embedded has the dual aim of designing language- and mathematics-integrated arrangements (in this case, the construction of meanings of the function-derivative relationship using the presented design principles) and developing an empirically grounded local theory of students' learning processes and the occurring language demands. For empirically specifying language demands, the methodological framework of Design Research with a focus on learning processes has proven valuable (e.g., in Prediger & Zindel, 2017).

Design experiments for data collection. Design experiments are the major method of data gathering in design research studies (Gravemeijer & Cobb, 2006). In the overarching project, four design experiment cycles were conducted with 16 pairs of students in Grades 10 and 11 (14-16 years old). This paper uses data from Cycle 3 in which design experiments were conducted in laboratory settings with nine pairs of 11^{th} graders. Two sessions of 45-60 minutes each were completely video-recorded for each pair of students (in total about 1000 minutes of video material). In this paper, the analysis focuses on a case study of two girls, Emily and Layla, and the first author as design experiment leader (in the following referred to as "tutor").

Methods for qualitative data analysis. The qualitative analysis of the transcripts was conducted with the aim of qualitatively tracing the students' conceptual learning pathway by locating the intermediate steps in the level model from Fig. 1. In a second step, the students' language resources and obstacles were extrapolated in the lexical, syntactical, and discursive dimension, following Schleppegrell (2007).

EMPIRICAL INSIGHTS INTO THE CASE OF EMILY AND LAYLA

Rediscovery of two-directional covariation and language resources

Emily and Layla had worked on similar headlines some weeks before. Meanwhile, they had learned to formalize the derivative and calculate it procedurally. So they had to rediscover the phenomenon of two-directional covariation: When Emily and Layla start to work on Task 1 (from Fig. 2), Emily falsely matches the headline with Graph 2:

65 Emily Yes and here [*hints to the headline "Less child births" and to the decreasing Graph 2*] the population growth just decreases, that means it was so much [*hints to the beginning of Graph 2*] […] and then it is less and less, well that means probably, how many people live at which time, and then there live simply less people.

66/7 Tutor/Layla Mhm.

68 Emily [*hints to the headline again*] I just remember, when we say, the growth becomes less, we had last time, that this can however become more, but not as much as before?

In Turn #65, Emily confounds population growth *f'* with the function *f*, which captures the population amount ("how many people live at which time") and transfers the increase to the wrong level. But in #68 she reminds herself that both levels can have different directions, even if she cannot yet express it. When the tutor invites her to be more precise about the axis labels, she formulates it more concisely:

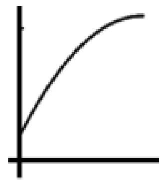
76 Emily There are fewer people born, and, therefore, the population grows not that fast anymore.

On this base, the students decide to choose Graph 3, but take much longer to discuss the axes' labels. They finally reject population growth and decide on population:

250 Emily Ok, the number of people increases, but there are not so many additionally; thus, not so many in a year are added

251 Tutor Mhm

252 Emily After all, the growth becomes less, as it does not increase so steeply [*hints to the end of Graph 3*]



254 Layla Thus, though, the population, however, increases, but simply not that much. And for this, the growth decreases, yes.

After a long discussion, the students have unpacked what decreasing growth means (in #68, 76, and 250); for this, they shift the level *f'* back to level *f* (see Fig. 1). During that period, they speak about *processes* of change in covariation approaches without condensing them into nouns. Then, the labelling of axes is requested, and as this requires nouns, this request serves as a scaffold to conduct a condensation by nominalization. It is only in #254 that Layla has condensed all information in the covariation phrases "*f* increases, even if *f'* decreases" on two levels without going back to correspondence approaches (see Fig. 1).

With respect to occurring language demands, this analysis shows that the students have many *lexical language resources* to express processes of change, e.g., "decrease/increase" (#65), "get/become less/more" (#68, 252), "not so many are added" (#250). The greater challenge is, however, to apply them with successive conciseness. This challenge is related to *syntactical and discursive demands* on making explicit the levels to which the change processes refer and to express their mutual relationships. As linguistically explained by Schleppegrell (2007), making something explicit requires sentences with explicit references (instead of unspecific "this" and "it"), which requires a condensation of processes into nouns, which function as lexical markers for objectified concepts: The long sentence "How many people live at which time, and then there are simply fewer people living" is later condensed into the nominalization that allows combination with a verb "population increases/decreases" (#254).

Language demands while constructing meanings for the second derivative

During their pathway, Emily and Layla quickly succeed in assigning the formal conditions about *f* and *f'* to the graphs and headlines as requested in Task 2 from Fig. 2. Their struggle begins again with Task 3, which requests matching $f''(x) > 0$ or $f''(x) < 0$.

320 Emily — Oh, God [second derivative]. I know only that you can, eh, the turning points […] eh, always zero, something […]

355 Tutor — The second derivate stands for the growth of the increase. [[*the German word for slope has two meanings, "slope" and "increase"; "increase" is preferred in this case as it is the more common everyday usage*]]

360 Emily — The growth of the increase?

361 Layla — Yes perhaps, how much that grows again?

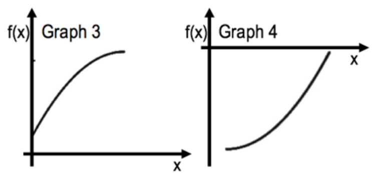
366 Emily — The growth, the growth of what? The function?

368 Emily — […] Well, we have a function and the derivative says always, what is the increase.

372 Emily — If it increases quickly or something like that?

With *f''*, Emily and Layla only associate procedural aspects of calculating turning points by using $f''(x) = 0$ in #320-354. When the tutor informs them about the standard interpretation as "growth of increase" (#355), they cannot make sense of it as they do not know how to refer the growth to something else ("the growth of what," #366). So far, they have not referred growth to any increase other than the increase of one level (amount, change, or change of change) they pointed out; they cannot cope with the double nominalization. In #368, Emily unpacks the nominalizations, which prepares her tentative interpretation of *f*' as "increases quickly" in #372. As this idea from #372 gets lost again, the tutor comes back to it:

390 Tutor — Ela, you just said that this is perhaps a statement about how strongly the increase increases. Look at Graphs 3 and 4. What are the commonalities of the increase?



391 Layla — **It** is positive.

402 Emily — Aha, because **this here** [*hints to Graph 4*] increases then, well, **that** goes more **That**, well, **here** is [*hints to Graph 3*], it grows but then **it** gets less […]

406 Emily — [And for Graph 4] it looks more as if **this** grows still steeper and not [like in Graph 3] **the increase** goes in direction 0, thus gets less.

414 Emily — Ok, both increase. But **here** [*hints to Graph 3*] The **increa**— well, **that** increases though, OK, that is what we assigned, because **this** gets less, thus, does not grow so steeply.

415 Tutor — **What** gets less?

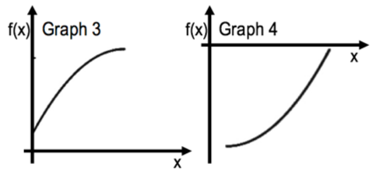
416 Emily — The increase

418 Emily — Ok, **it** increases **here** [*hints to beginning of Graph 3*] more steeply, that means, within the time, **it** becomes more. And **here** [*hints to middle of Graph 3*], in this section, less is added, even if the time interval is equal [*gestures a slope triangle*]. And, eh, **here** [*hints to the end of Graph 3*] the **increase** gets always less.

Again, the main language demands are not on the lexical level, but on the syntactical and discursive level: In these turns, all references to what increases or decreases (the subjects for the predicates) are marked in bold. The high amount of *underdetermined deictic expressions* (”it,” “this,” and “that”) for the addressed level shows the big need to become more explicit, because every underdetermined expression causes a risk to confound the levels f, f', and f''. Explicit navigation through the levels of Fig. 1 is only possible with explicitly articulated references to the levels. The tutors' prompts in #390 and 415 are intended try to support the students to make the references explicit. And, indeed, Emily can work with these prompts in #418 when she refers to different parts of the graph: at first underdetermined in language, then very precise (“the increase in this time interval,” #418). However, they have still not cracked the meaning of the formal condition $f''(x) < 0$, with reference to what f and f' mean.

457 Tutor OK, what can this mean for the second derivative. There is the option that $f''(x) < 0$ or $f''(x) > 0$. What can this mean, now?

458 Emily Perhaps, **this** [*hints to Graph 3*] is bigger than zero, because **it** gets less, than … and then **it** becomes zero? I mean, that **it** is first more and then **it** gets less. And here [*hints to Graph 4*] **it** is first less and then it is more. That is perhaps, anyway, that **this** is less than zero, and when **it** is less than zero, that means that the **increase** becomes less?



459 Layla Perhaps, the second [derivative] describes the **growth of the increase** and **here** [*hints to Graph 4*], **this** increases and there [*hints to Graph 3*] **it** decreases.

478 Emily [...] OK, we have said that when the **increase** is zero and then increases [as in Graph 4], then **it** [*she means the second derivative*] is bigger than zero, and when **it** [*she means f''*] is less than zero, than **it** decreases [*she means increase f' of f*].

It takes 20 more turns for Emily to correct the misinterpretation in #458 and assign the formal condition appropriately. #458 and 459 shows that shifting to the level of f' (change of change) again requires other language means in which the increase must be treated as object and therefore be nominalized. In #478, Emily finally succeeds in making sense of the second derivative by reaching the level of f'' (see Fig. 1), even if still with many underdetermined references and a restriction to positive f''.

Summing up, language demands already visible in the easier Task 1 become crucial obstacles in dealing with formal conditions for the second derivative in Task 3. Lexical means are not missing (all students in our data had sufficient lexical resources), instead, the main challenge is their concise use in sentences without underdetermined references. In the process of making the relationships explicit, nominalizations and the necessary objectifications go hand in hand (Schleppegrell, 2007).

DISCUSSION AND OUTLOOK

What does it mean to talk about amount and change conceptually? The case study of Emily and Layla has provided insights into students' pathways through productive struggle with the concepts of qualitative calculus: amount, change, and change of change. Requesting the connection of multiple representations is a main answer to

RQ2, and it is shown to help students to successively refine their thinking about the relationship of amount and change and express the phenomenon of two-directional covariation in successively concise ways. The major finding on RQ1 is that language demands in Grade 11 do not occur on the lexical level but on the discursive and syntactical level for establishing clear references and nominalizations for objectifications. This can give important hints for the further design of this and other teaching learning arrangements: Teachers' prompts for making references explicit are crucial for developing conciseness of language (another major answer to RQ2).

Future research is planned so that the methodological limits of the case study can be overcome by (1) increasing the sample, (2) varying the tasks and activity settings, and (3) transferring the research framework to other mathematical topics. As almost no research has investigated higher grades, the differences from research results in earlier grades found here motivate a further focus on the upper secondary level.

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References

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*(2/3), 135-164.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1-2), 103-131.

Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In J. van den Akker, K. Gravemeijer, S. McKenney, & N. Nieveen (Eds.), *Educational design research* (pp. 17-51). London: Routledge.

Hahn, S. & Prediger, S. (2008). Bestand und Änderung: Ein Beitrag zur Didaktischen Rekonstruktion der Analysis. *Journal für Mathematik-Didaktik, 29*(3/4), 163-198.

Moschkovich, J. N. (2010). *Language and Mathematics education*. Charlotte: Inform. Age.

Prediger, S., & Zindel, C. (2017). School Academic Language Demands for Understanding Functional Relationships: A Design Research Project on the Role of Language in Reading and Learning. *EURASIA Journal of Mathematics, Science and Technology Education, 13*(7b), 4157-4188. doi:10.12973/eurasia.2017.00804a

Schleppegrell, M. J. (2007). The Linguistic Challenges of Mathematics Teaching and Learning: A Research Review. *Reading & Writing Quarterly, 23*(2), 139-159.

Stroup, W. (2002). Understanding qualitative calculus: A structural synthesis of learning research. *International Journal of Computers for Mathematical Learning,* 7(2), 167-215.

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education, 25*(3), 279-303.

USE OF A LEARNING TRAJECTORY AS A CONCEPTUAL INSTRUMENT TO DEVELOP THE COMPETENCE OF PROFESSIONAL NOTICING

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The objective of this study was to characterize how a learning trajectory, relating to length magnitude and its measurement, could be used by pre-service kindergarten teachers to develop their professional competence of noticing the mathematical thinking of children. That is, how a learning trajectory could be used as a conceptual instrument, by means of two instrumented action schemes: by using the trajectory's learning progression model and tasks to interpret children's responses, and by suggesting new tasks that supported progress in comprehension. A total of 47 pre-service kindergarten teachers took part in the study. Results showed that the learning trajectory helped pre-service teachers to interpret children's mathematical thinking, and to make appropriate decisions in support of their students' progress.

THEORETICAL FRAMEWORK

Research on the acquisition of professional noticing of students' mathematical thinking has shown that the competence to interpret children's mathematical thinking can be acquired during initial teacher training programs, using tasks that allow to differentiate between degrees of understanding of a concept (Magiera, Van den Kieboom & Moyer, 2013; Schack, Fisher, Thomas, Eisenhardt, Tassell & Yoder, 2013; Son, 2013). However, the ability to make suitable decisions as to what actions to take is not an easy skill to develop. This led us to reflect on the need for a framework of reference that would help pre-service kindergarten teachers to structure their attention towards students' mathematical thinking. In this sense, a hypothetical learning trajectory of a mathematical topic would help pre-service teachers to identify their students' learning objectives, anticipate and interpret their mathematical thinking and contribute, with appropriate instructional decisions, to their progression in learning (Edgington, Wilson, Sztajn & Webb, 2016; Sztajn, Confrey, Wilson & Edgington, 2012).

A hypothetical learning trajectory can be understood as a conceptual instrument (Llinares, 2004) that adapts the theoretical framework of instrumental genesis (Drijvers, Kieran and Mariotti, 2010). In this theory, artifacts, defined as the tools themselves, are distinguished from instruments, which correspond to artifact that have come to embody a significant relationship between the user and the task to be performed. The use of an artifact introduces a cognitive activity of construction or evolution of utilization schemes in the subject; hence, students can approach an artifact differently and develop different instrumented action schemes (Rabardel, 1995). These instrumented

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 107-114). Umeå, Sweden: PME.

action schemes are typical of the instrumentation process, and are directly related to the use of the artifact with a view to carrying out an action or a task; that is, these schemes allow the student to understand the potentialities and restrictions of the artifact itself: they progressively constitute techniques enabling to effectively respond to mathematical activities.

In our research, a hypothetical learning trajectory is understood as an artifact that turns into an instrument when pre-service teachers make it their own, allowing them to solve the proposed tasks. Therefore, a pre-service teacher's *instrumentation of a learning trajectory*, takes place over two successive phases. The first is the development of *a first instrumented action scheme*: the learning progression model, facilitated as a hypothetical trajectory of learning, is used to interpret the characteristics of the student's understanding, based on the identified mathematical elements. In the second phase, *a second instrumented action scheme* is developed: the types of tasks facilitated in the trajectory and the learning progression model are used together to propose new tasks directed towards advancing all children's understanding, based on the interpretation made in the previous scheme.

For this research, we presented pre-service teachers with a learning trajectory of length magnitude and its measurement in childhood education adapted from Sarama and Clements (2009). In this trajectory, mathematical elements that define progression in learning about length magnitude are: recognising length, conservation and transitivity; those defining the measure of length are: unit of measurement-unicity, iteration, accumulation, relationship between the number and measurement unit, and universality of the measurement unit. Progression is organized into five inclusive levels that go from recognising length as a magnitude, to the construction of the concept of length measurement and initiation to the estimation of lengths (Callejo, Perez, Moreno, Sánchez-Matamoros, & Valls, 2017).

Our objective was to characterize the use, as a conceptual instrument, that pre-service kindergarten teachers make of a learning trajectory relating to length magnitude and its measurement, and how it facilitated the development of the teaching competence consisting in professionally noticing children's mathematical thinking. This use is characterized by changes in the development of the instrumented action schemes throughout the teaching module.

METHOD

Participants and context

The participants of this study were 47 students in the Kindergarten Schoolteacher Degree at the University of Alicante (Spain), who attended the subject "Learning Geometry" in the sixth semester. The training programme for this subject included a 10 hour long module on length magnitude and its measurement (5 face-to-face sessions) during which students had to solve a professional task (a teaching-learning situation providing three issues to reflect on the professional skills of identifying, interpreting and making instructional decisions). In order to solve the module's tasks, they disposed of the learning trajectory of length magnitude and its measurement for children aged 3 to 6 years (Sarama & Clements, 2009).

Data collection instrument

The data collection instrument consisted of the three professional tasks corresponding to the initial, intermediate and final sessions. These tasks were composed of teaching-learning situations (Table 1) and three professional questions:

Question 1. Justify the characteristics of children's understanding of each point, indicating the implicit mathematical elements.

Question 2. According to the characteristics of the children's understanding identified in Question 1, what level of understanding would you place them at? Justify your answer.

Question 3. Assuming that you are their teacher, define a learning objective and propose a task for these children to continue advancing in the understanding of length magnitude and its measurement.

Situation	Description of the teaching-learning situation	Mathematical elements
Initial	Four illustrations extracted from a video are provided. The teacher suggests that the children cut out a strip of paper as big as themselves (Illustration 1). The children try several times to get the strip at exactly their own height (standing, on the floor, standing against a cupboard ...) (Illustration 2). Then they decorate them. With the help of the teacher they compare the strip lengths two by two and the teacher puts them in order (Illustrations 3 and 4). (Adapted from van den Heuvel-Panhuizen & Buys, 2005)	Recognition (Illustration 1) Conservation (Illustration 2)
Intermediate	Four illustrations of a park outing are shown during which two teams measure the outline of a different tree selected by each team, using the piece of rope provided. Team A selects a tree with a thin trunk and measures it using the rope (Illustration 1). Meanwhile, the tree chosen by team B cannot be measured with the rope because it is thicker (Illustration 2). In response, children in both teams decided to surround each tree with their arms (team A: one girl, team B: four boys) (Illustrations 3 and 4, respectively). The teacher	Recognition (Illustration 1 and 2) Unicity (Illustration 4) Iteration (Illustration 4) Accumulation (Illustration 4)

	asks what would happen if two of the four children were replaced by another two. (Adapted from Alsina, 2011)	
Final	A teacher can be seen talking to some children. The teacher proposes making necklaces using strings of different lengths and shapes (rolled up, stretched and folded), and different beads of different kinds and sizes (macaroni, stars, etc.). Children choose different strings to make the necklace: Mario uses different sized beads (non-Unicity); Almudena chooses beads of the same size (Unicity) and inserts them leaving gaps between them (No Iteration); Elena and Luis use beads of the same size inserting them without leaving gaps (Unicity and Iteration). When the necklaces are made, a dialogue takes place between the teacher and the children. (Conservation and Accumulation) (Callejo et al., 2017)	Conservation Unicity Iteration Accumulation

Table 1: Description of the three teaching-learning situations and its mathematical elements

Data analysis

The data in this research consists in the answers of pre-service teachers to the three professional tasks, object of study. An inductive qualitative analysis was carried out (Strauss & Corbin, 1994) where a group of five researchers first analysed a small sample, then discussed codifications and their relationships with the evidence, leading to the creation of several categories. Once the categories were agreed upon, the rest of the data was added to review the initial system of categories and verify its validity.

We carried out this process of analysis in two phases. The first phase consisted in two steps: first, professional questions 1 and 2 were analysed together to characterize how the pre-service teachers had developed the first instrumented action scheme. Next, professional question 3 was analysed to characterize how pre-service teachers had developed the second instrumented action scheme.

In the second phase of analysis, changes in the use of the trajectory experienced by each of the pre-service teachers were identified along the three tasks: initial (of magnitude), intermediate (of measurement) and final (of magnitude and measurement), through the development of instrumented action schemes.

RESULTS

Results revealed that the learning trajectory helped students to master the interpretation of children's mathematical thinking; however, only a few were able to make appropriate decisions to support student progression. Moreover, pre-service kindergarten teachers used the learning trajectory as a conceptual instrument throughout the module in different ways. These different ways accounted for the five changes experienced in the use of the learning trajectory in relation to the development of instrumented action schemes (Figure 1).

In three of the five changes (changes 1, 2 and 3), the pre-service teacher did not use the trajectory as a conceptual instrument for either tasks, nor for the magnitude task (initial task) nor for the measurement task (intermediate task). They only gave general and rhetorical descriptions such as in the case of pre-service teacher Rosa:

> Rosa: "Illustration 2: They use the transitive property for indirect comparisons (in the first, the child stands up next to the strip and in the second, the child lies on top of the strip, with his feet on top of the strip and makes a sign with his head) .

However, in the magnitude and measurement task (final task), pre-service teachers used the trajectory in three different ways (changes 1, 2 and 3). Change 1 characterizes pre-service teachers who only use the progression model to interpret the understanding of some or all of the children, using elements of magnitude and/or measurement (first instrumented activity scheme). This was the case of Rosa, who developed the first instrumented action scheme for measurement:

> Rosa: Mario is at level 4, he knows how many macaroni he has used to make the necklace -property of accumulation-, iterates well because he leaves no gaps, no overlays ... Almudena is at level 3-4, she does not iterate correctly when leaving gaps, but does recognize the property of accumulation. Luis and Elena are at level 4, they iterate correctly and the accumulation property, "mine has 12 macaroni"

Change 2 characterizes pre-service students who partially developed both instrumented action schemes when interpreting the understanding of only some children, regarding only magnitude or measure, and only proposed tasks to facilitate children's learning progression. Finally, change 3 characterized pre-service teachers who achieved the instrumentation of the learning trajectory by interpreting the comprehension of all children, in relation to magnitude and length, and proposed tasks to advance children's learning progression, as in the case of Catalina:

> Catalina: Mario is at level 1. He recognizes the length magnitude, but does not compare the two strings [his and Luis']. Elena is at level 4, she chooses the longest string, correctly iterates the stars without leaving gaps or overlays. New task for Mario: Objective: Compare by displacement [conservation]. Task: Choose the smallest string [string C] and the largest string [string A], put them side by side and compare them. New task for Elena: Objective: Start acquiring the universality of the measurement unit. Task: Almudena and you have used the same string and the same material to make the necklace, which of the two necklaces is longer, that of Almudena or yours? Why?

Change 4 corresponds to those who initiate the module relating the characteristics of understanding with the mathematical elements identified (first instrumented action scheme), and they finish it, partially developing both instrumented action schemes.

Change 5 corresponds to pre-service teachers who initiated the module at the same stage students of change 2 finished at (partial development of both instrumented action schemes) and ended up developing both instrumented action schemes, that is, instrumentation of the learning trajectory.

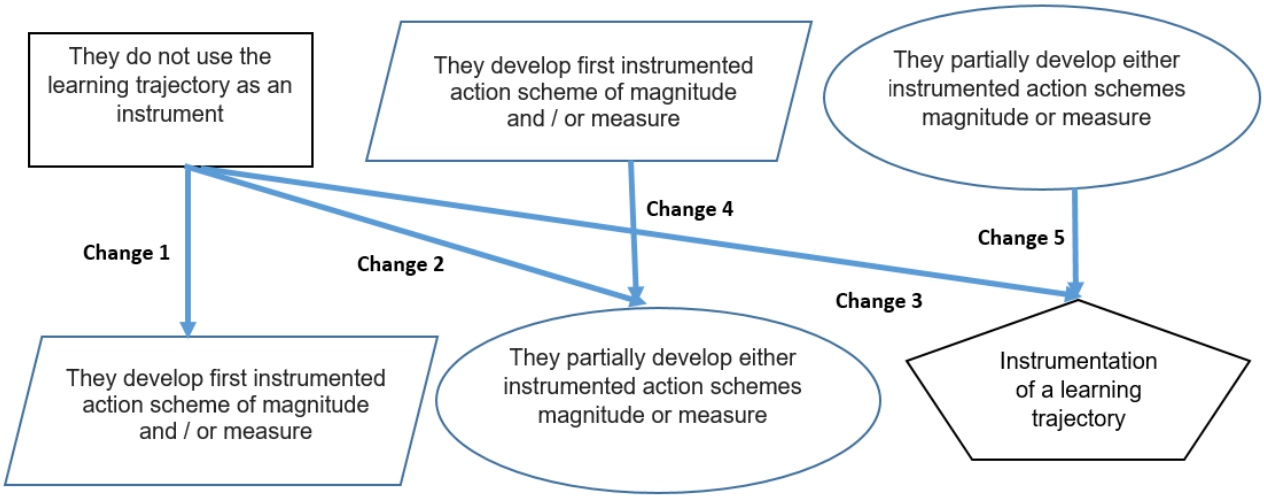

Figure 1: Changes identified in the use of the learning trajectory as a conceptual instrument

DISCUSSION

All pre-service teachers showed some progress throughout the module. The progress of pre-service teachers who first considered the learning trajectory as an artifact but were then able to initiate their first instrumented action scheme (change 1), could be explained by the fact that they considered the understanding of some mathematical elements (of magnitude and/or measurement, involved in the teaching-learning situations raised) as a conceptual advance. This fact is corroborated by previous research that has showed how recognising the understanding of mathematical elements, for specific concepts, could be considered as points of reference in teachers or pre-service teachers, when they learn about the mathematical thinking of children/students (Llinares, Fernández, & Sánchez-Matamoros, 2016). The impossibility of developing the second scheme of instrumented activity could be due to the fact that pre-service teachers tended to focus their actions on general teaching procedures rather than on students' conceptual progress, a fact already highlighted in previous studies such as that by Gupta, Soto, Dick, Broderick, & Appelgate, (2018).

Moreover, the progress of those who started at different stages (change 2 and change 4) but reached the same point, that is, initiated the development of both instrumented action schemes, could be justified by the following: they recognized the understanding of some mathematical elements of magnitude or measurement as a conceptual advance (Simon, 2006), and recognized that any proposed new tasks should advance the understanding of the concept. Finally, some pre-service teachers progressed from using the learning trajectory as an artifact to its instrumentation (change 3). Others progressed from the partial development of both instrumented action schemes to their instrumentation (change 5). These two progressions may be explained by the fact that these pre-service teachers understood the learning trajectory as a whole, allowing them to take into account the progressive construction of the concept, attaching importance to both magnitude and its measurement, thus adopting the most appropriate instructional decisions to help children advance in their understanding.

The use of the learning trajectory as a conceptual instrument is very positive since all pre-service teachers were able to initiate the development of the first instrumented action scheme. This means they learned to interpret characteristics of children's progression in understanding; they started to notice the understanding of a concept in a more structured and systematic way, attaching importance to the mathematical elements involved in different teaching-learning situations. Decision-making, however, still remains a challenge: it was shown that for the majority of pre-service teachers, disposing of tools was insufficient to make adequate decisions.

Acknowledgements

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References

Alsina, A. (2011). *Educación matemática en contexto: de 3 a 6 años*. I.C.E. Universitat de Barcelona. Hosori Editorial, S. L. (p.176).

Callejo, M.L., Pérez, P., Moreno, M., Sánchez-Matamoros, G., & Valls, J. (2017). A learning trajectory for length as a magnitude and its measurement: usage by prospective preschool teachers. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education, Vol. 2*, pp. 201-208. Singapore: PME.

Drijvers, P., Kieran, C., & Mariotti, M.A. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles y J.B. Lagrange (Eds.), *Mathematics education and technology: Rethinking the terrain* (pp. 89-132). New York: Springer.

Edgington, C., Wilson, P. H., Sztajn, P., & Webb, J. (2016). Translating learning trajectories into useable tools for teachers. *Mathematics Teacher Educator, 5(1)*, 65-80.

Gupta, D., Soto, M., Dick, L., Broderick, S.D., & Appelgate, M. (2018). Noticing and deciding the next steps for teaching: A cross-university study with elementary pre-service teachers. In: G.J. Stylianides y K. Hino (eds.), *Research Advances in the Mathematical Education of Pre-service Elementary Teachers, ICME-13, Monographs* (pp. 261-2759. Springer.

Llinares, S., Fernández, C., & Sánchez-Matamoros, G. (2016). How prospective teachers anticipate secondary students' answers. *Eurasia Journal of Mathematics, Science & Technology Education, 12(8)*, 2155-2170.

Llinares, S. (2004). La generación y uso de instrumentos para la práctica de enseñar matemáticas en educación primaria. UNO. *Revista de Didáctica de la Matemática, 36,* 93-115.

Magiera, M. T., Van den Kieboom, L. A., & Moyer, J. C. (2013). An exploratory study of pre-service middle school teachers' knowledge of algebraic thinking. *Educational Studies in Mathematics, 84(1),* 93-113.

Rabardel, P. (1995). *Les hommes et les technologies: approche cognitive des instruments contemporains*. Paris: Armand Colin.

Sarama J., & Clements D. (2009). *Early Childhood Mathematics Education Research. Learning Trajectories for Young Children.* London and New York: Routledge.

Schack, E., Fisher, M., Thomas, J., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education, 16*, 379–397.

Simon, M. (2006). Key Developmental Understanding in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8(4*), 359-371.

Son, J. (2013). How preservice teachers interpret and respond to student errors: ratio and proportion in similar rectangles. *Educational Studies in Mathematics, 84,* 49-70.

Strauss, A., & Corbin, J. (1994). Grounded Theory Methodology. In N.K. Denzin & Y.S. Lincoln (Eds.), *Handbook of Qualitative Research* (pp. 217-285). Thousand Oaks, Sage Publications.

Sztajn, P., Confrey, J., Wilson, P.H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41(5)*, 147–156.

van den Heuvel-Panhuizen, M., & Buys, K. (2005). *Young children learn measurement and geometry*. TAL Project. Freudenthal Institute, Utrech University and National Institute for Curriculum Development. Utrech. The Netherlands.

Wilson, P.H., Mojica, G.F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *Journal of Mathematical* Behavior, *32*, 103-121.

EYE-TRACKING FOR STUDYING MATHEMATICAL DIFFICULTIES—ALSO IN INCLUSIVE SETTINGS

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Eye-Tracking (ET) is a promising tool for mathematics education research. Interest is fueled by recent theoretical and technical developments, and the potential to identify strategies students use in mathematical tasks. This makes ET interesting for studying students with mathematical difficulties (MD), also with a view on inclusive settings. We present a systematic analysis of the opportunities ET may hold for understanding strategies of students with MD. Based on an empirical study with 20 fifth graders (10 with MD), we illustrate that and why ET offers opportunities especially for students with MD and describe main advantages. We also identify limitations of think aloud protocols, using ET as validation method, and present characteristics of students' strategies in tasks on quantity recognition in structured whole number representations.

INRODUCTION

Eye-tracking (ET) promises to allow for online recording of cognitive processes (Chen & Yang, 2014). Its potential as a tool for mathematics education as well as important theoretical and technical developments have led to an increased interest in ET in the PME community, which the number of methodological papers and a strongly increased occurrence of ET-related key phrases in recent PME proceedings hint at.

The potential that ET may hold has been illustrated in many studies (e.g., Lindmeier & Heinze, 2016; Obersteiner & Tumpek, 2015). Some studies isolated differences between certain groups when solving the same math problems, e.g., differences between strong and weak students in mathematics (Rottmann & Schipper, 2002). Such research responds to the significance to understand mathematical difficulties. Not least since inclusive education has gained significance, research on students with mathematics difficulties (MD) has attracted more and more attention; with the aim to understand knowledge and learning in a fine-grained way and to foster students individually and adequately (e.g., Moser-Opitz et al., 2016; Scherer et al., 2016). Even though ET is partially already used with students with MD, research has—to the best of our knowledge—not yet systematically analyzed what opportunities ET holds especially for understanding strategies and thoughts of students with MD.

Thus, the aim of our study is to investigate opportunities ET may hold especially for students with MD. Based on an empirical study with 20 fifth graders (whereof 10 with MD), we investigated the potential benefit of ET in tasks addressing quantity recognition in structured whole number representations. We investigated the benefit of ET as compared to think aloud protocols (TA) and also compared the opportunities of ET

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for students with and without MD. The results of our data analysis indicate that ET is especially advantageous for students with MD, whose strategies appear to be more diverse and, for several reasons, more difficult to explain. Our paper contributes to mathematics education research in three major ways: We illustrate that and why ET offers opportunities especially for students with MD and describe its main advantages. We identify limitations of TA for students with MD using ET as validation method. And, finally, we present strategies of students with and without MD in tasks on quantity recognition in structured whole number representations.

THEORETICAL BACKGROUND

Eye-Tracking (ET) and its use in mathematics education research

ET devices aim to identify gaze points by capturing eye movements and projecting from the fovea onto the surrounding scene. Video-based systems currently dominate the market; either in the form of head-mounted devices (ET glasses) or remote devices attached to a computer screen that displays visual stimuli (Holmqvist et al., 2011). ET devices promise to allow for online recording of cognitive processes (Chen & Yang, 2014). However, ET only provides a flickering view on "shadows" cast by brain processes in the form of eye movements. Accordingly, interpretation of ET data is non-trivial. It typically rests on the "eye-mind" hypothesis, which, as expressed by (Just & Carpenter, 1976) in the reductionist spirit of a brain-computer metaphor, posits that "the eye fixates the referent of the symbol currently being processed if the referent is in view. That is, the fixation may reflect what is at the 'top of the stack'" (p. 441). Data interpretation is challenging since the eye-mind hypothesis does not always hold (Holmqvist et al., 2011). Not all cognitive processes are tightly linked to visual stimuli. Also, foveal vision is not always required, e.g. when peripheral vision is sufficient.

Despite the challenges mentioned, ET is a potent tool for mathematics education research (MER). Powerful inferences are possible especially in specific, controlled settings—e.g., in "visually presented cognitive tasks" (Obersteiner & Tumpek, 2015, p. 257), and by using domain-specific interpretation—i.e. considering known semantics of fixated visual entities (Schindler & Lilienthal, 2017). The improved theoretical and computational means for interpretation as well as the advent of commercial, less intrusive, portable and increasingly affordable ET devices (Holmqvist et al., 2011) led to considerable interest in this technology; not least in the PME community. When analyzing the last five PME conferences, we found 25 ET-related papers, mostly ET studies and a smaller but increasing number of methodological papers dedicated to ET and the interpretation of eye movement data. We also carried out a full text analysis of the proceedings of the last five PME conferences and found a clear trend: While the wordings "eye tracking" and "eye movement" occurred only 20 times at PME37, this number increased to 208 at PME41.

In addition to improving ET devices and better means of interpretation, interest in ET in MER is fueled by its potential "as a method for identifying strategy use in mathematical tasks" (Beitlich & Obersteiner, 2015). This paper contributes to identify opportunities (and limitations) of ET for identifying strategy use in mathematical tasks. This is particularly important in comparison to methodological alternatives for the same purpose, including TA methods and response time tests.

Mathematics difficulties (MD) and their identification

Learning difficulties in mathematics are an important topic in practice and research and have attracted increased interest not least since inclusive education has gained significance. However, to date there is no consensus on a definition or term characterizing the group of students having difficulties in mathematics (Scherer et al., 2016). Certain researchers speak of *mathematical learning disabilities*, others of (severe) *mathematical difficulties*, depending on different national educational contexts and research traditions (see ibid.; Moser-Opitz et al., 2016). In this paper, we follow Moser-Opitz et al. (2016) who talk about students with mathematics difficulties (MD) and summarize their potential difficulties, comprising (see also Scherer et al., 2016): verbal counting (e.g., counting by groups and counting principles), grouping, degrouping, the base-10 number system and understanding the place value, understanding the meaning of operations, solving word problems; as well as factual knowledge, fact retrieval and (deficits in) working memory. To support students with MD in their mathematical learning, researchers and teachers aim to identify students' individual assets, difficulties, and strategies. Mathematics education, special education, and psychology use different methods for diagnosing: They consider, e.g., written products or tests, observations, or processing times of students when working on math problems. These methods have demonstrated their potential but are also limited because they consider an "outside" view on mental processes or use end results, which can make it difficult to distinguish between different processes leading to the same products. TA and ET, however, observe manifestations of internal processes and have great potential to identify strategy use. Despite its advantages and popularity, TA has drawbacks, especially when working with children with MD and other learning difficulties. In Concurrent TA (CTA) where participants are asked to verbalize their thoughts while performing a task, the additional cognitive load can be overwhelming (Ericsson & Simon, 1980), especially for students with MD for whom the task itself and verbalization can constitute a major cognitive effort. Instead, in Retrospective TA (RTA) verbalization occurs after completion of the task. Still, aspects that may affect verbalizations of students with MD in RTA may be, e.g., anxiety, difficulties with memory retrieval, introspection, or meta-cognitive reflection, or verbalization issues. We see that ET has two potential major advantages over TA: It can identify unconscious processes and there is no verbalization step that could absorb cognitive resources and potentially influence students' strategies. To the best of our knowledge, the potential of ET for better understanding strategies of students with MD has not yet been empirically investigated. We ask the following research questions: *What opportunities may ET offer for understanding strategies of students with MD?* We approach this question through an empirical study with students in an inclusive education setting. We use ET—in combination with TA and without—also address the questions

What opportunities does ET offer compared to TA? and *May the benefit of ET for understanding students' strategies be bigger for students with MD than for students without MD?*

METHOD

*Students. F*or answering the research questions, we use data from a research project with 20 fifth-grade students, ages between 9;11 and 11;11, in a German comprehensive school ("Gesamtschule"). The participating school was in a town of 80,000 inhabitants, situated on the edge of a German urban area. The study took place in the first weeks of fifth grade, after the students had finished the German primary school after grade 4. Students with mathematical difficulties (MD) were identified through qualitative diagnostic interviews addressing MD (following Schulz & Wartha, 2012) investigating, e.g., students' number sense and understanding of number and operations (see Moser-Opitz et al, 2016). Among the 10 students with MD, there were four with special educational needs (in learning, social and emotional development, and physical development).

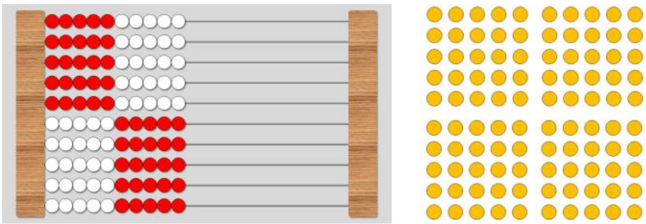

Figure 1: Number representations used in this study

Tasks. The students worked on tasks to determine the number of beads/dots on a 100-bead abacus and on a 100-dot square (Fig. 1). We first let the students determine the total number of beads in a 100-bead abacus (number of dots in the 100-dot square) and asked them to think aloud after they had given their answers (immediately after each task). Further, the students determined numbers (e.g., 7, 76, and 92) on the 100-bead abacus (the 100-dot square) without TA. In this set of tasks, we wanted to analyze students' strategies without potential interference of TA. Although structured number representations on the 100-dot square and abacus are addressed in second grade in Germany, we used this kind of tasks in grade 5, following Moser-Opitz et al. (2016) who point out, "some research (…) shows that low achievers in mathematics in the higher grades lack very basic competencies, such as counting in groups or understanding the base-10 system, even with small numbers" (p. 1f.). In previous studies, ET has proven to be useful to analyze students' strategies working with such structured representations. This is due to the fact that the representations comprise visually presented information and ET can help to understand how students capture such information. Lindmeier and Heinze (2016) analyzed students' strategies to determine numbers of dots/beads and found significant differences between first graders and adults, in particular different strategy use (e.g., counting, subitizing, or using structures). They conclude that their "study shows that eye-tracking data can be used to access different strategies when solving number tasks in structured representations" (p. 7). Rottmann and Schipper (2002) compared high and low achievers' use of the 100-dot square in addition and subtraction tasks using ET. Analyzing scanpaths, they found, e.g.: "Low achievers (…) use (…) [the material] in a way which turns out not to be helpful at all for them: their activities are either inappropriate or are unilaterally subordinated to the counting-all strategy" (p. 51).

Eye-Tracker. We used the ET glasses Tobii Pro Glasses 2 with a framerate of 50 Hz. They weigh 45 grams, are relatively unobtrusive, allow for reliable tracking, and are robust to students touching them. Even though tasks were presented on a computer screen, we decided not to use a remote eye-tracker to be able to capture students' eye movements when looking away from the screen, and also gestures, e.g. pointing.

Data analysis. We focus on the gaze-overlaid videos (videos taped by the ET glasses including the eye gazes as dot wandering around in the video; similar to Schindler & Lilienthal, 2017). The data analysis followed Mayring's (2014) qualitative content analysis in an inductive manner. The first three steps were (1) *description* of the student's *eye movements* in the video, (2) *paraphrasing* the content-bearing semantic elements in the description relevant for identifying student strategies *and transposing* them to a uniform stylistic level, and (3) *category development*, i.e. inductively assigning categories to the data with according descriptions/definitions. After having categorized all data, we went through the data once more in a *category revision* step; which included partially re-categorization. In a final subsumption step, we collected all instances matching every category. We also compared the informative content of ET and TA. For this purpose, we analyzed the TA for every student. We transcribed utterances and gestures (e.g., pointing) and then used the same steps as outlined above.

RESULTS

Comparing ET and TA

In the tasks to determine the number of beads (dots) on a 100-bead abacus (100-dot square), the students were asked to think aloud how they found out the number. When comparing eye movements and verbal descriptions of the students, three scenarios appeared (Tab. 1). 11 students' data (6 with MD, 5 without) were analyzed (the others were excluded, e.g., due to data loss or interference of the interviewer). Most cases matched scenario A, where ET provided more information about students' strategies.

	ET more informative	ET/TA equally informative	TA more informative
Scenario	A	B	C
Number of cases	6(5)	4(1)	1(0)

Table 1: Scenarios of informative content of ET vs. TA (numbers of students with MD in brackets)

Scenario A. Johanna, a student with MD and special needs in social and emotional development, in particular with anxieties, participated in this study (she herself was eager to take part). When determining the number of dots in the 100-dot square, her eyes glissaded over the first 5 dots in the first row and then over the second five in the first row, indicating that she grasped the number of dots in the first row. Then, she looked at the left edge of every row one after the other (beginning at the top) indicating that she was counting rows. She then said "100". When being asked "How did you do this?" by the interviewer, she answered "I don't know". When asked once more, she answered "5, 10, 15", which is hardly conclusive. In this case, ET appears to be more informative than TA for understanding the student's strategy. Johanna might have been anxious to explain her strategy (e.g., in order to avoid failure) or not used to explain her thoughts. In other cases, we experienced that the students had difficulties to express their thoughts (e.g., due to poor language skills or shyness). In yet another case, Jasmine did not mention a strategy that, as the ET revealed, she had used (counting 5s). When being asked about it, she said: "No, I first counted the 25s". Maybe she forgot about the strategy she used initially; or intentionally denied the strategy—possibly because she perceived that counting is not an expected strategy. ET as compared to TA was not only beneficial when students did not answer or denied a strategy. It furthermore often provided a greater level of detail and reflected processes that the TA only hinted at; for instance how (where and in which order) the students really counted or how they perceived ones. Besides, TA required the interviewer to ask follow-up questions which may guide students' strategies and implicitly transport certain norms with an impact on the students' behavior.

Scenario B. In four cases, ET and TA were equally informative. In Samuel's case, ET indicated that he first grasped the number of beads in the first row of the abacus (saccade over the 10 beads) and then counted the number of rows (subsequent fixations on the middles of the rows one after the other). Verbally, he described: "It is 10 beads per rod. And then I simply, like 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 (pointing to the rows one after the other in the middles)". His description contained the same information as the ET. Daniel and Elena also used pointing and speech explaining their strategies. It appears that pointing has the potential to compensate for flaws in verbalization (e.g., about the order of counted elements, the exact focus, etc.).

Scenario C. In one case, Simon's (a student without MD), eye movements were very brief (several saccades over the dot field) and thus partially ambiguous for the data analysis. In TA, he was able to express himself well and used additional pointing. In this case, TA and ET did not contradict each other, but—given the briefness of Simon's eye movements—TA provided less ambiguous information on his strategy.

Using ET videos only

Students' use of structures and strategy use. ET gave indications on whether students used, e.g., the structure of 10, 5, or 50. We found that less than half of the students in the MD group but nearly all in the control group used the 50-structure in the abacus (for quantities such as 54 or 68). Besides, in the MD group, there were three students that only used the structures of 5 (Nidal and Jasmine) or the structure of 10 (Ava) in the 100-dot square. Except for a few instances where ET data were ambiguous, we could assign strategies to eye movements. Strategies included, e.g., counting in structures (e.g. rows) and then determining the ones through, e.g., subitizing; or subtraction strategies (e.g. for 92: recognizing the 8 remaining beads and inferring 92). We found that 14 out of 20 students used (at least once) more than one strategy working on single problems, whereas the students in TA always only explained one single strategy (possibly due to certain norms in the math classroom).

Explanation of processing times. The ET data analysis gave hints why students' processing times (partially measured as reaction times) may be prolonged. Reasons for longer processing times include: use of different strategies (e.g., ensuring, or realizing that another strategy is advantageous), repeating the same strategies (e.g., re-counting, sometimes in different orders), use of time-consuming strategies (e.g., counting 5s or counting ones one by one), or slow execution of strategies (e.g., when counting). Besides, we noticed students looking around on non-meaningful entities, which may be caused, e.g., by stress (e.g., because students realize an issue).

Explanation of student mistakes. The students made several mistakes when determining the numbers of beads and dots. Analyzing ET videos helped us to understand why students made such mistakes and that the same wrong answers may have different reasons. For instance, the result "99" instead of 89 may appear because students wrongly grasp the number of rows, or because they make an error in a subtraction strategy. Even though ET cannot always entirely clarify students' inferences, ET appears to be a helpful tool to understand where mistakes originate.

DISCUSSION

The power of ET was already shown in several studies in MER. However, the potential it holds especially for students with MD was not systematically analyzed yet. For this reason, we conducted an empirical study with 20 students (10 with MD) with the aim to investigate the opportunities ET may hold especially for students with MD. We investigated the *benefit of ET as compared to TA* and found that in most cases, ET provided more detailed information, and appeared to be especially beneficial for analyzing strategies of students with MD. TA was in some cases equally informative as ET. In these cases, gestures and pointing typically helped the students to express their strategies. This hints at the significance of students' ability to express themselves for valid TA; which appears to be a generally important factor for analyzing strategies of students with, e.g., a different mother tongue and migration background. Our results suggest that students should be trained and explicitly asked to use pointing in TA to increase the informative content, when ET cannot be used. In the tasks without TA, ET revealed that many students used more than one strategy to solve the tasks, which they never reported in the TA condition. This indicates that students' verbal reports in TA may not reflect the variety of strategies students would have used without TA. We also compared *opportunities of ET for students with and without MD*. We found that for most students with MD (5 out of 6), ET provided more (detailed) information than TA, whereas this was only the case for a minority of the students without MD. Possible reasons are that students with MD used multiple and more diverse strategies more often, which increases the difficulty to verbalize. We assume that explanation difficulty but also anxieties (due to disadvantageous prior experiences) and normative aspects (e.g., hesitance to explain counting strategies because it may not be appreciated) may have an influence in this respect. However, our study is only a small scale study with only one kind of tasks. Its results cannot and should not easily be generalized. However, it hints at an important fact: That ET may be especially valuable for students with MD, where TA is particularly difficult. We believe our research to be a springboard for further research on this topic.

References

Beitlich, J. T. & Obersteiner, A. (2015): Eye-Tracking as a Method for Identifying Mathematical Strategies. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.) (2015). *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1*, 89-93. Hobart, Australia: PME.

Chen, Y.C. & Yang, FY. (2014) Probing the Relationship Between Process of Spatial Problems Solving and Science Learning: An Eye Tracking Approach. *International Journal of Science and Mathematics Education*, *12*(3), 579-603.

Ericsson, A. & Simon, H. A. (1980). Protocol Analysis: Verbal Aspects as Data. *Psychological Review*, *87*(3), 215-251.

Holmqvist, K., Nyström, M., Andersson, R., Dewhurst, R., Jarodzka, H., & Van de Weijer, J. (2011). Eye *tracking: A comprehensive guide to methods and measures*. Oxford: OUP.

Just, M.A. & Carpenter, P.A. (1976). Eye Fixations and Cognitive Processes. *Cognitive Psychology*, 8, 441-480.

Lindmeier, A., & Heinze, A. (2016). *Strategies for recognizing quantities in structured whole number representations – A comparative eye-tracking study*. Paper presented at 13th International Congress on Mathematical Education (ICME-13), 2016.

Mayring, P. (2014). Qualitative content analysis: theoretical foundation, basic procedures and software solution. Klagenfurt, Austria: Beltz.

Moser Opitz, E., Freesemann, O., Prediger, S., Grob, U., Matull, I., & Hußmann, S. (2017). Remediation for students with mathematics difficulties: An intervention study in middle schools. *Journal of learning disabilities*, *50*(6), 724-736.

Obersteiner, A. & Tumpek, C. (2015). Measuring fraction comparison strategies with eye-tracking. *ZDM—Mathematics Education*, 48(3), 255-266.

Rottmann, T., & Schipper, W. (2002). Das Hunderter-Feld—Hilfe Oder Hindernis beim Rechnen im Zahlenraum bis 100?. *Journal für Mathematik-Didaktik*, *23*(1), 51-74.

Scherer, P., Beswick, K., DeBlois, L., Healy, L., & Opitz, E. M. (2016). Assistance of students with mathematical learning difficulties: how can research support practice?. *ZDM*, *48*(5), 633-649.

Schindler, M. & Lilienthal, A. (2017). Eye-Tracking and its Domain-Specific Interpretation. A Stimulated Recall Study On Eye Movements In Geometrical Tasks. In Proceedings of PME 41, Vol. 4, 153-160.

Wartha, S., & Schulz, A. (2012). Rechenproblemen vorbeugen. Grundvorstellungen aufbauen. Zahlen und Rechnen bis 100. Berlin: Cornelsen.

A TEACHER'S REFLECTIVE PROCESS IN A VIDEO-BASED PROFESSIONAL DEVELOPMENT PROGRAM

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The purpose of the study reported herein was to closely observe the reflective process of one teacher who participated in a video-based professional development course for secondary school mathematics teachers. This process was initiated in a session where a group of teachers watched and analysed a Japanese TIMSS lesson, and continued with the teacher's decision to teach this lesson in her class. We analyse the teacher's reflective process in terms of Ricks' (2011) "process reflection" framework, and explore the factors that affected it. The results demonstrate the impact that facilitated peer discussion around a videotaped foreign lesson may have on teachers' reflection.

BACKGROUND AND RATIONALE OF THE STUDY

VIDEO-LM (Viewing, Investigating and Discussing Environments of Learning Mathematics) is a project that offers video-based professional development (PD) courses for secondary mathematics teachers in Israel since 2012. In this section, we briefly review the merits of using video for teachers' professional growth, and describe how the project utilizes this resource to achieve its goals. Then we focus on a specific videotaped lesson that served as a catalyst for the reflective case described further on.

Video-based professional development for mathematics teachers

The use of video for professional development of mathematics teachers has become prevalent during this era's technological advances. Sherin (2004) reviewed the benefits of using video: it provides detailed documentation of what happens in class, without the need to rely on memory; when watching a recorded lesson, one can move back and forth in time, freeze a picture and focus on it, and see certain sections again - perhaps from different points of reference. Consequently, video serves as a tool for increasing teachers' awareness of their teaching practices (Santagata, Gallimore & Stigler, 2005) and for developing their mathematical knowledge for teaching (MKT; Ball, Thames & Phelps, 2008). The main objectives for using video in professional development (PD) courses nowadays are: (1) dissemination of new curricula; (2) assessment and improvement of teaching skills; (3) advancement of teachers' proficiency to notice students' mathematical thinking; and (4) providing teachers with tools and language for reflection on their practice (Karsenty & Arcavi, 2017). The study reported herein is part of the VIDEO-LM project, which aims at the fourth objective.

The VIDEO-LM project

VIDEO-LM aims to develop reflective skills among mathematics teachers through

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deep, facilitated discussions about authentic videotaped lessons. During a PD session, teachers watch videos of lessons and discuss them using a special framework developed for this purpose, originating from Schoenfeld's "Teaching in context" theory (2010). Schoenfeld proclaimed that teachers' resources, orientations and goals can explain their real-time decision-making. Arcavi & Schoenfeld (2008) derived from these constructs some analytical tools, that serve to enhance mathematics teachers' reflection when observing videotaped lessons. These tools were modified by the VIDEO-LM project team into six components which comprise the "six-lens framework" (henceforth SLF), allowing the PD participants to share the same language when analysing lessons. The lenses are: mathematical and meta-mathematical ideas; goals; tasks and their enactment; classroom interactions; dilemmas and decision-making processes; beliefs about mathematics, its learning and its teaching (for a more detailed account of these lenses, see Karsenty & Arcavi, 2017). SLF differs in several significant aspects from other frameworks used in video-based programs: it focuses mainly on teachers' actions; the lessons presented in the courses are not necessarily considered as reflecting "best-practice", yet they must be rich enough to stimulate deep conversations; and the mathematical content of the lesson, rather than generic features of teaching, is at the centre of attention. VIDEO-LM discussions also endorse norms that decentre judgment and encourage reasoning about the filmed teacher's decisions.

The Japanese lesson

In VIDEO-LM sessions, PD participants usually watch teachers who are unfamiliar to them, sometimes even teachers from other countries. The lesson that instigated the reflective process to be discussed herein was a Japanese TIMSS lesson known as "Changing Shape without Changing Area" (http://www.timssvideo.com/67).

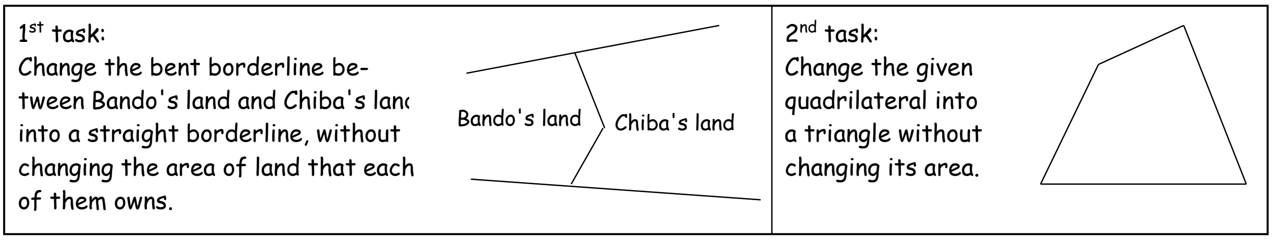

Figure 1: The problems introduced in the Japanese lesson

This lesson, filmed in 1995 as part of the TIMSS video project, is a geometry lesson for 8th grade. The lesson began with a brief reminder of the theorem that was learnt in the previous lesson: "*triangles with equal bases and between two parallel lines have equal areas*". Then, the teacher introduced task #1 (see Figure 1) on the board and gave the students 3 minutes to work on it on their own. When 3 minutes had passed, the teacher instructed the students to walk around the class, discuss the task with their peers, use hint cards if needed and consult with him and with an assistant teacher. This activity lasted for 15 minutes, and then some students presented their work on the board. Similar instructions were given for task #2. This lesson was chosen for VIDEO-LM courses because it contains many rich topics for discussions, such as the teacher's pedagogy, the intriguing tasks and the way the teacher facilitates them. In addition, task #1 is situated in everyday life, and both tasks are easy to understand but not easy to solve, requiring a non-trivial application of a geometric feature.

Rationale and research questions

The case study reported here was part of a larger study in which the discussions around the Japanese lesson were meticulously characterized (Karsenty & Schwarts, 2016). Here, we report in detail on a profound reflective process experienced by one teacher, Joelle, who participated in the project's pilot group. Joelle's story, which will unfold below, originally led to the hypothesis that viewing, analysing, and discussing such a lesson may affect teachers' reflectivity and practice. The purpose of this case study was therefore to follow more closely the emergence of a reflective process and its stages. As a result, the research question we relate to in this report is:

What characterizes a teacher’s reflective process following a PD session around a Japanese lesson? What are the factors that influence this process?

METHOD

Data collection and analysis

The reflective process was investigated using a qualitative method. Joelle is a secondary mathematics teacher in a public high school in Israel. When the study began, she had been teaching for 11 years. She participated in a VIDEO-LM PD course in 2012-13, and facilitated a VIDEO-LM course in 2013-14. Data were collected from several stations across Joelle's professional trajectory, and consist of: video recordings of all PD sessions in which she participated in 2012-13; audio recordings of all PD sessions that she facilitated in 2013–14; two semi-structured interviews with her, the first conducted at the end of the 2012-13 PD and the second several months later; and reflective essays submitted by participants of the 2013-14 PD course facilitated by her. For the purpose of this study, episodes were selected from the data according to Joelle’s references to the Japanese lesson. This material was transcribed. Data analysis included identifying in this body of data common themes that correlate with the four stages of the *process reflection framework* (see below), a framework that "integrates the myriad uses of reflection into a more applicable construct" (Ricks, 2011, p. 252).

Framework for data analysis: The process reflection framework

Dewey (1910) suggested that reflective thought is a process that begins with a surprising or problematic event. Schön (1983) combined Dewey's concepts with the idea of knowing-in-action, claiming that practitioners' knowledge of their practice is usually hidden within patterns of action. Ricks (2011) referred to their ideas in the context of mathematics education, arguing that reflection is essential to teachers' professional growth because it produces knowledge anchored in practice. Based on the ideas of Dewey and Schön, Ricks introduced a conceptual framework that classifies teacher reflections into two categories: (a) *incident reflection*, which includes specific events that are not related to future activity; and (b) *process reflection*, in which separate reflective events are combined into a coherent continuum that has purpose, direction and content. The process reflection is modelled in four stages, which constitute the *process reflection framework*: (1) experiential event – an event that leaves the practitioner surprised, confused or curious, and might therefore initiate a reflective cycle, since the beliefs or premises held were not sufficient to forecast the bewildering occurrence; (2) idea suspension and problem creation – this stage can develop only if the practitioner rejects an instant solution, and instead takes the time to recognize problematic characterizations of the event; (3) idea formation: ramify and refine – once the challenges have been defined, possible solutions can arise, and one of them is chosen in a justified manner; and (4) testing action with observation – the meaningful action that was chosen is executed and observed by the teacher, sometimes opening another reflective cycle.

FINDINGS: JOELLE'S IDENTIFIED STAGES OF PROCESS REFLECTION

In this section, we describe episodes from Joelle's practice using the reflective process framework, and explain why her utterances and actions constitute a particular reflective stage.

Stage 1: Experimental event

The story begins at the first PD session that Joelle attended. This session was part of a pilot course for 10 experienced teachers whose role was, besides regular participation, to act as an advisory forum and comment on the chosen videos and on SLF as a tool for video observation and analysis. The session was co-facilitated by the two researchers who designed the project. They screened a few minutes of the Japanese lesson, and asked the teachers to solve task #1 (Figure 1). Then, they focused the discussion around the lenses of **tasks** and **goals**. The participants were surprised by the task and the Japanese teacher’s pedagogy, and remarked that the lesson is very different from their common practice. Moreover, a few teachers failed to solve the task. These occurrences, which we identify as the experimental event, led to strong reactions on the part of teachers, either enthusiastic or reluctant ones. Joelle was a prominent opponent to using the lesson in future PD sessions, as expressed in her following words:

> First, well, Japan, that's ok, Japan... when I imagined, how in my class, and I have an excellent class, I would present such a task, there would be much more noise... much more of "what do you want, and what…" even in the beginning, when I am only presenting the task. "Teacher, what do you want and where did it come from?"… and afterwards, I don’t know how it will be here [in the Japanese lesson to be further screened], but in my class 3 or 4 students would try it, and the others, even good and strong students, would do nothing, because they are not used to such things at all...

Joelle further explained that in her opinion a lesson from another culture might alienate teachers, perhaps due to the gaps between Israeli students’ learning habits and those of Japanese students. One of the VIDEO-LM team members, Ana, who was present at the session, tried to convince Joelle to reconsider her opinion:

> Ana: This is a very good comment, but the discussion we want to raise is about what beliefs we bring and can convey to our students…

Joelle: I think, that you want to change the world too much. And I think that our world is not that bad, there is no need for revolutions here.

Joelle did not accept Ana's perspective, and highlighted her stance that the project team members “want to change the world too much”. She also claimed that there is no need to watch lessons from other countries, implying that ordinary Israeli lessons can provide good enough artefacts that more teachers would find relevant. This time, another teacher responded to Joelle's arguments:

Beth: On the one hand, I agree with you, but on the other hand I think that sometimes it's easier for us to look at something else entirely, because if I look at a teacher from here... I'll say, "Wow, I'll never be able to do that". But if I look at another culture, it's less threatening... it's less embarrassing even. It's also very thought-provoking, because it's really an interesting culture. And then the art of the discussion is to talk about what I can still take from this, in our culture, I do not have to take it as is.

Joelle: That's what was interesting for me, right.

Stage 2: Idea suspension and problem creation

As shown above, Joelle's reluctant response to the team member's comment has shifted towards a more affirmative reaction when responding to Beth’s comment. Joelle explained this shift in her view, in retrospective, during the first interview:

"What helped me is teachers who teach, in the field, not people from academia… It certainly influenced me, that people who teach at schools and deal with teaching and teachers said that this lesson is good."

At this stage, the second phase of the reflection process was initiated: it seems that Joelle suspended her strong beliefs against the lesson and started thinking about its possible affordances.

Stage 3: Idea formation: ramify and refine

A few weeks after the PD session described above, the project team asked Joelle to videotape a few lessons in her classroom, to enlarge the project's repertoire of lessons. Joelle agreed, offering to film lessons she planned to teach on the topics of probability and calculus, but most surprisingly, she also suggested to reconstruct the Japanese lesson in her class and film it. A possible interpretation of this suggestion is that Joelle had decided on an action that would help her clarify her beliefs. She referred to this decision in the first interview:

Joelle: At first, the Japanese class seemed mysterious to me because it is not like what we are doing, and then I said in the session, it is not applicable and it's not suitable for our curriculum, that was the problem... so precisely because it was so strange to me, I thought, "Who says you are right? Go and check it out, it's something worth checking".

Interviewer: But was there something... that influenced you?

Joelle: Yes, definitely. The fact that people were in favour of the lesson... and said, "It can be a good thing, it can be useful, and it is not so bad that there is a gap between our

> students' knowledge and what we expect from them". It certainly strengthened it... and once people justified their opinion then I was even more interested in checking it. It is not only the Japanese teacher who thinks this kind of lesson is profitable, teachers here believe it as well.

It seems that Joelle's motivation to teach the Japanese lesson in her class developed as a result of her peers' enthusiasm. This decision can be identified as the stage at which Joelle's incident reflections were collected into a cohesive process; she decided on an action that would assist her in examining her orientations. She used her experience, insights from peers and her agency in the classroom to revise beliefs through practice.

Stage 4: Testing action with observation

After conspicuously opposing the Japanese lesson, stating that "it does not suit Israeli teachers and students", and then rethinking its potential, Joelle finally taught it in a 7th grade advanced track classroom. In the next VIDEO-LM PD session, she recounted this experience:

> I tried to teach the Japanese lesson... I had many concerns, I was sure they [students] would not understand, nor would they connect, and I was very surprised... the kids were all very interested, I think they were more active than the Japanese [students], they had such a passion to solve it... some students came up with the solution pretty quickly on their own, but they did not stop, they said "So what? We want to find another solution!" Some parents later told me that their children came home and showed them... This was a very big surprise [that resulted] from this lesson.

Joelle's words indicate that her beliefs about the lesson were changed dramatically following her reconsideration of the lesson and the implementation in her classroom.

Another cycle?

In the second interview conducted with Joelle, several months after the first interview, she elaborated her reflection on the lesson that she had taught:

> The experience of the lesson was very positive... I thought they would tell me "What do you want from us? What is this thing?" but it was not like that at all. I would like to have a full program like this, with deep questions that will slowly bring us to something, but because I had only two interesting questions, I don't believe that there was much left. Yet, I don't think that it was pointless...

This excerpt reflects a complex position. It seems that the teacher is pleased with her own mental flexibility and daring, and thinks that the lesson was a positive experience that motivated students to participate in class. On the other hand, with the perspective of several months, it troubles her that such a lesson is not part of a teaching continuum rich in inquiry tasks related to the material being studied. However, the reflective cycle did not stop there. Joelle became a keen advocate of VIDEO-LM and of the Japanese lesson, and in the following year she facilitated a PD course in her hometown. Not only did Joelle present the Japanese lesson to the PD participants, but she also showed the reconstructed lesson that she had taught in her class. At the next meeting, more than half of the participants reported that they had taught it in their own classes. One of those teachers wrote about this experience in the course's final assignment, referring to Joelle's lesson:

> It was surprising to discover that when this lesson was given in an Israeli classroom, the students were as interested as the Japanese students, and that they were also eager to delve into the solution with great curiosity. Watching this lesson when it was "transferred" to Israel gave me confidence as a teacher to incorporate this type of teaching [in my class].

Joelle's concern that teachers would feel antagonism towards the Japanese lesson was refuted. Moreover, the "local" implementation of the lesson was compelling for many teachers, as evident from the quote above.

SUMMARY

The aim of this case study was to characterize one teacher's reflective process and to point to possible factors that might have influenced it. We argue that Joelle's reflective process was ignited by the disagreement among peers during the discussion in the PD session devoted to the Japanese lesson. This disagreement led to a productive peer discussion, in which different approaches and alternative beliefs were legitimized. According to Joelle's self-evidence, the various opinions triggered her to reconsider her strong orientation. This is a salient finding: the reflective process was initiated by other teachers' enthusiasm regarding the video, not by the agenda of the PD facilitators. Joelle decided on an action aimed at revisiting her beliefs, and taught the lesson in her class. After the lesson, which she considered to be successful, she still had some reservations, but these were based on a new experience and on a deeper examination. The reflective process Joelle went through led to a "chain reaction": not only that she presumably motivated reflective processes among the teachers in the PD course that she facilitated, but also these reflections were potentially reinforced by observing both the Japanese lesson and Joelle's local experience in her class.

Another factor that apparently influenced Joelle's reflective process was the substantial difference between the Japanese and the Israeli teaching cultures. This finding aligns with existing literature on cultural differences as a possible catalyst for teachers' learning. For example, Stigler and Hiebert (1999) state in their book *The Teaching Gap* (which also refers to the TIMSS Japanese lesson discussed here) that "teaching is a cultural activity... we are largely unaware of some of the most widespread attributes of teaching in our own culture" (p. 11). Regarding the experience of watching lessons from different countries, Stigler and Hiebert note that "looking across cultures is one of the best ways to see beyond the blinders and sharpen our view of ourselves" (p.13). Clarke and Hollingsworth (2000) claim that when teachers watch a lesson from a culture very different than their own, their assumptions about what is expected and accepted are no longer relevant, and they are more likely to contemplate their teaching. The case study reported in this paper illustrates how such contemplation takes place and how it can enhance reflective processes. In Joelle's case, the factors identified as triggers for a meaningful reflection were the act of watching a lesson from a different culture; the focus on the lenses of goals and tasks to analyse the filmed teacher's actions; and the power of a peer discussion that brings up different approaches. The combination of these elements led Joelle to a profound reflection process that affected not only her own practice, but also practices of other teachers, in a manner that reminds the effect of rings formation by a stone thrown in water.

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References

Arcavi, A., & Schoenfeld A. H. (2008). Using the unfamiliar to problematize the familiar: The case of mathematics teacher in-service education. *Canadian Journal of Science, Mathematics, and Technology Education*, *8*(3), 280–295.

Ball, D. L., Thames, M. H. & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Clarke, D. & Hollingsworth, H. (2000). Seeing is understanding: Examining the merits of video and narrative cases. *Journal of Staff Development*, *21*(4), 40-43.

Dewey, J. (1910). *How We Think*. Lexington: D.C. Health & Company.

Karsenty, R., & Arcavi, A. (2017). Mathematics, lenses and videotapes: A framework and a language for developing reflective practices of teaching. *Journal of Mathematics Teacher Education*, *20*, 433-455.

Karsenty, R. & Schwarts, G. (2016). Enhancing reflective skills of secondary mathematics teachers through video-based peer discussions: A cross-cultural story. Paper presented in the 13th International Congress on Mathematical Education (ICME-13), Hamburg, Germany, July 2016.

Ricks, T. E. (2011). Process reflection during Japanese lesson study experiences by prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education*, *14*(4), 251-267.

Santagata, R., Gallimore, R., & Stigler, J. W. (2005). The use of video for teacher education and professional development: Past experiences and future directions. In C. Vrasidas & G. V. Glass (Eds.), *Current Perspectives on Applied Information Technologies: Preparing Teachers to Teach with Technology* (Vol. 2, pp. 151-167). Greenwich, CT: Information Age Publishing.

Schoenfeld, A. H. (2010). How We Think: A Theory of Goal-Oriented Decision Making and its Educational Applications. New York: Routledge.

Schön, D. A. (1983). The Reflective Practitioner: How Professionals Think in Action. New York: Basic Books.

Sherin, M. G. (2004). New perspectives on the role of video in teacher education. *Advances in Research on Teaching*, *10*, 1-28.

Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap*. New York: Free Press.

MIDDLE SCHOOL STUDENTS' REASONING ABOUT VOLUME AND SURFACE AREA

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This study investigates students' reasoning skills in practical application of mathematics. Students were asked to explain the volume and surface area of a shoe box after enlargement. The data from two groups of students were collected and analysed. The first administration of the tasks was given to 253 Year 4-10 students to validate the usability of the items and marking rubric. The second administration involved 273 Year 8-10 students of a different cohort. The results show that students use a combination of linguistic, symbolic, and diagrammatic tools demonstrating different level of reasoning.

INTRODUCTION

Area and volume measurement provide a rich context for real-world applications of number processes and the development of spatial skills. In particular, spatial reasoning is seen as a crucial skill for functioning successfully in the built environment we live in and in ensuring future success. Despite being taught in primary and secondary levels, evidence shows that many students and prospective teachers alike have superficial understanding of both concepts such as: not understanding partitioning in terms of array and grid structure, conservation of area and volume, and confusing area with perimeter, volume with surface area, and volume with capacity (Owen & Outhred, 2006; Tan Sisman & Aksu, 2016). A lack of effective teaching instructions, an emphasis on formula memorisation and routine application of rules, combined with a lack of spatial visualisation and an inadequate integration of geometric knowledge contributed to students' inability to engage in mathematical reasoning in measurement situations. Research in the area of learning progressions has demonstrated that a framework describing student development of mathematical reasoning can lead to improvements in teaching and learning (Siemon et al., 2017).

THEORETICAL FRAMEWORK

Just as analysing students' errors and misconceptions allows one to comprehend how students internalise concepts and skills (Tan Sisman & Aksu, 2016), analysing how students explain the mathematics in practical situations allows one to determine factors that influence their thinking and reasoning abilities. When engaging in a task, five components are at work in an individual's working memory: *networks of propositions* (conceptual knowledge of mathematical concepts), *skills* (procedural knowledge needed to carry out the task), *imagery* (concrete objects, diagram or images associated with the event), *episodes* (events associated with the situation) and *attitudes*

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 131-138). Umeå, Sweden: PME.

(Clements, 2014; Gagné & White, 1978). How well a student reasons is influenced by the relationships between and among these components, as well as individuals' beliefs about their own mathematical ability to complete the task.

Mathematics can be seen as a deductive theory grounded in notions and axioms. Every concept can be defined minimally and sufficiently to specify that concept. Every concept mentioned triggers our memory to produce an 'image' - the collective mental pictures, actions, the corresponding properties and processes that are associated with the concept, together form a concept image (Vinner, 1991). Students' use of diagram allows ones to see their thought processes. Diagrammatic thinking then, relates to the actions of objectification, with diagrams as semiotic means of objectifying an abstract idea (Radford, 2008). Misconceptions occur when there is a disjuncture between personal concept image and definitions derived from experience and formal geometric knowledge deriving from axioms, definitions, theorems and proofs. For example, a child's first encounter of the word 'volume' may relate to the degree of loudness but conceptually in geometry, volume may mean: (1) the amount of matter that is contained within the boundary surfaces, (2) the amount of space occupied by an object in relation to other objects, and (3) the space caused by water displacement (Sáiz & Figueras, 2009). If rules are taught without understanding dimensionality, students are likely to assume that volume means 'length by width by height' of an object and that area means 'length by width' (which is procedural rather than conceptual thinking) and apply them indiscriminately (Owen & Outhred, 2006).

Teaching formula without understanding dimensionality is meaningless and can lead to computational errors caused by confusing the difference between 'directionality' and 'dimensionality' (Fernández & De Bock, 2013). The former refers to the different directions (orientation) a geometrical figure has whereas dimension refers to the magnitude of attributes or sometimes just the number of attributes. When asked to calculate the volume of a rectangular prism, Tan Sisman and Aksu (2016) found that only 29% of their 6th grade Turkish students gave the correct answer. Many failed to understand dimensionality and computed as 'length + width + height' (directionality), '3 x length + width + height' or multiply the dimensions of two measurement only. Similarly, when asked to calculate the surface area of a rectangular prism, only 6.3% could produce a correct response, the remaining cohort either added the length, width and height together or added all the dimensions and multiplied the product by six.

Inability to visualise three-dimensional (3D) objects can affect students' ability to reason. In a lesson study, teachers realised that teaching the concept of surface area is not about developing a formula (Lieberman, 2009). Rather, it is about getting their students to learn to visualise the 3D lateral surface area in two dimensions (2D) and make connections between height in 3D and width in 2D. Visualisation is a cognitive process in which objects are interpreted within the person's existing network of beliefs, experiences, and understanding (Phillips, Norris, & Macnab, 2010). In mathematics, diagrams are used schematically (*ibid*). Analysing how students reason about mathematical situations, their use of diagrams, language and symbols, can assist in determining individuals' networks of propositions, and attitudes towards mathematics. This will significantly contribute to our effort in determining a learning progression that promotes reasoning.

Research has shown that few students could engage in relational-inferential property-based reasoning, for example, students assumed that two bodies with equal surface area have the same volume, or that enlarging *n* times the linear magnitudes of a body enlarges *n* times its volume (Sáiz & Figueras, 2009). In this study, we attempt to unravel the range of students' reasoning abilities when considering changes in volume and surface area caused by enlargement. We adapted Wattanawaha's DIPT classification framework (Clements, 2014) by analysing students' reasoning based on the **D**imensions of thinking, whether students use 1D, 2D or 3D thought in their response, the degree of **I**nternalisation of visual images, the manner in which the student **P**resents the argument (the use of examples, linguistic, symbolic/algebraic, diagrammatic tools), and the **T**hought processes required to justifying their reasoning. Our aim is to categories the nature of students' geometric measurement reasoning to show the growth in networks of propositions, with specific focus on how a change in lengths affects volume and surface area.

METHOD

This study is part of a larger study, Reframing Mathematical Futures II (FMFII), where we have been developing a learning assessment framework to assist teachers to teach reasoning in geometric measurement. It is based on the premise that an evidence-based validated set of an assessment tools and learning tasks can be sued to nurture students mathematical reasoning ability (Siemon et al., 2017). The two items (Figure 1) were designed to assess students' ability to reason about the volume (GSZLV) and surface area (GSZLSA) of a geometric figure after enlargement Rubrics for scoring items were validated through both expert review and item analysis of student data. Figure 2 show how the DIPT was adapted in the analysis.

[GSZLV] Matt said that if you double the length of the edges of a shoe box, it will double in volume. Do you agree? Explain your answer (You may use diagrams if you wish).

Score	Description
0	No response or irrelevant response
1	Agrees with Matt with little/no explanation
2	Agrees with Matt giving reasons that only enlarge one or two dimensions
3	Disagrees with Matt with little/no reasoning, may say that it is quadrupled
4	Disagrees with Matt reasoning based on doubling all side lengths

[GSZLSA] Matt then says that if you double the lengths of the edges of a shoe box, its surface area will double. Do you agree? Explain your answer (You may use diagrams if you wish).

Score	Description
0	No response or irrelevant response
1	Agrees with Matt with little/no explanation
2	Agrees with Matt giving reasons that only enlarge one or two dimensions
3	Disagrees with Matt with little/no reasoning, may think increase is larger because of the 6 faces
4	Disagrees with Matt reasoning recognising that each face will increase area by 4 times so overall increase is quadrupled

Figure 1: Reasoning items on the volume and surface area of geometric figures after enlargement.

D – The scores 1, 2, 3 represent the evidence of dimensionality used in argument. For surface area, a 3 is given if students include all 6 faces in their argument

I – Static diagram use (additive thinking) scored 1, dynamic use to show change scored 2

P – Argument based mainly on words scored 1, symbolic scored 2 and diagrammatically 3

T – Argument with specific examples 1, general argument without examples 2.

Figure 2: Adapted DIPT framework for analysing students' reasoning.

The participants were middle-years students from across Australian States and Territories. Two groups of cohorts were involved. The first set of data – the trial data, was taken from 253 Year 4-10 students from two primary and four high schools across social strata and States to allow for a wider spread of data being collected. The teachers were asked to administer the assessment tasks and return the student work. The trial results were marked by two markers and validated by a team of researchers to ascertain the usefulness of the scoring rubric and the accuracy of the data entry. The second set of data – the project data, was taken from 273 Year 8-10 students from six high schools situated in lower socioeconomic regions with diverse populations. The project school teachers were asked to mark and return the raw scores instead of individual forms to the researchers. The project school teachers received two 3 days face-to-face professional learning sessions on spatial and geometric reasoning prior to the implementation of the assessment tasks. They also had access to a bank of teaching resources and four on-site visits to support their teaching effort.

FINDINGS

Tables 1 and 2 show the percentage breakdown of student responses for GSZLV and GSZLSA respectively in each year level. It is clear that more than 50% of the students are unable to coordinate more than two dimensions in the tasks. A large number of students in the trial schools either did not attempt the task or produced irrelevant responses. While this figure is much lower for the project schools, this could easily be explained by the professional learning the teachers in those schools have undertaken.

Overall in Years 8-10, 6% and 14.7% of trial and project school students respectively were able to reason fully about the volume task. For the surface area task, the figures were 4.2% and 8.4% respectively.

	Trial Data						Project Data			
Score	Yr 4	Yr 5	Yr 8	Yr 9	Yr 10	Overall	Yr 8	Yr 9	Yr 10	Overall
0	54.8	38.9	43	42.1	50	45.1	19.5	12.9	20.4	17.6
1	35.5	29.6	33.3	10.5	10	22.1	27.6	25.8	24.7	26
2	9.7	20.4	11.8	35	21.7	21	20.7	29	19.4	23.1
3	0	11	9.8	5.3	10	7.9	21.8	23.7	10.8	18.7
4	0	0	2	7	8.3	4	10.3	8.6	24.7	14.7

Table 1: Results for GSZLV

	Trial Data						Project Data			
Score	Yr 4	Yr 5	Yr 8	Yr 9	Yr 10	Overall	Yr 8	Yr 9	Yr 10	Overall
0	71	55.6	56.9	47.4	56.7	55.7	23	19.4	28	23.4
1	22.6	18.5	27.5	14	15	19.3	32.2	31.2	22.6	28.6
2	6.5	20.4	11.8	17.5	13.3	14.6	24.1	31.2	19.4	24.9
3	0	5.6	2	17.5	8.3	7.5	17.2	12.9	14	14.7
4	0	0	2	3.5	6.7	2.8	3.5	5.4	16.1	8.4

Table 2: Results for GSZLSA

The main interest here though, is not so much the number of students who were able to reason fully, but rather the nature of their reasoning. Using the adapted DIPT framework to analyse the trial school data (Wattanawaha reported in Clements, 2014), Table 3 and 4 shows a section of the DIPT analysis against the rubric scores for the assessment items where *n* is the number of students who achieved that rubric score.

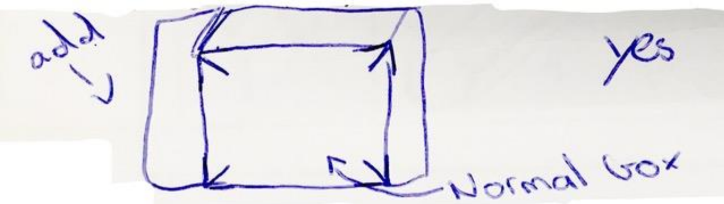
Score	*n*	DIPT Analysis
4	10	**D -** All students argued with 3D; **I -** 9 used diagram dynamically; **P** – All used symbolic argument; **T -** 9 used specific examples; 1 used general case 2L x 2W x 2H = 8LWH
3	20	**D** – 8 students argued with 3D, 5 with 2D, 5 with 1D; **I -** 9 used static diagram, 4 used a dynamic diagram; **P -** 8 argued linguistically, 9 attempted to argue symbolically, 3 diagrammatically; **T** – 9 used an example
2	53	**D** – 14 argued with 3D, 9 with 2D, 27 with 1D; **I** – 30 used a static diagram, no one used dynamic diagram; **P** – 28 argued linguistically, 19 symbolically, 6 diagrammatically; **T** – 22 attempted to use an example
1	57	**D** – 3 with 3D, 3 with 2D, 43 with 1D; **I** – 13 used diagram statically, 1 used diagram dynamically; **P** – 41 argued linguistically, 4 symbolically, 3 diagrammatically; **T** – 3 used an example

Table 3: DIPT Analysis for the volume task GSZLV.

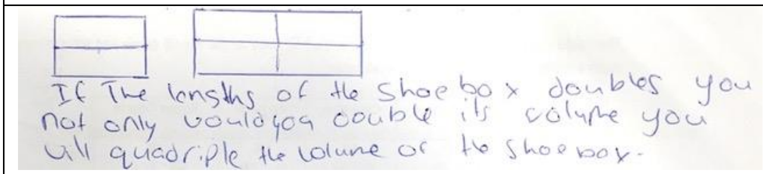
Score	*n*	DIPT Analysis
4	7	**D** - 4 students argued with 3D; 2 with 2D; **I** - 6 students used diagram dynamically; **P** – 3 students argued linguistically, 4 used symbolic argument; **T** - 6 students used specific examples; 1 student used general argument.
3	27	**D** – 5 students argued with 3D, 6 with 2D, 8 with 1D; **I** - 10 used static diagram, 4 used a dynamic diagram; **P** - 11 argued linguistically, 8 attempted to argue symbolically, 1 diagrammatically; **T** – 10 used an example
2	53	**D** – 2 argued with 3D, 12 with 2D, 37 with 1D; **I** – 18 used a static diagram, 3 used dynamic diagram; **P** – 43 argued linguistically, 5 symbolically, 4 diagrammatically; **T** – 22 attempted to use an example
1	44	**D** – no one used 3D, 2 argued with 2D, 26 with 1D; **I** – 10 used diagram statically, 1 used diagram dynamically; **P** – 22 argued linguistically, no one symbolically, 6 diagrammatically; **T** – 1 used an example

Table 4: DIPT Analysis for the volume task GSZLSA.

Students who scored more highly on the rubrics tended to be those who engaged in symbolic argument, used diagrams dynamically and provided examples to clearly show 3D thinking. There is a tendency for those in the upper year levels to score more highly although some Year 5 students did demonstrate 3 dimensional thinking. Note that when a student showed 2D thinking, it does not necessarily mean they did not have 3D thinking. Rather, they did not give evidence of it.

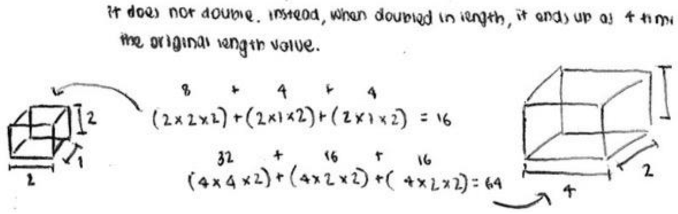


GSZLV rubric score = 1 (Yr 5)
D = 1 (1D thinking)
I = 1 (additive thinking)
P = 3 (diagrammatic argument)
T = 2 (general argument)



(The diagram here show adding of additional boxes)

GSZLV rubric score = 3 (Yr 9)
D = 2 (2D thinking)
I = 1 (additive thinking)
P = 1 (linguistic argument)
T = 2 (general argument)



GSZLSA rubric score = 4 (Yr 10)
D = 3 (recognition of 6 faces)
I = 2 (showed enlargement)
P = 2 (symbolic argument)
T = 1 (specific example)

Figure 3: Examples of DIPT analysis.

The language students used demonstrated across year levels a lack of understanding of the concept of volume:

> Yes because the shoe box weighs the same amount (Yr 10)
>
> If you double the edge of a shoe box there is no differences unless you double the perimeter (Yr 8)
>
> Yes only if there is nothing inside the box or the same amount of something (Yr 4)

and surface area:

> No because Matt is only extending the length, he is not adding any new edges (Yr 9)
>
> No you would have to double the length and width to make the surface area double (Yr 9)
>
> If the formula doesn't work for area or volume then it wouldn't work on a 2D or 3D shape (Yr 8)

Diagrams also indicated different levels of understanding as did the used of examples and related calculation. This suggest that more emphasis on visualising and working with 3D objects in measurement is needed to develop conceptual knowledge.

DISCUSSION

The DIPT analysis enabled students' geometric measurement reasoning to be analysed and categorized. The results show progression in thinking is not based solely on year level. Dimensional thinking is analysed based on the type of diagram, language or examples students used. Good students are skilled in providing an example with measurements, using diagrams to show perspective effectively and explaining their reasoning symbolically and linguistically. Poor students tended not to use these tools skilfully. They used diagrams statically, such as 2D shapes, nets, arrays, and divided rectangles, reflecting a lack of knowledge of the concepts. Note that the use of diagrams was not prevalent in younger grades, suggesting that this may be a skill that is learned at school. The number of students who did not attempt the tasks or made very cursory attempts is a reflection of their attitudes towards mathematical reasoning. When engaging in a problem solving task students need to retrieve appropriate information from their network of knowledge, have the skills to organise that knowledge and to present it in a coherent argument. This does have ramifications for classroom practice.

Success in mathematical reasoning relies on how individuals connect their web of knowledge and its relationships and use taken-as-shared mathematical language and representations to communicate and justify their thinking. This analysis has enable us to see students' current levels of mathematical reasoning about a measurement situation and the use of tools such as language, diagrams and example in argument. However, it has not explained how to move student from emerging use of these tools to become fluent in using them to engage in inferential property based analytical reasoning. Further research is needed in this area.

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References

Clements, M. (2014). Fifty years of thinking about visualisation and visualising in mathematics education: A historical overview. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics and mathematics education: Searching for common ground. Advances in mathematics education* (pp. 177-192): Springer, Dordrecht.

Fernández, C., & De Bock, D. (2013). Does the confusion between dimensionality and "directionality" affect students' tendency towards improper linear reasoning? In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Kiel, Germany: PME.

Gagné, R. M., & White, R. T. (1978). Memory Structures and Learning Outcomes. *Review of Educational Research, 48*(2), 187-222. doi:10.3102/00346543048002187

Lieberman, J. (2009). Using lesson study to develop an appreciation of and competence in task design. In B. Clarke, B. Grevholm, & R. Millman (Eds.), *Tasks in primary mathematics teacher education: Purpose, use and exemplars. Mathematics teacher education 4*. New York: Springer Science+Business Media, LLC.

Owen, K., & Outhred, L. (2006). The complexity of learning geometry and measurement. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education. PME (1976-2006). Past, Present and future* (pp. 83-115). Rotterdam: Sense Publishers.

Phillips, L. M., Norris, S. P., & Macnab, J. S. (2010). *Visualization in matheamtics, reading and science education*. Dordrecht: Springer.

Radford, L. (2008). Diagrammatic Thinking: Notes on Peirce's semiotics and epistemology. *PNA, 3*(1), 1-18.

Sáiz, M., & Figueras, O. (2009). A research-based workshop design for volume tasks. In B. Clarke, B. Grevholm, & R. Millman (Eds.), *Tasks in primary mathematics teacher education: Purpose, use and exemplars* (pp. 147-160). New York: Springer Science+Business Media, LLC.

Siemon, D., Horne, M., Clements, D. H., Confrey, J., Maloney, A., Sarama, J., . . . Watson, A. (2017). Researching and using learning progressions (trajectories) in mathematics education. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 109-136). Singapore: PME.

Tan Sisman, G., & Aksu, M. (2016). A study on sixth grade students' misconceptions and errors in spatial measurement: Length, area, and volume. *International Journal of Science and Mathematics Education, 14*(7), 1293-1319. doi:10.1007/s10763-015-9642-5

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht: Kluwer Academic Publishers.

USING SELF-VIDEO-BASED CONVERSATIONS IN TRAINING MATHEMATICS TEACHER INSTRUCTORS

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Videotaping is widely accepted as a useful tool for the professional development (PD) of teachers because it enhances their ability to perform an in-depth reflection on classroom events. Thus, a video-based discourse has the potential to increase teachers' awareness of their own teaching processes. The present study was conducted as part of a program that emphasizes a specially designed Video-Based Didactics (VBD) discourse. The study focuses on the professional development of two mathematics teachers who also serve as district instructors of mathematics teachers. We will describe how the VBD discourse contributes to the development of the two instructors in terms of the turning points between their different levels of awareness with regard to their classroom teaching and their teachers' instruction.

THEORETICAL BACKGROUND

Using video for the professional development of teachers

Over the last decades, videotaping has been widely accepted to be a useful tool for teachers' PD; however, the choices of what to focus on and what methods to use has changed over the years (Sherin, 2004; Santagata and Guarino 2011; Blomberg, Sherin, Renkl, Glogger & Seidel, 2014). Developing the ability to thoroughly reflect on classroom events is one of the main goals of the teacher training process, which is thought to be fostered by video-based observation (Borko et al., 2008; Sherin & Van Es, 2009). Teachers are commonly encouraged to observe either their own teaching videos or those of others (Borko, et al., 2008; Sherin & Han, 2004; Rosaen, Schram, & Herbel-Eisenmann, 2002). It was found that, compared with analyzing other teachers' videos, teachers who analyzed their own teaching experienced higher activation, manifested by higher immersion, resonance, and motivation. In addition, they became more aware of relevant components of teaching and learning. However, they were less self-reflective with regard to articulating critical incidents (Seidel et al., 2011). Moreover, use of video encouraged changes in their teaching habits because it helped teachers to focus on their analysis, view their teaching from a new perspective, trust the feedback they received, feel free to change their practice, remember to implement changes, and see their progress (Tripp & Rich, 2012).

Educational reformers have long advocated for increasing teachers' engagement in peer observation, feedback, and support (Vescio, Ross, & Adams, 2008), since such

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 139-146). Umeå, Sweden: PME.

collaboration has been linked to greater student achievements, an innovative climate, as well as an improved reform implementation and sustainability (Coburn, Mata, & Choi, 2013; Frank, Zhao, & Borman, 2004).

Leaders of teachers' video-based PD programs should be aware that "Video alone does not define a lesson, it must be embedded within an instructional approach in order to foster teacher learning" (Blomberg et al., 2014, p. 458). In addition, video-based discourse can foster teacher's awareness of the teaching processes in a classroom environment (Consuegra et al., 2016).

Theoretical framework

Teachers' professional growth involves climbing from lower to higher levels of awareness with regard to teaching. Here we adopted Mason's framework (1998) for such levels, developed for mathematics teachers, in our study of the PD of the two instructors of mathematics teachers:

1. **Awareness in action**: Awareness of the ability to choose, distinguish, differentiate, identify similarities, identify generalizations, see something as an example of something else, connect, adapt, visualize and more.
2. **Awareness in discipline** (knowledge of awareness in action): Awareness of the ability to examine how the teacher performs the actions mentioned at the previous level, while addressing discipline. For example, examining how teachers foster mathematical/physical thinking habits by formulating inquiry questions that encourage thinking or determining what methods they use for solving problems. Mastering this level indicates that the teachers know the importance of being aware of their actions in the classroom, and their consequences.
3. **Awareness in counsel** (knowledge of awareness in a discipline): At this level, self-awareness should be sensitive to the needs of others in order to build their own awareness in action and awareness in a discipline. Mastering this level may indicate professionality on behalf of the teachers as trainers of other teachers in their own discipline. (Mason, J., 1998)

THE VBD DISCOURSE APPROACH: PRINCIPLES AND PRACTICE

The widely agreed potential of using video analysis in teachers' PD has ushered in the development of a special video-based didactics for teachers' training. This led us to develop a program whose aim is to provide math (and physics) teachers with professional development by using video analysis effectively, both at pre-service and in-service levels as well as at the level of teachers-trainers' PD. The program utilizes an approach that emphasizes a specially designed Video-Based Didactic discourse (abbreviated hereafter as VBD discourse) generated purely for the teachers' PD and not for their administrative evaluation. This approach is based on the assumption that teachers' privacy and independence are highly important factors in teaching as a profession. It also takes into account the fact that although video analysis offers a great opportunity for achieving teachers' professional development, not all of them are comfortable in opening themselves up to such scrutiny (Sherin & Han, 2004).

Here we will describe a research study that aimed at revealing how using our VBD program changed the focus of attention and the levels of awareness of two mathematics teachers' district instructors.

PRINCIPLES OF THE VBD DISCOURSE

Our VBD discourse is based on the following rules:

A. Use of evidence – The VBD discourse is based on evidence from the teacher's class, especially excerpts of video recordings of the lessons. Regarding teachers' instructors, evidence is taken from their recorded discourses with the teachers. Both the trainer and the trainee are encouraged to use observations in order to formulate questions about their actual actions: the teacher as a trainee, about the teaching, and the trainer about the discourse conducted with the trainee. Similar to the constructivist approach for teaching (Honebein, 1996), discovery plays a central role in the VBD discourse and trainees who inspect their own teaching learn how to analyze, conceptualize, and evaluate it. The trainers follow a similar process regarding their guidance. **B. Ownership –** The teachers use their own devices (commonly, their own phone) to record their teaching. The video thus recorded belongs to them and is for their own personal use. **C. Autonomy –** The teachers choose the episode (we advise them to choose no more than 5-7 minute–long episodes from a whole lesson) on which the VBD conversation will be based. **D. Clear** role – In every VBD discourse the teacher who introduces the video evidence is the trainee and his partner is the trainer. Thus, even if the VBD conversation is held between colleagues of the same level of professionalism, the role played by each of them is clearly defined. They may switch roles from one session to another, but in each session their roles are clearly defined. **E. Shared professionality –** The subject matter and professional expertise should be shared by the trainer and the trainee, in order to enable a thorough content-related discourse. Thus, they both should have experience in teaching the same subject matter – either math or physics. **F. Mutual development –** The trainer and the trainee develop together their observation analysis tools. The development process is mutual even if the trainer brings to it a richer set of such tools. **G. Introspection** – Introspection is a basic condition for learning from self-viewing. The requirement to choose an episode from the whole lesson and provide it with a header initiates an introspection process. The introspection process addresses both the cognitive and the affective dimensions. **H. An inquiry focus –** The viewer of a video, being the trainee or assuming the trainer's role, is encouraged to adopt a curious eye rather than a judgmental one when looking at the segment for topics to be discussed during the discourse. The discourse itself is carried out as a common conversation between the trainer's and the trainee's interpretations of the video evidence at hand.

THE VBD TRAINING PROGRAM

The main goal of our program is to train teachers in both mathematics and physics in employing the principles described above (mainly F to H), when conducting a guidance discourse as trainers (according to principle D) with other colleague teachers. The steps of the program are as follows: (a) One teacher (the trainee) videotapes his or own lesson, chooses a short episode from that lesson (5-7 minutes), and provides it with a title. (b) The other teacher (the trainer) prepares the didactic discourse, based on this video segment, by watching it (usually more than once) with a critical, non-judgmental eye, and prepares inquiring questions that arose from his observation. We encourage the participants to look for instances that could be related to students' possible difficulties in understanding the math at hand. (c) The trainer and the trainee then meet (face to face or online) and discuss the inquiries that arose. The meeting is recorded and is a common property of the trainer and the trainee. Whenever needed during the discourse, both of them can refer back to selected parts from the classroom video. In this way, the mutual inquiry-based discourse focuses (1) on interesting teaching events that might have been overlooked during class, (2) on students' misconceptions expressed in the video, or (3) on the teacher's explanations and responses. This practice can be a one-time meeting or a continuous process and is designed to benefit teachers serving in both roles: as the subject of the inquiry and as the trainer.

RESEARCH METHOD

The purpose of our research was to study how the VBD discourse affected the professional development of two district instructors of mathematics teachers. The research method focuses on two case studies.

Research Population

Two district mathematics instructors, who were appointed by the Ministry of Education to be responsible for the PD of 10 mathematics school instructors (a school mathematics instructor is in charge of mathematics teachers' training from several schools and he works in coordination with his district instructor). Both of them participated during the last summer in our VBD discourse-training program for mathematics instructors and they are currently implementing the VBD discourse approach for training their group of school mathematics' instructors. Here we will discuss how the VBD discourse approach contributed to the PD of these two mathematics district instructors

Research Questions

In order to evaluate the effectiveness of the VBD discourse regarding the PD of the two mathematics' instructors, we initiated research that investigated the following questions: What support is provided by the VBD discourse to the mathematics teachers' instructors in their PD? What characterizes an effective VBD discourse?

In order to address these questions, we adopted Chapman's definition of an effective video-based discourse as one that generates new insights of new teaching ideas, strengthens one's awareness regarding the teaching-learning process, and creates critical incidents and significant turning points (Chapman, 2017).

Research Instruments

Our documentation covers the following five resources, from which the first two are part of the program and the others were used for our research:

Recording of: (1) Short video segments from the two teachers' instructors' mathematics lessons in their own classrooms. (2) A VBD discourse with each of two district instructors guided by one member of the project team. (3) A VBD discourse with one of the project team members guided by each of the two supervisors. (4) Interviews with the two instructors after the VBD sessions. In addition, the instructors also responded to a questionnaire.

DATA ANALYSIS

Each member of our research team independently analyzed each of the instructor's interviews and recorded discourses in which they played both the role of the trainer and the role of the trainee. The analysis focused on identifying the levels of awareness and the turning points between those levels. The results of the analysis of the members of the research team were then compared in order to evaluate their mutual agreement

FINDINGS AND THE FIRST INTERPRETATION

We will present here partial findings based on the above-mentioned resources.
The following is a short description of the part of an episode chosen by Nella (pseudonym) and part of the VBD discourse that took place based on that episode.

Episode 1: The diagonal. (From a lesson on the proof of: "The median length to hypotenuse in a right triangle is equal to half the hypotenuse".

Nella asked her students to think about another idea regarding how to prove the theorem. One of the students suggested adding an auxiliary construction of segments parallel to [each] leg, which meet at point E; then we get a rectangle and then we can draw the second diagonal (AE) passing through the midpoint of the permit (D). Nella drew the student's attention to the fact that his proposal included an impossible construction because he proposed constructing an auxiliary that has two properties: to construct AC||BE, AB||CE and the segment AE will pass through the point D, the middle of BC. She emphasized that auxiliary construction in geometry can only have one feature, and sketched on the board an illustration of the case in which the two conditions do not exist (Figure 1). After continuing a brief dialogue with the student, she suggested that an auxiliary construction be built differently, and led the students step-by-step to the theorem proof.

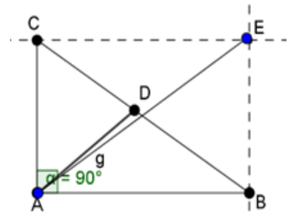

Figure 1: The diagonal

Part of the VBD discourse between Nella and one of the program's staff (following episode 1):

The first part of the VBD discourse *focused* on Nella's general pedagogy, manifested by the pleasant and relaxed atmosphere in her class and her didactical choice to conduct dialogues with her students about the different possibilities of proving the theorem. In addition, following Nella's remarks, the discussion dealt with the students' difficulties in using auxiliary constructions in geometry in general, and especially in the current exercise. Toward the end of the discourse, the following dialogue developed within the VBD discourse:

Trainer: There was something interesting in the dialogue with one student. He suggested drawing the parallel to the leg perpendicular and drawing the second diagonal so that it passes through point D. You told him that it is not a correct construction, because it has two characteristics: is this construction impossible?

Nella: He suggested building an auxiliary construction that has two properties, and I explained to him that it is impossible. Students have difficulties in understanding the roles of auxiliary construction.

Trainer: Let us think for a moment; as soon as we construct a parallel side to each of the triangle's perpendicular legs, what kind of square do you get?

Nella: A rectangle; in the classroom we understood that it was a rectangle.

Trainer: True, this is actually a rectangle where BC is a diagonal and D is the midpoint of BC. So I wonder whether the illustration you have shown on the board (Figure 1) is possible.

Nella: Ahh ... right ... so basically, the second diagonal BE must pass through point D. So ... the drawing I showed on the board (Figure 1) is not possible... Wow, I'm shocked; how did I miss noticing that it's impossible?

Trainer: I think this situation can happen to any teacher, especially when the student did not present the necessary reasons.

Nella: You're right ... I could actually continue with the idea of the student, and develop his direction in analyzing the proof, and only then introduce my idea of building an auxiliary construction ....how did I not see it, ...Wow it's amazing what happening here, ... It is not expected, this tool (VBD discourse) is powerful, I would not have thought about it, without this video and our discussion.

The VBD discourse led Nella to a turning point (Chapman, 2017), as expressed in the VBD discourse. This turning point aroused Nella's awareness both as a mathematics teacher and as a mathematics instructor, on several levels (Leaning on Mason, 1998). Here we present some of them:

Awareness in action as a mathematics teacher: Awareness of the importance of presenting a task solution in different ways, awareness of a variety of auxiliary constructions that can be integrated into a given task, awareness of students' difficulties in building auxiliary constructions in geometry, awareness of the properties of correctly built auxiliaries, and awareness of students' ideas.

Awareness in action as a mathematics instructor: Awareness of her action as a mathematics instructor: "I think about the processes I underwent in instructing math teachers, and I understand that this is not enough". Awareness of the power of re-observation in a mentoring process based on video by comparing observation without documentation: "Now after I've experienced the VBD discourse, I am aware of the power of this tool for me as an instructor… In my opinion, by observing the classroom while teaching without documentation, one cannot reach such deep insights". Awareness of the potential to develop points of view through VBD discourse, which helps math instructors to see something as an example of something else in the process of instructing teachers: "The situation that took place in my classroom is actually an example of a more general students' difficulty, which mathematics instructors should be aware of".

Awareness in discipline as a mathematics teacher: Awareness of why she chose to present the second proof based on a different auxiliary construction, and did not explore the student's suggestion of an auxiliary construction. Awareness of the importance of listening better to students' suggestions and of going through a proof trajectory different than her own.

Awareness in discipline as a mathematics instructor: Acknowledgement of the contribution of the VBD discourse to the development of mathematical and didactic teachers' knowledge: "The VBD discourse may be useful for professional development of mathematics teachers and mathematics teachers' instructors, because they can discuss mathematics and pedagogy ideas they have seen in the video…"

Awareness in counsel as a mathematics instructor: Acknowledgement of using VBD discourse as a training tool for enhancing the professional development of the teachers and instructors through knowledge of their awareness in action, and awareness in a discipline. "I learned from the VBD discourse to ask teachers, and instructors the following question: Did what happened in class (collaborative dialogue, flow with student ideas ...) characterize your lessons?"

CONCLUSION

In the present study, we presented some evidence for the professional development of two mathematics teachers' instructors during an effective VBD discourse. Participating in our VBD discourse-training program increased their awareness, both as teachers and as teachers' instructors. The findings of this study indicate that VBD discourse is a significant tool that contribute to the professional development of mathematics teachers and mathematics instructors.

References

Blomberg, G., Sherin, M. G., Renkl, A., Glogger, I., & Seidel, T. (2014). Understanding video as a tool for teacher education: investigating instructional strategies to promote reflection. Instructional Science, 42(3), 443-463.

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. (2008). Video as a tool for fostering productive discussions in mathematics professional development. Teaching and Teacher Education, 24, 417-436.

Chapman, Olive (2017). Mathematics Teachers' Perspectives of Turning Points in Their Teaching. In: Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). (2017). Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education (Vol. 1, 45-60). Singapore: PME.

Coburn, C. E., Mata, W. S., & Choi, L. (2013). The embeddedness of teachers' social networks: Evidence from a study of mathematics reform. Sociology of Education, 86(4), 311-342.

Consuegra, E., Engels, N., & Willegems, V. (2016). Using video-stimulated recall to investigate teacher awareness of explicit and implicit gendered thoughts on classroom interactions. Teachers and Teaching, 22(6), 683-699.

Frank, K. A., Zhao, Y., & Borman, K. (2004). Social capital and the diffusion of innovations within organizations: The case of computer technology in schools. Sociology of Education, 77(2), 148-171.

Honebein, P. C. (1996). Seven goals for the design of constructivist learning environments. Constructivist learning environments: Case studies in instructional design, 11-24.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. Journal of Mathematics Teacher Education, 1(3), 243-267.

Santagata, R., & Guarino, J. (2011). Using video to teach future teachers to learn from teaching. ZDM the International Journal of Mathematics Education, 43, 133–145. doi:10.1007/s11858-010-0292-3.

Seidel, T., Stürmer, K., Blomberg, G., Kobarg, M. & Schwindt, K. (2011). Teacher learning from analysis of videotaped classroom situations: Does it make a difference whether teachers observe their own teaching or that of others? Teaching and Teacher Education, 27(2), 259-267.

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. Teaching and Teacher education, 20(2), 163-183.

Sherin, M. G. (2004). New perspectives on the role of video in teacher education. In J. Brophy (Ed.), Using video in teacher education(pp. 1-28). Amsterdam, Netherlands: Elsevier.

Sherin, M. G., & van Es, E. (2009). Effects of video club participation on teachers' professional vision. Journal of Teacher Education, 60, 20-37

Rosaen, C. L., Schram, P., & Herbel-Eisenmann, B. (2002). Using hypermedia technology to explore connections among mathematics, language, and literacy in teacher education. Contemporary Issues in Technology and Teacher Education, 2(3), 297-326.

Tripp, T. R. & Rich, P. J. (2012). The influence of video analysis on the process of teacher change. Teaching and Teacher Education, 28(5), 728-739.

Vescio, V., Ross, D., & Adams, A. (2008). A review of research on the impact of professional learning communities on teaching practice and student learning. Teaching and teacher education, 24(1), 80-91.

ANALYSIS OF THE PCK OF AN ELEMENTARY MATHEMATICS MASTER TEACHER

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We present a case study investigating the Pedagogical Content Knowledge (PCK) of an elementary mathematics master teacher who implements a learner-centered approach to teaching fractions. Our data include two semi-structured interviews, various artifacts, and six videotaped lessons. In this study we identify ten elements of the participant's PCK and illustrate how these elements are enacted in the classroom through select lesson transcripts.

INTRODUCTION

Extending Shulman's (1986) Pedagogical Content Knowledge (PCK), mathematical knowledge for teaching (MKT) refers to the kind of special knowledge needed for the mathematics teacher to carry out teaching (Ball, Thames, & Phelps, 2008). Subject matter knowledge (SMK) as a subcategory of MKT was particularly useful as a theory to justify policy efforts in measuring the knowledge scientifically and identify effective mathematics teachers. While some researchers were critical of categorizing the knowledge of teaching and measuring each domain to quantify teacher competency, the complex nature of teaching and learning warrants research efforts to develop a framework for describing the kind of professional teacher competency that represents the practical application of knowledge, skills, and passion for facilitating meaningful learning for students. This study aims to (1) analyse the PCK of an elementary mathematics master teacher through the lenses of learner-centered instruction and (2) identify the way PCK is enacted during instruction. Two research questions guided this study: (1) What is the nature of the teacher's PCK? (2) To what extent is her PCK enacted in the classroom?

THEORETICAL BACKGROUND

Evolving Concept of PCK. The essence of the concept of PCK is that professional teaching requires something beyond mathematical content knowledge. Shulman (1986) explained it as a specific form of knowledge for *teaching*. Magnusson, Krajcik, and Borko (1999) argued that PCK is the combination of content knowledge and any other knowledge adapted for instructional situations. Longhran, Milroy, Berry, Cunstone, and Nulhall (2001) framed PCK as the teacher knowledge necessary to provide students with meaningful educational experiences and help achieve learning objectives. Niess (2005) explained that PCK is a way to connect the content and pedagogy. With PCK as a basis for the academic discourse on the kind and nature of teacher knowledge, some critical alternative views of mathematics teacher knowledge have

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emerged. Ball, Thames, & Phelps (2008) attended to the knowledge of teaching practice rather than teaching as a theoretical discipline. Additionally, they proposed an alternative construct (MKT) and emphasized the growth of teaching knowledge through experience and reflection. However, not only is MKT unclear about educational theories or the principles of teaching on which the framework is built, there are few instruments that can quantify MKT—especially when the knowledge evolves and grows over time.

Learner-centered Instruction. In the traditional paradigm of mathematics teaching, students are often on the receiving end while the teacher plays a central role, with the primary goal of delivering knowledge to students. In this framework, learning is essentially about acquiring knowledge from the teacher's predetermined paths and structures of knowledge, and effective teaching is recognized by the degrees of instructional time and coverage of curricular materials relative to the amount of disciplinary knowledge in school curriculum. In a reaction to the traditional paradigm, constructivist approaches mean to provide students with active learning opportunities to construct knowledge from meaningful contexts, socialization, and participation in the classroom community. Over time, the teacher as a facilitator of learning has challenged the traditional conception of teachers as the sage or authority. Active learning, discovery learning, or student-centered approaches to teaching have also shifted the focus of instruction, from teaching to deliver encapsulated knowledge to students to providing learning opportunities through which students can shape new views and learn from peers. In learner-centered instruction, students participate by sharing ideas, problem-solving, and/or group work. Therefore, teacher knowledge and skills that relate to asking questions, engaging students to explain their thinking, teaching reflection, motivating students to take interest in their learning, and synthesizing student views emerge as key teacher competencies. Given this, teacher knowledge of students becomes much more than anticipating student mistakes and responding. It involves understanding student thinking on certain math concepts at the individual level and designing lessons with multiple learning goals reflecting individual students and their areas of need.

This Study. An interest in the unique quality of teacher competency to facilitate student learning in the context of learner-centered approaches is what drives our inquiry into the unknown areas of practical teacher knowledge. In reaction to the work of theorizing the knowledge for teaching mathematics through psychometric methods, this study investigated the practice-based knowledge for teaching mathematics in the learner-centered classroom as an alternative view of constructing the mathematics teacher knowledge for teaching.

METHODOGY

This is an intrinsic case study (Stake, 1995) with the elementary mathematics master teacher's PCK as the primary interest. Through the analysis of teaching and student learning, the participant's interview data, videotaped lessons, and artifacts related to

PCK, this study aims to provide a deep understanding of (and insight into) the nature of a teacher's PCK and the way it is enacted in the classroom. To recruit a participant, a pool of interview candidates was created. Our selection criteria included the location of the school, documented teaching effectiveness, and consent to videotape lessons. We first contacted Mrs. Choi, who had the most teaching experience as master teacher, and upon her agreement to participate in our Fall 2016 study the observations and interviews took place in her classroom.

Data Collection

Interview. We conducted a semi-structured, audiotaped interview to access the teacher's beliefs about teaching mathematics. We began the first interview by asking what good teaching is and then asked follow-up questions. After reviewing the first interview data we prepared the next round of questions by seeking examples, evidence, and scenarios to clarify claims and unclear statements. Both interview sessions were untimed to capture as many participant comments as possible.

Artifacts. The participant provided (1) a teacher journal including lesson plans and student work samples and (2) her lecture materials. The journal data included 10 units of written notes on curriculum, sequences, and assessment data on student understanding, reflections, and action items for improvement. The lecture materials were originally created by the participant for her presentations at local professional development sessions and included 3 units of written documents on research-based instructional strategies—aligned with student needs and progress of learning—along with a detailed account of her teaching practice.

Videotaped lessons. The data included 6 sessions of teaching on fractions: two lessons were repeated three times for different groups of students. The first lesson taught fraction as a part-whole concept; the second lesson discussed fractions as division. We transcribed and analysed all six lessons, since the participants facilitated the same lesson plan differently in response to student needs and the analysis of the differences also served as data on teacher practice (Ronfeldt, 2011). Two cameras were used – one captured the teacher's teaching in the back and another captured student learning in the front—and the participant used a microphone for recording.

Data Analysis

Initially, the researcher (first author) reviewed the literature on MKT and attempted to characterize the participant's PCK (i.e., KCS and KCT) in the MKT framework. However, our initial coding adapted from the MKT instrument (Hill, 2008) did not work well with our interview data and artifacts. As a result, we used the participant's own categorization of her teaching practice instead of our initial codes, which included new codes such as "student motivation," "student misconception," "presenting tasks guided by specific math concepts," "assigning homework in specific order," "use of teaching materials," and "questioning." We then confirmed these six elements of PCK from the interview data and double-checked whether these elements could be corroborated by artifacts. Additionally, we sought to identify different elements not specified during interviews but evident in both the teacher's journal and lecture materials. As a result, we found a total of 12 elements of the participant's PCK. However, some of these elements were not mutually exclusive. For instance, the participant's strategies to encourage students to ask questions and prepare appropriate teaching materials intersected with her strategy to increase student interest in the lesson. Another example is that respecting students and getting to know students were too alike to separate, because the effort to get to know a person can be understood as respecting a person. Through consensus seeking among researchers, however, we combined some of these elements into one category. For example, "teacher's listening," "students' listening," and "believing in students' ideas" were merged into a single element, dubbed "listening to students." "Respecting students" had too many examples which were not associated with a unique element, so it was kept as a valid code. Due to the redundant nature of elements of PCK held by the participant we did not investigate the *frequency* of each element in the data; rather, we sought to confirm an element was evident in all three data sources – interviews, artifacts, and videos.

RESULTS AND DISCUSSION

Ten Elements of the Participant's PCK

The 10 elements include "listening to students," "preparing teaching materials," "leveraging student questions," "sequencing the lesson," "respecting students," "understanding about students," "improving student motivation," "creating low-risk and positive learning environments," "teacher questioning," and "building databases of students' mathematics." The following table provides representative evidence from interviews and artifacts that supports the elements as the participant's PCK.

Elements of PCK	Representative Evidence
1. Listening to students	Artifacts: I need to explain [to the class] why it is important to listen to each other's ideas and encourage them to confirm or challenge math ideas. [Teacher journal]
2. Preparing teaching materials	Interview: "I prepared a lot. Look at the stack of materials for tomorrow's lesson. Students need tasks to work on and need multiple contexts to apply their thinking. I also need to do some tasks myself before the lesson so that I can prepare good questions in advance."
3. Leveraging student questions	Artifacts: Some students' questions are not relevant to the concept I am teaching, but it is important to address the questions because my lesson somehow prompted a student to ask the question, and we never know whether the question is worthwhile in the end. [Teacher journal]

Table 1: Elements of PCK and anchored evidence from interview data and artifacts. (The data for the *next* seven elements are omitted due to the page limit.)

The Enactment of the Participant's PCK in Lessons

Next, we discuss each element and describe how the element is enacted in the classroom where appropriate.

Listening to students. Mrs. Choi stated that listening to students helped her better understand student thinking. She categorized valuable student comments and questions as (1) the statements made by nervous or unsure students, (2) mathematically correct yet unclear statements, and (3) mathematically incorrect statement. Mrs. Choi used her listening skills to better understand students, build trusting relationships, and ultimately create an intellectually safe classroom environment where students actively communicate mathematical ideas with little fear for embarrassment or belittling. Mrs. Choi often used phrases such as "Let's take a listen again," or "Can you run that by me one more time please?" or "Speaking of the great idea, can we listen to [Eunsoo's] idea again?"

Preparing teaching materials. Mrs. Choi prepared teaching materials but remained open for differentiation during and after each lesson in response to student understanding. Also noticeable was her attention to student ideas as a form of teaching material. In this way Mrs. Choi's lesson used a variety of materials that were prepared in advance or randomly selected, as well as suggested by students. For example, Mrs. Choi prepared a piece of a cookie (pre-cut in half) to demonstrate the whole and a half. Due to some students mistaking the half piece as a piece of the whole cookie, Mrs. Choi stopped using the cookie and used a random rectangular post-it as an alternative representation. Sticking it on the board she asked students, "If this post-it paper is 1, then how do we make 2 on the board? "How about one half?" Although the lesson plan had clear teaching procedures, Mrs. Choi tapped into student ideas to engage students in discussions about the meaning of 1 and fractions in the form $1/n$.

Leveraging student questions. Mrs. Choi stated that learning begins with student questions and that the teacher should facilitate learning in response to student-generated questions. Mrs. Choi actively used student questions as key teaching materials during lessons, and her PCK relative to student questions included (1) promoting the classroom culture to *ask* questions during lesson, (2) avoiding judging student questions, (3) providing contexts or materials to nurture student curiosity, (4) taking all questions seriously, and (5) thoughtfully addressing every question.

Sequencing the lesson. Mrs. Choi suggested that the teaching sequence in the textbook is not necessarily the best way to master a concept. Although she demonstrated knowledge of content, Mrs. Choi noted that teaching procedures should change in response to student progress as assessed through questions or other evidence. The following excerpt illustrates this as Mrs. Choi juggles student questions and sets the appropriate sequence of her lessons:

Interviewer: So, do mean you don't teach as the book prescribes or do you design a lesson based on student questions?

Mrs. Choi:	I believe textbook authors took student questions into writing the lesson anyway. So, my lesson should be very similar to theirs. So, I am not saying student questions are all there is to know. Any lesson just doesn't play out as the textbook's teaching procedures. So, I am not entirely relying on the book. But I like to use the exercise problems in the book.
Interviewer:	I am not sure if you're saying student questions are in line with the teaching sequence of the book. Or are you saying you gather student questions and rewrite the lesson?
Mrs. Choi:	Well, children want to learn what they are curious about. So, I tell them I will teach what they are curious, but we need to learn a new concept *before* we work on what they want to do. In this way, my lesson has a slightly different sequence from the book but it also reflects the traditional curriculum to a certain extent.

Respecting students. Mrs. Choi made efforts to ensure that her students knew she cared about their ideas. She regularly spoke phrases such as "precious idea," "your idea counts," and "I am confident you know something -- I just need to try harder to understand [your] ideas." She also suggested that her primary job is not evaluating students' ideas, but helping them make sense on their own or make connections in whole group conversations. As Mrs. Choi valued student ideas, the class was observed to be frank about their mistakes and comfortable expressing their opinions.

Understanding about students. Mrs. Choi regularly used open-ended tasks and implemented formative assessment by asking questions and encouraged student-student interactions. In the following excerpt, Mrs. Choi facilitates a mathematical conversation on fraction as division. She poses a question to complete a sentence "12 divided by ___ equals ___:

1	T:	How many do you want to divide 12 by?
2	S1:	Hmm… four.
3	T:	Let's divide 12 by 4 then. Go ahead. I am curious how you are going to show dividing 12 by 4.
4	S1:	(Drawing 12 cookies with three groups of 4)
5	T:	Okay. Do you agree with the way he drew the number 12 is divided by 4?
6	Ss:	Yes, that is correct.
7	T:	Correct? Okay, I see. Let's think about it. Wait a second. I have a question. "How many cookies are in each group?"
8	S1:	(…) Four?
9	T:	It is four. Let's talk about it together. So, what I hear is when you divide 12 in 4 groups, each has 4 cookies. Did I translate [S1's] thinking correctly?
10	Ss:	I think it is 16, isn't it? / There should be four groups, but there are three groups only. / I think it is four too. / It is confusing.
11	S2:	It is ¼ so we should have four chunks.
12	T:	Hold on. Let's listen to S2 again.

13	S2:	The whole thing is ¼, so it should be 4 chunks. But it is 3 now. That doesn't make sense.
14	T:	Anyone understands S2? Not sure? [S2,] can you explain it one more time?
15	S2:	One fourth should be one group out of four, but there are three groups only.
16	T:	Did you all follow what [S2] is telling us? He said four groups, right? Look what [S1] did. [S1] has fixed his drawing already? (laughter) That was quick. How many groups do you see now?
17	Ss:	Four groups.
18	T:	What does $1/4^{th}$ mean then?
19	Ss:	It is one out of 4.
20	T:	If [S1] eats one group of his cookies, how many cookies would he eat?
21	Ss:	Three.

In the excerpt, S1 mistakes dividing by a number as grouping with the number of objects in each group. Instead of correcting S1, Mrs. Choi asks the class to evaluate S1's idea and finds the class struggling to do so. Mrs. Choi then extends S2's idea on the ¼ as a group out of four groups and engages the class in developing the idea that 12 divided by 4 can be represented by grouping 12 into 4 groups. This excerpt illustrates Mrs. Choi's PCK, listening to student comments during lessons and co-constructing mathematical meaning of division as fraction through the iterations of questioning, listening, and responding.

Improving student motivation. Mrs. Choi expressed her belief that meaningful student-student interactions help students motivate themselves to learn. To this end, students get to reason with their classmates to develop new understanding.

Creating low-risk and positive learning environments. Mrs. Choi believed that all students had various levels of understanding and interest. This belief led her to accept student misconceptions and errors in the lesson as natural and significant steps in learning and developing new knowledge. In her lesson, she implemented wait-time consistently and responded to student misunderstandings by saying, "Can we try again together?" Other responses were "Let's keep thinking about it," or "I'd love to hear about it when you get it."

Teacher questioning. Mrs. Choi used her questions to view student thinking and create opportunities to reflect on other people's thinking, making connections in whole group discussions. She frequently checked for student comprehension and asked students to explain or contribute new ideas. She asked on a regular basis, "When I come up with an idea, I would be curious about other ideas and compare it to mine," and "This is a new idea, it is time to investigate different ideas and compare all."

Building databases of students' mathematics. Mrs. Choi recorded student ideas and reviewed the data to deepen the understanding of her students' thinking. She stated, "When you consider everybody's idea -- right or wrong, the lesson will get very messy. But you should be committed to listening to all students' ideas. There is no other way around it." In particular, Mrs. Choi showed interest in student errors and stated she could better support students by anticipating and responding to student misconceptions. During her lesson Mrs. Choi went to great lengths to record student comments, writings, and diagrams “as is” along with her own interpretations. She also made attempts to validate those interpretations by asking students, “This is how I understand your thinking, is this correct?”

CONCLUSIONS

We recommend future studies to dwell on the interconnected nature of PCK in practice and clarify the relationship in various elements of a teacher’s PCK, rather than categorizing the teacher’s practical knowledge in distinctive domains. Although we found 10 elements in the study representing Mrs. Choi’s PCK, a few elements were closely related. For example, when Mrs. Choi was listening to students there were other elements in play, such as understanding students or creating low-risk and positive learning environments.

Although it was clear from the data that Mrs. Choi’s PCK is still evolving, this study did not investigate exactly how Mrs. Choi developed her PCK. Our guess is that her reflective dispositions and commitment to professional practice played a significant role. For example, she has reviewed students’ course evaluations and used the data to improve her instruction. With that said, it is important to continue to explore various elements of PCK through case studies, investigate the pattern of developing PCK, and use this information to support our teachers.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers’ topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Lougran, J., Milroy, P., Berry, A., Cunstone, R., & Mulhall, P. (2001). Documenting science teachers’ pedagogical content knowledge through PaP-eRs. *Research in Science Education, 31*(2), 289-307.

Magnusson, S., Krajcik, J., & Borko, H. (1999). Nature, sources, and development of pedagogical content knowledge for science teaching. In J. Gess-Newsome & N. G. Lederman (Eds.), *Examining pedagogical content knowledge: The construct and its implications for science education* (pp. 95-132). Boston: Kluwer.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education, 21*(5), 509-523.

Ronfeldt, S. (2003). *Third grade math: Month by month guide.* Sausalito, CA: Math Solutions Publisher.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 75,* 4-14.

Stake, R. E. (1995). *The art of case study research.* Thousand Oaks, CA: Sage.

EXAMINING TEACHERS' DISCOURSE ON STUDENTS' STRUGGLE THROUGH FIGURED WORLDS

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We apply the lens of figured worlds on teachers' pedagogic discourse to understand their identity and practice in relation to offering students opportunities to struggle. The study involved 12 elementary mathematics teachers who were interviewed based on teaching vignettes - short stories exemplifying teaching that is high/low in students' opportunities for struggle. Two distinct figured worlds were identified: the world of "acquisition" and the world of "exploration". Teachers belonging to each of these worlds differed in their interpretations of identical vignettes depicting students' struggle, and their identities as teachers cohered with these interpretations. Implications of these results on attempts to reform teachers' practice towards explorative instruction are discussed.

BACKROUND

Figured world is defined as “a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others” (Holland, Lachicotte, Skinner, & Cain, 1998, p. 52). Artefacts and signs are attributed meaning that might differ from how those outsides of the figured world interpret them. People, actors in the figured world, have expectations for how events unfold and how others will behave in these events (Ma & Singer-Gabella, 2011). Identities are created and crystallized or consolidated in the process of participation in organized activity by the figured world. Within the figured world, people give distinct meaning to actions, results, objects, and events. They value certain actions and outcomes, while devaluing others. They position themselves in relation to these valued actions and outcomes thereby constructing an identity within this figured world.

Relying on the definition of identity as a collection of narratives (Sfard & Prusak, 2005), the theory of figured worlds makes it clear that such narratives are not created in a vacuum. They stem from social and cultural contexts that can be explicated through the examination of the figured world to which they belong.

Mathematics teachers' identity has received increasing interest in the past years (e.g. Beauchamp & Thomas, 2009). Though most of this literature has looked at the process of becoming a teacher, the lens of teachers' identity is also useful for examining change in teachers' practice. In particular, the tracing of teachers' identity narratives to particular figured worlds can be a powerful tool for understanding the challenges of change required by demand to "reform" instruction.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 155-162). Umeå, Sweden: PME.

Boaler and Greeno (2000) were the first to point out that "reform" and "traditional" teaching belong, in fact, to different figured worlds. They connected traditional teaching with the figured world of "received knowledge", characterized by procedure-oriented, teacher-centred instruction and the figured world of "connected knowledge" with discussions oriented, problem-solving instruction. Ma and Singer-Gabella (2011) have continued this line, showing that pre-service teachers indeed sway between the "traditional" and "reform" figured world as they go through their teacher-training program.

Over the years, accumulating evidence has shown that the "connected knowledge" figured world is more productive for mathematics learning (Schoenfeld, 2014). In particular, Hiebert and Grouws (2007) pointed to the importance of two aspects in such teaching: Explicit Attention to Concepts (EAC), and Students' Opportunity to Struggle (SOS). Based on Hiebert and Grouws' (2007) work, Stein and her colleagues (2017) presented a framework that classified teaching into relatively simple "types". A simple 2x2 matrix of high and low levels of SOS and EAC produces four quadrants: Q1 (High EAC, high SOS), Q2 (high EAC, low SOS), Q3 (low EAC, high SOS) and Q4 (low EAC, low SOS). In our own work (Shabtay & Heyd-Metzuyanim, 2017), we have shown that contrary to the naive expectation that teachers would generally aspire towards Q1 teaching, some teachers have strong objections to it, and in particular, resist offering opportunities for students' struggle. This finding has urged us to look more closely at the reasons for this resistance. In particular, we were interested to see how teachers' identities, as elicited from asking them to identify with a particular quadrant, draw upon the figured world of mathematics instruction.

METHOD

The "teaching vignettes" interview: Our method relies on a procedure we have tested in previously (Shabtay & Heyd-Metzuyanim, 2017), where we interview teachers on the basis of Stein et al.'s (2017) "teaching vignettes". These vignettes describe a typical lesson of each of the four quadrants. The Q1 teacher (High SOS, High EAC) gives her students a cognitively demanding task, walks around the classroom while students work on it in groups and asks questions to advance their thinking. She concludes with a discussion that points to the equivalence of different representations of rational numbers. The Q2 vignette (Low SOS and High EAC) describes a teacher that uses the same cognitively demanding task, but divides it into small steps and leads her students through them with leading questions. She attends to the concept through pointing out to the equivalence of fractions, decimals and percentages in the problem, but the students do not take a significant role in this explication. The Q3 (High SOS and Low EAC) vignette starts with the same task. The teacher lets students struggle but does not mediate the task in any way. Though two students present their solutions at the end of the lesson, fractions, decimals and percentages are not connected explicitly. The Q4 (Low SOS, Low EAC) vignette describes a teacher who provides a calculational task that is supposed to offer opportunities to practice converting between fractions, decimals and percentages.

Procedure and analysis

We conducted the "teaching vignettes" interviews with 12 teachers. To the questions about the vignettes, we added questions about their identity as teachers and learners of mathematics, their common practices in the classroom and their beliefs about good instruction. Each interview took about 55-80 minutes and was audio-recorded with the teacher's approval. The recordings were fully transcribed and analysed.

Analysis started by highlighting narratives where teachers talked about themselves and their practices and about the teachers depicted in the vignettes. The second stage included marking and tabulating the different dimensions of the figured world. According to the definition of figured world described above, these included statements regarding valued and non-valued actions, valued and non-valued outcomes, and non-valued outcomes. We also examined the teacher and students' roles as they were described in teachers' discourse. Finally, we collected similar statements regarding valued actions, outcomes and roles into tables that depict the different figured worlds identified in the interviews. These enabled us to map the teachers into those belonging to one figured world or the other. Most teachers could be mapped into one of the figured worlds quite neatly, while a few were categorized as "in between". These decisions will be further explained in the findings.

FINDINGS

We start by presenting two contrasting figured worlds, extracted from the interviews of two teachers, Orit and Hani. These contrasting figured worlds were seen both in the teachers' interpretations of the vignettes, as well as in their descriptions of valued and non-valued actions and outcomes.

Orit, a teacher with 18 years of experience, chose "Orna", the teacher exemplifying Q2 teaching vignette, as the teacher who resembles her best. She explained her choice as follows:

> "Because I felt like she (Orna-Q2) was doing it in a very structured way, she did not let them (the students) directly explore, (something) which would sound very nice pedagogically (as in) 'explore and get it out of the child'. But when I look at (my) whole class, and I know I have several groups of students and several levels, if I do this (let them explore) it will only resonate with the strong group. And all those below, I will lose them completely. Those in the middle, I'll lose them too ... "(Line 53).

We see in Orit's identification with Orna (Q2) not just explanations for why Orna's practice is better, but also for why the alternative practice is wrong. This is made even clearer in the ways in which Orit relates to Daphna (depicted in the Q3 vignette):

> "Daphna (Q3) simply ... Believe me, I don't understand. What is she doing? Telling them to 'think again'? Will that really help? If they don't know, she just frustrates them. They will continue to not know. Or they will copy from the good students. That's why I don't like all these explorative tasks ... Only the best students answer, and the rest of the students are left behind." (Line 65).

Looking at these two descriptions of the teachers through the lens of figured world, we see that valued actions include "doing things in a structured way", while non-valued actions include "letting students just explore", which only result in "frustrating students" and "losing the middle/weak students completely".

The ways Orit describes her identity, and the role she takes in the classroom, is very much aligned with this figured world. In the example below, pronouns indicating the role of her as a teacher and the role of her students are marked in bold:

> "First of all **I go** for something **we already learned**, that **we finished** and checked that it 'sits well' (in the students) before the new content will be learned.... and **I make the connections** ... and then, **I explain**, I don't... **I don't usually ask the children to explore**; I don't (just) give them the task. Before **I give** the task, **I first do the instilment** (Acquisition), and ... **(I) check that they understand**, and (I) don't go straight to an explorative task, 'come and explore'. **I start** from the lowest level "(Line 42).

As indicated by the bolded pronouns and verbs, Orit is mostly concerned with what **she** does. She is the one that "explains", "makes the connections" and "starts from the lowest level". Though there is some talk about making sure students "understand", this is only mentioned as a part of a gradual step-by-step process that is designed to make sure students are not just "left to explore". An important word figuring in Orit's pedagogical discourse is "instilment" (in Hebrew: 'Haknaya'). 'Haknaya' in Hebrew comes from the stem "to buy" or acquire. It denotes the period of the lesson where a teacher explains or demonstrates. Interestingly, it goes very well with the metaphor of learning as "Acquiring" knowledge and the teacher as "deliverer" of this knowledge (Sfard, 1998). With the lack of a good English translation for this word, we chose to name the figured world exemplified in Orit's talk as the figured world of Acquisition.

A very different figured world could be seen in the discourse of Hani. Hani debated whether she identifies with the Q1 or Q3 teacher, saying that she tends towards the Q3 teaching in classrooms that "let her do that", but clearly opting for the vignettes that described high students' struggle. She explained her sympathy with the Q3 vignette:

> (First)… I like that they (the students in the story) struggle alone. Second, there are different options for a solution. The fraction is presented in several representations. I don't like drill and practice, like teacher Sharon (Q4) does. (Int: Why?) Because I think it's technical and if there's no understanding, I'm not sure they'll remember the calculation. If it's up to me, I'd rather they do it the longer way, but with understanding. If there's understanding along the way, the calculation will be OK. (Line 45).

When asked why she did not like Orna's (Q2) type of teaching, Hani answered:

> She's just very structured. I prefer to give more freedom. She gives them too many scaffolds, where I think she could have trusted them more. (Line 49).

In Hani's discourse we find that valued actions include offering tasks with "different options for solutions", "giving students freedom", and "trusting" students. Non-valued actions include "technical calculations" and giving "too many scaffolds". Valued outcomes include students "understanding the calculations". However, Hani does not talk about "understanding the concepts" or "making connections". Thus her figured world seems to be more concerned with letting students struggle than with explicating mathematical concepts.

Similar to Orit, Hani also referred to students' level as determining her practice:

> "In classrooms where there are more difficulties, I act more like Nitsa (Q1). And in stronger classrooms, I'm more like Daphna (Q3), and if there's frustration, then I become more Nitsa. There are classes that are more open, and then I throw them more to the water, like Daphna does". (Line 43)

However, her choice was between Q1 and Q3 (both high SOS), thus she took it for granted that her students would be given opportunities to struggle. Also, she did not mention the difference between Q1 and Q3 as being related to the explication of mathematical concepts (as intended by the authors of these vignettes) showing, again, that the main focus in her figured world was on students' struggle.

Orit and Hani thus exemplify two quite contrasting figured worlds: one of Acquisition and one, which we termed Exploration. The later naming was based on the prominence of the world Exploration (Hebrew: Heker) in this discourse. As expected by the definition of figured worlds as "realms of interpretation", teachers' whose discourse belonged to the different figured worlds displayed different interpretations of the identical vignettes. While Orit was exasperated by the Q3 teacher who "just asked the students to 'think again'" and saw this as "only frustrating" students, Hani liked that this teacher "lets her students struggle". Notably, the emotional words used for describing students' reactions (such as "frustration") were nowhere described in the vignettes. Thus the teachers' "filled in" information about the depicted situation according to their figured world.

In our sample of 12 teachers: 5 teachers identified with Q2 and valued actions and outcomes according to the Acquisition figured world, and 4 teachers identified with Q3 or Q1, talking in ways that accord with the Exploration figured world. We found no relation between teachers' experience or the school in which they taught and their figured worlds. Also, most teachers were found to be coherent in their figured worlds meaning that their valued actions (e.g. practicing calculations) cohered with their valued outcomes (being able to follow procedures) and their role as a teacher (demonstrating the procedure and easing its enactment by students). However, a small group (of 3 teachers) were found to be "mixed". One such teacher was Sofi. On the one hand, Sofi declared her teaching mostly resembles the vignette depicting Q2 teaching. She justified this with "It's easier to teach a new subject step by step. Not all at once". However, Sofi also declared that:

> "But in elementary school it's not good if the teacher explains all the time. So I came to the conclusion, in elementary school for sure, that it's better to do Acquisition of 10 or 7 minutes and then give a task. A task that has many questions. But that's when we really really have a new subject. And then with the worksheets, let the children think, let the children talk."(Line 60).

Sofi's words reveal that she has some conflicts regarding the value of struggle. On the one hand, she values students thinking and talking. On the other hand, she declares that in her classroom, she first explains and does "Haknaya" (Acquisition) and only later lets the students work alone. In addition, although she declares she has realized that in elementary school students need to work alone, she structures her worksheets with incremental step-by-step tasks to avoid too much struggle.

Following is how Sofi relates to Daphna (Q3 teaching style):

> "I thought about my students, what would have happened if I gave them this task. And then I said (to myself) that I would have loved giving them this task without support. But unfortunately, I have such students that need mediation, instruction and so I understood that I don't really act like Daphna" (Line 62).

Notable in the discourse of Sofi about Q3 teaching are the conflicts between the two worlds. She talks somewhat regretfully ("unfortunately") about her students that "need mediation" and her words "so I understood that I don't really act like Daphna" indicate that she may have wanted to identify herself with teachers affording high-struggle, but had to admit she is not like them. Interestingly, like most of the other teachers in our sample, Sofi lays the responsibility for her teaching style on her students. She *would have* wanted to act like a Q3 teacher, had it not been for her students who "need instruction". This was a common theme in those teachers who chose not to identify with Q1 or Q3 teaching. The underlying message was that affording high struggle was "a nice pedagogical idea" (as phrased by Orit), but not suitable for "their" students.

To conclude, the overall analysis of the 12 teachers' discourse revealed the following blue-print for the two figured worlds of Acquisition and Exploration. As Table 1 reveals, offering students opportunities to struggle is interpreted differently through the two figured worlds. Valued actions connected with the word "struggle" in this world are letting students solve tasks without first explaining the steps and offering them freedom. Valued outcomes include students' "understanding" and "discovering on their own". In contrast, within the Acquisition world, letting students explore on their own is interpreted as "frustrating" and favouring only the strong students. In accordance with these valued actions and outcomes, teachers' roles and the ways they identified themselves differ significantly in the two worlds.

Teachers belonging to the Acquisition world talked mostly about what **they** do, focusing the responsibility of the teaching-learning interaction solely on their own shoulders. Teachers belonging to the Exploration world talked more about what **students** do. In their talk, the results of the teaching-learning interaction were divided more equally between them and their students.

Figured world	Valued actions	Valued outcomes	Roles and responsibility	Interpretation of Q2, Q3 vignettes
Acquisition	Explanations, Working gradually, Structured lessons	Students following procedures successfully	Teacher is the main actor. Students' roles are to listen and acquire	Q3 teaching is a nice "pedagogical idea" that is disconnected from the realities of classroom life
Exploration	Students struggle and discover by themselves. Students have freedom	Students understand the meaning behind calculations.	Both teacher and students share the responsibility for the learning process	Q2 teaching is too "structured", too "scaffolded" and too "technical"

Table 1: Comparison between Acquisition and Exploration Figured worlds

DISCUSSION

In this study, we applied the lens of figured worlds to better understand teachers' reported practices which afford students' opportunities to struggle with cognitively demanding tasks. Using the vignettes interview has proved successful for eliciting from teachers both narratives about their identity as teachers, and narratives about the depicted lessons of high/low student struggle. Our method of analysis, where we categorize teachers' discourse according to valued and non-valued actions, outcomes and roles proved beneficial for operationalizing the concept of figured worlds and enabling its application on transcripts of teachers' interviews.

Similarly to Boaler and Greeno (2000) and Ma & Singer-Gabella, (2011) we identified two distinct figured worlds in the discourse of our teachers. These are the Acquisition world and the Exploration world, which roughly can be connected to "traditional" and "reform" worlds identified in these previous studies. The fact that the Acquisition and Exploration figured worlds could be found in an Israeli sample, where the politics of reform in mathematics instruction are quite different than those found in the US, shows that this division is indeed a powerful one. Yet we have also found teachers whose figured worlds were not coherent and contained conflicts between valued actions belonging to the two worlds.

Another contribution of this study is the application of the theoretical lens of teachers' identity to the issue of reforming teachers' practice in the classroom, issues that have so far mostly been dealt with by examining teachers' beliefs (e.g. Stein,et al., 2017). We contend that examining the issues of reform or explorative practices through the lens of teachers' identity can offer insights into the reasons that such teaching practices are often found to be very resistant to change (Spillane & Zeuli, 1999). This, since not just particular beliefs need to be changed for teachers to change their practice. Their whole identity and the figured world on which it draws upon need to change too. Arguably, such a change is a deep and all-encompassing process, including both how teachers narrate themselves and how they interpret the teaching-learning world in which they engage.

References

Beauchamp, C. & Thomas, L. (2009). Understanding teacher identity: an overview of issues in the literature and implications for teacher education. *Cambridge Journal of Education*. 39, (2), 175–189.

Boaler, J. & Greeno, J. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics education* (pp. 171–200). Westport, CT: Ablex.

Hiebert, J. & Grouws, D. A. (2007). The effects of Classroom Mathematics Teaching on student' learning. *Second Handbook Of Research on Mathematics Teaching and Learning*, 1, 371-404.

Holland, D. Lachicotte, W. J. Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds.* Cambridge, MA: Harvard University Press.

Ma, J.Y. & Singer-Gabella, M. (2011). Learning to teach in the figured world of reform mathematics: Negotiating new models of identity. *Journal of teacher education*, 62 (1), 8-22.

Sfard A. & Prusak, A. (2005). Identity that makes a difference: substantial learning closing the gap between actual and designated identity. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 37-52. Melbourne: PME.

Sfard, A. (1998). On two metaphors for learning and on the dangers of choosing just one, *Educational Researcher*, 27(2), 4–13.

Spillane J.P. & Zeulli, J.S. (1999). Reform and teaching: Exploring patterns of practices in context of national and state mathematics reform. *Educational Evaluation and Policy Analysis*, 22, (1), 1-27.

Schoenfeld, A.E. (2014) What Makes for Powerful Classrooms, and How Can We Support Teachers in Creating Them? A Story of Research and Practice Productively Intertwined. *Educational Researcher*, 43 (8), 404–412.

Shabtay G. & Heyd-Metzuyanim, E. (2017). Teachers' discourse On students' conceptual understanding and struggle. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 177-184). Singapore: PME.

Stein, M. K., Correnti, R., Moore, D., Russell, J. L., & Kelly, K. (2017). Using Theory and Measurement to Sharpen Conceptualizations of Mathematics Teaching in the Common Core Era. *AERA Open*, 3(1).

GROUP THINKING STYLES AND THEIR MODELLING PROCESS WHILE ENGAGING IN MODELLING ACTIVITIES

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The current study investigated the relationship between students' thinking style and their modelling process and routes. Thirty-five eighth-grade students were examined. For the first stage, the students solved a word problem, and according to their solutions, they were assigned to one of two groups: a visual thinking style group and an analytic thinking style group. The two groups engaged in three modelling activities. Findings indicating differences in the groups' modelling processes in performing the three activities. The primary differences in the modelling processes were manifested in simplifying, mathematizing, and eliciting a mathematical model. In addition, the analytic thinking group skipped the real-model phase in the three activities, while the visual group built a real model for each activity.

INTRODUCTION

Thinking style and cognitive methods strongly affect student performance in many areas, largely determining significant differences in their performance, as demonstrated in empirical cognitive psychology studies (e.g., Cakan, 2000). Therefore, students' different thinking styles should be taken into account upon determining appropriate educational interventions (Sternberg & Zhang, 2005). Thus, teacher awareness of differential thinking styles is particularly important when students are exposed to modelling activities that offer them the opportunity to meet everyday challenges and demands and provide them with the abilities and competencies to deal with complex systems and real-word situations (Lesh & Doerr, 2003). Mathematical modelling is the process of translating between the real world and mathematics (Blum & Borromeo Ferri, 2009). Knowledge about students' modelling processes can ameliorate their teachers' interventions (Blum & Leiß, 2005). Given their potential, modelling processes have been studied widely (e.g., Blum & Borromeo Ferri, 2009). However, only a few scholars (e.g., Borromeo Ferri, 2010) have examined the modelling process of individuals having different thinking styles. Furthermore, almost no studies have focused on the modelling process with respect to thinking styles characterizing groups, where all modellers in each group have the same thinking style. This study aims to shed light on the influence of group thinking style on their modelling process and modelling route while engaged in modelling activities.

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FRAMEWORK

Mathematical thinking Style

A style is a way of thinking; it is not an ability, but rather a preferred way of using one's abilities (Sternberg, 1997). Thus, mathematical thinking styles denote how individuals prefer to learn mathematics, not how their mathematical understanding is assessed. In addition, it also is indicative of how the individual prefers to proceed with the mathematical task (Sternberg, 1997). Klein (cited in Borromeo Ferri & Kaiser, 2003) suggested three different thinking styles: the philosopher, who constructs on the basis of concepts; the analyst, who operates within a formula; and the geometer, who has a visual starting point. Similarly, Borromeo Ferri and Kaiser (2003), in their empirical study, suggested three thinking styles: the analytic, the visual, and the integrated. In the current study, we will follow the latter classification, focusing on the visual and the analytic thinking styles. The visual thinking style has been defined as thinking based on the shapes, drawings, and images presented in real situations and relationships (Campbell, Collis, & Watson, 1995). Students with a visual thinking style are characterized by a strongly image-oriented way of thinking when solving mathematical problems; this facilitates their obtaining, representing, interpreting, perceiving, and memorizing information, as well as expressing it (Borromeo Ferri & Kaiser, 2003). The analytic thinking style has been identified as thinking symbolically and formalistically, involving sorting and teasing out elements from their context. This style reflects a tendency to focus on the properties of objects and elements for classification into categories, preferring to use rules about categories and predicting behavior (Monga & John, 2007).

Modelling

Mathematical modelling means solving complex, realistic, and open problems with the help of mathematics, with the process that students develop and use in solving such problems termed modelling process. The modelling process is a cyclic, in which translating between the real world and mathematics transpires in both directions (Blum & Borromeo Ferri, 2009). The modelling processes from a cognitive perspective identified phases and transitions (Blum & Leiß, 2005). The phases comprise a situation model, a real model, and a mathematical model, mathematical results and real results. The transitions include several actions: understanding the problem and simplifying a situation model; presenting a real model; mathematizing, which leads to constructing a mathematical model; applying mathematical procedures; interpreting the mathematical results; and validating, in which mathematical results are validated in a real-life task. Various visual descriptions of the cyclic process-modelling cycle have been reported in the literature. The current research is based on Blum and Leiß's (2005) modelling cycle. Delineating the modelling process in detail, incorporating the various phases of the modelling cycle on an internal and external level, is referred to as the modelling route (Borromeo Ferri, 2007).

RESEARCH AIM AND QUESTION

Do and how groups of students with different thinking styles (visual or analytic) differ in their modelling process and their modelling routes while working on a sequence of modelling activities?

METHOD

Research participants and procedure

For the first stage of the study, a questionnaire for identifying participants' thinking style was administered to 35 students in an eighth-grade class. Based on the styles reflected in solving the questionnaire's tasks, students were then classified into three thinking style groups: analytic (14 students), visual (11 students), and integrated (10 students) thinking style groups. As the focus in the current study was the analytic and visual thinking style, we divided the students into two groups, based on their shared thinking styles. For each group, we selected five students (totalling 10 participants) with the assistance of their mathematics teacher in order to maximize matching variables (e.g., gender, mathematics abilities, socioeconomic status). Both groups (analytic and visual) were assigned three modelling activities in the course of three weeks, one activity per week. The modelling activities were adapted from the literature (e.g., Blum & Borromeo Ferri, 2009).

Data sources and analysis

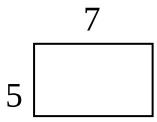
The data collected from two sources: Questionnaire and video recordings. Questionnaire: The study questionnaire comprised eight tasks for classifying students according to their thinking style. Some of these tasks were adapted from other studies (e.g., Lowrie & Clements, 2009), and some were designed by the researchers. The selected tasks were characterized by a variety of topic areas and a variety of possible solution strategies. An example the tasks is the Turf Problem (Lowrie & Clements, 2001. P. 86): *A husband and wife wanted to turf their backyard (put grass squares down). Before purchasing the turf, they had a ground pool put in their backyard. The pool was 3m wide and 5m long. Sensibly, they also paved an area 1m wide around the pool. If turf costs $10 per square meter, how much would it have cost to turf the backyard (150 m² in total) once the pool and the paving were finished*.

Video recordings: Video recording were made of the two groups working on the three modelling activities and were transcribed.

Questionnaire analysis: We used the constant comparative method (Glaser & Strauss, 1967) to analyze the problem-solving processes for each task in the questionnaire for each student. We adopted the categories described by Borromeo Ferri and Kaiser (2003): When illustrating and solving the mathematical problems, the visual thinking group was characterized by sketches, drawings, or graphs, while the analytical thinking style was expressed in a formula-oriented way that means that information from the text of a given problem, is expressed by means of equations. An example of students' answers classification for the Turf Problem can be seen in Figure 1:

Analytic style: 3+2=5m; 5+2=7m

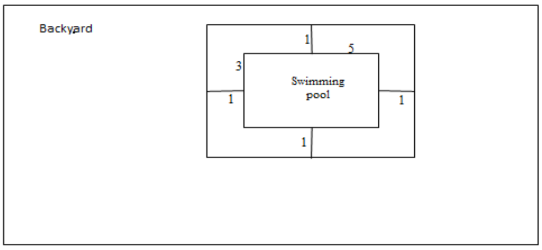
7*5=35;



150-35= 115m^2;

115*10= 1150$

Visual style:



5*7=35m^2

150-35=115m^2

115*10=1150$

Figure 1. Samples of students' solutions of the Turf Problem

Video recording analysis: We used the constant comparative method (Glaser & Strauss, 1967) to analyze the students modelling processes in three activities, taking into account the cognitive aspect of modellers' modelling cycle (Blum & Leiß, 2005).

FINDINGS

Modelling process between analytic and visual groups

The findings indicate that the analytic and visual groups demonstrated similar features in working on the three modelling activities, but differed in their modelling processes. Table 1 presents the general findings regarding the two groups' modelling processes.

Group		Analytic									Visual								
Modelling process		Simplifying	Real model	Mathematizing	Mathematical model	Working mathematically	Mathematical results	Interpreting	Real results	Validating	Simplifying	Real model	Mathematizing	Mathematical model	Working mathematically	Mathematical results	Interpreting	Real results	Validating
First activity Modelling cycle	1	-	-	√	√	√	√	-	-	-	√	√	√	√	√	√	-	-	-
	2	-	-	√	√	√	√	√	√	√	-	√	√	√	√	√	√	√	√
	3	-	-	√	√	√	√	√	√	√	-	-	-	-	-	-	-	-	-
Second activity Modelling cycle	1	-	-	-	-	√	√	√	√	-	√	√	√	√	√	√	√	√	√
	2	-	-	-	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
	3	-	-	-	√	√	√	√	√	√	-	-	-	-	-	-	-	-	-
Third activity Modelling cycle	1	-	-	√	-	√	√	-	-	√	√	√	√	√	√	√	-	-	√
	2	-	-	√	√	-	-	-	-	-	√	√	√	√	√	√	√	√	√
	3	-	-	√	√	√	√	√	√	√	-	-	-	-	-	-	-	-	-

Table 1: Modelling processes of the Analytic and Visual Groups in the Three activities

The analysis of the modelling processes of the two groups in the three activities revealed that the major differences between them were in the real model, simplifying, mathematizing, and mathematical model. Table 2 presents the differences between the two groups, illustrated by sample statements from the students' discussions while working on the problem.

Modelling process	Visual group	Analytic group
Simplifying	Students seek to illustrate the information in the situations by drawing and illustration. E.g., [5] Student 1: I can explain the situation; we have information about... [they drew illustration of shoes and body]. [6] Student 1: We can find the relation between us and the giants	Students simplified the situations by mathematizing, with skipping real model for the situations. E.g. [5] Student 2: We can calculate by ratio between width and length. [32] Student 3: The ratio between the length of the shoes and height of a person.
Mathematization	Students mathematize the situation by working in tables and lists. E.g., [10] Student 3: Make a table [16] Student 3: Your shoes 26 cm, here I write 26 cm [in the column of the shoes' length] your height is 160.	Students mathematize the situation by searching about formulas. E.g., [9] Student 4: The ratio between the length and the width … length 32 and width 12 [length and width of their shoes]. [11] Student 2: We should simplify the ratio … 32:12.
Mathematical model	The mathematical model illustrated by tables and lists	The mathematical model presented by formula

Table 2: Differences in Modelling Process Between Analytic and Visual Groups

Modelling cycles and routes in the analytic and visual groups

Analysis of the modelling processes of the two groups in the three modelling activities indicated that the analytic group went through more modelling cycles than did the visual group in each activity to obtain the final model, as presented in Table 1. In addition, the analysis indicated that the analytic group engaged in more skipping during the modelling phases than did the visual group. The groups' modelling processes are presented for the giant's shoes activity (Blum & Borromeo Ferri, 2009) only, due to space limitations. The modelling process of the analytic group can be split into three modelling cycles: the first cycle (C1.1, C1.2, C1.3, C1.4), the second cycle (C2.1, C2.B), and the third cycle (C3.1, C3.B, C3.3, C3.C, C3.4, C3.D, C3.5). Table 3 presents the modelling process and Figure 2 illustrates the modelling route of the analytic group.

Model-ling cycle	Process	Explanation
The first cycle ⟶	C1.1	Understanding the situation, simplifying through mathematizing by think about the relation between the width and the length of shoes 5.29: 2.37
	C1.2	Working mathematically: Find the ratio between the width and the length of one students; 32:12
	C1.3	Mathematical result: The ratio 8:3
	C1.4	Validating: Not helpful in solving the situation
The second cycle - - ▸	C2.1	Return to the situation, simplifying through mathematizing: Find the ratio between the length of student's shoes and her height.
	C2.B	Mathematical model: The height of person is four times the length of their shoes.
The second cycle ……▸	C3.1	Return to the situation, simplifying through mathematizing: Find the ratio between the average of their length of their shoes.
	C3.B	Mathematical model: The length of person is five times the length of shoes
	C3.3	Applying the models: 5.29*5
	C3.C	Mathematical results, the height of the giants is 26.45.
	C3.4	Interpreting to reality, it is almost 27 m
	C3.D	Realistic results 27 m
	C3.5	Validating the results in the situation, 27 m

Table 3: Modelling Process of the Analytic Group in The Giant's Shoes activity

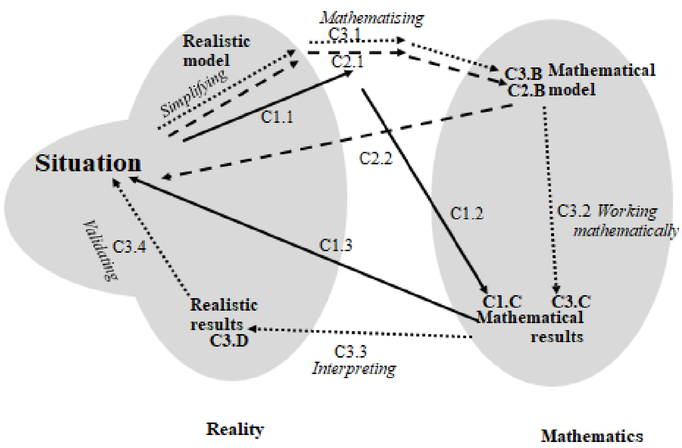


Figure 2: Modelling routes of the analytic group in the Giant's Shoes Activity

The visual group engaged in two modelling cycles: The group began with simplifying the situation through the use of drawing; they tried to draw a figure of shoes through their simplification to yield a real model (A) and thought about the numerical relationship between the giant's height and the length of his shoes, and this relation would be equivalent for ordinary people (C1.1); they began mathematizing by ordering their own shoe length and individuals' height measures, and the ratio between these measurements were recorded on a table they constructed (C1.2); they then elicited a mathematical model, indicating that the ratio between the length of the shoes and the height resembles the ratio of their own measures (C1.B), applied the results (C1.3), and each student received mathematical results resembling his\her ratio, they received different results because each had a different ratio (C1.C); thus, these results didn't resolve the problem (C1.4). The second cycle began with a mathematical model, comprising the average of the group's ratio calculations (C2.B), they applied it (C2.3) and received numerical results 32 (C2.C); this result was then transformed to a realistic result, indicating the giant's height as 32m (C2.D); they accepted this result (C2.5). Figure 3 illustrates the modelling route of the visual group.

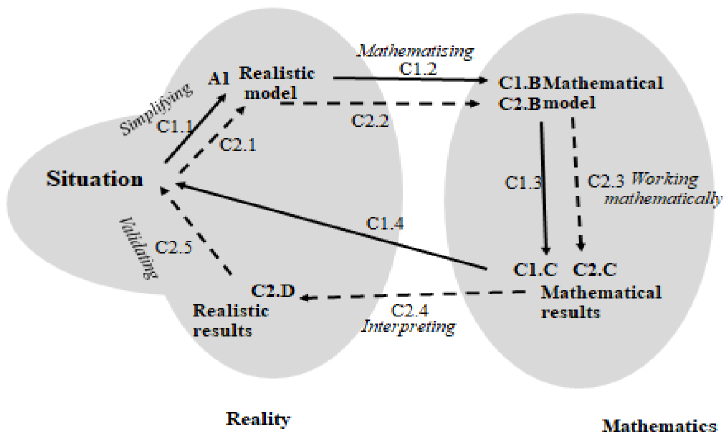


Figure 3: Modelling cycle of the visual group in the Giant's Shoes activity

DISCUSSION

The current study examined the modelling process and routes of two groups of eighth-grade students, an analytic thinking style group and a visual thinking style group. The findings revealed major differences in the two groups' modelling processes. The analytic group tried to simplify the three activities by mathematizing them, while the visual group tried to simplify the activities by drawing and illustrating. In addition, the findings revealed differences in the mathematizing process and in the illustration of the mathematical model. Upon examining the features of the process characterizing the analytic group when engaging in modelling activities, they were found to be similar to features activated in solving routine word problems as Klein (cited in Borromeo Ferri & Kaiser, 2003) reported that students having an analytic thinking style were more likely to search for structures, patterns, or formulas and their application, or briefly operate with formulas. According to the modelling cycle we identified that analytic group engaged in more skipping of modelling phases: In the three activities, they skipped the real model, while the visual group addressed this phase. These findings supported Borromeo Ferri's (2012) findings, she indicated that when analytic thinkers deal with modelling tasks, they preferred to change the real-world situation to a mathematical model and operated in a formalistic manner, while visual thinkers thought more in terms of the real world rather than of formal solutions, thus tending to present their thinking by means of pictures and drawings.

Finally, teachers' awareness of students' thinking styles can have an important role in designing effective interventions. We suggest expanding our work by examining more than a single group from each style in order to collect more information about modelling processes and routes of students with different thinking styles.

References

Blum, W., & Borromeo Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt?. *Journal of mathematical modelling and application, 1*(1), 45-58.

Blum, W., & Leiß, D. (2005). "Filling Up"-the problem of independence-preserving teacher interventions in lessons with demanding modelling tasks. In M. Bosch (Ed.), Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education (CERME 4) (pp. 1623-1633). Sant Feliu de Guíxols, Spain: Fundemi Iqs, Universitat Ramon Llull.

Borromeo Ferri, R. (2007). Modelling problems from a cognitive perspective. In C. Haines et al. (Eds.), Mathematical modelling: education, engineering and economics (pp. 260–270). Chichester: Horwood.

Borromeo Ferri, R. (2010). On the influence of mathematical thinking styles on learners modelling behavior. *Journal fur Mathematik- Didaktik, 31*(1), 99-118.

Borromeo Ferri, R., & Kaiser, G. (2003). First Results of a Study of Different Mathematical Thinking Styles of Schoolchildren. In Leone Burton (Ed.) Which Way in Social Justice in Mathematics Education? (pp. 209-239). London: Greenwood

Campbell, K. J., Collis, K. F., & Watson, J. M. (1995). Visual processing during mathematical problem solving. *Educational Studies in Mathematics, 28*(2), 177-194.

Cakan, M. (2000). Interaction between cognitive styles and assessment approaches. Unpublished doctoral dissertation, Louisiana State University, USA. Baton Rouge.

Lesh, R., & Doerr, H.M. (2003). Foundations of models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H.M. Doerr (Eds.), Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching (pp. 3–33). Mahwah, NJ: Erlbaum.

Glaser, B. G., & Strauss, A. L. (1967). Discovery of grounded theory: Strategies for qualitative research. Chicago: Aldine.

Lowrie & M. A. & Clements, K. (2001). Visual and nonvisual processes in grade 6 students' mathematical problem solving. *Journal of Research in Childhood Education, 16*(1), 77-93,

Monga, A. & John, D. (2007). Cultural Differences in Brand Extension Evaluation: The Influence of Analytic versus Holistic Thinking. *Journal of consumer research*, *33*(4), 529- 536.

Sternberg, R. J. (1997). Thinking styles. New York: Cambridge University Press.

Sternberg, R. J., & Zhang, L. F. (2005). Styles of thinking as a basis of differentiated instruction. *Theory into practice, 44*(3), 245-253.

A DUAL EYE-TRACKING STUDY OF OBJECTIFICATION AS STUDENT–TUTOR JOINT ACTIVITY APPROPRIATION

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The study develops late-Vygotsky's approach of the learning process as a progressive appropriation of an irreducibly collaborative student-tutor joint activity. Combining videography with dual eye-tracking data, we demonstrate how objectification begins with recurrent dialogue between a student and a tutor and continues with the student's egocentric speech that maintains the structure of the collective behavior, allowing the student to regulate her emerging mathematical ideas. The multimodal flow of conversation is conjoined and structured within the student-tutor dyad's joint visual attention that later transforms into mental joint attention, ready to be restored back to visual joint attention in case of difficulty or misunderstanding.

INTRODUCTION

Following the development of semiotic studies in mathematics education, we observe the shift from the consideration of mathematics education as an acquisition of rules and semantics of different ready-made semiotic registers, and the study of translations between them (Duval, 2006), towards an analysis of teaching-learning processes as an *in vivo* student-teacher activity that involves embodied interaction and the initial formation of signs for a student (Roth & Jornet, 2017). This shift towards processes of sign formation provokes us to draw some connections with the research on early childhood speech genesis. According to Bruner (1975), who was inspired by Vygotsky's ideas (Luria, 1986), a child's speech develops during joint activity with an adult and consequently the topic/comment structure of sentences repeat the structure of the speaker's attention within their activity. Joint attention was introduced by Bruner as an important mechanism of speech genesis. Claiming that the development of mathematical semiotic means is similar to speech development, we analyze joint attention between a tutor and a student through the use of dual eye-tracking. In this article, we pay attention to the ways joint attention is sustained during emerging mathematical meaning within dialogue and to the transition from joint visual attention to joint mental attention and back again.

THEORETICAL FRAMEWORK

The description of mathematical knowledge and discourse as multimodal is one of the most reliable and certain achievements of mathematics education research. It has been shown that visual inscriptions (Radford, 2010), gestures (Shvarts, in press) and movements (Abrahamson, Lee, Negrete & Gutiérrez, 2014) may appear to be ambiguous, and students need to be educated to perceive them in an adequate way. Once

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considered an integration of different semiotic systems and a system of translations between them (Duval, 2006), multimodality is now believed to precede the objectification of mathematical signs (Jornet & Roth, 2015; Shvarts, in press).

How does a student become able to structure and then to objectify multimodal flow in a cultural way? On the one hand, the student needs to be involved into the social practice (Radford, 2010). On the other hand, active constitution of the meaning is required by the student (Roth, 2008). Joint activity between a student and a teacher is neither limited to involvement in ready-made social practices nor to the active objectification of mathematics by the student herself. Teaching-learning processes involve mutual transformation (Roth & Radford, 2010), wherein each participant is an agent in the joint activity. Analyzing student-tutor interactions in embodied activity, Flood and Abrahamson (2015) demonstrated how the tutor repeats either the gesture or verbal entity of the student while developing another part of the semiotic node, verbal expression or gesture. Thus the irreducible character of joint activity to individual endeavors was demonstrated.

Following late-Vygotsky's ideas, Roth and Jornet noticed that the mediation between a teacher and a student by semiotic means is inappropriate in the theorization of learning, since there are no means to support this transfer: the prospective semiotic vehicles are in the process of constitution during learning (Roth & Jornet, 2017). The student and the teacher fall into a co-constitutive process of teaching-learning, where intersubjectivity supersedes individual minds. Introducing a novel (for his time) idea in speech development investigations, Bruner (1975) claimed that the structure of speech acts is determined by the structure of attention and activity. He proposed joint attention as a situation in which a child's and an adult's attentions are coordinated and the adult slowly develops the child's ability to understand speech through persistent interpretations of the child's intentions and their focus of attention. Following enactivist and phenomenological approaches to joint attention (e.g. Hutto, 2011), we suggest that these 'interpretations' by an adult should not be considered as a voluntary conscious conclusion concerning the child's activity, but rather the natural involvement in a joint action with the child. It is in multimodal embodied collaboration with others that cultural meanings and their semiotic vehicles emerge and then function within individual minds, thus being *appropriated* from joint activity into *intra*individual functions. We deliberately avoid the term *internalization* as it has become associated with the idea of individual constructions rather than the progressive transformation of social activity from *inter*individual spaces to an *intra*individual space (Roth & Jornet, 2017). Thus transforming from external speech between people through egocentric speech of a student with her own to inner speech, semiotic means keep their intersubjective nature and structure of joint attention. The first research question of this paper concerns the communicative elements that help the tutor-student dyad to sustain joint attention within a teaching-learning activity, allowing them to share cultural meaning as a coordination of multimodal resources in semiotic nodes. The second research question explores the progression from joint visual attention to joint mental attention during the collaboration.

METHOD AND LEARNING MATERIAL

Dual eye-tracking technology

Novel dual eye-tracking technology that allows synchronous recording of two participants' eye movements opens new horizons in the analysis of joint visual attention in co-actions. We used Pupil-Labs head-mounted eye-trackers to provide ecological settings where two people share a common space and can gesture and discuss manipulations on a shared monitor (see Figure 1a). The original software was elaborated for our analysis such that the video from the screen could be overlaid with two gaze paths, thus providing innovation compared to the static images under the gaze paths in previous studies (Shvarts, in press; Lilienthal & Schindler, 2017; Schneider et al., 2016). This technical solution allows a combination of qualitative frame-by-frame analysis of gaze paths with videography of gestures and verbal expressions.

All data were analyzed following the principles of the Vygotskian semiotic approach (Radford & Sabena, 2015) in search of meaningful patterns of *inter*individual regulation across student/tutor gaze paths and the dyad's multimodal utterance. Thus, the unit of our analysis was the intersubjective system of a tutor and a student in a teaching-learning activity rather than their individual teaching and learning processes.

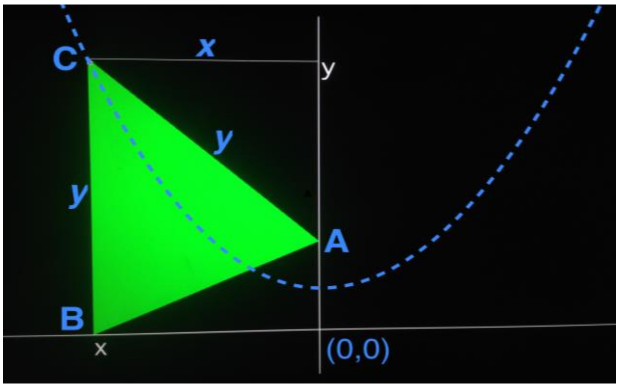

Figure 1a, 1b: Dual eye-tracking experimental setting (a), and (b) Stage 2 of the interactive learning material. Vertex C is manipulated by the student. Markers X and Y run along the axes. The triangle ABC is green when CB=CA. All blue inscriptions are given for clarity aims and are not represented for a student.

Learning material

We elaborated a computer-based interactive activity following the principals of action-based design (Abrahamson, 2014). The activity involves disclosing a parabola as a locus of points which are equidistant from a straight line (directrix) and a separate point (focus). This topic has been proven as an insightful source for the development of mathematical conceptualization from an embodied activity investigation (Brown, Heywood, Solomon, & Zagorianakos, 2013). There is a triangle on a screen (see Figure 1b), formed by Vertex A which is fixed to a point, Vertex B which runs along a horizontal line, and Vertex C which the participant can manipulate. While doing so, the student receives feedback from an interactive system via the changing color of a triangle: the triangle turns green when Vertex C is equidistant from the other two vertices (thus, the triangle is isosceles). Consistent with the idea that semiotic symbols need to be removed from an initial embodied activity (Abrahamson, 2014), there is only triangle at Stage 1 of the task and the students are required to "move the triangle in a way that it always will be green" and then to reflect on the rule that determines its color. Having objectified the triangle as isosceles, the student is forwarded to Stage 2 that introduces some mathematical symbolization. There are axes of the Cartesian plain, Vertex B of the triangle is marked X, and the projection of the manipulated Vertex C on the ordinate axis is marked Y. These symbols run along the axes while the triangle is manipulated. The students must find the formula of the curve that would be drawn by the manipulated vertex of the green triangle, thus expressing *Y* in terms of *X*. For this purpose they need to solve an equation that expresses the distance AY in two ways: using the right triangle AYC with the sides equal to *Y, X* and using the constant distance from the point (0,0) to Vertex A. All gestures are made with a thin pointer.

Four student-tutor pairs took part in the research, with each tutor passing through the corresponding learning prior to the study. The analysis of Stage 1 showed high temporal and spatial coordination of the tutors' perception with the students' actions, that allowed sustained joint visual attention during the students' manipulations of the triangle. Below we provide the analysis of one pair's interaction during Stage 2.

RESULTS AND DISCUSSION

The entire Stage 2 took 10 minutes and 18 seconds; in the first 3 minutes and 27 seconds there was a dialogue between the student and the tutor featuring an exchange of phrases and gestures. Then the tutor realized that the student needed broader support and switched to the mode of explanation. During the rest of the activity there are alternating episodes of the tutor's explanations and the student's thinking aloud. Below we provide parts of the transcription of these consequential episodes. The numbers in brackets are pauses, measured in seconds.

Let us focus on the analysis of the audio communication. There is a repeated pattern of the verbal utterances: one partner makes a statement and as soon as it is finished the other participant confirms it with a positive interjection such as "Yes" or "Ugu" (these confirmations are marked by bold in the transcriptions). This pattern is present almost during the whole activity and we could distinguish 39 positive interjections in about 7 minutes and 30 seconds of activity.

The analysis of the pauses between phrases shows that these confirmations follow the statement in a regular way (M = 613 ms, SD = 341 ms after excluding three outlying cases beyond two standard deviations), and a qualitative analysis reveals their essential communicative role. The long pause prior to the confirmation (3.2 seconds, Turn 18) inform the partner that there is a lack of understanding and the tutor prepares an additional explanation (Turn 19).

09 T: (0.3) And consequently we can say that all these distances, all these distances are [equal to] Y.

10 S: (1.2) Which distances, once more?

11 T: (0.4) This one is (0.2) Y ((Figure 2a))

12 S: (0.4) Yes =

13 T: = and this one is also Y

14 S: (0.9) Oh, yes.

15 T: (1.0) Thus we know this side ((Figure 2b))

16 S: (0.6) Ugu

17 T: (0.3) We know (0.8) the constant. (1.4) And <now> we can try somehow express <a a a> (1.2) this one

18 S: (3.2) ((the tutor looks at the eyes of the student)) Ugu

19 T: (0.3) using (0.8) right triangle

20 S: (0.6) <A a a>! using right triangle

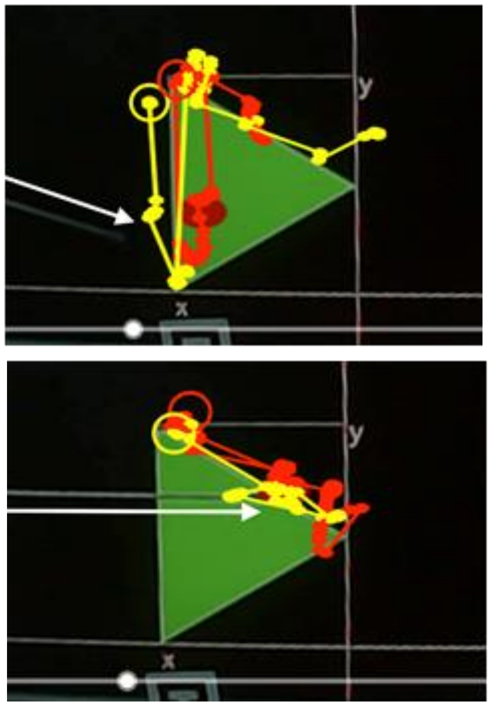

Figure 2a, 2b: Visual joint attention to the sides of the triangle while the tutor moves the pointer along them. Here and further the student's gaze path is yellow, the tutor's gaze paths is red and the white arrow is directed along the pointer

It is not a deliberate signal about a mistake since there was no question from the tutor and no interpretation or answer by the student. It is rather a change in the regular pattern of the activity, namely the absence of an expected confirmatory reaction that directs the tutor towards preparation of an additional guidance. These confirmatory interjections signify that the two participants keep joint attention during the entire dialogue as they refer in the dialogue to the same entities, while eye-tracking data reveal that the dialogue is accompanied by joint *visual* attention (see Figures 2a and 2b) as it joins the gesture, the verbal utterance and the visual inscription. Concordant with our previous findings (Shvarts, in press), the coincidence of different presentations appears as an active coordination between modalities. For example, Figure 2a shows that eye movements do not follow but rather anticipate the gesture along the side of the triangle thus joining visual and gestural expressions in anticipative perception.

The second piece of the transcription reveals the spontaneous thinking aloud of the student.

25	S:	(0.6) A sum of cathetuses (0.4) <is equal> to a square of hypotenuses (0.6) So it appears X-squared <is equal> (1.1) No (0.5) A moment (0.4) It appears Y-squared (0.5) equal X-squared plus unknown-squared ((Figure 3))
26	T:	(0.2) Ugu
27	S:	(0.6) So unknown-squared equal the root… >May I also have a pointer?< ((the tutor gives the pointer to the student))
28	T:	(0.8) Surely (0.9)
29	S:	((root)) of this one, so we are looking for this one side ((Figure 4a))
30	T:	(0.4) Ugu
31	S:	(0.3) So it appears this [side] is equal (0.8) a square root from (0.6) the square of this one [side] ((Figure 4b)) minus, I mean (0.15) square of X minus square of (1.0) Y. ((during second part of the sentence the student gives the pointer to the tutor and the tutor takes it))
32	S:	(0.9) No.
33	S:	(0.6) Yes. (0.6) S*quare of Y*
34	T:	(0.4) Aah… ((intensive breath in))
35	S:	(.) minus square of X
36	T:	(0.1) Yes! Yes! (0.5) You may write it down.

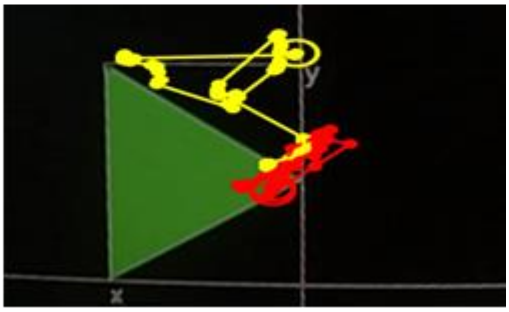

Figure 3: The absence of joint visual attention

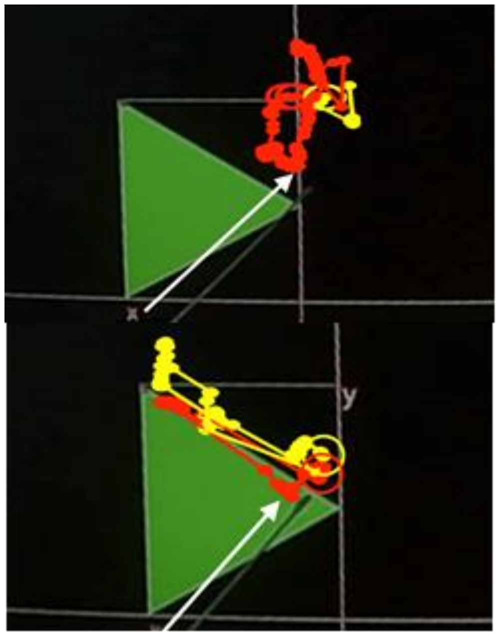

Figure 4a, 4b: Joint visual attention is restored when the student uses the pointer

This time it is the tutor who confirms each of the student's statements with an interjection. The pauses prior to the interjections are very short (0.2 to 0.4 seconds; see Turns 26 and 30), and again the absence of the next confirmation (Turn 32, the confirmation did not appear in 0.9 seconds) may signify for the student that there is a mistake or miscommunication. The statement–confirmation cycle is broken and the student herself immediately restores it (Turn 32). She, rather than the teacher, answers "No", as if there was a question. Thus the dialogue continues with almost the same regularity (and the student continues it further in Turn 33 with her "Yes"), but within the student herself. The tutor is actively present waiting for the student's final utterance, and expressively confirms it as soon as it is obtained (Turn 36 appears 0.1 seconds after the previous turn).

Thus the structure of the dialogue is kept during both presented episodes, while the relative participation of the student and the tutor is changing. At first the tutor provides the statements while the student confirms them, later the statements are provided by the student and confirmed by the tutor, and at the end the student may take on both roles appropriating the structure of joint activity.

The joint attention undergoes a serious change during the second episode. While continuation of the statement–confirmation cycles suggests that participants maintain attentional focus on the same entities, the video and eye-tracking data demonstrate that the student does not use the pointer and there is no longer any synchrony in relevant areas attendance (Turns 26 and 27; see Figure 4). Thus joint visual attention is transformed to joint mental attention. However, as soon as the student encounters a difficulty (Turn 29) she requests the pointer and restores joint visual attention with the tutor by using gestures (Turns 29 to 31). At the end of Turn 29 the student gives the pointer back to the tutor and at the very same moment produces an inadequate verbalization. Apparently, she do not only wants the tutor to follow her, but also needs the pointer for her own attentional regulation. We suggest that these two intentions are actually the same: by requiring the tutor's attention, she also organizes her own attention towards the visual inscription thus progressing from *inter*individual activity to *intra*individual.

CONCLUSIONS

The analysis of multimodal flow reveals high temporal and spatial coordination of the tutor and the student as they are involved in a teaching-learning joint activity. The statement–confirmation cycles help them to sustain joint attention and a break in the cycles allows restoring the joint attention in case of misunderstanding. We claim that their ability to anticipate their partner's gestures and verbal utterances and to direct the partner by minimal changes in activity (such as longer pauses) confirm the irreducible character of joint activity. While the verbal dialogue is supported by joint visual attention and necessary involves both participants at the beginning, later the student may appropriate the dialogic functions and regulate her visual attention on her own. Objectification is still in process, but the two participants may acquire partial independence: joint mental attention may supersede joint visual attention thus freeing the visual system for other tasks and serving future independence of the tutor's and the student's activity. However, in case of difficulty or misunderstanding, intensive embodied coordination is restored and joint visual attention again guarantees the co-constitution of mathematical meaning.

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References

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction*, 2(1), 1–16.

Brown, T., Heywood, D., Solomon, Y., & Zagorianakos, A. (2013). Experiencing the space we share: Rethinking subjectivity and objectivity. *ZDM - International Journal on Mathematics Education*, 45(4), 561–572.

Bruner, J. S. (1975). The ontogenesis of speech acts. *Journal of Child Language*, 2(1), 1–19.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1–2), 103–131.

Flood, V. J., & Abrahamson, D. (2015). Refining Mathematical Meanings Through Multi-modal Revoicing Interactions: The Case of "Faster." Paper Presented at the Annual Meeting of the American Educational Research Association, 1(c), 29–32.

Hutto, D. D. (2011). Elementary Mind Minding, Enactivist-Style. In Joint Attention: New Developments in Psychology, Philosophy of Mind, and Social Neuroscience (pp. 307–341). United States: MIT Press.

Jornet, A., & Roth, W.-M. (2015). The joint work of connecting multiple (re)presentations in science classrooms. *Science Education*, 99(2), 378–403.

Lilienthal, A. J., & Schindler, M. (2017). Conducting dual portable eye-tracking in mathematical creativity research. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education, Vol. 1, (p. 231). Singapore: PME.

Luria, E.A. 1986. Moi otets A.R. Luria [My Father, A.R. Luria]. Moscow: Gnozis

Radford, L. (2010). The eye as a theoretician: seeing structures in generalizing activities. For the Learning of Mathematics, 30(2), 2–7.

Radford, L., & Sabena, C. (2015). The question of method in a vygotskian semiotic approach. In Approaches to qualitative research in mathematics education (pp. 157–182). Springer Netherlands.

Roth, W.-M. (2008). The dawning of signs in graph interpretation. In L. Radford, G. Schubring, & Seeger F. (Eds.), Semiotics in mathematics education: Epistemology, History, Classroom, and Culture (pp. 83–102). Rotterdam: Sense.

Roth, W.-M., & Jornet, A. (2017). Understanding educational psychology: a late Vygotskian, Spinozist approach. Springer.

Roth, W.-M., & Radford, L. (2010). Re/thinking the Zone of Proximal Development (Symmetrically). Mind, Culture, and Activity, 17(4), 299–307.

Schneider, B., Sharma, K., Cuendet, S., Zufferey, G., Dillenbourg, P., & Pea, R. (2016). Using Mobile Eye-Trackers to Unpack the Perceptual Benefits of a Tangible User Interface for Collaborative Learning. ACM Transactions on Computer-Human Interaction, 23(6), 39–62.

Shvarts, A. (in press). Joint attention in resolving the ambiguity of different presentations: a dual eye-tracking study of the teaching-learning process. In N. Presmeg, L. Radford, W.-M. Roth, & G. Kadunz (Eds.), Signs of signification: Semiotics in mathematics education research. Dordrecht: Springer.

TEXTBOOK EFFECTS ON THE DEVELOPMENT OF ADAPTIVE EXPERTISE

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During the last two decades research on the development of adaptive expertise has gained growing research interest. While a number of studies investigated the effects of different instructional approaches, the state of knowledge regarding the impact of learning resources in this field is quite limited. This study provides new insights into the relations of textbook quality and students' adaptive use of strategies in multi-digit addition and subtraction. By reanalysing longitudinal data of 1404 students from grade 1–3, we found quality discrepancies in the textbooks' opportunities to learn as well as substantial effects of these on the students' actual strategy use. Thus, mathematics textbooks can be regarded as meaningful classroom factor predicting the development of students' adaptive expertise.

THEORETICAL BACKGROUND AND EMPIRICAL FINDINGS

In the last two decades, a vigorous research interest in the genesis of individual adaptive expertise developed in mathematics education and psychology. Although valuable progress has been made, especially by the implementation of reform-based instructional approaches, little is known yet about the role of learning resources in this domain. According to Mullis and colleagues (2012), textbooks can be considered as the most important teaching materials used by primary school mathematics teachers. However, relational or even causal research on textbooks is limited (Fan, 2013), and the findings about effects on students' achievements are inconsistent (van Steenbrugge et al., 2013; Törnroos, 2005).

Adaptive expertise in multi-digit addition and subtraction

Since end of the 1990's, there has been a development towards a broad consensus about the importance of an adaptive or flexible use of strategies in arithmetic computation as a learning goal in primary school mathematics. A comprehensive definition of adaptivity is given by Selter (2009, p. 624):

> Adaptivity is the ability to creatively develop or to flexibly select and use an appropriate solution strategy in a (un)conscious way on a given mathematical item or problem, for a given individual, in a given sociocultural context.

In the context of multi-digit addition and subtraction up to 1000 we distinguish between mental calculation, written algorithms and informal strategies (e.g., Heinze et al., 2009). Here, informal strategies cannot unambiguously be assigned to mental or written methods due to a fluent transition (Fuson et al., 1997). An overview of the most common solution strategies is given in Table 1 (Selter, 2001). Adaptive expertise in

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 179-186). Umeå, Sweden: PME.

this context is indicated by an adaptive use of strategies to find efficient solutions to given arithmetic problems.

Stepwise	Split	Compensation	Simplifying	Indirect Addition
123 + 456 = 579	123 + 456 = 579	527 + 398 = 925	527 + 398 = 925	701-698 = 3
123 + 400 = 523	100 + 400 = 500	527 + 400 = 927	525 + 400 = 925	698+3 = 701
523 + 50 = 573	20 + 50 = 70	927 – 2 = 925		
573 + 6 = 579	3 + 6 = 9			

Table 1: Most common types of strategies for addition and subtraction (the table shows examples for addition; there are corresponding versions for subtraction for all strategies but indirect addition).

First longitudinal studies revealed that adaptive expertise is accompanied with a broader conceptual knowledge or a deeper understanding of base-ten number conceptions (e.g., Fuson et al., 1997). In a row of reform-based teaching experiments, the development of students' strategy use was examined, yielding first insights, like positive effects of an early emphasis on the flexible use of strategies in lessons and of comparing and contrasting the efficiency of different strategies (e.g., Klein, Beishuizen & Treffers, 1998; Rittle-Johnson & Star, 2007). A theoretical framework for the genesis of adaptive expertise was developed by Siegler and his associates (cf. Siegler & Lemaire, 1997). The framework distinguishes four dimensions, for which are shown that changes in any of them can improve speed and accuracy of strategy choice overall. The four dimensions are: *strategy repertoire*, the knowledge of different types of strategies; *strategy distribution*, the knowledge of relative frequencies these strategies are used; *strategy efficiency*, the ability to perform strategies quickly and accurately; and *strategy selection*, the ability to flexibly select a strategy on a given problem.

Nevertheless, empirical studies repeatedly reported a lack of adaptivity in students' actual strategy use (e.g., Selter, 2001; Heinze et al., 2009; Torbeyns & Verschaffel, 2016). Although a shift towards the weight of informal strategies is perceptible in mathematics education, like the adoption in curricula and standards, students still tend to favour one strategy per operation, which is applied to almost each type of problem. One reason for the missing success might be a lack of learning opportunities or at least a lack of quality of learning opportunities in the mathematics classroom.

Textbooks as learning resources

For the evolution of adaptive expertise, learning resources like textbooks could play an important role. As mediator between the official and the implemented curriculum, textbooks translate the abstract curriculum into concrete operations for teachers and students to carry out (Valverde et al., 2001). Thus, as the adaptive use of strategies was adopted in many curricula and standards, textbooks should offer opportunities to learn for this domain. In turn, the textbook content influences the teachers' instruction, as they're the most important learning resource for primary school mathematics teachers (Mullis et al., 2012). In the Trends in International Mathematics and Science Study (TIMSS) 2011 86 % of the German and 75 % of all primary school mathematics teachers declare to use the textbook as a basis of instruction (Mullis et al., 2012). Moreover Krammer (1985) found differences in the implemented teaching practices of teachers using distinct textbooks. Schmidt et al. (2001) found a relationship between the space a topic covers in a textbook and the instructional time teachers dedicate to this topic in the mathematics classroom. Those topics which are not included in the textbooks used are unlikely to appear in classroom (Schmidt et al. 1997). Furthermore, there are some indications that textbooks have an effect on student achievement. Schmidt et al. (2001) found a direct relation between the amount of space allocated to covering a topic and the size of achievement gain on that topic for the eighth grade TIMSS data of the United States. In line with that, Törnroos (2005) showed an effect of the number of opportunities to learn in mathematics textbooks for the TIMSS test on the students' achievement in TIMSS. Using the sample of our study presented below, Niedermeyer et al. (2016) found substantial differences of four different textbooks on students' arithmetic achievement from the end of the first grade till the end of the second grade. On the other hand van Steenbrugge et al. (2013) did not find any differences in the performances of elementary school students using distinct mathematics textbooks. All in all, there are some indications that characteristics of mathematics textbooks affect the student achievement. In the special field of adaptive expertise, some studies suggest such an effect on the students' strategy choice (cf. Fagginger Auer et al., 2016; Heinze et al., 2009). While Fagginger Auer and colleagues showed the relation of the textbook used and the student's strategy profile in a multilevel latent class analysis, Heinze et al. found a relation between the adaptivity of student's strategy use and their textbook's instructional approach to teach adaptive use of strategies.

According to Fan (2013), there is still a lack of evidence-based relational and causal research on the effect of mathematics textbooks. Although the studies reported may give valuable indications, their scope remains cross-sectional (except for Niedermeyer et al., 2016). However, for the investigation of a prolonged process like the development of adaptive expertise, there's a need for longitudinal data. Also, previous research on textbooks is mainly small-scale and often includes textbooks representing different curricula, with the result that the effects of curricula and textbooks are confounded. Furthermore, to our knowledge, there are neither any qualitative textbook analyzes developing a scale to classify textbook quality, nor any studies examining the effects of textbook quality on student achievement (Fan, 2013). Most existing studies of textbooks' contents refer to comparisons between books, often between those of different countries (Fan, 2013).

RESEARCH QUESTIONS AND METHODS

The present study aims at contributing to the mentioned research gaps. In contrast to most former investigations on textbooks effects, we use longitudinal data (grade 1-3) of a large sample. By comparing textbook series following the same curriculum, we can circumvent the problem that curricula and textbooks are confounded. Another research desideratum we address is the development of a textbook quality scale and the examination of the effects of textbook quality on student achievement. Since a global measure for textbook quality seems barely feasible and adaptive expertise is an important learning goal, this study focuses on the subdomain of adaptive expertise. Consequently, we address the following research questions: (1) Do textbook series differ with respect to the quality of their opportunities to learn for adaptive expertise? (2) Which effect does the textbooks' quality concerning adaptive expertise have on students' adaptive expertise at the end of grade 3?

Research context

The basis for our analysis is an existing data set from a large three-year longitudinal study with primary school students from one federal state in Northern Germany. The overall sample consists of 2330 students from 127 classes. It comprises student data from the beginning of grade 1 (at the age of 6 years) to the end of grade 3. The original aim of the study is to address students' development in arithmetic. Of this sample, about 1700 students from 82 classes use one of the four most common mathematics textbooks: "Denken und Rechnen", "Einstern", "Flex und Flo" and "Welt der Zahl". The distribution of the classes over the textbooks is relatively even. Our analysis is based on the subsample of 1404 students from the 82 classes, who use one of the four textbooks and worked on the tasks examining adaptive expertise at the end of grade 3.

Instruments, data collection and analysis

For the purpose of a quality-based scale of the four textbook series we derived categories from the previously mentioned dimensions of adaptive expertise by Siegler and Lemaire (1997). Since strategies for multi-digit addition and subtraction are taught in grade 2 (number domain up to 100) and grade 3 (number domain up to 1000), we analyzed both books of each series by three independent and trained persons on the basis of the categories derived. A uniform scoring was reached by consensus method. In each category, we set up a ranking and compiled a relative overall scale of the books' quality for both grades by weighting each category equally. The following four categories were derived and scored: *(1) Strategy repertoire*: 0 points if a strategy was not treated, 1 point for incidentally introduced strategies (e.g., as a "trick"), 2 points for explicitly introduced strategies, and 3 points if explicitly introduced strategies were additionally illustrated in different representations; (2) *Strategy distribution*: 0 points for strategies not introduced or introduced only once (as this is included in the preceding category), 1 point for strategies introduced and additionally presented by means of another task or problem, and 2 points for strategies introduced and additionally presented more than once by means of other tasks or problems; (3)

Strategy efficiency: Since strategy efficiency evolves and enhances by growing experience and practice, our analysis criterion would have been practicing problems and task, to foster this ability. As we didn't find any variance between the textbooks regarding their amount of exercises, we did not include strategy efficiency as a category to our analysis; (4) *Strategy selection*: Former research has shown that an explicit comparison of solution strategies is conducive for an adaptive use of strategies (cf. Rittle-Johnson & Star, 2007). Therefore, we assessed dichotomously whether exercises explicitly demand a strategy comparison as a systematic component (i.e., a recurring content) of the books, or not.

Student and teacher data were collected by different tests and questionnaires. Data for controlling the learning prerequisites of the students (basic numerical skills, basic language skills, general cognitive abilities) were measured with approved standardized instruments at the beginning of grade 1. Data for the individual learning progress were collected at the end of grade 1 and 3 with grade-specific arithmetic tests. The arithmetic test at the end of grade 1 was scaled using Item Response Theory. The arithmetic test at the end of grade 3 included four problems (482+218, 473+398, 381-99, 702-698) each suggesting the use of specific strategies as efficient solutions (e.g., indirect addition for 702-698). For each problem, the written calculations or notes of the students were coded by trained research assistants with partial credit scoring for inefficient (0 points), partly efficient (1 point), or efficient (2 points) strategy use. Purely mentally computed solutions without notes were considered as the use of an (internalized) efficient strategy. About 22 % of the solutions with notes were double-coded, providing solid Cohen's κ's of .83–.90. The resulting scale for students' adaptive expertise (0–8 points) shows an acceptable reliability (Cronbach's $\alpha = .71$).

We conducted multilevel analyses which take into account the nested structure of the sample (students in classes). We included the variables for learning prerequisites at school entrance on the individual level and cognitive ability also as aggregated value on the class level (as an indicator of group composition). To account for the arithmetic development we included the grade 1 arithmetic test scores on individual level. Teachers beliefs (whether they're rather constructivist or not), a scale combining teacher experience and qualification, as well as the previously described textbooks quality for grade 2 and 3 were included on class level. Missing data on independent variables were handled by the Full Information Maximum Likelihood method (FIML). Due to sample selection we had no missing data for the adaptive expertise score.

RESULTS

The analysis of textbook quality yielded scores from 10 to 24 points per book regarding the dimension *repertoire*, 4 to 10 points per book regarding *distribution* and 0 to 1 point per book regarding *comparison*. While the books of the series "Welt der Zahl" are ranked first in all categories but one, those of "Einstern" are ranked last with only one exception. The relative overall scales of the books' quality were derived by an equally weighting and averaging of the three category rankings. The final mean ranks range from 1.00 (“Welt der Zahl”) to 2.33 (“Einstern”) in grade 2 and from 1.33 (“Welt der Zahl”, “Flex und Flo”) to 3.33 (“Einstern”) in grade 3. In relation to each other, the opportunities to learn for adaptive expertise of the “Welt der Zahl” series are of the highest quality on this scale, those of “Einstern” of the lowest, the other two are in between. With respect to research question 1 we have found substantial differences in the textbooks’ qualities.

The outcomes of the multilevel analysis with students’ adaptive expertise as dependent variable are shown in Table 2. In model 1 a substantial effect of arithmetic prior knowledge on individual and a small significant effect of the class composition regarding basic cognitive abilities on class level can be seen. The model 2 also includes the teacher variables which have no significant effects. The largest significant effect appears by including the textbook quality scales of grade 2 and 3 in model 3. The effect of the textbook quality in grade 2 is substantial, whereas the textbook quality in grade 3 has no additional effect. The inclusion of the textbook quality led to a substantial increase of the explained variance from model 2 to model 3 ($\Delta R^2 = 11.4$).

	Model 1	Model 2	Model 3
Level 1 (students)			
Basic cognitive abilities	.01 (.01)	.01 (.01)	.01 (.01)
Linguistic preconditions	-.01 (.02)	-.01 (.02)	.01 (.01)
Mathematical preconditions	-.02 (.01)	-.02 (.01)	-.02 (.01)
Arithmetic prior knowledge	.48** (.06)	.48** (.06)	.48** (.06)
Level 2 (class)			
Basic cog. abilities (aggregated)	.14** (.04)	.13** (.04)	.12** (.04)
Arithmetic prior knowl. (aggr.)	-.30 (.18)	-.25 (.18)	-.30 (.19)
Teacher qualification		-.32 (.17)	-.25 (.15)
Teacher beliefs		.04 (.26)	-.01 (.20)
Textbook quality (grade 2)			-.50** (.19)
Textbook quality (grade 3)			-.09 (.16)
Intercept	-1.60 (1.11)	-1.10 (1.24)	-1.89 (1.23)
Explained within class variance	11.1 %	11.1 %	11.1 %
Explained between class variance	24.4 %	31.2 %	42.6 %

* p <.05, ** p <.01,

Table 2: Multilevel regression for individual and classroom covariates and textbook quality on students’ adaptive expertise at the end of grade 3

DISCUSSION

While the results presented are plausible according to the assumption that arithmetic prior knowledge and the quality of opportunities to learn are predictors for students’ adaptive expertise, they provide new insights regarding the role of textbooks as learning resources in this field. We have not only shown effects of the textbooks on students' adaptive expertise, but also, that these effects can be explained by the quality of the textbooks' opportunities to learn. The results reveal that a fine-grained and theory-based content analysis of mathematics textbooks is a fruitful approach to understand the impact and effects of textbooks as learning resources. Our findings supplement the results of Schmidt et al. (2001) and Törnroos (2005). Following this approach the outcomes may be replicated for other mathematics domains.

Our results have important implications. The present study shows that students using a certain textbook could be disadvantaged in comparison to students using another textbook. Therefore a textbook permission, based on theory based quality indicators, could lead to an improvement of textbook quality and avoid a disadvantage of students caused by a specific textbook. Furthermore, teachers should be trained how to use textbooks, so they are able to reflect on the quality of textbooks' learning opportunities and, if necessary, compensate inadequate representations of the curriculum.

There are several limitations of our study. Since we reanalysed an existing data set we were not able to administer specific instruments for our research. In particular, the questionnaires do not provide fine-grained data on the implementation of the teaching content or the teacher knowledge. Despite of these limitations the data set has the advantage that it covers a large sample taught by the same curriculum and allows multilevel analysis with an adequate explanatory power. Furthermore we were able to assess and examine the effects of textbook quality on student achievement for a specific domain. Accordingly, we were able to supplement and further develop existing research on the effects of textbooks on students' learning.

References

Fagginger Auer, M. F., Hickendorff, M., van Putten, C. M., Béguin, A. A., Heiser, W. J. (2016). Multilevel latent class analysis for large-scale educational assessment data. Exploring the relation between the curriculum and students' mathematical strategies. *Applied Measurement in Education, 29*(2), 144-159.

Fan, L. (2013). Textbook research as scientific research. Towards a common ground on issues and methods of research on mathematics textbooks. *ZDM: The International Journal on Mathematics Education, 45*(5), 765-777.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I. Carpenter, T.P., Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130-162.

Heinze, A., Marschick, F., Lipowsky, F. (2009). Addition and subtraction of three-digit numbers. Adaptive strategy use and the influence of instruction in German third grade. *ZDM: The International Journal on Mathematics Education, 41*(5), 591-604.

Klein, A. S., Beishuizen, M., Treffers, A. (1998). The empty number line in dutch second grades. Realistic versus gradual program design. *Journal for Research in Mathematics Education, 29*(4), 443-464.

Krammer, H. P. M. (1985). The textbook as classroom context variable. *Teaching and Teacher Education, 1*(4), 273-278.

Mullis, I. V. S. (2012). *TIMSS 2011 international results in mathematics*. Chestnut Hill, Mass.: TIMSS & PIRLS International Study Center.

Niedermeyer, I., van den Ham, A-K., & Heinze, A. (2016). Effects of textbooks on mathematics teaching and learning in German primary schools. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (pp. 363-370). Szeged (HU): PME.

Rittle-Johnson, B., Star, J. R. (2007). Does Comparing Solution Methods Facilitate Conceptual and Procedural Knowledge? An Experimental Study on Learning to Solve Equations. *The Journal of Educational Psychology, 99*(3), 561-573.

Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H. A., Wiley, D. E., Cogan, L. S., Wolfe, R. G. (2001). *Why schools matter: A cross-national comparison of curriculum and learning*. San Francisco: Jossey-Bass.

Schmidt, W. H., McKnight, C. C., Valverde, G. A., Houang, R. T., & Wiley, D. E. (1997). *Many visions, many aims: A cross-national investigation of curricular intentions in school mathematics.* Dordrecht: Kluwer.

Selter, Ch. (2009). Creativity, flexibility, adaptivity, and strategy use in mathematics. *ZDM: The International Journal on Mathematics Education, 41*(5), 619-625.

Selter, C. (2001). Addition and subtraction of three-digit numbers. German elementary children's success, methods and strategies. *Educational Studies in Mathematics, 47*(2), 145-174.

Siegler, R. S., & Lemaire, P. (1997). Older and Younger Adults' Strategy Choices in Multiplication. Testing Predictions of ASCM Using the Choice-No-Choice Method. *Journal of Experimental Psychology: General, 126*(1), 71-92.

Torbeyns, J., & Verschaffel, L. (2016). Mental computation or standard algorithm? Children’s strategy choices on multi-digit subtractions. *European Journal of Psychology of Education, 31*(2), 99-116.

Törnroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation, 31*(4), 315-327.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book. Using TIMSS to investigate the translation of policy into practice through the world of textbooks.* Dordrecht: Kluwer Academic Publishers.

Van Steenbrugge, H., Valcke, M., & Desoete, A. (2013): Teachers views of mathematics textbook series in Flanders. Does it (not) matter which mathematics textbook series schools choose? *Journal of Curriculum Studies, 45*(3), 322-353.

BRAZILIAN HIGH SCHOOL TEXTBOOKS: MATHEMATICS AND STUDENTS' SUBJECTIVITY

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We analyze the nationally approved high school mathematics textbooks in Brazil from a perspective of governmentality and subjectivity, important notions in recent socio-cultural-political studies of mathematics education. A Foucauldian discourse analysis on how financial mathematics and interdisciplinarity are displayed in the books was carried out. We show how the mathematics is entangled with ideas of the utility of mathematics and moral directions of behavior on how to become a good capitalist, consuming and caring citizen. This points to how the mathematics teaching and learning suggested in the textbooks go beyond the mathematical contents, but simultaneously normalize the students' conducts and form their subjectivity.

INTRODUCTION

Mathematics textbook analysis has recently boomed as an area of research given the significance of the textbook as an element to mediate the curriculum and guide teaching and learning (Fan et al., 2013). In socio-cultural-political studies of mathematics education (Planas & Valero, 2016), the textbook has been a way of studying not only the images of mathematics (Dowling, 1998), but also how different types of students are positioned, thus making available different images for student's identity (Doğan & Haser, 2014). From a perspective on the cultural politics of mathematics education (Valero, 2017), textbooks are conceived as important technologies of power, through which ideals of the desired mathematically competent student are put forward. In these ideals, characteristics of mathematics as a school subject as well as of the child as a mathematics learner are articulated. Textbooks as a key element of the curriculum offer strong cultural theses about who the child is expected to become, "making legible and administrable the child as future citizen" (Popkewitz, 2004, p. 5). These theses unfold in the meeting of the student with pedagogical practices, conducting the conduct of students towards becoming particular types of subjects. The question that emerges when analyzing textbooks from this perspective is which are the images put forward about what constitutes the good mathematical learner.

We present some of the results in the Brazilian project "Discursive Networks in Brazilian High School Mathematics Textbooks" (Silva, 2016), which aims at analyzing the constitution of the student through the discursive network in mathematics textbooks. Textbook analysis helps us to understand the mechanisms that operate in the materiality of the textbooks and how they contribute to the fabrication of

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specific kinds of subjectivities in mathematics classrooms. In this paper, we examine the notions of mathematically competent student that emerge in textbooks when two quite current educational ideas are made part of textbooks: the idea that learning *financial mathematics* is important for developing a sense of finances and economy; and the idea that *interdisciplinarity* is a key strategy to bring mathematics into context as a strategy to promote meaning for students. Even though these are two different ideas that the curriculum should use and contribute to, the analysis of textbooks allows to show their connection in the making of cultural theses for the mathematics learner in Brazil.

A FOUCAULDIAN TOOLBOX FOR TEXTBOOK ANALYSIS

One of Foucault's interest was "to create a history of the different modes by which, in our culture, human beings are made subjects" (Foucault, 1982, p. 208). He wanted to understand how discourses produce specific kinds of subjects and how forms of knowledge (including mathematics) influence ways of living in the world. Technologies of power guide our understanding of ourselves and normalize collective practices, leading to the articulation of very specific forms of being 'man' (Walshaw, 2016). In mathematics education, this research approach has contributed to what Planas and Valero (2016) called the *socio-cultural-political axis* that has emerged in the last 10 years of research in PME. Stinson and Walshaw (2017, p. 1412) emphasized the importance of such type of approach in mathematics education, because "such a perspective refutes closure and keeps the possibilities for mathematics teaching and learning open to multiple and uncertain interpretations and analyses". In this sense, these studies provide insights that complement our understanding of mathematics learning processes focused on the contents of mathematics and on learning.

Foucault's toolbox allows to articulate theory and methodology in a way that leaves to the researcher the possibility of building analytical forms of working with empirical materials. For the analysis of textbooks, the theoretical/methodological strategy that we have articulated based on Foucault traces the elements of textbooks that, when repeated systematically, articulate descriptions and narratives about financial mathematics and interdisciplinarity as part of mathematics education. Furthermore, the textbooks offer narratives about how these elements are to be incorporated in children's actions and forms of being a student in mathematics classes. These are to be found in discourse.

Foucault's notion of discourse (Foucault, 1972, pp. 48-49) "is not a slender surface of contact, or confrontation, between a reality and a language (langue), the intrication of a lexicon and an experience". Rather, it is "a group of rules proper to discursive practice. These rules define not the dumb existence of a reality, nor the canonical use of a vocabulary, but the ordering of objects". Therefore, analyzing discourse is about recognizing "practices that systematically form the objects of which they speak". To make visible the discourse that emerges in mathematics textbooks, we use the concept of *statement*, which is "a seed that appears on the surface of a tissue of which it is the constituent element. The atom of discourse" (Foucault, 1972, p. 80). Statements have four characteristics: A referent, a subject, an associated field, and a specific materiality for dealing with things actually expressed. These are repeated and reproduced, and are activated through techniques, practices and social relations (Fischer, 2001, p. 202).

In the next section, we will enter into the context of Brazilian high school mathematics textbooks as a way to explore the practices and positioning of them in relation to the government and the curriculum; and then will proceed with the analysis of the statements that navigate in the textbooks concerning financial mathematics and interdisciplinarity, in order to identify the cultural theses of children that they build.

TEXTBOOKS IN BRAZIL

The corpus of the analysis is the six textbook collections approved by the Brazilian National Textbook Program (BNTP). Each collection contains three textbooks, each one for a high school year. Each collection has different authors and publishers. Eighteen books in all have been analyzed. The BNTP makes a call for the production of textbooks. Proposals from different authors in publishing houses are examined and evaluated according to criteria for quality textbooks in each subject. Since in Brazil there is no national curricular guidelines, these criteria define many elements of the factual mathematics curriculum. High school textbooks assessments occur every three years. If a proposal is approved, the Federal government guarantees the distribution of the textbooks to all Brazilian public schools. Teachers receive a books summary (Brazil, 2014) so that they can choose which collection they want to use in their classes. All the students in the country receive the collection chosen by their teacher. There is a large public investment in textbooks. In 2015, approximately 7.5 million high school books were bought and distributed by the government (Brazil, 2017). This is a lucrative business; therefore, it is important to produce high quality textbooks. Publishers and authors make a big effort to have their textbooks approved. This involves at least two things: (i) writing textbooks that fall within the criteria of the BNTP, and (ii) producing textbooks with a language that is attractive to the teachers, so that teachers choose a given collection.

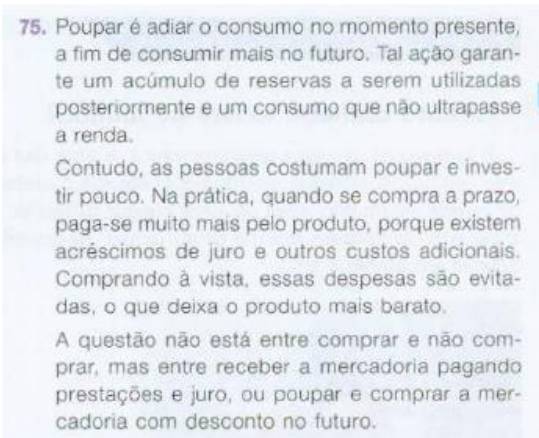
Given this context, analyzing the mathematics textbooks is equivalent to analyze the current, common or most accepted discourse about mathematics education, since these are the materials that guide classroom practices in the country. In other words, these textbooks represent the "order of discourse" for the current mathematics teaching and learning in Brazil. In what follows we will focus on how the notions of learners appear entangled in notions of financial mathematics and interdisciplinarity. We build on the analysis of financial mathematics in the master's thesis of Camila Coradetti (2017), and the analysis of interdisciplinarity in the master's thesis of Ludiane Berto (2017). These two theses were associated to the project "Discursive Networks in Brazilian High School Mathematics Textbooks". We thank them for their work and contribution to our understanding of the overall problem of the research project.

THE CAPITALIST CONSUMING LEARNER

In the 6 textbook collections, there appears a specific chapter called "Financial Mathematics". The emergence of this particular element within the discursive network of the textbooks is related to the criterion of the BNTP to offer contextualization for mathematics that connects to everyday practices and social phenomena of relevance for the population (Brazil, 2014).

Figure 1 introduces financial mathematics with a situation of buying a Smartphone that costs R$ 1299,00, to be purchased in 1 or 12 equal payments. The buyer thinks that she already has saved R$ 200,00. The book announces that "the knowledge of simple financial operations such as loans, financing, discounts, interest rates and investment income are of great importance for a full citizenship". Some of the elements in this presentation —costs, savings, credit, consumption and citizenship— are found in many exercises in the books.

Figure 1: Leonardo (2013, p. 8).



75. Poupar é adiar o consumo no momento presente, a fim de consumir mais no futuro. Tal ação garante um acúmulo de reservas a serem utilizadas posteriormente e um consumo que não ultrapasse a renda.

Contudo, as pessoas costumam poupar e investir pouco. Na prática, quando se compra a prazo, paga-se muito mais pelo produto, porque existem acréscimos de juro e outros custos adicionais. Comprando à vista, essas despesas são evitadas, o que deixa o produto mais barato.

A questão não está entre comprar e não comprar, mas entre receber a mercadoria pagando prestações e juro, ou poupar e comprar a mercadoria com desconto no futuro.

Figure 2: Souza (2013, p. 83, v. 2).

Figure 2 shows an exercise that starts with an explanation about savings provided in terms of possibilities for future consumption. The idea that consumption and expenses should not surpass income appears repetitively. The situation of credit is presented: "The issue is not between buying or not buying, but between getting the merchandise paying for credit and interests, or buying with a discount in the future". The problem continues with a situation of buying a LED TV with different forms of credit.

"Financial planning" is a concept found in an activity that proposes the elaboration of a spreadsheet for the control of a family's finances. Finances are compared to health: Taking care of finances is like taking care of health and being happy. "It may seem simple, but it requires planning and caution" (Leonardo, 2013, p. 23). The activity asks children in groups to make a budget of income and expenses for a family.


Pesquisa e ação Planejamen

Ter uma vida financeira saudável e equilibrada pode parecer simples, mas requer planejamento e cautela. E a receita para isso não é nada complicada: basta que os gastos da família sejam menores ou iguais à renda familiar. Mas, como saber se a renda familiar é suficiente para os gastos realizados ao longo de um mês?

Vamos elaborar uma planilha de planejamento financeiro para uma família. O controle financeiro se dá no equilíbrio entre o consumo (gastos essenciais fixos, gastos variáveis e gastos imprevistos) e a renda familiar.

Figure 3: Leonardo (2013, p. 23).

The overall analysis showed that many activities had two options for the student to make the best choice, without a discussion about the need to carry out such consumption. Even when there was guidance on the importance of saving money, the goal of accumulating funds was almost always to buy something in the future. For instance: “saving is delaying consumption at the present time in order to consume more in the future” (Souza, 2013, p. 83) and “buying a car, owning the house or carrying out the dream trip are achievements that usually require a lot of work and investment time” (Souza, 2013, p. 58). The action of consuming is part of becoming a citizen. This is present in explicit formulations about responsibility and the justification for the importance of paying taxes and the existence of tax systems. In addition, there are explanations about the national financial system and the importance of consuming products to activate the economy, generate jobs, and contributing to the country’s economic growth. Information in these texts is presented numerically, and students are asked to calculate with those numbers and to use certain procedures in the calculations.

In the textbooks, there is an articulation of tasks in which mathematical concepts are presented and procedures are to be performed, images that appeal to consumption of electronic devices and other products of importance to the consumer and to a modern society, and moral rules about being a caring, responsible and happy citizen individually and in the family. There are three statements that circulate in the textbooks concerning financial mathematics: (i) it is necessary to instruct for good decision-making, (ii) investment and savings are practices for capital accumulation, (iii) and citizenship is linked to consumer training.

THE CARING LEARNER

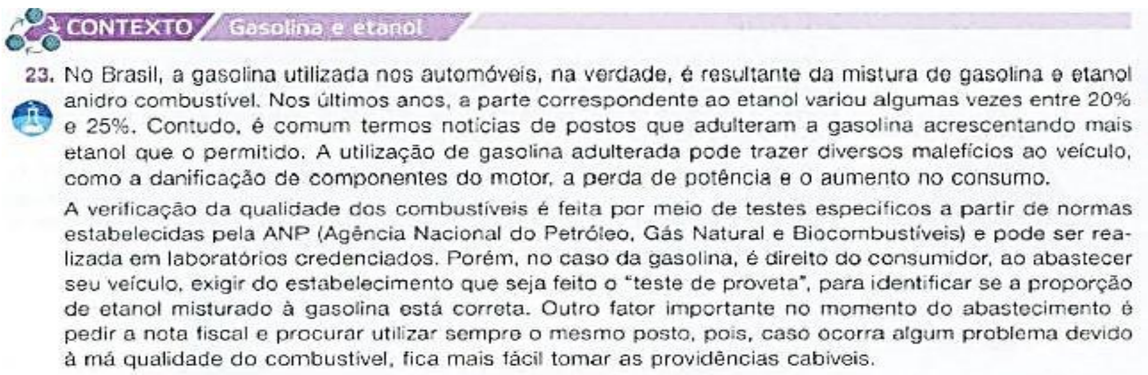
In the evaluation guidelines of the BNTP a criterion is interdisciplinarity. Therefore, this element appears explicitly in almost each section and topic in textbooks. In some problems, there is the mention to the disciplines within which mathematics is applied. Justifications about the importance of interdisciplinarity are explicitly connected to citizenship. Mathematics as a tool to solve problems and make models in the situations of the problems are strongly connected to moral rules about what is good and desirable as both a mathematical behavior and the behavior of the person.

There are interdisciplinary activities with geography, biology, chemistry and other disciplines in the curriculum. In this one, the sustainable use of water is the focus (Figure 4). Information about the use of water is presented in volume of water or use of volume of water per unit of time. "Not to leave the faucet leaking avoids a water waste of approximately 50 lt. of water per day". The following images not only indicate amounts of water use but also bring recommendations of behavior such as "avoid long showers", "using a bucket" for washing the car instead of a hose.

In other types of activities topics such as obesity appear a similar entanglement between explanations of the subject —how fat tissues are built in the body—, with mathematical models and calculations —calculation of corporal mass index and the understanding of "normal weight, under-weight and overweight—, and moral instruction on keeping a healthy weight through the regulation of consumption and burning of fat.

Figure 4: Souza (2013, p. 62, v.1).

In other types of exercises with chemistry, the composition of gasoline in Brazil is discussed (Figure 5). Gasoline quality control and how consumers should pay attention to the gasoline they use in their vehicles is the focus: "be a good consumer and do not be fooled" is part of the explanation of the importance of mathematics and chemistry in conducting a test to determine the amount of ethanol in gasoline. Body care activities have also been found as orientations for food education, associated to incentives for the practice of physical exercises, through explanations that value the importance of certain habits to build a healthier life. Information is presented numerically, and the problems prompt students to use certain procedures on the information to answer to the questions posed in the context of the problems. In other words, it became evident the articulation of mathematics to morals about the good life of citizens.


CONTEXTO Gasolina e etanol

23. No Brasil, a gasolina utilizada nos automóveis, na verdade, é resultante da mistura de gasolina e etanol anidro combustível. Nos últimos anos, a parte correspondente ao etanol variou algumas vezes entre 20% e 25%. Contudo, é comum termos notícias de postos que adulteram a gasolina acrescentando mais etanol que o permitido. A utilização de gasolina adulterada pode trazer diversos malefícios ao veículo, como a danificação de componentes do motor, a perda de potência e o aumento no consumo.

A verificação da qualidade dos combustíveis é feita por meio de testes específicos a partir de normas estabelecidas pela ANP (Agência Nacional do Petróleo, Gás Natural e Biocombustíveis) e pode ser realizada em laboratórios credenciados. Porém, no caso da gasolina, é direito do consumidor, ao abastecer seu veículo, exigir do estabelecimento que seja feito o "teste de proveta", para identificar se a proporção de etanol misturado à gasolina está correta. Outro fator importante no momento do abastecimento é pedir a nota fiscal e procurar utilizar sempre o mesmo posto, pois, caso ocorra algum problema devido à má qualidade do combustível, fica mais fácil tomar as providências cabíveis.

Figure 5: Context section "gasoline and ethanol" (Souza, 2013, p.121, v.3).

Two main statements that circulate in the textbooks concerning interdisciplinarity and students are: (i) interdisciplinarity contributes to the formation of conscious and politically correct citizen-consumers, and (ii) interdisciplinarity is fundamental to for being able to take care of the self, of the other and of the environment.

CONCLUDING REMARKS

So, what have we found in the textbooks? We have found a lot of activities that constitute a kind of manual for citizenship and morality. Among other things, there is a set of rules that constitute a condition of existence that standardizes and normalizes ways of life. School mathematics is a tool through which the child can acquire concepts, and do calculations while becoming also better citizens. The analyses have showed that there are many characteristics and that the teaching proposed by textbooks goes beyond mathematics, normalizing conducting the conduct of the students.

The students have to be good capitalist consumers to be good citizens and they are supposed to understand the importance of caring form themselves and for others. In this sense, the mathematics curriculum is a powerful instrument to govern people, so the instructions in textbooks operate a process of subjectivation in line with what the government has already explored to regulate actions to maintain order and progress. The mathematics curriculum and mathematics learning, as shown in these textbooks, is not only about mathematics; it is about politics, culture, and power.

References

Berto, L. F. (2017). *Enunciados sobre interdisciplinaridade em livros didáticos de matemática do ensino médio* (master's thesis). Federal University of Mato Grosso do Sul, Campo Grande, Brazil.

Brazil (2014). *Guia de livros didáticos: PNLD 2015 – matemática - ensino médio*. Brasília: Ministério da Educação, Secretaria de Educação Básica.

Brazil (2017, December 11) Fundo Nacional de Desenvolvimento da Educação. *Dados estatísticos do PNLD*. Retrieved from: http://www.fnde.gov.br/programas/livrodidatico/livro-didatico-dados-estatisticos

Coradetti, C. A. L. M. (2017). *Um olhar contemporâneo para a matemática financeira presemte nos livros didáticos do ensino médio* (master's thesis). Federal University of Mato Grosso do Sul, Campo Grande, Brazil.

Doğan, O., & Haser, Ç. (2014). Neoliberal and nationalist discourses in Turkish elementary mathematics education. *ZDM*, *46*(7), 1-11. doi:10.1007/s11858-014-0605-z

Dowling, P. (1998). *The sociology of mathematics education: Mathematical myths, pedagogic texts*. London: Falmer Press.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: development status and directions. ZDM, 45(5), 633-646. doi:10.1007/s11858-013-0539-x

Fischer, R. M. B. (2001). Foucault e a análise do discurso em educação. *Cadernos de Pesquisa*, (114), 197-223.

Foucault, M. (1972). *The archaeology of knowledge*. New York: Pantheon Books.

Foucault, M. (1982). The Subject and the Power. In H. L. Dreyfus & P. Rabinow (Eds.). *Michel Foucault: Beyond structuralism and hermeneutics* (pp. 208-226). Chicago: University of Chicago Press.

Leonardo, F. M. (2013). *Conexões com a Matemática* (2nd ed., Vol. 3). São Paulo: Editora Moderna.

Planas, N., & Valero, P. (2016). Tracing the Socio-Cultural-Political Axis in Understanding Mathematics Education. In A. Gutiérrez, G. C. Leder & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education. The Journey Continues* (pp. 447-479). Rotterdam: Sense Publishers

Popkewitz, T. (2004). The Alchemy of the Mathematics Curriculum: Inscriptions and the Fabrication of the Child. *American Educational Research Journal*, 41(1), 3-34.

Silva, M. A. (2016). Investigações Envolvendo Livros Didáticos de Matemática do Ensino Médio: a trajetória de um grupo de pesquisa. *Jornal Internacional de Estudos em Educação Matemática*, 9(3), 36-54.

Souza, J. R. (2013). *Novo olhar matemática* (2nd ed., Vol. 1 and Vol. 2). São Paulo: FTD.

Stinson, D. W., & Walshaw, M. (2017). "Theory at the crossroads": Mapping moments of mathematics education research onto paradigms of inquiry. In E. Galindo & J. Newton (Eds.), *Proceedings of the39th PME-NA* (pp. 1407–1414). Indianapolis: Hoosier Association of Mathematics Teachers Educators.

Valero, P. (2017). Mathematics for all, economic growth, and the making of the citizen-worker. In T. S. Popkewitz, J. Diaz y C. Kirchgasler (Eds.). *A political sociology of educational knowledge: Studies of ex-clusions and difference* (pp. 117-132). New York: Routledge.

Walshaw, M. (2016). Michel Foucault. In E. De Freitas & M. Walshaw (Eds.). *Alternative theoretical frameworks for mathematics education research: Theory meets data*. New York: Springer.

DESIGNING FOR GUIDED REINVENTION OF MATHEMATICAL CONCEPTS

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In this theoretical paper, I discuss two ways of promoting guided reinvention for mathematical concepts. I begin by discussing our elaboration of the rationale for Freudenthal's construct of guided reinvention. I then use two constructivist constructs, generalizing assimilation and reflective abstraction, to distinguish two types of conceptual development. Finally, I explain and exemplify our approaches for designing instruction to promote guided reinvention through generalizing assimilation and through reflective abstraction. These approaches are potentially complementary to approaches grounded in problem solving.

Guided reinvention is a construct deriving from the work of Freudenthal (1973) that is central to the Realistic Mathematics Education (RME) research and development program (Gravemeijer, 1994). In this article, I review Freudenthal's definition and rationale for guided reinvention. I then extend that rationale, and discuss theoretical approaches to designing for guided reinvention developed by our Learning Through Activity research program (Simon, Kara, Placa, & Avitzur, in press). These design approaches build on two constructivist constructs, one of which LTA researchers elaborated in prior work, and provide instructional approaches that are different than, but complementary to, problem-solving instruction in general, and an RME approach in particular. Thus, inspired by the notion of guided reinvention, LTA has generated instructional design approaches that differ considerably from what is typical of RME.

WHAT IS GUIDED REINVENTION AND WHY IS IT IMPORTANT?

Freudenthal's construct and rationale

Freudenthal proposed the idea of guided reinvention to indicate his hypothesis that under the proper conditions (represented by the term "guided"), students could come to mathematical ideas through their mathematical work ("reinvention") as opposed having the ideas "imposed" (Freudenthal, 1991, p. 47). In guided reinvention, mathematics educators carefully plan instructional situations designed to engage students in mathematical activity that is likely to result in them arriving at the intended understandings.

Freudenthal (1991) argued that guided reinvention is more motivating and enjoyable than traditional school mathematics, that it results in more lasting learning, and that it gives students the experience of "mathematics as a human activity" (p. 47).

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LTA elaboration of the rationale for guided reinvention

In addition to the rationale given by Freudenthal, LTA researchers promote guided reinvention for the following reasons.

First, students build new ideas on the basis of their prior knowledge. Every individual's prior knowledge is somewhat idiosyncratic, even though there may be significant overlap. As a result, the mental process that can build the new concept from a somewhat idiosyncratic conceptual foundation varies from individual to individual. It is virtually impossible to specify the exact parameters of the needed mental activity for even one student, let alone thirty students in a classroom. One solution to this dilemma (guided reinvention) involves creating instructional situations that promote the foundational activity hypothesized to lead to the intended learning, but which allow students some variation in mental process and rate of learning based on their prior knowledge. The LTA approach to guided reinvention engages students in experiences that foster their development, while allowing their learning processes to govern what they focus on, what they wrestle with, and when they make particular abstractions. Note that although the focus here is on adaptation to the idiosyncratic nature of students' prior knowledge, we assume significant overlap in both the prior knowledge and the developmental processes of individuals in a class.

A second reason for utilizing guided reinvention is that the development of a *new* mathematical concept involves the making of an abstraction, that is, construction of a more advanced concept through coordination of available concepts. (more on this below in the discussion of reflective abstraction.) "Giving" students the idea associated with the concept, what Freudenthal called "imposing," can interfere with their opportunity to make the abstraction. In such situations, students may use the idea (e.g., a procedure for solving certain types of tasks) without understanding the logical necessity of that idea. For example, Steffe (2003) described a student, Jason, developing a scheme for commensurate fractions. Jason was asked to find a way to partition a unit bar such that he could pull out either 1/2 or 1/3. Had someone told Jason to multiply the denominators to find the number of parts involved in a solution, he could have solved many such tasks, but it is not clear he would have come to the understanding involved. Even giving him an articulate justification for the strategy would likely not remedy the missed opportunity to allow him to abstract the concept involved.

A third reason for utilizing guided reinvention is that student engagement in guided reinvention is useful from the standpoint of ongoing formative assessment in both a research setting and classroom instruction. Students who make the abstraction in question have reached a point in the development of their ideas to do so. Students who do not may still need more of the experiences provided or need different experiences to be ready to make the abstraction. It is the researchers' or teacher's job to figure out what further experiences are needed. In the LTA research program, we accept the challenge of fostering guided reinvention of target concepts. If students do not make abstractions compatible with our instructional goals (successfully reinvent), we take it as an indication that our instructional design is still not adequate.

TWO TYPES OF MATHEMATICS CONCEPT LEARNING

The LTA approach to guided reinvention is framed by two constructivist constructs that account for conceptual learning, generalizing assimilation and reflective abstraction.

Generalizing assimilation

Assimilation (Piaget, 1952) is the process by which learners' existing knowledge structures their experiences. Perception, recognition, interpretation, and understanding are all a function of learners' assimilatory structures. For the purpose of this discussion, I focus particularly on the recognition and understanding aspects of assimilation.

In discussing generalizing assimilation, let us start with a non-mathematical example. A child has a concept of *apple* based on a range of experiences with red apples. Each time she sees a red apple, albeit different varieties, she recognizes it as an apple. That is, she assimilates it to her concept of apple. One day, she is given a yellow fruit that otherwise looks and tastes like an apple. She assimilates it to her concept of apple. In this case, she assimilated an instance of the concept that was somewhat beyond the extant concept. The result of doing so is a more general and more useful concept. By "more useful," I mean able to assimilate a broader range of relevant experiences. This process of extending a concept through assimilation, is *generalizing assimilation*.

When generalizing assimilation is applied to mathematical concepts, the assimilation of new experiences can lead to modification of the existing concept resulting in a change in the learner's understanding. Thus, generalizing assimilation results in modification of what is recognized as an example of the concept and, in doing so, modifies the concept itself.

An important point about generalizing assimilation is that it applies to conceptual learning that is available through *extension of an existing concept*. But what if such a concept does not exist? What if a new concept must be constructed? This leads to the discussion of reflective abstraction.

Reflective Abstraction

The construct of reflective abstraction provides a framework for understanding the construction of new (to the learner) mathematical concepts using prior concepts as the raw material for that construction. Piaget (1980), who proposed the construct, wrote, "[Reflective abstraction] alone supports and animates the immense edifice of logico-mathematical construction" (p. 92). According to Piaget, such construction is based on reflection on one's own (mental and/or physical) activity. He described reflective abstraction as a coordination of actions.

LTA researchers saw an opportunity to elaborate reflective abstraction as a basis for instructional design in mathematics. This elaboration was grounded in LTA's research on fraction learning (Simon, Placa, & Avitzur, 2016). I highlight some of the features of that elaboration. This emerging elaboration of reflective abstraction is discussed and warranted in detail elsewhere (e.g., Simon, Kara, Placa, & Avitzur, in press).

- Reflective abstraction derives from an activity called on by the learner to (successfully) meet her goal.
- An activity is set of (goal-directed) actions already available to the learner that are called on sequentially.
- Reflective abstraction involves a coordination of actions.
- A coordination of actions produces a single, higher-level action that can be used in place of the sequence of actions that made up the original activity.

PROMOTING GUIDED REINVENTION

I now discuss the LTA approach to promoting guided reinvention through generalizing assimilation and through reflective abstraction. Readers might be familiar with the approach to the former, as it is compatible with other approaches. I include it for completeness and to emphasize the differences between the two. The LTA approach to the latter is novel.

Promoting guided reinvention through generalizing assimilation

A decision to promote generalizing assimilation is based on the hypothesis that the instructional goal can be met through modification of an existing concept. The instructional strategy is consistent with variation theory (Ling & Marton, 2011). The researcher begins by posing a task which is beyond the current range of examples on which the student's concept is based, but which is close enough to potentially be assimilated to that concept. Often, the instructional goal cannot me achieved through a single generalizing assimilation, so a sequence of tasks is used to provoke a progression of small changes in the assimilatory structure. We offer an example from Simon, Kara, Norton, and Placa (in press) of an attempt to achieve an instructional goal through generalizing assimilation. The example, which shows an intervention that was only partially successful, is instructive, because it illustrates not only the approach, but the limitation of the approach as well.

The research was focused on promoting a concept of multiplication that is useful for conceptualizing both whole-number and fraction multiplication. In developing a concept of whole-number multiplication, US students predominantly develop a multiple-groups concept. Because they conceive of multiplication as involving multiple groups of a certain size, it is difficult for students to make sense of multiplication by a fraction. Our instructional goal was to foster a concept of multiplication in which the multiplier specifies the number of composite (or fractional) units in the quantity. Our conjecture was that students who have a multiple-groups concept of whole-number multiplication, through generalizing assimilation, could expand their concept of multiplication to include multiplying by a mixed number and then expand the resulting concept to include multiplication by a fraction.

We began with a task *Here is a unit* [referring to a bar on the screen in Java Bars[1]]*; make a bar that is 4 units long. Now make a bar that is two times as long as the bar you just made.* Kylie (10 years old) iterated the unit 4 times and then iterated the resulting bar twice. She had assimilated the activity into her whole-number multiplication concept. The researcher then took the first step in promoting generalizing assimilation. *Here is a bar that is 6 units long. Can you make me a bar that is 6 times 3 1/2?*[2] Kylie iterated the bar three times, partitioned the original bar in half, and attached half of the original bar to the end of the bar she had created. The researcher gave her five more tasks with mixed-number multipliers. She solved them in similar fashion. Kylie had assimilated multiplication by a mixed number to her prior concept of multiplication (generalizing assimilation).

Our conjecture was that because Kylie was representing appropriately the result of the fractional part of the multiplier, she would have no difficulty assimilating a task in which the multiplier was a fraction. This conjecture proved to be wrong. When given a task in which the multiplicand was multiplied by 1/5. She expressed that she had no idea what to do with the bar on the screen. Our interpretation of the data was that, multiplication for Kylie was an operation of making multiple copies. She had expanded her conception to include making multiple copies that included a partial copy. However, a fraction multiplier did not fit with her conception of making multiple copies. We seemed to have found a limit to what we could accomplish with Kylie through our attempts to promote generalizing assimilation. We reasoned, therefore, that we needed to promote the goal concept, that is through reflective abstraction.[3]

Promoting guided reinvention through reflective abstraction

To promote guided reinvention through reflective abstraction, LTA researchers build on the LTA elaboration of reflective abstraction previously discussed.[4] Central to the LTA approach is specification of an activity *that students currently have available* that can be the basis for the abstraction specified in the learning goal. The activity is expected to afford a particular coordination of actions (coordination of concepts). Once this activity has been identified, we design a task sequence to both elicit the intended student activity and lead to the eventual coordination of actions on the part of the students. In an LTA task sequence, how to solve the tasks is *not* what is being learned. Ideally, the task sequence allows the students to solve each task correctly using an activity available to them. Rather, it is through successful engagement in the task sequence that coordination of actions occurs resulting in the new concept. Following is an example.

One aspect of the LTA study of fraction learning focused on what is often referred to as "fraction of a set." More accurately the focus was on developing anticipation of how to find the fraction, *a/b*, of a whole number quantity, *n* (where *b* divides *n*). I describe a task sequence developed in the context of work with Kylie, once again using Java Bars. Pre-assessment demonstrated that Kylie was unable to solve a task such as 3/5 of 10. The first task was: [Given an unmarked bar] *The bar on the screen is 10 units long. Make a new bar that is 3/5 of the original bar. How long is the new bar?* Based on prior work, students were expected to partition the bar into 5 parts, pull out one part, and iterate it three times. They then would use whole number division to determine that each of the 5 parts is 2 units long. Using multiplication, they determine that the three-part (3/5) bar is 6 units long. The activity which involves creating a 3/5 bar (partitioning, pulling out, and iterating) and using whole number multiplication and division to evaluate the bar produced was selected, because the activity was assumed to be available to the student, and a coordination of its component actions could lead to the goal concept. Data bore out these conjectures.

Kylie solved the initial task as expected. She made a bar, partitioned it into 5 parts, pulled out 1 part, and iterated it 3 times. She looked at the bars and announced the answer as 6 units. She explained how she evaluated the size of the bar produced. She continued this strategy for another couple tasks. The researcher then gave her the tasks, "5/6 of 12" and "5/4 of 12," which she was able to do mentally.

We explained Kylie's learning as follows (Simon, Kara, Placa, & Avitzur, in press). Through progressive coordination of the actions that made up her activity, Kylie began to determine the size of a part while partitioning and the size of the new bar while iterating. We exemplify this hypothesis with the task 3/8 of 32. As she created her 8/8 bar, she would have been thinking of each eighth as 32 divided by 8 or 4 units, and while she was iterating the 1/8 three times, she would have been thinking of it as 3x4 or 12 units. Finally, she reached a point in which her actions were coordinated; she no longer needed to carry out the original activity. She knew (anticipated) that taking 5/6 of a number meant taking a sixth of the number and multiplying the result by 5.

CONCLUSIONS

The LTA approach to promoting guided reinvention differs from the approach envisioned by Freudenthal. Whereas the latter was more focused on problem solving and "mathematics as a human activity," the former provides a complementary approach that may prove particularly useful for fostering difficult concepts and working with struggling students. Here I emphasize two aspects of the preceding discussion.

Promoting reflective abstraction versus generalizing assimilation

Variation theory has become a popular basis for instruction in the mathematics education community. However, an implication of the work discussed here is that variation theory, while valuable, may not be sufficient for promoting a full range of conceptual goals. Further, it suggests that the distinction between generalizing assimilation and reflective abstraction may offer a useful theoretical lens for anticipating when a variation-based approach will be successful and when it will not.

Engineering conditions for reflective abstraction

An assumption underlying reflective abstraction is that it is a learning process through which the individual learner goes. Instructional interventions cannot make reflective abstraction happen. Rather, reflective abstraction happens through the learner's activity and only when the learner is ready to make the abstraction. The LTA approach is based on research into this inherent ability and is an attempt to engineer task sequences that increase the likelihood that students engage in activities through which they can make important abstractions. When this engineering is successful, the students' elicited goal-directed activity provides both the raw material for making the abstraction (coordination of actions) and the tendency to attend to relevant features of the situation. The LTA approach to engineering for promoting reflective abstraction contrasts with those constructivist-based instructional approaches that focus on provoking perturbations.

Notes

[1] We used JavaBars (Biddlecomb & Olive, 2000), a computer application in which students can partition a bar, pull out a part, and iterate a bar or a part of a bar.

[2] There was prior agreement that the first number in the product a x b is the multiplicand and the second number is the multiplier.

[3] In the next section, we do not continue the example discussed in this section. It is too complex to be discussed in an eight-page paper. We refer the reader to the cited research report.

[4] The LTA approach to promoting guided reinvention through reflective abstraction is discussed in full in Simon, Kara, Placa, & Avitzur (in press) and Simon, Placa, and Avitzur (2016).

References

Biddlecomb, B., & Olive, J. (2000). *JavaBars* [Computer software]. Retrieved from http://math.coe.uga.edu/ olive/welcome.html.

Freudenthal, H. (1973) *Mathematics as an educational task*. Dordrecht, The Netherlands: Riedel.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer Academic Publishers.

Gravemeijer, K. P. E. (1994). *Developing Realistic Mathematics Education*. Utrecht, The Netherlands: CD-Beta press/Freudenthal Institute. Piaget, J. (1952). *The origins of intelligence in children*. New York: International Universities Press.

Ling, L. M., & Marton, F. (2011). Towards a science of the art of teaching: Using variation theory as a guiding principle of pedagogical design. *International Journal for Lesson and Learning Studies*, 1(1), 7-22.

Piaget, J. (1952). *The origins of intelligence in children*. New York: International Universities Press.

Piaget, J. (1980). *Adaptation and intelligence: Organic selection and phenocopy*. Chicago, IL: University of Chicago Press.

Piaget, J. (2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). Sussex: Psychology Press.

Simon, M. A., Kara, M., Norton, A., & Placa, N. (in press). Fostering construction of a meaning for multiplication that subsumes whole-number and fraction multiplication: A study of the Learning Through Activity Research Program. Special issue of the *Journal of Mathematical Behavior*.

Simon, M. A., Kara, M., Placa, N., & Avitzur, A. (in press). Towards an integrated theory of mathematics conceptual learning and instructional design: The Learning Through Activity theoretical framework. *Journal of Mathematical Behavior*.

Simon, M. A., Placa, N., & Avitzur, A. (2016). Two stages of mathematical concept learning: Further empirical and theoretical development. *Journal for Research in Mathematics Education, 47*, 63–93.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *The Journal of Mathematical Behavior, 22*, 237-295.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*, 129–162.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, DC: Falmer.

STRUCTURING STUDENTS' MATHEMATICAL CONVERSATIONS WITH FLOWCHARTS AND INTENTION ANALYSIS – AFFORDANCES AND CONSTRAINTS

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This paper is based on a study that compares how student groups from three different cultural contexts solve the same mathematical problem. In this paper, the focus is on the methodology. More specifically, we describe the use of flowcharts and intention analysis to reveal important qualities in student discussions. The aim is to show how these tools may support the analyses of students' mathematical discussion. We also emphasise the shortcomings of the tools in the light of our experiences from the study. We show that in order to better understand why certain interaction happens, it is important to extend the analysis to contextual factors.

BACKGROUND

Problem solving together with peers is strongly advocated as a working manner by the new Finnish National Core Curriculum (FNBE, 2015) implemented during the last academic year 2016-2017. However, we know little about how students of different ages manage to work productively in small-group situations. We also assume that there are differences in classroom cultures and socio-mathematical norms that influence the ways in which students are fostered to work together with peers.

The paper is a part of the VIDEOMAT-project that aims to examine what kind of knowledge about similarities and differences between different classroom cultures researchers and teachers can gain from discussing and comparing videotaped classroom activities together (Kilhamn & Röj-Lindberg, 2013). The material that the present paper draws on consists of three videotaped problem-solving sessions where students between the age of 12 and 13 years solve three different mathematical problems together with peers in small groups. The groups represent classrooms in three cultural contexts: the Swedish speaking part of Finland, Sweden and the US (see Smedlund, 2016).

It is extremely important to develop proper tools in order to analyse the character of small-group discussions that reveal both how productive the discussions are mathematically and how the different students participate and contribute to the solution. Furthermore, it is important to know if, and in that case what, students learn from these sessions. There are also social factors that are important to focus on in group work, for example how students experience the cooperation and how they treat each other in these situations. Consequently, a great deal of effort has been made on developing

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proper analytical tools for small-group discussions in the field of mathematics education.

Sfard and Kieran (Kieran, 2001; Sfard, 2001; Sfard & Kieran, 2001) developed an interactive flowchart to map the discussions of students. Ryve (2006) suggested adding an analytical construct of contextualisation as well as analysing different types of mathematical discourses, further improving the interactive flowcharts. He states that this is beneficial in at least two ways, namely by offering "the researcher an opportunity to scrutinise his or her own arguments for the interpretations of the students' immediate intentions", and "opportunities to form an opinion of the presented interpretations and the underlying argumentation supporting these interpretations". (Ryve, 2006, p. 203).

In the data analyses of the three groups in our study, tools developed by Sfard and Kieran (Kieran, 2001; Sfard, 2001; Sfard & Kieran, 2001) and Ryve (2006) were used. The focus of this paper is on the work of analysis, not on the results. More specifically, the aim of this paper is to investigate and report the strengths and shortcoming of these tools when analysing students' mathematical discussions in groups that represent different cultural-educational contexts. We also present our implemented improvements to the interactive flowcharts when analysing our data. These improvements even further assist the transparency and understanding of the method and thereby contribute to the field of mathematics education.

FLOWCHARTS AND INTENTION ANALYSIS

Sfard and Kieran (2001, p. 58) make use of interactivity flowcharts in the hope of evaluating the interlocutors' interest in creating dialogue with their partners. The information gained from these flowcharts allowed the researchers to make conjectures about the discourse that preoccupies their interlocutors. An interlocutor is defined as a person taking part in a conversation or discussion. Addressing and/or reacting to somebody's former utterances with any particular action is defined as communication. Audible and public, or silent and private utterances are produced for the sake of communication and are never stand-alone as isolated events (Sfard & Kieran, 2001).

Sfard and Kieran start their analysis by categorising utterances as *reactive*: a reaction to a previous contribution of a partner, or *proactive*: a wish to evoke a response in another interlocutor. A systematic construction of the flowchart using arrows (see Figure 1 & 2) allows consecutive utterances to be related to utterances that have already happened or utterances that are yet to come. The arrows, a metaphor for a speaker's intentions communicated indirectly, make the speaker's intentions for interaction visible. Sfard and Kieran aim to avoid the pitfalls usually associated with the notion of intention. Intention is born within the act of communication and therefore there are intentions within utterances. These intentions can be regulated before, during and after the act of communication (Sfard & Kieran, 2001).

Ryve (2004, p. 173) noted that the methodological tool as developed by Sfard and Kieran was not sufficient and needed further development. Ryve stated that there was a need for distinguishing between different types of *mathematical content* in different utterances. In addition, there was a need for a methodological device for constructing the interactive flowcharts. Ryve (2006) assumes that all human behavior is intentional and suggests contextualisation for making the interpretations of students' immediate intentions more explicitly constructs of the analyst. The researcher should, hence, argue for a contextualisation of a task and reflect with questions such as "Could other utterances supporting this hypothesis be found? Could utterances serving as evidence against this interpretation be found?" By testing the interpretation in relation to the cognitive, situational and cultural contexts, the researcher aims to make his/her interpretations explicit to both himself/herself and to the reader.

The tools described above which were developed by Sfard and Kieran and complemented by Ryve, hereby referred to as the SKR-tool, served as a starting point for our analysis.

METHODOLOGY

The context of the study

During the VIDEOMAT-project, students took part of four algebra lessons and a fifth problem-solving lesson. The instructions from the researchers in the VIDEOMAT-project to the teachers was that the fifth lesson would consist of working with the three mathematical problems without further instructions on how the groups should be formed or how the pupils should work with solving the problems (Kilhamn & Röj-Lindberg, 2013). The instructions from the teachers to the students were similar in all three countries and included working together solving the problems and cooperating. Every student should be able to understand the solutions that their groups had worked out. Only the third problem, called the matchstick problem, was analysed and in this paper examples from the Swedish-speaking group from Finland are used (Smedlund, 2016). The matchstick problem is as follows: "Four squares in a row consist of 13 matchsticks, how many squares can be built this way using 73 matchsticks?"

The data processing

The selection of the groups was based on the visibility of each participant in the group and also on the audio quality. After selecting one Swedish, Finnish and American group according to these criteria, the video recording was observed by the first author at 0.5x speed to be able to transcribe everything clearly. If something was unclear, the observation was repeated as many times as needed to be able to transcribe each utterance. A three-step situated discourse analysis (similar to Radford, 2000) of transcripts was used as follows: 1. Valuing each utterance as equally important 2. Contextualising utterances 3. Including pauses and hesitations. Pauses and hesitations in our transcripts are handled as consecutive utterances from the same person and thus split into separate utterances, for example: 47a "Yeah because, look when you've made

52 one only has to put three matchsticks to make a square." 47b "Do you understand?" 47c "Look, now you've made 53". This approach was chosen because these different utterances have separate intentions, which would be difficult to distinguish if they were counted as a single utterance (see Figure 2).

After the transcripts were completed, the video recording was observed at normal speed to identify possible body language and actions that took place. These actions were noted in a protocol to further grasp the problem-solving situation that had unfolded. This in turn offered parts of the information needed to interpret the context of the situation, based on *cognitive*, *situational* and *cultural* aspects for the students (cf. Ryve, 2006). The descriptions of the context, described individually for each student, were based on the communication between students and instructions from the teacher. We added a complement to the contextualisation using an analysis of the level of co-operation, which is further discussed in a later section.

When there was a clear understanding of the individual contexts, the process of interpreting each individual utterance in relation to the context started. This was achieved with the interaction flowchart tool and each utterance was described as either proactive (arrow forward in the discussion), reactive (arrow backwards in the discussion) or pro- and reactive (arrow both forward and backwards in the discussion). Furthermore, utterances were categorised as on-task (lines) or off-task (dotted lines).

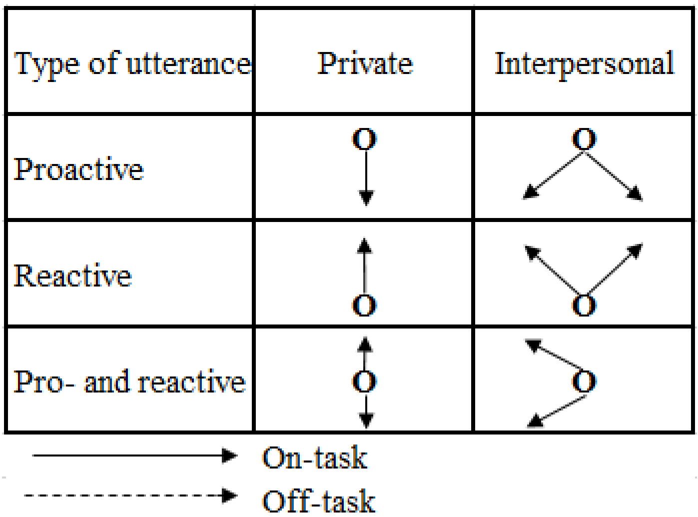

Figure 1: Revised figure from Sfard & Kieran (2001)

We also categorised different suggestions for solving the problem (cf. Ryve, 2006). During the data analysing process, we noticed several shortcomings and added tools for example for the contextualisation and structuring of the flowcharts described in a later section. Also the revision of the figure above is explained below.

THE AFFORDANCES OF THE TOOL

The SKR-tool is used to present classroom activities and discourses in text format. Our experience is that the SKR-tool described above is very useful in the analysis for the reasons outlined below. The flowcharts give both the analyst and reader an explicit view of the discourse. This enables the analyst to highlight and present significant moments to the reader, in our case moments where solution methods are suggested to and received by the group. Using contextualisation to strengthen the intention analysis is effective and enables the analyst to describe his or her interpretations more clearly. The SKR-tool is very time-consuming, especially as it requires constantly comparing each utterance with the context. However, in our experience, it is worth building such an explicit interpretation of a situation where the analyst has a chance to display his/her point of view to the reader. It strengthens the transparency and the interpretations of the immediate intentions.

In our study, we focused on the students' suggestions of methods for solving the matchstick problem. For example, a student notices a structure in the problem, a re-occurrence of adding three matchsticks to make a new box and this, in turn, leads the student to finding and then sharing a divisive method of solving the problem with the group. How the group meets this suggestion or insight is mapped with help of the interactive flowchart. These suggestions/insights are either accepted or rejected and this results in further discussions, further explanations or attempts to receive the group's attention. By studying the occurrences, using the contextualisation for each group member and categorisation of different problem-solving methods, it was possible to follow other successful solutions and less successful solutions.

THE CONSTRAINTS AND IMPROVEMENTS OF THE TOOL

Next, we describe the constraints of the SKR-tool and the improvements we implemented during our study.

First, as shown in Figure 1 (see above), the flowchart developed by Sfard and Kieran (2001) has been improved by adding "pro- and reactive arrows" because experiences from our study show that utterances can both reflect on something said earlier and seek further reactions from other interlocutors.

Second, one characteristic that hindered following the flowcharts in Sfard and Kieran's (2001) and Ryve's (2006) papers was that the transcriptions were often presented apart from each other, which made it necessary to go back and forth through the paper to understand the situation. This is a problem considering that the aim of this tool to facilitate a construction of a clear view of a specific situation. Therefore, in our study, the flowcharts were joined with the transcripts so that the reader can follow the situation more clearly (see Figure 2). Colour coding the arrows in the flowchart according to the type of problem-solving method that the students were working with further enhances the clarity of the described situation. In Figure 2, orange utterances relate to a divisive method of solving the problem and green utterances relate to drawing matchsticks to attain a solution of the problem. This simplifies the process of analysing how the different suggestions are handled in the group. Students are communicating, but they are working with different methods of solving the problem. Bodil is not interested in Casper's suggestions something that is clearly visible in Figure 2.

Third, the contextualisation can be further enhanced by evaluating every member's individual motivation to cooperate and by analysing the cooperation level of the group as a whole. Questions such as "Do they want to work together or not?" and "How does their willingness to cooperate impact the suggested solutions?" are important to pose to deepen the conceptualisation. The level of cooperation in each group was, in our study, therefore analysed by using Sahlberg and Berry's (2003) matrix of cooperation. Adding a description of the level of cooperation enhances the contextualisation and is useful, for both the analyst and the reader.

An example of this is the group work in the Finnish-Swedish group, where cooperation did not work out. Bodil (B) was not at all interested in Casper's (C) method. We can assume it is because Bodil was content with finding one solution to the problem, and another method seemed unnecessary to her. But just as likely it might be that Bodil and Casper just could not work together due to personal reasons. Understanding the students' motivation for working together and whether the situation is beneficial for co-operation is crucial for the kind of research questions where one wants to understand aspects of problem solving in groups.

#	A	B	C	T	Utterances	Actions
45a			O		Twenty one.	Casper thinks and turns towards Bodil
45b			O		Seven times three, seven, seven more squares. Seven more squares!	Casper leans over Bodil's paper and points
46		O			Nope.	Bodil leans back
47a			O		Yeah because, look when you've done 52 you only need to add three matchsticks for a square.	Casper points at and turns Bodils paper around
47b			O		Do you get it?	
47c			O		Look now you've done 53.	Casper points at Bodils paper
48		O			I've done 55.	Bodil leans in
49			O		Have you?	
50a		O			Mmm.	
50b		O			It's 55 matchsticks that I've used.	
51a			O		Oh no.	Casper leans back with his hand on his forhead

Figure 2: Flowchart from the Swedish-speaking group in Finland (Smedlund, 2016)

DISCUSSION

The SKR-tool was a starting point for charting different suggested solution methods and their reception in our study. To improve the tool, we first added the possibility of marking "pro- and reactive arrows" as an utterance could have both proactive and reactive intentions. Having to remember utterances and compare them to the interactivity flowcharts in the SKR-tool was a weakness that we amended by adding the transcripts next to the interactivity flowchart. This addition enables the reader to follow the unfolding situation with more ease. We used colours to highlight different solution methods that students were discussing in the utterances to further clarify the mathematical productiveness in the discussions. Using video to complement the text format could further strengthen the analyst's presentation of his/her findings, as well as give the reader/viewer even further insight into the described situation. Finally, assessing the level of cooperation (Sahlberg & Berry, 2003) of the groups was useful for understanding and more explicitly making the interpretations of intentions a construct of the analyst.

This tool still has restrictions. This is a very time consuming methodological tool. It is important to remember that the episodes that we analysed only lasted about 10 minutes each, yet the hours spent observing, transcribing and interpreting them resulted in small segments of the discourse being presented. Moreover analysing a short video-recorded segment demands that the analyst stays true to his/her contextualisation. Knowing each student might make the interpretation of their intentions more complete. However, then one might over-interpret and assume intentions of utterances based on the student's usual behaviour outside this situation. The SKR-tool is useful for making the interpretations explicit. Ryve's addition of contextualisation for checking each interpreted intention makes it easier to have an explicit representation of the analyst's interpretations. It also makes the analysis more coherent. However, to be certain the contextualisation is "good enough", one should maybe ask different researchers to check the contextualisation before going into intention analysis. Furthermore, especially in cross-cultural studies it is extremely important to gain insight into the cultural-educational context and how it relates to the socio-mathematical norms of the classroom which students are used to working in. Hence, various mathematical aspects and different social aspects can be revealed with the tool. While questions like "how are the suggested methods of solving the problem received?" are possible, questions like "why did the students solve the problem using this method?" are impossible to answer without further background information not suited for this tool.

The methodological improvements and further considerations of the SKR-tool strengthens the possibilities of applying and further improving it in future studies. The findings from these kind of analyses can be used to improve both researchers' and teachers' understanding of essential aspects of mathematical problem solving in groups.

References

Finnish National Board of Education. (2015). *Grunderna för läroplanen för den grundläggande utbildningen 2014 [National core curriculum for basic education 2014]*. Tampere: National Board of Education

Kieran, C. (2001). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges, *Educational Studies in Mathematics, 46*(1-3), 187–228.

Kilhamn, C., & Röj-Lindberg, A.-S. (2013). Seeking hidden dimensions of algebra teaching through video analysis. In B. Grevholm, P. S. Hundeland, K. Juter, K. Kislenko & P.-E. Persson (Ed.), *Nordic research in mathematics education, past, present and future (p. 299–328)*. Oslo: Cappelen Damm.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics*, *42*(3), 237–268.

Ryve, A. (2004). Can collaborative concept mapping create mathematically productive discourses? *Educational Studies in Mathematics 56*(2), 157–177. DOI: 10.1023/B:EDUC.0000040395.17555.c2

Ryve, A. (2006). Making explicit the analysis of students' mathematical discourses – revisiting a newly developed methodological framework. *Educational Studies in Mathematics 62*(2), 191–209. DOI: 10.1007/s10649-006-4834-0

Sahlberg, P., & Berry, J. (2003). *Small group learning in mathematics*. Åbo: Samfundet för Pedagogisk Forskning i Finland

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, *46*(1-3), 13-57.

Sfard, A., & Kieran, C. (2001). Cognition as Communication: Rethinking Learning-by-Talking Through Multi-Faceted Analysis of Students' Mathematical Interactions. *Mind, Culture, and Activity*, *8*(1), 42–76. DOI: 10.1207/S15327884MCA0801_04

Smedlund, J. (2016). *Det är som fyra, fyra och tre, tre, tre* [It's like four, four and three, three. three]. Unpublished master's thesis. Vasa: Fakulteten för pedagogik och välfärd, Åbo Akademi. http://urn.fi/URN:NBN:fi-fe2016053113082

SCHOOLS' STRATEGIES FOR PROMOTING GIRLS' PARTICIPATION IN MATHEMATICS

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Fewer girls than boys in England participate in post-compulsory mathematics. Previous studies have shown the significance to girls of their mathematics lessons and teachers, of cultural constructions of gender and mathematics, of career perceptions and family 'science capital'. A multiple case-study project investigated institutions with unusually high participation by girls in mathematics. Focus groups and lesson observations were used to explore school pedagogy and culture. Common factors were: early preparation for demanding mathematics, a departmental ethos which encouraged student-teacher interactions in and out of lessons, teachers who explicitly and repeatedly confirmed that girls would succeed at mathematics A-level, appreciation of mathematics as opening doors to many careers.

INTRODUCTION

There is a considerable body of research showing concern for the social, economic and institutional injustices that result from women's unequal participation in mathematics (Ceci & Williams, 2010; Forgasz & Mittelberg, 2007; Hyde & Mertz, 2009). Many such papers also argue that their nation's economic advantage relies on increasing the proportion of the population with science, technology, engineering and mathematics (STEM) skills. From these perspectives, girls who do not follow advanced mathematics courses are a potential source for recruiting more STEM-skilled workers, and hence their participation deserves scrutiny. Within this research, we particularly note studies that investigate the disinclination of some girls' (and boys') to study mathematics at a higher level (Archer et al., 2012; e.g. Mendick, 2005; Mujtaba & Reiss, 2016). This work has established a range of inter-related factors that influence individual students' study and career intentions, intersecting with gender in ways that lead to unequal participation. Our study builds on this prior research to consider girls' participation in mathematics starting from the different viewpoints of the culture and practice in schools with high participation.

Using a multiple case-study approach in the English policy context, where participation within the academic track can be measured by the choice of "A-level" subjects at age 16, we found little evidence of specific initiatives to attract girls to study mathematics. Instead, a common feature of these successful schools was a strong culture of encouraging all students to aspire to study mathematics, operationalised through a co-ordination of informal careers guidance, teacher relationships and

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pedagogic strategies. The findings suggest that schools can encourage girls by focusing on stable teacher relationships and early, supported classroom challenge.

THEORETICAL FRAMEWORK

We base our work on the understanding that choices and preferences made by individual students are constructed within the discourses of classrooms, schools and wider society (Smith, 2010). Attitudinal surveys show that students' beliefs about the gender stereotyping of mathematics vary between countries (Hyde & Mertz, 2009) and within different cultures in one country (Forgasz & Mittelberg, 2007). Thus, we see the knowledge produced about individual factors associated with participation in mathematics as indicating a range of psychological and sociological constructs that can be mobilised into gendered patterns both by local cultural practices and by wider discourses of mathematics, society and adolescent identity.

In the English context, mathematics is compulsory until the "GCSE" examination at age 16; thereafter students on the academic track choose three or four "A-level" subjects. In 2017, 24% percent of A-level students chose Mathematics and 4% chose Further Mathematics, however this reduced to 18% and 2% for girls, who are 54% of the cohort. Research suggests a range of factors that affect students' intentions to study mathematics at A-level and could be influenced by school practices. Participation is most strongly associated with high prior- and high relative- attainment in mathematics: the latter particularly affecting girls, who tend to perform well over their eight (or more) GCSE subjects (Noyes & Adkins, 2016). Contributory attitudinal factors for all students include enjoyment of lessons, perceived teacher-competence, perceived self-competence, intrinsic interest in mathematics and awareness of the utility of mathematics for supporting access to other areas; successive surveys find that girls score these lower and that they affect girls' choices more markedly (Brown, Brown, & Bibby, 2008; Mujtaba & Reiss, 2016). This suggests an important cultural influence resulting from schools' pedagogic practices and career guidance. Mujtaba and Reiss also found that fewer girls than boys, aged 13 and 15, report receiving advice and encouragement to study mathematics (and physics) and that such advice is influential for them, particularly when it is received from a trusted family- or teacher- source. Archer, DeWitt and Wong (2014) review school-level strategies for recruiting girls into STEM subjects, such as school science projects led by universities and visits from female role-models, and note that where their impact has been evaluated, they appear more successful in sustaining an early STEM interest than in changing minds. These authors call for less emphasis on elite aspirations in STEM interventions, arguing that explicit diversity in the messages promoted to girls makes their participation easier to negotiate. Our appreciation of the complexity of girls' choices but also of the possibility of supporting them underpins our research interest in school structures and relationships. We asked:

- In schools which are successful in recruiting girls into mathematics, are there any intentional strategies addressing girls' participation? How are these conceived, operationalised and evaluated by teachers?
- What messages are current in the school culture about who does mathematics?
- Are there aspects of mathematics pedagogy, of careers or teacher guidance that support girls' participation in studying mathematics? How is this support conceived and operationalised?

THE STUDY

A multiple-case study methodology was chosen in order to explore "hypothesised variations" (Yin, Clarke, Cotner, & Lee, 2006, p. 114) of school type and size, and to produce detailed, contextual information about the practices of mathematics teaching and recruitment in each school and the beliefs of teachers and students. Five sites were identified as having girls' participation in mathematics, using a combination of criteria:

- relatively high proportions of girls entered for both Mathematics and Further Mathematics A-levels according to Department for Education 2012-13 data;
- ensuring some diversity in region and school type, including one school where classes are single-sex to 16 (as girls' participation is higher in single–sex schools) and one 16-18 year college;
- preferring schools with a non-selective intake (for greater generalisability);
- willingness to participate.

Data was collected in two phases, spaced a year apart. In the initial phase, at each site one of the authors conducted: one 50-minute focus group of 3-5 mathematics teachers exploring the strategies considered significant for retaining girls in mathematics; one focus group with year 12 or 13 female A-level mathematics students exploring their experiences of mathematics classrooms, their perceptions of mathematics as a gendered subject and their reasons for choosing whether or not to continue; (if possible) a focus group with year 11 girls likely to study mathematics; observation of one or two A-level or GCSE mathematics lessons focussed on features considered important by teachers and students. Second-phase visits comprised an interview with each lead teacher investigating the stability of the cultural practices identified in the analysis, collecting data related to transition between year 12 and 13, and gathering evidence of any new initiatives or further reflection on girls' participation.

Teachers' and students' accounts were emphasised in our design, since we acknowledge that teaching (for teachers) and choosing subjects (for students) are highly reflexive practices, for which reasons are sought and articulated to oneself and others. Nevertheless, this approach runs the risk of foregrounding explanations that are dominant by being popularly or powerfully accepted. Focus group discussions were thus chosen to gain several perspectives on the same feature and to gain insights into emerging shared meanings. Other explanations were explicitly sought in the teacher focus groups, and coherence tested through triangulation with lessons observations, student records and respondent validation. Data was collected in the form of field notes, transcriptions, and quantitative data on mathematics class size, module choices and mathematics GCSE and A-level grade profiles by gender.

During analysis, each case was summarised to identify what the participants reported as local strategies affecting girls' participation, and where there was agreement or not between teachers and students about practices and the effect of those. Case data was coded by how accounts of these practices matched factors derived from the literature. Both authors then worked across the cases to consider strategies that had elements in common. This established three thematic strategies common to the schools, although operationalised in different ways. Further case reports were written using these themes and sent to the school (teacher) contacts for validation.

The case study sites are outlined in Table 1, showing their type and size and their decile for girls' participation from the year preceding the study. To meet all criteria we chose sites that (initially) performed in the top three deciles of all schools and in the top two deciles of state schools.

	Area	Gender	Size of A-level cohort	Decile for % of Girls completing Maths A-level (state sector only), years -1 to +1.		
School A	Town	Mixed	100-150	10(10)	8 (9)	9 (9)
School B	Inner city	Girls to 16	Under 100	9 (10)	7 (8)	8 (9)
School C	Conurbation	Mixed	Over 300	8 (9)	8 (9)	8 (9)
School D	Outer city	Mixed	100-150	10(10)	10(10)	10(10)
College E	City	Mixed	100-150	8 (9)	8 (9)	4 (4)

Table 1

FINDINGS AND DISCUSSION

We found no mathematics initiatives aimed specifically at girls in the case study sites. Teachers were aware that, nationally and internationally, girls were under-represented in advanced mathematics but had not examined their school data by gender or noticed its relative success. This meant that in focus groups they were often thinking through what they had done to raise achievement and interest, and recalling past conversations about aims and effects on different groups of students. A common feature of all sites was that teachers gave accounts of collectively-agreed intentions and strategies to recruit *both* girls and boys to mathematics A-level and these extended beyond the most able students. All schools set by prior attainment and it was explicitly considered part of the role of higher-set teachers to develop relationships with their classes that would encourage transition to A-level. Our analysis showed these strategies were based on three themes: pathway career thinking, robust emotional encouragement, and flexible cognitive support for working with challenge. Each of these school strategies can be traced as contributing to factors identified in the literature as supporting girls' participation. In the focus groups, girls reported a sense of progression to mathematics A-level, rather than specifically gender-based encouragement, typified by: "We're good at it, we enjoy doing it, why wouldn't we?"

Encouraging pathways thinking before year 11

Teachers in the case studies promoted mathematics as a subject that has wide applicability and contributed to a range of career pathways, thus emphasising diversity. For instance, year 12 girls reported that teachers "kept on saying it would open up opportunities. It's an all-round subject. Goes with everything". Some mathematics teachers had influential sixth-form pastoral roles which they used to promote mathematics, emphasising the value of statistics, in particular, for its connections to social and life sciences. Students considering joining College E to study science or technology were guided in preliminary individual interviews to take mathematics as a companion subject, thereby making mathematics more attractive to a wide range of students. In addition, school teachers made explicit connections with A-level content in their lessons with 14-16 year olds beyond the top sets. This was reported by students as teachers aiming to inspire interest and "show everyone can do it" (year 13 student).

Awareness of the utility of mathematics is associated in the literature with participation but as an extrinsic motivation. In these schools, the appeal to utility was expressed through a message of wide and multiple applicability rather than access to specific or elite courses. Choosing mathematics was thus presented by (and to) students as a way of honouring the scope of their own current and future interests. In this culture it became also an intrinsic motivation. This approach of inclusivity, that maintains a close relation to girls' existing aspirations, contrasts with the messages promoting a narrow mathematics 'pipeline' warned against in Archer et al. (2014).

Although an unintended variation, we noted that all the case study schools drew from catchments with large minority ethnic communities. In several focus groups, girls or teachers referred to the high value such families placed on mathematics and sustained hard work within a career-focussed pathway, a value that was reflected in the approach of the mathematics department. Staff and students also pointed to the presence of well-respected and dynamic female teachers among those teaching top-set GCSE and A-level classes. These close-at-hand connections between mathematics, family and social relationships were reported as giving it a broad appeal. We suggest that they also strengthened access to the informal 'grapevine' knowledge about careers and pathways that comprises what Archer et al. (2014) call invaluable 'family capital' in science or mathematics.

Specific, repeated, evidence-based, personal and collective encouragement

Across the settings, girls reported that as individuals and as a friendship group they felt actively and repeatedly encouraged to take A-level mathematics, and that their teacher was overtly confident they would succeed. Students ascribed this to perceiving that teachers knew the students' feelings and ways of working, and could thus offer personal guidance based on evidence not just of prior attainment but of student identity. In some schools, a departmental policy of teacher continuity explicitly aimed to create this relationship of trust. There was a close match between the teachers' and students' accounts of the relationship, and this was described in terms of teachers knowing individual students (girls and boys) well:

> Teacher A: that's why it's important I've taught them for so long; they know I care about them, and they care when they do badly, that they upset me, and stuff.
>
> Year 11: Teacher A is like that – she really wants to know what you enjoy doing and what affects you and the things that matter to you.

Some students questioned whether recruitment for A-level was intentional and suggested it rose as a natural consequence of a valued pedagogic relationship: for example "I just think the way that she teaches, it does encourage you. Like without her deliberately trying" (year 11 student). In contrast, teachers described an ongoing, specific, in-and-beyond-the-classroom emphasis on "building up confidence" for girls to take A-level. The same student's teacher reported: "I am spending a lot of time, a lot of lunch times, just talking to the girls. And they have got the 'can I do A-level' attitude. 'Am I capable of it?'" The evidence from these cases suggests, first, that the teachers do work at relationships that seem natural and, second, that such approaches are successful because they permeate teachers' actions in and out of class.

The notion of 'building confidence' was a common feature of teacher talk in all these schools, associated with their caring role and girls' classroom behaviour. Our analysis suggested that girls presented themselves as cautious in their choices, rather than unconfident: they used the combination of teachers' opinions and their own experience as evidence for themselves and others to decide whether their preferred approaches to mathematics would lead to success at A-level. This adds a nuance to previous findings (e.g. Brown et al., 2008) that girls' experience and enjoyment of mathematics lessons are important in determining their choices. In these schools, we could not identify any common features of classroom time or management. Instead, the experience these girls described as enjoyable (and that we observed) was the opportunity to build class-teacher and pupil-pupil relationships. These relationship were personal and trusted, explained through examples of how teachers had already helped them to develop strategies to overcome mathematical difficulties, and would continue to do so. They allowed them to imagine future participation within familiar ways of working and practices of self. Girls and teachers contrasted this with boys' risk-taking choice behaviour, choosing subjects without determining the probability of success.

In the four schools visited, the departmental scheme for 14-16-year-olds included unusual depth of mathematics and/or additional mathematics qualifications offered to higher sets. Girls and teachers cited this extended curriculum as giving credible evidence that girls had succeeded at demanding mathematics and should continue. The certification was important, but the most important effect appeared to be the experiences of struggle, support and success.

Flexible opportunities for students to build and check understanding

The third feature identified from our case studies is related to the previous two. As well as the inclusive pathways approach to A-level choice and the attention to personal evidence-based encouragement, classroom teaching offered multiple and flexible opportunities to meet mathematical difficulties and it gave messages that students should expect to develop deep and satisfying understanding over repeated encounters.

There has been much discussion of girls' (and boys') unease in a mathematics culture when it is possible to succeed without understanding (Boaler, Altendorff, & Kent, 2011; Solomon, 2007). In these schools the dominant message was to challenge that culture: all students should experience mathematics problems where they have to think for themselves in order to succeed. This was sometimes explicitly stated as a strategy to build mathematical resilience (Lee & Wilder-Johnston, 2017). The only intentional gender-related strategy reported in the mixed schools was to select quieter students to answer whole class questions, because teachers recognised that classroom talk was often sustained by boys. The girls also reported this strategy, but ascribed it low impact in encouraging participation. They valued more highly when teachers managed lessons so as to facilitate low-key teacher-student and student-student conversations in which girls could check their personal understanding. Several girls identified teachers who were good at explaining ideas in a variety of ways, rather than just repeating the same explanation, showing the value they placed on teachers who could combine their knowledge of students with good pedagogic knowledge of mathematics. Girls talked about experience of challenge, of pace and of competition, but not about feeling pressured to go faster than they could understand.

CONCLUSION

The three themes we introduce above were common across the case studies though implemented differently in each local context. Our study suggests three broad but achievable recommendations for schools. Firstly, teachers throughout the school should be familiar with A-level syllabuses and content so that they can perform their leading role in overtly orienting students towards participation. Secondly, teachers should have a repertoire of mathematics activities and strategies that allow students to experience challenges and seek help without a whole-class audience. Finally, it is important that mathematics teachers, parents and teachers of other subjects give overt messages to individuals and friendship groups that they expect girls (and boys) to succeed in mathematics, but that this will sometimes require persistence and hard work, as well as short-term failures.

References

Archer, L., DeWitt, J., Osborne, J., Dillon, J., Willis, B., & Wong, B. (2012). Science Aspirations, Capital, and Family Habitus How Families Shape Children's Engagement and Identification With Science. *American Educational Research Journal, 49*(5), 881–908.

Archer, L., DeWitt, J., & Wong, B. (2014). Spheres of influence: what shapes young people's aspirations at age 12/13 and what are the implications for education policy? *Journal of Education Policy*, *29*(1), 58–85.

Boaler, J., Altendorff, L., & Kent, G. (2011). Mathematics and science inequalities in the United Kingdom: when elitism, sexism and culture collide. *Oxford Review of Education*, *37*(4), 457–484.

Brown, M., Brown, P., & Bibby, T. (2008). "I would rather die": attitudes of 16-year-olds towards their future participation in mathematics. *Research in Mathematics Education*, *10*(1), 2–18.

Ceci, S., & Williams, W. M. (2010). Sex Differences in Math-Intensive Fields. *Current Directions in Psychological Science*, *19*(5), 275–279.

Forgasz, H., & Mittelberg, D. (2007). The Gendering Of Mathematics In Israel And Australia. In J.-H. Woo, H.-C. Lew, K.-S. P. Park, & D.-Y. Seo (Eds.), *Proc. 31st Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 233–240). Seoul, Korea: PME.

Hyde, J., & Mertz, J. E. (2009). Gender, culture, and mathematics performance. *Proceedings of the National Academy of Sciences*, *106*(22), 8801–8807.

Lee, C., & Wilder-Johnston, S. (2017). The Construct of Mathematical Resilience. In U. Xolocotzin (Ed.), *Understanding Emotions in Mathematical Thinking and Learning* (pp. 269–291). Elsevier.

Mendick, H. (2005). Mathematical stories: why do more boys than girls choose to study mathematics at AS-level in England? *British Journal of Sociology of Education*, *26*(2), 235–251.

Mujtaba, T., & Reiss, M. (2016). Girls in the UK have similar reasons to boys for intending to study mathematics post-16 thanks to the support and encouragement they receive. *London Review of Education*, *14*(2), 66–82.

Noyes, A., & Adkins, M. (2016). Studying advanced mathematics in England: findings from a survey of student choices and attitudes. *Research in Mathematics Education*, *18*(3), 231–248. https://doi.org/10.1080/14794802.2016.1188139

Smith, C. (2010). Choosing more mathematics: happiness through work? *Research in Mathematics Education*, *12*(2), 99–115.

Solomon, Y. (2007). Not belonging: what makes a functional learner identity in the undergraduate mathematics community of practice? *Studies in Higher Education*, *32*(1), 79–96.

Yin, R. K., Clarke, C., Cotner, B., & Lee, R. (2006). *Case Study Methods*. Routledge Handbooks Online.

EFFECTS OF SOCIOECONOMIC STATUS ON MIDDLE SCHOOL STUDENTS' MATHEMATICS ACHIEVEMENT IN CHINA

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Socioeconomic status (SES) plays an important role in influencing students' academic achievement. This paper presents a large-scale quantitative study to explore the diverse effects of SES on Chinese middle school students' mathematics achievement. According to the data analyses on over 25,000 middle school students in one province, it showed that parents' educational level, occupation and household resources significantly influence students' mathematics achievement. The results of interaction analysis suggested that the impact patterns of SES depend on region and gender. Compared with girls, boys' mathematics achievements were more heavily influenced by their SES.

INTRODUCTION

Family usual is learning and living localities for children while providing the most important environmental influence on children's early development. Moreover, the family will continue to be a strong factor of academic achievements after they entering school. Chapin (1928) defined SES as "*the position that an individual or family occupies with reference to the prevailing average standards of cultural possessions and participation into group activities*". In a meta-analysis reviewing nearly 200 studies, White (1982) summarized that three components of SES were focused on traditional researches, namely occupation, education, and income. Independent influences of the three components were also proposed by some empirical studies. However, most studies only measure elements like home resources and mother's educational level rather than parents' occupation or father's educational level. Level of home resources was recommended by Sirin (2005) as the fourth core factor of SES, which reflects a student's learning condition that the family can provide. Similarly, indicators of resources at home were also included in PISA' index of economic, social and cultural status (ESCS) together with occupation and education level of parents.

Conger and Donnellan (2007) reviewed two important theoretical approaches that illustrate the social caution of socioeconomic influence, which can be used to explain the impact of SES on academic achievement. The first paradigm named the family stress model (FSM) proposes the negative influences of family economic hardship. The stress on lower-SES family will lead to parents' emotional and behavioural problems as well as more frequent conflicts, which in turn result in students' learning difficulties. The second perspective titled family investment model (FIM) draws attention to the positive impact on students' academic achievement in higher-SES fam-

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ily. It is suggested that students in high-SES family receive abundant learning materials, living resources, and parental motivation, which promote their mathematics achievement.

Although the impact of SES on academic has already been confirmed theoretically and practically, the strength may vary widely among different cultures (OECD, 2012) or in different samples. In other words, some demographic features like gender, grade, nationality, and regions may moderate the impact size. The aforesaid two theoretical approaches reveal the fact that the pathway from SES to academic achievements is complex and indirect. According to the diversity of circumstances in different school regions and students' personalities, exploring the effect of SES can help to deeply understand the environmental factors of students' developing.

METHOD

Participants

The participants in the present study were sampled from a large-scale quantitative study from Collaborative Innovation Center of Assessment toward Basic Education Quality in China. All the students were from 11 cities and 90 districts, covering all the administrative divisions of Zhejiang. 25,029 students (52.6% boys and 47.4% girls) in this study were randomly selected from 501 schools in grade 8, which were highly representative to the middle school students in the province in southeastern China.

Measures

Parent questionnaire. Occupational and educational level of each parent and indicators of cultural and educational resources at home were investigated using parent questionnaire. Parents' educational level was measured with a 7-point scale: 1 = *never entering school*, 2 = *primary school*, 3 = *middle school*, 4 = *high school*, 5 = *junior college*, 6 = *university*, 7 = *postgraduate*. Parents' occupational level was evaluated with a 5-point scale: 1 = *agricultural workers, laborers and other elementary occupations*, 2 = *clerical support workers and self-employed entrepreneurs*, 3 = *business managers and administration managers*, 4 = *government associate professionals*, 5 = *teaching professionals, health professionals, and science professionals*. After investigating parents' educational level and occupational level separately, the largest value of two parents was used as the final variable coding.

Indicators of family resources were calculated by students' household items and cultural/educational possessions including the numbers (or none) of bedrooms, rooms with a bath or shower, computers can be used for school work, study rooms as quiet place to study, times of family trip in the past year, books to help with school work and technical reference books.

Some demographic variables of these students were additionally required as controlling variables or moderate variables, including gender, number of children at home (1 = *only one child*, 2 = *more than one child*), number of parents at home (1 = *single* *parent family*, 2 = *normal family*), and school region (in city, in town, or in country). The school region information was recoded into two dummy variables for data analyses.

Mathematics achievements test. Mathematics achievements were measured by using a sub-test from the Mathematics Competencies Test Bank (Guo, Cao, Yang, & Liu, 2015) in Grade 8. The sub-test measured students' capacity in four contents including Function, Equations & Inequalities, Geometry, and Statistics & Probability. The four content categories play an important role in Chinese mathematics curriculum. Students were given 100 minutes to finish the test.

Statistical procedure

Item response theory. The item response theory (IRT) analysis was used to achieve students' Rasch-scaled achievements estimates and level of family resources with the one-parameter model and implemented by ConQuest software (Wu, Adams, & Wilson, 1997). The estimates of mathematics achievement were then transformed into a scale with average value as 500 and SD as 100.

Correlation analyses and hierarchical regression analyses. The association of the estimated mathematics achievements of each student and the SES variables were tested using correlation analyses and hierarchical regression. For the hierarchical regression analyses, all the demographic variables were included in the first step and the three SES variables were involved in the second step. Variables within each step were chose following stepwise rule.

Interaction analyses. To examine potential dissimilarity of the relationships between each SES variables and students' mathematics achievements, interaction effects were tested using general linear modeling. Each interaction was checked one by one separately and the statistical significance was defined based on F test.

RESULTS

Correlation and regression

The correlations among estimated mathematics achievement and three SES variables were showed in Table 1. Students' mathematics achievement significantly correlated with their parents' educational level and occupational level (r = 0.297 and 0.259, respectively). The mathematics achievement also correlated with their family resources significantly (r = 0.297). The relationship between mathematics achievement and parents' occupational level seem to be weaker compared with the other two components. Furthermore, three SES variables highly correlated with each other, especially for the association between parents' educational level and occupational level (r = 0.610).

	Mathematics achievement	Parents' educational level	Parents' occupational level
1. Mathematics achievement	-		
2. Parents' educational level	0.297**	-	
3. Parents' occupational level	0.259**	0.610**	-
4. Level of family resources	0.297**	0.488**	0.475**

Note. **, $p < .01$.

Table 1: Correlations among mathematics achievements and SES variables.

Following the stepwise rule, all variables were remained in the final model. The regression analysis showed that controlling for gender, region, and other demographic variables, the three SES variables can significantly influence students' mathematics achievement (see Table 2). Involved in the model in the second step, SES variables totally explain additional 5.8% of the variance. When changing the order of the two steps, SES variables entirely explained 12.1% of the variance and ΔR^2 of demographic variables was only 0.019. Consistent with the correlation analyses, the independent effect of parents' occupational level was smaller than the effects of the other two variables.

	Variables	Unstandardized coefficients	Standardized coefficients (β)	ΔR^2
Step 1	Gender	5.891***	0.036***	
	Number of parents	14.819***	0.044***	
	Number of children	-9.870***	-0.060***	
	Region1 (city = 1)	4.875**	0.024**	
	Region2 (country = 1)	-18.968***	-0.114***	0.082***
Step 2	Parents' educational level	9.994***	0.135***	
	Parents' occupational level	3.629***	0.047***	
	Level of family resources	15.699***	0.145***	0.058***
			$R^2 = 0.140$	

Note. **, $p < .01$; ***, $p < .001$.

Table 2: Hierarchical regression models using demographic variables and SES variables to predict students' mathematics achievement.

Interaction analyses

Among the three tested models (see Table 3), only the interaction between school region and parents' educational level was statistically significant (F = 2.236, p = 0.008). The results suggested that the relationship between the parents' education and students' mathematics achievement differed from students studied in different school regions.

	Main effect	Interaction effect	*F* value	*p* value
Model 1	gender, number of parents, number of children, school region, EDU, OCC, FR	Region*EDU	2.236	0.008
Model 2		Region*OCC	0.673	0.716
Model 3		Region*FR	0.906	0.404

Note. EDU = Parents' educational level, OCC = Parents' occupational level, FR = Level of family resources.

Table 3: The interaction effect between school region and SES variables on mathematics achievement.

The detailed patterns were performed in Figure 1. It is obvious that students' average mathematics performance increased parallel with their parents' educational levels except for postgraduate degree. Students in cites who have postgraduate degree parents acted equally with students whose parents only receive university degrees. However, both in towns and countries, postgraduate parents' children performed worse than some other groups. Postgraduate parents' children at most achieved counterparts with parents who completed primary school or middle school learning especially in countries.

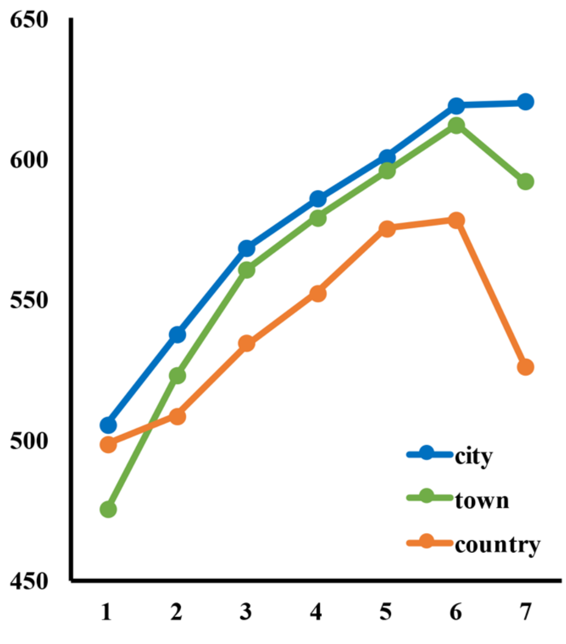

Figure 1: Effect of parents' educational level for students in different school regions.

After testing the models involved interaction between gender and SES variables (see Table 4), the interactions between gender and parents' educational level, and that between gender and occupational level were both statistically significant ($p = 0.001$ for gender*EDU, $p < 0.001$ for gender*OCC).

	Main effect	Interaction effect	*F* value	*p* value
Model 1	gender, number of parents, number of children, school region, EDU, OCC, FR	Gender *EDU	3.604	0.001
Model 2		Gender *OCC	5.458	<0.001
Model 3		Gender *FR	1.091	0.296

Note. EDU = Parents' educational level, OCC = Parents' occupational level, FR = Level of family resources.

Table 4: The interaction effect between gender and SES variables on mathematics achievement.

The detailed patterns were visually displayed in Figure 2. Referred to the interrelation between students' mathematics performance and their parents' occupational level, boys were more sensitive to the variation of parents' job-related characteristics. The gender's moderate effect on parents' educational level was similar with the moderate effect of school region. Postgraduate parents' male children performed worse than counterparts with parents who completed university studies and female children's mathematics performance keep rising aligned with parents' educational level growing.

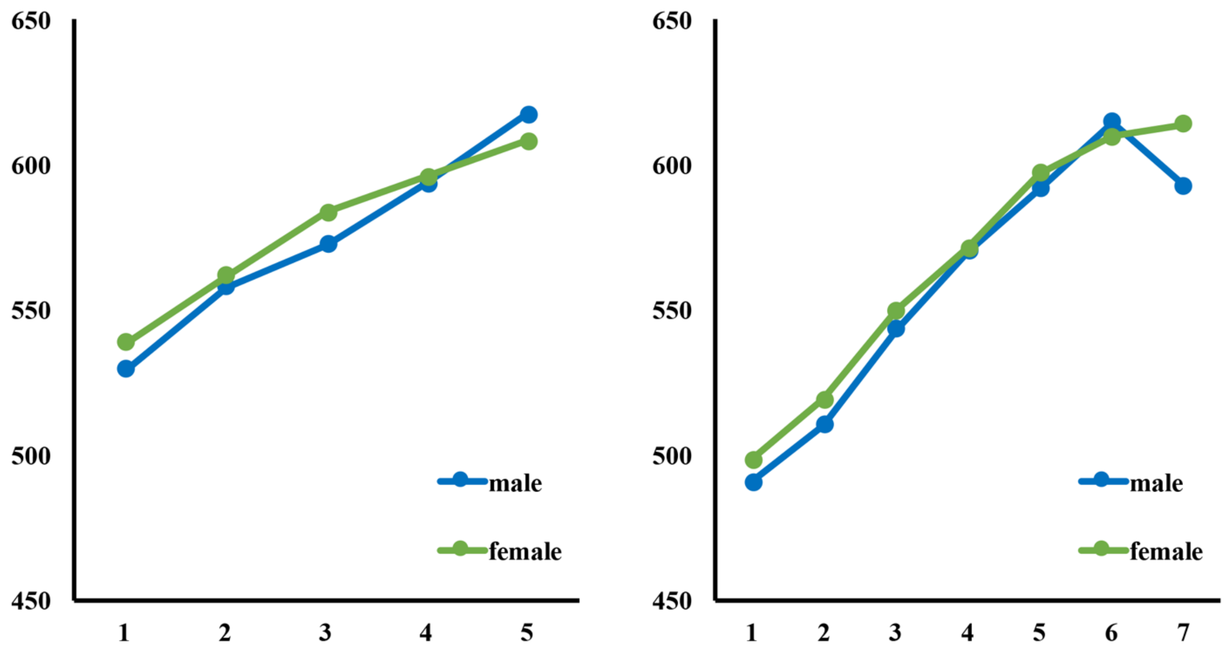

Figure 2: Students' gender moderated the effects of parents' occupational level (in the left) and educational level (in the right).

DISCUSSION & CONLUSION

Our results indeed showed that Chinese middle school students' family environmental factors including parents' education, occupation, and home resources significantly influence their mathematics achievement, and the correlation values were consistent with previous studies (Sirin, 2005; Van Ewijk & Sleegers, 2010). The interaction analyses have suggested the different impact of SES variables relied on school regions and students' gender. Moreover, the effect strength of parents' occupational level tends to be weaker when compared with the other two factors. It is compatible with previous researches on Chinese students, which showed that parents' educational level and family income are strongest factors (Wang, Li, & Li, 2014). The students' family background can explain 12% of the variation in their academic achievement and is essential influencer. Some studies also confirmed the long-term effect of early family socioeconomic status (Crane, 1996; Wang, et al., 2015). Consequently, researchers should not ever ignore the impact of SES variables.

Although the increasing of parents' educational background has a positive impact on urban students, the postgraduate degree of parents in rural areas has a negative impact. In countries and towns, children with parents having postgraduate degree performed worse than those parents completed undergraduate education (in countries) or middle school education (in towns). This kind of contrast can be supported by family stress model. With the acceleration of developing in big cities like Beijing and Shanghai, people with high degree are willing to find a job in big cities rather than stay in countries. Therefore, parents with high degree who stay in rural areas bear stronger pressure, which transforms into a negative impact on their children's development.

In addition to school region, students' gender also regulated the effect of SES variables. Understanding students' gender characteristics can help educators to provide efficient tutoring to boys and girls. Our results suggested that male students were more sensitive to the variation of parents' educational level and occupational level. This consequence may be due to parents' dissimilar expectations for boys and girls. Chui and Wong (2017) have found that parents in China are stricter towards male children and they tend to be happier when female students overachieve in academic examinations. As a result, boys are easily affected by parents' attitude, pressure, and investment.

In conclusion, we should understand the influence of family socioeconomic status correctly. Educational researchers and the whole society need to guide parents to avoid biased expectation of their children, and provide suggestions about reasonable investment in education.

References

Chapin, F. S. (1928). A quantitative scale for rating the home and social environment of middle class families in an urban community: a first approximation to the measurement of socio-economic status. *Journal of Educational Psychology, 19*(2), 99.

Chui, W. H., & Wong, M. Y. (2017). Avoiding Disappointment or Fulfilling Expectation: A Study of Gender, Academic Achievement, and Family Functioning among Hong Kong Adolescents. *Journal of Child and Family Studies*, *26*(1), 48-56.

Conger, R. D., & Donnellan, M. B. (2007). An interactionist perspective on the socioeconomic context of human development. *Annu. Rev. Psychol.*, *58*, 175-199.

Crane, J. (1996). Effects of home environment, SES, and maternal test scores on mathematics achievement. *The Journal of Educational Research*, *89*(5), 305-314.

Guo, K., Cao, P., Yang, F., Liu, J. (2015). Research on Assessment of Mathematics Competence Based on Curriculum Standards[基于课程标准的数学学科能力评价研究]. *Journal of Mathematics Education*, (2), 17-21.

OECD (2013). *PISA 2012 Results: Excellence through Equity (Volume II): Giving Every Student the Chance to Succeed*. Paris: OECD.

Sirin, S. R. (2005). Socioeconomic status and academic achievement: A meta-analytic review of research. *Review of educational research, 75*(3), 417-453.

Van Ewijk, R., & Sleegers, P. (2010). The effect of peer socioeconomic status on student achievement: *A meta-analysis. Educational Research Review*, *5*(2), 134-150.

Wang, L., Li, X., Li, N. (2014). Socio-economic status and mathematics achievement in China: a review. *ZDM*, *46*(7), 1051-1060.

Wang, Z., Soden, B., Deater- Deckard, K., Lukowski, S. L., Schenker, V. J., Willcutt, E. G., ... & Petrill, S. A. (2017). Development in reading and math in children from different SES backgrounds: The moderating role of child temperament. *Developmental science*, *20*(3).

White, K. R. (1982). The relation between socioeconomic status and academic achievement. *Psychological bulletin*, *91*(3), 461.

Wu, M. L., Adams, R. J., Wilson, M. R. (1997). ConQuest: Multi-aspect test software [computer program]. *Camberwell, Vic.: Australian Council for Educational Research*.

WICKED PROBLEMS IN SCHOOL MATHEMATICS

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The paper concerns climate change controversies and teachers' facilitation of pupils' critical mathematics perspectives through wicked problems. The data was collected in a research partnership with three teachers and their tenth grade pupils. A particular focus is directed towards how controversies can influence teachers to make different versions of a quiz, and this is discussed in relation to the teachers' value perspectives. The teachers´ choices of questions, numbers, and graphs are connected to their facilitation of action or critical thinking. In the dialogues, the teachers challenged each other's choices, and the controversies and value aspects were made explicit.

INTRODUCTION

During the last decade, a socio-political turn in mathematics education has received a growing interest among researchers and practitioners (Gutiérrez, 2013). This turn links mathematics and mathematics education with complex, socio-political questions and is a core aspect of Critical Mathematics Education (CME). Barwell (2013) argued that socio-political issues like climate change deserve attention by mathematics education, because mathematics is used to describe, predict, and communicate climate change. Climate change is sometimes referred to as a wicked problem, characterised by vague problem-formulations, multiple solutions difficult to define as right or wrong, no central authority, use of multiple time spans, and disagreement on who should bear the costs (Levin, Cashore, Bernstein, & Auld, 2012). Wicked problems often involve controversies that spark disagreement, and for climate change, typical examples are: Does (anthropogenic) climate change exist? If yes, what causes climate change?

In political debates about climate change, biased use of mathematics happen. Some interest-organisations and blogs, for example wattsupwiththat.com, selectively use graphs and mathematical representations to support their arguments. Citizens who regard mathematics as neutral and value-free may then be exposed to a political standpoint without being aware of it. Several researchers (e.g. Atweh, 2012; Ernest, 2009; Mellin-Olsen, 1987; Skovsmose, 1994) challenge the view on mathematics as objective, value-free, and non-political. Skovsmose (2014, p. 116) problematized this view on mathematics and mathematics education by underlining how important it is "to address critically mathematics in all its forms and application". Therefore, mathematics education plays an important role in educating citizens to become able to recognize political use of mathematics, and discussing climate change can serve as a powerful topic to achieve this. In Norway, the Education Act (1998) underlined that "pupils and apprentices shall learn to think critically and act ethically and with environmental awareness". This is a broader environmental perspective, but it is further

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specified in the political guidelines (Ministry of Education and Research, 2016) that pupils should be enabled to critically reflect on climate change, to understand and to believe in, and to acknowledge the responsibility to take actions. The focus on critical reflections and understanding is a common task in schools, while to believe in and take actions represent a more activist perspective. This raises interesting questions in respect to the normative aspect of climate change. For example, can critical reflection conflict with more normative and activist perspective in the curricula? Ho and Seow (2015) explored this conflict, and found that scholars in social studies disagree whether the purpose of teaching climate issues is to develop independent and critical thinking, or to advocate for certain values and environmental change. If mathematics teachers implement climate change in the classroom, then this conflict is an important aspect to consider. Abtahi, Gøtze, Steffensen, Hauge, and Barwell (2017) found in a research survey that mathematics teachers expressed concerns about political and conflicting perspectives in climate change, and how this could affect their neutrality as teachers.

So, there is a profound emphasis in CME on mathematics as political and subjective and not neutral or value-free, and climate change is set forward as a potent topic for giving attention to these aspects in mathematics teaching. Yet, little research has focused on climate change in the mathematics classroom (Barwell, 2013). The focus of this paper is therefore on how controversies and teachers' value perspectives (can) influence teachers' facilitation of pupils´ critical mathematics perspectives.

CONCEPTUAL FRAMWORK – CONTROVERSIES AND VALUES

The field of CME is characterised by addressing social justice, the role of mathematics in society, and the importance of addressing how mathematics is used critically (Skovsmose, 1994). Nicol, Bragg, Radzimki, He, and Yaro (2017) explored different contexts for social justice and mathematics, including environmental ones, and found dialogue as a useful tool to engage with the discomfort and the potential controversies that can take place using such contexts. Atweh (2012) urged mathematics educators and researchers to include controversial topics in the classrooms. A controversial political issue is defined by Hess (2009, p. 37) as "questions of public policy that spark significant disagreement", and as open questions with many different and legitimate answers. In climate change, there are multiple controversies: Is there climate change? If yes, what are the causes (natural or anthropogenic factors)? What are the effects (on oceans, ice, weather, food, health)? What should be done about it (avoid emission or to cope with the impacts)? Who should bear the burden (rich vs poor country/people, polluter vs polluted)? To some of these questions there is a broad consensus by scientists, but in the public sphere and in politics they can be regarded as controversies. Including climate change in their teaching can therefore be challenging for teachers due to for example personal values or parents' opinions. Social context plays a role when deciding if a topic is controversial or not. In Norway, young people express positive attitudes towards reducing oil production. However, as a nation largely depended on the income from oil and gas extraction, this causes much debate and controversies among citizens and politicians. Yasukawa (2007, p. 10) claimed that CME has the "potential to provide people with the skills and inclination to question how mathematical information and methods are created, presented and used to construct the social and cultural world in which we live". To enable pupils to do this, they must do more than "pure mathematics". A critical mathematics perspective can be connected to mathematical literacy, defined as "the capacity to identify and understand the role that mathematics plays in the world, make well-founded judgments, and use and engage with mathematics in ways that meet the needs of one's life as a constructive, concerned and reflective citizen" (OECD, 2003, p. 24). These competences go beyond calculation and formal methods, and can connect mathematics with topics like climate change.

According to Gray and Bryce (2006), many science teachers avoid political interests and values when teaching certain topics. In mathematics education research, several have argued for a more ethical and value-based teaching (e.g. Atweh, 2012). Values can have a range of perspectives in climate change, such as cultural preference for equitable division of resources, individual interests vs collective ones, and level of altruism. People that hold specific values interpret information accordingly, as shown in beliefs about climate change by democrats and republicans in USA (Corner, Markowitz, & Pidgeon, 2014). Ernest (2009) argued that human values play a significant role in mathematics education, which problems and concepts we include for example, and that any choice is an act of valuation. He elaborated on the values of absolutist mathematics and of the values of social constructivist mathematics, and claimed that a social responsibility of mathematics is needed. In this paper, we see values as a foundation for the perceptions of how we should behave in society, and values can be guidelines for how people view controversies such as climate change. The concept of reflective knowing is relevant when dealing with values and controversies. Skovsmose defined this as "the competence needed to be able to take a justified stand" (1994, pp. 100-101). This competence goes beyond mathematical literacy as defined by OECD, and includes an ethical dimension with norms and values. Reflective knowing can involve evaluation and discussion of potential social consequences of climate change from a mathematical perspective. Skovsmose emphasised, in addition to the ability to evaluate mathematical facts and scientific information, the ability to "receive outputs" and to "provide inputs" to the system (1994, p. 101).

METHOD – A RESEARCH PARTNERSHIP

The data for the overall project was collected through a one-year research partnership with a researcher from teacher education and three mathematics and natural science teachers from lower-secondary school. The research partnership was inspired by action research and a collaborative research agreement based on equity, respect, and dialogue was established. The teachers were invited based on the researcher's knowledge about their engagement with climate change issues, a strategical sampling. The data material consists of video recordings of seven partnership meetings and 42 classroom interactions, audio recordings of interviews with the teachers and the pupils, the pupils' and the teachers' written materials, and observation and field notes.

The teachers explored different ways to facilitate pupil's critical mathematics perspective in iterative loops. They let the pupils collaborate on real-life data from their own excursions and from local community issues, work with specific tasks in the classroom (e.g. calculating average), discuss graphs, make posters, and attend a climate change exhibition. The principal made the exhibition mandatory for the teachers and their pupils. The teachers made a quiz for the pupils to hand out to by-passers on the exhibition, but they chose to make different versions of two of the quiz-questions. This stood out as an opportunity for investigating the teachers' reflections on what seemed as potential controversies and different value perspectives.

The data was transcribed, coded, and categorised, patterns were identified and analysed, all by using NVIVO. The authors analysed emerging findings as a group and controversies and values were singled out. The teachers' quotes were checked with the coded material for consistency of perspectives. In the analysis, the focus is on two of the quiz-questions, on the multiple-choice answers the teachers made (in particular on the use of numbers and graphs), and on the teacher´s utterances in a succeeding discussion about the quiz and CME.

RESULTS AND DISCUSSION

The teachers expressed concerns on the mandatory exhibition, one being to engage pupils in an activity just before their final exams. They therefore chose to make a quiz themselves, and participate with a group of five volunteers from each class. According to the teachers, the intention with the quiz was to get the by-passers on the exhibition to reflect on climate change issues with fact-based questions. The quizzes had five questions, each with four possible answers, one correct and three distractors. Kim made five questions. Max then used three of the same questions but chose to replace two of them. The following two questions are one of Max's own questions and one of Kim's questions that Max chose not to use, both with the correct answers marked:

> Max: How much sea ice was it in the Arctic in 2016 compared to a normal level?
>
> a) □ + 10% b) □ equal c) □ -10% d) ☒ -36%
>
> Kim: How much has the Earth's average temperature increased since 1998?
>
> a) ☒ Almost nothing b) □ approx. 1°C c) □ approx. 2°C d) □ approx. 4°C

Both questions concern key topics in climate change: Max's question concerns sea-ice levels in Arctic, and the four alternatives, +10 %, 0 %, -10 % and -36 %, suggest both an increase and a decrease of sea ice. The correct answer, -36 %, is the most extreme choice. Kim's question concerns the global temperature, and the correct answer and the three distractors all suggest a small increase in the average temperature change. The correct answer, almost nothing, is at one end of the scale. Kim had included a graph on the backside of the quiz to justify the correct answer (Figure 1, graph on the left).

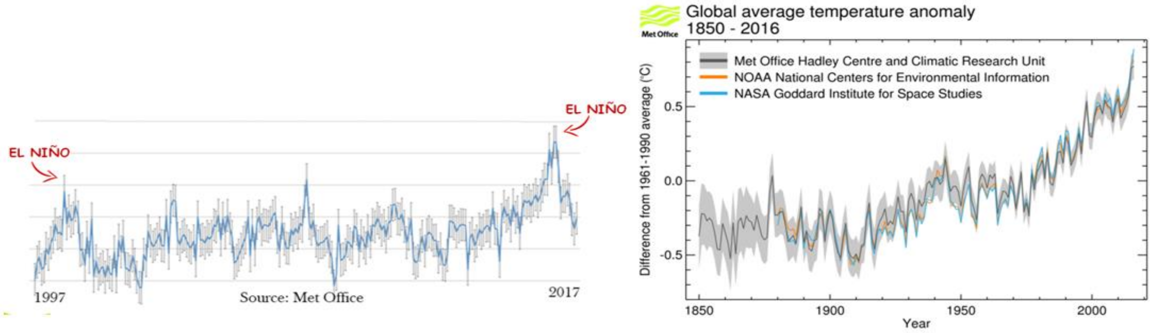

Figure 1: Two graphs showing global temperature. The graph on the left with a 1997-2017 time span, and the graph on the right with 1850-2016 time span.

Controversial aspects and value perspectives: the case of Max

The correct answer stands out from the three alternatives because it is not a round number, a multiple of ten, and by the difference between the alternatives (10 versus 26). Max elaborates on the intention of the question and the alternatives by saying: “so it is a bit to illustrate that it has changed … a lot.” By the choice of numbers and by the way the correct answer stands out from the others, Max implicitly brings forward that the temperature has changed a lot. Location can be essential when discussing ice melting, while Artic experience an alarming ice melting, Antarctic is more status quo. When Max and Kim reflect on the sea ice question, the following dialogue takes place:

> Max: … so, maybe it is deliberately cheating when leaving out Antarctic, I admit that …
>
> Kim: Did you try to influence?
>
> Max: Yes, I did.

In the dialogue, Max refers to excluding Antarctic in the question as “cheating”, followed by “I admit that”. The wording reflects a confession, as if Max has distorted the facts. The utterance can reflect mixed feelings for the choices Max did, and Kim pursue this when asking if Max tried to influence. In the subsequent conversation, Max said that it would be crucial to include ice melting in other areas in a classroom interaction with pupils in order to give a more nuanced picture. The dialogue between the teachers clarifies the value aspect of Max’ choice of question, and Max expresses that discussing the topic with others is enriching for own understanding and teaching. The value perspective, the wish to influence, occurs also in other statements by Max: “They should reflect, that is one aspect, but another thing I would like is that they should act … it is important that they make conscious choices about consumption.” Max considers mere reflection as insufficient, pupils “should act” as well. This can be interpreted as a teacher’s *facilitation of action*. Max’s perspective relates to Skovsmose’s reflective knowing and output-perspective in which reflective knowing is something more than just reflection. This perspective is also in accordance with how political guidelines promote a normative view by emphasising that pupils should believe in and take actions towards a sustainable development. When asked if and how climate change controversies can affect teaching, Max says: “I think it is important that ... in a way, they also face this in school … that there are different views on things."

Controversial aspects and value perspectives: the case of Kim

Kim's argument for choosing the distractors for global average temperature, 1°C, 2°C, and 4°C, was to use common figures from media. Like Max, Kim chose to place the correct answer, "almost nothing", at one end of the scale. Although the distractors are commonly used, projections from the Intergovernmental Panel on Climate Change estimate a temperature increase in a range from 0.1°C to 0.3°C per decade (IPCC, 2007). Another reason Kim gave was: "I have tried to make questions that … where they get answers that they do not think is right, in a way. To get them to … oh, that was not what I expected." Exaggerated estimates of temperature change is a typical misconception in climate change (Gowda, Fox, & Magelky, 1997). The emphasis on making questions with unexpected answers to trigger pupils' reflection can be interpreted as Kim's aim to challenge the temperature misconception. Climate change is defined by a change that lasts for an extended period of time, and most scientist do not consider temperature increase as a controversial issue. What may lead to controversies is when someone deliberately uses short or very long time spans (thousands of years), to justify arguments claiming no temperature increase. When the teachers discuss the graph, Max comments Kim's reason for choosing the specific time span, 1998-2017:

> Max: But it was a reason why you chose 1989 and not 1889.
>
> Kim: Yes, yes, this is the last 20 years.

By saying that "it was a reason" why Kim chose 1989 and not 1889, Max challenges Kim's choice, and asking for a reason could uncover Kim's motive for choosing that particular time span. It can be questioned, as Max did, why the graph shows these years, and why Kim chose to use this graph. Giving attention to an overestimated temperature rise can indicate something about Kim's values. This is strengthen by the choice of words used for the correct answer: "almost nothing." Highlighting that the temperature rise is close to nothing can be understood as an attempt to reveal misconceptions and trigger pupils' critical reflection. However, the time span choice can also be understood as an argument against climate change. Kim later presented a graph with a longer time span (Figure 1, graph on the right). Using different graphs indicates that Kim is aware of the potential value-laden perspective of short time spans.

Kim highlighted controversies as important: "I emphasise all the controversies … I try to get the pupils to think. Let them form their opinion. And, I think climate change is an excellent opportunity to do this." Kim takes the position as a facilitator of discussions by getting "the pupils to think" and to "let them form their opinion". This can be interpreted as an emphasis on taking a neutral stand, to not reveal own opinion, and act more like a *facilitator of critical thinking*. Kim emphasises that the pupils should take a stop and reflect on the perceived reality, because the reality may not necessarily be correct. Furthermore, Kim emphasised pupils' ability to be critical with comments such as "they should be critical. A critical mathematics perspective, I hope they learn this. A lot of it".

CONCLUDING COMMENTS

In line with Ernest's (2009) argumentation for human values' influence on mathematics, we find that values and controversies in climate change can influence teachers' facilitation of CME. Max and Kim chose different questions to emphasise particular issues in climate change. While Max highlighted the extensive ice melting, Kim emphasised how small the temperature change was, and both used mathematics to do this. Their different questions and reflections reveal controversies and value aspects. Controversies is perceived as a challenge when teachers consider using climate change in mathematics education. Political guidelines and curricula can put teachers in a challenging and risky situation. Mathematics has traditionally been regarded as a neutral subject with little controversy. This can make it extra challenging to include wicked problems like climate changes in mathematics lessons. The tension between facilitating action versus critical thinking is revealed in the teachers' practise and dialogues. Max emphasises to enable pupils to understand causal relationships that empower reflected choices, by which they can act and create a better future. Kim emphasises to help pupils find their own answers and critically engage in how mathematics is used in argumentation. The teachers see themselves as agents with an important message. Facilitating action can be questioned if it promotes a political agenda with a desire to influence pupils' political standpoints. Facilitating the pupils' critical independent decisions, without revealing own viewpoint, may disclose that choosing certain graphs and numbers is not always neutral. We find that the teachers' question to each other made the controversies and value aspects more explicit. In our opinion, climate change is a major challenge that should be included in mathematics education, both in research and practice. Exploring how teachers find opportunities in the tension between facilitating action and facilitating critical thinking is crucial to address. We consider it as an opportunity, as well as a challenge, to engage pupils in wicked problem within the frames of the school context.

References

Abtahi, Y., Gøtze, P., Steffensen, L., Hauge, K. H., & Barwell, R. (2017). Teaching climate change in mathematics classroom: An ethical responsibility. *Philosophy of mathematics education journal*, 32, 1-18.

Atweh, B. (2012). Mathematics education and democratic participation between the critical and the ethical: A socially resonse-able approach. In O. Skovsmose & B. Greer (Eds.), *Opening the Cage* (pp. 325-341). Rotterdam: Sense.

Barwell, R. (2013). The mathematical formatting of climate change: critical mathematics education and post-normal science. *Research in Mathematics Education, 15*(1), 1-16.

Corner, A., Markowitz, E., & Pidgeon, N. (2014). Public engagement with climate change: the role of human values. *Wiley Interdisciplinary Reviews: Climate Change, 5*(3), 411-422.

Education Act. (1998). *Lov om grunnskolen og den vidaregåande opplæringa* LOV-1998-07-17-61. Retrieved from https://lovdata.no/dokument/NL/lov/1998-07-17-61

Ernest, P. (2009). Values and the social responsibility of mathematics. In B. Sriraman, P. Ernest & B. Greer (Eds.), *Critical Issues in Mathematics Education* (pp. 207-216). Charlotte: Information Age Publishing.

Frankenstein, M. (2010). Critical Mathematics Education: An Application of Paulo Freire's Epistemology. *Philosophy of mathematics education journal,* 25, 1-24.

Gowda, M. R., Fox, J., & Magelky, R. (1997). Students' understanding of climate change: Insights for scientists and educators. *Bulletin of the American Meteorological Society, 78*(10), 2232-2240.

Gray, D. S., & Bryce, T. (2006). Socio-Scientific Issues in Science Education: Implications for the Professional Development of Teachers. *Cambridge Journal of Education, 36*(2), 171-192.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37-68.

Hess, D. E. (2009). *Controversy in the classroom*: *The democratic power of discussion.* New York: Routledge.

Ho, L. C., & Seow, T. (2015). Teaching Controversial Issues in Geography: Climate Change Education in Singaporean Schools. *Theory & Research in Social Education*, *43*(3), 314-344.

IPCC (2007). *Climate Change 2007: The Physical Science Basis. Summary for Policymakers*. Cambridge: Cambridge University Press. Retrieved from http://www.ipcc.ch/pdf/assessment-report/ar4/wg1/ar4_wg1_full_report.pdf

Ministry of Education and Research (2016). *Fag - Fordypning – Forståelse - En fornyelse av Kunnskapsløftet*. (Meld. St. 28 2015–2016). Oslo: Ministry of Education and Research.

Levin, K., Cashore, B., Bernstein, S., & Auld, G. (2012). Overcoming the tragedy of super wicked problems: constraining our future selves to ameliorate global climate change. *Policy Sciences, 45*(2), 123-152.

Mellin-Olsen, S. (1987). *The politics of mathematics education* (Vol. 4). Dordrecht: Reidel.

Nicol, C., Bragg, L. A., Radzimski, V., He, T., & Yaro, K. (2017). *Teaching mathematics for justice: Pedagogies of discomfort, contradictions and dialogue.* In B. Kaur, W. K. Ho, T. L. Toh & B. H. Choy (Eds.), Proc. 41st Conf. of the Int. Group for the Psychology of Mathematics Education (Vol. 3 pp. 297-304), Singapore: PME.

OECD (2003). *The 2003 Assessment Framework: Mathematics, Reading, Science and Problem Solving Knowledge and Skills*. Paris: OECD.

Skovsmose, O. (1994). *Towards a Philosophy of Critical Mathematics Education*. Dordrecht: Springer.

Skovsmose, O. (2014). Critical Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 116-120). Dordrecht: Springer Netherlands.

Yasukawa, K. (2007). An agenda for mathematics education in the decade of education for sustainable development. *Nordic Studies in Mathematics Education, 12*(3), 7-24.

OBJECTIFING TINGA: A CASE OF CHILDREN INVENTING THEIR OWN DISCOURSE ON FRACTIONAL QUANTITIES

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Much has been written about differences between ontogenetic and the historical development of mathematics. In this paper, I present a case of learning that may inform our vision of what have happened when people began thinking in terms of fractional quantities for the first time. The case is taken from a large research project, in which I follow the development of the discourse of rational numbers. The study provided insights into both historical and ontogenetic development of that discourse.

Much has been written about differences between ontogenetic and the historical development of mathematics. Such differences are clearly visible in the “commognitive” models of these two developments (Sfard, 2015) that were corroborated in empirical research on early numerical thinking (Lavie & Sfard, 2016). There are exceptions, however. In this paper, I present a special case of learning that may be interpreted as quite close in its trajectory to historical development. The case is taken from a large research project, whose aim is to map the development of the discourse of rational numbers (the project has been conducted with Anna Sfard). It is the story of a second-grader named Amir, who lacked formal knowledge of fractions and created a new mathematical object while grappling with the task of fair sharing. Amir continued building this object also after he began learning about fractions in school. Two years have passed until his idiosyncratic discourse coalesced with the canonical discourse of rational number. In this paper, after presenting the framework that guides me in my research, I follow that process from its beginning.

THEORETICAL BACKGROUND

Among all the topics in school curriculum, rational numbers hold the dubious distinction of the most difficult to teach, the most mathematically complex and the most cognitively challenging for students in many countries (OECD, 2014). Its learning has been vigorously studied by mathematics education researcher for at least thirty years now. It was agreed that rational numbers should be characterized as a set of related but distinct constructs rather than as a homogenous single one (Behr, Lesh, Post, & Silver, 1983). In spite of the popularity of this model, its critics sustained that the division into different interpretations of the rational number is insufficient for describing children's construction of the concept (Olive & Lobato, 2008). Indeed, only few of the many studies, if any, offered a satisfactory panoramic picture of the learning of rational numbers (Lamon, 2007).

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In the hope to overcome these weaknesses, I adopt theoretical perspective known as commognitive (Sfard, 2008), according to which mathematics is a form of communication – a discourse. The discourse of rational numbers is identifiable by four characteristics: its *special words* and their use, its *visual mediators* and their use, the *narratives* that are *endorsed* by the discourse community, and *discursive routines*, which are patterns of discursive actions a person tends to perform. Terms such as *three quarters*, or symbols, such as $^3/_4$ are used as *signifiers* of *mathematical objects*. This last term refers to the signifier itself together with its multiple interconnected *realizations*, that is, other entities (terms, or visual mediators) that in some contexts may be considered as its equivalents. For example, in the activity of baking a cake, the signifier: "¾" is realized by a cup with an appropriate amount of sugar, whereas in activities with number line it is realized as a specific point on this line. Artefacts are exchangeable in their roles as the principal signifiers and realizations. When one tries to bake a double-size cake, ¾ turns into a realization of the aforementioned cup of sugar, and it may now be doubled (2 x ¾) before any operation on the cup and the sugar is performed. The object one creates for a given signifier depends on this person's experiences and understandings.

According to the commognitive model of the historical development, mathematical discourses originated in practical activities that these discourses came to refine and extend. The discourse on rational numbers might well have its roots in early human attempts to implement comparisons or equal sharing of continuous quantities – length, area, etc. It might have began with an introduction of a new signifier to be used as a noun that, given a certain whole, enabled constructing the the required part. The discourse that evolved around this new signifier brought about an improvement of the activities. Subsequently, the signifier underwent objectification, that is, became a name of a mathematical object. The process involved several discursive moves, among them (1) *saming*, the act of associating one signifier with many realizations and (2) *reification*, turning narratives about processes into ones about objects.

Today, it is through the process of learning that children gradually become participants of historically established discourses. The way it happens is bound to deviate from the historical trajectory because the formal discourse is introduced to children readymade and the learner is not required to invent new signifiers or to decide how to use them in discourse. Rather than an act of an independent construction, learning rational numbers is the activity of individualizing the historically established discourse of rational numbers. It begins with the learner's exposure to new words and symbols appearing in such everyday expressions as "half an hour" or "quarter to six", and proceeds with the introduction of the formal discourse of rational numbers in school. One of the teachers' most challenging tasks is to help the child to associate this discourse with daily activities. As has been shown in research, many students graduate without succeeding in seeing connections between rational numbers and real-life tasks. As I will argue below, these connections may be there from the very beginning if fractions appear in learning the way they entered history: as a by-product of attempts to implement a familiar practical activity.

#	Code	Task	additional questions
1	1-to-4	Name numbers between one and four	Are there more such numbers?
2	Story	Tell me a story using the word *half*	Repeat with *quarter* and *third*
3	sharing marbles	Danny has four packs of 12 marbles. He wants to divide them equally between his three friends. How many marbles would each friend get?	
4	4-pizzas-for-3	Three children shared 4 pizza equally. How much pizza each child got?	If a number of slice is stated: What is the size of a slice?
5	5-bisquits-for-3	How would you divide equally five biscuits among three kids?	

Table 1: Examples of tasks

STUDY DESIGN

The longitudinal research project from which this case study is taken aims at mapping the learning of rational numbers over a period that begins prior to their introduction in school and ends when the student may be expected to be already a competent participant of the discourse. To this end, I have been repeatedly interviewing 12 pairs of children from grades 1st to 6th (in Israeli school, fractions are introduced in the third trimester of the 3rd grade or in the beginning of the 4th). During the interviews, I have been engaging the students in activities that occasion the use of rational numbers, but can also be successfully executed without these artefacts. For these activities, a battery of 25 tasks was designed. Some of the items are presented in Tab. 1. For two years, the participants had to cope with all 25 tasks every four month. The interviews, held in Hebrew, have been recorded and transcribed. For this report, some episodes have been translated into English.

THE CASE OF AMIR

Amir and his interview partner Noa joined the study as second-graders, when they did not have any formal acquaintance with fractions. In interview 1, Amir was already familiar with the words *half* and *quarter* and seemed to use these words properly in a sentence (this became clear when he coped with Task 3). He could also relate to half and quarter as quotients (as in the case of 2 or 4 children sharing one snack) and as an operator (“I won half of Dan's marbles”). He did not recognize half and quarter on the number line. Amir was also familiar with the word *third*, but he identified this word (as did other participants of our study) with *three quarters*. Amir, in the absence of a word suitable to describe the part resulting from dividing a whole into three equal parts, invented the word *tinga* that corresponded in its later use to *third*. The initial event and the subsequent process of objectification will now be presented as a series of four discursive developments: (1) naming, (2) saming, (3) reifying and (4) coalescing of *tinga* with the *third*. The developments will be shown with the help of the six episodes involving tasks from Tab.1 and taken from consecutive interviews.

Naming. Task 4 raises the need to talk about one part obtained from an equal division of continuous quantity into three parts. In interview 1, which took place when Amir was in the 2nd grade, he coped with Task 4 as follow: (1) he gave each child one pizza; (2) divided the remaining pizza into quarters, (3) gave each child a quarter, (4) divided the remaining quarter into three "small parts" and gave each child one of them. Amir's answer was "Each child got one pizza and quarter of a pizza and this small part". Four month later, in his second encounter with the task during interview 2, Amir, still the second-grader, handed out a whole pizza to each diner as before. Then the following discussion took place:

1.	Amir:	How do you call that thing… I don't really know how to describe it. Do you happen to have an erasable board? [*talks to the interviewer*]
2.	Noa:	Draw it in your paper.
3.	Amir:	There is that thing, I do not remember what it is called? But it is a little more than a quarter. [*draws a circle*]
4.	Noa:	Third?
5.	Amir:	No, not third, not half.
6.	Noa:	Eighth?
7.	Amir:	Lets' say this is a circle and it has three halves and this adds to this. How do you call it? [*divides the circle into quarters and "add" to one of the quarters, shades it and points at it*]
8.	Intr:	Lets' say we call it John. How did you get this part?
9.	Amir:	Oh! Tinga! So we will give a tinga to every child.

Amir realized that there might be a name for what he called before the "small part", but he did not know that name. He dismissed Noa's suggestion of the third, because for him, this meant three quarters.

The invented word *tinga*, which Amir eventually used as a name of the part, reappeared in the same interview as a name of an action rather than object. In response to Task 5 (see Tab. 1), Noa put one biscuit in each of the three plates, and then, after splitting the remaining two biscuits into quarters, added two quarters to each plate. Amir was watching her and eventually asked: "What will you do with the last one (quarter)?" While Noa' answer was slow to come, Amir walked away and said quietly to himself "Now you will do *tinga*". The verbal clause *do tinga* indicates that Amir was using *tinga* for both the activity and *the physical* product (not quantity) of fair sharing into three parts.

Saming. The task involving 3 pizzas and 4 children returned in interview 3. Once again, Amir began by allotting half a pizza to each child, but then he had hard time trying to divide the circle representing the remaining pizza into three equal parts. Inspired by his partner Noa, who chose to represent pizzas as rectangles, he started again with pizzas of that shape. Using the rectangular model he divided the remaining pizza into three parts and stated "Each will get one and tinga". Amir's use of the word *tinga*, therefore, went beyond the specific case of the circular pizza. As could be seen from what happened when Amir was presented with Task 3, this use was even broader:

1. Amir: How many marbles exactly? or with halves?
   [*Draws four circles, divides them into two*]
2. Amir: Oh wait a minuet! I'm on to something! Do you remember the tinga?
   [*Draws another four circles*]
3. Intr: Yes.
4. Amir: Signe… I don't know how to draw tinga here …
   [*Tries to divide the first circle into three equal parts*]
5. Intr: Why are you using tinga here?
6. Amir: Because, he has three friends, right?
7. Intr: Yes
8. Amir: So each friend will get tinga, tinga, tinga, tinga
   [*marks each circle when saying "tinga"*]

Amir was using the signifier *tinga* while sharing three pizza among four people and also while dividing four bags of 12 marbles among three children! He thus associated the signifier *tinga* with two quite distinct – discrete and continuous - realizations: (1) the portion of pizza that each person got; (2) The set of marbles that each child got. This was a clear case of a rather extensive saming.

Reifying. During the first three interviews, when Amir and Noa were asked to name numbers between one and four (Task 1), they answered "two and three", and to the researcher question "Are there any other numbers?", they responded with the unanimous 'no'. In interview 4 that took place when the children were in the third grade, their response changed.

1. Intr: Do you know any other numbers between one and four?
2. Noa: Two and a quarter, two and a half, two and three quarters
3. Amir: Two and tinga

Here, Amir offered *two-and-tinga* as an optional *number* between one and four. This was the first time Amir was using tinga without associating it with either concrete objects or activity.

Another step towards the reification of the signifier tinga was observed in interview 5. Still the third-graders, Amir and Noa opened that interview with an announcement: "We changed the name! Tinga is now a *thirdth*" (I am using this distorted English version of the word for $^1/_3$ to signal the non-standard use of its Hebrew counterpart, *shlish*; the use was modelled on the Hebrew regular names for the other parts). While asked for his reasons for the change, Amir said:

> Amir: Tinga is when you can divide a circle into three parts. You call a circle that is divided to eight parts [*draws a circle and divides it into eight parts; points to one of the obtained parts*] *eighth*. So I just said *thirdth*… all are ending with *th*.

The children decided to name the obtained part according to the number of the resulting parts of the activity. At this point, the signifier was also disassociated from the activity. From now on, Amir referred to the activity as "dividing into three equal parts", and to the obtained portion as "thirdth".

Coalescing of *tinga* with the *third*. Interview 6 was conducted in November, three weeks after Amir and Noa, now the fourth-graders, started learning fraction in school. In the very beginning of the interview Amir announced: "Tinga is a third".

DISCUSSION

We have already noted earlier in this article that one cannot infer from the ontogenetic development to the historical development of the discourse of fractions (or vice versa). Historical development of discourses involves acts of creation, the results of which are later individualized in the process of learning. The case of Amir gave us a rare glimpse into learning involving acts of creation not unlike those that might have taken place during the historical development of the discourse on fractions. Here, like in history, an idea for something like a fraction emerged out of the necessity for a discourse that would mediate and refine activities of equal sharing. Of course, there were differences. Unlike our ancestors who invented fraction, Amir already knew words for certain parts, *half* and *quarter*, and was familiar with some of their everyday uses. Still, for these two words to inspire the idea of *tinga* he had first to become aware of several facts. First, he needed to realize that underlying the words *half* and *quarter* there was the action of splitting into a constant number of equal parts (at this point, “equal”, rather than implying identical quantities, referred to congruent shapes). Second, he had to know that the name of the part resulting from an equal splitting should depend on the number into which the whole was split and on nothing else, not even on the question whether the whole was a discrete set or a continuous object. The replacement of *tinga* with *thirdth* came with yet another insight, according to which it would be useful to coin the name of the part so that it hints at the number into which the whole had to be split. All these realizations constitute critical steps on the way toward fractions. They were all made by Amir and Noa on their own, with no significant help from the interviewer.

There is much to admire in this special case of learners arriving at the more-or-less canonical version of a historically established discourse on their own. Because of the contingency of acts of invention, however, and in the light of time constraints imposed by school timeframes, it would be a mistake to ask teachers to reproduce this success. The main value of our study is in the insights it offers about both historical and ontogenetic development of the discourse on fractions.

References

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. *Acquisition of Mathematics Concepts and Processes*, 91–126. article.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. *Second Handbook of Research on Mathematics Teaching and Learning, 1,* 629–667. JOUR.

Lavie, I., & Sfard, A. (2016). How children individualize numerical routines – elements of a discursive theory in making (in Hebrew). *Studies in Mathematics Education, 4,* 22-68 .

Olive, J., & Lobato, J. (2008). The learning of rational number concepts using technology. *Research on Technology and the Teaching and Learning of Mathematics: Research Syntheses, 1,* 1–53. JOUR.

Organisation for Economic Co-operation and Development. (2014). A profile of student performance in mathematics. In PISA 2012 results: What students know and can do. Student performance in mathematics, reading and science (Rev. ed., Vol. I, pp. 31–144).

Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing.* book, New York: Cambridge University Press.

Sfard, A. (2015). *Learning, commognition and mathematics*. In D. Scott & E. Hargreaves (Eds.), The Sage Handbook of Learning (pp. 129 - 138). London: Sage.

INVESTIGATING SELF-EFFICACY EXPECTATIONS AND MASTERY EXPERIENCES ACROSS A SEQUENCE OF LESSONS IN MATHEMATICS

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While a few longitudinal studies have provided empirical data on the relationship between students' mathematics self-efficacy expectations (SEE) and mastery experiences (ME) at the macro-level (across months or years), there is little empirical evidence for the theoretically proposed process of SEE change at the micro-level (across lessons or learning events). We aimed to address this research gap using a micro-analytic design with repeated measures across mathematics lessons involving grade 6 (n=81) and grade 10 (n=100) students. Path models with cross-lagged effects illuminated the relationship between SEE and ME: the dominant direction was from SEE to ME, but the effects from ME to SEE were also substantial. Implications for both theory and practice are discussed.

INTRODUCTION

Self-efficacy expectations (SEE) are important because they are related to a number of positive learning behaviours and outcomes in mathematics, such as learning strategies, perseverance, and choice of career paths (Zimmerman, 2000). Understanding how mathematics SEE change and develop during classroom settings is important both to further our understanding of the construct and to inform classroom practice. Mastery experiences (ME) are established as the dominant source of SEE through both theory and empirical studies (Usher & Pajares, 2009). According to social cognitive theory (Bandura, 1997) students develop their SEE iteratively on the basis of appraisals of their ME, which highlights the importance of students' ME during lessons in mathematics.

While some longitudinal studies have provided empirical support for a reciprocal relationship between SEE and mathematics performance (see Talsma, Schüz, Schwarzer, and Norris, 2018), these studies have generally included long gaps between measurement occasions, e.g. months or years. While informative, such longitudinal designs do not facilitate investigation into the theoretically proposed process of SEE change, including cycles of ME and SEE appraisals across hours or days. We found only one previous study employing a micro-analytic approach to investigating mathematics SEE and performance, however this was in the context of an automated algebra tutor rather than a natural classroom setting (Bernacki, Nokes-Malach, & Aleven, 2015). Furthermore, while social cognitive theory posits levels of perceived task difficulty might influence students' SEE, investigations on mathematics SEE rarely focus on this

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 243-250). Umeå, Sweden: PME.

(see Street et al., 2017). As such, we do not know whether or how task difficulty may influence the process of SEE change during classroom learning. In order to address the above gaps in the research we designed a study including multiple measurement occasions across lessons in mathematics, using multidimensional, item-by-item matched measures of SEE and ME. We investigated the following research question: *What is the relationship between self-efficacy expectations and mastery experiences across a sequence of regular lessons in mathematics?*

THEORETICAL FRAMEWORK

According to social cognitive theory (Bandura, 1997) SEE are individuals' confidence regarding their capability to perform on future tasks or events. Self-efficacy is sometimes referred to as *beliefs* and sometimes as *expectations* – we have chosen the term expectations in order to bring attention to the focus on future attainment. There are four theoretically proposed *sources* of SEE (Usher & Pajares, 2009), where the dominant source is ME. Students' previous experiences of success or failure in mathematics form a basis for the formation of subsequent mathematics SEE. Importantly, ME are students' *interpretations* of experiences, not their objective performance. According to Bandura (1997), ME are appraised in regard to the perceived difficulty of the task – while optimal challenges are important in order to facilitate experiences of success, succeeding on hard tasks is likely to boost students' SEE to a larger degree than succeeding on easy tasks.

According to Bandura (1997) it is beneficial to hold somewhat optimistic SEE. This leads to positive learning behaviours, which are in turn related to positive performance or learning outcomes. The reciprocal relationship between mathematics SEE and ME is central to the theoretically proposed process of self-efficacy change, yet has been investigated empirically in only few longitudinal studies we are aware of. A recent meta-analysis on the reciprocal effects between academic SEE and performance (Talsma, Schüz, Schwarzer, and Norris, 2018) found the effect from performance on SEE was three times stronger $\beta = .21$) than the effect from SE on performance ($\beta = .07$). Out of 347 studies identified the authors found only 2 including children's mathematics SEE and performance measures, highlighting the need for more such studies.

In two studies Hannula and colleagues investigated Finnish school students' mathematics SEE and performance across several years. Hannula, Maijala, & Pehkonen (2004) found a main effect from students' SEE on their performance, while reciprocal effects were supported in a subsample of the data. In contrast to the 2004 study, Hannula, Bofah, Tuohilampi, and Metsämuuronen (2014) found the effect from ME on SEE was stronger than the effect in the opposite direction. The difference in magnitude was reduced in the subsequent time lag, when students were older. In contrast to the long time lags commonly used, Bernacki et al. (2015) applied a micro-analytic design to investigate how learners' SEE related to their performance in algebra, using an intelligent tutoring system. Sequential prompts elicited SEE every 5 minutes on average, after students received automated feedback on their performance. In a path model Bernacki et al. (2015) found that previous performances predicted SEE, but not the other way around. Correlational analyses indicated relationships between SEE and subsequent performance indicators.

While the above studies all included direct measures of performance, Phan (2012) investigated the relationship between students' ME and the growth of their SEE across one year. In contrast with both theory and empirical studies, Phan found a negative relationship between these constructs (β = -.26). Phan questioned whether grade 3 and 4 students had enough personal learning experiences in mathematics to formulate their SEE. An alternative explanation is a faulty conceptualisation of the ME measure, which had failed to achieve construct validity in previous studies (see Usher & Pajares, 2009).

METHODOLOGY

We employed a micro-analytic design, including measures of students' SEE and ME for each of a sequence of 3-4 lessons when they were introduced to a new topic in mathematics, as students' SEE are most likely to change as they engage with new tasks. Furthermore, this design was chosen as previous studies have found that longer time lags are associated with weaker cross-lagged effects (Talsma et al., 2018). Participants were students in four Norwegian grade 6 classes (n=81) and five grade 10 (n=100) classes (mean age 11 and 15, respectively). This was a subsample from a larger study (see Sørlie & Söderlund, 2015), where schools were strategically selected in order to be representative of the Norwegian population of state school students. Talsma et al. (2018) reported the effect from SEE on performance was not supported in the case of children. Including school age students in our study enabled investigations into this theoretically proposed relationship operationalised at a micro-analytic level. 93% of students consented to take part after receiving written information.

Measures

Our study included three measures. The *Self-Efficacy Gradations of Difficulty* (SEGD) is a multidimensional measure that we developed in an earlier study (Street et al., 2017) and adapted in two versions for the present study. The SEGD long version is focused on mathematics as a subject, while the short version focuses on lessons in mathematics. Prior to the first lesson, the students filled out the long version of the SEGD (15 items; Cronbach's alpha .92), while at the beginning of every lesson they filled out the short version of the SEGD. This included 9 items on students' SEE for learning (example item *I can learn something new, if I get lots of help from the teacher*), problem solving (example item *I can solve all the medium difficulty tasks*), and self-regulation (example item *I can persevere when faced with very hard tasks*) in mathematics, where students respond on a scale from 0 to 10 (Cronbach's alpha from .92 to .94). The *Mastery Experiences Gradations of Difficulty* (MEGD) is a multidimensional measure developed for the present study, to measure students' ME of learning, problem solving, and self-regulation in mathematics. The MEGD is modelled on the short version of the SEGD, and each of the 9 ME items are matched with a SEE item (example item *I did not give up when faced with very hard tasks*), in regard to both facet-specificity and difficulty (Cronbach's alpha from .93 to .96). Previously, larger effects between SEE and ME have been found in studies where measurements are aligned with theoretical tenets (Talsma et al., 2018). In our study the measures of SEE and ME are all in congruence with the main conceptualisations of the constructs. Furthermore, we include students' self-reports of ME rather than objective performance measures which may differ from their perceptions of success, a limitation from previous studies (Talsma et al., 2018).

Analyses

We used SPSS for data management and MPlus for structural equation modelling. We inspected the correlation matrices and considered autoregressive models for each construct separately, before specifying models including cross-lagged paths. We used the following parameter estimates to determine model fit: The chi square (χ^2/df <3 acceptable), the Root Mean Square Error of Approximation (RMSEA <.08 acceptable), the Comparative Fit Index (CFI) and the Tucker-Lewis index (TLI) (CFI/TLI > .90 acceptable), the Standardized Root Mean Square Residual (SRMR<.10 acceptable) (Schermelleh-Engel, Moosbrugger, and Müller, 2003). In addition we screened whether parameter estimates were meaningful and within bounds, e.g., standard errors and residual variances. For further model comparisons we applied ΔRMSEA (.015), ΔCFI (-.010), and ΔSRMR (.030) (Chen, 2007) as cut-offs. For tests of invariance, we considered in addition the $\Delta\chi^2$.

RESULTS

The factor structure of our measures

For all three measures, we specified a series of six alternative empirical models on the basis of theory and previous studies (Street et al., 2017), with the main aim to inspect whether models including latent facets (i.e., learning, problem solving, and self-regulation) or latent levels (i.e., easy, medium difficulty, and hard tasks) fit best. For both SEE and ME a Latent Levels Model, including three levels of difficulty and correlated uniquenesses for facet-specificity, was chosen as the best-fitting measurement model, consistent with previous findings (Street et al., 2017).

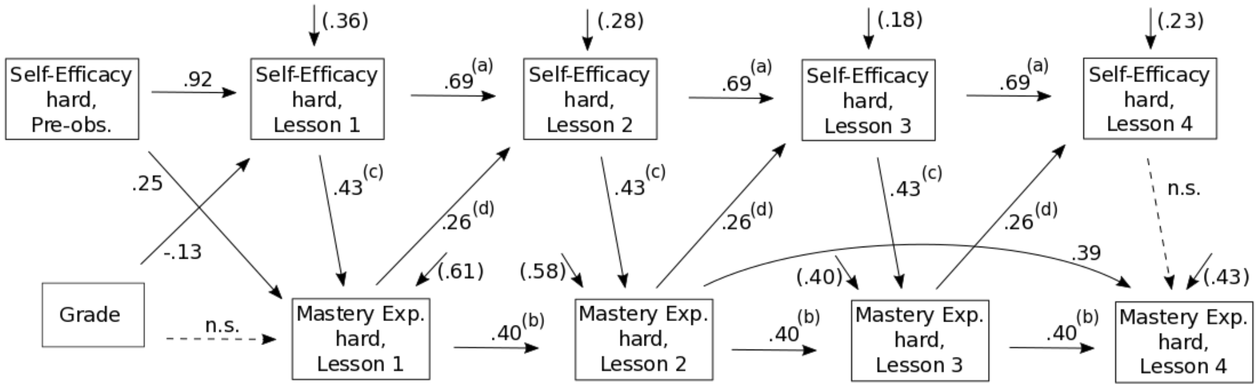
In order to investigate measurement invariance for each of our repeated measures, we specified three models, one for each level of difficulty, which each included a parcel based on the measurement model described above (SEE / ME for easy, medium difficulty, and hard tasks). All models were either strongly or partially strongly invariant, indicating it is reasonable to use these to investigate changes over time.

The relationship between students' Self-Efficacy Expectations and Mastery Experiences across lessons in mathematics

In order to investigate our research question we specified three cross-lagged path models, one for each of the levels of difficulty. These models included autoregressive paths for SEE and ME, and cross-lagged paths representing the effect of SEE at the beginning of a lesson on ME at the end of the same lesson, and the effect from ME at the end of a lesson on SEE at the beginning of the subsequent lesson (see Figure 1). Furthermore, we included grade level and students' mathematics SEE prior to the first lesson as covariates. We first tested models specifying simplex change processes for both SEE and ME, however these did not result in "acceptable" fit indices. As modification indices suggested adding an additional path, we specified models with a path for students' ME from lesson 2 to lesson 4.

For each of the levels of difficulty, we compared reciprocal effects models (including cross-lagged paths in both directions) with unidirectional models (paths from SEE to ME only, or vice versa). Consistently, the reciprocal effects models fit best. Finally, we included in the best fitting models (see Figure 1) equality constraints on autoregressive and cross-lagged paths, in order to investigate relative construct stability over time, as well as the dominant direction of the cross-lagged effects. Figure 1 displays the model structure and unstandardised parameter estimates for the hard level SEE and ME constructs, as an illustrative example. Some tendencies, displayed in this model, were common across all three levels of difficulty.

Figure 1: Cross-lagged path model of mathematics Self-Efficacy Expectations and Mastery Experiences
Hard Level of Difficulty constructs



$\chi^2_{(33)} = 79.79$, $p < .001$. CFI=.95, TLI =.93, RMSEA=.09, SRMR=.08

Self-Efficacy hard, Pre-obs. = Students' mathematics Self-Efficacy Expectations for hard tasks, prior to Lesson 1
Self-Efficacy hard, Lesson 1-4 = Students' mathematics Self-Efficacy Expectations for hard tasks, for the upcoming lesson
Mastery Exp. hard, Lesson 1-4 = Students' mathematics Mastery Experiences on hard tasks, from the last lesson

(a) & (b) indicate equality constraints on stability paths, (c) & (d) indicate equality constraints on cross-lagged paths.
Paths are unstandardised, estimates in brackets are standardised (STDYX)

First, for all three level constructs students' mathematics SEE predicted their lesson 1 SEE and ME (positive relationship), and grade level predicted their lesson 1 SEE (negative relationship). Furthermore, students' SEE and ME were significantly stable over time, and stability paths did not differ significantly between the lessons. Restraining the stability paths for each construct as equal, we could compare their relative stability: In the case of all level constructs, students' SEE (path a) were significantly more stable than their ME (path b). For each level of difficulty, we tested also whether the magnitude of the cross-lagged effects differed across time, and the dominant direction of these effects. In case of the easy level construct, there were no differences across time for either of the cross-lagged effects, while in the case of the medium and the hard level constructs the path from lesson 4 SEE to ME differed significantly from the previous 3 lessons (path c). Comparing the magnitudes, we found the effect from SEE on ME (path c) was stronger than the effect from ME on SEE (path d), for all three level constructs.

DISCUSSION

Our study indicates students' mathematics SEE are more stable than their ME. It makes sense theoretically that students' self-perceptions are more stable than their interpretations of success at the lesson-level of specificity and in relation to novel tasks, given that ME are reliant to a large degree on the type and difficulty of the task, and effects of ME on SEE are moderated also by cognitive appraisal. Future studies should investigate the stability of SEE and ME at different levels of specificity, e.g. task versus lesson versus subject. An important finding of our study was that the effects from students' SEE to their ME (β_{easy}= .48, β_{medium}= .60, β_{hard}= .43) were significantly stronger than the effects in the opposite direction (β_{easy}= .26, β_{medium}= .31, β_{hard}= .26). In contrast, Talsma et al. (2018) concluded that the overall effect of performance on SEE was significantly larger than the opposite effect, and that the effect of SEE on performance was non-significant in the case of school age children. Importantly, our study includes student self-reported ME, more apt for investigating the theoretically proposed process of SEE and ME changes, which may partially explain these differences. Previous studies have indicated time lag as a moderator of the cross-lagged effects between SEE and ME. We cannot conclude whether the shorter time lag from SEE to ME (generally 45-60 minutes) than the lag from ME to SEE (generally several hours or days) contributed toward the difference in magnitude. Future studies including similar time lags for both cross-lagged effects could further illuminate this relationship.

If students' ME are more readily influenced than their SEE, and the effect from SEE to ME is stronger than the reverse effect, a practical implication is that interventions might do well to start by facilitating positive ME (e.g. by ensuring students work on appropriately challenging tasks with appropriate levels of scaffolding), and then, importantly, support students' appraisals of these experiences, in order that their ME may exert larger influence on their SEE. While previous recommendation for practice often highlight the dichotomy of whether interventions should target SEE or ME, we thus suggest focusing on ways to strengthen the effect of ME on SEE, for instance through specific, task-focused feedback highlighting student progress. This approach may be particularly salient at a time when students are engaging with new mathematical topics, and their self-perceptions are more amenable to input.

Our study is unique as it illuminates the relationship between SEE and ME across a series of lessons in mathematics, while previous longitudinal studies have generally included time-lags of weeks or months, and used objective performance measures rather than students' ME. The functional properties of the relationship between SEE and ME might differ in a micro-analytic as opposed to a macro-analytic time frame. In our study construct stability was higher in association with hard tasks, while cross-lagged effects were stronger in association with medium difficulty tasks (in absolute terms). This indicates task difficulty is important to SEE change, and that engaging with moderately difficult tasks in school might facilitate such changes. Future studies should investigate also how teachers may influence the relationship between mathematical tasks and changes in SEE through classroom quality.

References

Bandura, A. (1997). *Self-efficacy: The exercise of control*. New York: W. H. Freeman & Co.

Bernacki, M. L., Nokes-Malach, T. J., & Aleven, V. (2015). Examining self-efficacy during learning: variability and relations to behavior, performance, and learning. *Metacognition and Learning*, *10*(1).

Chen, F.F. (2007). Sensitivity of goodness of fit indexes to lack of measurement invariance. *Structural Equation Modeling: A Multidisciplinary Journal*, *14*(3), 464-504.

Hannula, M. S., Bofah, E. A., Tuohilampi, L., & Metsämuuronen, J. (2014). A longitudinal analysis of the relationship between mathematics-related affect and achievement in Finland. In *Proc. of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 38, pp. 249-256).

Hannula, M.S., Maijala, H., & Pehkonen, E. (2004). Development of understanding and self-confidence in mathematics; grades 5-8. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 17-24). Bergen, Norway: PME.

Phan, H. P. (2012). The Development of English and Mathematics Self-Efficacy: A Latent Growth Curve Analysis. *The Journal of Educational Research*, *105*(3), 196–209.

Schermelleh-Engel, K., Moosbrugger, H., & Müller, H. (2003). Evaluating the Fit of Structural Equation Models : Tests of Significance and Descriptive Goodness-of-Fit Measures. *Methods of Psychological Research Online*, *8*(2), 23–74.

Sørlie, K., & Söderlund, G. (2015). Regionale forskjeller i mestringsforventninger. In G. Langfeldt (Ed.), *Skolens kvalitet skapes lokalt. Presentasjon av funn fra forskningsprosjektet "Lærende Regioner."* Oslo: Fagbokforlaget.

Street, K. E. S., Malmberg, L. E., & Stylianides, G. J. (2017). Level, strength, and facet-specific self-efficacy in mathematics test performance. *ZDM - Mathematics Education*, *49*(3), 379–395.

Talsma, K., Schüz, B., Schwarzer, R., & Norris, K. (2018). I believe, therefore I achieve (and vice versa): A meta-analytic cross-lagged panel analysis of self-efficacy and academic performance. *Learning and Individual Differences, 61*(January), 136–150.

Usher, E.L. & Pajares, F. (2009). Sources of self-efficacy in mathematics: A validation study. *Contemporary Educational Psychology, 34*(1), 89-101.

Zimmerman, B. J. (2000). Self-Efficacy: An Essential Motive to Learn. *Contemporary Educational Psychology*, *25*(1), 82–91.

MATHEMATICS IN DISGUISE: EFFECTS OF THE EXTERNAL CONTEXT OF MATHEMATICAL WORD PROBLEMS

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Both the content and the context of a mathematical word problem (WP) influence its solution process. We focus on the external context beyond the problem text, e.g. the classroom and the cover sheet of the WP. The few studies analysing influences of the external context commonly focus on the mathematical solution. In contrast, we report two experiments analysing processes beyond solutions. First, we found that a mathematical external context increases physiological arousal indicated by electrodermal activity, but not self-reported state anxiety. In contrast, eye movements during WP reading did not differ between a mathematical and a problem-solving external context. This indicates that the external context can initiate a variety of processes and emphasizes its relevance for mathematical WP.

INTRODUCTION

When solving mathematical tasks, it is not only the content, but also the context that matters. Students might be capable of performing a mathematical operation, but they are not necessarily able to apply it in a real-life context (Greer, Verschaffel, & Mukhopadhyay, 2007; Carraher, Carraher, & Schlieman, 1985). In many educational systems, the focus of national curricula shifted towards putting mathematics into context during the last two decades (e.g. CCSSI, 2017). Consequently, modeling tasks and word problems (WP) have gained importance in mathematics education, since they allow educators to integrate a mathematical operation into a real-life context.

Efforts to implement mathematics into a real-life context are also reflected in large-scale studies like the *Program for International Student Assessment* (PISA; OECD, 2016) or the *Trends in International Mathematics and Science Study* (TIMSS; IEA 2013). For example, the definition of *mathematical literacy* in PISA refers to “an individual’s capacity to formulate, employ, and interpret mathematics in a variety of contexts [and] assists individuals to recognize the role that mathematics plays in the world […]” (OECD, 2016, p.67). Accordingly, PISA uses mathematical WP that employ contexts referring to students’ everyday life. They include a brief narrative introduction about a real-life topic (e.g. *Climbing Mount Fuji*). Students are then asked to solve a realistic problem, for example “Using Toshi’s estimated speeds, what is the latest time he can begin his walk so that he can return by 8 pm?”. Notably, the contextualization only takes place on the level of the WP, i.e. in the form of text and illustrations. We will refer to this as the *internal context* of a mathematical WP. In contrast, we refer to the *external context* of a mathematical WP as the situation in

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which it is presented. This includes the classroom setting, the cover sheet of a test, or the emotional and motivational framing of the assessment, e.g. if it is timed or graded.

External context effects

One of the most popular studies addressing the external context was conducted by Carraher, Carraher, and Schlieman (1985). The researchers asked children working at a market how much a number of items would cost, requiring a multiplication. Later, the researchers presented them with the same multiplication at home, in the form of a WP describing a market situation, and in a formal mathematical task. Children solved 97% of the tasks correctly at the market. At home, the same children only solved 70% of the WP and 37% of the formal mathematical tasks, even though the problems were structurally equivalent. In our terms, Carraher et al. (1985) first changed the external context by assessing the items at home and then adapted the internal context by removing the market scenario. Still, they did not present children with the *exact same* mathematical problem in both external contexts. Similarly, most research on context effects focuses on internal context variation or the variation of both contexts at the same time (Fleischer, Wirth, & Leutner, 2014, Johns, Schmader, & Martens, 2005). In contrast, studies are rare that systematically manipulate the external context (see Dewolf, Van Dooren, & Verschaffel, 2011, for an overview).

Furthermore, research commonly focuses on effects on mathematical achievement but does not analyze in detail how the solution process is affected. The few exceptions focus on the solution process from a mathematical perspective, for example what mathematical models are constructed or how reasoning refers to mathematical arguments (Dewolf et al., 2011). For example, Dewolf et al. (2011) presented 151 children (10 to 12 years old) with a mathematical WP either during a mathematics class or a religion class. Students followed different approaches to solve the problem. In the mathematics class, they preferred calculations and offered numerical answers, whereas in a religion class, students favored verbal descriptions and non-mathematical arguments. Dewolf et al. (2011) conclude that students resort to socio-mathematical norms about WP solving when a mathematical context is induced.

Following a different approach, Fleischer et al. (2014) took into account possible influences of mathematical self-concept and *mathematical anxiety* (MA) on external context effects. In their study, 515 ninth-graders solved items from PISA. To vary the external context, the researchers presented the same items to two groups of students, but in different booklets. One booklet introduced the test as a mathematical test, the other as a problem-solving test. The context presented to the students had a significant effect on achievement in favor of the problem-solving context. Most interestingly, this effect was moderated by students' self-concept and MA. Students reporting a lower mathematical self-concept or higher MA showed higher deficits in the mathematical context compared to the problem-solving context. Fleischer et al. (2014) conclude that context effects prevent some students from using their full cognitive potential for solving the tasks, and that self-concept and MA might regulate this effect.

The study by Fleischer et al. (2014) indicates that there are several processes that might be activated through a variation of the external context. The activation of socio-mathematical norms affecting the mathematical solution process might be only one of these processes. To some extent, the external context seems to initiate affective and motivational processes related to MA and beliefs about the self.

The present study

In the present study, we conducted two experiments to analyze two possible processes of external context effects beyond mathematical solution strategies. Fleischer et al. (2014) found that MA moderates external context effects. MA can influence mathematical achievement through a number of cognitive and physiological processes (Dowker, Sarkar, & Looi, 2016). Students with high MA generally report higher levels of state anxiety in tests or classes, which can in turn impact achievement (Ashcraft & Moore, 2009; Goetz, Bieg, Lüdtke, Pekrun, & Hall, 2013, Dowker et al., 2016). To assess whether state anxiety is higher during a mathematical context, we assessed physiological arousal, measured through *electrodermal activity* (EDA) and self-reported state anxiety during a mathematical and a problem-solving context. EDA are fluctuations in skin conductance caused by sweat gland activity, which in turn is caused by an activation of the sympathetic nervous system (Boucsein, 2012). Its assessment can supplement common self-reports of state anxiety (Strohmaier, Schiepe-Tiska, & Reiss, 2017c). We expected that the mathematical context would lead to an increase in self-reported anxiety and EDA (Hypothesis 1a and 1b).

Furthermore, Fleischer et al. (2014) identified self-concept as a moderator of context effects. Moreover, they proposed that the external context could influence cognitive information processing. Naturally, WP contain a certain amount of text. Therefore, reading is a key process during the solution process of mathematical WP (Daroczy, Wolska, Meurers, & Nuerk, 2015). Recently, we proposed to adopt parameters of eye movements used in research on reading to characterize cognitive processes during complex WP solving (Strohmaier, Lehner, Beitlich, & Reiss, 2017a). Research on eye movements commonly distinguishes two types of eye movements. During *fixations*, the eyes rest on a certain position and retrieve information. Between fixations, the eyes move rapidly and vision ins suppressed. This is called a *saccade*. More difficult WP are read slower, fixations last longer, readers show more saccades and re-read text passages more often (Strohmaier et al., 2017a). These parameters are also related to mathematical self-concept (Strohmaier, Schiepe-Tiska, Müller, & Reiss, 2017b). Accordingly, we expected that a mathematical context would lead to an increase in fixation times, number of saccades, and ratio of regressions per saccade and a decrease in reading speed (Hypothesis 2).

EXPERIMENT 1

Method

Participants were 86 undergraduate students (age *M* = 23,2, *SD* = 4.1; 62% female). The experiment consisted of three phases in a within-subject design. The first phase consisted of a 6-minute relaxation exercise as a baseline measure. For the second and third phase, participants were asked to solve two 10-minute tests. Both tests consisted of six WP from TIMSS and PISA and were either introduced and labeled as a mathematical test or a problem-solving test. Items were chosen to be challenging, which was confirmed by a mean solution rate of 42%. The order and the assignment of the external context to the two tests alternated between participants. Self-reported state anxiety was assessed immediately after each phase with one item according to Goetz et al., (2013) ("I feel anxious", rated on a 4-point Likert scale). EDA was assessed during the three phases with an *Empatica E4 wristband* (for a detailed description of the EDA data processing, see Strohmaier et al., 2017c). As a measure for EDA, we counted spontaneous fluctuations in skin conductance per minute (NS.SCR.freq, Boucsein 2012). As an implementation check, students were asked after the tests how much they felt both tests had to do with mathematics on a 7-point Likert scale. An ANOVA for repeated measures and paired *t*-tests were conducted for analyses.

Results

In table 1, descriptive results are displayed. Self-reports of state anxiety differed significantly between the three phases ($F(2,170) = 6.97$, $p < .05$, $\eta^2 = .08$), but no pairwise difference between the mathematical and problem-solving context emerged ($t(85) = -0.54$, $p = .59$). EDA differed significantly between the three phases ($F(2,170) = 14.62$, $p < .001$, $\eta^2 = .15$). Furthermore, EDA was significantly higher in the mathematical context than in the problem-solving context ($t(85) = 2.24$, $p < .05$). Reports about how much the test had to do with mathematics were significantly higher in the mathematics condition ($t(85) = 6.85$, $p < .001$). The mean solution rate did not differ significantly between the tests ($t(85) = -1,13$ p = .26).

	Phase		
	Relaxation exercise	Problem-solving context	Mathematical context
Self-reported anxiety	1.36 (0.58)	1.64 (0.75)	1.61 (0.76)
EDA	15.4 (12.7)	19.3 (15.9)	20.8 (16.0)
Solution rate (%)		43.4 (26.0)	40.3 (22.3)
Relation to math		4.55 (1.47)	5.84 (1.28)

Note. Standard deviations are given in parentheses. n = 86.

Table 1: Self-reported anxiety, physiological arousal, solution rate, and reported relation to math during a mathematical test in a problem-solving and a mathematical external context

EXPERIMENT 2

Method

Participants were 118 undergraduate students (age M = 24.4, SD = 3.8; 63% female). They solved nine WP from PISA on a *SMI RED500* remote eye tracker. The mean solution rate of the items was 75%, which indicates that the WP were easier than in experiment 1, but still offered an acceptable challenge. The items were randomly assigned to two blocks, one of which was introduced as a mathematical test, the other one as a problem-solving test. There was no time limit on the tests. Eye movements were recorded and analyzed according to Strohmaier et al. (2017a). Four measures of eye movements that characterize the reading process of WP were used: Mean fixation duration, saccades per word, ratio of regressions per saccade and reading speed. Paired t-tests were conducted for analyses.

Results

None of the reading parameters differed significantly between the two contexts (p between .09 and .82). Descriptive results indicate a marginal trend towards fewer saccades and a faster reading pace in the mathematical context ($t(117) = -1.68$, $p = .10$; $t(117) = 1.73$, $p = .09$). Solution rate did not differ significantly between the two contexts ($t(117) = -0.29$, $p = .78$).

	Context	
	Problem-solving context	Mathematical context
Mean fixation duration (ms)	235 (25)	236 (25)
Saccades per word	2.24 (0.76)	2.13 (0.70)
Regressions per saccade	.326 (.060)	.327 (.061)
Reading speed (words/minute)	160 (53)	170 (50)
Solution rate (%)	74.4 (20.2)	73.8 (19.5)

Note. Standard deviations are given in parentheses. n = 118.

Table 2: Eye movement parameters during reading and solution rate of mathematical word problems in a problem-solving and a mathematical external context

DISCUSSION

External context effects

Contrary to hypothesis 1a, self-reports of state anxiety did not differ significantly between a mathematical and a problem-solving context. Nevertheless, hypothesis 1b was confirmed. The mathematical context caused a higher physiological arousal then the problem-solving context. This indicates that the external context influences arousal and state anxiety during WP solving. An explanation for the absence of an effect on self-reports might be that they refer stronger to long term beliefs about mathematics

(Goetz et al., 2013; Strohmaier et al., 2017c). Alternatively, physiological arousal might not solely be caused by MA, but could also be a result of increased effort or motivation. Anyway, the physiological reaction could in turn influence the solution process of the test. For example, physiological arousal interacts with working memory capacity (Ashcraft & Moore, 2009).

Considering hypothesis 2, we found no influence of the external context on eye movements during WP reading. It is possible that the variation of the external context was not as strongly induced as in the other experiments, since the eye tracker only allowed an introductory slide, but no cover sheet. Furthermore, items were solved orally and the tests were not timed, which might counteract a mathematical context. Possibly, the mathematical context was not induced as strongly as necessary. Moreover, external context effects could be stronger for more challenging problems like in experiment 1, since they require more strategies and cognitive resources that might be affected by context effects. Another assumption is that the process of reading is independent of the external context. Although the reading process of WP is related to mathematical thinking, it might depend mostly on the mathematical content or internal context of the problem.

Our study included two experiments investigating if the external context of a mathematical WP influences the solution process of mathematical WP beyond differences in the mathematical solution. Our findings supplement prior research (e.g. Dewolf et al., 2014), because it illustrates that the external context not only influences the mathematical solution. Rather, physiological arousal was identified to be higher in a mathematical context, possibly affecting cognitive resources.

Application

Regarding efforts to put mathematics into a real-life context, we propose several implications. WP and modelling tasks are commonly used to embed mathematical operations into a real-life context. This only addresses the internal context, while the external context has not received much attention in educational practice. For example, cross-curricular activities could break up the strict separation of contexts, in which mathematical problems are only solved during mathematics classes. This could have two positive effects. First, students that are strongly affected by the external mathematical context, for example because of MA, might be more successful when the external context is changed. Second, continuous context variation should break up the lines between subjects and reduce social norms and preconceptions about mathematical problems, which could decrease external context effects on the long run. A different remedy is provided by Johns et al. (2005). They found that educating girls about negative effects of the external context decreased gender differences in mathematics test. Transferring these findings by telling students about possible effects of arousal during mathematics tasks could help to decrease its influence.

In standardized tests, the external context can often not be controlled for practical reasons. Still, there are ways to consider possible effects of the external context. For example, Fleischer et al. (2014) illustrate that students' characteristics like MA or mathematical self-concept moderate these effects. When interpreting results, differences between these characteristics should be regarded to check for possible underlying context effects. Secondly, including a questionnaire testing for students' perception of the external context can give an indication if external context effects were present. If no information about the external context of the test is available, possible effects should be discussed.

Limitations

Our experiments considered two specific processes, while a variety of processes could be activated by the external context. We took into account effects on state anxiety and reading patterns based on previous research, but these effects can only be generalized to a limited account. Furthermore, we used WP from PISA and TIMSS that reflect a recent understanding of mathematical abilities and tested undergraduate students. Further research should take into account situations in the classroom, for example using WP from textbooks and testing during regular classes.

Conclusion

Summing up, this study provides innovative approaches that take into account effects of the external context on WP solving. We focused on two processes that go beyond the mathematical solution of the WP. This adds to a small number of research that has investigated effects of context variation on achievement and mathematical solutions. We hope that this impulse will initiate future efforts to understand and consider external context effects in mathematics education.

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References

Ashcraft, M. H., & Moore, A. M. (2009). Mathematics anxiety and the affective drop in performance. *Journal of Psychoeducational Assessment, 27*(3), 197–205. doi:10.1177/0734282908330580

Boucsein, W. (2012). *Electrodermal Activity* (2 ed.). New York: Springer.

Carraher, T. N., Carraher, D. W., & Schlieman, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology, 3,* 21–29.

CCSSI. (2017). *Common core standards for mathematics*. Retrieved from http://www.corestandards.org/Math/Practice/#CCSS.Math.Practice.MP1

Daroczy, G., Wolska, M., Meurers, W. D., & Nuerk, H. C. (2015). Word problems: a review of linguistic and numerical factors contributing to their difficulty. *Frontiers in Psychology, 6,* 348. doi:10.3389/fpsyg.2015.00348

Dewolf, T., Van Dooren, W., & Verschaffel, L. (2011). Upper elementary school children's understanding and solution of a quantitative problem insinde and outside the mathematics class. *Learning and Instruction, 21*, 770-780. doi:0.1016/j.learninstruc.2011.05.0

Dowker, A., Sarkar, A., & Looi, C. Y. (2016). Mathematics anxiety: What have we learned in 60 years? *Frontiers in Psychology, 7*, 508. doi:10.3389/fpsyg.2016.00508

Fleischer, J., Wirth, J., & Leutner, D. (2014). Effekte der kontextuellen Einkleidung von Testaufgaben auf die Schülerleistungen im analytischen Problemlösen und in der Mathematik. *Zeitschrift für Pädagogische Psychologie*, *28*(4), 207–227.

Goetz, T., Bieg, M., Lüdtke, O., Pekrun, R., & Hall, N. C. (2013). Do girls really experience more anxiety in mathematics? *Psychological Science, 24*(10), 2079–2087. doi:10.1177/0956797613486989

Greer, B., Verschaffel, L., & Mukhopadhyay, S. (2007). Modelling for life: mathematics and children's experience. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education* (pp. 89-98). New York: Springer.

International Association for the Evaluation of Educational Achievement [IEA] (2013). *TIMSS 2015 assessment frameworks*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center.

Johns, M., Schmader, T., & Martens, A. (2005). Knowing is half the battle: teaching stereotype threat as a means of improving women's math performance. *Psychological Science, 16*(3), 175–179. doi:10.1111/j.0956-7976.2005.00799.x

OECD. (2016). *PISA 2015 assessment and analytical framework: science, reading, mathematic and financial literacy*. Paris: PISA, OECD Publishing.

Strohmaier, A. R., Lehner, M. C., Beitlich, J. T., & Reiss, K. M. (2017a). *Reading mathematical word problems - the relationship between eye movements, task difficulty, and achievement.* Manuscript submitted for publication.

Strohmaier, A. R., Schiepe-Tiska, A., Müller, F., & Reiss, K. (2017b). *Mathematical reading patterns - The influence of self-concept and situational context*. Paper presented at the Annual Meeting of the American Educational Research Association (AERA), San Antonio (TX), USA.

Strohmaier, A. R., Schiepe-Tiska, A., & Reiss, K. M. (2017c). *Electrodermal Activity as an indicator for state anxiety in mathematics tests: an application of the control-value theory*. Manuscript submitted for publication.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making Sense of Word Problems*. Lisse: Swets & Zeitlinger.

TASK DESIGN TO PROMPT MAKING SENSE OF PRE-CALCULUS CONCEPTS USING DYNAMIC TECHNOLOGICAL TOOLS

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This study aims at exploring how learning tasks designed according to the Method of Varied Inquiry approach may enable students to discern between mathematical objects inherent in simulation of a real-world phenomenon and to become aware of the mathematical concepts within the tasks. The Method of Varied Inquiry approach is a combination of two theoretical perspectives: the logic of inquiry and the variation theory. A learning experiment was conducted with 15-year-old students in a high school in Turin, Italy. Findings shows that the tasks enable the students to discern between critical aspects of a real-world phenomenon and the mathematical relationships inherent to it. Insights for improving task design and role of the teachers were made.

INTRODUCTION

Calculus is considered to be one of the most important topics in mathematics and is included in several curricula worldwide. Calculus ideas are inherent in many other topics and real-life situations. Nurturing calculus thinking can result in the productive integration of citizens into modern society (NGSS, 2013). This type of thinking is essential for dealing with the challenges that citizens face in the 21st century (Artigue & Blomhøj, 2013). Furthermore, encouraging students to model mathematically real-life situations may help them integrate into society as citizens who are able to make intellectual decisions (Blum, 2002).

Often, as seen at least in Italy and in Israel, calculus is taught in senior high schools in a formal way as a set of rules and strategies for investigating functions and computing areas bound between functions. This kind of teaching-learning activity concentrates essentially on the formal world of mathematics, which poses a barrier for the sense of mathematical statements (Arzarello, 2016). Several scholars have criticized this kind of teaching, claiming that it cannot guarantee boosting the understanding of calculus concepts, and even found it to be a barrier for understanding calculus when it is learned at a university level (Thompson, Byerley, & Hatfield, 2013). In addition, official Italian and Israeli curricula stress the necessity for presenting examples where mathematical models of different phenomena are emphasized.

To overcome the barrier existing between the formal world and a real-life situation, Arzarello (2016) suggested a ‘virtuous cycle’ model (Fig. 1). The model consists of four formal and informal intertwined aspects: (1) aspects of the real-life situation

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represented in the formal system; (2) treatment within a formal system/conversions between systems; (3) interpretation of the results of the formal system in the real-life situation; and (4) interpretation/theorization of the real-life situation through the theoretical lens. Since the formal and informal aspects are closely intertwined in mathematical reasoning, Arzarello (ibid.) argued that a major teaching goal should be to operationalize this virtuous cycle in classroom practices.

The main question remains as to how to apply the virtuous cycle in classroom practices, ensuring a deep understanding of mathematical concepts. This question should guide the discussion of this study, as inspired by the logic of inquiry approach (Hintikka, 1999), which generally viewed scientific inquiry and knowledge construction as a question-answer process, and the variation theory (Marton, Runesson, & Tsui, 2004), which defines learning as a change in the way something is seen, experienced or understood. We designed a set of tasks to be applied by the Method of Varied Inquiry method– a learning-teaching approach developed by Arzarello (2016) – which may facilitate the conceptual learning of pre-calculus concepts and engage students in scientific inquiry. The aim of this study is to explore how design principles may enable students to become aware of mathematical concepts inherent in a task.

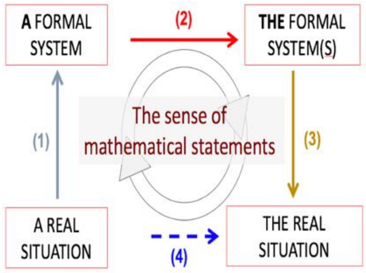


Figure 1. Virtuous Cycle (Adapted from Arzarello (2016a))

THEORETICAL FRAMEWORK

This study is guided by the Method of Varied Inquiry (MVI) approach, which is a combination of two theoretical perspectives: the logic of inquiry (Hintikka, 1999) and the variation theory (Marton et al., 2004).

The main idea behind the logic of inquiry approach involves seeking rational knowledge by questioning (Hintikka, 1999). Hintikka conceived the process of seeking new knowledge as an interrogative process between two players. The first player (the inquirer) has the role of asking questions, and the second player has the role of answering and is called the verifier (or oracle). The former is the seeker of knowledge who tries to prove a conclusion to be reached from prior experiences or even from theoretical premises. The latter is considered the source of knowledge.

In order to design educational situations that may promote inquiry processes, Arzarello (2016) referred to the variation theory. The variation theory (Marton et al., 2004) defines learning as a change in the way something is discerned, i.e., seen, experienced or understood. According to this theory, meanings emerge as the learner focuses his awareness on the object of learning. In this case, some aspects of the object appear at the forefront of his attention. Yet, not all aspects are discerned at the same time or in the same way. In order to understand an object of learning in a certain way, various specific critical aspects must be discerned by the learner. To facilitate the discerned object of learning, Marton et al. (2004) proposed four interrelated functions (or patterns) of variation to be taken into account when designing educational tasks: (a)

Contrast: "…in order to experience something, a person must experience something else to compare it with"; (b) Generalization: "…in order to fully understand what 'three' is, we must also experience varying appearances of 'three'…"; (c) Separation: "In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant"; and (d) Fusion: "If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously" (Marton et al. 2004, p. 16).

In the MVI approach, Arzarello (2016) proposes that drawing students' attention to critical aspects, asking to vary them, and observing their effects on the phenomena may foster students' inquiry processes. The main idea of the MVI is creating challenging situations by varying some aspects of the phenomena (real-world or mathematical) while keeping the others invariant. Exploring various aspects of the same phenomena may lead the students to grasp the intended object of learning.

DESIGN PRINCIPLES OF THE TASKS

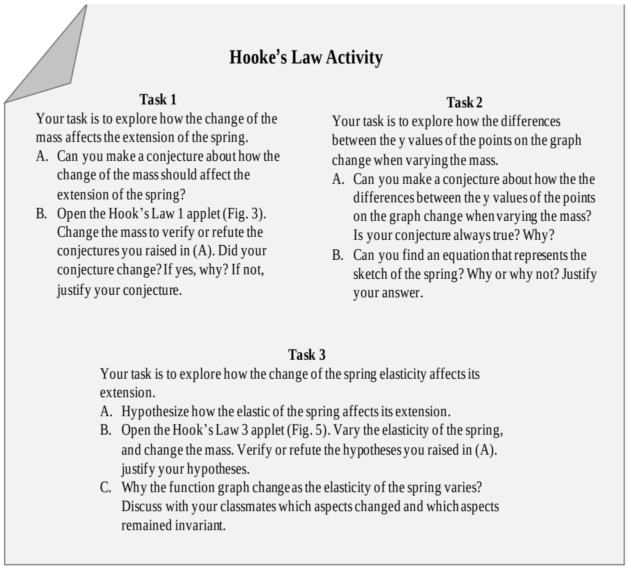
The design principles of the tasks were motivated by the virtuous cycle and the MVI approach. Each task contains a simulation of real-world phenomenon and mathematical representations of the dynamic aspects of the phenomenon. The main idea of the design is creating different situations by varying some aspects of the phenomenon while keeping the others invariant. For example, the first task may request students to explore the mathematical model of the stretch of a spring (as in the well-known Hooke's Law). In this situation, the mass and length of the spring vary, and the elasticity of the spring is invariant. The second task requests students to explore the same situation as in task 1, but this time with a new representation added. In accordance with the virtuous cycle, we assumed that adding a new mathematical representation to the previous one may draw the students' attention to a new representation and prompt connections between "A formal system" with "The formal system." In the second layer of the exploration, the elasticity of the spring is varied, and the students explore how this new situation affects the mathematical model. Exploring various aspects of the same phenomenon may lead the students to grasp the intended object of learning, namely, linear function and its properties.

Task	Mathematics	VC	Critical aspects
1	Propositional relationship	(1)	Mass-length relationship
2	Linear function based on the first differences	(1), (2), (3)	Delta length, straight line, mass-length relationship
3	Similarities and differences between the family of linear functions	(2), (3), (4)	Delta length, straight line, family of linear function graphs

Table 1: Mathematical ideas, the virtuous cycle and critical aspects of the phenomenon

Task	Contr-	Separa-	Generalization	Fusion	Variant	Invariant
1	X	Mass	X	X	Mass-length	Delta length
2	X	Mass	X	X	Mass-length	Straight line Delta length
3	X	X	Different situations of elasticity	Mass and elasticity	Mass, length, elasticity, slope	Linear line

Table 2: The task's analysis according to varying functions



Hooke's Law Activity

Task 1

Your task is to explore how the change of the mass affects the extension of the spring.

A. Can you make a conjecture about how the change of the mass should affect the extension of the spring?

B. Open the Hook's Law 1 applet (Fig. 3). Change the mass to verify or refute the conjectures you raised in (A). Did your conjecture change? If yes, why? If not, justify your conjecture.

Task 2

Your task is to explore how the differences between the y values of the points on the graph change when varying the mass.

A. Can you make a conjecture about how the the differences between the y values of the points on the graph change when varying the mass? Is your conjecture always true? Why?

B. Can you find an equation that represents the sketch of the spring? Why or why not? Justify your answer.

Task 3

Your task is to explore how the change of the spring elasticity affects its extension.

A. Hypothesize how the elastic of the spring affects its extension.

B. Open the Hook's Law 3 applet (Fig. 5). Vary the elasticity of the spring, and change the mass. Verify or refute the hypotheses you raised in (A). justify your hypotheses.

C. Why the function graph change as the elasticity of the spring varies? Discuss with your classmates which aspects changed and which aspects remained invariant.

Figure 2. Hooke's Law written tasks

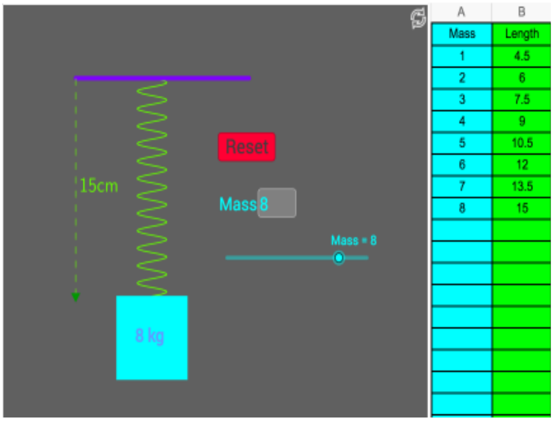


Figure 3. Two representations of Hooke's Law

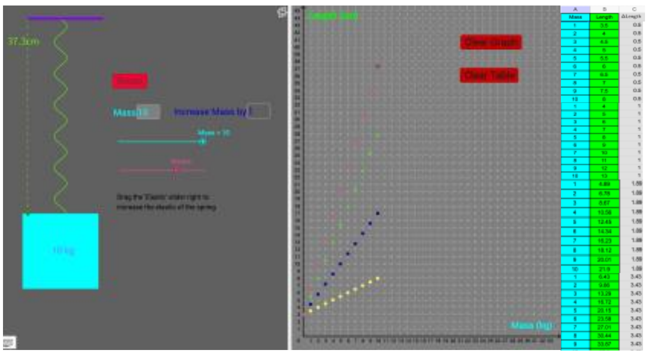


Figure 5. Two variables are varying simultaneously

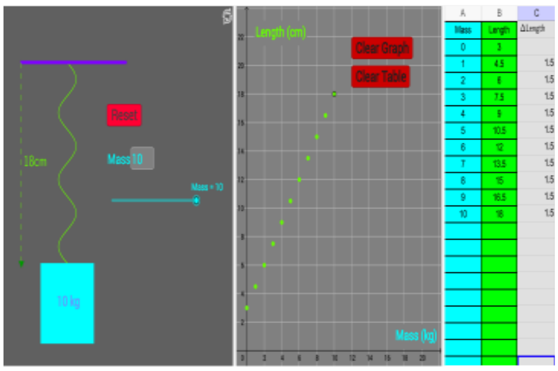


Figure 4. Three representations of Hooke's law

STUDY DESIGN

To study how the design principles reflected in the students' learning, a learning experiment was held in a high school in Turin (Italy) in which two senior teachers applied the task in their classrooms. Each classroom comprised 18 15-year-old students. We followed four lessons of the teachers as they taught each lesson for 1.5 hours. The students were required to work in small groups, sharing the task worksheet and a single computer. The students and their actions on the computers were video recorded. Regarding data analysis, segments of the recorded clips of the students' actions using the computers were identified. Finally, the effects of the task design, and the transition between the representations and the real phenomena were considered.

RESULTS

Observing the mathematical relationship between a spring's length and its mass

This excerpt illustrates how varying the mass drew the students' attention to the relationship between mass values and spring length.

9	Patricia	We start from 2 and it's 6 cm. Now let's double it; for 4...it's 9...
10	Anna	So, this means that....
11	Patricia	This is not directly proportional, because we double one, the other should double too!
12	Carlota	Try to set 1...
13	Patricia	First, we put 2, and it was 6 cm, then we put 4, which is double, and it should have been 12 cm if it were proportional! Instead it was just 9...
14	Dina	So, it stretches 1.5 every kilo!
15	Carlota	It's not directly proportional...

The students conjectured that the relationship between mass and length is directly proportional. In line [9], Patricia set two values for mass and examined the relationship between length values. This action suggests that the students connected the real-world phenomenon with the numeric representation of mass-length. After refuting the conjecture, Carlota changed the mass value and set it at 1. She concentrated on the length of the spring that corresponded to mass=1. Although in line [14], Dina noticed that the rate of change of length values is constant, Anna and Carlota continued to maintain that the relationship is not directly proportional.

Observing the increment of a linear function graph

Adding a graphical representation to the numerical and real-world phenomenon drew the students' attention to the invariant aspect of the function graph. In the following excerpts, the students tried to come up with a conjecture as to why and how the straight line increased. Doing so, they shifted alternatively between the three representations while focusing on how the variant parameters – mass versus length – reflected on the invariant parameters – linear function graph and delta length.

48	Patricia	Why do the y-values change? Because the y-values stand for the length of the spring... if the mass increases, the length increases, so that's why they increase!
49	Anna	So, why does the length change?

50	Carlota	*Increasing the mass values by dragging the slider.*
51	Patricia	It increases because... Observe how the y-values change on the points... why do they change?
52	Anna	Because it's a proportional relationship [points to the table of values]. If one increases, the other does too!
53	Patricia	Yes, because the y-value depends on the x-value… So, if you increase the x, which is the mass, consequently you increase the y, which is the length! But in which way does it change?

In line [48], Patricia focused on the linear line on the screen. In an attempt to answer her question regarding the change in y-values, she named the x-axis and y-axis by referring to the real-world phenomenon. These actions suggest that Patricia focused first on the graphic representations and then on the real-world phenomenon. Changing the mass values using the slider drew Patricia's attention to the point that moves along the straight line. In an attempt to answer Patricia's question [51], Anna [52] referred to the table of values to describe the relationship between mass and length values. Patricia in line [53] summed up the insight obtained at this stage, and progressed one step further by asking the question "in which way does it change"?

Observing the delta length

To answer Patricia's question, Anna referred to the delta length columns in which its values were constant. Once she points to the delta length column, she explained why the values are constants.

55	Anna	It's always 1.5 because by adding 1 kilo every time… the stretching is 1.5! If we had added half a kilo, it would be 0.75!
56	Carlota	I know, but how can we explain this?
57	Anna	Let's write this, saying that if we add half a kilo
58	Carlota	Eh, but the difference is always 1.5, why?
59	Anna	Because we always add 1 kilo! If we add half a kilo, the difference would be different!

Anna's words "it's always 1.5" suggests that she paid attention to invariant values in the delta length column. She justified the invariance of the delta length values referring to the constant variation of the mass. Her statements in [55] suggest that she explained the rate of change of the spring length by referring to the real-world phenomenon. Her statement "If we had added half a kilo, it would be 0.75" suggests that Anna continued to focus on how changing the value of the variant parameter – the mass – affect the values of the invariant parameter – delta length. Furthermore, this shift in her focus suggests that she started becoming aware of the rate of change of the linear function.

Observing the invariant when several parameters vary

The third task was designed to help the students become aware of the inclination in the linear function, and allow them to distinguish between different situations of linear function that differ in terms of their slope. Initially, the students set the value of elasticity at 5, thereafter, they varied the mass values using the slider. In the Cartesian system, a straight line was displayed. Next, they set the elasticity at 10, thereafter 15, and repeated the same action by dragging the slider. Finally, three straight lines were displayed on the screen.

101	Anna	When the elasticity was 5, it increased by half, half, half, half. All the values change in the same way. Here [points to the table of values] one, one, one.
102	Carlota	As the elasticity increases, the length increases.
103	Anna	As we increase the elasticity by five, the length will increase by 0.5.
104	Patricia	The change in length, not the length itself.
105	Anna	For every value of elasticity, the differences between length values are constant. And here [points to the Cartesian system], we get linear functions.

After varying mass values and elasticity of the spring, Anna focused her attention on the table of values. She noticed that the differences in spring length were constant. She also noticed that the differences were not always congruent as in the previous tasks. In this task, the differences in spring length were constant and congruent for a certain value of elasticity. In line [102], Carlota focused her attention on the real-world phenomenon. She stated the relationship between elasticity of the spring and its length. While Anna and Carlota focused on the entire length of the spring, Patricia focused on the differences in spring length. Anna's statement in line [105] suggested that she connected between the whole representations. She started from the representation of the real-world phenomenon and continued to the table of values. In this connection, she had explained the differences by mathematical means – the effect of elasticity of the spring on the spring stretch. Thereafter, she connected the numeric representation with the graphical one. Her statement "the differences between length values are constant. And here, we get linear functions" suggests that she characterized linear function according to a constant difference.

CONCLUDING REMARKS

This study aims at exploring how learning tasks that were designed according to the MVI approach may enable students to discern between mathematical objects inherent in a real-world phenomenon and to become aware of the real-world phenomenon and its mathematical representations. The results of the study suggest that the design principle has the potential of supporting the educational aims set for the study. As shown in the results, the design of the activities allows the students to discern between the critical aspect of the phenomenon and the mathematical relationships. However, discerning between the learning objects was not sufficient enough to enable them to become aware of them and to endow the mathematical ideas with their cultural meanings, as was intended by the designer. Our results showed that the students discern between the learning objects and endow them with personal rather that cultural meaning (Mariotti, 2009), as is expected by the designer. For example, discerning that the mass-length relationship is not directly proportional did not comply with the designer's intention, which aimed at helping the students discern and become aware of the linear proportionality between mass and length.

To derive benefits from the design principle proposed in this study, teachers should play an essential role in mediating the personal meanings of the students with the cultural meaning intended by the activity designer (Bussi & Mariotti, 2008). In this context, the findings of this study should help teachers become aware of the personal meanings students may have. Knowing these personal meanings in advance may help teachers to plan their lessons in a way that bridges students' personal meanings to the cultural meaning, without preventing the inquiry processes of the students.

References

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM, 45*(6), 797–810.

Arzarello, F. (2016). How to promote mathematical thinking in the classroom. *Lecture delivered at the 3rd Italian Summer School in Math Education*, UMI-CIIM, Bardonecchia, 2016.

Blum, W. (2002). ICMI Study 14: Applications and modelling in mathematics education – Discussion document. *Educational Studies in Mathematics, 51*(1), 149–171.

Bussi, M. G. B., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. *Handbook of International Research in Mathematics Education*, New York, 746–783.

Hintikka, J. (1999). The role of logic in argumentation. In: *Inquiry as Inquiry: A Logic of Scientific Discovery* (pp. 25–46). Springer.

Lead States, N. (2013). *Next generation science standards: for states, by states.* The National Academies Press, Washington, DC.

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: the role of the teacher. *ZDM, 41*(4), 427–440.

Marton, F., Runesson, U., & Tsui, A. (2004). *The space of learning.* Mahwah, NJ: Lawrence Erlbaum.

Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*(1–2), 124–147.

MAKING SENSE OF THE TEACHING OF CALCULUS FROM A COMMOGNITIVE PERSPECTIVE

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Examining the discourse through the lens of commognition theory allowed an investigation of how teachers of mathematics teach elementary calculus. Analysing the teachers' word use and narratives provided insights into the specialisation of the mathematical language used in the discourse. Analysing the visual mediators, routines and meta-rules used in the classroom discourse, but more importantly, how and when they were used, explained the modes of mediation used in teaching elementary calculus.

INTRODUCTION

This paper reports on a discursive analysis of mathematical discourse on elementary calculus through the lens of the commognitive framework (Sfard, 2008). Given the microscopic nature of commognitive analyses and the word count limitations, one case out of nine was selected for this paper. Thus, this case study is part of a more extensive (doctoral) study, which seeks to research how teachers of mathematics teach elementary calculus in England. Elementary calculus is part of school (post-16) or college mathematics curriculum in the United Kingdom (UK) and many other countries. The object of enquiry is how mathematics teachers teach the derivative. The unit of analysis is the discourse of the teacher, primarily, though the classroom discourse is also considered in as far as it provides the social context of the teacher's discourse. It is a discursive analysis, therefore, a qualitative study. In the following sections, a brief introduction to the commognitive framework is outlined, followed by the methodology explaining the discursive approach to data analysis. This is then followed by a discussion of selected findings, and finally, a summary of conclusions and implications.

THE COMMOGNITIVE FRAMEWORK

Sfard (2008) presents the commognitive framework for the study of (mathematical) thinking. Commognition is a term founded by Sfard, which conceptualises thinking as a 'form of communication' with oneself. Thus, cognition plus communication constitute commognition (p.570). Thinking is construed as individualisation of interpersonal communication. Thus, thinking processes and interpersonal communication are facets of the same phenomena. Discourse is the core unit of analysis. Discourse can either be non-specialised discourses – 'colloquial discourse' or 'literate discourses' (Sfard, 2008, p.299) which are artefact-mediated mainly by symbolic tools designed specifically for communication. Mathematics is regarded as a form of discourse, which is

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characterised by four commognitive constructs: *word use*; *visual mediators*; *endorsed narratives;* and *routines*.

Word use refers merely to the kinds of words used in the discourse. Narratives are utterances in the discourse, thus, made up of words, any written or spoken text used to construct or endorse other narratives. In literate mathematical discourse, endorsed narratives include processes such as defining, estimating, abstracting, conjecturing, proving and generalising (Sfard, 2007, 2008). Visual mediators are visible objects, including symbolic artefacts such as formulae, graphs, drawings and diagrams that are created and used to enhance mathematical communication. According to Sfard (2007), visual mediators are the means through "which participants of a discourse identify the object of their talk and coordinate their communication" (p. 571). Noticing and categorising visual mediators is important in commognitive analyses. Routines are the "well defined repetitive patterns" (Sfard, 2007, p.572) in teachers' actions in classroom discourse. Didactical and mathematical routines can be noticed in the use of mathematical words, visual mediators and narratives, i.e., can be observed in the processes of "creating and substantiating narratives" (p.572) about say, differentiation. Routines are the meta-rules that govern when and how these visual mediators and narratives are used. Meta-rules, if formulated, take the form of meta-level narratives – "propositions about the discourse rather than its objects" (Sfard, 2007, p.573).

The commognitive framework allows for the study of the discursive developments of individual students and the discursive practices of the teacher.

METHODOLOGY

Data collection and participants

Nine teachers of mathematics (and their classes) took part in the study. However, this paper reports on data sets from one of the participant teachers. Peter is a male teacher of mathematics in a college who had been teaching post-16 mathematics for more than three decades. He has a first-class honours degree in mathematics and a Post-Graduate Certificate in Education (PGCE), both from the UK. It was mainly because of his long teaching experience why Peter was chosen for this study.

Data sets for the case study include two audio-recorded interviews with the teacher and one video-recorded lesson observation in which the teacher discussed tangents, gradients and differentiation. The teacher was interviewed first, prior to teaching the observed lesson on calculus and secondly, after teaching the lesson. The lesson observation video data and the interviews audio data were transcribed with respect to the participants' utterances and actions. The primary focus of the study is the teacher's utterances and actions.

Method of analysis

The analysis uses a priori characterisation of discourse comprising the four main commognitive constructs of *word use*, *narratives*, *visual mediators* and *endorsed routines* (Sfard, 2008). For the analysis of word use, the extent to which the teacher uses specialised mathematical terminology in his mathematical discourse is examined. This focuses on the teachers' literate and colloquial word use in differential calculus. The analysis explores the visual mediators incorporated in the discourse and examines how the discourse makes use of the multiple mathematical, visual mediators. A key focus is an analysis of the transitions between different visual mediators, signified by the presence of both, verbal and visual realisations - words or symbols that 'function as nouns' (Sfard, 2008, p.155). For the investigation of routines, the analysis focuses on the meta-rules with respect to analyses of word use, visual mediators, and endorsed narratives in terms of how and when they are used (Sfard, 2008). For the analysis of narratives, attention is given to both written and spoken text about definitions, proofs, and facts related to differentiation. The focus is on the meta-level narratives that were particularly pertinent to the teacher's word use, visual mediators, and routines within the mathematical discourse. The meta-rules are important in the analyses of narratives as they regulate practices when the participants generate and substantiate mathematical meaning (Güçler, 2013).

FINDINGS AND DISCUSSION

In the analysis below, I discuss three findings of the study: the teacher's approach to introducing the 'derivative'; inconsistency in the teacher's use of calculus words; and ambiguity with calculus symbolism. The excerpts and the numbering of the utterances, as presented in the discussion, are all extracted from the original transcripts of the data sets.

How do teachers introduce the idea of differentiation?

In the pre-lesson interview with the teacher, Peter explains the necessity and importance for promoting conceptual and 'relational' understanding, i.e., "knowing both *what to do* and *why*" (Skemp, 1976, p.20, italics mine) of differentiation. Talking about his approach to introducing differentiation, Peter said: [interview transcript]:

46 Teacher: I want them to have at least a feel of what we are trying to do, what differentiation means rather than just state that, right, when you start with x^2 you get $2x$. Right, they will get it, but what does it mean? I just want them to have a feel of what it actually means.

48 Teacher: I don't see how you can start saying, right, $y = x^2, \frac{dy}{dx} = 2x$ ... you know... I certainly won't be using the $\frac{dy}{dx}$. I don't think… I certainly ... I mean I can't believe I will be using that notation today. If I do, I haven't planned to anyway.

The word 'certainly' is used twice in [48]. The teacher's view, as expressed here, is that it would be inappropriate to use the $\frac{dy}{dx}$ notation in the first lesson on differentiation. Notice the teacher's didactical objective in [46], *what it (differentiation) means* is repeated at least three times in [46] alone. The word 'what' is repeated four times and the word 'mean' three times.

Orton's studies (Orton 1983_a and 1983_b) found that students had 'instrumental' knowledge' (Skemp, 1976) of differentiation; they could carry out the standard calculations/rules in differentiation very well, but they did not have adequate knowledge of where the standard rules come from. They lacked the relational understanding of how and why the methods or rules work. Peter explains the need for the substantiation of the narrative: If $y = x^2$, then $\frac{dy}{dx} = 2x$ rather than starting with the standard rules for differentiation. This belief may explain his approach to introducing differentiation in the lessons that followed the interview.

The teacher gave out the graph of $y = x^2$ with the following instructions [lesson transcript]:

26 Teacher: Now I want you to locate the point on the graph where x equals one. Can you locate the point $x = 1$? y will also be one as well, and I want you to

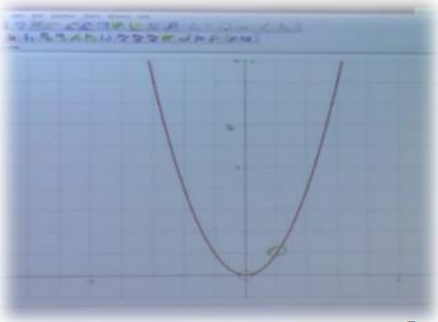

Fig.1: The graph of $y = x^2$

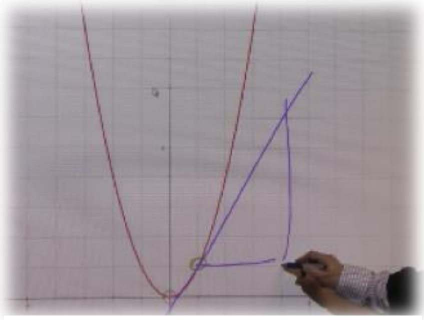

Fig.2: Drawing tangents

draw with a ruler the tangent. I want the tangent to be as long as you like, a straight line. You're doing this by eye, by no other way, by eye.

30 Teacher: I want you to imagine you're traveling around this curve… $x = 1$ is about there, isn't it? Make your line long and bold. Now I want you to measure the gradient of that line.

The mathematical object of the discourse is differentiation. The teacher's approach is to construct the definition of derivative by exemplar (Viirman, 2013), using the visual mediators [Fig.1 & Fig.2] of the graph of $y = x^2$. Substantiation of differentiation is done through estimating the derivative using tangents [Fig.2], rather than starting with the standard rules for differentiation.

Inconsistency in word use

I look at two phrases that Peter used repeatedly in his discourse: 'gradient of a curve' and 'tangent'.

The gradient of a curve: Routines are the meta-rules governing the repetitive discursive actions of participants of the discourse (Sfard, 2008). To identify any well-defined didactical practices or repetitive patterns in the teacher's actions, i.e., routines, it is important to identify the object of the discourse (Nardi et al., 2014, p.185), the 'discursive objects' (Sfard, 2008, p.166). Here is Peter introducing the mathematical object of his lesson, [lesson transcript]:

4 Teacher: And I want to pose a problem to you, and the problem is this… [*Teacher writing on the white board* – “The gradient of a curve”].

5 Teacher: What do we mean by that? That's my first question to you. Now we all know, I hope what is meant by the gradient of a line.

7 Teacher: How do you measure the gradient of a line then? How do you measure the gradient?

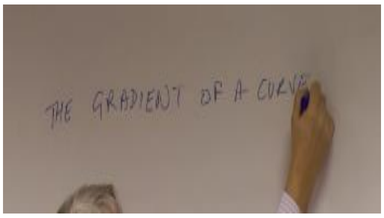

Fig.3: The title of the lesson

9 Teacher: Right, so my question to you is what do we mean by the gradient of a curve?

The teacher’s narrative ‘gradient of a curve’, which is signified both verbally [4] and visually [Fig.3], is inconsistent with literate mathematical discourse. An endorsed narrative describes ‘the gradient of a curve *at a point’*. The teacher’s narrative ‘the gradient of a curve’ is, therefore, colloquial discourse. But what is this discursive object here framed as ‘the gradient of a curve’? Notice, the teacher begins by asking the ‘what’ gradient question and he did not get a satisfactory answer from the students; he changed the question to the ‘how’ to measure the gradient of a line, and then asked about the ‘what’ gradient of a curve. The questioning suggests that by knowing ‘how to’ measure the gradient of a line (operational), that would lead the students to knowing ‘what is’ the ‘gradient (object) of a curve’; it doesn’t say ‘at a point’. What the teacher refers to, in the discourse, as ‘gradient of a curve’, is indeed, the gradient function or derived function.

Tangent: To understand the routine for constructing the object of the derivative, it is important to observe and analyse the processes of creating and substantiating narratives (Sfard, 2007). Together with the use of visual mediators, the analysis of narratives would enable us to identify the types and characteristics of the routine procedures. Using the graph of the function $y = x^2$, the teacher talks about the tangent:

20 Teacher: Is there anywhere on that curve where you definitely, already know its gradient?

21 Student: $x - axis$

22 Teacher: Good, would you all accept that the x-axis is a tangent to the curve? What is the gradient of the $x - axis$?

23 Student: Zero

24 Teacher: Zero. A tangent, you did this in mechanics, is sort of the direction in which you are instantaneously traveling.

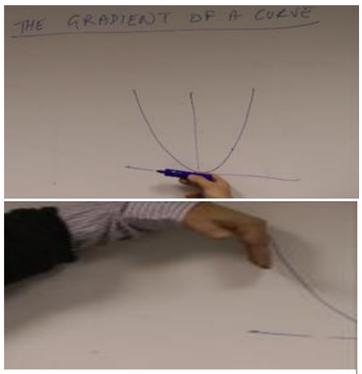

Fig.4: Teacher’s sketch diagrams

25 Teacher: The direction in which you're going there [*Teacher pointing at the graph on the board*] is the instantaneous direction, the tangent of the curve.

In [20] - [25] the teacher constructs a definition of the tangent by exemplar (Viirman, 2013) by illustrating its key properties with a specific example of the visual mediator, the graph of the function $y = x^2$, and such routines are characteristic of, and prevalent in mathematical discourse. However, notice that the teacher describes ‘instantaneous direction’ as ‘the tangent of the curve’ [25]. Although objectified, treating the mathematical concept of direction as a mathematical object, the narrative is inconsistent with literate mathematical discourse. An endorsed narrative describes direction as the slope or gradient of the tangent. Thus, the narrative ‘the tangent of the curve’ here should be substantiated to mean ‘the gradient of the tangent’.

Ambiguity with calculus symbolism

In calculus discourse, symbolic artefacts, such as the $\frac{dy}{dx}$, are integral to the thinking and communication process (Sfard, 2007). Apparent in the teacher’s discourse are visual mediators: written words, graphs [Fig.1 & Fig.2], deictic language [25] and gesturing [Fig.4] and symbolism. However, there is some ambiguity in the teacher’s use of calculus symbolism. Here is one of the teacher-student dialogue from the lesson [lesson transcript]:

85 Teacher:	So, let's make a note of this, [*writing on the board*] If $f(x)$ is x^3, it means $f'(x)$ is $3x^2$.
86 Student:	What is that dash mean?
87 Teacher:	It means the derivative, the gradient function. That's the notation I have used here.
88 Student:	What does the derivative mean?
89 Teacher:	It means the gradient function, the gradient of the curve is $2x$, of x^2. It's not a constant, is it?
90 Student:	No
91 Teacher:	The gradient, a constant?
92 Student:	No
93 Teacher:	It's a function of x
95 Teacher:	We call it a gradient function. We call it the derivative. There are other names as well, is that ok?

The use of symbolic signifier f' ‘f-dash’ in [86 - 87] by the teacher, poses some challenges for the students. The question in [86] suggests that the student is having some difficulties with symbolic realisations, which seems to be exacerbated by the teacher’s response with specialised calculus vocabulary [88]. In substantiating the narrative, the teacher switches between visual and vocal signifiers, from symbols [85] to specialised words – derivative, gradient function [87]. However, these specialised calculus words added to the student’s difficulty with calculus – the meaning of the derivative [88]. The teacher reiterates his earlier narrative [87] in [95], linking the words ‘derivative’ and ‘gradient function’. Notice that the teacher’s routine is to construct a definition of derivative by exemplar (Viirman, 2013). Thus, by illustrating the properties [89 – 94] of the object of the discourse with a specific example. However, note the inconsistency in word use of gradient in [91-93], the teacher’s utterances in [91] and [93] are in fact contradictory; the gradient is indeed a constant! The word derivative could refer to the derivative function (a function) or the derivative at a point (a constant). This dualism was not substantiated in the lesson; it was not made explicit for the student. A commognitive study with calculus students by Park (2013), found that 'most students did not appreciate the derivative at a point as a number and the derivative function as a function' (p.624). In calculus discourse, such ambiguity is compounded by calculus symbolism.

The teacher's use of visual signifier f' draws upon historically established mathematical discourse in calculus symbolism (Sfard, 2008). Symbolic mediators such as $\frac{dy}{dx}$ or $f'(x)$ have a dual role. On the one hand, $f'(x)$ can be an objectified narrative for 'the derivative of $f(x)$', and an operational narrative for 'the process of differentiation' on the other. Such a symbolic signifier is what Gray & Tall (1991) described as 'procept' (Tall, 1992b, p.4). In the mind of a literate mathematician, a procept can evoke either a process or a concept, and it all happens subconsciously (Tall, 1992b). The term procept refers to "the amalgam of process and concept in which process and product are represented by the same symbolism" (Tall, 1992b, p.4). The "duality (as process or concept), flexibility (using whichever is appropriate at the time) and ambiguity (not always making it explicit which we are using)" (p.4) in the use of calculus procepts presents challenges for many students. Calculus symbolism and vocabulary has been found to present challenges for both students and teachers (Tall, 1992a). Given the flexibility and the duality of use of such procepts, it is essential that teachers make it explicit enough for students to develop the necessary flexible thinking and understanding to be able to deal with the possible ambiguity of use (Tall, 1992b).

CONCLUSION AND IMPLICATIONS

Mathematical discourse on calculus involves specialised mathematical language and visual mediators. In calculus, symbolic realisations are an important aspect of visual mediation, so are graphical representations. For example, notice that the graph [in 20-23] is used, not as a mere auxiliary means for conveying a pre-existing thought, but as a way of communicating. Thus, visual mediators are integral to commognition, i.e., the thinking and communication process in the discourse, contrary to the common understanding of tool use.

The classroom discourse involves multiple visual mediators, but more importantly, the didactical routines show evidence of constant shifts between different signifiers or modes of mediation. We see shifts between symbolic signifiers (e.g.f') and specialised mathematical words (derivative); shifts between verbal mediation (e.g. use of deictic language) and visual mediation (e.g. the graph, the teacher gesturing), in [25] for instance. For the teaching of calculus, Tall (1992a) argues for the need for versatile transitions between representations, graphics, numerics and symbolics (p.9). Such representations resemble Sfard's 'realisations' (p.154) of the signifiers which could be spoken words or written words or visual symbols. Nardi et al. (2014) explain the importance of symbolic realisations in mathematical discourse, that symbolic mediation brings 'generative power' (Sfard, 2008, p.159) and 'powerful manipulative ability' (Tall. 1992a, p.9) of the discourse.

There is also evidence of some inconsistency in the teacher's use of calculus words, and some ambiguity in the use of calculus symbolism in the classroom discourse. This suggests that difficulties with calculus persist, both for students and teachers alike. Therefore, mathematics teachers and educators should always pay particular attention to the specialised vocabulary and symbolism in the calculus discourse.

References

Güçler, B. (2013). Examining the discourse on the limit concept in a beginning-level calculus classroom. *Educational Studies in Mathematics*, *82*, 439–453.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education*, *16*(2), 182-198.

Orton, A. (1983a). Students' understanding of integration. *Educational Studies in Mathematics*, *14*, 1–18.

Orton, A. (1983b). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14*, 235–250.

Park, J. (2013). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematical Education in Science and Technology*, 44*(*5), 624–640. Online: https://doi.org/10.1080/0020739X.2013.795248

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences, 16*(4), 565-613.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing*. New York, NY: Cambridge University Press.

Skemp, R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching*, *77*, 20–26.

Tall, D. (1992a). Students' Difficulties in Calculus. *Proceedings of Working Group 3 on Students' Difficulties in Calculus, ICME-7,* (pp.13–28). Québec, Canada.

Tall, D. (1992b). Mathematical Processes and Symbols in the Mind. In Z. A. Karian (Eds.), *Symbolic Computation in Undergraduate Mathematics Education* (pp. 57-68). Washington, DC: Mathematical Association of America.

Viirman, O. (2013). The functions of function discourse – University mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology*, *45*(4), 512–527.

KINDERGARTEN TEACHERS' KNOWLEDGE OF STUDENTS: THE CASE OF REPEATING PATTERNS

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Knowledge of students' conceptions and competencies is an important element of teachers' knowledge for teaching mathematics. This study reports on kindergarten teachers' knowledge of children's abilities to complete two repeating pattern tasks: extending repeating patterns and comparing two repeating patterns. These tasks had previously been implemented with children. Results indicated that on the extension task, teachers tended to underestimate children's ability to complete the task, but on the comparison task, they tended to overestimate children's abilities.

INTRODUCTION

Knowledge of students - their conceptions, misconceptions, and ways of thinking - is an essential element of knowledge needed for teaching mathematics. Shulman (1986), for example, suggested that pedagogical content knowledge (PCK) includes knowing "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons…" (p. 9-10). Others (e.g., Ball, Thames, & Phelps, 2008) described the relevance of being able to anticipate and predict what examples students might find confusing or difficult and what tasks students might find interesting or motivating. Knowledge of students' conceptions and processes can impact on teachers' decision-making, allow teachers to attend to individual students, and influence students' learning outcomes (Carpenter, Fennema & Franke, 1996). This study investigates kindergarten teachers' knowledge of children ages 4-6 years, within the context of repeating patterns.

Repeating patterns are patterns with a cyclical repetition of an identifiable 'unit of repeat' (Zazkis & Liljedahl, 2002). For example, the pattern ABBABBABB… has the minimal unit of repeat ABB. Exploring repeating patterns may promote children's appreciation of underlying structures (Starkey, Klein, & Wakeley, 2004). Specifically, recognition and analysis of patterns, such as when comparing two patterns, may promote development of algebraic thinking (Clements & Sarama, 2007), providing children the opportunity to observe and verbalize generalizations. Engaging in extension tasks, where children predict what comes next in a pattern, can promote deductive reasoning skills (Greenes, Ginsburg, & Balfanz, 2004). Studies found, however, that teachers do not always provide worthwhile patterning opportunities for children, and when children engage spontaneously in patterning, teachers may fail to capitalize on the child's interest, missing out on an opportunity to extend children's interest and

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knowledge in patterning (Waters, 2004). Perhaps teachers could be more aware of children's abilities to engage with repeating pattern activities.

This study is part of a larger study which investigated kindergarten teachers' knowledge for teaching repeating patterns. Previously, we found that teachers are able to complete various pattern tasks, such as defining repeating patterns, extending repeating patterns, and comparing repeating patterns (Tirosh, Tsamir, Levenson, & Barkai, in press). The current study investigates two questions related to teachers' knowledge of children: What are kindergarten teachers' estimates of children's abilities to complete an extension task and a comparison task? Are teachers' estimates in line with children's competencies?

BACKGROUND

One of the most frequent activities implemented with children is to extend a given repeating pattern. Papic et al. (2011) reported that many children succeed at these tasks without necessarily recognizing the unit of repeat. Instead, they use the "matching one item at a time" strategy, also known as the "alternation strategy," especially successful with simple ABAB patterns. Rittle-Johnson, et al. (2013) found that some children reverted to producing an ABAB pattern while others could not produce more than one unit of repeat correctly when extending an ABB pattern. Similarly, Swoboda (2010) found that for some four-year old children, continuing a pattern means duplicating the unit of repeat once, and no more. In other words, both the complexity of the unit of repeat, and the number of times the unit is repeated, may contribute to task complexity.

Comparing two patterns is a task often thought of as more abstract (Rittle-Johnson et al., 2013). Papic et al. (2011) describe an incident where a child spontaneously claimed that a blocks pattern he created was similar to a flower pattern because one is "blue, yellow, yellow, blue, yellow, yellow" and the other is "curved, spiky, spiky, curved, spiky, spiky." When asked to elaborate on their similarity, the child responded that, "There is one curved and one blue, and then there's two spiky and two yellow, that's the same pattern" (p. 255). Papic et al. took this claim as evidence of the child's emergent recognition of an ABB pattern and the child's readiness to consider structure. In another study, Waters (2004) observed a young girl who created a necklace out of game materials and described her necklace as "diamond, funny shape, diamond, funny shape" (p. 326). When asked if she could describe her necklace using numbers, she replied "1, 1, 1, 1" and would not accept the teacher's suggestion of describing the pattern numerically as "1, 2, 1, 2." Waters concluded that the child was unable to describe her pattern using a numerical representation, perhaps because of her young age. It could also be that the child was attempting to describe that each material was used once and then exchanged with another material, also used once.

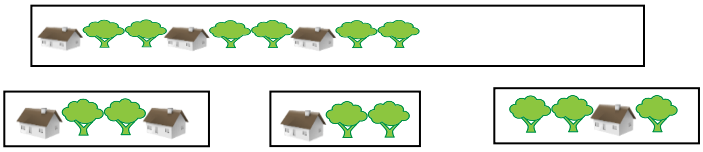
Directly related to the current study is Tsamir et al.'s (2017) study of kindergarten children's interactions with two repeating pattern tasks. The first task consisted of presenting children with two pictorial linear patterns, along with possible ways of continuing each pattern, and then asking them to choose which of the continuations was appropriate. Both patterns had an ABB structure, however Pattern One (see Figure 1) consisted of three complete repetitions of the unit of repeat (ABBABBABB), and Pattern Two (see Figure 2) included three instances of the repeating unit and in addition the first two elements of the unit of repeat (ABBABBABBAB). In general, results indicated that between 60-80% of children chose continuations based on the minimal unit of repeat (MUR). Choosing MUR continuations for Pattern Two was more difficult than for Pattern One. In addition, it was sometimes difficult for children to accept a continuation that did not end the pattern with a complete unit of repeat

The second task in Tsamir et al.'s (2017) study consisted of showing children pairs of patterns and asking them in what ways the patterns were the same and how they were different. The first pair of patterns consisted of two strands of colored beads, each made up of the same colored beads, but the first strand had an ABB structure, and the second an AB structure (see Figure 3a). The second pair consisted of two strands, each with different colored beads, but both strands had the same ABB structure (see Figure 3b). Qualitative analysis of children's responses led to three levels of structural knowledge. Children who made no mention at all or did not refer at all to the unit of repeat, were assigned Level 0; children who used a "matching one item at a time" strategy were assigned Level 1; children who were able to abstract the unit of repeat, were assigned Level 2. Results of that study indicated that approximately 20% of children performed at Level 2 on both tasks. However, 38% of children showed some structural awareness (Level 1) when engaging with the first pair of strands, and only 23% showed this same level when comparing the second pair of strands.

METHODOLOGY

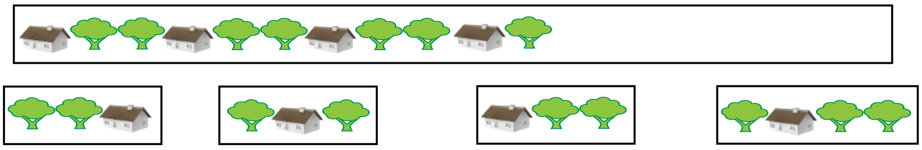
Participants in this study were a group of 36 kindergarten teachers who participated in the study related to kindergarten teachers' knowledge for teaching patterns (Tirosh et al., in press) and were currently participating in a professional development program devoted to patterning concepts. All teachers had a first degree in education and were teaching in public kindergartens. Informal interviews with some of the teachers revealed that most of the patterning activities taking place in their kindergartens consisted of children drawing borders or frames for pictures, albeit borders which were made up of repeating patterns.

Before the program began, teachers were presented with the same two tasks implemented in Tsamir et al.'s study (2017) described above (see Figures 1 and 2). For each pattern, teachers were presented with continuations based on the minimal unit of repeat (MUR) and other continuations, and were asked to choose ways for continuing the pattern. In our previous study with teachers (Tirosh et al., in press) it was found that most teachers chose extensions based on the MUR, although less teachers chose an extension when it ended the pattern in an incomplete unit of repeat. Teachers were then asked: What part of the kindergarten class will choose continuations based on the MUR – (1) almost none / (2) a few / (3) about half / (4) many / (5) almost all of the children?



1a: MUR 1b: MUR 1c: Other

Figure 1: Pattern one ending with a complete unit of repeat and possible continuations



2a: Other 2b: MUR 2c: Other 2d: MUR

Figure 2: Pattern two ending with a partial unit of repeat and possible continuations

For the second task, teachers were presented with two pictures, each a picture of the two strands of beads shown to children (see Figures 3a and 3b). First, teachers were asked to write down for each pair what was the same and what was different. Results from our previous study (Tirosh et al., in press) found that nearly all teachers referred in some way to the fixed structure of the core unit. Teachers were then asked: Do you think kindergarten children will be able to tell what is different and what is the same for each pair of strands in relation to the structure of the pattern?

Figure 3a: same colors, different structure

Figure 3b: different colors, same structure

Data analysis

Teachers' assessment of children, were compared to the results of Tsamir et al.'s (2017) study of kindergarten children. For the extension task, children's successes were configured in terms of the percentage of children (0%-100%) who succeeded. Teachers' assessment, however, was given on a 1-5 scale (see above). The 1-5 scale used by teachers was reconfigured to reflect the 0%-100% scale in the following way. The lowest score on both scales was 1 and 0% respectively and the highest score was 5 and 100% respectively. We transformed the 1-5 scale by using the linear equation: $y=25(x-1)$ where x represents the scale used for teachers and y the scaled used for children. This method of reconfiguration was also used in a previous study which compared teachers' knowledge of students' number conceptions to children's actual conceptions (Tsamir et al., 2014). We then compared teachers' estimates of how many children would succeed at a task, with results of children's actual performances.

For the second task, most teachers responded with a simple 'yes', children can tell what is similar and what is different, or 'no', they cannot complete this task. Some were less decisive and responded with: 'some', 'a few', 'most', 'a bit difficult', 'maybe', and 'doubtful'. Responses were arranged in decreasing order of agreement as follows: 'yes', 'most', 'some', 'a few', 'maybe', and 'no'. The arrangement was suggested by one author and validated with two other authors who agreed 100% with the scale. That being said, we acknowledge that the terminology is subjective and thus refrain from quantifying the degree to which teachers agreed or disagreed with the statement of children's ability. After this, we noted how many teachers wrote the same estimate for both pairs of strands. That is, did teachers think that both pairs of strands presented the same degree of complexity or did teachers think that it would be more difficult for children to notice and tell about the pattern structure in one of the pairs? For example, a teacher that wrote 'yes' for one pair of strands, and 'most' for the second pair was categorized as believing that it would be easier for children to relate to pattern structure in the first pair than in the second pair. Finally, teachers' responses were compared to the actual results of children's performances (as reported in the Background).

RESULTS

Regarding the extension task, teachers believed that little more than half of kindergarten children would be able to choose MUR extensions for each pattern (see Table 1). When comparing teachers' estimates of MUR extensions (P1 extensions a and b, P2 extensions b and d) to other extensions (P1 extension c, P2 extension a and c), it seems that teachers believed it would be more difficult for children to know when an extension was not in line with the MUR than when it was.

Pattern	Extension	Teachers' estimates based on a 1-5 scale		Teachers' estimates translated to percent	Percent of children who succeeded
		M	SD		
1	a (MUR)	2.97	1.16	49.25	65
	b (MUR)	3.17	1.18	54.25	79
	c (other)	2.82	1.11	45.5	76
2	a (other)	2.55	.90	38.75	69
	b (MUR)	2.61	.90	40.25	59
	c (other)	2.52	.83	35.5	57
	d (MUR)	2.62	.89	40.5	61

Table 1: Comparing teachers' estimates with children's performance.

When comparing teachers' estimations to the findings of Tsamir et al.'s (2017) study (see last column of Table 1), we see that teachers consistently underestimated children's ability to choose MUR extensions. However, teachers did correctly assess that children would have greater success extending Pattern One than Pattern Two. In other words, teachers knew that extending the pattern which ended in a partial unit would be more difficult than extending the pattern that ended in a complete unit.

Regarding the comparison task, for the first pair of strands (same colors, different structures), 54% of the teaches wrote 'yes, children would be able to complete this task', and for the second pair (different colors, same structures), 67% wrote 'yes'. This is in line with actual results, which, as described in the background, showed that less than a quarter of the children were able to abstract the structure of the patterns. On the other hand, for each pair of strands, only 10% of the teachers wrote a direct 'no'. In other words, teachers believed that at least some of the children would be able to relate to structure when comparing patterns. This suggests an overall positive view of children's ability to complete this task. This view may be a bit more positive than actual results, which showed that only about half of the children exhibited some structural awareness when comparing the strands of beads.

A closer look at teachers' views of the different strands of beads may be seen in Table 2, which presents how many teachers believed children would perform the same for both pairs of strands, and how many thought pointing out similarities and differences would be more difficult in one pair over the other. As can be seen, most teachers believed that both strands of beads presented the same level of difficulty. Amongst teachers who thought there was a difference, more teachers believed that the first pair of strands would be more difficult for children to compare than the second. However, looking back at the children's study, we found that children showed more awareness of pattern structure when comparing the first pair of strands (same colors, different structures) than when comparing the second pair of strands (different colors, same structure). In other words, noticing and mentioning the pattern structure was more likely to occur when the patterns had different structures than when they had the same structure, opposite of what teachers' estimated.

Pattern pairs have…	the same degree of difficulty			different degrees of difficulty	
Children…	can complete the task	cannot complete the task	Some can complete the task	The first pattern pair is more difficult	The second pattern pair is more difficult
Frequency	19 (49)	3 (8)	3 (8)	12 (30)	2 (5)

Table 2: Teachers' estimates of the degree of difficulty when comparing patterns

Summary

In general, teachers tended to believe that children would perform better on the comparison task than on the extension task. In comparison to children's actual competencies, teachers underestimated children's ability to choose appropriate ways of continuing a pattern. However, they did recognize which pattern would present less confusion to children when asked to extend the pattern. Teachers did not recognize that one pair of patterns would be more difficult to compare than the other, and they might have overestimated children's ability to abstract the structure of patterns when asked to compare two patterns.

DISCUSSION

Teachers' estimations of children's abilities varied according to the task given (extension versus comparison tasks), as well as the way a pattern was presented (e.g., presenting two patterns with the same structure, but one ending with a complete unit of repeat while the other ends part way). These findings indicate that teachers are well aware that even within the same mathematical content, different tasks, as well as different examples, may present different challenges for children. This is an important and positive aspect of teachers' PCK (Ball et al., 2008).

Knowing that different tasks present different challenges is a first step. The next step is to know which examples and tasks are more difficult to contend with and why. We offer a few possible reasons for this gap in teachers' knowledge of children. First, in our previous study with children (Tsamir et al., 2017), we investigated children attending various kindergartens, and not necessarily these teachers' classes. Thus, it might be that if teachers were to assess children in their own classes, their assessments would be more accurate. Another reason might be due to the teachers' lack of experience with a variety of repeating patterns. In our previous study (Tirosh et al., in press), when teachers were asked to draw repeating patterns, nearly all teachers drew ABC patterns that ended in a complete unit of repeat, hinting at little experience with patterns that do not end with a complete unit of repeat. When asked to compare two patterns, there were no differences in the way teachers compared patterns with the same structure and the way they compared patterns with different structures. Thus, it might be that teachers' estimations of children's abilities stemmed from their own experiences and ways of engaging with repeating patterns. Finally, recall that teachers' reports of repeating pattern activities in their classrooms consisted only of children drawing borders made up of repeating patterns; it seems that these teachers had little experience observing children engaged with various pattern activities.

All of the above possibilities point to a need for professional development that widens teachers' example space of repeating patterns, as well as promotes their knowledge of children. In the same manner that students' mathematics knowledge is a starting point for teachers to build upon (Shulman, 1986), so too, we need studies of teachers' knowledge, including their knowledge of students, in order to plan for meaningful professional development. This study contributes toward this goal.

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References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special?. *Journal of Teacher Education, 59*(5), 389-407.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20.

Clements, D. H., & Serama, J. (2007). Early childhood mathematics learning. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 461-555). New York: Information Age Publishing.

Greenes, C., Ginsburg, H. P., & Balfanz, R. (2004). Big math for little kids. *Early Childhood Research Quarterly, 19*(1), 159-166.

Papic, M., Mulligan, J., & Mitchelmore, M. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237-269.

Rittle-Johnson, B., Fyfe, E. R., McLean, L. E., & McEldoon, K. L. (2013). Emerging understanding of patterning in 4-year-olds. *Journal of Cognition and Development, 14*(3), 376-396.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher, 15*(2), 4-14.

Starkey, P., Klein, A., & Wakeley, A. (2004). Enhancing young children’s mathematical knowledge through a pre-kindergarten mathematics intervention. *Early Childhood Research Quarterly, 19*(1), 99-120.

Swodboda, E. (2010). Natural differentiation in a pattern environment (4 year old children make patterns). In V. Durand-Guerrier, S. Soury-Lavergne, and F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 2657-2666). Lyon, France: Institut national de recherche Pedagogique.

Tirosh, D., Tsamir, P., Levenson, E., & Barkai, R. (in press). Preschool teachers’ knowledge of repeating patterns: focusing on structure and the unit of repeat. *Journal of Mathematics Teacher Education*, DOI 10.1007/s10857-017-9395-x.

Tsamir, P., Tirosh, D., Levenson, E., Barkai, R. & Tabach, M. (2017). Repeating patterns in kindergarten: Findings from children’s enactments of two activities. *Educational Studies in Mathematics, 96*(1), 83-99.

Tsamir, P., Tirosh, D., Levenson, E., Tabach, M., & Barkai, R. (2014). Developing preschool teachers' knowledge of students' number conceptions. *Journal of Mathematics Teacher Education, 17*, 61-83.

Waters, J. (2004). Mathematical patterning in early childhood settings. In I. Putt & M. McLean (Eds.), *Mathematics Education for the Third Millennium* (pp. 565-572). Townsville: Mathematics Education Research Group of Australia.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49*(3), 379-402.

THE USE OF NONSTANDARD PROBLEMS IN AN ODE COURSE FOR ENGINEERING STUDENTS

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We report on the design and use of 'nonstandard' problems in an ordinary differential equations (ODEs) course for engineering students. The focus of the paper is on the analysis of the development of students' mathematical discourse and conceptual understanding of Existence and Uniqueness Theorems (EUTs) from a commognitive theory perspective. Our analysis so far shows how students use familiar mathematical routines in new situations furthering their knowledge and understanding. Nonstandard problems have been a useful tool to gain insights into students' learning of mathematics.

INTRODUCTION AND BACKGROUND

The importance of ODEs in undergraduate (UG) mathematics education is widely acknowledged (Rasmussen & Wawro, 2017). However, fewer than two dozen empirical studies were published in top journals since 2004 which is "somewhat surprising given the centrality of differential equations (DEs) in the undergraduate curriculum" (ibid., p. 555). It is known that students experience difficulties with ODEs and even with the understanding of the very notions of a differential equation and its solutions (Arslan, 2010; Raychaudhuri, 2008). EUTs are among very few theoretical results included in standard ODEs courses for engineering students. Roberts (1976) emphasised that teaching EUTs makes engineering students aware of situations where solutions to initial value problems (IVPs) may not be unique or may not even exist.

Even though students experience difficulties with conceptual understanding of EUTs and their correct application (Raychaudhuri, 2007), "traditional content of DEs courses can be improved by including more activities aimed at enhancing the student's understanding of basic concepts such as DE, solution to a DE and existence and uniqueness theorem" (Arslan, 2010, p. 887). Recently, Klymchuk (2015) pointed out that students may form a habit of applying formulas or rules without checking conditions/constraints because assessment questions are often formulated so that these are automatically met. "But in real life problems not all functions and equations behave so nicely and ignoring conditions and constraints might lead to significant and costly errors" (ibid., p. 63). We use nonstandard problems to help students understand and correctly apply EUTs for DEs viewing such problems as tasks "for which students had no algorithm, well-rehearsed procedure or previously demonstrated process to follow" (Breen, O'Shea & Pfeiffer, 2013, p. 2318). Our intention is to provide challenging experiences for students studying towards mathematics-intensive degrees.

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RESEARCH SETTING AND METHODOLOGY

The research took place in an ODEs course for engineering students in their fourth year of study. The activity formed an assessed piece in the final part of the course when students acquired sufficient theoretical knowledge and good computational skills. Participation in the research was voluntary. Tutorials were attended by 50-65% of the total number of students enrolled. The lecturer of the course, one of the authors of this paper, devised a set of six problems to challenge students' conceptual understanding of the EUTs. Students were requested to work on the problems individually first (in the tutorial and at home) before discussing their solutions in small groups. In addition, each group presented their solution to one of the problems to the class. Students' written solutions (referred to as "scripts") were collected and photocopied. Group discussions were audio-recorded and transcribed. Students' scripts and group discussions of two groups, G1 and G2, (out of five recorded) form the basis of analysis for this paper. The code S12 is used to identify the student #2 in G1, etc.

We situate ourselves in an interpretative framework for data analysis using qualitative methods such as coding, interpreting and categorising (Cohen, Manion, & Morrison, 2008) to characterise students' discourse and how discourse develops. To aid and make meaningful our analysis we consider the theoretical constructs of narrative and routine of commognitive theory (Sfard, 2008). A scholarly mathematical discourse such as a definition or theorem is a narrative endorsed by the community of mathematicians. The EUTs formally introduced by the lecturer to her students represent such a mathematical narrative. Narratives are endorsed (or rejected) with the help of routines, repetitive patterns produced in creating or substantiating a narrative. Sfard distinguished three types by their aim. (1) The aim of explorations is to produce or substantiate a narrative and thus further a mathematical discourse. Explorations are further divided into construction, substantiation and recall. (2) Deeds are actions aimed at a physical change in objects. (3) Participation in rituals has the aim of creating or sustaining social bonds. Sfard also distinguishes the 'how' of a routine (the process or course of action) from the 'when' (the circumstances that evoke a person's actions and those that signal its completion), the applicability and closure conditions of a routine.

In this paper we pose the following research questions: How do students engage with nonstandard problems related to EUTs? In what ways do such problems further students' understanding of new mathematical concepts?

THE TASKS

We present the analysis of the first two problems, P1 and P2, of the assessment. All problems required the correct application of the EUTs. The theorem required for solving P1 and P2 states that if coefficients of a linear DE are continuous on a given interval, there exists a unique solution of the initial-value problem on this interval.

Students encountered 'how to verify' techniques for *particular* solutions to a DE but the requirement of P1(a) is for the *general* solution. This is an unusual problem for engineering students who are, to our knowledge and experience, almost never asked that. We make this claim by referring to standard UG engineering textbooks. It is easy to verify that the given function is a solution to the given DE (and students were able to do this) but to show that it is the general solution one has to explain the role of the arbitrary constant (method M1). To avoid this discussion, it is also possible to derive the general solution using an integrating factor or variation of constants (method M2). These are the two possible correct solutions for this problem with M1 being the method that the lecturer anticipated her students to use.

1. (a) Verify that

$$y(x) = \frac{2}{x} - \frac{C_1}{x^2}$$

is the general solution of a differential equation

$$x^2 y' + 2xy = 2.$$

(b) Show that both initial conditions $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solution. Does this fact violate the Existence and Uniqueness Theorem (EUT)? Explain your answer.

Figure 1: Formulation of Problem 1.

In P1(b) one may erroneously believe that two different initial conditions (ICs) give rise to the same solution $y = \frac{2}{x} - \frac{1}{x^2}$. However, since the coefficients $p(x) = \frac{2}{x}$ and $q(x) = \frac{2}{x^2}$ are not defined at $x = 0$ and are continuous on (-∞, 0) and (0, + ∞), but not on any interval including zero, there are two disjoint intervals of existence of solutions, each containing one of the initial points. The formulation of P1(b) is nonstandard for engineering students and the problem was intentionally stated in this form to create a conflict or consternation. If students go ahead and show that a given function formally satisfies the DE without paying attention to discontinuity of coefficients and the function itself at zero, they may miss the point altogether. It is for this reason the lecturer explicitly asked, "Does it violate the EUT?" and "Explain your answer".

In P2(a) students were asked to verify that a given function is the *general* solution of a second order linear DE. Again, there are two correct methods. The first is to substitute the function and its derivatives into the DE and discuss the representation for the general solution as a linear combination of two linearly independent solutions (method M3), and the second method is to integrate the DE using the substitution and reducing the order (method M4). This is an unusual problem as one cannot be sure that the given function is the general solution by merely substituting it into the DE. Usually, second order linear DEs with variable coefficients are not discussed in detail in ODE courses for engineering students but in this case the EUT can be successfully applied.

P2(b) required students to verify that the ICs cannot be satisfied. Most students did this correctly. In P2(c) it was necessary to notice that the ICs were defined at $x = 0$ which is the point of discontinuity of the coefficients of the DE. Hence, the existence and uniqueness of solutions cannot be guaranteed even though a solution may still exist and

2. (a) Verify that

$$y(x) = C_1 + C_2 x^2$$

is the general solution of a differential equation

$$xy'' - y' = 0. \qquad (1)$$

(b) Explain why there exists no particular solution of equation (1) satisfying initial conditions

$$y(0) = 0, \qquad y'(0) = 1.$$

(c) Suggest different initial conditions for this differential equation so that there will exist exactly one particular solution of a new initial value problem. Motivate your choice.

Figure 2: Formulation of Problem 2.

be unique if the conditions of the EUT are not satisfied. There are at least two ways to modify the ICs: (i) to change the initial point from $x = 0$ to any other value and use the EUT, or (ii) to modify the ICs at zero and show by direct inspection that the solution exists. In the latter case one also has to prove that the solution is unique.

ANALYSIS OF PROBLEM 1

Our analysis of P1(a) shows that students were able to demonstrate that a given function satisfies the DE as the extracts from the group discussion provided below show. However, all nine students (four in G1 and five in G2) failed to explain in their discussions and in written solutions that were submitted (with the exception of one student), why this solution was the *general* solution.

S12: Since we got the solution, I just took the derivative of that and put it into the original equation, to see that two equals two, and … that was my verification.

S13 and S14 confirmed having done the same as S12 while S11 responded,

S11: So I was the only one who actually did any work, [laughter] so I actually integrated the whole thing, and ended up with the right expression, so I think your way of doing it is a lot easier.

S13 answered “A bit more efficient at least” and all moved on to P1(b). Thus three students only showed that a given function satisfies the DE. One student integrated the DE obtaining the general solution which is a correct method (M2). However, the incomplete solution is accepted as correct (and complete) without further discussion. The discussion in G2 followed a slightly different pattern with S22 stating that it was “obvious” to differentiate the function, “put it into the DE and see if it’s correct as usual.” Three students (S23, S24 and S25) integrated the DE, for example,

S25: I did the same as you did, using the integrating factor, multiplying and then I just solved the equation because it’s solvable ….

S24: I also solved the equation by the integrating factor, but I think it's more easy just to derivate it once and put it into the original equation and see if it's correct.

We see S24 echoing the same sentiment as S11 that method M1 is easier. However, one student intercepted to say,

S25: But there could be more solutions, they are not general solutions.

S24 answered "Yes, maybe" and all moved on to P1(b). Thus, three students reported on obtaining the general solution by integration and one of them stated that the verifying-by-substitution method was easier. Remarkably, S25 seemed to be aware that there may be a problem with it. The written (handed-in) solution of this student did contain a discussion of the constant and was the only correct solution using M1 submitted for final assessment to the lecturer. However, in the group discussion this point was not elaborated, and students agreed on the two approaches being equivalent – when they are not.

Analyses of students' scripts showed how students changed their solutions following the discussions with peers and the presentation. From a correct solution using integration (M2) to obtain the *general* solution - and thus proving what had been asked - to a verifying-by-substitution method (M1) omitting the discussion of the constant (and hence incomplete). From the analysis of the dialogue we deduce that this change occurred because students thought that M1 was easier and more efficient and not because students (with the exception of S25) realised that they were moving into a new discourse that had the aim of extending their conceptual understanding of the difference between the notions of a particular and the general solution of a DE.

P1(b) had a clear reference to the theorem to be used. The correct solution involved verification that both solutions for two different ICs were given by the same formula but these solutions were defined and continuous on two different intervals. G1 discussed P1(b) - considered "the hard part" by S11 - as follows.

S12: Undefined at zero, so we get two different curves and both solutions work. We do not have a continuous curve which happens to intersect at these two points, it's two curves that will be correct in this small area.

One student in G2 explained correctly the effect of the discontinuity.

S25. Yes, but there is a discontinuity at $x = 0$, so it [the theorem] only guarantees that for $x > 0$ and $x < 0$.

Other students in G2 provided explanations such as "So you have to have two unique solutions, one on each interval" (S24), "for two different initial conditions" (S22), and "one for the left part and one for the right part of 0" (S24). S21 appeared to struggle saying, "There could happen the chance maybe that they are the same perhaps." This indicates that the student did not understand that by the definition any solution to a DE should be at least a continuous function. P1(b) probed students' understanding of what violation implies. Four students agreed and stated explicitly that it did not violate the EUT. S21 stated, "It violates at $x = 0$" which prompted the reply,

S22. Well, but that's not the question. The question is that if you have two identical particular solutions, does that violate the theorem. And it doesn't.

When S21 repeated the question, S22 gave a fuller explanation that convinced S21.

Analysing students' scripts of P1(b) we found that ten students provided complete and correct solutions. This number increased to fourteen after students had the opportunity to discuss their solutions in small groups. We tend to believe that students had time to reflect on the conditions of the EUTs and benefited from the discussion.

ANALYSIS OF PROBLEM 2

The formulation of P2(a) is similar to that of P1(a) but the nature of the problem is different due to the higher order of the DE. Most students in G1 and G2 verified by substituting the function and its derivatives in the DE that it is a solution, but failed to show that it is the general solution, not completing M3. S11 described obtaining the general solution by integration (M4) with "It's only me that's stupid, obviously, cos I derived the entire expression" while S13 concluded it to be "inefficient." Students agreed that "there are different ways of showing the same thing" (S11) but considered the (incomplete) method M3 as "more efficient" (S11) or "better" (S13). The opportunity to expand their discourse was cut short at that point. In G2, S24 explained that he solved the problem by substituting the given function into the DE, and S21, S22 and S23 agreed. S25 argued that the Wronskian should be used for showing linear independence of two solutions, 1 and x^2, thus verifying that the given function is the general solution. However, S25 hesitated about the validity of this approach noticing that the Wronskian vanishes at $x = 0$ and not realising that the DE is not defined at this point. This important detail was pointed out by S24.

S25: Yes. This is what I arrive at, but I think it's strange, that it's a general solution when it's still not valid. I am not too confident in this.

S24: But the original equation $xy''\text{-}y' = 0$, it isn't defined for $x = 0$.

Not all students were able to follow this argument as can be seen in students' scripts. The exception was S25 who correctly used the Wronskian (M4) in his solution.

To explain why no particular solution can be found for the given ICs in P2(b) both groups provided similar arguments. Most students (S11, S12, S23, S24) used the ICs, obtained $C_1 = 0$ and a contradiction for the constant C_2 $(0 \cdot C_2 = 1)$. S22 reflected on the general solution considering a parabola which has an extremum at $x = 0$ where the derivative should always vanish, leading to the contradiction. S11 was expanding his discourse by looking for a reason "why it was this way", leading him to explore:

S11: I did the same thing as well but I tried thinking why is it this way, and my sort of conclusion was that it's in the bottom of a parabola, where the derivative always is 0.

In the final scripts three students applied ICs, obtained inconsistent system of linear equations and correctly used the EUT to explain this, while nine students only acknowledged the contradiction and provided no explanation. Two students employed a geometric argument to show that the ICs cannot be satisfied.

P2(c) provided an opportunity for students to practice the new mathematical discourse of EUTs. Students in both groups offered solutions saying they "just guessed" (S11, S12) or "made them up" (S23, S24). S11 reflected on the properties of parabolas to obtain ICs that worked. Reflecting on the multiple ways of formulating ICs students agreed that it suffices to shift the initial point to $x \neq 0$. S25 drew on the EUT to round off the discussion in G2, "I used the existence and uniqueness theorem because of discontinuity at $x = 0$, so no guarantee there, but for all other x there is a solution guaranteed", with S22 and S24 voicing agreement.

DISCUSSION

Nonstandard problems stimulated lively mathematical discussions in which students gave accounts of their solutions and collectively explored different approaches. For P1(a) students discussed two different ways they believed should verify that the given expression was the general solution. While one of the two approaches was correct, the other *could have* resulted in a correct explanation if students had recalled the definition of the general solution. Students experienced certain difficulties with the correct mathematical meaning of *particular* and *general* solutions - confirming previous research (Arslan, 2010; Raychaudhuri, 2007). We conclude that P1(a) has to be modified to call students' attention to the details of the definitions and theorems.

In P1(b), in addition to correctly produced solutions, students unexpectedly worked out collectively what does violation of the conditions of EUTs mean. They developed new understandings in this context (not foreseen by the lecturer at the stage of the task design).

P2(a) echoed some of the difficulties that students had in P1(a) with the meaning of the *general* solution. For P2(b) several students approached the solution via graphical representation which had not been discussed in this context in the course. While some students used linear algebra reasoning to arrive at the contradiction, others explained it using geometric argument. For P2(c) students correctly used the EUT to shift the ICs from $x = 0$.

In summary, we observed students' difficulties with the definitions of the particular and general solution, something the lecturer had not anticipated. We have also perceived successful use of visual mediators by students to explain why the solution to the given problem could not exist.

CONCLUSION AND FURTHER WORK

In this study we analyse students' mathematical understanding as a result of engagement with nonstandard problems, from individual scripts that show students' initial understanding to more advanced discussions in small groups and improved final solutions handed in for assessment. Students used a number of different recall routines. They employed integration of DE to obtain the general solution in P1(a) and P2(a). After group discussions most students switched to the recall routines of differentiation and substitution (appropriate for particular solutions). Thus, contrary to our expectations, students used familiar recall routines in a context where further modifications were needed. We are, therefore, going to consider how to modify formulations of P1(a) and P2(a), in particular the routine prompts used in these problems.

Our task design reflected a new, unfamiliar to students, narrative around EUTs. The lecturer used familiar words in an unusual context and described tasks differently. In our ongoing analyses we will focus on the theoretical constructs of narratives and routines including the applicability and closure conditions of routines to further our understanding of the teaching and learning process. More detailed analyses will be included in the conference presentation.

References

Arslan, S. (2010). Do students really understand what an ordinary differential equation is? *International Journal of Mathematical Education in Science and Technology, 41,* 873–888.

Breen, S., O'Shea, A. & Pfeiffer, K. (2013). The use of unfamiliar tasks in first year calculus courses to aid the transition from school to university mathematics. In B. Ubuz, C. Haser & M. Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (2316–2325), Antalya, Turkey: ERME.

Cohen, L., Manion, L. & Morrison, K. (2011). *Research Methods in Education*. London: Routledge.

Klymchuk, S. (2015). Provocative mathematics questions: drawing attention to a lack of attention. *Teaching Mathematics and Its Applications, 34,* 63-70.

Rasmussen, C. & Wawro, M. (2017). Post-calculus research in undergraduate mathematics education. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (551-581). Reston, VA: National Council of Teachers of Mathematics.

Raychaudhuri, D. (2007). A layer framework to investigate student understanding and application of the existence and uniqueness theorems of differential equations. *International Journal of Mathematical Education in Science and Technology, 38,* 367-381.

Raychaudhuri, D. (2008). Dynamics of a definition: a framework to analyse student construction of the concept of solution to a differential equation. *International Journal of Mathematical Education in Science and Technology, 39,* 161-177.

Roberts Jr., C.E. (1976). Why teach existence and uniqueness theorems in the first course in ordinary differential equations? *International Journal of Mathematical Education in Science and Technology, 7,* 41-44.

Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

ENACTMENT OF INQUIRY-BASED MATHEMATICS TEACHING AND LEARNING: THE CASE OF STATISTICAL ESTIMATION

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This paper investigates the enactment of inquiry-based mathematics teaching (IBMT) and inquiry-based mathematics learning (IBML). The focus is on how two teachers enacted the same task into their classrooms and on how this enactment framed students' mathematical activity concerning the notion of statistical estimation. Through the use of the sociodidactical tetrachedron, our study shows that IBMT and IBML are framed by different factors such as the selection and use of artefacts, the newly established and existing classroom norms and the meanings that teachers and students attribute to statistical estimation and to inquiry-based teaching and learning.

INTRODUCTION

Inquiry in mathematics – described through terms IBML and IBMT - can be defined loosely "as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work" (Artigue & Blomhøj, 2013, p. 797). As regards its classroom implementation, inquiry is considered an opposing approach to teacher-centred ones integrating a combination of at least two of the following three characteristics: opportunities for students to generate several options and solutions; discuss together; and make justified decisions (Chan, 2006). Existing empirical studies indicate the benefits of inquiry in the teaching and learning of mathematics and science in terms of students' outcomes and teaching quality at all educational levels and systems (Bruder & Pescott, 2013). However, integrating inquiry in the real classrooms is a rather complicated task bringing to the fore a number of issues such as the nature of the designed tasks, the provided resources, the teaching management and the students' learning (Artigue & Blomjoi, 2013). Thus, teachers aiming to enact inquiry in their teaching face a number of concerns at the level of design and implementation and according to their decisions and actions inquiry is likely to take different forms with respect to the surrounding (e.g., institutional) conditions. To address this issue, we adopt a sociocultural perspective to study the teaching activity of mathematics teachers in relation to students' activity while enacting inquiry in their classroom. The teachers in our study worked collaboratively on an open-ended task engaging students in developing strategies to count the number of people in a demonstration based on an air photograph. We focus on the notion of statistical estimation that appeared to be central in the classroom activity. We address the following research question: How are IBMT and IBML enacted into the mathematics classroom?

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THEORETICAL FRAMEWORK

Elaborating on the conceptualization of IBML and IBMT, Artigue and Blomjoi (2013) indicated a number of concerns that need to be addressed as a basis for integrating inquiry in classroom teaching. These include: the ‘authenticity’ of tasks and students’ activity in terms of connection with real life activities; the epistemological relevance of the tasks from a mathematical point of view; the modelling dimension of the inquiry process and the extra-mathematical sources of rationality; the experimental dimension of mathematics; the students’ autonomy and responsibility for producing and validating answers; the guiding role of the teacher and teacher–student(s) communication in the classroom. In our attempt to study IBMT focusing on the teacher’s decisions and actions and IBML focusing on students’ actions and behaviours, we use the sociodidactical tetrahedron (SDT) (Rezat & Strässer, 2012).

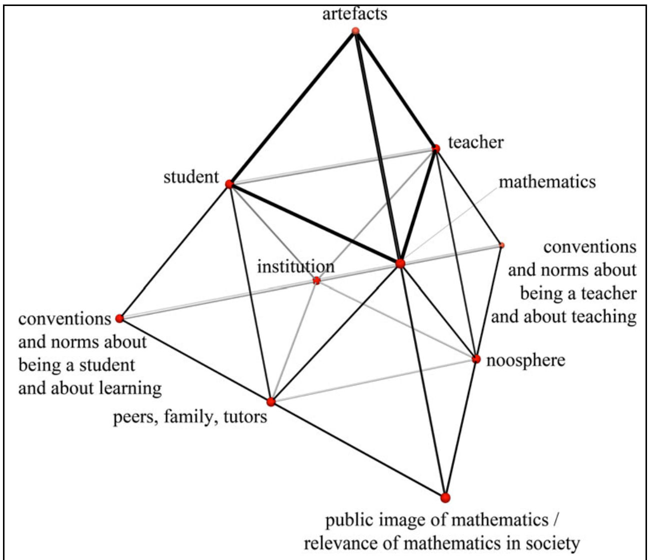

Fig. 1: The sociodidactical tetrahedron.

SDT allows us to focus on the classroom interaction and explain how IBMT and IBML are enacted in the classroom by taking into account the sociocultural setting. Rezat and Strässer (2012) used the Engeström’s triangle and distinguished two activity systems: (a) the teaching of mathematics taking as subject the teacher and (b) the learning of mathematics taking as subject the students. So, they extended the classical didactical triangle (teacher – student – mathematics), first to the didactical tetrahedron (teacher – student – mathematics – artefacts), and finally based on EMT to the *Sociodidactical Tetrahedron (SDT)* (Fig. 1). The community of teachers is made up of teachers and mathematics educators, the *noosphere* that has created certain images about what is mathematics teaching and learning. The community of students belongs to *peers, family* and possibly *tutors*. In the base of the STD, the two vertices represent *conventions and norms about being a student and about learning* and *conventions and norms about being a teacher and about teaching* while in the third vertex the role of mathematics in relation to the division of labour in the society and the public image of it (*public image of mathematics*) are considered. The communities include both teachers’ and students’ communities as well as the *institution* that is represented as another point in the STD. In our study, we focus on the relations between the teacher and the students with the mathematics content (the statistical estimation), the artefacts (the materials provided by the teacher and those used by the students), the public image of mathematics in the society (the authentic workplace setting), the noosphere (the meaning of IBML and IBMT as they are discussed in the community of mathematics education researchers and teachers), the classroom norms and conventions (how teachers and students conceive mathematics teaching and learning in the classroom with regard to different approaches to teaching and learning) and the institution (curriculum, school schedule, school rules).

The notion of statistical estimation in our study refers to the informal inferential reasoning process in which students make arguments to support inferences about unknown populations based on observed samples (Zieffler et al., 2008). A statistical inference is formed by (a) a statement of generalization “beyond the data,” (b) use of data as evidence to support this generalization, and (c) probabilistic (non-deterministic) language that expresses some uncertainty about the generalization (Makar & Rubin, 2009). Here, we explore IBMT and IBML when the main mathematical idea is the statistical estimation.

METHODOLOGY

The reported study took place in the Mascil context (www.mascil-project.eu) that targeted mathematics and science teachers’ professional development through the integration of inquiry-based learning and workplace into their teaching. To this end, professional development (PD) groups of in-service mathematics and science secondary teachers have been established. Each group, supported by a teacher educator, participated in cycles of designing, implementing and reflecting during a period of a school year. Before and after each implementation of the designed lessons professional development (PD) meetings took place. In this paper we focus on two mathematics teachers (Vangelis and Eirini) who worked in lower secondary schools in Athens and were members of the same PD group. The teachers collaborated in the transformation and implementation of the mascil task *Counting People* that was proposed to the group by the teacher educator in the 3rd PD meeting. Both implemented the task after adapting it to the Greek context in their 9th Grade classes (14 years old) for two teaching hours each. In both cases, the students were separated into groups (4-5 students) and all groups worked collaboratively for the solution of the problem. We chose to focus on the cases of Vangelis and Eirini because they expressed different perspectives in relation to IBMT and this allowed us to address issues related to our research question.

The *Counting People* task, as was transformed by Vangelis and Eirini, engages students to devise their own plan for counting the number of people in a particular antiracist demonstration that took place in Athens in front of the Parliament House. In the beginning of the two lessons both teachers provided the students with one photo (see Fig. 2) showing people demonstrating in three streets in front of the Parliament House (the Vas. Georgiou str., Vas. Amalias str., and Othonos str.) where the crowd density varies. Students are asked to adopt the role of a journalist and to provide information about the number of demonstrators in the photo. Key steps for a possible solution can be: (i) relating the counting of people with calculating the area of the specific streets; (ii) devising a plan to estimate the area of the three streets (the teachers did not provide a scale); (iii) estimating the total number of demonstrators in the three streets. From the above, it seems that statistical estimation becomes a central inquiry issue in this task.

Fig. 2: The photo.

Fig. 3: The map of the area.

The data consists of (a) the audio recordings of three 2- hour PD meetings (one for designing teaching (3rd meeting) and two (4th and 5th meeting) for reflecting)); (b) a video-taped lesson for each teacher; (c) audio taped students' group discussions in each classroom; (d) teachers' reflective interviews. Concerning the data analysis, first from the transcribed lessons and the group discussions we identified key actions that each teacher and the students had undertaken in relation to IBMT and IBML and the statistical estimation. Then through the analysis of the PD meetings and the reflection interviews we looked for the teachers' perspectives of IBMT and IBML and statistical estimation. At the next stage of analysis, we focused on the interaction between the teacher's and students' actions in relation to the elements of the STD. Finally, the two cases were contrasting to identify key differences on how IBMT and IBML were enacted in their classes.

RESULTS

Teachers' perspectives

In the 3rd PD meeting the teacher educator introduced the *Counting People* task. Vangelis (V) realized at once that statistical estimation was a central mathematical notion in this task: *"You can approach it through statistics [...] estimating the number of a group of animals for example ... you take a sample and consider how many they are in areas of high or low density respectively."* Vangelis had a strong background in statistics something that helped him to connect the task directly with central statistical methods and concepts such as statistical estimation and sample representativeness. Eirini acknowledged statistical estimation as a central learning objective but she related it with the idea of approximation *"It is reasonable to have great deviation in the results since everything is a matter of approximation"* (5th PD meeting).

Both teachers realized the task's potential to promote inquiry. However, the two teachers seem to have different approaches of what inquiry-based teaching is about. Eirini characterizes her teaching as 'guided inquiry' and she adds *"I leave my students to negotiate up to a point, when I realise that they go beyond the problem, I intervene. I always hear them trying to understand their difficulties and then I try to guide them in finding the solution"* (Eirini's interview). Vangelis, on the other hand, characterizes his teaching as 'open inquiry' by arguing that he minimizes his interventions and allows students to develop their own strategies. In the 4rth reflective PD meeting, Vangelis explained why he chose to leave students work without strict guidance: *"I chose not to guide them because every student has his own pace, others work quicker and others slower"*.

Setting up the task

Eirini and Vangelis set up the task in different ways in relation the selected artefacts. Eirini gave students only printed materials, a photo of the demonstration (Fig. 2) and a printed Google map of the area under investigation (Fig. 3). Vangelis provided the students only the photo (Fig. 2) and he said to them *"You can search in the internet for the appropriate resources to handle the task"*. All groups in Vangelis' class tried to locate the area under consideration in the Google earth maps.

Teachers' and students' enactment

Below we compare teachers' and students' enactment in the two classrooms in relation to two main mathematical ideas related to statistical estimation: The scale selection and the density anticipation.

The scale selection was one of the main students' objectives in both classrooms implementations. Students in Vangelis' class developed various strategies to address the given problem (e.g., they developed a plan to find the scale in a map representation of the area by using the length of a car as a unit measurement or used the scale that appeared automatically in the bottom of the screen in a specific map representation in Google maps). This scale showed the length in meters of a particular length in the map. In a subsequent PD meeting, Vangelis mentioned *"I could not manage to hide this representation"* expressing this way his intention to make the activity more exploratory for his students. In Eirini's class all groups developed the same plan for the scale selection. Eirini observed very closely her students' group work. The following is a typical example of the discussion between Eirini and her students while they were trying to define a unit measurement and estimating the scale in the printed map shown in Fig. 2.

1. E: What mathematical notion is relevant to this activity?
2. St1: Find the area of the road […] the scale.
3. E: How do we find the scale?
4. St1: I need to have a real object.
5. E: Can you identify a real object in the map representation [Fig. 2]?
6. St2: Let's see … The length of a car or a bus.
7. E: Better a car, not a bus.
8. St2: We have to know the dimensions of a real car and the dimensions of it in the photo ... But which car? … It could be a Volvo or a Smart; it could be 3 meters long or 4 meters long.
9. E: Discuss it with your group and decide on that.

As we can see in the above extract, Eirini guides her students to the desired mathematical object (line 1) and she provides hints to facilitate the process (lines 3, 5, 7). At the same time, we see the emergence of some informal indications of statistical inference in St2' attempts to identify an appropriate object for modelling the situation (line 8). The expressions of uncertainty highlight St2's encounter with the early steps of this statistical notion.

The density anticipation of demonstrators per one square meter is another central issue that it came up in students' discussions in both classrooms. Vangelis interfered very little in students' discussions while his students exploited many statistical ideas even though in an informal way. We present the following extract as a typical one showing students' collaboration in Vangelis' classroom and how Vangelis handled this discussion.

10.	St3:	If we take the half [area of the three streets] with 5 persons per square and the other half with 6 persons per square so we will be more close to…
11.	V:	Why?
12.	St3:	Because some people may be fatter and other thinner… (laughs!)
13.	St2:	Why don't we estimate how many people can fit in a square meter [he means to actually define a square meter and see how many fit]?
14.	St5:	Why don't you try first with 5 and then with 6 so we consider something in between.
15.	St4:	We don't care so much for who [could be the representing sample]. We can say that all persons are like St3. We just want an approximation.
		[The group stops for a little to talk and observes another group of students who simulated the problem by forming 1 square meter with a measuring tape on the floor, standing inside to find out the number of people that fit in this area. This group estimated 6 persons per 1 square meter. Then Vangelis asked]
16.	V:	Every square meter in the photo can have 6 persons?
17.	St3:	No, there are some empty spaces [spaces with no demonstrators in Fig. 2].

As we can see in the above extract, the students negotiated a lot about their choice for the number of people in a square meter (lines 12-15). After Vangelis inquiry question (line 16) students located spaces of low density. Statistical ideas that came up in students' discussions where the features of the persons (fat/thin) implying the representativeness of a sample or suggestions to take elsewhere 5 and elsewhere 6, implying a mean value of a high and a low density.

The extract below comes from one of Eirini's group of students who were asked to describe what they did while estimating the number of demonstrators in Fig. 2: *"E: How did you estimate the population? St: We first found the area in the streets where there were demonstrators. Then we agreed that two persons fit well in a square meter. [...] This is our sense, without [doing] measurements."* Eirini asked her students to report on how they calculated the density anticipation without involving them in a systematic exploration of the problem as in the case of Vangelis class. As we can see in the above extract the students in Eirini's class made their estimation based on intuitive approximations.

Tetrahedron	Vangelis' implementation	Eirini's implementation
Institution	Statistical estimation is not included in the official curriculum; authentic realistic tasks are not usually part of the Greek curriculum and textbooks.	
Teachers and mathematics	Strong background in statistics; related statistical estimation with selection of the best sample representativeness; appreciation of the multiple approaches in a statistical investigation.	Related statistical estimation with the idea of approximations; appreciation the multiple approaches in a statistical investigation.
Teachers and noosphere	Encouragement of the students to follow their own paths; limited intervention.	Guiding instruction on the basis of students' responses.
Teachers and artefacts	On line resources (photo, maps).	Printed resources (photo, map).
Teachers and conventions	Students explore and share their ideas; students have their own learning pace; emphasis is given on the mathematical processes.	Students express their ideas; emphasis is given on the mathematical concepts and properties.
Teachers, mathematics and society	Encouraging students to link their strategies with methods used by professionals.	Encouraging students to link the task question to other professions.
Students and mathematics	Extended use of probabilistic language; development of early steps of stochastic thinking through systematic experimentation; enactment of simulations.	Limited use of probabilistic language; informal indications of statistical inference in an intuitive way; use of data-request processes in a deterministic context.
Students and artefacts	Selecting artefacts beyond those proposed by the teacher.	Using the artefacts proposed by the teacher.
Students and conventions	A mathematical problem has one correct answer; the teacher verifies the correctness of a solution.	

Table 1: The analysis of the two implementations through SDT.

In the end of the task implementation both teachers asked the groups to present their results. Eirini addressed the question: "*Why do we have different answers?*" and then she referred to the particularity of situations with estimations. Vangelis did the same and even though students insisted on asking him *"What is the correct answer?"* Vangelis refused to answer which value was close to the actual situation. Later on, Eirini asked her students to discuss which professions may have to deal with the problem of estimating a population. Vangelis encouraged his students to inquire further the estimations given by news sites and suggested them to explore the methods used by the journalists and comparing to their own.

Table 1 summarizes the results from the analysis of the two different implementations by using the SDT.

CONCLUSION

In this paper we studied how IBMT and IBML were enacted in two classrooms. We see similarities between the two implementations such as the use of an open-ended task and the existence of norms allowing students to generate their solutions. The above characteristics are crucial as reported by Chan (2006). However, our analysis based on SDT brought to the fore differences in the two classrooms concerning the teachers' perspectives and actions and the students' mathematical activity. In particular, the different meanings teachers attributed to IBMT (open or guided inquiry) or to statistical estimation (e.g., connected to sample representativeness or the idea of approximation) and the selection of different artefacts (printed or on-line) promoted diverse possibilities for students' mathematical exploration. For example, Vangelis' students selected artefacts beyond those proposed by him and developed early steps of stochastic way of thinking while Eirini's students developed indications of statistical inferences but in a fragmental way and they were working mostly in a more deterministic context. Additionally, in both classrooms a number of concerns about the nature of mathematical solutions and the role of extra-mathematical sources of rationality are raised (Artigue & Blomhøj, 2013).

References

Artigue, M. & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM*, 45(6), 797–810.

Bruder, R. & Pescott, A. (2013). Research evidence on the benefits of IBL. *ZDM*, 45(6), 811–822.

Chan, C. M. E. (2005). Engaging students in open-ended mathematics problem tasks—a sharing on teachers' production and classroom experience (Primary). Paper presented at the 3rd ICMI-East Asia Regional Conference on Mathematics Education, East China Normal University, Shanghai, China.

Zieffler, A., Garfield, J., DelMas, R., & Reading, C. (2008). A framework to support research on informal inferential reasoning. *Statistics Education Research Journal,* 7(2), 40-58, http://www.stat.auckland.ac.nz/serj

Makar, K., & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal,* 8(1), 82–105.

Rezat, S. & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: artifacts as fundamental constituents of the didactical situation. *ZDM*, *44*(5), 641–651.

THE INTERPLAY OF INFORMATIVE ASSESSMENT CRITERIA AND CONTINUOUS FEEDBACK WITH MATHEMATICS STUDENTS' LEARNING ORIENTATIONS

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Many researches have suggested that making assessment criteria visible supports learning. On the other hand, others have claimed that too much clarity in assessment criteria and feedback could lead to instrumentalism: superficial observance of criteria without deeper thinking. Due to this ambiguous body of knowledge, we wanted to investigate what type of mathematics learning occurs during a course which provides clear assessment criteria and continuous feedback, combined with a technology enhanced learning environment based on self-assessment and reflection of learning.

BACKGROUND

In the fall semester 2017, one of the authors of this article was giving a course of the didactic of mathematics for pre-service teachers in Helsinki. In the didactic course, students were provided continuous and informative feedback and clear assessment criteria, i.e. detailed descriptions of what type of activities were to be connected to which grade. The students got to choose which grade they were willing to work for. At the beginning of the course, the students and the teacher discussed the criteria and how they were connected to the course's learning goals. According to e.g. Hattie & Timperley (2007), Stefani, Clarke & Littlejohn (2000), Roberts, Park, Brown & Cook (2011) such pedagogy clearly supports learning in a positive way, as it strengthens reflection skills, learner ownership and autonomy. During the course, the students strived for the highest grades and they reached the goals of the course well. They reported that the assessment system was very motivating, clear and fair, and that it pushed them to work harder.

However, some literature (e.g. Hume & Coll, 2009) suggest that students should not be provided with exact information about what to do to gain a specific grade. This idea is further elaborated by Torrance (2007, 2012), who claims that too much transparency, by which he means clarity on learning objectives, could lead to instrumentalism. In that case, the students might just superficially follow the criteria, and use any feedback they receive to mechanically correct their performance instead of really going deeper in their thinking. Morrison & Joan (2002) claim that instrumentalism leads to "teaching the test" perspective. Bloxham & West (2004) describe how over-specification of assessment criteria may narrow down students understanding of learning goals. On the other hand, leaving the assessment criteria unclear doesn't help students (nor the

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 299-306). Umeå, Sweden: PME.

teacher) to see how the criteria are ought to be met. This way the teacher's power over students may increase, and the student's role becomes again reliant.

This kind of a situation can be avoided by combining the socio-constructivist view of feedback with the cognitive view (Evans, 2013). In her article, Evans made a thematic analysis of assessment feedback in higher education. She embraces the tensions of beneficial and not so beneficial assessment feedback practices, but provides also an extensive list of the attributes that have been proved to make continuous feedback useful. In her list, effective assessment feedback a) is ongoing and an integral part of assessment, b) is explicitly guided, c) emphasises feed-forward instead of feed*back*, d) engages students in and with the process, e) attends to support learning, not personal attributes, and f) involves training in assessment feedback as an integral part.

Despite the claimed benefits of feedback, there might be challenges in pedagogies that allow students to set their grade goals in advance, expect teachers to define the criteria to each grade and finally wait students to perform certain tasks to achieve the criteria. This might weaken students' ability to set useful learning tasks themselves and to identify the depth and connections of the tasks (Torrance, 2011, 2007; Hume & Coll, 2009; Evans, 2013), leading to instrumentalism. In Torrance's (2007, p. 282) words: "transparency of objectives coupled with extensive use of coaching and practice to help learners meet them is in danger of removing the challenge of learning and reducing the quality and validity of outcomes achieved."

How to measure whether learning has occurred 'deeply' or through some kind of instrumentalism? In this article we conceptualize different learning styles with the concept of learning approaches. They have been broadly divided into 'deep' and 'surface' learning approaches by, for example, Biggs (1987, 2012) and Entwistle (1991). As deep approach refers to an intention to truly understand the topic to be learned with an intrinsic motivation (Diseth, 2003), surface approach is linked with the intention to complete the task and not so much with the intention to grow as a learner (Biggs, 1987). These two approaches model the diversity of different learner orientations in our course context.

To avoid instrumentalism, we should know what is in the other end of the continuum. Are deep learning approach and instrumentalism opposites? Can learning turn deep, if it is guided by specific instructions, constructed mainly by someone else?

THE DIGITAL SELF-ASSESSMENT PROJECT

In the Department of Mathematics and Statistics in the University of Helsinki, teachers in first-year courses have started to emphasise clarity in assessment criteria combined with continuous feedback and extensive student autonomy. By these means, they wish to elicit deeper and more complex thinking. The Digital Self-Assessment (DISA) model aims to create a digital assessment model for large university level courses, based on self-assessment. The model seeks to encourage students to constantly reflect on their own learning and take more responsibility for it.

In the DISA model, students receive extensive feedback from teachers, peers, themselves and a software designed specifically for the course model. The assessment criteria are made visible and transparent through a learning objectives matrix. The aim of the model is to support student autonomy, motivation and depth of learning, as well as their self-regulation and reflection skills. The model can be used in teaching large courses, and it has been piloted in two mathematics courses (Linear algebra and matrices, two instances, 130 and 400 participants, respectively).

Each week, students were given a set of problems to solve. For digital tasks, instant automatic feedback was offered. Others were manual tasks completed with pen and paper. For a subset of the manual coursework, the students received written comments from the teachers or peers. For solving the problems, students were offered guidance by peer tutors in drop-in sessions.

Instead of a final exam, the students set their grades themselves at the end of the course via a simple questionnaire, based on the learning objectives matrix. The students assessed their mastering of each topic and awarded themselves a grade for the course. The students were also asked to write down why they chose that specific final grade. Before the final self-assessment, a similar self-assessment was practised twice during the course.

RESEARCH TASK AND RESEARCH QUESTIONS

Our earlier results concerning the DISA model imply that the model supports students in using deep learning approach, and study for themselves, not for an exam (Nieminen, Rämö, Häsä, & Tuohilampi, 2017). Bearing in mind Torrance's (2007, 2012) critique, we became interested in investigating how the assessment criteria and continuous feedback interact with students' learning. The exact research questions of this study are:

1. How do deep learning and surface learning orientations distribute across the students taught with the DISA model?
2. How did the students perceive the transparent assessment criteria and extensive feedback in the DISA model?

METHOD

After a large first year linear algebra course in the fall 2017 with a little over 400 participants, a digital survey was conducted. The course was part of a comparative DISA research project, so the participants were divided into two groups: approximately 200 hundred students participated in a regular course exam while 183 students set their own course grade with a digital self-assessment sheet. The data used in this paper consists of the survey data for those in the self-assessment group who answered the survey and gave their permission to use the data in research (n = 155).

The survey consisted of qualitative and quantitative questions. Deep and surface learning approaches were tested with a validated questionnaire from the HowULearn project (Parpala, Lindblom-Ylänne, Komulainen, & Entwistle, 2013), both consisting of four items (α = .76 and α = .75) on a 5-Likert scale. The open ended questions concerned the student perceptions on the assessment methods in the DISA model; the questions were based on the interview questions by Mumm and colleagues (2015).

To create student profiles based on the reported levels of deep and surface learning approaches, a cluster analysis was conducted. Based on our previous study (Nieminen et al., 2017) we used a solution of clusters as a base for k-means-analysis with an Euclidean distance. Ward's algorithm was chosen for clustering algorithm to decrease the differences among the clusters, and the scores of the variables were standardized to Z-points before the analysis.

To describe how instrumentalism and deep learning were perceived by the students, a qualitative content analysis (QCA) was conducted, based on the model of Schreier (2012). First, a coding frame was created so that only the answers concerning the perceptions on transparent assessment criteria and extensive feedback were selected. This resulted into 166 analysis units consisting of single answers. These open answers were then divided into three categories; those concerning some kind of an 'instrumentalism' of learning and those concerning 'deep learning', and those concerning both of these. This phase was heavily influenced by the researcher's earlier knowledge about these concepts. Finally, a data-driven QCA was conducted to all these three categories.

RESULTS

How do deep learning and surface learning orientations distribute across the students taught with the DISA model?

Deep learning approach (M = 3.83, SD = .72) was reported to be higher than surface learning approach (M = 2.22, SD = .81) after the course (t (153) = 27,83, p = .000).

The results of the cluster analysis are shown in Table 1.

		Deep learning approach		Surface learning approach	
Cluster	N	Mean	Std. Dev.	Mean	Std. Dev.
1	24	3.05	.48	1.71	.45
2	15	2.80	.58	3.60	0.67
3	51	3.84	.45	2.76	.40
4	64	4.36	.41	1.66	.40
Total	154	3.83	.72	2.22	.81

Table 1: Mean values of surface learning and deep learning in four clusters.

The four clusters were named according to their features: 1) Little surface oriented and little deep oriented learning (*disoriented*), 2) A lot of surface oriented and little deep oriented learning (*surface approach orientation*), 3) A lot of deep oriented as well as surface oriented learning (*mixed orientation*), 4) A lot of deep oriented and little surface oriented learning (*deep approach orientation*).

How did the students perceive the transparent assessment criteria and extensive feedback in the DISA model?

The data analysis with QCA resulted in three different categories: 1) instrumentalism, 2) deep learning and 3) mixed perceptions. Here we present each category with citations from the data; all the citations are marked with brackets showing the learning approach cluster in which the respondent belongs to. This is done to further describe the cluster formation.

The answers of the students that reflected some sort of instrumentalism also reflected untrained reflection skills. These kinds of answers dealt primarily with extensive feedback and not that much with transparent learning objectives. Although self-assessment as a method was used precisely to enhance reflection (see Nieminen et al., 2017), the extensive feedback supporting it was sometimes seen as something that guides but that does not encourage to deepen the understanding on your own learning. Some students felt that the feedback only 'pointed out your own mistakes' as these examples show:

> They [assessment methods] did not particularly support learning but perhaps gave a better idea of what to practice more. (cluster 2)

> Self-assessment is also useful, since the objective assessment of yourself is hard, but it's useful, so that you know how to put your energy into learning the right things. (cluster 4)

In our data, hurry was seen as a cause of instrumentalism. Hurry was also seen as something that reduced the power of our learning environment designed to enhance reflection. The next quote from the data is an example of this:

> Now, however, I was worried about how I could show my excellent skills in the course without doing a lot of tasks. When there was a huge amount of other courses and submissions alongside, there was no time left for the tasks. . . The exam would have been much easier and it would have been a lot less work for me. (cluster 4)

On the other hand, some of the answers were coded as representing some kind of a deep approach to learning. In these answers formative, extensive assessment methods were seen as something that enabled a deeper approach to learning. Feedback received from various sources was linked to building an image of yourself as a learner. Two of the respondents described their learning as follows:

> Self-assessment helped me to reflect on myself as a learner of mathematics. (cluster 4)

> I also liked that the teaching assistant did not directly say the answer, but asked auxiliary questions or said remarks, so that I could realise how to solve a task and learn this way. (cluster 4)

Students sought for objectivity as they gained feedback on their own learning; this was clearly seen in the answers that reflected a deep learning approach. Transparent learning matrix helped the students to form their own personal goals, as is seen in the answer of one of the respondents:

> Being in the self-assessment group motivated to be more aware of your own goals and the work you had done than before. (cluster 4)

> Self-assessment makes it possible and maybe even forces you to consider your own skills. Then you have to face your level of expertise and take a stand on it. When you know what you are capable of and should be capable of, you can set your own goals and strive for them effectively. (cluster 4)

The answers that were coded to represent both instrumentalism and deeper learning shed more light on the issue. Some of the students described reflective and deep learning in their answers, yet the same answers showed elements of instrumentalism too. This data is not deep enough to examine the level of reflection behind these comments; that would require, for example, student interviews. It does, however, cast some light on cluster number 3 of our analysis, that represents the students that make use of both deep and surface learning approach. The complicated connection of instrumentalist learning and the motivation to truly understand the content of the course is seen in the comment below:

> Thanks to self-assessment, I have had a very strong motivation to make as many tasks as possible and to understand things as well as possible. In addition, self-assessments have made it clear what needs to be learned during the course and what needs improvement. On the other hand, self-assessing the grade for the course has created some pressure on taking the course. (cluster 3)

CONCLUSIONS

Four clusters with different learning orientations were found in the student group studying according to the DISA model. There were two groups with ambiguous orientations (disoriented and mixed orientations), one with lots of surface orientation and one with lots of deep learning. The latter one was clearly the one having the greatest number of students. All in all, deep learning orientation was remarkably more present in students' answers than surface orientation.

These results imply that students do not perceive surface learning and deep learning as mutually exclusive. The students also seem to focus on the what question of learning (knowledge) instead of a more holistic picture, even though still having deep learning orientation. Thus, clarity in assessment criteria might not be evidently good or bad. Our results support its use with careful consideration, allowing students to be active in the construction of knowledge (see Hume & Coll, 2009). Students should not become reliant of the teacher's specifications, while teachers should not turn their attention to provide evidence for meeting the criteria, instead, a combination of the socio-constructivist and the cognitive view of feedback should be used (Evans, 2013).

According to our qualitative data, the students in the first year mathematics course have trouble with "knowing what they know". This might imply that clear assessment criteria are especially useful in forming deeper kind of learning in this context; university mathematics is a new kind of a context for most of the students in this first year course. Also, personalized and formative feedback was perceived as an important form of support, as was the case, with, for example, Roberts, Park, Brown & Cook (2011). These results show 'instrumentalism' in a different light, since it can be seen as something that is required for deeper understanding of university mathematics to form.

Before the course, we tried to ensure that all the feedback was to support and guide learning during the class, but this was not always the case. Some students felt that extensive feedback was represented the 'true' level of learning. We intended that this feedback was supposed to be used as a base for further reflection. However, whether it is formative, extensive assessment that leads to this kind of possible *assessment as learning,* as Torrance (2007) suggests, is questionable, since the same formative assessment is also seen as crucial for deep learning to strive in our data. We could identify mechanisms of deep learning basing on performing the tasks following carefully described learning goals. Further, definitions of feedback often include or even require the idea of bridging the gap between desired and actual performance (Evans, 2013). This definition entails the desired performance exactly defined.

Torrance (2007) suggested that the core of instrumentalism is establishing transparent criteria for a course and then providing information about how the students can meet these criteria. In the DISA model, student autonomy is supported by letting the students to do the assessment by themselves, with support of extensive feedback. Is this autonomy the key to transform instrumental learning into deeper kind of understanding? In our model, student reflection is promoted, so that students would not just take the learning criteria as given but rather explore them with a critical view. Students in the model are expected to take more responsibility on their own learning. However, as the clustered learning orientations imply, an idea of learning as 'an act of social and intellectual development' (Torrance, 2007, p. 293) is not always reached in our model. Further analysis is needed to investigate the mechanisms connecting deep learning and instrumentalism in the field of learning mathematics. One has to ask how useful is the idea of setting instrumentalism and deep learning on the same continuum?

References

Biggs, J. B. (1987). *Student Approaches to Learning and Studying. Research Monograph*. Australian Council for Educational Research Ltd.

Biggs, J. (2012). What the student does: teaching for enhanced learning. *Higher Education Research & Development*, *31*(1), 39-55.

Bloxham, S., & West, A. (2004). Understanding the rules of the game: marking peer assessment as a medium for developing students' conceptions of assessment. *Assessment & Evaluation in Higher Education*, *29*(6), 721-733.

Diseth, Å. (2003). Personality and approaches to learning as predictors of academic achievement. *European Journal of Personality*, *17*(2), 143-155.

Entwistle, N. J. (1991). Approaches to learning and perceptions of the learning environment. *Higher education*, *22*(3), 201-204.

Evans, C. (2013). Making sense of assessment feedback in higher education. *Review of educational research*, *83*(1), 70-120.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of educational research*, *77*(1), 81-112.

Hume, A., & Coll, R. K. (2009). Assessment of learning, for learning, and as learning: New Zealand case studies. *Assessment in Education: Principles, Policy & Practice*, *16*(3), 269-290.

Morrison, K., & Joan, T. F. H. (2002). Testing to destruction: a problem in a small state. *Assessment in Education: Principles, Policy & Practice*, *9*(3), 289-317.

Mumm, K., Karm, M., & Remmik, M. (2016). Assessment for learning: Why assessment does not always support student teachers' learning. *Journal of Further and Higher Education*, *40*(6), 780-803.

Nieminen, J. H., Häsä, J., Rämö, J., & Tuohilampi, L. (2017). Replacing Exam with Self-Assessment: Reflection-Centred Learning Environment as a Tool to Promote Deep Learning. In Press.

Parpala, A., Lindblom-Ylänne, S., Komulainen, E., & Entwistle, N. (2013). Assessing students' experiences of teaching–learning environments and approaches to learning: Validation of a questionnaire in different countries and varying contexts. *Learning Environments Research*, *16*(2), 201-215.

Roberts, K. D., Park, H. J., Brown, S., & Cook, B. (2011). Universal design for instruction in postsecondary education: A systematic review of empirically based articles. Journal of Postsecondary Education and Disability, 24(1), 5-15.

Schreier, M. (2012). Qualitative content analysis in practice. Sage Publications.

Stefani, L., Clarke, J., & Littlejohn, A. H. (2000). Developing a student-centred approach to reflective learning. Innovations in Education and Teaching International, 37(2), 163-171.

Torrance, H. (2012). Formative assessment at the crossroads: Conformative, deformative and transformative assessment. Oxford Review of Education, 38(3), 323-342.

Torrance, H. (2007). Assessment as learning? How the use of explicit learning objectives, assessment criteria and feedback in post-secondary education and training can come to dominate learning. 1. Assessment in Education, 14(3), 281-294.

VARIATION OF STUDENT ENGAGEMENT BETWEEN DIFFERENT ALGEBRA TASKS

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In this study, we analyse how 7th grade students' engagement during small group work differed in two consecutive algebra lessons: in the first lesson students solved equations and in the second lesson they created equations for other small groups to solve. Data was collected by videorecording the work of two groups in both lessons. Through directed content analysis, categories indicating student engagement were formed based on previous research and refined during analysis. The analysis revealed a change from individual engagement to collaborative engagement between lessons and an increase in many passive students' engagement. Task characteristics which may affect the type and amount of engagement are discussed.

INTRODUCTION

Transition from arithmetic to algebra is a difficult point in school mathematics (e.g. Kieran, 1992) often resulting in a decline in student engagement. Therefore, it is important to find ways to engage students when transitioning to algebra. Nyman and Kilhamn (2015) found that a group of Swedish teachers tried to engage students in algebra mostly through contextual or organisational methods. They concluded that it is important to find ways to engage through the content itself and to study which characteristics of algebra tasks are related to engagement.

Open problems have been reported to be potentially engaging tasks (e.g. Sullivan, Mousley, & Zevenbergen, 2006). In this study, we are comparing a typical equation solving lesson and a lesson in which students create equations. The latter open problem solving activity is called Reversed Equation Solving. The research question is what kind of differences in cognitive engagement and peer-to-peer interaction emerge between these lessons for two small groups. We also discuss how characteristics of the tasks may relate to differences in engagement and interaction.

ENGAGEMENT

The quality or level of engagement has generally been found to have a profound effect on learning outcomes (see review by Fredricks, Blumenfeld, & Paris, 2004). Engagement is a hot topic in the scientific discussion which shows from a broad range of recent studies related to engagement: questionnaire development and defining the construct of engagement and it's dimensions (see overview of a recent special issue by Fredricks, Filsecker, & Lawson, 2016), teachers' and students' views of engagement

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 307-314). Umeå, Sweden: PME.

(Nyman & Kilhamn, 2015), general factors related to continuation and decline of engagement (Henningsen & Stein, 1997).

Shortly, engagement can be defined as the extent to which a student is actively involved with the content of a learning activity (Helme & Clarke, 2001). A common conceptualization is that engagement comprises three distinct, but interrelated dimensions: behavioral, emotional and cognitive engagement (Fredricks et al. 2004). In this paper, we mostly concentrate on cognitive engagement although observation of peer-to-peer interaction is also related to the behavioral dimension. Fredricks et al. (2004) define cognitive engagement as student's level of investment in learning. According to them, it includes being thoughtful, strategic, and willing to exert the necessary effort to learn and overcome challenges.

Engagement is often analysed from questionnaire or self-report data. Several studies have concluded that an important next step would be to study engagement by observing students over a sequence of lessons (Fredricks et al., 2004; Helme & Clarke, 2001). We are trying to tackle that challenge by developing further methods to capture indicators of cognitive engagement through video-study.

Fredricks, Wang, et al. (2016) studied indicators of cognitive engagement described by students and teachers in interviews. They reported thinking hard, connecting ideas, trying to understand ideas, persisting and self-monitoring as indicators of cognitive engagement. Helme and Clarke (2001) studied engagement from mathematics lesson videos. They used a framework where indicators of cognitive engagement specific to mathematics include questioning, completing peer utterances, exchanging ideas, giving explanations, justifying and gestures. The indicators of engagement used by Helme and Clarke (2001) have similarities with categories used in studying student interaction. For example, Asterhan and Schwarz (2009) used categories for dialogical moves which overlap with the categories by Helme and Clarke. Also differences exist as Asterhan and Schawrz’s categorisation is more detailed and includes, for example, challenging as a separate category. Nevertheless, when students participate in collaborative activities, visible indicators of cognitive engagement are related to students’ interactional moves.

METHODS

Context and data

The reported study is a part of the Finnish national Flexible Equation Solving programme (2014–2019). Two consecutive lessons of different nature from the 9-lesson pilot study for 7th graders in 2015 were selected for further analysis:

Lesson 4: Equation Solving (ES)
Lesson 5: Reversed Equation Solving (RES)

In the lessons, students were seated in groups and were asked to work on the assignments together (teacher facilitation was mostly absent in these small groups). During ES (lesson 4) students solved equations in groups whereas in RES (lesson 5) students created equations in groups, shared them on the blackboard with their names, solved each other's equations and compared their work. The equations were created by choosing a starting point (e.g. 5 = t) and operating on both sides (more about Reversed Equation Solving in Tuomela, 2016).

In this study, we focus on two small groups of four students who worked actively during both lessons. The both groups consisted of one mathematically strong, one weak and two average students. Data was collected by video recording the work of the two groups using two video cameras. The time for group work was 18 minutes in lesson 4 and 15 minutes in lesson 5.

Data analysis

The research strategy was to 1) create categories describing students' engagement during interaction, 2) form engagement profiles for each student and group and 3) interpret the differences between lessons. In line with directed content analysis (Hsieh & Shannon, 2005), indicators of cognitive engagement were defined based on previous research (Helme & Clarke, 2001; Fredricks, Wang, et al., 2016; Asterhan & Schwarz, 2009). Definitions, examples and coding rules for the categories were collected in a coding agenda which is summarized in Table 1.

Category	Description
Ask	Asking a task or working strategy related question.
Help	Helping a peer. Typically stating an answer or showing a notebook.
Idea	Sharing ideas, suggesting next steps or reflecting on mathematics.
Conc	Concentrating. Mumbling calculations aloud or resisting distractions.
Resp	Task related short response like yes, no, nodding or a simple opinion.
Chal	Challenging. Showing signs of disagreeing or asking for explanations.
Just	Justifying an idea or statement.

Table 1: Indicators of cognitive engagement. Short descriptions of categories.

The coding agenda was refined during the analysis. Formative checks of reliability and coding iterations were done. Throughout the process, definitions and coding rules for the categories were discussed between researchers.

RESULTS

We found two different working modes in small group work: students concentrating on their individual work and students working collaboratively thinking together. First, we elaborate on the two working modes using example episodes. Then, we examine how the amount of indicators of cognitive engagement changed from lesson 4 (ES) to lesson 5 (RES).

Individual and collaborative working modes

The first episode illustrates individual working mode. In the episode, students in group A were solving equations during lesson 4 (ES).

1	Anna:	I don't get it.. [unintelligible]
2	Eve:	Poor you… [indifferent tone]
3	Eve:	I like these a lot, these where we need to calculate
4	Anna:	How can the first one be solved? (Ask)
5	Eve:	I don't know. I jumped to this block, because here I can multiply all numbers by two. [pause]
6	Eve:	That would be also... [unintelligible] [pause] (Conc)
7	Eve:	[mumbling by herself] 6a... [pause] (Conc)
8	Eve:	[mumble] Add both sides... [mutters calculations aloud] (Conc)
9	Anna:	Don't do it so fast...
10	Eve:	Oh, sorry… [grinning indifferently] [long silence]
11	Eve:	[mumbles calculations] ...Yes! (Conc)
12	Eve:	What, did you drop off the sled somewhere? [sly grin] (Ask)
13	Anna:	May I look at it…? (Ask)
14	Eve:	Sure, feel free... [still grinning] (Help)
15	Anna:	Well, here was the mistake… (Idea)

This example shows that although the students were working in a group, they were engaged mostly to their own work (turns 5, 6-8, 11). When they were talking, authority was clearly present because they asked for help (4, 13) and provided help (14) in a simple copying manner instead of sharing their thinking, reflecting on mistakes or making decisions together on the same level.

The second episode illustrates collaborative working mode. In the episode, students in group B were creating an equation during lesson 5 (RES).

1	Anni:	Let's take turns to pick one [transformation]. (Idea)
2	Anni:	Let's do four of em'. (Idea)
3	Anni:	So I will… multiply by seven. (Idea)
4	Lassi:	Add… No… Leo's turn. (Idea)
5	Lassi:	What shall we do? (Ask)
6	Leo:	Multiply… (Idea)
7	Anni:	No… No but… (Resp)
8	Anni:	You shouldn't multiply anymore… (Chal)
9	Anni:	It just becomes the same kind of. (Just) [12 utterances skipped]
10	Anni:	So $7x + 6 = 21 + 6$ [erases right side] so 27 [writes $7x + 6 = 27$] (Idea)
11	Anni:	[Anni and Suvi compares their work] Like this. (Help)

12 Suvi: What…? Aaaa!!! [notices a mistake: $7x + 6$ became $13x$] (Resp)
13 Anni: Because they cannot be comb… (Just)
14 Suvi: Why don't they put those separately? [points right side of equation] (Ask)
15 Anni: Because here it doesn't have either... [points left side] (Just)
16 Lassi: So is it…? [simultaneously with Suvi] (Ask)
17 Suvi: How about that one? [simultaneously with Lassi] (Ask)
18 Anni: There isn't. (Help)
19 Anni: You cannot combine them because there is x. (Just)
20 Suvi: Oh, that's true! [hitting her palm to her forehead and laughing] (Resp)
21 Anni: I made the same mistake earlier!! Really! [laughing] (Idea)
22 Lassi: Right then, this is… Is this now a good equation? (Ask)
23 Anni: No, still 2 (transformations) (Resp)
24 Suvi: I haven't decided yet…
25 Anni: And Lassi neither.
26 Anni: Right. Substract, multiply, divide, add… (Help)
27 Anni: Let's not add… (Idea)
28 Anni: Let's agree you cannot use the same transformation twice in a row. (Idea)

Throughout this episode students frequently shared their ideas (1-4, 6, 10, 27-28), justified (9, 13, 19) and asked questions (5, 14, 16-17, 22) in a productive way that built their understanding or moved forward the assignment. While doing this, they often used the words "Let's" and "we" showing how they were working on it together (1-2, 5, 27-28). They also made sure that everyone in the group became involved in the process (1, 4, 24, 25).

The chosen episodes also illustrate the different nature of the two small groups. The most active person in group B (Anni) was an empathic leader who involved others (1) as well as regulated group actions and atmosphere (2, 8, 21, 23, 25, 26-28). In contrast, the most active person in group A (Eve) was not so sensitive towards other group members (2-3, 10, 12) and concentrated mostly to her own work.

Changes in indicators of cognitive engagement

The amount of indicators of cognitive engagement for students in group A as well as for groups A and B are presented in Table 2. To save space, the individual student data from group B was omitted. According to table 2, lesson 5 (RES) contained more interactions related to collaboration (Ask, Help, Idea, Resp) and less muttering calculations aloud (Conc) than lesson 4 (ES). This implies a change from individual working mode towards collaboration. In other words, type of engagement changed as the groups were more engaged to collaborative work in lesson 5 (RES) than in lesson 4 (ES). This happened regardless of the previous amount of collaboration and the different climate in the groups.

		Ask	Help	Idea	Conc	Resp	Chal	Just	Total Engagement
Eve	ES	7	14	6	30	2	2	3	64
	RES	13	13	13	3	17	2	0	61
	RES-ES	6	-1	7	-27	15	0	-3	-3
Kim	ES	5	1	2	9	1	0	0	18
	RES	21	3	5	5	13	0	0	47
	RES-ES	16	2	3	-4	12	0	0	29
Anna	ES	11	0	3	4	3	0	0	21
	RES	10	3	6	1	5	1	0	26
	RES-ES	-1	3	3	-3	2	1	0	5
Tuomas	ES	0	0	0	0	0	0	0	0
	RES	4	5	9	1	3	0	0	22
	RES-ES	4	5	9	1	3	0	0	22
Group A	ES	23	15	11	43	6	2	3	103
	RES	48	24	33	10	38	3	0	156
	RES-ES	25	9	22	-33	32	1	-3	53
Group B	ES	46	14	46	46	36	8	5	201
	RES	45	21	81	42	63	10	10	272
	RES-ES	-1	7	25	-4	27	2	5	71

Table 2: Amount of indicators of cognitive engagement in each category.

During ES, two students showed no indicators of cognitive engagement and three students showed about 20. These five students are considered passive. Three of them showed 20-50 indicators of engagement more during RES. It should also be noted that the increased engagement for groups is mostly due to the awakening of these passive students. Table 2 indicates the changes for the passive students of group A (Kim, Anna and Tuomas). Both groups had also students whose total engagement did not increase much (Anna and Eve for group A), although the type of engagement changed.

DISCUSSION

Two clear differences in students' engagement were found when observing two small groups during two different kind of algebra tasks. Firstly, students' type of engagement changed from individual to collaborative. They started sharing ideas, opinions, and questions during RES (lesson 5) when compared to ES (lesson 4). Secondly, three passive students during ES became clearly more engaged during RES.

Considering the type of engagement, as suggested in this study, is important because previous studies have found that in effective small groups students use talk in which they share emerging ideas, explore each other's ideas and challenge ideas (Mercer & Howe, 2012). This means that effective small groups engage in collaboration. Thus, the distinction between individual and collaborative engagement helps to make sure that students are not only engaged but engaged in collaboration. Furthermore, interventions like Reversed Equation Solving combined to discussion about individual and collaborative type of engagement could be used to raise teachers' awareness of different types of student engagement and of engagement-supportive practices as called for by Skilling, Bobis, Martin, Anderson and Way (2016).

The results imply that only the task (without much teacher facilitation) can dramatically change the nature of peer-to-peer interactions from individual working mode into collaboration and awaken passive students. Several characteristics of Reversed Equation Solving may account for this. Firstly, the assignment requires the students to create something of their own and allows use of creativity. Secondly, it requires the students to make decisions together and to agree on next steps. Thirdly, students publish their work to the whole classroom and get feedback of their work from other students. Fourthly, when students create their own tasks, the difficulty level is adjustable, while on the other hand whole-class publishing encourages the students to try something novel or tricky. Also previous studies suggest that engagement is related to novelty of task (Helme & Clarke, 2001) and supporting student autonomy (Skilling et al., 2016). Henningsen and Stein (1997) emphasize that it is important to use demanding tasks and to maintain engagement in them instead of using easier tasks. RES is a good example of this in the context of algebra. Thus, this study contributes to the conquest of finding out how to engage through content and discovering characteristics of algebra tasks related to engagement (Nyman & Kilhamn, 2015).

Video analysis methods allow studying engagement in detail both in group and individual level. The results reveal the importance of looking at individual changes. If only changes in group level had been observed, then it would have gone unnoticed that it is actually due to only three students becoming more engaged and the rest not becoming more engaged (although type of engagement changed). Thus, this study points to the importance of considering the complexity of individual and social processed underlying engagement as called for by Järvelä, Järvenoja, Malmberg, Isohätälä and Sobocinski (2016). Video study of engagement has also challenges. Regarding passive students it could be asked if the change was actually increase in engagement or just engagement becoming visible. This kind of analysis did not reveal any information about those students' level of engagement who were silent.

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References

Asterhan, C. S., & Schwarz, B. B. (2009). Argumentation and explanation in conceptual change: Indications from protocol analyses of peer-to-peer dialog. *Cognitive science, 33*(3), 374–400.

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of educational research, 74*(1), 59–109.

Fredricks, J. A., Filsecker, M., & Lawson, M. A. (2016). Student engagement, context, and adjustment: Addressing definitional, measurement, and methodological issues. *Learning and Instruction, 43*, 1–4.

Fredricks, J. A., Wang, M-T., Linn, J. S., Hofkens, T. L., Sung, H., Parr, A., & Allerton, J. (2016). Using qualitative methods to develop a survey measure of math and science engagement. *Learning and Instruction, 43*, 5–15.

Helme, S., & Clarke, D. (2001). Identifying cognitive engagement in the mathematics classroom. *Mathematics Education Research Journal, 13*(2), 133–153.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for research in mathematics education, 28*(5), 524–549.

Hsieh, H. F., & Shannon, S. E. (2005). Three approaches to qualitative content analysis. *Qualitative health research, 15*(9), 1277–1288.

Järvelä, S., Järvenoja, H., Malmberg, J., Isohätälä, J., & Sobocinski, M. (2016). How do types of interaction and phases of self-regulated learning set a stage for collaborative engagement? *Learning and Instruction, 43*, 39–51.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 390–419). New York: Macmillan.

Mercer, N., & Howe, C. (2012). Explaining the dialogic processes of teaching and learning: The value and potential of sociocultural theory. *Learning, Culture and Social Interaction, 1*(1), 12–21.

Nyman, R., & Kilhamn, C. (2015). Enhancing engagement in algebra: didactical strategies implemented and discussed by teachers. *Scandinavian Journal of Educational Research, 59*(6), 623–637.

Skilling, K., Bobis, J., Martin, A. J., Anderson, J., & Way, J. (2016). What secondary teachers think and do about student engagement in mathematics. *Mathematics Education Research Journal, 28*(4), 545–566.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education, 4*(1), 117–143.

Tuomela, D. (2016). Task potential of Reversed Equation Solving. In H. Silfverberg & P. Hästö (Eds.), *Proceedings of Annual Symposium of the Finnish Mathematics and Science Education Research Association 2015.* Turku, Finland: Finnish Mathematics and Science Education Research Association.

INTERPLAY IN STUDENTS' THINKING MODES AND REPRESENTATION TYPES OF LINEAR ALGEBRA IN A DGS

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This paper reports the interplay of students' thinking modes and representation types of linear algebra when they interact with a dynamic geometry system (DGS). The participants of the case study are two undergraduate linear algebra students and the data obtained from task-based interviews in the context of linear combination, linear independency-dependency, and basis and dimension has been analysed according to a theoretical lens of students' thinking modes and representation types of linear algebra. According to the findings, students often switch between thinking modes in a DGS, and the most common thinking modes and representation types are synthetic-geometric mode and geometric representation, while the least common are analytic-structural thinking modes and abstract representation.

BACKGROUND

Linear algebra is offered as a main course in addition to calculus courses in many teaching programs in the fields of mathematics, science and engineering. Due to the nature of linear algebra, it is predominantly axiomatic and proof-based. However, as linear algebra instructors, we exploit geometry, which is a powerful tool used to visualize the process in the teaching of linear algebra. However, geometry should be used carefully in a balanced way, otherwise students' perceptions of linear algebra concepts may be limited to geometrical concepts (i.e. geometric vectors) (Gueudet-Chartier, 2004); that is, they cannot make generalizations regarding abstract vector spaces, where students tend to think practically. For instance, for the *core* concepts of linear algebra, such as basis, similarity and linear transformation, students prefer to use the elementary algebra of geometric vectors and other related computations that the lecturer uses rather than using mathematical definitions of the concepts (Montiel, Wilhelmi, Vidakovic, & Elstak, 2012).

Recently, in order to construct certain (geometric) key notions and prepare an infrastructure for the theory of vector spaces in linear algebra researchers have referred to the use of dynamic geometry systems (DGSs) (Donevska-Todorova, 2015, 2016; Turgut, 2015, 2017). For instance, a careful (semiotic) potential analysis of a number of tools and functions of a DGS substantially evoked construction of the mathematical meaning of linear transformation (Turgut, 2017), but it also contributed to students moving from practical thinking to theoretical thinking on the notion of parameter (Turgut & Drijvers, 2016). However, a DGS only provides a geometric context and, to date, it is not yet established how a DGS could be used as an assessment tool for stu-

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 315-322). Umeå, Sweden: PME.

dents' existing knowledge, and whether a DGS context could be a barrier for students to discuss the theory of vector spaces. In other words, the focus is on how students transfer and discuss their knowledge of linear algebra while using a DGS and, consequently, the following research question is considered: what are the thinking modes and representation types of linear algebra used by students in a DGS regarding the concepts of linear combination, linear dependency/ independency, basis and dimension?

THEORETICAL FRAMEWORK

In order to answer the research question above, two interrelated theoretical insights are referred to; students' thinking modes of linear algebra (Sierpinska, 2000), and the representation types they use in learning (Hillel, 2000). According to the epistemological analyses that she implemented, Sierpinska (2000) concludes that students tend to think practically rather than theoretically by highlighting the difficulties they face during the process and she defines two processes accordingly; (i) *practical thinking* and (ii) *theoretical thinking*. Practical thinking refers to a sort of 'in-action' thinking (Sierpinska, 2000). On the other hand, theoretical thinking occurs when students start to think 'about-the-action' itself and make reasoning about semiotic meanings regarding the object.

Sierpinska (2000) identifies three thinking modes that correspond to these two processes; (i) synthetic-geometric thinking (SGT), (ii) analytic-arithmetic thinking (AAT), and (iii) analytic-structural thinking (AST). SGT is based on practical observations concerning the geometric characteristics and properties of the action. However, this does not deal with how they are formed as a mathematical notion or as an object. AAT is a process corresponding to numeric and algebraic calculations and computations regarding the action, while AST deals with the dialectics between an epistemological and a visual viewpoint for the construction of an object. Consequently, AST is a conceptualization process of the object as a mathematical object. In conclusion, synthetic thinking relates to individuals' visual comprehension abilities, while analytical thinking modes relate to generalization and de-contextualization of one or more interrelated processes through reasoning (Turgut & Drijvers, 2016). The following example can be given for the thinking modes summarized above. If one student only thinks visually about vectors then, through visualization, this process refers to the SGT mode. If the student finds their addition and multiplication with scalars and uses matrix representations or algebraic features of elementary operations of vectors, it concerns AAT. Finally, if the student considers the given vectors as an element of the related vector space, it refers to AST (Dogan-Dunlap, 2010). What should be noted here is that different thinking modes are not independent of each other. For example, students can make use of SGT while applying AST.

METHODOLOGY

Within the scope of the study, 74 undergraduate linear algebra students were asked to solve six open-ended problems. Later, the researcher identified seven of these students as participants of the study using a purposeful sampling method based on observations focusing on (i) their mathematical knowledge and skills, (ii) their communication skills, and (iii) their different ways of thinking, describing and expressing. However, only four of them volunteered for the study, and for brevity, the case of two students is presented: these are S1 and S2, both twenty-year-old females, who were sophomore students enrolled in a department of mathematics education at a government university, located in western Turkey. The interviews were conducted with S1 and S2 using a laptop with GeoGebra software installed. The students, who participated in task-based interviews, had an average level of knowledge regarding linear algebra concepts. The interviews were conducted after certain topics were covered in regular class lectures; linear equations systems, vector spaces, sub spaces, linear combination, span, linear dependency-independency, basis, dimension and determinants. Data was collected through task-based interviews using a single task designed in such a way as to reveal the students' thinking modes regarding linear combination, linear independency dependency, basis and dimension concepts. GeoGebra software was used as a DGS context. Field notes were taken during the interviews and the processes were video-recorded. In addition, screen recording software was used to provide information concerning their reasoning processes.

The Task and Data Analysis

The aim of the task is to encourage the students to speak about the bridge regarding the following situations: (i) three vectors on the same plane; (ii) the plane formed by the end points of vectors; and (iii) the vectors not on the same plane. It also aims to explore the relationships between the formation of these ideas and concepts, such as linear combination, linear dependency-independency, and basis and dimension. Following this aim, specific tools and functions of a DGS were used as follows. First, two *sliders*, a and b, were constructed and, using these parameters, three points, $A=(-1, 3, a)$, $B=(3, -4, 3)$ and $C=(b, 2, -1)$ were constructed. Through these points and through the origin, three vectors $\mathbf{u}=(-1, 3, a)$, $\mathbf{v}=(3, -4, 3)$, $\mathbf{w}=(b, 2, -1)$ were sketched and a plane generated by these vectors. Associated matrix generated by **u**, **v** and **w** (as appears 'matris1' in Figure 1) were also computed. The sliders here change the positions of the vectors and they lie out of the plane for certain values while they are within the same plane for other values. Activating Algebra, Graphics and 3D Graphics windows, the task was delivered to the students as shown in Figure 1, and the task was proposed: *Explain what is happening in the Algebra, Graphics, and 3D Graphics Windows synchronically from the perspective of linear combination, linear dependency-independency, and basis and dimension concepts using sliders and 3D rotate tools*.

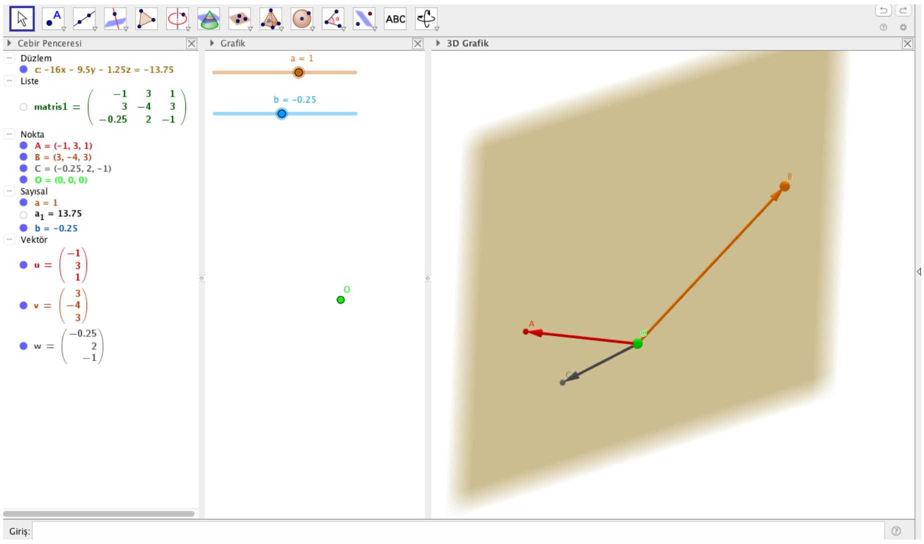


Figure 1: GeoGebra interface regarding the task

The discourse obtained from clinical interviews was transcribed and all the elements attached to the discourse, such as drawing, dragging, and students' productions were analysed synchronically. In the study, initially, students' thinking modes of linear algebra (Sierpinska, 2000) were coded one by one, and later extra data obtained from other data types, such as photographs, gestures, and explanations written on paper were also coded according to the framework of the representation types (Hillel, 2000). The findings section presents analyses regarding the coding process and other extra elements, such as video capture of the analyses, screen shots, drawings and so on. For sake of page limitation, mathematically rich discussions have been selected and are presented.

FINDINGS

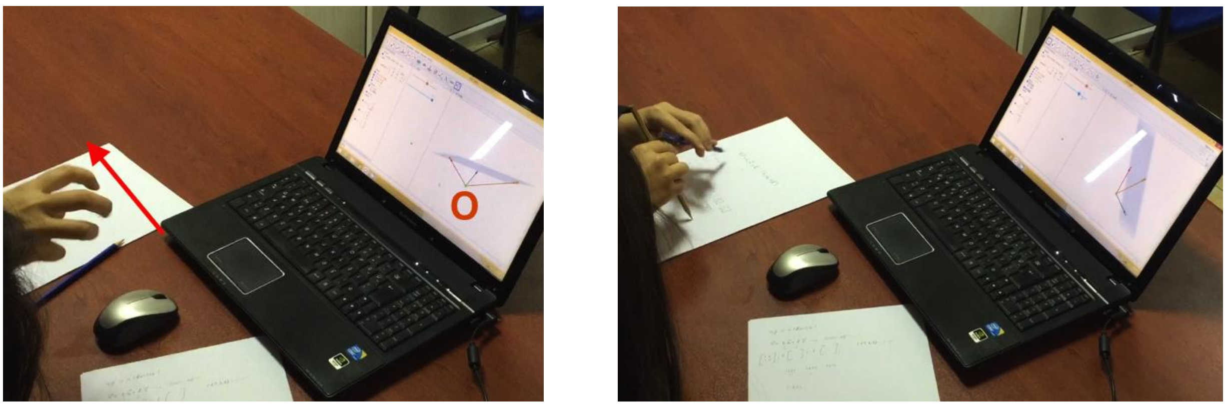
To begin with, the students began following the instructions for the task and described what was happening in the DGS interface and to talk about changes due to the use of the dragging tool and the objects built on the screen, rather than mathematical justifications. S2 requested more detailed information regarding the steps to follow by asking questions about the roles of the sliders and the 3D rotate tools and asking which to use first. Later, using the 3D rotate tool, S2 started to explore the view the image of the given figure in different positions. Table 1 displays the discourse used while the participants dealt with the task.

In this part of the discussion, in terms of S1's exploration with the 3D rotate tool, she realized that the given three vectors are on the same plane (#36). The geometric representation she used can be considered as a sign of an SGT process. Later, her mathematical explanation (#43) as to why three vectors were on the same plane and her explanation of the parallelogram rule (#45) are signs of S2's transition to an AAT mode and use of an algebraic representation type. The explanation given by S2 is (again) a repetition of S1's explanation.

#	Utterer	Discourse
36	S1	… [after exploring the different positions of the given figure] Here it is, only a 2 dimensional plane…
…	…	…
43	S1	Well, it is possible to define one in terms of the others
44	R	Why? ...
45	S1	It seems a vector added to another… [*gesturing*]. I remember we have the parallelogram rule for addition…
…	…	…
47	S2	Because they are on the same plane, each can be obtained by the others' linear combination…
48	R	Therefore…
49	S1	Dependent.
50	R	Try different positions by dragging the sliders.
…	…	…
60	S1	It seems the point O went up like this [*gesturing, see* Figure 2a].
61	R	... What is its difference from the first one?
…	…	…
64	S2	They have become linear independent; they are not on the same plane…
…	…	…
67	S1	For example, it is like … I changed this and became like that while it was like that [*showing with pencils, see* Figure 2b]
68	R	So, what can you tell by the space spanned by these three vectors?
69	S1	... Now it is three-dimensional. At first, we did not think like that. We saw when we changed the sliders.

Table 1: Initial discussion regarding the task

For instance, it is repetition of S1's utterance (#43) using a mathematical language. S2 can be said to have an AAT mode and use of algebraic representation (#47). As the discussion continued, S1 was observed to be aware of the fact that the vectors are linear dependent (#49). Using a gesture, she showed that the position of point 'O' changed as she tried different values; in other words, point 'O' goes out of the plane (#60, Figure 2a). At this point it can be seen that S1 has an SGT mode and use of geometric representation (#67, Figure 2b).



(a) (b)

Figure 2: (a) S1's description regarding the *O*, (b) S1's description of the process

However, S2 explains the mathematical situation due to the changing position of point 'O' by stating that it is because the set of given vectors is linear dependent, which can be considered an AST mode and an abstract representation. In the last part of the discussion, S1 explains that the spanned space due to the vectors being independent in $\mathbf{R}^3$ and, in the previous case, it was $\mathbf{R}^2$ due to the position of the given vectors. Even though stating that $\mathbf{R}^2$ is not mathematically correct, the explanations given are traces of an AST mode and abstract representation. In the later part of the discussion, the researcher asks an extra question by pointing to the algebra window in order to reveal how the concepts expressed by the participants would be shaped when they leave the DGS context. In other words, the aim of asking this question is to obtain more detailed information about what kind of meaning the participants give to linear dependency and linear independency concepts. Table 2 shows the second part of the discussion regarding the task.

#	Utterer	Discourse
72	R	If I don't have the images of the given vectors, how can I examine whether they are linear independent and linear dependent?
73	S2	We try to write them in terms of the others.
...	...	...
76	S1	We calculate its determinant of the given vectors.
77	R	What about its definition ... is determinant the practical one?
78	S2	… The use of definition takes a lot of time.
79	R	… What will the result of the determinant show me?
80	S2	If its determinant is 0, I call that linear dependent, but not here [*pointing to the 3D Graphics window*]…
81	R	... So in the first case?
82	S1	Zero.
83	S2	... Since it gives the volume.

Table 2: The second part of the discussion regarding the task

Initially, S2 answers the question asked by the researcher by stating #73 and using an AAT mode and algebraic representation. On the other hand, S1 is in SGT mode and prefers to use geometric representation (#76). Although the researcher repeats the definition, S2 states that using the definition takes a long time (#78) and she thinks that she can characterize the set of given vectors as linear dependent or independent using determinant notion. These explanations can be considered as proof of the fact that she is *not* in an AST or an AAT process. S2 states a relationship between the determinant value and volume and, by preferring this representation type, implies that she is in an SGT mode and using geometric representation.

CONCLUSIONS AND DISCUSSION

The present study aims to answer the following research question: what are the students' thinking modes and representation types of linear algebra in a dynamic geometry environment regarding concepts such as linear combination, linear dependency and independency, basis and dimension? According to to the data from the task-based interviews conducted with two linear algebra students, the students have different thinking modes and are able to make rapid (quicker than expected) transitions among these thinking modes. In addition, within this study, the close relationship between their thinking modes of linear algebra and the representations used (Donevska-Todorova, 2014) are experimentally replicated. The findings reveal that the most commonly-used thinking mode is SGT and that the most commonly-used representtation type is geometrical representation, while the least commonly-used thinking mode is AST and, accordingly, the least commonly-used representation type is abstract representation. The students are unable make generalizations and their failure generally is de-contextualization of proposed concepts from the DGS. In fact, they often prefer practical thinking to theoretical thinking. The reason why students often prefer practical thinking might be the fact that the dynamic geometric environment provides the characteristics of $\mathbf{R}^2$ and $\mathbf{R}^3$. For instance, since students could not be provided with contexts regarding abstract vector spaces during the interviews, it may be thought that the dynamic geometry environment might be an obstacle for students to adopt theoretical thinking processes, which is similarly discussed in another context (Kuzle, 2017).

Dogan-Dunlap (2010), focuses on students' thinking modes through a dynamic module, and finds that the SGT thinking mode and geometrical representations are also prominent (see Table 3, p. 2148) compared to others. Moreover, Dogan-Dunlap (2010) notes that the graphical tools used do not change the AAT modes and that students are able to apply different thinking modes in the same context. Similar findings are also evident in the present study. For instance, generally, while the student participants explain linear combination in the DGS by way of an AAT mode and algebraic representation, they explain the notion of span through an SGT mode and geometrical representation. However, it can also be seen that they use SGT and geometrical representation for linear dependency and independency, and use determinant rather than definition to explore whether a vector set is linear independent or not. Finally, they use different thinking modes and representations while expressing basis and dimension concepts.

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References

Dogan-Dunlap, H. (2010). Linear algebra students' modes of reasoning: Geometric representations. *Linear Algebra and its Applications, 432*(8), 2141–2159.

Donevska-Todorova, A. (2014). Three modes of description and thinking of Linear Algebra concepts at upper secondary education. In J. Roth & J. Ames (Eds.), *Beiträge zum Mathematikunterricht* (pp. 305–308). Münster: WTM-Verlag.

Donevska-Todorova, A. (2015). Conceptual understanding of dot product of vectors in a dynamic geometry environment. *The Electronic Journal of Mathematics and Technology, 9*(3), 192–209.

Donevska-Todorova, A. (2016). *Thinking modes, with or without technology?* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Gueudet-Chartier, G. (2004). Should we teach linear algebra through geometry? *Linear Algebra and its Applications, 379*, 491–501.

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra* (pp. 191–207). Dordrecht: Springer Netherlands.

Kuzle, A. (2017). Delving into the nature of problem solving processes in a Dynamic Geometry Environment: Different technological effects on cognitive processing. *Technology, Knowledge and Learning, 22*(1), 37–64.

Montiel, M., Wilhelmi, M. R., Vidakovic, D., & Elstak, I. (2012). Vectors, change of basis and matrix representation: onto-semiotic approach in the analysis of creating meaning. *International Journal of Mathematical Education in Science and Technology, 43*(1), 11–32.

Sierpinska, A. (2000). On some aspects of students’ thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209–246). The Netherlands: Kluwer Academic Publishers.

Turgut, M. (2015). Theory of semiotic mediation in teaching-learning linear algebra: In search of a viewpoint in the use of ICT. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the 9th Congress of European Society for Research in Mathematics Education* (pp. 2418–2424). Prague: Charles University and ERME.

Turgut, M. (2017). Students’ reasoning on linear transformations in a DGS: a semiotic perspective. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Congress of the European Society for Research in Mathematics Education*. Dublin, Ireland: DCU Institute of Education and ERME.

Turgut, M., & Drijvers, P. (2016). *Students' thinking modes and the emergence of signs in learning linear algebra*. Paper presented at the 13th Congress on Mathematical Education, Hamburg, Germany.

SAME UNIT COORDINATION: A CONCEPTUAL SCREENER FOR MIXED UNIT COORDINATION AND BASE-10, PLACE VALUE REASONING

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This quantitative study corroborates a conceptual linkage implied by Tzur et al.'s (2013) model of progression in schemes for multiplicative reasoning. We demonstrate that the Same Unit Coordination (SUC) scheme serves as a conceptual screener for the Mixed Unit Coordination (MUC) scheme—and hence for base-10, place value (PV-B10) reasoning. Solutions to written word problems designed to indicate each scheme, given to 200 fourth and 351 fifth graders, largely supported our hypothesis that the SUC scheme is a necessary but insufficient requisite—for the MUC scheme. We discuss implications of these findings for teaching and learning PV-B10.

Teaching and learning place value and base ten (PV-B10) is challenging for teachers and students (Verschaffel, Greer, & DeCorte, 2007). We posit that investigating students' multiplicative schemes can help understand this challenge. Specifically, we test the hypothesis that the *Same Unit Coordination* (SUC) scheme, found in the multiplicative reasoning progression postulated by Tzur et al. (Tzur et al., 2013), might serve as a conceptual screener for the *Mixed Unit Coordination* (MUC), which underlies PV-B10 reasoning. Yet, this linkage had to be corroborated statistically (Kilpatrick, 2001). In this study we examine how students' current ability to use a particular scheme (MUC), including the related ability for PV-B10 reasoning, depends on their current ability to use a more rudimentary scheme (SUC) in the progression.

Studying how students' MUC scheme may depend on their SUC scheme can help explain challenges in learning PV-B10. Key to such understanding is one's ability to operate not only on units of 1 but also on different composite units: 1s, 10s, 100s, etc. (Ulrich, 2015, 2016). For example, to find the difference between 25 and 78, one would have to keep track of accrual of 10s *and* of 1s: 35-is-1-ten, 45-is-2, … 75-is-5; and three more 1s will be 78—so the difference is five 10s and three 1s, or fifty-three 1s (e.g., Klein, Beishuizen, & Treffers, 1998). This differs from a much simpler problem, of finding the difference between 20 and 70. To solve this simpler problem, one could keep track of the accrual of 10s between those two numbers: 30-is-1-ten, 40-is-2, … 70-is-5, so the difference is five 10s or fifty 1s. These two problems illustrate, respectively, the operations on a mix of units (10s and 1s) versus operations on a single type of composite unit (10s only). Research has repeatedly demonstrated the difficulties involved in learning to operate at 3 levels (Ulrich, 2015, 2016).

CONCEPTUAL FRAMEWORK

We use a constructivist lens for explaining additive and multiplicative reasoning as

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levels of units coordination (Hackenberg, 2007; Steffe, 1992), clearly distinguishing operating on units of one (1s) and on composite units. It draws on Piaget's (1985) contention that a child's schemes for assimilating and recognizing a problem afford and constrain her solution (i.e., reasoning ≠ observable performance). In additive reasoning, a child can mentally coordinate the *same type of unit* (e.g., two 10s + three 10s = five 10s), while operating on two levels of units (1s and composite). However, multiplicative reasoning requires coordinating *3 different levels of units*: number of composite units in a compilation, items in each composite unit (1s per unit), and a total number of 1s (Ulrich, 2015). For example, consider the problem: "Box A has 5 towers, 4 cubes each; Box B has 12 cubes. If Dan puts all 12 cubes into towers of 4 and returns the towers to Box B, how many towers in all would be in both boxes?" Reasoning multiplicatively at 3 levels of units, a child can set a goal to find *how many composite units* would be in a compilation consisting of so many towers (5) *and* the additional 1s. She would accomplish her sub-goal, of figuring out how many composite units (towers of 4 cubes each) are produced by twelve 1s (cubes), by decomposing 12 into 3 units (towers) of four 1s (cubes) each. This would allow her to add those 3 composite units (4s) with the given 5 composite units (4s) to obtain 8 composite units (towers of 4).

The example above illustrates the first four schemes in the multiplicative reasoning progression (Tzur et al., 2013). The first, *multiplicative double counting* (mDC), underlies the coordination of 1s (e.g., 12 cubes) with composite units of 4 (e.g., 4 cubes per towers) into a compilation of 3 composite units (e.g., towers of 4). The second, *same unit coordination* (SUC), underlies additive operations on composite units (e.g., add 3+5 towers). While operating additively, the child conceives of those being composite units (e.g., towers made of 4 cubes), not simply units of 1. The third scheme (not a focus of this study), is *unit differentiation and selection* (UDS). In our example, this scheme underlies the child's setting of the sub-goal, as she had to differentiate the 12 (cubes) from the 5 (towers) and/or the unit rate of 4 (cubes-per-tower), select the 12 as input for her operation, and select 4 as the 'factor' by which to operate on (decompose) 12. The fourth scheme, *mixed unit coordination* (MUC), underlies the solution to the example above in its entirety, by coordinating the operations on 1s and composite units to produce the looked for compilation of composite units.

From this depiction of MUC, the claim that SUC is a conceptual screener seems apparent: A child needs to anticipate that adding the same units (e.g., towers of 4) requires converting the given 1s. It is also apparent why SUC is but a screener, as UDS would have to be coordinated with SUC into a single, multi-step activity regulated by the global goal (find total of composite units) and the sub-goal it triggers (find the number of composite units made of the given 1s). Importantly, this depiction helps to explain reasoning in PV-B10. For example, consider a child who solves the problem: "A school bought 2 boxes (100 apples each), 4 bags (10 apples each), and 19 single apples; how many apples does the school have in all?" A child reasoning with MUC can convert the number of apples in *two boxes* by anticipating 100 would be 10 units of 10 units of 1, and take two such units (hence, 200); similarly anticipating 4 bags as four units of 10 units of 1 apple (hence, 40 apples), and add all 1s (including 19) to arrive at 259 apples. A child withot MUC may add 1s and 10s (e.g., 20+40+19=79), or add all given units as if they were 1s (e.g., 19+4+2=25)—two common, erroneous solutions.

METHODS

This study is part of a larger project to promote and study the impact of elementary teachers' shift toward a student-adaptive pedagogy (AdPed) on student outcomes. We developed and validated two written assessments—one for the mDC scheme (Hodkowski et al., 2016) and one for the MUC scheme (forthcoming). The latter, used in the present study, contains six word problems designed to suit students learning English as an additional language, including one SUC screener and 4 MUC problems.

In the SUC screener Problem (#2), we intended for students to bring forth a strategy by which to find the difference (#2a) or sum (#2b) of given numbers of composite units in two separate compilations (4 towers of 3 cubes, 9 towers of 3 cubes). We assigned "1" to the student if both responses were correct (5 towers and 13 towers, respectively), and "0" otherwise. Students then proceeded through the 4 MUC word problems.

In Problem #3 (5 towers of 4 cubes each + 12 cubes), we intended for students to use MUC with small numbers (scored "1" if answering 8 towers). In Problem #4, we intended for students to do the same (9 bags of 6 candies each + 48 candies) with larger numbers (scored "1" if answering 17 bags). In Problem #5, we intended for students to operate, in a 'missing addend' task, on the *difference in 1s* between two given compilations of composite units (6 lines of 7 desks each, which a teacher needed to extend into 11 lines of 7 desks each). We asked them to determine the correctness of a hypothetical student's statement that 30 extra desks are enough to get to 11 lines (scored "1" if answering 35 extra desks were needed). In Problem #6, we intended for students to bring forth their MUC scheme in solving a PV-B10 problem with 100s (boxes of apples), 10s (bags of apples), and 1s (single apples). Given that School A has 2 boxes + 4 bags + 19 apples and School B has 1 box + 16 bags + 11 apples, students had to find which school had more apples and how many (scored "1" if answering School B has 12 more apples). To assess students' comprehension of problem statements, in each MUC word problem (#3-6) we included sub-questions that required students to fill in blanks with given information. For example, in Problem #5, students had to fill a given in the blank: "Each line has ___ desks." Chrobach's-α (0.84) for all MUC items indicated good internal consistency.

Setting and Participants

Participants were 4th graders (N=200) and 5th graders (N=351) from four different elementary schools in two public school districts, located in the metropolitan area of a large US city (ages ~9-11 years). About 85% of the students in our study identified as students of color, and ~70% were learning English as an additional language.

Data Collection and Analysis

We report results from three administrations of the written MUC assessment: Spring

2016, Fall 2016, and Spring 2017. Each assessment took place during one class period of a regular school day (~40-50 minutes). A graduate research assistant (GRA) read out loud each problem and sub-questions, one page at a time, while students followed silently. The GRA monitored students solved only problems on a page at issue. Once completing all questions on that page, the GRA told them to flip to the next page, monitoring they do not flip back to problems in previous pages. Table 1 disaggregates the assessment totals by student grade and administration date. We analysed data from all 551 available assessments, which include students who have been assessed twice or three times, because they reflect their ability (or lack thereof) to bring forth SUC and MUC schemes in far-apart administrations. This larger number allowed us to further test hypotheses about linkages between the SUC and MUC schemes.

Grade	Spring 16	Fall 16	Spring 17	Total
4	54	90	56	200
5	80	168	103	351
Total	134	258	159	492

Table 1: Numbers of students taking the MUC assessment by grade and date.

To increase reliability, six GRAs entered the student responses into a spreadsheet in pairs, one reading the responses out loud and the other entering those as is (no scoring, yet). The first GRA monitored and verified that responses were entered correctly for every student. Using that raw-data spreadsheet, scoring of responses was calculated in a different spreadsheet set to do so automatically (with scores as noted above).

Our analysis tested three main hypotheses pertaining to (a) each distinct MUC problem and (b) the total of correct responses a student gave to all 4 MUC problems. Table 2 recaps the hypotheses and the statistics used to test each ("Yes SUC" refers to students who could solve both SUC problems, "No SUC" otherwise; ">" implies outperform).

Hypothesis	SUC	Each MUC Problem	All MUC Problems
"Yes SUC" > "No SUC"		Mann-Whitney (np) Chi-square	ANOVA (total: 0-4) Cramer's V (number correct)
5^{th} graders > 4^{th} graders	Chi-square	Mann-Whitney (np)	ANOVA (total: 0-4)

Table 2: Study hypotheses and statistics to test them.

RESULTS

We analyse data to support the claim that the SUC scheme serves as a conceptual screener for the MUC scheme. First, we note that, of all participating students (N=551), more than half (54%) did not yet construct the SUC scheme. Chi-square analysis (χ^2=5.8, p=0.16) confirms our hypothesis that significantly higher percentage of 4^{th} graders (61%, N=200), versus 5^{th} graders (50%, N=351), would lack the SUC scheme. With this in mind, we turn to findings linking SUC and MUC.

Table 3 provides data for all students, for 4^{th}, and for 5^{th} graders. For each group, we begin with a line showing the average number of students who correctly solved each

MUC problem (#3-6) and the average of their total score for all MUC problems. Below that line, in two additional lines, we distinguish data of "Yes SUC" and "No SUC."

All students

Line 'All-a' shows a low percentage (25%) of success on the 4 MUC problems (#3-6), that is, one MUC problem on average—which indicates absence of that scheme. Data for distinct problems show a decrease in successful solutions, ranging from 40% in Problem 3 to the mere 9% in Problem 6. A chi-square test shows this decreese is statistically significant: #3 vs. #4 (χ^2=105.6, p<.0005), #4 vs. #5 (χ^2=63.0, p<.0005), and #5 vs. #6 (χ^2=70.4, p<.0005). The remarkably low success on problem #6 is alarming, as this problem requires using the MUC scheme to solve a PV-B10 task.

Line	MUC Problems →	3	4	5	6	Total
All-a	All Total	40%	29%	20%	9%	25%
All-b	All No SUC	23%	16%	11%	7%	14%
All-c	All Yes SUC	61%	44%	30%	12%	37%
4-a	4th Total (N=200)	27%	23%	12%	5%	17%
4-b	4th No SUC	15%	12%	2%	4%	9%
4-c	4th Yes SUC	44%	38%	27%	5%	29%
5-a	5th Total (N=351)	48%	33%	25%	12%	30%
5-b	5th No SUC	28%	19%	18%	9%	19%
5-c	5th Yes SUC	69%	47%	31%	15%	41%

Table 3: Percentages of students solving MUC problems correctly.

Lines 'All-b' and 'All-c' confirm our main hypothesis about MUC: "Yes SUC" students outperformed "No SUC" students. The Mann-Whitney statistic shows this for distinct MUC problems: 23% of "No SUC" vs. 61% of "Yes SUC" in Problem 3 (Z=9.18, p<.0005), 16% vs. 44% in Problem #4 (Z=7.23, p<.0005), 11% vs. 30% (Z=5.36, p<.0005) in Problem #5, and meagre 7% vs. 12%" for Problem #6 (Z=1.88, p=.06). ANOVA for all 4 MUC problems further confirms this: "No SUC" merely solved 14% of MUC problems vs. 37% for "Yes SUC" ($F_{1,550}$=94.7, p<.0005).

Grade-4 students

Results for 4th graders echo those for all participating students (albeit lower averages). Line '4-a' shows a very low percentage (17%) of successful solutions to the 4 MUC problems: on average, 4th graders could not solve even one MUC problem—indicating absence of that scheme. Distinct MUC problems show a decrease from 27% in Problem 3 to the meagre 5% in Problem 6. A chi-square test shows this is statistically significant for #3 vs. #4 (χ^2=33.5, p<.0005), #4 vs. #5 (χ^2=43.1, p<.0005), and for #5 vs. #6 (χ^2=2.7, p<.0005). The remarkably low success on problem #6, which indicates use of the MUC scheme to solve a PV-B10 task, is alarming particularly because those students have been taught PV-B10 for 4-5 years. Notably, our findings help explain this difficulty: to adequately reason in a PV-B10 number system students need to construct the MUC scheme, which seems dependent on the SUC scheme.

Lines '4-b' and '4-c' confirm our main hypothesis for 4th graders: "Yes SUC" students outperformed "No SUC" students. The Mann-Whitney statistic shows this for distinct MUC problems #3-5: 15% of "No SUC" vs. 44% of "Yes SUC" in Problem 3 ($Z=4.6$, $p<.0005$), 12% vs. 38% in Problem #4 ($Z=4.22$, $p<.0005$), and 2% vs. 27% ($Z=5.12$, $p<.0005$) in Problem #5. ANOVA for all 4 MUC problems further confirms this: Mere 9% of "No SUC" vs. (still alarming) 29% of "Yes SUC" ($F_{1,199}=35.1$, $p<.0005$).

Grade-5 students

Results for 5th graders further echo those discussed above. Line '5-a' shows a very low percentage (30%) of correct solutions to the 4 MUC problems: on average, 5th graders merely solved one MUC problem—indicating absence of that scheme. Distinct MUC problems show a decrease from 48% in Problem 3 to the rather low 12% in Problem 6. A chi-sqaure test shows this is statistically significant: #3 vs. #4 ($\chi^2=66.1$, $p<.0005$), #4 vs. #5 ($\chi^2=26.7$, $p<.0005$), and #5 vs. #6 ($\chi^2=40.8$, $p<.0005$). The low success on problem #6 seems even more alarming than in 4th grade, as 5th graders have been taught PV-B10 for 5-6 years. Yet, barely 1-in-10 5th graders could solve the PV-B10 problem. As noted for grade-4, these findings help explain this difficulty: Both SUC and MUC schemes are needed to adequately reason and solve PV-B10 word problems.

Lines '5-b' and '5-c' confirm our main hypothesis for 5th graders: "Yes SUC" students outperformed "No SUC" students. The Mann-Whitney statistic shows this for distinct MUC problems #3-5: 28% of "No SUC" vs. 69% of "Yes SUC" in Problem 3 ($Z=7.63$, $p<.0005$), 19% vs. 47% in Problem #4 ($Z=5.63$, $p<.0005$), and 18% vs. 31% ($Z=2.94$, $p=.003$) in Problem #5. ANOVA for all 4 MUC problems further confirms this: Mere 19% of "No SUC" vs. (more sensible) 41% of "Yes SUC" ($F_{1,350}=54.0$, $p<.0005$).

Between-grade comparison

It seems plausible that SUC is a screener at one grade-level, but later it diminishes. Demonstrating it serves a similar role in 4th and 5th grades lends support to the claim it is a *conceptual* screener. We found that although 5th graders are more likely than 4th graders to have constructed the SUC and MUC schemes—the difference between "No SUC" and "Yes SUC" remains, albeit "shifted upward." Here, for the results shown in Table 3, we provide analysis of significance, using the Mann-Whitney statistics for distinct problems and ANOVA for the total of all 4 MUC problems.

All 4th vs. All 5th: #3 ($Z=2.62$, $p=.009$), #4 ($Z=3.53$, $p<.0005$), #5 ($Z=5.04$, $p<.0005$), and #6 ($Z=2.91$, $p=.004$); Total ($F_{1,550}=27.0$, $p<.0005$).

"No SUC" 4th vs. 5th: #3 ($Z=2.65$, $p=.008$), #4 (non-significant), #5 ($Z=4.03$, $p<.0005$), and #6 (non-significant); Total ($F_{1,550}=12.9$, $p<.0005$).

"Yes SUC" 4th vs. 5th #3 ($Z=3.70$, $p<.0005$), #4 (non-significant), #5 (non-significant), and #6 ($Z=2.22$, $p=.026$); Total ($F_{1,550}=F=8.96$, $p=.003$).

Taken together, these results support our claim that SUC is a necessary but insufficient precursor for MUC. Whereas 5th graders outperformed 4th graders on SUC and MUC, at each grade-level the gap between the "No SUC" and "Yes SUC" is sizeable. In fact, the gap seems even more pronounced in 5th grade. These results, including the extremely low figures for Problem #6 (PV-B10), amount to our main claim: reasoning with the MUC scheme seems dependent on the availability of the SUC scheme.

Number of successfully solved MUC problems

To provide further evidence of the screening role of SUC, we compare the "No SUC" to the "Yes SUC" group on the total number of problems students could solve correctly (Table 4). Using crosstabs analysis with the Cramer's-V statistics shows the difference between "No SUC" and "Yes SUC" is significant (Cramer's V = 0.41, $p<.0005$). Conceptually, a crucial difference lies between students who solved at most 1 problem (indicating no MUC scheme), and those who solved at least 2 problems (indicating evolving or available MUC scheme). Nearly 85% of students without SUC could solve *at most 1 MUC problem* (with 66.2% unable to solve any MUC problem). In contrast, nearly half (48.2%) of "Yes SUC" students solved *at least 2 MUC problems*.

MUC Correct Problems	0	1	2	3	4	Total
No SUC	196 66.2%	54 18.2%	27 9.1%	14 4.7%	5 1.7%	296 100.0%
Yes SUC	70 27.5%	62 24.3%	71 27.8%	37 14.5%	15 5.9%	255 100.0%

Table 4: Number of MUC problems solved correctly, "No SUC" vs. "Yes SUC."

DISCUSSION

In this study, we corroborated a claim about conceptual linkages, found in previous, qualitative research that 'mapped' conceptual progressions in students' schemes for multiplicative reasoning (Steffe, 1992; Tzur et al., 2013). We demonstrated that the second Same Unit Coordination (SUC) scheme serves as a conceptual screener for the fourth scheme—Mixed Unit Coordination (MUC). We explained this linkage in that MUC is postulated to arise through coordinating SUC and the third scheme in the progression, UDS (developing a written, large-scale assessment for UDS is forthcoming). Three, interrelated findings supported our claim that SUC is a conceptual screener for MUC: (a) ~50% of students in our population were yet to construct the SUC scheme, (b) "Yes SUC" students outperformed "No SUC" students on each distinct and on all 4 MUC problems, and (c) nearly 85% of "No SUC" students lacked the MUC scheme—solving at most 1 MUC problem (66% solved none).

Along with the important contribution of corroborating that SUC is a conceptual screener for the MUC scheme, this study provided a step toward explaining the difficulties many students (and teachers) seem to face when reasoning and solving place value, base-ten (PV-B10) problems. As we explained in the Conceptual Framework, adequate reasoning in a PV-B10 number system requires the MUC scheme, including a 3-level units coordination (Ulrich, 2015, 2016). The non-routine Problem #6 we used in our assessment could help reveal conceptual 'blocks' indicative of the lack of MUC scheme as manifested in the other 3 MUC problems. To solve Problem #6, a student would have to clearly distinguish between 1s, 10s, and 100s, convert units to obtain a total of 1s for two different quantities (271 and 259), and find the difference between those quantities (12=10+2). Using the SUC scheme is needed to add/ subtract composite units of 10, of 100, and so on. Accordingly, our findings highlight the *futility of teaching PV-B10 to students who are yet to construct the SUC scheme,* and thus likely also the MUC scheme. This implies the need to provide elementary teachers with professional development that enables them to (a) distinguish multiplicative schemes (SUC, MUC) based on students' problem solving and (b) adapt goals and activities for students' learning to construct those schemes – instead of following a scripted curriculum based on grade-level standards. Simply put, teachers need to tailor PV-B10 curriculum to where students are conceptually (with SUC and MUC schemes as conceptual requisites) – not 'tailor students to the curriculum'.

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References

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior*, 26, 27-47.

Hodkowski, N. M., Hornbein, P., Gardner, A., Johnson, H. L., Jorgensen, C., & Tzur, R. (2016). Designing a stage-sensitive written assessment of elementary students' scheme for multiplicative reasoning. In M. B. Wood, E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1581-1587). Tucson, AZ: The University of Arizona.

Kilpatrick, J. (2001). Where's the evidence? *Journal for Research in Mathematics Education, 32*(4), 421-427.

Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic versus graudal program design. *Journal for Research in Mathematics Education*, 29(4), 443-464.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development* (T. Brown & K. J. Thampy, Trans.). Chicago: University of Chicago.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, *4*(3), 259-309. doi:10.1016/1041-6080(92)90005-Y

Tzur, R., Johnson, H. L., McClintock, E., Kenney, R. H., Xin, Y. P., Si, L., . . . Jin, X. (2013). *Distinguishing schemes and tasks in children's development of multiplicative reasoning.* PNA, 7(3), 85-101.

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics*, *35*(3), 2-7.

Ulrich, C. (2016). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics*, *36*(1), 34-39.

Verschaffel, L., Greer, B., & DeCorte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557-627). Charlotte, NC: Information Age.

FACILITATING CONCEPTUAL ENGAGEMENT WITH FRACTIONS THROUGH SUSPENDING THE USE OF MATHEMATICAL TERMINOLOGY

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In this study we sought to establish whether an instructional sequence focused on fractions as measures was effective in supporting a group of South African Grade 3 students' understanding of fractions. The sequence is centred on a story that utilises 'nonsense' words to describe fractions. The students in this study had already been introduced to fraction terminology and symbols, but struggled in the initial lessons of this sequence and in the pre-test to use these with understanding. This paper focuses on how this sequence's suspension of the use of the mathematical terminology in favour of these 'nonsense' words helped to facilitate students' deep engagement with the concept of fractions during these lessons.

In this paper we describe the results of a study investigating the effectiveness of an instructional sequence with the aim of facilitating students' understanding of the inverse order relation of unit fractions. This sequence of four lessons, designed by Cortina, Višňovská and Zúñiga (2012), proposes an alternative starting point to teaching fractions: using fractions as measures rather than equipartition as the context in which the concept of fractions is introduced.

The lesson sequence is centred on a story about the origins of standardised measurement. A particularly notable feature of the lesson sequence is the use of 'nonsense' words, words with no established meaning, to describe fractions rather than the accurate mathematical terminology. For students who have not previously been exposed to the mathematical names for fractions, this sequence delays the use of the mathematical vocabulary. For the students in this study, it represented a suspension of the use of the terminology they had already encountered in the vocabulary and symbols of fractions at school. The focus of this paper is on the influence that this choice to suspend the use of accepted mathematical terminology had on the engagement of the students in the lesson activities and we propose that this played an important role in the progress students demonstrated in reasoning about the relative sizes of fractions.

In order to show this, we reflect on moments of interaction with the students as they worked through the activities in the lesson sequence. We focus in particular on several moments in which students grappled with the use of the mathematical terminology related to fractions (e.g. 'half' and 'quarter') and argue that in these moments the effort to encourage accurate use of the words detracted from the desired focus of the task at hand. We contrast this with the work of the students after the 'nonsense' terminology

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was introduced through the story. In addition, we report on the results of the pre- and post-tests to show how students progressed in understanding of the relative sizes of unit fractions. Significantly, this improvement was based on an assessment that utilised the mathematical vocabulary and symbols and not the ‘nonsense’ words and created symbols used in the story and the activities.

USING MEASUREMENT AS A CONTEXT FOR TEACHING FRACTIONS

Cortina, Višňovská and Zúñiga (2014) argue that equipartition can be a didactical obstacle to teaching fractions. They explain that equipartition has been incorrectly considered by many to be “either the only or the most advantageous way to introduce students to the topic” (Cortina, et al., 2014). Specifically, they identify three fixed images of fractions that an equipartition approach develops: “fraction as a result of acting on an object (fraction as fracture); fractions as ‘so many out of so many’; [and] fraction included in a whole (pp. 4-5). This approach is limited when students need to “find meaning in uses of fractions that are inconsistent with these images” (p. 7).

In order to support students in reasoning about the relative size of fractions, Cortina and Višňovská and Zúñiga (2012) have proposed an alternative starting point to teaching fractions: using ‘comparing’ instead of ‘fracturing’. Their resulting instructional design, which uses length measurement activities as the vehicle for fraction learning, was the focus of this research. There is ongoing research into the effectiveness of this design that points to the value of taking such an approach (see Cortina & Visnovska, 2016).

As Lamon (2012) explains, when students start working with natural numbers, measurement takes its simplest form in the counting of separable objects. When they begin to encounter fractions, the measurement of continuous quantities becomes possible (Lamon, 2012). This is done by segmenting the quantity to form whole units, then subdividing the whole units and iterating the resulting part units in order to measure. In subdividing the unit into fractional pieces, the degree of precision of the resulting measurement is increased (Lamon, 2012). Measurement contexts can thus provide particularly fertile ground for developing the concept of fractions. It is on this activity of subdivision of whole units into fractional part units that Cortina et al.’s (2012) instructional design rests.

THE ROLE OF WORDS IN CONCEPT FORMATION

Vygotsky (1987) writes “direct instruction in concepts is impossible [and leads to] mindless learning of words” (p. 170). Concept formation, he explains, involves all basic intellectual functions and is impossible without the use of words as signs or “functional tools” (Vygotsky, 1986, p. 107) that drive the formation of concepts. Development of the physiologically based intellectual processes, e.g. memory or perception, does not lead to higher forms of intellectual ability. It is verbal thinking that is necessary for the qualitatively “radical change” (p. 109) that makes thinking in concepts, such as fractions, possible.

He distinguishes between phases that lead to thinking in real concepts and argues that transition from one stage to the next is reliant on a child's verbal interaction with adults (Vygotsky, 1986). In the first phase, syncretic heap, a child groups objects randomly and words do not hold stable meanings (Vygotsky, 1986). This can be seen in the students' seemingly random use of the words 'half' and 'quarter' to describe any part of a whole during the first lesson. As Berger (2006) explains, children use words that they initially do not fully comprehend, but as they use it in communicating with adults, the meaning of the word and its associated concept evolves. In other words, the concept "undergoes substantial development for the child as [they] use the word or sign in communication with more socialised others" (Berger, 2005, p. 155). In mathematics, Berger (2005) argues, the individual is required to construct the concept such that its meaning agrees with how it is used in the mathematics community.

Berger (2006) advocates for activities that allow for idiosyncratic uses of mathematical words and symbols in the early stages of concept formation. She explains (p. 17):

> …[it] is not *how* (emphasis in original) the student uses the signs but rather *that* (emphasis in original) [they] use the signs. Through this use, the student gains access to the 'new' mathematical object and is able to communicate (to better or worse effect) about it. And…it is this communication with more knowledgeable others which enables the development of a personally meaningful concept whose use is congruent with its use by the wider mathematical community

Berger therefore seemingly argues that it is necessary that the mathematically accepted terminology must be used to allow students to come to a whole understanding of the concept of fractions. Vygotsky's notion of 'signs', however, can be understood to be broader than one specific word per concept. He writes that signs can be understood as an "auxiliary means of solving a given psychological problem" (1978, p. 52). For example, a word can be used to aid someone in remembering something and in this way the "sign acts as an instrument of psychological activity" (p. 52). It should hold therefore that if the mathematical vocabulary becomes a stumbling block to students' conceptual engagement in a task, the introduction of a new 'sign' such as, in the case of our study, a nonsense word that can serve the same psychological purpose as the original word could allow this to be overcome. Once the concept itself is therefore better formed, the original word can be substituted back, but with more conceptual clarity. We propose that this could facilitate conceptual development supporting more accurate use of the accepted terminology.

METHODOLOGY

The broader study, of which this paper represents a part, took a design research approach, as developed by Gravemeijer and Cobb (2013). Our goal was to explore the "innovative learning ecology" (p. 75) proposed by Cortina et al. (2014) in their instructional sequence. Accordingly, our retrospective analysis presented here, focuses of the use of vocabulary in the classroom in order to offer a proposal as to how the sequence works to support students' learning (Gravemeijer & Cobb, 2013).

Three South African Grade 3 classes, of 36 students each, participated in the instructional sequence, which was facilitated by the first author, with the second author in attendance for two of the lessons. The four lessons were run during the normal school day, one on each of four consecutive days. The lessons were video recorded for later analysis and the first author maintained a journal of field notes to record additional observations.

Critical incident analysis (Flanagan, 1954; Butterfield, Borgen, Amundsen & Maglio, 2005) was used to identify excerpts of the video recordings that were relevant to the research focus. These were transcribed in rich detail for further analysis. In this study we selected the moments in which fraction terminology and nonsense terminology were used by the students. In addition, students completed a pre-test and post-test assessing their understanding of the inverse size order relation of fractions. Their responses were summarised and analysed for recurring patterns in the type of errors made. Each item was analysed for patterns in responses, and each student's work was analysed for shifts in performance from pre-test to post-test. There were 83 students who completed both the pre-test and the post-test.

We recognise that in having chosen to position the first author as facilitator of the sequence, the credibility of the findings could be questioned. To offset this risk, a rich audit trail is available for scrutiny that includes the lesson videos as well as the students' test scripts. The design research approach itself also enhances the credibility in that it necessitates a strictly scripted lesson delivery. This removes much of the subjectivity in decision-making within the lesson. Furthermore, the students' test responses were analysed in addition to the lesson transcripts as a form of triangulation.

THE LESSON SEQUENCE

The overarching goal of the lesson sequence is that students come to make sense of the inverse order relation of unit fractions. Each lesson in the sequence works towards achieving this. In the first two lessons, students explore measuring length using their bodies and are prompted to think about how this differs when using small or large units. By the end of the first lesson, they should be aware of the challenge in communicating measurements when using body parts of different sizes to carry out the measurements. This leads to the second lesson, in which students come to recognise that it is more suitable to measure with a standardised unit. Students are provided with sticks of identical lengths with which to measure. Through the activities, students usually become aware that there is a remaining space not covered by a whole unit, and experience the challenge of finding a way to accurately communicate the length of this remainder.

During these two lessons, students attempted to use fraction names to describe the remainders. This was expected as the students were familiar with fraction names, but their use of these, and their assumption as to the accuracy of their descriptions, limited the realisation of the lesson aims. As an example, when asked to measure the length of their (identical) desks with the stick, the students needed to make the observation that the number of whole units measured was the same for all groups, but that the stick did not allow for an accurate answer as to how long the remainder was. What transpired was that students were convinced that they were communicating the length with precision by using a fraction name to label the remainder part (many said half, when a quarter was the closest unit fraction describing the remainder as a fraction of the stick). The lesson momentarily took a turn towards assisting the students in naming the remainder appropriately in terms of the fraction name.

As an example, one group of students entered into the following exchange with the second author:

MG:	And, how long is your desk?
Student 1:	Four and a half.
MG:	Is it a full half?
Student 1:	Yes.
MG:	How much is a half? [holds stick out to student and student touches the stick at approximately half of its length] Yes! Was it that much?
Together:	[as they re-measure the desk together] One…two…three…four…
MG:	[pointing to the remaining length of the desk] So, is it a full half?
Student 1:	No.

This exchange was similarly repeated with other students and groups of students. This was not, however the aim of the sequence, nor a part of its design, so the focus was quickly turned to simply acknowledging that there was much disagreement and no accurate way of finding and describing the remainder. In this way the desired consensus was reached that a better system than a single stick was needed

It is in the second lesson that students are told a traditional story in which an ancient potter experiences difficulty in measuring to make her pots accurately and visualises using a standard stick to measure rather than body parts. In the third lesson, the story continues with the character finding a solution to the problem of measuring the remainder by carefully constructing subunits of the stick to be used to measure the remaining lengths. In the story, these subunits are given special names – what we refer to as 'nonsense' words. The sub-units are called 'obele', or 'smalls', each with special characteristics. An 'otibele', translated as 'a small of two', is a length that fits exactly twice into the length of the stick. This is a 'half' but it is never referred to as such. A list of nonsense words with the translations was given to each student (from a small of two to a small of ten representing all unit fractions from $^1/_2$ to $^1/_{10}$).

The students adopted this terminology with delight and excitement as indicated by their repeatedly saying the words aloud and smiling as they read them. Their confused use of the fraction names entirely disappeared for the remaining lessons. It is at this point that the students became engaged in particularly rich conceptual work with fractions. They constructed units that fit exactly *x* times into the unit. For example, they constructed a single unit that would fit exactly twice into the length of the stick. Students were given straws so they could easily through trial and error cut them to the needed length. The straws were shorter than the length of the stick unit, so that students could not simply fold them in half to judge the length of a small of two. They repeated this process until they had a set of 9 subunits of decreasing size. The students were continually prompted to realise that the more times a unit fits into the whole, the smaller it is.

In the fourth lesson, students used their unit sticks and this set of smaller subunits to measure objects and realised that this solved the problem of measuring remainder lengths and communicating the lengths. Until this point, the word ‘fraction’ was not mentioned. Students took joy in naming the lengths using the nonsense words and did so in a conceptually accurate manner. They indicated understanding that a ‘small of ten’ was smaller than a ‘small of nine’.

At the conclusion of the lesson sequence students’ attention was drawn to the fact that they had been working with fractions, and that these subunits could also be named using the accepted mathematical fraction names. This was merely mentioned, and was not explored further. However, as is shown in the following section, the students made great gains in their standard test performance despite all conceptual work having been done using nonsense terminology.

ASSESSMENT RESULTS

Students showed impressive gains in their understanding of the inverse order relation of unit fractions when their pre- and post-test responses were compared.

One item asked the following: “Thembi gets half ($^1/_2$) a candy bar and Angi gets one-fifth ($^1/_5$) of a candy bar. Colour in how much candy they each get.” This item was accompanied by two identical rectangles representing these candy bars for the students to colour in. A response was considered correct if the portion coloured in was within one-tenth of the accurate amount. In the pre-test, 32 students drew the half and fifth correctly.

There were 17 students who coloured in a fraction that was more than one-tenth too large or too small, although these students correctly indicated that the half was larger than the fifth. Twenty students reversed the size order relation, indicating that a fifth was larger than a half. In the post-test, of the 17 students who were inaccurate in their drawings, only 4 students over- or underestimated by more than one-tenth. Most notable were the twelve students who reversed the size order relation in the pre-test, but who drew the fractions correctly in the post-test.

This item was followed by a question asking which child received more, Thembi or Angi. There were many students who did not answer this question (only 27 answered this in the pre-test, and 59 in the post-test), however of those who answered, an interesting observation was made that pointed to some of the confusion the students had regarding the inverse order relation. Half of the students in the pre-test provided answers that were incongruent with their drawings. That is, if their drawing showed that Angi received more, they indicated on this question that Thembi received more. This was the case for 4 students. Interestingly, eleven of the students who drew the fractions correctly named Angi as the child receiving more. In the post-test, only 3 of these students persisted in this error.

The final item on the test asked students to circle the fraction which was larger of a set of four pairs of fractions: $^{1}/_{2}$ or $^{1}/_{4}$; $^{1}/_{5}$ or $^{1}/_{3}$; $^{1}/_{4}$ or $^{1}/_{8}$; and $^{3}/_{4}$ or $^{3}/_{3}$. The figure below shows the number of students, in the pre-test and the post-test, answering correctly. A dramatic increase is evident, and there were only 15 of the 83 students who persisted in making the same errors in both tests.

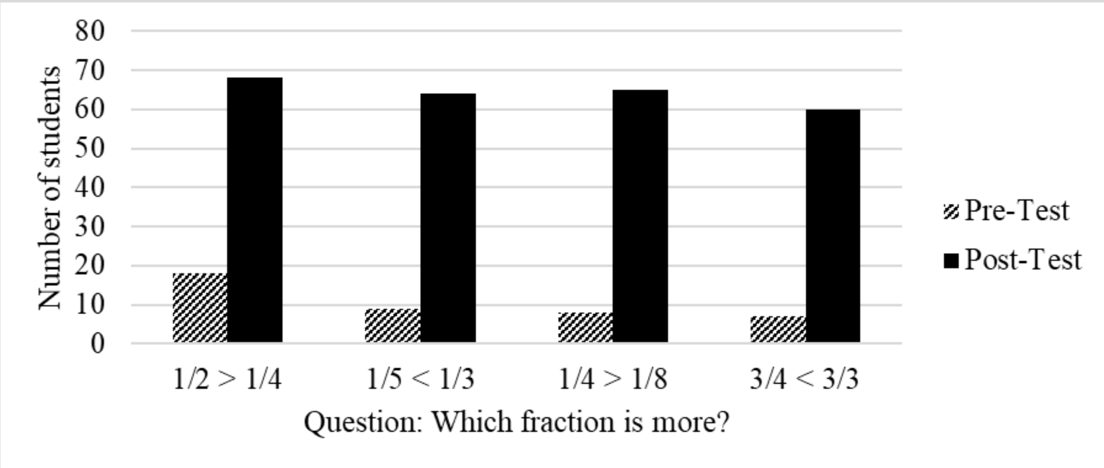

Figure 1: Number of students answering correctly.

This increase provides clear evidence of the students having improved in their understanding of the inverse order relation of unit fractions.

CONCLUSION

During the first two lessons, prior to the introduction of the nonsense words, it was clear from the students' use of the mathematically accepted fraction names, that their understanding of the concepts underlying the words was still developing. Their use of the words revealed their syncretic heap thinking (Vygotsky, 1986) and while this is a part of the development of the concept, it detracted from the work of this specific lesson sequence.

The design called for the students to suspend their use of the mathematically accepted terminology for the duration of the conceptually-driven activities of the third and fourth lesson. It was significant to observe, therefore, that the students improved, not only in their understanding of the inverse order relation of unit fractions, as was the goal, but that they were also able to demonstrate this understanding on an assessment that used the standard terminology and symbols.

This suggests that there was value in suspending the use of standard terminology, and temporarily replacing it with words that were not linked to any emerging conceptual knowledge, to allow students to engage in deep conceptual work independent of their grappling with the definitions of the technical terminology. The fraction concept itself became better formed for the students such that the original words could be substituted back and, as the students' assessment responses indicated, with increased conceptual clarity.

The use of this nonsense terminology can therefore be understood to be a feature of this lesson sequence that contributed to facilitating conceptual development that supported more accurate use of the accepted terminology.

References

Berger, M. (2005). Vygotsky's theory of concept formation and mathematics education. In H.L. Chick & J.L. Vincent (Eds.), *Proc. 29th Conf. of the Int. Group for the Psychology of Mathematics Education* (pp. 153-160). Melbourne, Australia: PME.

Berger, M. (2006). Making mathematical meaning: From preconcepts to pseudoconcepts to concepts. *Pythagoras, 63*, 14-21.

Butterfield, L.D., Borgen, W.D., Amundsen, N.E. & Maglio, A.T. (2005). Fifty years of the critical incident technique: 1954-2004 and beyond. *Qualitative Research, 5*(4), 475-497.

Cortina, J.L., Višňovská, J. & Zúñiga, C. (2012). Alternative starting point for teaching fractions. In J. Dindyal, L.P. Cheng & S.F. Ng (Eds.), *Proc. 35th Conference of the Mathematics Education Research Group of Australasia* (pp. 210-217). Singapore: MERGA.

Cortina, J.L., Višňovská, J. & Zúñiga, C. (2014). Equipartition as a didactical obstacle in fraction instruction. *Acta Didactica Universitatis Comenianae Mathematics, 14*, 1-18.

Cortina, J.L. & Višňovská, J. (2016). Reciprocal relations of relative size in the instructional context of fractions as measures. In C. Csikos, A. Rausch & J. Szitanyi (Eds.), *Proc. 40th Conf. of the Int. Group for the Psychology of Mathematics Education* (pp. 179-186). Szeged, Hungary: PME.

Flanagan, J.C. (1954). The Critical Incident Technique. *Psychological Bulletin, 51*(4), 327-358.

Gravemeijer, K. & Cobb, P. (2013). Design research from the learning design perspective. In Plomp, T. & Nieveen, N. (Eds.), *Educational Design Research* (pp. 72-113). Enschede: Netherlands Institute for Curriculum Development.

Lamon, S.J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd ed.). London: Lawrence Erlbaum Associates.

Vygotsky, L. (1978). *Mind in Society: The Development of Higher Psychological Processes.* Cambridge, MA. Harvard University Press.

Vygotsky, L. (1926/1986). *Thought and Language*. Cambridge, MA: The MIT Press.

Vygotsky, L. (1987). The development of scientific concepts in childhood. In R.W. Rieber (Ed.), *The collected works of L.S. Vygotsky: Volume 1, Problems of General Psychology* (pp. 167-242). New York: Plenum Press.

INVESTIGATING LEARNERS' FRACTION UNDERSTANDING: A LONGITUDINAL STUDY IN UPPER ELEMENTARY SCHOOL

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We longitudinally followed 201 upper elementary school learners in the crucial years of acquiring rational number understanding. Using latent transition analysis we investigated their conceptual change from an initial natural number based concept of a rational number towards a mathematically more correct one by characterizing the various intermediate states learners go through. Results showed that learners first develop an understanding of decimal numbers before they have an increased understanding of fractions. We also found that a first step in learners' rational number understanding is an increased understanding of the numerical size of rational numbers.

INTRODUCTION

There is broad agreement in the literature that a good understanding of the rational number domain is highly predictive for the learning of more advanced mathematics (e.g., Siegler, Thompson, & Schneider, 2011). It is therefore worrying that many elementary and secondary school learners and even (prospective) teachers face serious difficulties understanding rational numbers. An often reported source for the struggle with understanding rational numbers is the natural number bias, i.e., the tendency to (inappropriately) apply properties of natural numbers in rational numbers tasks (e.g., Christou & Vosniadou, 2009; Gomez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Obersteiner et al., 2014; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vosniadou, 2013).

The literature reports at least three aspects of the natural number bias. The first aspect involves the numerical size of numbers. Learners often consider a fraction as two independent numbers, instead of a ratio between the numerator and denominator. This incorrect interpretation of a fraction can lead to the idea that the numerical value of a fraction increases when the numerator, denominator, or both increase, just like it is the case with natural numbers. For example, 1/8 can be judged larger than 1/6, just like 8 is larger than 6. Similarly, in the case of decimal numbers, some learners have been found to wrongly assume that, just like it is the case with natural numbers, longer decimals are larger, while shorter decimals are smaller. For example, these learners judge 0.12 larger than 0.8, just like 12 is larger than 8 (e.g., Vosniadou, 2013).

The second aspect concerns the effect of arithmetic operations. After learners did arithmetic with mostly natural numbers only in their first years of schooling, some

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 339-346). Umeå, Sweden: PME.

learners have been found to apply the rules that hold for natural numbers also to rational numbers, also in cases where this is inappropriate. These learners assume for example that addition and multiplication will lead to a larger result, while subtraction and division will lead to a smaller result. For example, they think that 5 * 0.32 will result in an outcome larger than 5 (e.g., Christou, 2015).

The third aspect is density. Contrary to natural numbers that have a discrete structure (each natural number has a successor number; after 5 comes 6, after 6 comes 7, …), rational numbers are densely ordered (between any two rational numbers are always infinitely many other numbers). This difference in structure of both types of numbers leads to frequently found mistakes such as thinking that there are no numbers between two pseudo-successive numbers (e.g., between 6.2 and 6.3 or between 2/4 and 3/4 (e.g., Merenluoto & Lehtinen, 2004).

A lot of research on learners' transition from natural to rational numbers has been described from a conceptual change perspective. This perspective argues that since children encounter natural numbers much more frequently than rational numbers in daily life and in the first years of instruction, they form an idea of what numbers are and how they should behave based on these first experiences with and knowledge of natural numbers. So, to overcome the natural number bias, a conceptual change revising these initial natural number based understandings is required. Conceptual change is considered to be not an all or nothing issue but a gradual and time-consuming process, with qualitatively different intermediate states between the initial and the correct understanding (e.g., Vosniadou, 2013).

While the natural number bias has generated a lot of research, empirical evidence on the development of learners' understanding, i.e. their conceptual change from a natural-number-based towards a mathematically more correct concept of a rational number, is scarce. Nonetheless, it is important to investigate in detail how this development occurs. If general patterns can be found, a learner's profile at a certain measurement point can predict its further development. From an educational perspective, such profiles can help teachers to provide effective instruction that is adapted to the specific knowledge and needs of their learners (Schneider & Hardy, 2013).

THE PRESENT STUDY

In the present study, we longitudinally followed the development of rational number understanding of upper elementary school learners in the crucial years of acquiring rational number understanding. The aim of this study is to make a theoretical contribution to the research field by characterizing in detail the intermediate states of learners' conceptual change from an initial natural number based concept of rational numbers towards a mathematically more correct one and by investigating whether these intermediate states have a consistent character across students or not.

METHOD

Participants were recruited from four elementary schools and 11 classrooms in Flanders, Belgium. In total 201 learners from fourth (*n* = 113) and fifth grade (*n* = 88) participated in this study and 50.2% of the participants were boys. Data were collected following the ethical guidelines of the university.

Learners' rational number knowledge was measured three times over the course of two school years, spanning a total time of 15 months: at the beginning (= Time 1, learners were in 4th and 5th grade), and end of Spring of the first school year (= Time 2, learners were in 4th and 5th grade) and at the end of Spring in the second school year (= Time 3, learners were by that time in 5th and 6th grade). According to the Flemish curriculum, learners should have acquired all knowledge about rational numbers that is measured in our test instrument at the end of the 6th grade.

To measure learners' rational number understanding, we used the Rational Number Knowledge Test (RNKT). This test was already used and validated in previous research investigating the relation between learners' spontaneous focusing on quantitative relations and their rational number understanding (Van Hoof, Degrande, et al., 2016). Table 1 displays examples of items for all three aspects.

Density	Size	Operations
How many numbers are there between 0.74 and 0.75?	Which is the larger number? 0.36 or 0.5	Is 21 : 0.7 bigger or smaller than 21?
What is the smallest possible fraction?	Order the following numbers from small to large 4/7 2/6 5/10	2/6 + 1/3 = …

Table 1: Examples of both fraction and decimal test items from the Rational Number Knowledge Test per aspect.

ANALYSIS

Data were analyzed using latent transition analysis (LTA). LTA is a longitudinal data analysis technique designed to detect unknown groups of participants and to model change in group membership over time through transition probabilities (Nylund, 2007). In our study, the groups can be interpreted as developmental states in learners' conceptual change, characterized by a specific answer pattern. Our LTA analyses were conducted in the statistical software Mplus version 7.2. We estimated the model parameters using the maximum likelihood estimation with robust standard errors. We restricted the number and nature of the states to be the same over the three measurement points, reducing the number of parameters to be estimated and making it possible to compare the results across measurement points (Schneider & Hardy, 2012). There were no missing data.

RESULTS

We opted for the six-state solution, based on the lowest AIC and BIC values and because it is the simplest model that still allows to differentiate between (un)successful conceptual change in all three aspects of rational number understanding.

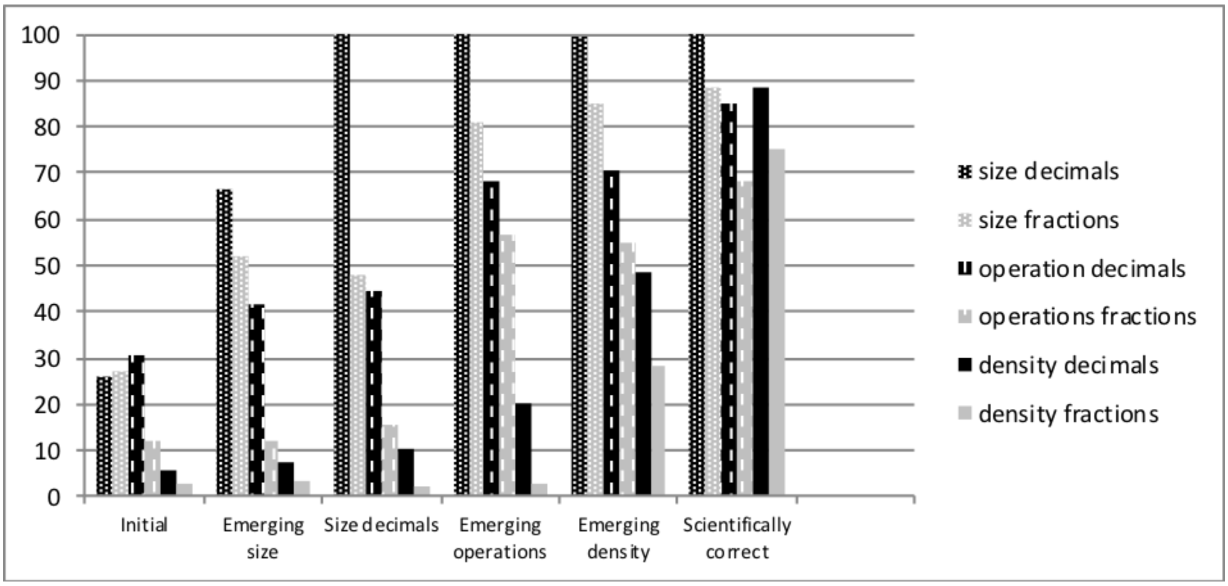

Figure 1: Accuracy levels (in %) on all aspects of the RNKT per state.

The mean accuracy scores on all subtests of the RNKT per state (see Figure 1) show, first, that learners in the '*Initial*' state are characterized by an initial natural number based understanding of rational numbers. They have a very low accuracy on all subtests, with a maximum subtest score of only 30.7% on decimal operation tasks. Second, learners in the '*Emerging size*' state have low accuracy scores on almost all subtests. Contrary to the 'Initial' state, they already have some understanding of the size of fractions (mean accuracy = 52.2%) and of decimals (mean accuracy = 66.7%). On all other subtests, they score below 50% accuracy. Third, learners in the '*Size decimals'* state are characterized by having a good understanding of the size of decimal numbers, performing almost perfectly on these items. Their mean scores on all other subtests are below 50%. Fourth, learners in the *'Emerging operations'* state have developed a good understanding of the aspect of operations. Moreover, they developed a good understanding of the size of fractions, but still have a natural number based idea of the structure of rational numbers. This is shown by their accuracy scores on decimal density tasks (mean accuracy = 20.4%), but especially on fraction density tasks (mean accuracy = 2.8%). Fifth, learners in the *'Emerging density'* state also have a good understanding of the size and operations aspect, but moreover developed already some understanding of the dense structure of rational numbers (mean accuracy decimal density tasks = 48.4% and mean accuracy fraction density tasks = 28.3%). Sixth, learners in the 'Mathematically more correct' state show a good understanding on all subtests, with a minimum subtest score of 68.1% on fraction operation tasks.

Interestingly, in every profile (except in the 'Initial' profile on size tasks) and in all three aspects of the natural number bias, learners score remarkably higher on the decimal than on the fraction tasks, indicating that understanding the size of decimals, operations with decimals, and decimals' density is easier to achieve than understanding these three aspects for the fraction counterpart.

As shown in Table 2, the number of learners in each state change over time. Both from Time 1 to Time 2 and from Time 2 to Time 3, a clear shift towards a better understanding of rational numbers is found: While half of the learners had an 'Initial' or 'Emerging size' state on Time 1, this dropped to only 13% at Time 3.

	Begin Spring Year 1			End Spring Year 1			End Spring Year 2		
	Grade4	Grade5	Total	Grade4	Grade5	Total	Grade5	Grade6	Total
Initial	42	4	46	16	2	18	7	2	9
Emerging size	40	12	52	26	13	39	13	4	17
Size decimals	22	21	43	55	23	78	9	3	12
Emerging operations	3	27	30	5	32	37	65	59	124
Emerging density	6	19	25	9	9	18	12	6	18
Mathematically more correct	0	5	5	2	9	11	7	14	21

Table 2: Number of learners in each state over time

As a second step in our LTA, we further characterized the general trend from the initial natural number based idea of a rational number ('Initial' state) to the mathematically more correct one ('Mathematically more correct' state). Therefore we had a look at the Latent Transition Probabilities (LTP) (see Table 3). Overall, the states stayed more stable from Time 1 to Time 2 compared to the stability over Time 2 to Time 3. This is not surprising given that there was less time between Time 1 and 2 than between Time 2 and 3. Further, the 'Emerging operations' state stands out as being the most stable state. Learners who are in this group at Time 1 have 89% chance of staying in this group at Time 2. In the same line, learners who have the 'Emerging operations' state at Time 2 have 94% chance of having the same state at Time 3. This suggests that once learners at the end of elementary education have developed a good understanding of the operations and size with rational numbers, they most often do not develop further and hence do not yet have a good understanding of the dense structure of rational numbers. If we take a look at the highest latent transition probabilities, as they indicate the transitions that occur most frequently, we see that from Time 1 to Time 2 learners from both the ‘Initial’ state and the ‘Emerging size’ state at Time 1 have a high chance of ending up in the ‘Size decimals’ at Time 2. This suggests that learners with an initial natural number based understanding of rational numbers at Time 1 first have an increased understanding of the size of decimal rational numbers. In the transition from Time 2 to Time 3, learners from both the ‘Emerging size’ and the ‘Size decimals’ state have a very high chance of ending up in the ‘Emerging operations’ state. This shows that learners who have an initial natural number based understanding of rational numbers, except for the size of decimal numbers, are very likely to develop an increased understanding of operations with rational numbers (both decimals and fractions) and the size of fractions in a next step, while they still have an initial natural number based understanding of the dense structure of rational numbers.

T1 \ T2	Initial	Emerging size	Size decimals	Emerging operations	Emerging density	Mathematically more correct
Initial	0.33	0.12	0.37	0.00	0.18	0.00
Emerging size	0.06	0.29	0.53	0.00	0.10	0.02
Size decimals	0.00	0.16	0.70	0.13	0.01	0.00
Emerging operations	0.00	0.00	0.00	0.89	0.04	0.07
Emerging density	0.00	0.12	0.06	0.17	0.45	0.20
Mathematically more correct	0.00	0.00	0.00	0.20	0.20	0.60
T2 \ T3	**Initial**	**Emerging size**	**Size decimals**	**Emerging operations**	**Emerging density**	**Mathematically more correct**
Initial	0.33	0.22	0.16	0.28	0.00	0.00
Emerging size	0.05	0.13	0.12	0.70	0.00	0.00
Size decimals	0.01	0.00	0.10	0.72	0.13	0.04
Emerging operations	0.00	0.00	0.00	0.94	0.00	0.06
Emerging density	0.00	0.00	0.00	0.10	0.37	0.53
Mathematically more correct	0.00	0.00	0.00	0.29	0.00	0.71

Table 3: Latent transition probabilities from Time 1 to Time 2 and from Time 2 to Time 3.

While a large group of learners who have a good understanding of operations first go through the early states of a good understanding of size, no such developmental path is found in the transition probabilities in the group of learners with (good) understanding of density. Very few learners of these qualitatively different group go through previous states. This suggests that the two states 'Emerging density' and 'Mathematically more correct' describe qualitatively different learners who understand density as opposed to the rest of the learners who do not see the dense structure of rational numbers.

CONCLUSION AND DISCUSSION

Our results add to our current theoretical understanding of the several different intermediate states going from learners' initial natural number based concept of rational numbers towards a mathematically more correct one. The finding that six different profiles can be distinguished in learners' rational number understanding shows that although all learners in our sample received similar rational number instruction, substantial individual differences could be found at every time point in learners' conceptual understanding of fractions and decimal numbers. It should be noted however that although we found several rational number understanding profiles and differences in learners' learning trajectories, we also found that the number of rational number profiles (n = 6) and transition paths (n = 56, of which only 11 were frequent) was much smaller than the number of participants in this study (n = 201). This indicates that learners' conceptual change from an initial to a more correct concept of rational numbers is constrained along certain patterns, and general developmental paths can be described. Based on the trends that we observed, we can characterize the development from the initial natural number based to the mathematically more correct idea of rational number as follows: First, learners develop a good understanding of the size of decimal numbers, followed by a good understanding of the size of fractions. Once learners have a good understanding of the size of rational numbers, they develop an understanding of operations with rational numbers (first decimals, then fractions). A qualitatively different group of learners also develops its understanding of the dense structure of rational numbers (first with decimals, then with fractions), without necessarily going through the profiles of good understanding of size and operations. These findings are consistent with the integrated theory of numerical development (Siegler, et al., 2011), which states that understanding the numerical sizes of fractions forms a crucial step in the understanding of fractions.

We continue with an important educational implication. From the theoretical background, we know that the process of conceptual change is gradual, time-consuming, and far from easy. Still, while instruction aimed at conceptual change in mathematics needs a lot of effort, research has shown that it can be successful under appropriate conditions. For example, curriculum designers should focus on a deep exploration and understanding of a few concepts instead of superficially covering a great amount of material (Vosniadou, 2013). The results of the present study show that a first step in learners' rational number understanding is a good understanding of the size of rational numbers. Therefore, we would suggest that instruction in the beginning explicitly focuses on the numerical size of rational numbers before introducing the more advanced content, such as operations with rational numbers.

Important to note is, finally, that the notion of natural number bias should not only be associated with its adverse effect of learners' prior knowledge on their further learning (Vamvakoussi, 2015). Using natural number knowledge acts as a facilitator too, namely in contexts that are compatible (congruent) with natural number knowledge. However, there is a need for a stronger awareness of the possible negative consequences of introducing rational numbers without an explicit attention for both the similarities and differences with natural numbers.

References

Christou, K. P. (2015). Natural number bias in operations with missing numbers. *ZDM Mathematics Education, 47*, 747-758.

Christou, K. P. & Vosniadou, S. (2009). Misinterpreting the use of literal symbols in Algebra. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis, *Proc. 33rd Conf. of the Int. Group for the Psychology of Mathematics Education*, (Vol 2, pp. 329-339), Thessaloniki, Greece: PME.

Gómez, D., Jiménez, A., Bobadilla, R., Reyes, C., & Dartnell, P. (2014). Exploring fraction comparison in school children. *Enero, 10,* 1-10.

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: Towards a systemic model of the processes of change. *Learning and Instruction, 14,* 519–534.

Nylund, K. L. (2007). *Latent transition analysis : Modeling extensions and an application to peer victimization* [Doctoral dissertation]. Retrieved from http://www.statmodel.com/download/Nylund%20dissertation%20Updated1.pdf

Obersteiner, A., Moll, G., Beitlich, J. T., Cui, C., Schmidt, M., Khmeliska, T., Reiss, K. (2014). Expert mathematicians' strategies for comparing the numerical value of fractions – Evidence from eye movements. In P., Liljedahl, S., Oesterle, C., Nicol & D. Allan (Eds.), *Proc. 38th Conf. of the Int. Group for the Psychology of Mathematics Education*, (Vol 4, pp. 338-345). Vancouver, Canada: PME.

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction, 28*, 64-72.

Schneider, M., & Hardy, I. (2013). Profiles of inconsistent knowledge in children's pathways of conceptual change. *Developmental Psychology, 49*, 1639-1649.

Siegler, R. S., Thompson, C., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*, 273-296.

Vamvakoussi, X. (2015). The development of rational number knowledge: Old topic, new insights. *Learning and Instruction, 37,* 50–55.

Van Hoof, J., Degrande, T., McMullen, J., Hannula-Sormunen, M., Lehtinen, E., Verschaffel, L., & Van Dooren, W. (2016). The relation between learners' spontaneous focusing on quantitative relations and their rational number knowledge. *Studia Psychologica, 58,* 156-170.

Vosniadou, S. (Ed.). (2013). *International handbook of research on conceptual change* (2nd ed.). New York, NY: Routledge.

EMERGENT PROPORTIONAL REASONING: SEARCHING FOR EARLY TRACES IN FOUR- TO FIVE-YEAR OLDS

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More and more research suggests that proportional reasoning emerges already at a very early age in childhood. The present study aimed to investigate these abilities in four- to five-year-old children. A five-item proportional reasoning task involving discrete quantities was administered in 389 children. On average children could solve about one item correctly. An error analysis allowed to identify five answer profiles, showing that four- to five-year olds already systematically and meaningfully attempt to make sense of one-to-many and many-to-many correspondences. Our findings advocate for the presence of precursors of the one-to-many correspondence notion in most four- to five-year-old children.

INTRODUCTION

The development of proportional reasoning has been widely studied. Lesh, Post and Behr (1988) view proportional reasoning as a pivotal concept of children's elementary school arithmetic and all that is to follow. Proportional reasoning is not only essential in the learning of numerous advanced mathematical topics such as algebra, geometry, statistics, or probability, people also encounter it in numerous daily life situations (e.g., if you need two cups of sugar in a recipe for four people, you'll need four cups of sugar in a recipe for eight people). The concept of proportion is traditionally thought to be hard to apprehend for children. Resnick and Singer (1993, p.107) even said it is "one of the stumbling blocks of the middle school curriculum".

Development of proportional reasoning

From a traditional Piagetian perspective the development of proportional reasoning is a rather late achievement. Piaget and Inhelder (1975) state that children cannot reason proportionally until the age of 11, because proportions are relations between quantities and, consequently, involve second-order reasoning or understanding the "relation between relations". They argue that children have to achieve a formal-operational level of cognitive functioning to be able to think about proportional relations (Inhelder & Piaget, 1958). Many studies support this theory about the development of proportional reasoning (e.g., Schwartz & Moore, 1998; Noelting, 1980). Along this line of reasoning, Piaget and colleagues consider additive reasoning as prior to multiplicative reasoning (Inhelder & Piaget, 1958). This claim is supported by several studies revealing that children misuse additive strategies to solve multiplicative problems, especially when they have not yet received thorough instruction on multiplicative reasoning (e.g., Kaput & West, 1994; Karplus et al., 1983; Noelting, 1980). However,

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other studies documented the inverse mistake, namely that of older children erroneously using multiplicative reasoning in additive problems or over-using both methods (Van Dooren et al., 2005, 2010). These latter studies imply that proportional reasoning may develop differently than suggested by Piaget and associates.

Other empirical findings also support the idea that some proportional reasoning abilities may already be present at a much earlier age than assumed in the Piagetian view, even though the full concept of proportional reasoning may only be acquired later.

Kouba (1989) presented six- to eight-year-old children with proportional reasoning problems such as “six cups and five marshmallows in each cup; how many marshmallows are there?”. Children paired objects and counted, creating one-to-many correspondences (e.g., to find the total number of marshmallows they pointed five times to each cup while counting). Other children used dealing or sharing strategies. Both strategies aim to establish a one-to-many correspondence. Kouba reported that 43% of the children used appropriate strategies This level of success is modest, possibly due to the use of difficult ratios (e.g., 7:1, 8:1 and 9:1) (Kouba, 1989). Becker (1993) and Carpenter et al. (1993) used easier ratios (e.g., 4:1, 3:1 and 2:1) and indeed obtained considerably higher success rates in five-year olds, respectively 81% and 71%.

Nunes and Bryant (2010) point out that many children already use the schema of one-to-many correspondence even before being taught about multiplication and division in school. Indeed, Frydman and Bryant (1988) revealed that some children already at the age of five can use one-to-many correspondences to create equivalent sets. In their task the children were asked to give equivalent sets (of sweets) to the recipients (dolls) but the units in the sets were of a different value. One doll liked her sweets in single units, whereas another doll liked her sweets in double units. The children had to construct equivalent sets by giving two single sweets to the first doll and one double sweet to the second doll. The researchers observed that children between five and seven years became progressively more competent in dealing with one-to-many correspondences.

Boyer et al. (2008; Boyer & Levine, 2012) presented six- to nine-year-old children with a proportional equivalence task. Children were asked to match a given mixture of juice and water presented in continuous quantities with a target mixture. Results indicated that they can match equal proportional mixtures The accuracy decreased as the scaling magnitude between the equivalent proportions increased (e.g., 1/4 and 2/8 is easier than 1/4 and 3/12).

Finally, Resnick and Singer (1993) also found that young children are able to reason about proportions. In one of their tasks five- to seven-year-old children had to feed fish of different lengths. Five- to six-year-old children (the earliest ages tested) could successfully perform proportional reasoning for discrete food (beads) and six- to seven-year-old children for continuous food (strips of ribbon). More specifically, they tended to give proportionally larger amounts of food to larger fishes.

The present study

The major goal of the present study was to investigate the proportional reasoning ability of four- to five-year-old Flemish children who were in their third year of kindergarten. We were especially interested in the presence of early traces of proportional reasoning and in the nature of their reasoning in case it is incorrect. The study fits in a larger research project in which we longitudinally follow a large sample of children through kindergarten and the first years of primary school, in order to map the development of their proportional reasoning as well as several other key early mathematical competencies.

METHOD

Participants

Seventeen schools (31 classrooms) were selected to represent the range of socio-economic backgrounds in Flanders. Parents of 410 four- to five-year-old children gave an informed consent to participate in the study. Data of 21 children (5,12%) are missing due to children moving away and changing schools, ultimately leading to a final group of 389 participants (200 boys, 189 girls) with complete data. At the moment of testing, their average age was five years, three months (range four years, two months to five years, eleven months).

Design and materials

Children were individually tested during three sessions of 30 minutes in the Autumn of 2017, when they were in the first semester of their third kindergarten year. Children completed a large number of tasks addressing a variety of core mathematical competencies that were targeted in the longitudinal project. In this paper, we focus on the assessment of children’s proportional reasoning ability, and more specifically on the tasks involving two discrete quantities, which were administered in one session that took approximately 10 minutes. The results for the two other tasks addressing proportional reasoning – involving continuous quantities or a mixture of discrete and continuous quantities – are not included in this paper due to length restrictions.

Our task was designed to measure children’s ability in reasoning about the proportional relation between two discrete quantities. An example item is shown in Figure 1. Mathematically speaking, in each item, children had to construct a set B equivalent to a comparison set A by putting the elements in set B in the same ratio as the elements in set A. In other words, the ratio on the left side of the expression has to be equivalent to the ratio on the right side in order to express the same, proportional functional relationship. Frydman and Bryant (1988), Kouba (1989) and Resnick and Singer (1993) used similar tasks in their studies. The two discrete quantities we used were puppets (wooden figures) and grapes (wooden green ovals). For example, in one of our items we asked the child: If one puppet got 2 grapes (set A: 1 puppet/2grapes) how many grapes would you have to give to these 4 puppets (set B: 4 puppets/number of grapes unknown) for it to be fair?

Instruction: "All puppets are equally hungry. If I give two grapes to this puppet, how many grapes do you have to give to these puppets for it to be fair? You can give the grapes to the puppets so that it's fair."

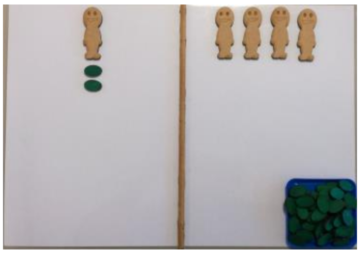


Figure 1: Instruction and material of item 1.

Three item features were taken into account in order to design the set of items for this task: the unknown quantity (puppets or grapes), the presence of a one-to-many correspondence or a many-to-many correspondence, and an increase or decrease of the involved numbers when going from A to B. Systematically varying these three features generated eight items (Figure 2). Ratios 1:2, 1:3 and 1:4 were used across the items.

Because of the shorter attention spans in young children and limited available test time, we only administered the five - presumably - easiest items. The three more difficult items will be administered in the next test waves of our longitudinal study. Item order was determined by presumed increasing difficulty based on a rational task analysis (Figure 2).

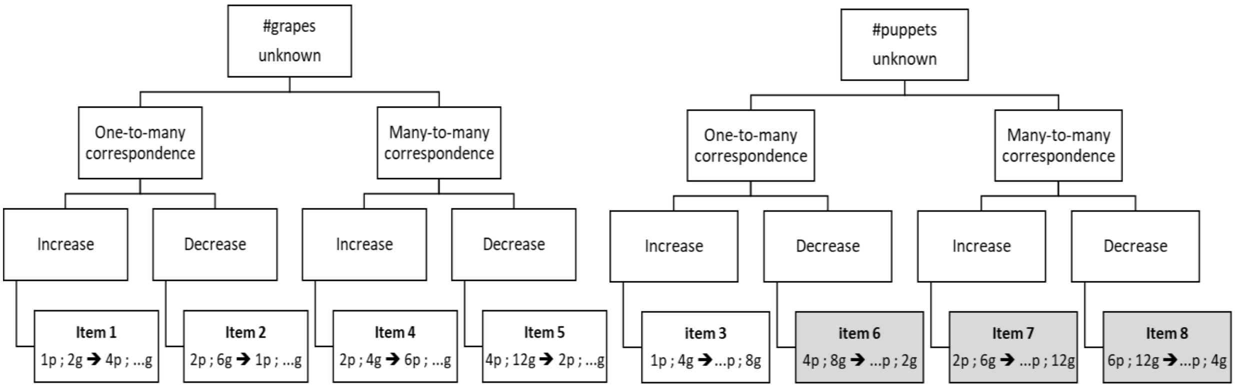


Figure 2: Items created for the proportional reasoning task with discrete quantities (p = puppets, g = grapes). The first 5 items (white boxes) were assessed in this study.

RESULTS

Not surprisingly, the task as a whole was difficult for most children at this age. On average children solved 1.07 (sd = 0.868) out of five items correctly. Two children (0.5%) got a maximum score of five, four children a score of four (1.0%), 13 children (3.3%) a score of three, 77 children a score of two (19.8%) and 196 children (50.4%) a score of one, leaving a substantial number of children (97, 24.9%) who solved none of the items correctly.

Item 1, a one-to-many correspondence situation with an increase and the number of grapes unknown, was the easiest one. Most children (276, 71.0%) gave the correct answer. Item 3, a one-to-many correspondence with an increase and the number of children unknown, turned out to be the second easiest item with 82 children (21.3%) answering it correctly. The other three items (item 2, 4 and 5) were far more difficult, with only 3% to 6% correct answers (Figure 3). Item 2, a one-to-many correspondence with a decrease and the number of grapes unknown, was a lot more difficult than item 1 and even more difficult than item 3. A decrease in the involved numbers apparently made it more difficult to reason correctly than anticipated in our rational task analysis.

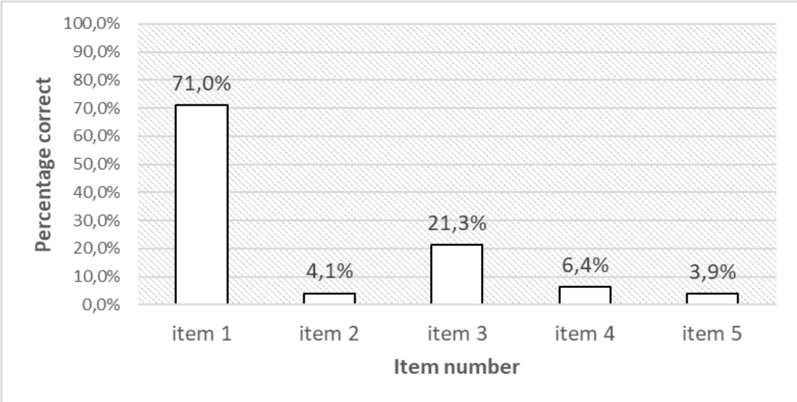


Figure 3: Percentage correct per item.

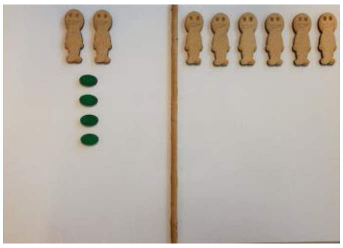
Answer categories

Given that we wanted to show the early traces of proportional reasoning in children, and that we particularly wanted to qualify the nature of these traces, we did not simply count the number of correct answers, but also carried out a systematic error analysis per item. Based on this systematic analysis, we came to the following major answer categories:

- "Correct answer": the correct number of grapes (or puppets) is put in set B.
- "One-to-one correspondence error": a child gives one grape to each puppet in set B.
- "Many-to-one correspondence error": the wrong application of the one-to-many correspondence. A child takes the number of the questioned variable in set A and gives that number to every entity in set B (e.g., 2 puppets in set A altogether have 4 grapes so I give 4 grapes to each puppet in set B).
- "Left-right correspondence error": a child gives the same number of the questioned variable in set A to the puppets in set B (e.g., altogether, in set A 2 puppets have 4 grapes, so I give 4 grapes to the group of puppets in set B too).
- "Additive error": children apply the difference between puppets and grapes in set A to set B (e.g., in set A two puppets have four grapes, so the difference between puppets and grapes is two. I have to give eight grapes to the puppets in set B to create the same difference). This error is frequently described in the proportional reasoning literature (e.g., Inhelder & Piaget, 1958).

In Table 1, these five answer categories are exemplified by means of Item 4. This table also shows how often each of them occurred.

Item 4



	Frequency	Percentage
Correct answer: 12 grapes	25	6.4%
One-to-one correspondence error: 6 grapes	49	12.6%
Many-to-one correspondence error: 24 grapes	157	40.4%
Left-right correspondence error: 4 grapes	28	7.2%
Additive error: 8 grapes	4	1.0%

Table 1: Examples of answer categories (for item 4) and number of occurrences.

Answer profiles

Based on this error analysis, answer profiles for the task as a whole were made. Children were assigned to a profile if they made the same type of answer on at least three of the five items.

Following this procedure, five profiles could be distinguished (Table 2). One hundred and twenty-nine (33.2%) children belong to the many-to-one correspondence answer profile, which makes it the most common profile. The one-to-one correspondence answer profile got assigned to 32 (8.2%) children. About five percent of the children belonged to the left-right correspondence answer profile and the correct answer profile. None of the children had an additive answer profile.

Answer profiles	Frequency	Percentage
Correct answer profile	19	4.9%
One-to-one correspondence answer profile	32	8.2%
Many-to-one correspondence answer profile	129	33.2%
Left-right correspondence answer profile	20	5.1%
Additive answer profile	0	0%

Table 2: Occurrence of the answer profiles.

CONCLUSION AND DISCUSSION

This study focused on finding early traces of the development of proportional reasoning, both in terms of children’s correct answers and in terms of the nature of their errors. Four- to five-year-old children were asked to solve five items on proportional reasoning with discrete quantities, long before they had received any instruction on proportional reasoning. Because of children’s young age and lack of instruction, the low average of about one item correct out of five items was not surprising.

A first remarkable result was that almost three-quarters of the children solved item 1 correctly. This item consists of a one-to-many correspondence being given, and an increase of children whose number of grapes is unknown. The high accuracy on this item suggests that many children at this early age do already have a notion of one-to-many correspondence, confirming the results of studies of Frydman and Bryant (1988) and Kouba (1989).

Despite the high accuracy on item 1 and the insight in one-to-many correspondence that many children showed, few children were able to solve one or more of the other items correctly. Several item characteristics such as the presence of a *many*-to-many correspondence (instead of one-to-many), or a *decrease* (instead of increase) of quantities apparently made it more difficult to reason correctly. Even the contextual variation (which has no mathematical implications as such) that the number of puppets (instead of grapes) is unknown, made the item considerably more difficult.

Because we also wanted to get a better view on the nature of the children's emergent proportional reasoning, a systematic analysis of their answers and answer profiles was conducted. This systematic exploration of wrong answers led to surprising results. Although Piaget and colleagues consider additive reasoning as emerging earlier than multiplicative reasoning (Inhelder & Piaget, 1958), we could not categorize any four- to five-year-old child in the additive answer profile. At the same time, we did find three other notable profiles, the most important of which was the many-to-one correspondence answer profile (in about one-third of the children). These children already have a notion of one-to-many correspondence but apply it incorrectly. This strategy can be seen as a meaningful initial attempt to make sense of the one-to-many and many-to-many correspondences.

In sum, the findings stated above advocate for the emergence of a notion of one-to-many correspondence in most four- to five-year-old children, which seems an important early step in the development of proportional reasoning. We were also able to show that many children exhibited specific types of reasoning that are erroneous, but that nevertheless can be considered as meaningful initial attempts to make sense of the one-to-many or many-to-many correspondence. These types of reasoning can be valuable starting points for instruction in the early years of primary education, where proportional reasoning may already be addressed in playful and meaningful contexts.

References

Becker, J. (1993). Young children's numerical use of number words: counting in many-to-one situations. *Developmental Psychology, 19*, 458-465.

Boyer, T.W., Levine, S.C., & Huttenlocher, J. (2008). Development of proportional reasoning: where young children go wrong. *Developmental Psychology, 44*, 1478-1490.

Boyer, T., & Levine, S.C. (2012). Child proportional scaling: Is 1/3 = 2/6 = 3/9 = 4/12? *Journal of experimental Child Psychology, 111*, 516-533.

Carpenter, T.P., Ansell, E., Franke, M.L., Fennema, E., & Weisbeck, L. (1993). Models of Problem Solving: A study of Kindergarten Children's Problem-Solving Processes. *Journal* *for Research in Mathematics Education, 24,* 428-441.

Frydman, O., & Bryant, P.E. (1988). Sharing and the understanding of number equivalence by young children. *Cognitive Development, 3*, 323-339.

Fujimura, N. (2001). Facilitating children's proportional reasoning: A model of reasoning processes and effects of intervention on strategy change. *Journal of Educational Psychology, 93*, 589-603.

Inhelder, B., & Piaget, J. (1958). The growth of logical thinking from childhood to adolescence. London: Routledge.

Kaput, J.J., & West, M.M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). New York: State University of New York Press.

Karplus, R., Pulos, S., & Stage E. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 45-89). New York: Academic Press.

Kouba, V. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education, 20*, 147-158.

Lesh, R., Post, T., & Behr, M. (1998). Proportional Reasoning. In Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: Lawrence Erlbaum Associates & National Council of Teachers of Mathematics.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part 1. Differentiation stages. *Educational studies in Mathematics, 11*, 217-253.

Nunes, T. & Bryant, P. (2010). Paper 4: Understanding relations and their graphical respresentation. In T. Nunes, P. Bryant, & A. Watson (Eds.), Key understanding in mathematics learning. (Accessed by the 14th of December, 2013 in http://www.nuffieldfoundation.org/sites/defaukt/files/P4.pdf)

Piaget, J. & Inhelder, B. (1975). *The origin of the idea of chance in children.* Norton, New York.

Resnick, L.B., & Singer, J.A. (1993). Protoquantitative origins of ratio reasoning. In T.P. Carpenter, E. Fennema, & T.A. Romberg (Eds.), *Rational numbers: An integration of research (*pp. 107-130). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schwartz, D.L., & Moore, J.L. (1998). On the role of mathematics in explaining the material world: Mental models for proportional reasoning. *Cognitive science, 22*, 471-516.

Singer-Freeman, K., & Goswami, U. (2001). Does half a pizza equal half a box of chocolates?: Proportional matching in an analogy tasks. *Cognitive Development, 16,* 811-829.

Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2005). Not everything is proportional: Effects of age and problem type on propensities for overgeneralization. *Cognition and Instruction, 23,* 57-86.

Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication … and back. The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction, 28*, 360-381.

COUNTING ON – LONG TERM EFFECTS OF AN EARLY INTERVENTION PROGRAMME

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This paper reports the long-term results of an intervention study with 134 six-year-old students from seven preschool-classes in northern Sweden to evaluate whether the Think, Reason and Count in Preschool-class programme (TRC) could prevent at-risk students from becoming low-performing students in mathematics. Whereas the pre-test score revealed that the intervention and the control group preformed equally, scores on the delayed follow-up-test in Grade 3 showed that the intervention group performed better than the control group and that at-risk students had closed the performance gap between themselves and their not-at-risk peers.

INTRODUCTION

Because mathematics performance prior to starting primary school has correlation to later mathematics performance (Duncan et al., 2007), low-performers in early mathematics education tend to remain low-performers if they do not receive appropriate support (Geary, 2013; Morgan, Farkas & Wu, 2009; 2011). Sayers, Andrews & Björklund Boistrup (2016) highlights evidence of that correlation and underscore that certain basic mathematics skills can predict later arithmetical competence and that these factors indicates a cross-culturally common phenomenon. In response, efforts to prevent future low performance in mathematics among low-performing students should be made before the students begin their formal education (Morgan et al, 2011). On a broader scale, to improve students' overall performance in mathematics in educational systems, the most effective way is to reduce the number of low-performing students in general (OECD, 2016).

In Sweden, the continued increase of low-performing students in mathematics demands for methods preventing at-risk students from developing into low-preforming students (OECD, 2016). In general, the Swedish school-system is obliged to support students at risk of not achieving the national education goals. As researchers have suggested, if key components in mathematics could be addressed early in education, then low-performing students might remedy or at least not fall further behind (Gersten, Jordan & Flojo, 2005). The most successful method of preventing further low-performance has been early interventions before students begin their formal education (McIntosh, 2008; Duncan et al., 2007; Nunes, Bryant, Sylva & Barros, 2011). Furthermore, a critical point in mathematical development is the transition from informal (i.e. preschool) to formal mathematical education (McIntosh, 2008). In the

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Swedish school-system this transition occurs in preschool-class (age 6). Studies have shown that using the *Think, Reason and Count in preschool-class* (TRC) programme, developed by Sterner, Helenius and Wallby (2014), can improve students' performance in mathematics (Sterner, 2015). As an added benefit, teachers who use the TRC programme become more aware of their students' mathematical development which helps them to identify at-risk students (Vennberg, 2015). In response to those findings, the purpose of our study was to examine whether the TRC programme affects students' long-term performance in mathematics?

BACKGROUND

Low-performing student and at-risk students

Swedish students' mathematics performance in the Programme for International Student Assessment (PISA) has gradually declined and Sweden's proportion of low-performing students has the highest increase in comparison to all other countries that participated in PISA 2003-2012 (OECD, 2016). PISA scores show that 28% of Swedish students score below Level 2–the baseline level of proficiency in mathematics–and are thus considered to be at great risk of being or becoming low-performers. Van Luit & Van de Rijt's (2005) standardised Early Numeracy Test, ENT is often used to identify students at risk of becoming low-performing students. ENT scores are grouped at five levels (i.e. Levels A-E), of which Levels D and E represent the first quartile of the lowest-performing students. Fuchs & Fuchs (2005) have suggested that students that perform in that quartile on standardised tests are at risk of experiencing difficulties with mathematics. In Sweden, attempts have been made to understand the declining scores in international evaluations (e.g., PISA and Trends in International Mathematics and Science Study) and why nearly 10% of students fail the Swedish national tests in mathematics administrated in Grade 3, 6, 9 or earn failing grades in mathematics, if not both (Swedish National Agency for Education, 2016). Because students can encounter obstacles in their mathematics development during the transition from informal to formal mathematical learning (McIntosh, 2008), it is imperative to focus on Swedish preschool-class, at the threshold of that transition.

The Swedish context

All children in the Sweden are required to begin attending compulsory school from the year they turn 7 and continue attending for 9 years. Preschool-class was introduced as a separate, optional form of schooling in the Swedish school system in 1998 (Swedish National Agency for Education, 2001) to bridge informal learning in preschool and the formal learning in compulsory school and to link these school forms differences in pedagogic, tradition and culture. Beginning in the Fall 2018, preschool-class, in its unique form, will be compulsory in Sweden and added as a separate form of schooling within the school-system (Prop. 2017/18:9). At the time of the study reported here, the preschool-class did not yet have a syllabus for mathematics. Nonetheless, the core content of the TRC programme aligns with the mathematics content of the syllabus in the Swedish National Curriculum that currently regulate and control the preschool-class assignment (Swedish National Agency for Education, 2016a).

Early interventions

Research in international settings has shown that early intervention in students' mathematics education can benefit their development in mathematics (McIntosh, 2008; Duncan et al 2007; Nunes et al, 2011). A review of early numeracy interventions (Mononen, Aunio, Koponen & Aro, 2014) has shown that many interventions can aid at-risk students, although clear evidence of capacity to close the performance gap to their not-at-risk peers remains lacking. Nevertheless, it is suggested that the longer the effect of the intervention, the greater the odds that it can prevent students' difficulties in mathematics. In this study reported here, we used the TRC programme (Sterner et al., 2014) as an intervention. The TRC programme aims at bringing forth the mathematical abilities that are necessary to learn and perform mathematics (e.g. Kilpatrick, Swafford & Findell, 2001; NCTM, 2001). The TRC programme is evidence based and builds upon structured activities in mathematics in which students, both individually and in groups, meet, use, develop and reason about different representations of numbers. The activities are to be implemented with a specific teaching model. The TRC programme draws upon research that has been proven to be effective for students-at-risk but is designed for regular teaching in all preschool-classes. Three design principles were combined to support teaching: structured activities with specific content, a modified circular teaching–learning structure based on the model of concrete–representational–abstract sequence of instruction and reasoning about the students' own documented work (Sterner & Helenius, 2015).

AIM

Studies from various countries have concluded that early interventions are crucial to prevent at-risk students from developing into low-performing students. However, results on whether such interventions have any positive long-term effects remain inconclusive. Therefore, we aim to investigate eventual longitudinal effects on students' mathematical performance in Grade 3 (i.e. at 9 years old) after an extensive whole-class intervention in preschool-class (i.e. at 6 years old). We sought to answer three questions. **RQ1**: Does implementing the TRC programme have any effect detectable in difference between pre- and post-test score? **RQ2:** To what extent does the TRC programme affect students' long-term performance as measured by the Swedish national tests in mathematics in Grade 3? **RQ3**: How does pre-test performance levels (A-E) in mathematics prior to formal schooling influence the mathematics scores on such national test in Grade 3?

METHOD

Participants and Procedures

The research design comprised a pre-test, an intervention, a post-test and a delayed follow up-test. The sample comprised 149 students from seven preschool-classes in four schools in a midsize town in northern Sweden. Fifteen students who did not participate in all three tests were excluded from the analyses. Hence, the data analyses included 134 students. The schools were chosen with the help of the municipality administration in order to achieve an equivalent selection based on socio-economic backgrounds factors. The classes were divided into an intervention group of four classes (n=79) and a control group of three classes (n=55). The mathematics teaching and content in the intervention group shifted into that of the TRC programme only, and the teachers attended a teaching development programme focused on the underlying mathematical theories and ideas of the TRC programme.

Measures

We assessed the students' mathematical performance at three time points: prior to the intervention in November in preschool-class (pre-test, T1), immediately after the intervention in June in preschool-class, (post-test, T2) and 3 years after the intervention in March, when the students were in Grade 3 (delayed follow-up-test, T3). The students' performance was assessed T1 and T2 by using the ENT (Van Luit et al., 2005) which is used in many countries to identify at-risk students and in studies comparing different countries (Aunio & Niemivirta, 2010). The ENT consists of two comparable parts, called A and B; ENT A was used as a pre-test (T1) and ENT B as a post-test (T2). The criterion to identify students at risk of becoming low-performers was a score in the 25th percentile (i.e. Levels D and E) at T1. To determine whether the intervention helped to increase the students' performance in mathematics in the long-term, we compered scores at T1 and T2 to scores on the Swedish national tests in mathematics in Grade 3 (T3). The Swedish national test in mathematics is compulsory for students in Grade 3, and its purposes are several. It is not only to support an equal assessment of students' knowledge but also to provide a basis for analysing the extent to which knowledge requirements are met among schools across the nation. Although the test does not give a complete picture of students' knowledge in mathematics, because not all areas in the syllabus are tested, the design is based on the curriculum, the syllabus in mathematics and the related knowledge requirements that describe the lowest acceptable level (PRIM-gruppen, 2016; Swedish National Agency for Education, 2016b).

Method of analysis

An initial control analysis revealed that the exclusion of students that did not complete all three tests did not affect the comparability of the two groups. To determine whether the intervention had any detectable effect, a difference score was calculated as the progression between the pre- and post-test (T1 and T2). An independent t-test was conducted to measure whether any significant difference emerged in the progress of the two groups. An additional t-test was conducted to control for any difference in national test scores (T3) between the groups. Next, the groups were divided into four sub-groups according to the levels of their performance in mathematics at T1; sub-group *Low* comprised all at-risk students (i.e. Levels D and E). A graph showing the average progress across the tests for each subgroup was constructed, which allowed us to investigate the group of at-risk students in greater depth. Because the number of at-risk students were small (*n*=10 and *n*=18) the difference was analysed by a non-parametric test, the Wilcoxon signed-rank test, which is considered to be robust with small sample sizes and eventual differences in variance. All statistical analyses were performed with the Statistical Package for the Social Sciences version 24.0.

RESULTS

Result of the initial *t*-test revealed a significant difference in progression between T1 and T2 in favour of the intervention group (t = -2.098, df = 132, p = .038, g = .368). Results of the subsequent *t*-test indicate a significant difference between the two groups on the national test score as well (t = -2.113, df = 88.141, p = .037, g = .397), as shown in Figure 1a.

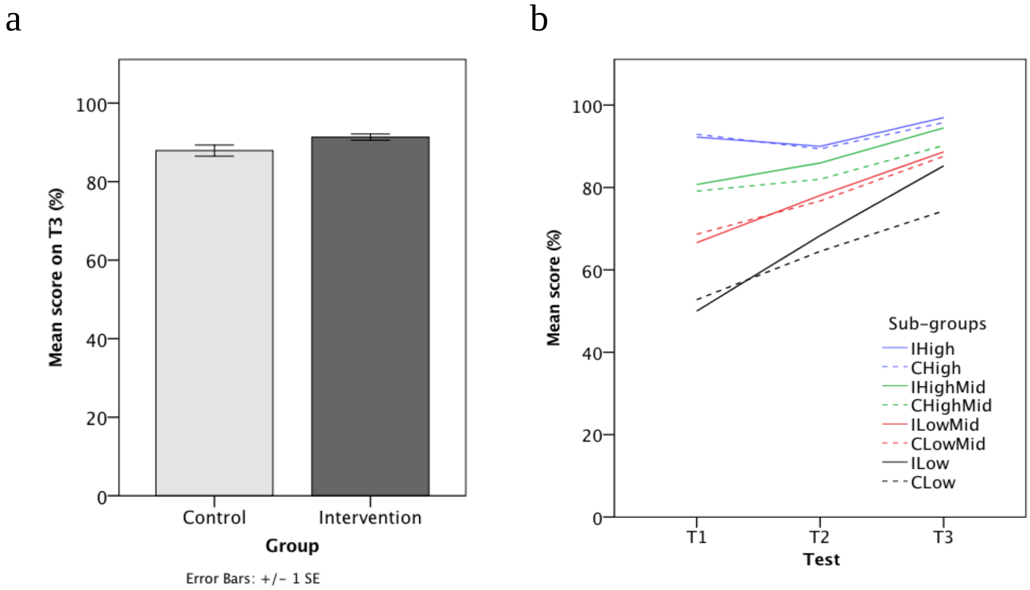

Figure 1. (a) Mean scores for the national tests, T3. (b) Progress in scores on the three tests for each sub-group. *Note*. ILow=Intervention group, low-performers; CLow=Control group, low-performers; etc.

We more closely investigated the difference in T3 by dividing the two groups into sub-groups based on the performance levels used to classify at-risk students in T1. A closer look at these sub-groups shows that the main difference between the groups in T3 is related to the at-risk students (Figure 1b). Results of a non-parametric test showed that the only significant difference in T3 scores occurred between the sub-groups with at-risk students–that is, ILow and CLow (Z = -2.376, p = .016). Another compelling result is that the at-risk students in the intervention group caught up to their not-at-risk peers and partly closed the performance gap, whereas the distance between the lowest quartile and the second-lowest quartile was constant over time in the control group. Those results indicate that the TRC programme works as intended and seems to have a lasting effect on the students' performance in mathematics, as measured by scores on the national test in Grade 3. Clearly, such improvement among at-risk students does not negatively affect students who perform at higher levels. In fact, some indications (Figure 1b) suggest that even students at the higher performance levels might benefit from the intervention as well as their at-risk peers.

DISCUSSION

We examined whether the early and extensive TRC programme, *Think, Reason, and Count in preschool-class,* exerted any detectable long-term effect on Swedish students' performance in mathematics as measured by the Swedish national test 3 years after the intervention. As in a previous study (Sterner, 2015), the intervention affected students' performance and short-term progress, and the progress indicated by the difference in pre- and post-test scores was greater in the intervention group. That result could have stemmed from the fact that at-risk students were identified earlier (Vennberg, 2015) and that additional support was provided at a critical transitional point in the development of their mathematical thinking (McIntosh, 2008). Attention given to mathematical reasoning, which is a part of the TRC programme, could also have exerted an effect. Indeed, the ability to reason mathematically is one of the core competences of mathematics (Kilpatrick et al., 2001; NCTM, 2001), and practicing that ability can be the key component essential for mathematical progression (Gersten et al., 2005; Norqvist, 2016). This could be a key component in mathematics that could be addressed early so that low-performing students might remedy or at least not fall further behind their not-at-risk peers. The intervention group also performed better than the control group from the long-term perspective, as measured by scores on the national test in Grade 3. That result could derive from a factor other than the intervention; however, a close look at the data did not indicate any distinct decline or increase in individual class performance, which could signify an extraordinary poor or excellent learning environment. The preschool-classes were also chosen to ensure a similar representation of socio-economic background factors in the two groups, factors which did not change during the 3 years between T2 and T3. Although countless other factors could have influenced the students, but there is an indication that the TRC programme seems to have had a long-term effect on the participating students' national test scores. Furthermore, the data suggest that the TRC programme affected at-risk students in the long-term. In the intervention group, such students had caught up with their not-at-risk peers, while such progress had not occurred in the control group. At-risk students in the control group remained low-performers in Grade 3, which confirms Duncan's (2007) and Sayers's (2016) conclusion that the mathematics-related knowledge students bring to school determines their later performance and grades in mathematics. Regarding practice, our findings imply that teachers and principals need to dedicate time to implement the TRC programme in whole-class mathematics work in preschool-class. Such action could be the successful prevention before formal education to prevent future low-performing students as suggested by the results of several earlier studies (e.g., Duncan et al., 2007; McIntosh, 2008; Morgan et al, 2011; Nunes, et al., 2011). Additionally, since the longer the effect of the intervention lasts the greater the odds that it prevents students from facing difficulties in mathematics, the TRC programme could improve the overall performance in mathematics in educational systems, because the most effective way to do that is to reduce the number of low-performing students (OECD, 2016).

CONCLUSION

Our findings indicate that consciously, systematically application of the TRC programme in preschool-class can improve students' long-term performance in mathematics. In particular, participation in the TRC programme improved the possibilities for at-risk students to perform at the same level as their not-at-risk peers and such progress seems to have lasted. More detailed analyses of the scores of at-risk students on Sweden's national test in mathematics can elaborate the differences in performance between and within the groups of at-risk and not-at-risk students, as well as identify factors that help at-risk students to avoid becoming low-performers later in their mathematics development.

References

Aunio, P. & Niemivirta, M. (2010). Predicting children's mathematical performance in grade one by early numeracy. *Learning and Individual Differences*, 20(5), 427–435.

Duncan, G., Dowsett, C., Claessens, A., Magnuson, K., Huston, A., et al. (2007). School readiness and later achievement. *Developmental Psychology*, 43(6), 1428–46.

Fuchs, D., Fuchs, L.S. (2005). Responivness to-intervention: A blueprint for practitioners, policymakers, and parents. *Teaching Exceptional Children*, 38(1), 57-61.

Geary, D. (2013). Early foundations for mathematics learning and their relations to learning disabilities'. *Current Directions in Psychological Science*, 22(1), 23–27.

Gersten, R., Jordan, N., & Flojo, J. (2005). Early identification and intervention for students with mathematics difficulties. *Journal of Learning Disabilities*, 38, 293-304.

Kilpatrick, J., Swafford, J., Findell, B. (Eds). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academic Press.

McIntosh, A. (2008). *Förstå och använda tal- en handbok.* Göteborgs Universitet: NCM. [In Swedish]

Mononen, R., Aunio, P., Koponen, P., Aro, M. (2014). A review of early numeracy interventions for children at risk in mathematics. *International Journal Early Childhood Special Education.* 6(1), 25-54.

Morgan, P.L., Farkas, G. & Wu, Q. (2009). Five-year growth trajectories of kindergarten children with learning difficulties in mathematics. *Journal of Learning Disabilities*, 42, 306–321.

Morgan, P.L., Farkas, G. & Wu, Q. (2011). Kindergarten children's growth trajectories in reading and mathematics: Who falls increasingly behind? *Journal of Learning Disabilities*, 44 (5), 472-488.

National Council of Teachers of Mathematics, NCTM. (2011). *Principals and standards for school mathematics.* Reston, VA: Author.

Norqvist, M. (2016). *On mathematical reasoning: being told or finding out* (Doctoral thesis). Umeå: Umeå Universitet.

Nunes, T., Bryant, P., Sylva, K. & Barros, R. (2011). The relative importance of two different mathematical abilities to mathematical achievement. *British Journal of Educational Psychology*, 82, 136-156.

OECD. (2016). Low-performing students: Why they fall behind and how to help them succeed. Paris: OECD publishing.

Prop. 2017/18:9. *Skolstart vid sex års ålder*. Stockholm: Utbildningsdepartementet. [In Swedish]

Sayers, J., Andrews, P., Björklund Boistrup, L. (2016). The Role of Conceptual Subitising in the Department of Foundational Number Sense. In Meany, T., Helenius, O., Johansson, M., Lange, T., Wennberg, A. (Eds). *Mathematics Education in the Early Years*. (pp. 371-394). Cham. Switzerland: Springer.

Sterner, G. (2015). *Tal, resonemang och representationer- en interventionsstudie i matematik i förskoleklass* (Licentiate thesis). Göteborg: Göteborgs Universitet. [In Swedish]

Sterner, G. & Helenius, O. (2015). Number by reasoning and representations– the design and theory of an intervention program for preschool class in Sweden. In Helenius, O., Engström, A., Meaney, T., Nilsson, P., Norén, E., Sayer J., Österholm, M., (Eds), Development of mathematics teaching: design, scale, effects. *Proceedings from MADIF9: The 9th Swedish mathematics education research seminar*. Linköping: SMDF.

Sterner, G., Helenius, O & Wallby, K. (2014). *Tänka, resonera och räkna i förskoleklass* [Think, reason and count in preschool-class]. Göteborg: NCM. [In Swedish]

Swedish National Agency for Education. (2001). *Att bygga en ny skolform för 6-åringar. Om integrationen förskoleklass, grundskola och fritidshem.* Skolverkets rapport nr. 201. Stockholm: Liber. [In Swedish]

Swedish National Agency for Education. (2016). *Statistik och utvärdering.* Retrieved 2017-10-07, http://skolverket.se/statistik-och-utvardering [In Swedish]

Swedish National Agency for Education. (2016a). *Lgr11, Curriculum for the compulsory school system, the preschool-class and the recreation centre 2011* (Revised 2016). Stockholm: Skolverket: Fritze.

Swedish National Agency for Education. (2016b). *Nationellt prov i matematik i årskurs 3.* Retrieved 2017-10-20, https://www.skolverket.se/bedomning/nationella-prov/alla- nationella-prov-i-skolan/arskurs-3/nationellt-prov-i-matematik-i-arskurs-3 [In Swedish]

PRIM-gruppen. (2016). *Att konstruera ett nationellt prov i matematik.* Retrieved 2017-10-20 http://su.se/primgruppen/matematik. Stockholms Universitet. [In Swedish]

Van Luit, J. & Van de Rijt, B. (2005). *The Early Numeracy Test*. Doetinchem, The Netherlands: Graviant.

Vennberg, H. (2015). *Förskoleklass- ett år att räkna med: förskoleklasslärares möjligheter att följa och analysera elevernas kunskapsutveckling i matematik* (Licentiate thesis). Umeå: Umeå Universitet. [In Swedish]

RITUALISED AND EXPLORATORY GRAPHING ROUTINES IN MATHEMATICAL MODELLING: THE *DIGOXIN* TASK

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The project we report from in this paper explores whether and how, biology students' competence and confidence in – as well as appreciation for – mathematics in their discipline can be improved through greater integration of mathematics and biology in their study programme. Here, we examine biology students' mathematical discourse as they engage with a biology-related Mathematical Modelling (MM) activity, the Digoxin task. We report commognitive analyses of data collected during sessions in which biology-related MM activities were introduced to undergraduate biology students (four sessions with 12 first-semester students). We focus on the interplay between students' ritualised and exploratory engagement with the activity, particularly concerning graphing routines, and consider pedagogical implications.

Much of university mathematics teaching is taking place in the context of study programmes not aimed primarily at mathematics students, but rather at students in other fields of study such as engineering or the natural sciences. Still, although research into the mathematical needs of students in such programmes has been conducted for quite some time (e.g., Kent & Noss, 2003), in much university mathematics education research the specialisms of participating students is little more than the backdrop of the studies (Nardi, 2016; Biza, Giraldo, Hochmuth, Khakbaz & Rasmussen, 2016). The present paper belongs to a small but growing number of studies investigating the mathematical education of non-mathematics specialists, and is concerned particularly with mathematical modelling (MM) for biology students.

The increased importance of mathematical methods in biology is placing new demands on biology education, causing some researchers to suggest a greater integration of mathematics and biology in the curriculum (Brewer & Smith, 2010; Steen, 2005). Research on the use of MM in university biology education (e.g., Chiel, McManus & Shaw, 2010) indicates that engagement with MM activities can contribute to more positive attitudes towards, and self-perceived competence in, both biology and mathematics. In this paper, we report parts of a collaborative project between two Norwegian national Centres for Excellence in Higher Education in which we investigate patterns in Biology students' MM activity, known in the language of the discursive perspective we espouse in our analysis – the theory of commognition (Sfard, 2008) – as *ritualized* or *exploratory routines*.

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Specifically, our aim in this paper is to investigate the interplay of *ritualized and exploratory participation* in the students' MM activity in the context of a particular task, the *Digoxin* task. First, we introduce the theoretical underpinnings of our study.

THE RITUAL-EXPLORATION DYAD IN COMMOGNITIVE THEORY

According to the commognitive perspective 'it is by reproducing familiar communicational moves in appropriate new situations that we become skillful discursants and develop a sense of meaningfulness of our actions' (Sfard, 2008, p. 195). Communication through written or spoken language, and manipulation of physical objects and artefacts, are the main means to the discursive ends of teaching and learning. Consequently, what distinguishes a discourse is a community's *word use*, *visual mediators*, *endorsed narratives* and *routines* (ibid., p. 133–135). Specifically, routines are repetitive patterns in the discourse. Sfard defines three types of routines: *explorations*, *deeds* and *rituals*. A *routine* is called an *exploration* if its aim is the "production of an *endorsed narrative*" (p. 298). *Deeds* are *routines* that involve practical action, resulting in change in objects, either primary or discursive (p. 241). Additionally, there are *routines* that "begin their life as neither *deeds* nor *explorations* but as *rituals*, that is, as sequences of discursive actions whose primary goal […] is neither the production of an endorsed narrative nor a change in objects, but creating and sustaining a bond with other people" (p. 241). For Sfard, *rituals* are a "natural, mostly inevitable, stage in routine development" (p. 245). Hence, to some extent, *rituals* function as predecessors of *explorations*. Heyd-Metzuyanim, Tabach and Nachlieli (2015), have provided further means of distinguishing *ritual* and *exploration*. They write, for instance, that where "ritual participation is often focused on manipulation of mathematical symbols […] explorative participation talks about mathematical symbols in an *objectified* way" (ibid, p. 548), and where ritual participation emphasizes human action, "explorative participation concentrates more on mathematical objects and narratives (or truths) as existing in the world, *alienated* from any human action" (p. 549).

METHODOLOGY

Context and participants of the study

The research design of the larger project comprises cycles of developmental activity (planning, implementation, reflection, feedback) which are theoretically informed, contribute to the emergence of theory and take place in a partnership between teachers (in this case, a university mathematician) and mathematics education researchers (Goodchild, Fuglestad & Jaworski, 2013). The project, which is ongoing, is conducted at a well-regarded Norwegian university where biology students take a compulsory mathematics course in the first semester of their university studies. This is a generalist course catering to students from about twenty different natural science programmes, providing few opportunities for focusing on issues specific to biology.

The main part of the first cycle of the project, from which we report here, consisted of four three-hour sessions with a group of 12 volunteering students, concurrent with their mandatory first-semester mathematics course. The research team comprised three mathematics education researchers and one mathematician. All sessions were taught by a research mathematician with extensive experience of MM and consisted of brief lectures introducing various aspects of MM, followed by group work on MM tasks set in a biological context. The teaching was conducted in English, but most student group work and student contributions to group discussions were in Norwegian. There was little intervention from the research team in the students' work on the tasks.

Data collection and analysis

All whole-group and small-group activity during the sessions was video and audio recorded, and then transcribed. In addition, much of the written material produced by the students was also collected. The first author, who was present at all four sessions, then produced condensed descriptive accounts of these since working with such condensed accounts makes potential patterns in the activity of the discursants more easily discernible. Both authors then examined these accounts, cataloguing episodes where one or more students focused on one particular routine, for instance, graph construction. We also looked for signs of ritual or exploratory engagement, making use of the general characterizations given by Sfard (2008), as well as the distinctions provided by Heyd-Metzuyanim, Tabach and Nachlieli (2016). For instance, we looked for signs of objectified discourse: whether the students talked about mathematics as performing operations on symbols, or in terms of properties of mathematical objects. We also tried to discern the aims of the routine use: could we, for example, see signs in the students' discourse of engaging with the routine out of a sense of its relevance for solving the problem, or was it performed mainly because the students were expected to do so (out of obligation to the research team)? Further signs of ritualized routine use could be, for instance, a strong reliance on external sources for substantiation, rigid rule following and mimicking previously encountered routines without regard for relevance to the problem at hand. We also identified broader themes around which to organize the analysis. Next, the first author returned to the raw data in order to produce preliminary analytical accounts of the sessions. These accounts included data excerpts of all episodes relevant to the chosen themes, together with a preliminary analysis, including, for instance, the previously produced classification of ritualized and exploratory routine use. Finally, we selected a smaller number of episodes representative of the themes. In this paper, we will focus on one episode that illustrates the interplay of students' ritualized and exploratory routine use, and how ritualized mathematical routine use can stand in the way of a more exploratory engagement with the larger MM task. To this end, we draw on data from the first part of the third session, where the students were working on a task concerned with modelling the decay in the body of *Digoxin*, a drug used to treat heart disease.

The *Digoxin* task

The *Digoxin* task consisted of three parts: "(**a**) For an initial dosage of 0.5mg in the bloodstream, [a table given to the students] shows the amount of digoxin a_n remaining in the bloodstream of a particular patient after n days, together with the change Δa_n each day. Plot Δa_n versus a_n and explore the graph. Suggest a simple model based on a difference equation of the form $\Delta a_n = k_3 a_n$, where k_3 is a positive constant. What is your choice of k_3? (**b**) Now our objective is to consider the decay of digoxin in the blood stream to prescribe a dosage that keeps the concentration between acceptable levels so that it is both safe and effective. Design a simple linear model describing the following scenario: we prescribe a daily drug dosage of 0.1mg and know that half the digoxin remains in the system in the end of each dosage period. (**c**) Consider three different options where the initial one-time dose of medicine received by the patient is a_0 = 0.1mg, 0.2mg or 0.3mg. What are your conclusions? What would you recommend if you were this patient's doctor?"

For more detail on how this task fits within the broader set of activities in the sessions, and an analysis of the development of the mathematical discourse of one of the groups (the group labelled B in this paper) see (Viirman & Nardi, 2017). In what follows, we will look more closely at the work of two of the groups (labelled B and C) as they engage with part (a) of the task. In this part, after plotting the data, the students were expected to identify a linear relationship between the change and the amount of digoxin remaining, and then use the graph to estimate the proportionality constant. We note that there is a misprint in the task formulation: the constant k_3 will be negative, not positive, a fact that was highly confusing for the students, as we will see.

EPISODES FROM THE *DIGOXIN* DATA: GRAPHING ROUTINES

The *Digoxin* task was intended as a continuation of a task from the previous session, concerning the growth of yeast in a petri dish. As reported in (Viirman & Nardi, 2017) Group B (all three groups, in fact) experienced great difficulty with that task due to ritualized engagement with a previously established graphing routine, using time as the independent variable. Hence, in the *Digoxin* task, it was clearly stated that the students should plot change against amount, and both groups B and C acknowledged this when beginning work on the task. Still, there was profound disagreement in both groups concerning how to construct the graphs, and in what follows, we will trace how they handled this disagreement, and how different forms of ritualized and exploratory routine use influenced their success with the task.

Having agreed that the graph should plot Δa_n against a_n, the students in Group B all set out drawing their own graphs. Soon, however, the established graphing routine starts getting in the way:

B2:	Should n go on the x- or y-axis? (…)
B4:	But you don't have to. It's just Δa_n against a_n. Not n.
B2:	Isn't it good to include it?

B4: You can't have three. You can't have that and that [indicating Δa_n and a_n].

B3: You could plot them in the same. It will just be two different lines. Then we will see them against one another.

B2: Yes, compare them. (…)

B3: Since one is only plus and the other only minus.

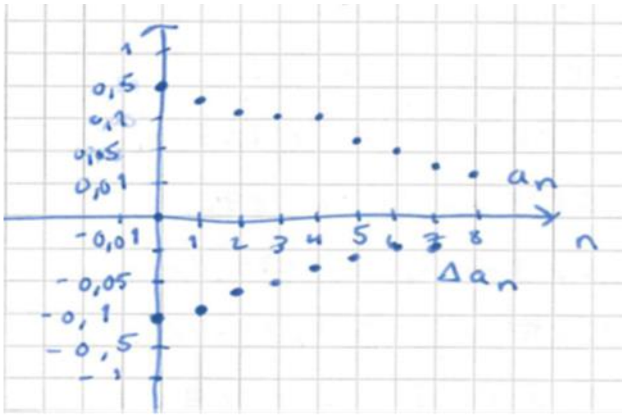

Figure 1: Graph by student B2.

After more arguing, B2 and B3 decide to do a plot with time on the *x*-axis, and both Δa_n and a_n on the *y*-axis (see Figure 1). B1 is unconvinced, however: "Isn't Δa_n just a function of a_n?" B3, continuing to argue her point, says that a_n and Δa_n will not interact, since one is positive and the other negative, meaning that they will end up in different parts of the coordinate system.

Here, we can see how the established graphing routine hinders a more exploratory engagement with the task. At the same time, the students are not just blindly mimicking what they are used to doing. Instead, they make a serious attempt to fit what is being asked of them within their established routine. This leads to reasoning that is creative but mathematically unsound. Still, their engagement with the graphing routine suggests a limited grasp of the connection between the graph and the model, and the question of how to estimate the value of k_3 from this type of graph is never raised. Instead, they again engage in an inventive but mathematically unsound routine, doing pointwise estimates directly from the table of data by dividing Δa_n by a_n for each consecutive value of *n*. Since the data displays an almost perfectly linear pattern, this in fact gives a good estimate of the value of k_3. They do not appear particularly confident in their work, however. When realizing that the task asks for a positive constant, they do not attempt to justify their method, not even when B1, who had been working independently, building on her observation that Δa_n was a function of a_n, presents her graph, which shows a linear relationship between Δa_n and a_n with the corresponding slope. Rather, they try to come up with a way of adjusting the calculations so as to make the result positive, and when this fails they just opt for dropping the minus sign, in order to align their answer with what they see as expected.

In their work, Group 3 display a similar disagreement regarding an acceptable way of constructing the graph. However, where B1 makes no attempt at convincing her fellow group members of her method, students C1 and C2 argue their respective positions strongly. (Two more students, C3 and C4, were also present, but much less active.) C2 quickly abandons the idea of involving time in the graph, whereas C1 persistently argues for including it:

C2: But it says plot difference in a_n versus a_n. To plot those two against each other.

C1: Yes, but why have they given us *n*, then?

C2: I don't know! (…) There have been some trick questions included before.

C1: I think that we're supposed to use it.

However, C1 also states that "you should plot a_n against Δa_n", suggesting that she is facing similar difficulties to B2, B3 and B4 with fitting this notion to her established graphing routine. Slightly later, she does indeed suggest the same type of graph as the one they constructed, with time on the *x*-axis, and both a_n and Δa_n on the *y*-axis.

Having drawn their respective graphs, the students turn to estimating the constant k_3. Like Group B, they do not consider using the graph for this purpose. Instead, C1 suggests using the model they have been provided and solving for k_3, again similarly to Group B. In doing so, they make an algebraic error, meaning that the values they calculate are in fact reciprocals of k_3. When realizing that the task prescribes that the constant should be positive, at least C2 reacts quite differently to Group B:

C2: No. It can't be, because that [the amount of Digoxin] is supposed to decrease, and it cannot be negative.

C4: Yes, but the graph can't look like this, because here [pointing at the graph] is the amount, the amount in the blood after this time?

C2: Yes, this [points at the x-axis] is the amount. And this [points at the graph] is the difference in amount. There's less and less difference in amount.

C4: But where is the time?

C2: It's not time. It's milligrams. This [points at the x-axis] is milligrams, and this [points at the y-axis] is difference in milligrams.

C1: I think you should have *n* on the x-axis.

C4: But the difference in milligrams changes drastically, so there shouldn't be a linear graph here. It should go like this almost [indicating with his hand a curve decreasing slowly at first and then more and more rapidly].

C2: No. This just shows that it decreases, in a way. I think.

C4: But it doesn't!

C2: Yes, it does! I have plotted it! I have plotted it here, it's all there. (…) [points at the formula], [the change] will always be negative because it always decreases. And then that [the amount] must be negative to get a positive number. So, for k_3 to be positive, he would have to have negative milligrams in the blood. And that is impossible, so the answer must be negative.

Here, C2 engages with the task in an exploratory manner. Notably, when authority (the task formulation) is in disagreement with his conclusions (the constant is negative), he resorts to substantiations both from the real-world ("negative milligram in the blood"), and from the mathematical characteristics of the model (the graph, the formula), to argue the validity of his point. At the same time, he apparently does not consider the graph as anything more than an illustration of the data: "This just shows that it decreases, in a way." He does not attempt to use it to estimate the value of the constant, or to judge whether the calculated value is reasonable. We also note how C1 and C4 persist in arguing for using time as the independent variable.

In the whole-class follow-up, the discrepancy between how the lecturer and the students viewed the graph as a discursive tool becomes apparent. Upon noticing the graph that C3 drew, the lecturer asks him to explain how they intended to use it. C3 responds that they could see that there was a process of decay. The lecturer asks him to elaborate further: "What else? The shape of the graph, say?" C3 responds that it was linear, and the lecturer then asks about the slope. From this exchange, we can detect a difference in the way the lecturer and C3 engage with the graph. For the lecturer it is a tool that is key to engaging with the task; C3 sees it simply as an illustration.

TOWARDS A PEDAGOGY OF HIGHLIGHTING CONNECTIONS

In this paper, we have examined a particular episode of two groups of biology students engaging with a MM task, showing how different forms of engagement influenced their success in solving the task. The difficulties resulting from a ritualized engagement with graphing routines (Viirman & Nardi, 2017) were present also here, but we could also see development, with some students (B1 and C2 in particular) being able to engage with the already established routines in an exploratory manner, adapting them to the task at hand. In the case of Group B, this led to the group being able to at least partially solve the task (see also Viirman & Nardi, 2017), whereas Group C were not able to resolve the conflict between ritualized and exploratory routine use. At the same time, when faced with results seemingly at odds with the formulation of the task, Group B were unable to justify their routine use, whereas student C2 engaged with substantiation in an exploratory manner. Moreover, he did this using both real-world/biological and purely mathematical narratives, suggesting an ability to navigate between the two discourses.

There were further indications of exploratory engagement with the task. Even those students who insisted on using time as the independent variable in their graphs did so without simply mimicking the established routine. Rather, they engaged creatively with it, inventing a new routine, which at the same time fit their expectations of what a graphing routine should look like, and the requirements of the task. Unfortunately, this new routine proved mathematically unviable. Similarly, lacking the mathematical tools to utilise the graphs for the estimation of the constant, both groups invented a numerical method for doing this. In the case of the *Digoxin* task, this method turned out to produce acceptable results. Later in the same session, however, they applied it on a much messier data set in the context of another task, and the results were highly unsatisfactory. The perceived need for developing such a graph-use routine also points to the important role of the teacher when engaging inexperienced students in MM activities. For the lecturer, an experienced mathematician, it was obvious that the graph, properly constructed, is a carrier of information that can substantially support solving the task. For the students, however, it seemed to be merely a data illustration. They saw no connection between the request to plot and explore the graph, and the estimation of the constant k_3. Were it not for this connection deficit, the students might as well have done pretty well on the *Digoxin* task. If our commognitive analyses are anything to go by, a pedagogy that steers attention to such connections is much needed.

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References

Biza, I., Giraldo, V., Hochmuth, R., Khakbaz, A., & Rasmussen, C. (2016). *Research on Teaching and Learning Mathematics at the Tertiary Level: State-of-the-art and Looking Ahead. ICME-13 Topical Surveys*, Springer International Publishing AG Switzerland.

Brewer, C., & Smith, D. (Eds.). (2011). *Vision and change in undergraduate biology education: A call to action*. Washington DC: American Association for the Advancement of Science.

Chiel, H. J., McManus, J. M., & Shaw, K. M. (2010). From Biology to Mathematical Models and Back: Teaching Modeling to Biology Students, and Biology to Math and Engineering Students. *CBE-Life Sciences Education*, *9*(3), 248-265.

Goodchild, S., Fuglestad, A. B., & Jaworski, B. (2013). Critical alignment in inquiry-based practice in developing mathematics teaching. *Educational Studies in Mathematics, 84*(3), 393-412.

Heyd-Metzuyanim, E., Tabach, M., & Nachlieli, T. (2016). Opportunities for learning given to prospective mathematics teachers: between ritual and explorative instruction. *Journal of Mathematics Teacher Education, 19*, 547-574.

Kent, P., & Noss, R. (2003). *Mathematics in the university education of engineers*. London: The Ove Arup Foundation.

Nardi, E. (2016). Teaching mathematics to non-mathematicians: what can we learn from research on teaching mathematicians? *13th International Congress on Mathematical Education* (ICME13, TSG2). Hamburg, Germany.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing*. New York, NY: Cambridge University Press.

Steen, L. A. (Ed.) (2005). *Math & Bio 2010: Linking undergraduate disciplines*. Washington DC: The Mathematical Association of America.

Viirman, O., & Nardi, E. (2017). From ritual to exploration: The evolution of Biology students' mathematical discourse through Mathematical Modelling activities. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Conference of European Research in Mathematics Education (CERME)* (pp. 2193-2200). Dublin, Ireland: ERME.

SYMBOLISATION AND OBJECTIFICATION THROUGH SOCIAL INTERACTIONS FOR MEANINGFUL LEARNING OF MATHEMATICS

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From the very beginning of their history, mathematical objects have been developed in close relationship with the symbols they use. Starting from an epistemico-historical analysis of the development of algebraic notation, this article proposes a theoretical reflection on the interdependence between objectification and symbolisation that is specific to the mathematical thinking. Based on recent Radford's recent definitions of learning and mathematical objects, it aims to develop the importance of symbolisation activities organised into chains of significations and of social interactions in mathematics learning conceived as a social process of objectification. It finally proposes an example of a classroom activity illustrating the theoretical principles.

INTRODUCTION

Right from the start, mathematics has developed in close relation to the symbols it uses. From the first markings denoting quantities on stone tablets to formal representations of imaginary numbers, the process of creating symbols has been inseparable from the emergence of mathematical objects and their development. From a learning perspective, ever since the dissemination of Vygotsky's work in mathematical education, sociocultural approaches have considered mathematics discourse and its objects to be mutually constitutive (Font, Godina & Gallardo, 2013). Hence symbols, defined as any formal or informal written marking used to communicate mathematical reasoning, are elements in mathematics discourse; they are the mediating tools for communication, i.e. signs in the sense used by Vygotsky. For Vygotsky, it is indeed the appropriation of these signs that essentially marks the transition from elementary to more advanced activities. In the classroom, research literature has for many years highlighted the difficulties students face in using mathematical symbols such as letters, the equals sign, or the minus sign (Kieran, 1992; Vlassis, 2004). These difficulties are at least partially connected with teaching practices in which symbols are always presented in their definitive form and with the underlying implicit idea that they must be considered independently from the concepts they represent. This viewpoint often leads students to regard mathematics as the manipulation of meaningless symbols.

The aim of this article is to present a reflection on the interdependence of mathematical objects and their symbolisation starting from an epistemico-historical analysis of the development of algebraic notation. This reflection is largely based on recent work by

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 371-378). Umeå, Sweden: PME.

Radford and by Font et al., which offer new approaches in terms of mathematical objects and learning. The originality of this article lies in the links that are established between these works on the one hand and the semiotic and activity theories on the other, leading to the structuring of activities into chains of signification rooted in specific classroom practices. It will make these principles more concrete by presenting a generalisation activity in the field of elementary algebra that is planned on the basis of this structure; this will also highlight the importance of social interactions and the teacher's interventions during the activity to encourage the emergence of mathematics.

THE HISTORY OF ALGEBRA: A CONCEPTUAL (R)EVOLUTION RESULTING FROM A MAJOR SYMBOLIC DISCOVERY

The history of algebraic notation in the West is an illuminating example of the interdependence of symbolisation and conceptualisation in the development of mathematics. We will take a quick look at the history of algebra in order to understand the emergence of mathematical objects over the course of history. According to several authors (Harper, 1987; Kieran, 1992; Ifrah, 1994) the development of algebraic notation occurred in three main phases. The first phase was that of *rhetorical language,* in which natural language alone was used to solve problems which were very often related to agriculture, economic transactions or some other concrete situation. The second phase saw the development of a *syncopated language*. Diophantus' major innovation was the idea of *arithme:* an indeterminate quantity of units. The conceptual change introduced by Diophantus with the *arithme* is that this unknown quantity is to be taken into account in calculations. His symbolic innovations consisted of abbreviated words. They proved necessary due to the limitations of writing at the time, as effective techniques for copying mathematical manuscripts more quickly. Diophantus' advances took place in the context of solving problems related to the grouping of numbers (cubes, squares, etc.). This period saw increasingly extensive use of mathematical symbolism, which allowed ever more sophisticated operations to be developed that would have been impossible to carry out in words. Finally, in the third phase, *symbolic language* brought a radical change through the work of François Viète (late 16th century), with letters also starting to be used as parameters, i.e. as given quantities. Thanks to this symbolic language, it became possible to express general solutions and to use algebra as a tool to demonstrate the rules governing numerical relations. Ifrah points out that this is what made possible the emergence of other mathematical concepts, such as that of functions – a discipline which today constitutes one of the foundations of applied mathematics – as well as the algebraisation of analysis and the rise of analytical geometry.

THE IMPORTANCE OF SYMBOLISATION IN THE LEARNING OF MATHEMATICAL OBJECTS

The preceding analysis shows how mathematics is a human and social construct which undergoes constant change, and in which symbolism has played a key role in the emergence of more and more sophisticated mathematical concepts, making it possible to solve increasingly complex problems set in a given socio-cultural context. While our purpose here is certainly not to assimilate the ontogenesis of mathematical objects with a kind of phylogenesis, we will follow Radford (1997) in postulating that a certain parallelism exists between the two; however, this has more to do with the order of the process rather than the knowledge of the facts in themselves – a process in which symbolisation and conceptualisation interact in order to solve a specific problem in a given socio-cultural context. This section first presents a redefinition of the learning process on the basis of a definition of mathematical objects proposed by Radford (2008) and then explores the importance for learning, of symbolisation activities and, more broadly, of signs and chains of signification in the emergence of mathematical objects.

Mathematical objects and learning

According to Radford (2008, p.223), 'learning is not about constructing or re-constructing a piece of knowledge, but rather about actively and imaginatively endowing the conceptual objects that the students find in his/her culture with meaning'. This is what he calls a process of objectification, that is to say a social process of progressive awareness of a cultural object, for example a figure, shape or number, the general characteristics of which we gradually perceive at the same time as we give it meaning. This definition involves clarifying the nature of mathematical objects. According to Radford (2008, p.222), 'mathematical objects are fixed patterns of reflexive activity incrusted in the ever changing world of social practice mediated by artefacts'. This understanding of mathematical objects is in fact quite close to Vygotsky's definition of concepts (1997) according to which 'from the psychological angle, a concept is at any stage of its development an act of generalisation'. Both Radford and Vygotsky put an emphasis on action ('activity' in Radford / 'act' in Vygotsky) as well as on generalisation ('pattern' in Radford / 'generalisation' in Vygotsky). Moreover, both seem to suggest that a concept/object is not monolithic, but may be composed of several levels of development: Radford (2008, p.226) adds that 'the conceptual object is an object made up of layers of generality'. In the context of this article, we will consider mathematical concepts and objects in a very similar way. Radford also speaks of 'conceptual objects' and does not make a clear distinction between the two ideas.

Another important point here is that Radford places special emphasis on the idea that objects are 'incrusted' in mediated social practices. He argues that access to mathematical objects is only possible via the social and mediated activity that requires them. In other words, social interactions and the mediating tools of communication, such as symbols, are consubstantial with learning. Similarly, Font et al. (2013) note that mathematics is a human activity and that the entities involved in this activity (i.e. objects) emerge from the actions and discourse through which they are expressed and communicated. The authors use the term 'emergence' intentionally in this context to emphasise the fact that these objects emerge from the practices of individuals. They must not be regarded as independent of people, of the language used to describe them or of their representations. This is why, in the view of Radford (2011), it is incorrect to say, as several authors have done (for example, Duval, 2000; Sfard, 2000), that access to objects is only possible via their representations. Access to these objects, according to Radford (2011), is not just a question of representation: rather, such access is only possible through the social and mediated activity that requires it.

Symbols, signs and mathematics classroom activity

From the point of view of symbols and signs, this emphasis on the mediating tools of an activity leads us to turn to semiotic theory in order to refine our understanding. Like Radford (2013), we would draw attention to the fact that the idea here is not to produce a mere amalgam between semiotics on the one hand and socio-cultural approaches on the other, but to use the foundations of the former to express (or even transform) the presuppositions of the latter. Thus, on the basis of semiotics, in this section we offer a definition of a sign and a reflection on the development of the chains of signification that flow from this definition. In the work of semioticians (Lacan/Saussure, cited by Gravemeijer, 2002), a sign is generally taken to consist of a 'signifier' and a 'signified'. This definition of a sign is modelled by the circle representing these two inseparable dimensions of a sign as shown in Figure 1 below.

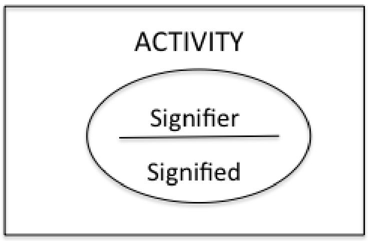

Figure 1: Proposed definition of a sign including the idea of activity.

However, taking into account the importance of activity as explained in the previous point, this conception of a sign cannot be considered adequate. It should be remembered that for Radford (2008), it is mediated social activity that allows access to mathematical objects, while for Font et al. (2013), objects emerge from individuals' practices. In sociocultural approaches, a sign is never an entity in itself: it exists and makes sense in the context of a specific activity, and is produced in order to achieve a given objective (Radford, 1998). In this article, we will retain the basic definition of an activity put forward by Radford (1998). He claims that an activity has two important characteristics: 1) it is mediated by signs and therefore embedded in a culture, and 2) it is focused on a goal. Thus, from our point of view, a sign is defined by a signifier and a signified in close relation to each other, mediating a given activity focused on a goal. This is why we propose to add to the representation of a sign proposed by semiotics by setting it in the midst of the activity in which it is produced (see Figure 1).

Signs and chains of signification

In the context of learning activities, Gravemeijer and Stephan (2002) have defined different stages of symbolisation constituting a chain of signification (see also Presmeg, 2006) in which the basic component is the sign. The development of signs in a chain of signification implies that the new signified encompasses the original sign, while the signification of the original sign changes in a progressive process of mathematisation that is increasingly abstract. This development of the sign is made necessary by activities of increasing complexity, as in the history of mathematics where symbolisations developed in step with the problems that were addressed and inversely. These chains of signification constitute a framework which we believe to be suitable for structuring a mathematical activity a priori by defining beforehand a structured learning trajectory on the basis of symbolisations of increasing complexity which permit different levels of generality of the mathematical objects. The advantage of this process, according to Presmeg (2006), is that at every point in the chain there is the possibility for students to go back, including to the very first actions. It should be noted that Gravemeijer and Stephan (2002) emphasis the idea that the development of signs emerges with the development of activities in the classroom. Surprisingly, however, their initial schematic presentation of a chain of signification fails to take this dimension into account. This is why we have adapted it to our definition of a sign, including signs in the context of evolving activities. In the next point we offer an example of a chain of signification that we have adapted to this context and which creates a structure for a learning environment on generalisation (see Figure 3).

AN EXAMPLE OF A CLASSROOM ACTIVITY

On the subject of 'activity', Radford (2016) makes a distinction between the activity as planned and the activity as it unfolds. He believes that an activity cannot be reduced to a description on paper, just as a symphony cannot be reduced to its score. For Radford, the score, as the activity described on paper, is something 'general' which presents 'potential'. But mathematical objects will only become objects of consciousness, feeling and thought when this general aspect is deployed and transformed into something 'sensible'. The 'singular' is the appearance of the 'general' through the mediation of human activity. Thus, starting from the same 'general' entity (i.e. the written description of an activity), the activity that takes place in the classroom can lead to very different results depending on how the students and teachers engage in the discussions, how agreements and disagreements are managed, etc. and ultimately depending on the richness of the interactions that take place within the classroom.

Description of the activity

In our experiment we used a generalisation activity called 'Antoine makes some mosaics' based on the 'manufacturer problem' (Bednarz, 2005), in two classes in early secondary education (grades 7 and 8). This type of environment is considered potentially rich for developing algebraic thinking. The main mathematical objects involved in this activity were the formula and the variable. The situation was presented using pictorial representations: a mosaic composed of coloured squares inside and a border of white squares. Figure 2 below shows two mosaics given as examples to the students.

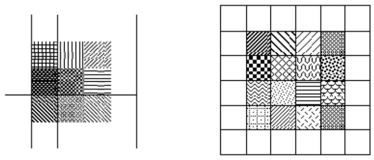

Figure 2: Two models shown to students in ‘Antoine makes some mosaics’.

The questions which accompanied this activity involved asking students to find a way to determine the number of white border squares for any number of coloured squares along one side (*n*). The situation implied a progression along a chain of signification starting with concrete material (small cubes) used to work out the number of white border squares (*n* = 5) (question 1). Students were then asked to produce a calculation, initially for a small number of squares (*n* = 7) (question 2) and then for a larger quantity (*n* = 33) (question 3). Finally, students had to find a general solution, which must be expressed first in everyday language (question 4) and then in mathematical language (question 5). Several ‘formulas’ could emerge, reflecting different visual presentations. Thus the activity was structured according to a chain of signification presented in Figure 3 below:

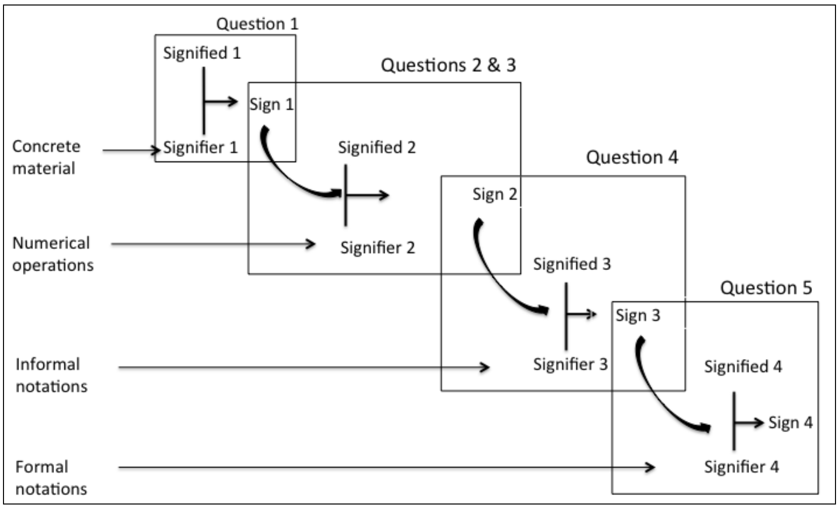

Figure 3: Chain of signification for the activity ‘Antoine makes some mosaics’.

The social interactions at the heart of the dialectic between objectification and symbolisation

It is obviously impossible to discuss all the results of these experiments in this article. Figure 4 below presents two formulas very frequently produced by students in response to question 5. These formulas corresponded to the visual presentation of the border of white squares in terms of 2 (*n* + 2) + 2*n*, i.e. two long sides (2 x (*n* + 2)) and two short sides (2 x *n*).

	Formulas produced by students for question 5
1	N+2+N+2+N+N=
2	a+b+a+b

Figure 4: Examples of formulas produced by students for question 5.

In example 1, the formula, although still contextual, correctly identifies the variable as 'indeterminate' (Radford, 2014), and the status of the formula as an act of generalisation. In example 2, by contrast, students are not yet fully aware of the variable: they express the two different lengths using different letters. In the classroom, this process of awareness did not occur spontaneously in the groups: it was only achieved later on, thanks to the teacher's input, followed by discussions within these groups, the checking and refining of hypotheses using cultural artefacts (returning to actions with the material, and to the meaning of the operations produced in questions 2 and 3), etc. Radford (2008) emphasises in this regard that 'the investigation of the students' and teachers' interactions and use of semiotic means of objectification is indeed a methodological strategy to account for the processes of learning in the classroom. It provides a broad, but sufficiently specific, frame with which to track students' progressive acquisition of cultural forms of mathematical being and thinking' (p. 227).

FINAL CONSIDERATIONS

These observations which have just been briefly described highlight the value of developing activities which, both in the way they are structured beforehand and in their occurrence in the classroom will encourage the simultaneous emergence of symbolisation and mathematical objects in a social process of progressive awareness. Note that authors such as Warren and Copper (2009) speak more of a teaching-learning trajectory than of a purely learning trajectory, in order to bring out the idea that the act of teaching is as important as the learning trajectory that has been planned beforehand. Thus, these activities designed according to the principles discussed in this article will allow students to assign meaning to mathematical objects, but also to broaden their understanding in a variety of usage contexts, thus encouraging levels of generality that are increasingly abstract and detached from the initial settings. In no case is there any question of confining students to their informal attempts and symbolisations. On the contrary, the aim is to use these as a lever in a social and gradual process of objectification-symbolisation in close interaction, similar to that which has occurred naturally in the history of mathematics.

References

Bednarz, N. (2005). Parler les mathématiques. *Vie pédagogique, 136*, 20-23.

Duval, R. (2000). Basic issues for research in mathematics education. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the Twenty-fourth International Conference for the Psychology of Mathematics Education (Vol. I)*, (pp. 55-69). Hiroshima.

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics, 82*(1), 97-124.

Gravemeijer, K., & Stephan, M. (2002). Emergent models as an instructional design heuristic. In K., Gravemeijer, R., Lehrer, B., van Oers, & L., Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 145-169). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Kieran, C., (1992), The learning and the teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on Mathematics Teaching and Learning* (pp. 390-419). New York: Mac Millan.

Presmeg, N. (2006). Semiotics and the "connections" standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics, 61*(1-2), 163-182.

Radford, L. (2016). Mathematics and Mathematics classroom activity through the lens of a metaphor. In M. Iori (Ed.), *La Matematica e la sua Didattica/ Mathematics and Mathematics Education. In occasion of the 70 years of Bruno D'Amore* (pp. 439-446). Bologna: Pitagora Editrice.

Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal,* 26, 257-277.

Radford, L. (2013). On semiotics and education. *Education & didactique, 7*(1), 185-204.

Radford, L. (2011). Vers une théorie socioculturelle de l'enseignement-apprentissage: la théorie de l'objectivation. *Éléments, 1*, 1 – 27.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics education: epistemology, history, classroom, and culture* (pp. 215-234). Rotterdam: Sense Publishers.

Radford, L. (1998). On signs and representations, a cultural account. *Scientia Paedagogica Experimentalis, 1*, 277-302.

Sfard, A. (2000). Symbolizing Mathematical Reality Into Being - Or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms. Perspectives on discourse, tools, and instructional design* (pp. 37-98). Mahwah, NJ: Lawrence Erlbaum Associates.

Vlassis, J., (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction, 14*(5), 469-484.

Warren, E. & Cooper, T. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. *Mathematics Education Research Journal, 21*(2), 76-95.

THE APPLIED KNOWLEDGE OF TRAINEES AS INDUSTRIAL CLERKS SOLVING PROBLEMS WITH VOCATIONAL AND NON-VOCATIONAL CONTEXT

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Following literacy frameworks, one aim of educational policy is to prepare students for their professional lives. Vocational training for industrial clerks, which relies on mathematical competence, is of high interest for mathematics education. One specific question is which role school-acquired mathematical competence plays in vocational training. It is suggested that mathematical competences connected to vocation-related mathematics may be a link between demands of general and vocational education. We designed problems that mirror these different demands and conducted an interview study with trainees as industrial clerks. The results show that the trainees' knowledge application differed for the different types of demands so that the approach enabled us to reveal qualitative differences between general and vocation-related mathematics.

THEORETICAL BACKGROUND

Following common literacy frameworks, one specific goal of general education is to prepare students for their professional lives. However, problems of transition from general to vocational education are known in many countries irrespective of the specific educational system (cf. OECD, 2000). In Germany, for example, there are growing difficulties of matching training place supply with training place demand across almost all vocational fields and there is a high dropout rate (25%) in vocational training with approximately one third of the dropouts during the first year (BIBB, 2017). Accordingly, it is important to gain more insight into the demands of vocational training and into the conditions for vocational preparation that are set within general school education.

Empirical investigations that address the transition from school to vocational training from a subject-specific perspective would help to understand the specific challenges trainees experience when transitioning to vocational education. However, educational frameworks for school education and vocational education are mostly not strictly commensurable. From the view of mathematics education, a starting point to investigate the transition problems is to focus on vocational training programs in which mathematical competences are of central importance. Vocational education standards show that particularly merchants and specifically industrial clerks face rich mathematical demands in professional training (e.g., KMK, 2002 for the German situation). These professions are, therefore, at the center of our attention.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 379-386). Umeå, Sweden: PME.

Due to the major structural differences of the systems of school and vocational education we focus the individual level of students and trainees. We ask in which way the differences of the two systems in terms of mathematical demands can be seen as a continuity or discontinuity from the perspective of the individual learner. In particular, we ask what kind of mathematical knowledge do trainees as industrial clerks apply when dealing with parallel mathematical problems in a vocational and a non-vocational context, so that possible discontinuities can be elicited.

Characterizing the educational context of this study

The German education system allows different ways to gain access to vocational training. After at least nine years of compulsory education in primary and secondary school, the students should possess the necessary competences to start a vocational training. However, for specific occupational fields, 10 or 12 years of schooling are required as access to vocational training. Vocational education in Germany is organized in the so-called dual system combining theory (vocational school) and practical work (individual training with a training supervisor at the workplace).

Approximately 60% of the training contracts each year fall within the commercial field. The dual vocational training for industrial clerks, which we focus in our study, is constantly over years among the most important training tracks in the commercial field (BIBB, 2017). The vocational training standards require a substantive part of the 3-year training time to be used for economics and mathematics contents (KMK, 2002). These contents fall into the economical categories of cost accounting, performance calculation, depreciation, and financing/investment. From an inner-mathematical point of view, they can be seen as applications of the rule of three and percentage calculation, which are contents belonging to lower secondary mathematics education. The difference between both implementations of the same mathematics is that in vocational education, the contexts are specific for industrial clerks, what is reflected amongst other in a specific terminology such as *break-even point*.

Although, the context where we investigate this transition from school to vocational training is coined by a specific national education system, this research is of wider interest from two perspectives: First, it specifically focuses on the (non-)compatibility of educational frameworks that built on ideas of literacy vs. frameworks that are oriented more narrowly on vocational demands, what is pertinent to many educational systems. Second, from a broader perspective, the approach can serve as a model how mathematics education can address questions with relevance for educational policy decisions related to incommensurable conceptions of (mathematics) education by refocusing from the system to the individual level.

To sum up, if general mathematics instruction should prepare for vocational training, the commercial field in general, and in particular industrial clerks, can be seen as especially important to provide a good starting point for research in transition conditions. Overlooking a successful transition to vocational training, it can lead to a better understanding of goals of general education against the mathematical requirements that are given through the specific demands of vocational trainings.

From general mathematical competences to vocational competences

Educational contexts of general and vocational education differ in many aspects, resulting in fundamentally different task requirements in general education and vocational training. Nickolaus et al. (2013) tried to characterize the transition from general education into vocational training in Germany by contrasting and synthesizing the models of competence from a domain-specific point of view. On the one hand, future trainees leave school with “general mathematical competences” (in the sense of literacy-conceptions) according to the educational standards for mathematics as a subject. On the other hand, vocational training aims at “vocational competences” (in the sense of action competence) defined by specific demands that are necessary for the pursuit of a profession (e.g., Winther & Achtenhagen, 2009).

According to literacy-conceptions, however, future trainees should already acquire mathematical competences at secondary school that *prepare* for the successful participation in vocational training. The theory of cognitive flexibility supports the practice of integrating differing contexts into mathematical education in order to support the acquisition of rich, applicable, and flexible (mathematical) knowledge and competences (Spiro et al., 1988). Hence, students are expected to possess initial “vocation-related mathematical competences” so that it is an aim of general education that basic tasks related to vocational contexts can be solved (Nickolaus et al., 2013).

From a theoretical point of view, the “vocation-related mathematical” competences are seen as overlapping the other competence areas: They can be understood as being part of the general mathematical competences, subject to general education. Equally, they should serve as the base of the vocational competences that will be developed during vocational training. From a theoretical perspective, such “vocation-related mathematical competences” should facilitate transition processes as they function as link between competence acquisition in school and in vocational training. From an empirical perspective, it is unclear if this theoretical differentiation is relevant in the sense that “vocation-related mathematical competences” can actually be delineated as a being to a certain extent self-contained and not necessarily part of general mathematical competences.

RESEARCH QUESTIONS

Among others, literacy frameworks for mathematics secondary education aim for the preparation for professional life, especially vocational education. Thereby, the vocational field of industrial clerks offers a high proportion of mathematical contents and is predestined to investigate conditions of transition from secondary education into vocational training. Furthermore, vocation-related mathematical competences were seen as being a potential mediator between general mathematical competences acquired in school and vocational competences to be acquired in vocational training. A first quantitative study from Siebert and Heinze (2014) points to an empirical separability of general mathematical and vocation-related mathematical competences for trainees. What remains unexplained so far is whether this separability can indeed be attributed to the application of different competences that were acquired in general and vocational educational contexts. Hence, we aim at addressing this possible explanation through the following research question:

- What knowledge do trainees as industrial clerks apply when dealing with mathematics-related problems in a vocational context?

METHOD

We implemented a three-stage research procedure aiming at the applied knowledge of trainees as industrial clerks in different (vocational and non-vocational) demands. At first, we developed three pairs of contextualized mathematical problems. Each pair comprises one problem with a vocational context of industrial clerks and one problem with an isomorphic mathematical structure and the same mathematical content in a non-vocational context. Both problems can be solved by applying general mathematical competence acquired in secondary school. Second, these problems were solved by $N = 42$ trainees as industrial clerks. The results of the testing informed the selection of a subgroup of trainees with high differences in test scores between the two types of problems and a control group with low differences. Third, these trainees participated in a subsequent guided stimulated recall interview study with the aim of investigating whether the differences in testing results can be explained by difference in the kind of knowledge applied.

Problems used in the assessment

As the starting point, we picked three complex problems of the collection for the final examination for trainees as industrial clerks from the German Chamber of Industry and Commerce. There was one problem consisting of three and two problems consisting of two subtasks. These problems contained mathematical and technical contents appearing during the first two years of vocational training for industrial clerks and, therefore, focused on vocation-related mathematical competences. We modified the tasks to receive parallel problems with different outer-mathematical contexts but kept the inner-mathematical content and structure.

For example, problems 3 and 6 deal with different tasks for the application of percentage calculation. Problem 3 looks at selling computer tables from the perspective of an industrial clerk. Price calculation, production costs and profit deviation are key terms for solving that problem. In contrast, problem 6 asks for the same calculations while regarding and comparing numbers of unemployment in two different districts from an external perspective. Thereby, the applied commercial terms are replaced by questions about, for example, the deviation of unemployment numbers so that other contexts of general interest, that are not located in the specific vocational field, were realized.

To check whether the designed problems are – as intended – parallel, the complexity and the demanded competences for the problems were classified using the framework of the German national educational standards. The classification of all six problems was realized with a multi-step classification process by experienced raters including the co-authors of this research report and led to an almost complete accordance between each of the three problems with vocational context and their parallel problems with non-vocational context. Thus, despite minimal deviation regarding the requirements of one pair of problems the parallelism of the problems in respect to mathematical demands can be assumed.

Test administration

The six problems were administered to a group of N = 42 trainees as industrial clerks from second (N_1=18) and third (N_2=24) year in a German vocational training track. Further, we applied a dichotomous 0-1-scoring to each subtask. Since we wanted to compare the three problem types, we standardized the score for each problem due to the different number of subtasks. Table 1 shows the descriptive results of the standardized test scores. The scores for the pairs of parallel problems (pairwise and in total) indicate that the vocational (problems 1-3) and non-vocational (problems 4-6) context has an influence on the solution rates and therefore offer a solid basis for the stimulated recall interview study.

M *(SD)*	Problems with vocational context	Problems with non-vocational context
Problems 1 & 4	.71 (*.24*)	.58 (*.30*)
Problems 2 & 5	.42 (*.25*)	.51 (*.37*)
Problems 3 & 6	.60 (*.35*)	.62 (*.36*)
Total (*N*=42)	.56 (*.17*)	.56 (*.25*)

Table 1: Mean (standard deviation) for the standardized scoring of trainees as industrial clerks (*N*=42).

In preparation for the interview study, we selected nine trainees from second and nine from third year of vocational training. These trainees either reached a significant better or a significant worse scoring on the three problems with vocational context (six trainees each). In addition, we selected a control group of six trainees showing only small differences. This theoretical sampling led to a heterogeneous group of test persons for the interviews.

Guided interviews and qualitative content analysis

For the implementation of stimulated recall interviews focusing on the applied knowledge of the trainees as industrial clerks, we developed a partially standardized questionnaire. In addition to questions about the solution process and noticed difficulties, we asked which knowledge they used to solve the problem and where they learned that knowledge (e.g., secondary school, vocational school, training supervisor). Due to time limitations, we restricted the questions related to the individual solution process to the three problems with vocational context. Additionally, the interviews included questions about the perceived differences and similarities between the parallel problems in order to investigate if the differences (or parallelities) are relevant for the individual trainees.

According to the qualitative method of content text analysis (Schreier, 2012), we set up two main categories of classification for the evaluation of the interviews. The category *solution process* deals with the self-reported knowledge the trainees applied while solving the problems. This category aims at analyzing the applied knowledge indirectly. Concurrently, the category *knowledge acquisition* relates to the source of the applied knowledge directly. Both categories could occur either as a *vocation-related* or a *general education specification*. Furthermore, we set a third specification (*unknown*) for answers not falling into one of the two other specifications.

For nine interviews (50%), the coding was conducted by two trained raters independently. High values of interrater reliability (percentage agreement: $p_0 = .91$, Cohen's Kappa: $\kappa = .89$) indicate a high level of objectivity of the coding.

RESULTS

The qualitative content text analysis reveals insights into the applied knowledge of the trainees when working on the problems. Some of the answers showed a solution process with a vocation-related specification indicated by specific terms from the vocational field of industrial clerks:

Interviewer: How did you proceed?

P7: Well, then … I tried to … calculate the break-even-point, where the proceeds are equal to the costs.

Here, the vocation-related specification is directly marked by using the specific commercial terms (break-even-point, proceeds, costs) that are not part of secondary education. In a similar way, the vocation-related specification can appear while answering the questions referring to the question of knowledge acquisition:

Interviewer: OK, what knowledge did you use to solve the problem?

P10: Well, you need to know, from what, how you calculate the profit, that it is sales minus costs.

Here, the linking to vocational school as the place of knowledge acquisition happens indirectly just with background information about the structure of vocational training standards for industrial clerks. Elsewhere, test persons give the link to vocational school as knowledge acquisition as an answer to the same question directly through naming the relevant field of instruction (business processes and accounting):

P4: Um, that what we had in instruction right now. … I don't know. Business processes and accounting.

Otherwise, P8 gives the link to secondary education (Gymnasium) as the place of knowledge acquisition:

P8: And, well, um, the remaining … that has been more math. … Standard math … Standard calculation. So rather from *Gymnasium*.

If the stated solution process of a test person for a problem or the stated knowledge acquisition falls into the category of vocation-related specification such as the examples P7, P10 and P4 above, we consider the applied knowledge for that person and problem as vocation-related. For the three problems (seven subtasks) with a vocational context, 67% of the solutions of the 18 trainees rely on vocation-related knowledge. Two of these seven subtasks were hardly solved so that noticeably lower proportions of vocation-related knowledge was identified (since no knowledge was available at all). By excluding these outliers, the value for vocation-related knowledge rises to 76%.

In addition, for each pair of parallel problems we asked for perceived differences and similarities. In 65% of the cases, the trainees detected the implemented similarities regarding the mathematical structure and differences regarding the contexts. Nevertheless, half of the trainees showed clear differences in the solution rates between the three problems with vocational contexts and the three problems with non-vocational contexts. The interview could hence substantiate that the differences did not result from a lack of insight into the structure of the problems. It indicates that these problems rather pose different demands for the trainees what leads to the different solution rates. Thus, a distinction between general mathematical competences and vocation-related mathematical competences is substantiated by the findings.

DISCUSSION

Our study addresses questions that stem from the immensurability of educational policy frameworks at the transition from general to vocational education. Vocation-related mathematical competence – as competences acquired in general education, but with connection to vocational demands – were suggested from a theoretical perspective as a potential link on the individual level. The study hence aimed at investigating whether they can be seen as a relevant construct also from an empirical point of view. Therefore it was investigated what kind of knowledge trainees as industrial clerks apply when they solve parallel problems with vocational and non-vocational contexts.

The vocational training standards include a high amount of mathematical contents, which can inner-mathematically be classified as content of lower secondary education. However, in vocational education, the contexts differentiate from general education contexts (while maintaining mathematics) and are specific for industrial clerks. Further, theory of situated cognition suggests the inseparability of knowing and action (Brown, Collins, & Duguid, 1989). As a consequence, it is not unlikely that such identified differences between the implementations of the same mathematics leads to the acquisition of separate areas of competences that ground in contextuallized knowledge.

The results based on a qualitative content text analysis indicate that the differentiation between general and vocation-related mathematical competences is relevant for trainees. In accordance with the theory of situated cognition, we could observe that they applied a specific body of knowledge when working on the mathematical problems in a vocational context that do not require, from an objective perspective, specific knowledge beyond what is subject to instruction in general education. It, thus, sets ground to further characterize transitions from general to vocational education with a focus on domain-specific competence development, which is considered a highly fragile and complex transition from the individual, but also institutional point of view.

References

BIBB [Federal Institute for Vocational Education and Training] (2017). *Datenreport zum Berufsbildungsbericht 2017 [Data report to the national report on vocational training]*. Bonn: BMBF.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational researcher*, *18*(1), 32-42.

KMK (2002). *Rahmenlehrplan für den Ausbildungsberuf Industriekaufmann/Industriekauffrau vom 14.06.2002 [National curriculum for the training of industrial clerks]*.

KMK (2003). *Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss vom 04.12.2003 [National educational standards in mathematics at lower secondary level]*.

Nickolaus, R., Retelsdorf, J., Winther, E., & Köller, O. (2013). *Mathematisch-naturwissenschaftliche Kompetenzen in der beruflichen Erstausbildung. Stand der Forschung und Desiderata* (ZBW, Beiheft 26). Stuttgart: Steiner.

OECD (2000). *From initial education to working life.* Paris: OECD.

OECD (2014). *Education at a glance 2014. Country note: Germany.* Paris: OECD.

Schreier, M. (2012). *Qualitative content analysis in practice*. Los Angeles: SAGE.

Siebert, U., & Heinze, A. (2014). Die Lernausgangslage von Auszubildenden: Erste Ergebnisse des Projekts ManKobE. In J. Roth & J. Ames (Eds.), *Beiträge zum Mathematikunterricht 2014* (Vol. 2, pp. 1131-1134). Münster: WTM.

Spiro, R. J., Coulson, R. L., Feltovich, P. J. & Anderson, D. K. (1988). *Cognitive Flexibility theory: Advanced knowledge acquisition in ill-structured domains.* Technical Report No. 441.

Winther, E., & Achtenhagen, F. (2009). Measurement of vocational competencies. A contribution to an international large-scale assessment on vocational education and training. *Empirical research in vocational education and training*, *1*(1), 85-102.

STUDENTS' READINESS TO APPROPRIATE THE DERIVATIVE - META-KNOWLEDGE AS SUPPORT FOR THE ZPD

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To what extent are students ready to appropriate a mathematical concept when this is introduced? We studied the Zone of Proximal Development (ZPD) of ten students in grade 10 before being taught the derivative. We used task-based interviews to study their pre-derivative knowing (e.g. functions, slope), and how they handled problem situations that required the derivative. Most were able to apply numerical or graphical approximations. Some students saw commonalities among the tasks, and felt a need for a more precise tool. By describing goals of a new tool, they showed to have meta-knowledge without knowing the concept itself. This meta-knowledge assisted them to appropriate the derivative as a tool they could use flexibly and confidently in non-routine situations. Other students' appropriation took more than a year.

INTRODUCTION

The derivative is an important mathematical concept in advanced strands of secondary schools (NCTM, 2000), but also one students typically struggle with (e.g., Zandieh, 1997). The derivative was developed by Isaac Newton and Wilhelm Leibniz; the latter introduced the notation *dy*/*dx* that we still use today. The concept builds on everyday experiences (increase, growth, speed) and is a tool to describe change of quantities. It is a flexible concept, as it can be represented symbolically, graphically, numerically and verbally. In non-mathematical areas, it is useful to model phenomena. Astronomer/physicist Newton described the derivative as a tool, *"... whereby I can calculate curves and determine maxima, minima, and centers of gravity"* (Boyer, 1959, p. 207).

The derivative is a cultural artifact developed and refined by mankind. Being based on human thoughts and ideas, we indicate this artifact as a *concept*, similar to concepts such as 'energy' in physics, or 'democracy' in society. Mathematical concepts, such as the derivative or logarithms, are typically learnt in schools, whereby students are taught cohort-wise. In this paper we describe a study of this cohort-wise treatment and we focus on student's readiness to learn about a concept when it is taught.

Our study was carried out in The Netherlands, where the derivative is introduced in grade 11 to students in the pre-university science stream. In the prior year, a number of related procedures are taught in mathematics and physics classes. Students learn to find the rate of change for linear equations and relate this to the slope of the graph. In physics classes, students learn about distance, velocity and acceleration. There, students are taught a graphical procedure to calculate instantaneous velocity for non-linear situations by drawing a tangent line to the graph, and determine the rate of

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 387-394). Umeå, Sweden: PME.

change to this line (known as the *tangent method*). In grade 10, the derivative is neither mentioned in textbooks nor in classrooms.

THEORETICAL FRAME

There are different frameworks to study how students learn mathematical concepts. Within PME, there are many voices using terms such as abstraction, conceptual understanding, encapsulation, interiorization, internalization, objectification, reification, and so forth. The present paper is too small to do justice to ontological and epistemological underpinnings of these authors. Inspired by Moschkovich (2004) and others, we use the neo-Vygotskian term *appropriation*. Appropriation expresses a person's process of gaining ownership over something external, and making it one's own. Radford (2001) says: *"we do not mean that students' knowledge appropriation is achieved through a kind of crude transfer of information coming from the teacher. As we see it, knowledge appropriation is achieved through the tension between the students' subjectivity and the social means of semiotic objectification"* (p.241). In this paper, we follow Vygotsky (1978) by taking appropriation as a dynamic meeting of culture, social environment and cognition. Aspects that play a role are: a person's subjective experiences, socially mediated processes, and cultural concepts and tools. The process of appropriation is hard to capture by a researcher, but the result of a successful appropriation can be observed: it is when a student has ownership over a concept as a personalized cultural tool, when (s)he can use it flexibly, confidently and strategically in a variety of non-routine situations, without being prompted. We emphasize the use-value of the concept, which is based on the duality of the derivative being both a concept and a tool. A successful appropriation implies that a student knows what the concept is (factual knowledge), how it can be used (procedures, algorithms), can make connections among its representations and with other concepts (conceptual knowledge), and has meta-knowledge of it (background knowledge of its aims, limitations, when/where/for what purpose it can be used).

The appropriation of mathematical concepts is socially and culturally embedded and mediated. It happens in the Zone of Proximal Development (ZPD), which is "… *the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with a more capable peer"* (Vygotsky, 1978, p. 86). According to Lerman (2001, p. 103), *"the zpd would be better conceptualized not as a physical space, in the sense of the individual's equipment (either cognitive or communicative), but as a symbolic space involving individuals, their practices and the circumstances of their activity."*

A social environment of the ZPD is the classrooms, where there are rules, norms, a qualified person responsible for students' learning, and learning goals set by institutions (a ministry, a school board). Students are supposed to listen to explanations from others (from a teacher live in class, from a video, etc.), to read texts (written by others), to do tasks. The drawback of this institutionalized environment is that much mathematics learning occurs in situations that are far from ideal: teachers rush to finish the curriculum, they teach the way they were being taught themselves, and/or they teach to the test, and so forth (Nolan, 2013). Additionally, there are strange mathematical conventions (e.g. *x*, *f'*), there is Latinized jargon (derivative, secant), and in too many mathematics classrooms there are myths that mathematics is only for 'brainy' kids, or for boys (Boaler, 2015). It may occur that the ZPD will not comprise appropriation of a concept, but rather panic or alienation (Williams, 2016).

Each student has a ZPD. Numerous researchers have studied it by observing students during their learning process (e.g., Moschkovich, 2004; Radford, 2001; Williams, 2016). We contend that direct observing students 'while in the zone' does not capture all. For example, such studies may miss (1) earlier learning, (2) differences between surface and deep learning, and (3) students' tendency to forget some of their learning. Therefore, we studied the ZPD longitudinally, by researching students' readiness (the zone before the ZPD) and their knowing after they were taught a mathematical concept. We included the study of retention (until a year after being taught), which enabled us to retrospectively deduct to what extent the appropriation of the concept indeed happened, and thus, had been in the ZPD. The research question was: what are critical aspects of students' readiness that assist them in appropriating the derivative?

METHODS

We opted for a qualitative description and analysis of students' work before and after they were taught the derivative. We had a sample of convenience of ten pre-university students (6 boys, 4 girls) who took mathematics at advanced level. Their pseudonyms are: Andy, Bob, Casper, Dorien, Elly, Karin, Maaike, Nico, Otto, and Piet. Among them, weaker students were underrepresented because we looked for students who most likely would move up from grade 10 to grade 12 without delay.

We studied students' textbooks, their notebooks, their work on classroom tests, and on three occasions, we administered a task-based interview. The *April interview* took place when the students were still in grade 10. After the summer holiday, the students were taught the derivative at the beginning of grade 11. We followed up with the *November interview*. A year later when the students were in grade 12, we administered another *November interview*. In this paper, we focus on the April interview and students' readiness for the derivative. The two November interviews enable us to study the resulting students' long-term appropriation of the derivative.

The April interview was designed to provide in-depth information about students' readiness to appropriate the derivatives. The students were given tasks unfamiliar to them, which could be solved using the derivative, but also without. The tasks were situated, so, the variables had a situated meaning (e.g. time, price). The tasks offered different mathematical representations (graphs, symbols, tables), whilst the mathematical terms derivative, slope or differentiation and the symbols *f'* and *dy*/*dx* were absent to avoid directing the students (in case they had heard of the derivative). The interview protocol prescribed that a student, after having completed a task, was repeatedly asked to check the answer through other methods. In this way, we were assured to observe a range of activities. Both subsequent November interviews contained the same tasks, with a few adaptations to reduce inter-interview effects. In this paper we will focus on the two tasks that are summarized here:

Barrel: A barrel contains a liquid that runs out through a hole at the bottom. The volume of the liquid in the barrel decreases over time, expressed as $V = 10(2 - {}^{1}/_{60}t)^2$ with V in m^3 and t in *min*. Additionally a table and the V-t-graph is given. Students are asked to calculate the out-flow velocity at t=40.

Monopoly: For a company the revenue function is $R(q) = -0.5q^2 + 12q$ and the cost function is $TC(q) = 0.03q^3 - 0.5q^2 + 4q + 15$ with q the amount of sold products. Additionally, the graphs of the functions are given. Students are asked to find at what production level the costs and the revenue will increase at the same rate.

The interviews were transcribed and both authors analysed the transcripts. From the April interview, we coded methods used when solving the tasks, identifying graphical, numerical and symbolic methods; we used a simple scale to describe the quality of usage (method is accurate, inaccurate, or only mentioned). Also, we coded whether students (1) mentioned commonalities among tasks, (2) expressed limitations in their repertoire, (3) recognized a need for a new tool, and (4) made connections among aspects of the derivative (rates of change, slope, velocity). From both November interviews, we coded frequency and quality of the use of the derivative.

RESULTS

Below we report on five students, Otto, Piet, Dorien, Elly and Andy, whom we selected because of their illuminating differences. We first report on their approaches to the tasks in the April interview, when they weren't able yet to use the derivative. We add a summary of their subsequent development, as reflected in their answers to the two November interviews, when they were in grades 11 and 12 respectively.

Otto's reasoning before and after being introduced to the derivative

When given the Barrel task, Otto calculated the value $V(40)$ by inserting t=40 into the formula, and found 17.7778. According to him, this matched with the coordinates in the graph. When prompted for a way to check his calculation, he replied that he could solve the equation $17.7778 = 10(2 - {}^{1}/_{60}t)^2$, which should then yield t=40. So, he mistook the volume of the liquid for the outflow velocity (as if $V(40) = V'(40)$).

For the Monopoly task, to find where the two formulas increase at the same rate, Otto equated them. Then he said: *"Equal… that is not the same as equal increase."* To solve this task in another way, he looked globally at the graphs searching for a spot where the graphs ran approximately parallel. We coded this as graphical approximation. On another instance in this interview, he used an interval to approximate the increase at one point, and mentioned (without carrying it out) that he could use the slope of a secant. In this interview, he did not solve one task accurately. His approaches were disjointed; he didn't see commonalities among the tasks.

In the November interview when in grade 11, again Otto did not accurately solve any of the tasks. He named a few methods, including the derivative. When trying the derivative, he was insecure and repeatedly said: *"the derivative gives the formula for the tangent"* as a mantra, without relating this concept to instantaneous change or velocity in one point. A year later when in grade 12, he related the derivative again primarily to graphical aspects (tangents and slopes), but now additionally, he related it to $\Delta y/\Delta x$. This helped him to use the derivative to solve a few tasks.

We deduct that before the derivative was introduced, Otto struggled to see differences between a function's value and its increase, and how a slope in a graph is related to change of values. As a result, Otto's appropriation of the derivative took more than a year. At the moment it was introduced, Otto's ZPD allowed him to learn the derivative at a surface level, as expressed in his mantra. Fortunately, in the ensuing year, he managed to further appropriate the derivative, but his knowing remained fragmented.

Piet's reasoning before and after being introduced to the derivative

When given the Barrel task, Piet calculated the average outflow velocity over the interval [40, 120]. He explained that his calculation is for a linear relation, and that the described outflow went gradually slower and wasn't linear. When prompted for another method to check the first, he calculated the average outflow over [0, 40], which was another interval method. Also for the Monopoly task, he used an interval. Frequently in this interview, he talked in graphical terms: *"here, the graph goes steeper and here less steep"* or *"here they go the same"* while gesturing.

In the November interview when in grade 11, Piet again used the interval method and twice he used the derivative. A year later he had developed a strong preference for the symbolic representation and he solved all tasks using the derivative. In his explanations, he talked in graphical terms (secant, slope, steepness).

We deduct that before the derivative was introduced, Piet was aware of differences between a function's value and its increase, of linear and non-linear equations, and that his interval method was inaccurate. As a result, Piet was able to appropriate the derivative when it was introduced. Piet's ZPD embraced the derivative, although he did not yet use it in all instances. A year later, he used it flexibly and confidently.

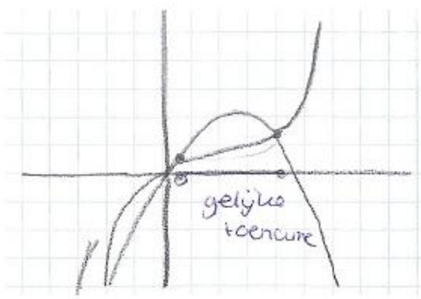

Figure 1: Dorien's graphs showing that two functions have equal increase (in Dutch: *gelijke toename*) between their intersection points

Dorien's reasoning before and after being introduced to the derivative

When given the Barrel task, Dorien started by reading the value 17.5 in the graph at *t*=40. She reasoned that during 40 *min*, 22.5 *l* must have flown out. She said: *"This is, I think, not the velocity at this point"* and she said that what she calculated was an average velocity: *"I can calculate average velocity. But at this one point it is different. I don't know how to calculate the velocity at one moment".* For the Monopoly task, she used graphs and reasoned correctly that the intersection points yield an interval (drawn on the *x*-axis), on which the average increase of both functions is equal, see Figure 1.

In all tasks, Dorien used graphs, tables and functions, mostly accurately. At several instances, she mentioned the difference between average velocity and velocity at one point. Seeing the final task in the interview, she exclaimed: *"Oh no! Not yet another one! There we go again! That you must know the velocity in one point and not the average velocity."* Apparently, she recognized a commonality in the tasks, and limitations in her knowing, and formulated a goal of a tool that she needed.

In the November interview when in grade 11, Dorien used the derivative for all but one task (where she applied a physics formula). She clearly was able to use the derivative as a tool for solving situated tasks. However, she hadn't been taught the chain rule yet, and therefore some of her answers were inaccurate. A year later, the derivative had become a tool that she used flexibly and confidently. We deduct that the appropriation of the derivative was in her ZPD when she was in grade 10.

Elly's reasoning before and after being introduced to the derivative

Both in the April and the first November interview, Elly showed little understanding of functions, neither of slopes nor of rates of change; she solved not one task at least to some degree. After being introduced to the derivative, her knowing was at a surface level: *"if you have a formula like* $f(x) = x^2+3x+20$*, then you take down the powers and subtract one, so* $2x + 3$ *and 20 is cancelled".* This, she used when prompted for a derivative of a given function, but for her the concept remained unrelated to tangents, change or velocities, etc. In the final interview, when Elly was in grade 12, she said *"If you put the derivative equal to zero, you get the rate of change, isn't it?"* showing insecurity and (erroneous) surface knowledge. She had not appropriated the derivative.

Andy's reasoning before and after being introduced to the derivative

The second student who didn't appropriate the derivative was Andy. Contrary to Elly, in the April interview, he showed a very rich repertoire of numerical and graphical methods to solve the tasks. He used his graphic calculator (GC) to calculate average rates of change on small intervals such as [39, 40] and [40, 41], relating these to the slope of a tangent becoming the slope of a secant. He used the expression *dy/dx* confidently (a notation in the GC-screen). To find where costs and revenue increased at the same rate, he moved the cursor over the graphs, and read off where *dy/dx* was the same in both graphs. In no way, he was short of tools. In the subsequent November interviews, Andy again used the GC, plotting graphs, reducing intervals to [40, 40.001]. He had discovered a new option in the GC, the Calc-dy/dx-option and used that frequently. Although his notebook showed that he practiced many tasks on derivatives, in the interviews he stuck to his GC-repertoire. His ZPD was GC-focused.

Synthesis of the ten cases

In Table 1 we ordered the students according to the frequency of accurately using the derivative in the first November interview. The students form three groups.

	Interview April Grade 10 (8 tasks)					Subsequent Derivative Use	
	Interval method	Secant method	Small inter-val	Tan-gent method	Other	Interview Nov. Grade 11 (6 tasks)	Interview. Nov. Grd 12 (6 tasks)
Dorien	●●◉	-	-	-	-	●●●◉○	●●●○○
Casper	●●◉○	-	-	-	-	●●●	●●●●●
Nico	●●○	○	-	-	-	●●◉◉○	●●◉◉◉
Piet	●●◉◉	-	-	-	-	●●○	●●●●●
Otto	◉	○	-	-	◉	◉◉◉	●●◉○
Maaike	-	-	-	-	-	◉◉	●●●●◉
Bob	-	-	-	●◉	-	◉○	●●●●○
Karin	●◉	-	-	-	-	◉○	●●◉◉○
Andy	●●○	●	●	●●○	-	-	●
Elly	-	-	-	-	-	-	-

Legend: ● accurately used ◉ inaccurately used ○ mentioned, not used - not mentioned

Table 1: Methods used before being taught the derivative, and subsequent use of it.

The group at the bottom consisted of Elly (see above) and Andy (see above), who did not appropriate the derivative in the period covered by our study.

The middle group consisted of Otto (see before), Maaike, Bob, and Karin. After being taught the derivative, they used the derivative inaccurately for some tasks. They lacked confidence, made few connections (e.g. between slopes and $\Delta y/\Delta x$), and talked in mantras (see the case of Otto). At the moment of being introduced to the derivative, it was not in their ZPD. However, a year later, they could use it mostly accurately.

The third group consisted of Dorien (see before), Casper, Nico, and Piet (see before). In the first November interview, they used the derivative for several tasks mostly accurately. This meant that at the moment the derivative was introduced in class, they were ready for it: their ZPD comprised the appropriation of it.

CONCLUSION, DISCUSSION, RECOMMENDATIONS

In this study, we searched for critical aspects of students' readiness that assist them in appropriating the derivative. We argued that a successful appropriation can be observed longitudinally studying students using the derivative flexibly and confidently as a tool without being prompted. Our research design has weaknesses (possible inter-student communication between interviews, students remembering tasks from earlier interviews), so our results need some caution. In our study, we could identify a few students (Dorien, Casper, Nico, Piet), whose appropriation of the derivative was in their ZPD when it was introduced in class. Some other students (Otto, Maaike, Bob, Karin) did not appropriate it when it was introduced, but in the ensuing year they did. Finally, there were students (Elly, Andy), who didn't appropriate the derivative. This diversity points at critical aspects of readiness. First, the ZPD is different between students and a cohort-wise introduction of a concept creates inequities; then appropriation and alienation can occur within one classroom (Williams, 2016). Second, when comparing between the students in our study, the students who appropriated the derivative quickly had in common that they mastered cognitive foundations of the derivative (slope of a line, different representations, difference between value and increase). Third, a critical aspect of these four students' readiness was that they had meta-knowledge of the derivative without knowing the concept itself: they saw commonalities among tasks, they described limitations in their methods, and expressed the need for a more precise tool. Our study shows that students can "know about", in which a yet unknown concept begets a tool-based meaning, and that students' meta-knowledge of a mathematical concept preceded their ZPD.

We confirm Lerman's (2001) conceptualization of the ZPD as a symbolic space, which involves persons, their social environment, and practices. Practices to support the ZPD can include activities aiming at meta-knowledge; for example activities for which the students yet lack a tool. Such activities can create a need for a new tool, and give meaning to a new concept. Such activities are based on the concept-tool duality.

References

Boaler, J. (2015). *The elephant in the classroom.* London, UK: Souvenir Press.

Boyer, C. (1959). *The History of Calculus & its Conceptual Development.* New York: Dover.

Lerman, S. (2001). Cultural, discursive psychology: a sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics, 46,* 87–113.

Moschkovich, J. N. (2004). Appropriating mathematical practices: Learning to use functions through interaction with a tutor. *Educational Studies in Mathematics, 55,* 49-80.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Nolan, K. (2012). Dispositions in the field: Viewing mathematics teacher education through the lens of Bourdieu's theory. *Educational Studies in Mathematics, 80*(1-2), 201-215.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics, 42*, 237-268.

Vygotsky, L. (1978). *Mind in society.* Cambridge, MA: Harvard University Press.

Williams, J. (2016). Alienation in mathematics education: Critique and development of neo-Vygotskian perspectives. *Educational Studies in Mathematics, 92*, 59–72.

Zandieh, M. (1997). *The evolution of student understanding of the concept of derivative*. Doctoral Dissertation. Eugene, OR: Oregon State University.

CROSSING THE BOUNDARIES BETWEEN SCHOOL MATHEMATICS AND WORKPLACE THROUGH AUTHENTIC TASKS

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This paper reports a case study research aiming to connect the teaching and learning of mathematics in upper secondary education with the workplace. Four 10th grade students engaged in authentic tasks from the merchant navy context concerning the navigation of a ship through the use of original tools (e.g., the nautical map). We use the notions of activity system and boundary crossing to study students' construction of meanings for geometrical concepts. Results indicate that the students took a new look at the school taught geometry, by adopting the workplace perspective (perspective taking) and addressing authentic workplace problems through the lens of school mathematics (perspective making).

INTRODUCTION

Students' preparation for their professional life constitutes one of the aims of mathematics education (FitzSimons, 2014). However, the relationship between mathematics and workplace is complex and has been studied in many research studies over the last 30 years. Almost all these studies converge to the conclusion that mathematics used by professionals has some unique characteristics emerging from the cultural nature of the workplace which puts an indelible mark on the mathematical ideas developed in it (Millroy, 1992). These unique elements make workplace mathematics different from those in typical education enabling them to be identified as a distinct genre within the workplace discourse practice (Williams & Wake, 2007). Besides, several studies in the workplace reveal that apprentices or novices face a lot of difficulties in understanding the mathematical practices involved in authentic work situations. These difficulties indicate the limitations of school and academic knowledge as well as the complexity of linking this knowledge and workplace *(skill gap)* (Fitzsimons, 2014). Even though, studies have revealed how professionals engaged in mathematics when handling breakdown situations (e.g., Pozzi, Noss and Hoyles, 1998) we do not know if these situations could provide a context for students' mathematics meaning making at schools. While the teaching exploitation of the workplace in vocational education has received considerable attention over the past years (Bakker et al., 2014), only recently has the connection between general education and workplace been an area of research focus (e.g., Psyharis & Potari, 2017). The reported study aims to contribute in this direction by connecting the merchant navy context (ship navigation) and the teaching and learning of mathematics in upper secondary education. It was carried out through

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the close collaboration between a mathematics teacher (who acted also as a researcher) and a ship captain.

THEORETICAL FRAMEWORK

By perceiving the sociocultural nature of learning process, we have adopted the third generation of Activity Theory, as the theoretical framework for our study, considering two interacting activity systems (Fig. 1) as unit of analysis (Engeström et al., 1995). In the present research, we study the interaction between teaching and learning mathematics in school (students, teacher) and merchant navy workplace (captain, ship navigation). We acknowledge Activity Theory an appropriate theoretical background as it treats the learning process ingrained in a system of *object*-oriented, *tool*-mediated and *rule*-defined actions without neglecting its individual and communal dimension. The latter elements are also highlighted by the *subject*; the *community* and the *division of labor* which include the structural elements of the activity systems (see Fig. 1). The categories of *objects, goals, tools/artifacts* and *rules* are shaping mathematics within workplace activity into a recognizable kind of mathematic practice, in direct connection with the workplace context. Among two distinct activity systems emerge sociocultural differences as subjects engage in new practices unfamiliar to them. From this perspective, difficulties in using the typical school knowledge in realistic workplace situations can be considered as an example of discontinuity (Bakker & Akkerman, 2014).

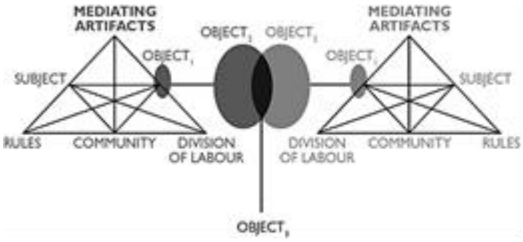

Fig. 1: Interacting activity systems.

The aforementioned discontinuities are defined by Bakker and Akkerman (2014) as *boundaries*. These authors use the term *boundary crossing* to refer to the interactions between the subjects in order to establish or restore communication among the activity systems. Within the bidirectional boundary crossing the tools/artifacts utilized by the subjects to bridge the two activity systems are defined by Star και Griesemer (1989) as *boundary objects*, i.e. objects that both inhabit several intersecting words and satisfy the informational requirements of each of them. Bakker and Akkerman (2014) consider boundary crossing as a cognitive process that can be described through the possible activation of four learning mechanisms: *identification* of the intersecting practices; *coordination* of practices by developing tools/objects to establish effective communication between them; *reflection* while subjects become aware of their own perspectives by redefining them in relation to the perspectives of others (perspective making) as well as by taking a new look to their own perspectives through the eyes of others (perspective taking); and *transformation* leading to changes in the existing practices, even the emerging of a new hybrid practice. We recognize boundary crossing perspective to be a fruitful alternative to understanding knowledge transfer between contexts, as we consider reasonable the assumption that students' engagement into workplace situations can highlight the discontinuities between typical mathematical knowledge and its use in work (Wake & Williams, 2001).

In the present study, we consider two activity systems: teaching and learning of mathematics in upper secondary schools (geometry) and merchant navy (ship navigation). We focus on the interaction of the two systems and the students' learning as they engage in authentic workplace activities. Boundary crossing offers us the tools to describe learning as meaning generation in terms of the students' bidirectional moves from one system to the other. Our research questions are: "How do students construct meanings for mathematical concepts when working on authentic ship navigation tasks?" "What is role of the workplace context - including authentic tools, rules and practices - in this process?"

METHODOLOGY

The research constitutes the pilot study of the first author's PhD aiming to explore the potential of authentic workplace situations as a context for mathematics learning in upper secondary schools [1]. It took the form of a case study with four 10th grade students and took place in a secondary school in Athens where the first author (who is also a teacher) works. A professional captain was invited to participate in all phases of the study. The practitioner contributed to the familiarization of teacher and students with the workplace, being bearer of nautical knowledge. He judged the correctness and compatibility of the activities with the workplace (designing phase) and also the solutions proposed by students in terms of the professional practice (implementation phase). The teacher acted as an agent of mathematical knowledge bringing back to the forefront the typical mathematics, supporting students' inquiry of mathematical concepts, encouraging them to express their ideas and strategies, and asking for refinement and revision when appropriate.

The implementation part of the research was divided into three phases. The students have been engaged in authentic activities from the workplace of the Merchant Navy (ship captain). In the first introductory phase, after the captain presented the main features of the workplace (e.g., nautical map, professional's tools); the students were asked to design the ship's course on the map so as to get familiar with the workplace context. In the second phase, the professional familiarized the students with the measures and tools he uses to find the position of the ship on the map. In particular, he showed them how to take *bearing* (a straight line of sight connecting the ship and visual prominent landmark on the shore), *range* (distance from the ship to an object represented as a point through radar) and *horizontal angle* (angle which has vertex the ship and sides two straight lines linking the ship with two landmarks) with the use of ruler, divider, protractor and parallel rulers (i.e. two rulers moving in parallel lines). Thus, the aforementioned measures correspond to the geometrical notions of straight line, circle, knowing radius, and inscribed angle, respectively. Then, the students were engaged in solving a series of realistic problems (e.g., *Avoid Obstacle*, *Safe Passage*). In the third phase, the captain gave to the students, six measures (two bearings, two ranges and two horizontal angles) and asked them to find the ship's position on the nautical map using as many as possible ways. In the results section, we present how the students propose a solution to the task *Safe Passage* concerning the navigation of a boat through a hazardous area.

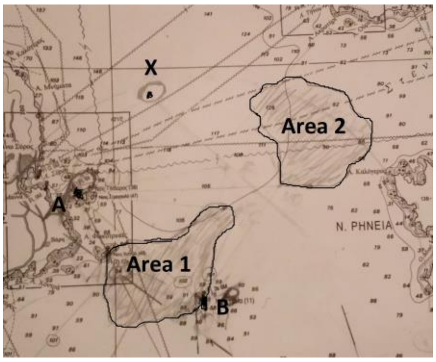

Fig. 2: The problem.

Safe Passage, that constitutes a typical duty/process in the navy context, was given to the students as an open geometrical problem. Starting from a point X the ship must pass through a dangerous (hatched) area with underwater obstacles (that are not visible). On the map the captain had marked landmarks A and B as reference points (Fig. 2). The students were asked, to use landmarks A and B and the aforementioned measures (bearing, range and horizontal angle) and tools in order to find a way to keep track of the ship's course to ensure its safe passage. This is a kind of problem not typical of those that students encounter in school. However, the students could make sense of it in the context of school mathematics.

The collected data consisted of: transcriptions of videos recording the main phase of task implementation; teachers' personal notes; teachers' resources and materials (lesson plan, worksheets, ppts, digital files); outcomes of the students' activities on the nautical map. The analysis was carried out in two levels. After carefully watching the videos and examining the corresponding transcripts, critical episodes were selected and coded under a grounded theory approach (Charmaz, 2006). The episodes were related to the construction of meanings that arose as students tried to "decode" workplace practices, understand and use the captain's tools, or respond to the activities modelling the realistic problems faced by the professional during the course of the ship. In the second level, the analysis focused on the boundary crossing of students between the two activity systems (teaching and learning of school mathematics and merchant navy) in relation to the meaning generation identified at the first level. In this paper, the levels of analysis described above concern the notion of inscribed angles that appeared to be central during the students' activity in the implementation of *Safe Passage*. The analysed episode falls into the learning mechanism of *reflection*.

RESULTS

Initial solution

The task was given to the students by the professional. He marked points X, A and B on the map and asked the students to find a method to keep track of the ship's course through the hazardous area. Students started solving the problem by drawing the segment AB. After that, they drew a line from point X parallel to segment AB as a safe route of the ship (Fig. 3). One student proposed to measure the distance between the lines drawn from X and the segment AB so as to keep track of ship's course. This was a correct solution from a mathematical point of view.

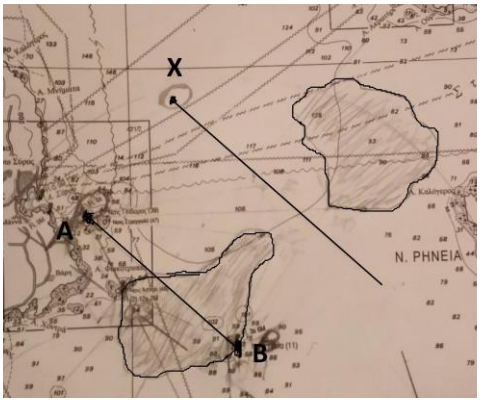

Fig. 3: Initial solution.

However, since AB in reality is an imaginary straight line (without visible landmarks), the students realized that it was not possible in the workplace context to measure the distance between the ship and AB with the use of the available measures/tools (i.e. from the ship). Student S2 communicated plainly his concern on this issue by wondering in which way he could be sure that the ship follows a safe course. Student S1 recognised that it was not possible to use straight lines (bearings) and distances (ranges) to keep track of the ship's course as there were no visible landmarks in the hazardous area. He proposed, for first time, to use inscribed (horizontal) angles.

1	S3:	Let's move like this. [He draws a straight line from X parallel to the segment AB]
2	S2:	How shall I know if I am far or near the hazardous area?
3	S3:	You are right. I have no landmarks to take measures. [Dangerous area is not visible]
4	S1:	[to S2] We must use horizontal angles… Bearings or ranges won't work.

The students' realization that they could not use their school taught geometrical knowledge to address the problem indicates a discontinuity between school mathematics and workplace. The constraints posed by the professional's measures/tools and the workplace rules (ship's safety) led the students to abandon their initial (mathematically accepted) solution discovering that it was ineffective in the new context. The fact that the students used terms from the professional's language during their exploration (lines 3 and 4) indicates that they had become familiar with the workplace tools.

Find a safe point

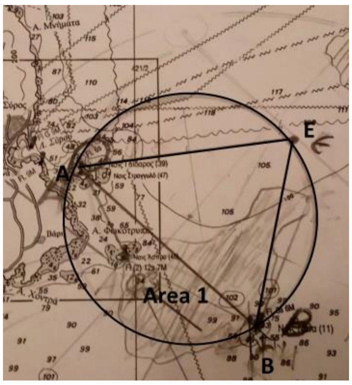

Fig. 4: Inscribed angle.

Later on, the teacher's intervention ("*How can I determine the dangerous waters?"*) was crucial for the students to overcome their difficulties in finding a strategy to exploit the concept of inscribed angle they had recognised as relevant. He gave them the hint to mark a safe point on the chart and the students marked the point E (Fig. 4). That was not only a safe point between the two dangerous areas, but also a point on the line they drew in their initial solution. They still used as reference points A and B, though this time student S3 suggested to draw a circle passing through the points A, E and B and to measure the corresponding inscribed/horizontal angle (Fig. 4). The part of the dangerous area near the shore (area 1, see Fig. 4) was somehow contained in the designed circle. In that way students had come up with the idea to keep track of the ship's course through the use of an inscribed/horizontal angle. However, at this phase the students were experimenting with distinct safe points (e.g., E) without having formed a comprehensive strategy.

5 T: Ok, why don't you mark on the chart a safe point for the ship?

6 S1: [He marks point E] Wait, the circle should pass through all three points. [He refers to points A, E, B]

7 T: So, how can I determine the dangerous waters? [Students had designed the circle and the inscribed angle AEB]

8 S4: As the area surrounded by the arc of a circle. [He refers to arc AEB]

Through the design of the circle passing through points A, B and E the students achieved to orient accurately the dangerous area. Taking the role of a professional helped students to cross the boundaries between the two activity systems, using authentic tools (horizontal angle) as boundary objects, satisfying workplace rules (safety) and trying to fulfil a workplace demand (safe passage). Their choices were influenced jointly by the perspectives of both the school (accuracy in defining the dangerous area through a circle) and the workplace (tools, constraints, norms). Teacher's intervention helped them to connect the value of an inscribed/horizontal angle with the cyclical sector that defines the dangerous area.

Final solution

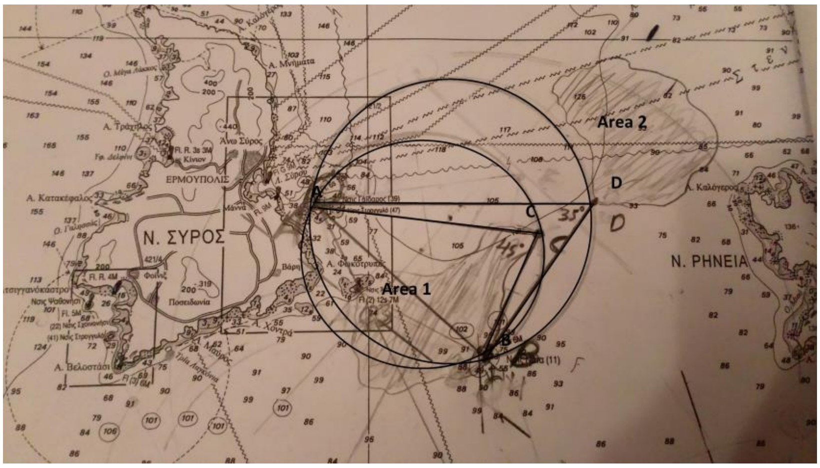

Fig. 5: Final solution.

In the final phase of their exploration, the students were engaged in finding a measure indicating that the ship sails in a safe area. For this they had the idea to design the point C (the ending point of the dangerous area near the shore, area 1, Fig. 5), the circle passing through A, C, and B and the inscribed/horizontal angle ACB. Their strategy was to use the value of the inscribed/horizontal angle (45°) to define the dangerous area oriented by the designed circle. With the intention to provoke students' mathematical reasoning, the teacher asked if the value of the inscribed/horizontal angle had to be bigger or smaller than the angle ACB to ensure the ship's safe passage. Student S1 suggested using the point D (the beginning point of the dangerous area away from the shore, area 2, Fig. 5) as they used the point C before. Thus, the students designed a new circle passing through the points A, D and B and the inscribed/horizontal angle ADB so as to have a visual representation of the 'safe' area for the ship's course. Measuring the inscribed/ horizontal angles (from the points C and D) and observing the difference in their values (45° and 35° respectively), the students accepted that as the radius of the circles (passing through A and B) increases the corresponding inscribed angle decreases (Fig. 5). In this way, they developed a strategy to check whether the ship is following a safe course or not based on the value of the inscribed/horizontal angle having as vertex the position of the ship and sides defined by the lines connecting the vertex with the points A and B respectively.

9	T:	Fine, the horizontal angle is 45°. To be safe, you need a wider or a narrower angle?
10	S3:	Wider. [Wrong answer]
11	S4:	No, for bigger radius, the angle becomes narrower. [After students make the second circle and measure the angle at D]
12	S1:	The second angle is 35°. It is narrower.
13	S2:	We must keep track of the horizontal angle.
14	S2:	The limits are from 35° to 45° for a safe passage.

Taking a global view of the above incidents, the students achieved to find a solution to the need to keep track of the ship's course based on the measure of inscribed/horizontal angles. In the end, the captain recognized that the students' final solution was identical to the one used in the workplace. To achieve this, the students reinvented the notion of the inscribed angle and constructed meaning for the alteration of it, a geometrical relationship that it was not taught at school. Thus, not only they used their existing knowledge in the new context but also they developed meanings for new geometrical concepts. Acting like professionals and through the teacher's and the captain's help, the students took a new look at the school taught geometry, by adopting the workplace perspective *(perspective taking)* and addressing authentic workplace problems through the lens of school mathematics *(perspective making).*

CONCLUSIONS

In the current study we adopted the perspective that the integration of authentic workplace situations into mathematical teaching can enrich students' mathematical knowledge. Our analysis focused on how the merchant navy context motivated students to cross the boundaries between school mathematics and professional space. The realistic context (original workplace problem), the practitioner's tools (measures) and the workplace constraints (rules) acting as boundary objects revealed the insufficiency of school taught knowledge, highlighting a discontinuity between formal mathematics and the genre of mathematics developed in the workplace. Throughout their exploration for solving an authentic task the students were influenced by the school perspective (parallel lines), they moved to an intermediate model which jointed the school and the workplace perspective (horizontal/inscribed angle) and they gave the final solution taking the workplace perspective (official professional's practise). Through a reflection process they reinvented geometrical notions (inscribed angle), while they gave meaning to new geometrical relationships (alteration of inscribed angles). The professional acted as an agent of the workplace knowledge by bringing to the forefront the workplace context. The analysis indicates also the teacher's critical role in provoking students' mathematical reasoning and helping them overcome difficulties through inquiry-based questions and crucial interventions.

Notes

[1]. The study is inspired by the European project Mascil (http://www.Mascil-project.eu) that aims to promote the integration of inquiry-based learning and workplace in the teaching and learning of mathematics and science.

References

Bakker, A., & Akkerman, S. F. (2014). A boundary-crossing approach to support students' integration of statistical and work-related knowledge. *Educational Studies in Mathematics*, *86*(2), 223–237.

Bakker, A., Groenveld, D., Wijers, M., Akkerman, S. F., & Gravemeijer, K. P. E. (2014). Proportional reasoning in the laboratory: an intervention study in vocational education. *Educational Studies in Mathematics*, *86*(2), 211–221.

Charmaz, K. (2006). *Constructing grounded theory*. London ; Thousand Oaks, Calif: Sage Publications.

Engeström, Y., Engeström, R., & Kärkkäinen, M. (1995). Polycontextuality and boundary crossing in expert cognition: Learning and problem solving in complex work activities. *Learning and Instruction*, *5*(4), 319–336.

FitzSimons, G. E., (2014). Commentary on vocational mathematics education: where mathematics education confronts the realities of people's work. *Educational Studies in Mathematics*, *86*(2), 291–305.

Millroy, Wendy L. (1992). *An ethnographic study of the mathematical ideas of a group of carpenters*. Reston, Va: National Council of Teachers of Mathematics.

Pozzi, S., Noss, R., & Hoyles, C. (1998). Tools in practice, mathematics in use. *Educational Studies in Mathematics*, *36*(2), 105–122.

Psycharis, G. & Potari, D. (2017). Mathematics teachers' learning at the boundaries of teaching and workplace. In G. A. Stillman, W. Blum & G. Kaiser (Eds.) *Mathematical Modelling and Application: Crossing and Researching Boundaries in Mathematics Education* (pp. 301-311). ICTMA Series Springer.

Wake, G. D., & Williams, J. S. (2001). Using college mathematics in understanding workplace practice. *Summative Report of Research Project Funded by the Leverhulme Trust, Manchester University, Manchester.*

Williams, J. S., & Wake, G. D. (2007). Metaphors and models in translation between college and workplace mathematics. *Educational Studies in Mathematics*, *64*(3), 345– 371.

DOES THE COGNITIVE DEMAND OF A PROBLEM INCREASE WHEN THE ANSWER IS AN INEQUALITY?

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This study is inspired by the observation that in school mathematics inequalities rarely appear as a problem-solving outcome. We call such problems inequality-tasks, while problems in which an answer is attained in the form of an equality we call equality-tasks. We hypothesised that inequality-tasks require higher cognitive demand as compared to equality-tasks. We examined this hypothesis using short geometry verification problems with students who differed in their levels of general giftedness and expertise in school mathematics. We employed Event Related Potential methodology to confirm the hypothesis. Analysis of neuro-cognitive measure leaded to new research questions and hypothesis about insight-related moment of answer verification and activation of working memory associated with high cognitive demand.

BACKGROUND

Obstacles related to equalities and inequalities

The importance of the concepts of equalities and inequalities has been addressed by mathematics education researchers (Dreyfus & Eisenberg, 1985) as well as by cognitive psychologists (Lyons & Beilock, 2011). The topic of numbers comparisons has also fascinated the neurocognitive community (De Smedt, Noël, Gilmore & Ansari, 2013).

Cognitive studies have related to the understanding that the development of mathematical competence includes the basic ability to represent the relations ‘‘greater than’’ or ‘‘less than’’ between distinct numerosities (Lyons, & Beilock, 2011). However, the results obtained in these studies revealed a dependence on the number format used (De Smedt et al., 2013).

According to the collection of papers presented at PME research Forum (Bazzini & Tsamir; 2004), equalities constitute a cognitive obstacle to understanding inequality. Difficulties related to manipulating and interpreting inequalities are connected to students' previous experiences, which are mainly focused on the equation and on equality constructs. For example, students treat the inequalities in the same algorithmic manner as equations in spite of the distinctions between symmetrical (for equalities and equations) and asymmetrical (for inequalities and inequations) relationships.

In secondary school, inequalities are mainly taught as a (compartmentalized) topic in algebra classes and ignored as problem-solving tools in other fields of mathematics (Bazzini & Tsamir, 2004). While geometry in school mathematics is considered an

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important resource for the development of students' visual and logical reasoning, argumentation and proof skills (Hanna & De Villers, 2012), inequalities are extremely rare both in proof and computational tasks. The exception can be seen in the topic of the triangle inequality, which receives sporadic attention in the geometry curriculum. Our analysis of school geometry textbooks revealed that less than 0.1% of problems include inequalities.

Neurocognitive research in mathematics education

Neuro-imaging research focuses on the underlying brain structures (the magnitude of brain activation as well as the brain topographies) associated with different types of mental activities, including mathematical processing. The following findings are only short examples that illustrate this observation:

Through focus on the magnitude of brain activation, the neural efficiency hypothesis affirmed that brighter individuals and experts in a field display lower (i.e., more efficient) brain activation while performing cognitive tasks (Neubauer & Fink, 2009). However, this effect is task-difficulty dependent. Focus on the localisation of brain activation associated with mathematical processing demonstrates that parietal brain parts have an important role in mathematical cognition and that fact retrieval is more associated with the left hemisphere while number comparison or approximation is of a more bilateral nature (Dehaene, Piazza, Pinel, & Cohen, 2003). The prefrontal and parietal cortex are recruited in advanced topics like algebra, geometry, or calculus (Anderson, Betts, Ferris & Fincham, 2011) and the posterior cortex is thought to be involved in the mental representations of objects (Zacks, 2008). Research on mathematical insight (Jung-Beeman et al., 2004; Leikin, Waisman & Leikin, 2016) demonstrated that increased activation in the PO8 (PO8-PO4) electrode site is associated with the moment of insight. Furthermore (and importantly for our study) Event-Related Brain Potential (ERPs) studies identified three main positive parietal components associated with mathematical processing: P100, P200 and P300. The P100 and P200 components are usually interpreted as corresponding to early visual processing (Dunn, Dunn, Languis, & Andrews, 1998) while P300 is assumed to be connected to working-memory updating (Kok, 2001).

THE STUDY

Hypothesis and research questions

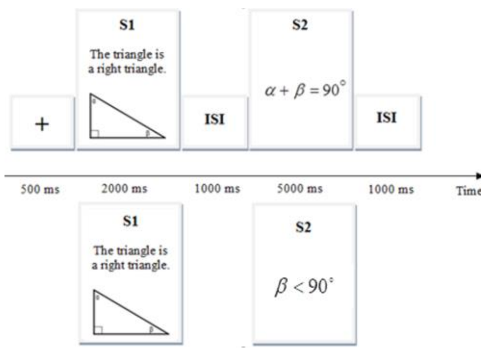
Our review of school textbooks indicates that, inequalities very rarely appear in school geometry. Thus, tasks that include inequalities can be considered unconventional and thus are of high cognitive demand. We employ ERP methodology to examine this hypothesis.

Our study explores differences in mathematical processing associated with short geometry verification problems in which the answer is presented in the form of an equality (“equality-tasks”) vs. in the form of an inequality (“inequality-tasks”). We ask how mathematical processing differs between solving equality-tasks as compared to inequality-tasks. We examine (1) behavioural measures: Accuracy of responses (Acc) and reaction time (RT) and (2) electrophysiological measures: ERP amplitudes, P100, P200 latencies and amplitudes, and late potentials scalp topographies amplitudes. We also ask (3) how these differences depend on the level of problem-solving expertise and the level of general giftedness of the study participants.

Participants

Seventy Hebrew-speaking, right-handed male high school students (16-17 years old) from the northern region of Israel participated in this study voluntarily. All the students passed a sampling procedure in the framework of a larger study (for details see Leikin et al., 2016). This procedure was directed at the identification of levels of (a) general giftedness (G – generally gifted; NG – non generally gifted), and (b) expertise in school mathematics (EM – experts; NEM – non-experts). The students belonged to four major research groups: G-EM (N=19); G-NEM (N= 20); NG-EM (N=15); NG-NEM (N=16). The study received the approval of the Helsinki Committee, the Israel Ministry of Education, and the Ethics Committee of the University of Haifa.

Materials and Data collection



S1 – Introducing a situation; S2 – Question presentation;
+ – Fixation cross; ISI – Inter Stimulus Interval

Figure 1: The sequence of events and task examples.

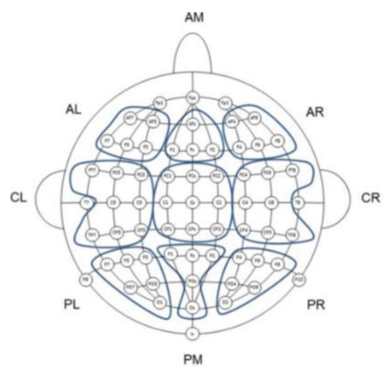


Figure 2: The selected electrode sites used for the analysis.

A computerized test (Alpha-Chronbach = .760) was designed with 60 tasks using E-Prime software (Schneider, Eschman & Zuccolotto, 2002). Each task was presented in two windows with different stimuli (S1 – Task condition; and S2 – Suggested answer) that appeared consecutively (see Leikin et al., 2016). In the S1 window, participants received a geometric figure with angle values marked by the Greek letters α and β. The connection between α and β was determined by a geometrical theorem. At S2, the participants had to determine the correctness of a statement about α, β or the relationship between α and β as an inference of the given properties in the S1 drawing (Figure 1). The participants were required to press the appropriate button on the keyboard according to their decision about the correctness of the S2 statement. 36 of 60 tasks included an equality and 24 of 60 included an inequality to be verified.

Scalp voltages were continuously recorded using a 64-channel BioSemi ActiveTwo system (BioSemi, Amsterdam, The Netherlands) and ActiveView recording software.

Two flat electrodes were placed on the sides of the eyes in order to monitor horizontal eye movement. A third flat electrode was placed underneath the left eye to monitor vertical eye movement. During the session electrode offset was kept below 50 μV.

DATA ANALYSIS AND STATISTICS

Behavioural measures

Accuracy (Acc) was determined according to the percentage of correct responses on equalities and inequalities separately. Reaction time (RT) was calculated as the mean time for answer verification. Repeated measures ANOVA was used to examine differences in Acc and RT associated with the type of the task as a within subject factor. Additionally, effects of G and EM characteristics were examined as between-subject factors.

Electrophysiological measures

ERP waveforms were analysed offline using the Brain Vision Analyzer software (Brain-products) and were time-locked to the onset of S2 (see Leikin et al., 2016). For statistical analyses, the ERPs were topographically aggregated (by using the mean absolute values) to obtain nine electrode sites (Figure 2).

Early components: P100 and P200 were analyzed over posterior electrode sites where they reached their maximum. Peak latencies (ms) and amplitudes (μV) of P100 (92-195 ms) and P200 (200-280 ms) were analyzed using repeated measures ANOVA, taking type of the task (equality and inequality) and laterality (left, middle and right) as within-subject factors, whereas effects of EM and G characteristics were examined as between subject factors.

Late potentials: The mean absolute ERP amplitude of late potentials in the 280-330 ms and 330-700 ms, which are considered as P300-like components, were examined in the nine electrode sites. Additionally, we performed special examination of the mean absolute ERP amplitudes in the PO4-PO8 and PO3-PO7 electrodes.

The effect of task type and caudality (anterior, central and posterior) and laterality (left, middle and right) were examined as within-subject factors (and the factor of the time (280-330 ms and 330-700 ms) for PO electrodes), whereas effects of EM and G were examined as between subject factors. We performed pairwise comparisons when significant interaction was found. We applied Greenhouse–Geisser correction for sphericity deviation and Bonferroni correction for pairwise comparisons when appropriate. With respect to the study hypothesis and research questions we report findings on caudality, laterality EM and G factors as associated with task type.

FINDINGS AND DISCUSSION

Differences in behavioural measures

Examination of the behavioural measures confirmed the research hypothesis. The higher cognitive demand of the inequality-task is indicated by significantly lower accuracy and longer reaction times. This is in line with studies in mathematical education indicating that students in general encounter difficulties when coping with inequalities (e.g. Bazzini & Tsamir, 2004). Table 1 summarises the examination of behavioural measures in the study.

Measure	Equality-tasks Mean (SD)	Inequality-tasks Mean (SD)	*F* (1,66)
Acc (%)	86.4 (7.9)	79.5 (10.8)	47.930***, η_p^2 = .421
RT (ms)	1547.7 (406.6)	1788.9 (423.6)	92.940***, η_p^2 = .585
*** *p*<.0001; Acc – Accuracy, RT – Reaction time			

Table 1: Behavioural results for the task type.

Electrophysiological data

Analysis of earlier potentials (see Figure 3) demonstrated that P100 amplitudes were significantly higher when solving inequality-tasks than when solving equality-tasks [7.7(SD = 3.2) µV vs. 8.0(SD = 3.4) µV; *F* (1, 66) =5.636, *p* < .05, η_p^2 = .078]. At the same time for P200 we found a reversed effect: amplitude of P200 was significantly lower when verifying inequalities than when verifying equalities [4.9(SD = 2.7) µV vs. 5.4(SD = 2.7) µV, respectively; *F* (1, 66) =5.757, *p* < .05, η_p^2 = .080].

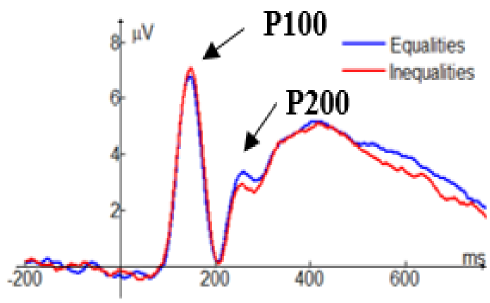


Figure 3: Example of graph of the mean absolute amplitude in PL.

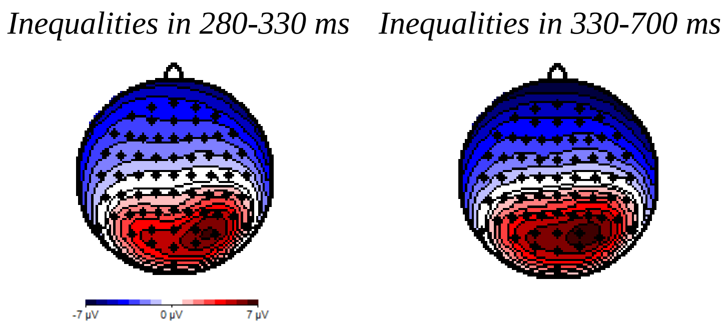


Figure 4: Scalp topographies for inequalities in the 280-330 ns and 330-700 ms.

Note here that P100 is thought to be modulated by attention such as early visual selection, whereas P200 may be connected to selective attention and feature detection processes as well as early sensory stages of item encoding (Dunn, Dunn, Languis, & Andrews, 1998). Accordingly, we suggest that the larger amplitudes of P100 for the inequality-tasks may be due to the “unexpected” form of the statement, which lead to reduced automatic early processing in comparison to the equalities. The inverse differences in P200 amplitudes may point to the differences in encoding processes of equalities and inequalities. For an explanation of such differences in P200 more focused investigation is necessary.

Late potentials in the 330-700 ms (in the nine electrode sites)

In the 330-700 ms effect of task type [F (1, 66) = 5.313, $p < .05$, $\eta_p^2 = .075$]: Similarly, to the amplitude of P200, the mean absolute amplitudes were lower in the inequality-tasks vs the equality-tasks.

Obtained late potentials in the 330-700 ms seem to be compared with P300 like component that may be embedded in the 280-700 ms (Dunn et al., 1998). Interestingly significant interaction of the task type with the G and EM factors was found [F (1, 66) = 4.211, $p < .05$, $\eta_p^2 = .060$]. When EM students verified correctness of both equalities and inequalities, the mean absolute amplitudes of G [3.8(SD=1.5) μV and 3.3(SD=1.4) μV for equalities and inequalities, respectively] students were lower than of NG [4.3(SD=1.5) μV and 4.1(SD=1.4) μV for equalities and inequalities, respectively] students. In contrast, when NEM students verified correctness of equalities the mean absolute amplitudes of G and NG students were similar [3.8(SD=1.5) μV and 3.7(SD=1.5) μV, respectively], whereas when NEM students verified correctness of inequalities, the mean absolute amplitudes of G students were higher than those of NG students [4.0(SD=1.4) μV and 3.3(SD=1.4) μV, respectively].

The significant differences in the electrical activity between equalities and inequalities was achieved only in G-EM [$p = .018$, 95% *CI* [.082, .834]: 3.8(SD=1.5) μV) group.

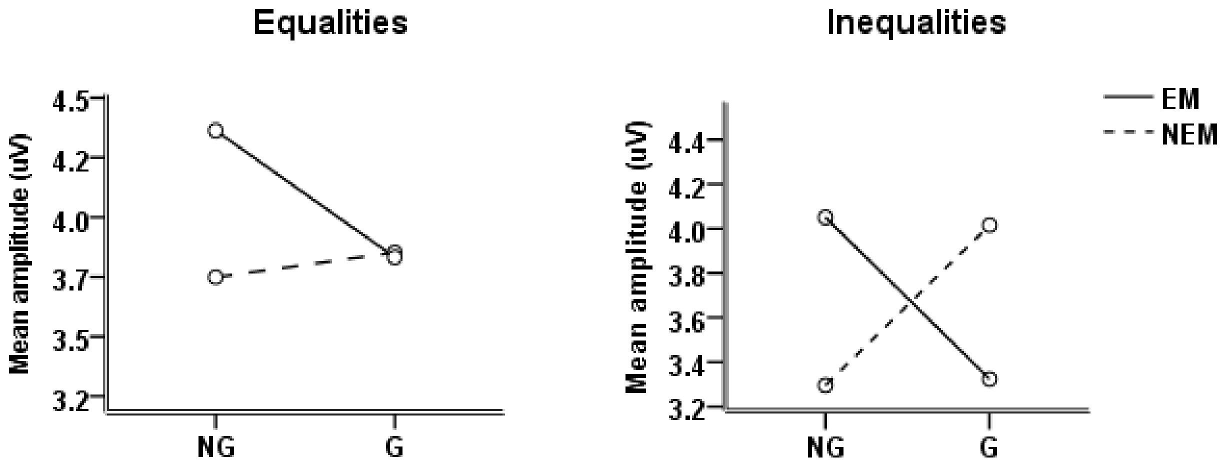


Figure 5: The mean absolute ERP amplitude in the 330-700 ms associated with solving equality and inequality-tasks.

Separate ANOVAs were performed for each type of task (see Figure 5). The analysis showed significant interaction of the G and EM factors when solving inequality-tasks [F (1, 66) = 4.580, $p < .05$, $\eta_p^2 = .065$] with effects similar to those described above.

These interactions confirm the research hypothesis about the higher cognitive demand of the inequality-tasks. Consistently with our previous finding and suggestions (Leikin et al., 2016), we assume that these effects and differences in mean absolute amplitudes can be connected to neuro efficiency in the G-EM group of participants. The lower ERPs in NG-NEM tasks reflect reduced attempts when coping with cognitively demanding tasks in this group of students. The higher ERPs in G-NEM and in NG-EM students may demonstrate that these groups of students “do not give up” when solving cognitively demanding tasks.

The examination of electrical activity in PO electrodes. As mentioned earlier, based on our previous observation of the insight-related activations of PO8-PO4 electrodes (Leikin et al., 2016) we performed a separate examination of PO electrode sites (PO-3, 4, 7, 8: see Figure 4). During 280-700 ms we found significant interactions of task type with G and EM factors [F (1, 66) = 6.964, $p < .01$, $\eta_p^2 = .095$], which was similar to the interaction described early for the overall activity on nine electrode sites. Additionally significant differences in mean absolute ERP amplitudes associated with verifying equalities and inequalities were seen in the G-EM [$p = .002$, 95% *CI* [.369, 1.550]) and NG-NEM [$p = .006$, 95% *CI* [.280, 1.567]) groups with higher activation associated with verifying equalities.

Additionally we found significant interaction of the time factor with the type of the task and laterality [F (1, 66) = 5.438, $p < .05$, $\eta_p^2 = .076$]. During 280-330 ms the significant differences [$p = .000$, 95% *CI* [.415, .1.367]) between mean absolute amplitudes evoked by solving equality-tasks and inequality-tasks were achieved in PO4-PO8 electrodes with higher activation related to equalities. In contrast, during 330-700 ms the similar significant differences [$p = .008$, 95% *CI* [.122, .793]) were found in PO3-PO7 electrodes.

Following Jung-Beeman et al., (2004), Leikin et al. (2016) argued that activation of PO4 and PO8 electrodes is associated with the insight moments of mathematical processing. Thus, we hypothesise that during 280-330 ms verification of both equalities and inequalities involves insight-related moment, which is replaced by analytical activity with shift to PO3-PO7.

CONCLUSION

The behavioural measures strongly confirmed our hypothesis that inequality-tasks embed higher cognitive demand than equality-tasks. We were intrigued by the findings that verifying inequalities evoke lower ERPs than verifying equalities, since we could predict that higher cognitive demand will be reflected in higher ERPs. However, the findings appear to be consistent with Kok (2001) who stated that working-memory updating caused by evaluation of complex stimuli and the intensity of cognitive processing affected by task difficulty is reflected in a decrease in P300 amplitude. Thus the lower electrical potentials during 280-700 ms (P300-like component) associated with verification of inequalities can indicate higher cognitive demand of inequality-tasks.

Acknowledgements

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References

Anderson, J. R., Betts, S., Ferris, J. L., & Fincham, J. M. (2011). Cognitive and metacognitive activity in mathematical problem solving: prefrontal and parietal patterns. *Cognitive, Affective, & Behavioral Neuroscience*, *11*(1), 52-67.

Bazzini, L. & Tsamir, P. (2004). Algebraic equations and inequalities: issues for research and teaching. Research Forum 02. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 139-161). Bergen, Norway: PME.

De Smedt, B., Noël, M. P., Gilmore, C., & Ansari, D. (2013). How do symbolic and non-symbolic numerical magnitude processing skills relate to individual differences in children's mathematical skills? A review of evidence from brain and behavior. *Trends in Neuroscience and Education*, *2*(2), 48-55.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. *Cognitive Neuropsychology*, *20*(3-6), 487-506.

Dreyfus, T., & Eisenberg, T. (1985). A graphical approach to solving inequalities. *School Science and Mathematics*, *85*(8), 651-662.

Dunn, B. R., Dunn, D. A., Languis, M., & Andrews, D. (1998). The relation of ERP components to complex memory processing. Brain and Cognition, 36(3), 355-376.

G. Hanna and M. DeVillers (Eds.) (2012) *Proofs and Proving. ICMI Study-19 Volume*. Springer.

Jung-Beeman, M., Bowden, E. M., Haberman, J., Frymiare, J. L., Arambel-Liu, S., Greenblatt, R., & Kounios, J. (2004). Neural activity when people solve verbal problems with insight. *PLoS Biology*, *2*(4), 500–510.

Kok, A. (2001). On the utility of P3 amplitude as a measure of processing capacity. *Psychophysiology*, *38*(3), 557-577.

Laborde, C. (2005). The hidden role of diagrams in students' construction of meaning in geometry. In J. Kilpartrick, C. Hoyles, O. Skovsmose & P. Velero (Eds.), *Meaning in mathematics education* (pp. 159-179). New York, NY: Springer.

Leikin, R., Waisman, I., & Leikin, M. (2016). Does solving insight-based problems differ from solving learning-based problems? Some evidence from an ERP study. *ZDM, 48*(3), 305-319.

Lyons, I. M., & Beilock, S. L. (2011). Numerical ordering ability mediates the relation between number-sense and arithmetic competence. *Cognition*, *121*(2), 256-261.

Neubauer, A. C., & Fink, A. (2009). Intelligence and neural efficiency. *Neuroscience & Biobehavioral Reviews*, *33*(7), 1004-1023.

Schneider, W., Eschman, A., & Zuccolotto, A. (2002). *E-prime Computer Software* (Version 1.0). Pittsburgh, PA: Psychology Software Tools.

Zacks, J. M. (2008). Neuroimaging studies of mental rotation: a meta-analysis and review. *Journal of Cognitive Neuroscience*, *20*(1), 1-19.

A CONCEPT ANALYSIS OF THE NOTION *CONCEPT*: CONTRIBUTIONS OF AN ANALYSING TOOL

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The word 'concept' is used with several meanings in mathematics education. In order to obtain a coherent theoretical framework, a concept analysis of the notion concept *is performed for some current frameworks in the field. The taken approach includes creating a tool for analysing views on* concept*, based on a literature review in philosophy. This tool uses three distinctions of views on* concept*: mental versus abstract, subjective versus intersubjective, and molecular versus holistic. Examples from texts in mathematics education are given, where the three distinctions are present. Further, the taken approach offers a perspective on* concept *that simplifies comparisons between frameworks.*

INTRODUCTION

It is not unusual that a word has different meanings in different contexts. In order to get a coherent theoretical framework, one might study the meanings of words, both in explications and within textual contexts. Concept analysis could be used as an integrated part of a research process, maybe not even mentioned, or it could be a study in itself. The purpose of such studies might be to solve conceptual problems, to create new concepts or to contribute to theory development (Nuopponen, 2010a, p. 6).

There are few methodological texts concerning concept analyses in mathematics education, and such analyses are not always explicitly described. One common approach is to, from a certain goal and delimitation, start with a literature review and then make a text analysis, where different views on a concept are interpreted, compared and categorised (Nuopponen, 2010b, p. 6). Yoon (2006) describes this process as:

> I began by collecting a large number of excerpts from the […] literature in which any of the three terms were used, which I then sorted into categories of distinct usages. This process gave me 11 distinct categories of term usage. I then created definitions to describe each of these categories (Yoon, 2006, p. 32)

Since the overall project in my research is to develop a framework for analysing concepts in mathematical problem solving, the object for my concept analysis is the notion *concept* itself. It would be an overly wide project trying to find all usages of *concept* within the field of mathematics education. Instead, the focus is on some current frameworks for conceptual understanding, found through a literature review that will not be described here. A possible approach for the concept analysis would then have been that of Yoon (2006). However, in an early phase of the study it became clear that such an approach did not offer sufficiently understanding for comparing

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 411-418). Umeå, Sweden: PME.

views on *concept* in different frameworks, it was hard to relate different views to each other. Based on that insight, I realised that some kind of pre-understanding was needed. Now, since views on *concept* are frequently discussed within philosophy and cognitive science, a literature review was conducted within these fields, resulting in a tool for analysing views on *concept*. From the characteristics in this tool, keywords were developed that might be seen in certain formulations. The procedure for finding views on *concept* in the educational texts consisted primarily in searching for these formulations. The purpose in this paper is to exemplify how the analysing tool may contribute to the concept analysis. The question is if the analysing tool can facilitate comparisons between views on *concept* in different frameworks. In the paper, some preliminary results from the study are chosen with the aim of showing the usage of the analysing tool.

THREE DISTINCTIONS

The analysing tool is the result of a configurative literature review, searching for key references and the broad lines within the discussion of *concept* in philosophy and cognitive science. The selection of references is based on how clear the description of *concept* is and if they contribute with new perspectives. The tool makes three distinctions, represented in the matrix in Figure 1. The first distinction is between concepts seen as mental and concepts seen as abstract in a Platonic sense, the second is between concepts seen as subjective and concepts seen as intersubjective, and the third is between concepts seen as molecular and concepts seen as holistic. These three distinctions are described further below.

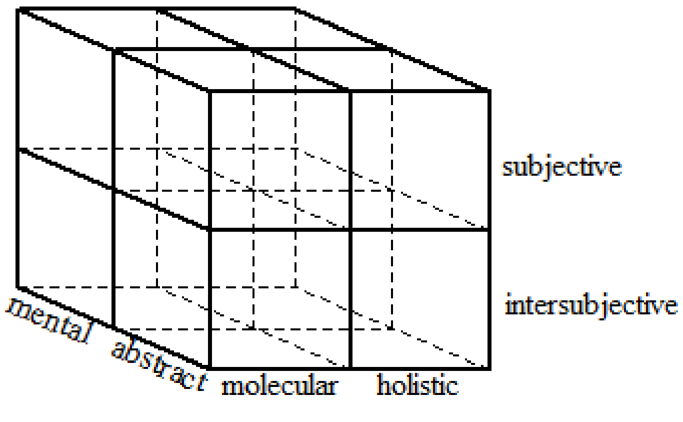

Figure 1: The three distinctions in the analysing tool, represented as a 3D matrix.

In analysing views through the usage of words in texts, the method is to search for keywords and formulations indicating the different views. These keywords together with the matrix in Figure 1 constitute the analysing tool for the concept analysis.

Concepts seen as mental or as abstract

Mental views on *concept* are common in contemporary empiricism (Jenkins, 2008, p. 120), and cognitive science (Murphy, 2004, p. 1; Carey, 2009, p. 4). In an educational context, Piaget, *e.g.*, explicates *concept* as a mental representation (Furth, 1969, p. 53). Keywords for a mental view are terms like ‘conceptual representation’ and ‘constructing concepts’, and also expressions indicating a Piagetian view, like ‘acquiring concepts’. Abstract views on *concept*, on the other hand, are common within philosophies of language (Katz, 1972/1999, p. 133; Zalta, 2001, p. 345), where concepts might be senses (independent of the mind) that form the meanings of words. In educational contexts, such a view might appear in a social theory of learning. The word ‘abstract’ is the antonym to ‘concrete’, and in educational contexts, mental objects are often seen as abstract. Here, however, ‘abstract’ is used only in the meaning of non-mental, as, *e.g.*, in a Platonic sense. Keywords for an abstract view are terms indicating that concepts are meanings of words and also formulations claiming that concepts are building blocks in formal mathematics. Further, expressions like ‘talk about a concept’ indicate an abstract view, since in a mental view we are using concepts in our thinking, we do not think about them.

Concepts seen as subjective or as intersubjective

Regardless of whether concepts are seen as mental or abstract, there is also a distinction between a subjective concept (or ‘individual concept’) and an intersubjective concept. A subjective concept might be a personal representation (Carey, 2009, p. 354), or the abstract content of a representation (Zalta, 2001, pp. 344-345). The difference between these two views do not have practical consequences for education; in both cases, the child develops his or her own concepts. Keywords for a subjective view are seen in formulations like ‘our own way of constructing concepts’ and ‘the students’ concepts’. An intersubjective concept, on the other hand, might either be abstract and something that several people share through the usage of language (Katz 1972/1999, p. 133), or a mental representation partly integrated in a collective understanding (Potter & Edwards, 1999, p. 448). In both cases, our personal understanding could be evaluated against the intersubjective concept. Keywords for an intersubjective view are seen in formulations indicating that concepts are developed in a culture of, *e.g.*, mathematicians or teachers, like ‘the concept *vector* in the curriculum’, and expressions like ‘the concept rational number’ (indicating that there is just one such concept).

Concepts seen as molecular or as holistic

Two different views on conceptual structures are used in the tool. First, the molecular view claims that concepts are hierarchically structured, that some concepts are more basic than others, and that complex concepts could be defined from basic ones. The assumption that definitions are central for conceptual understanding is embraced by empiricists (Jenkins, 2008, p. 127) and some philosophers of language (Katz, 1972/1999, pp. 127-128). In mathematics education, van Hiele (1957/2004, pp. 64-65), *e.g.*, claims the importance of developing the capacity of using definitions in formal reasoning and proofs. Keywords for a molecular view are seen in formulations like ‘every concept is a basic concept or well-defined by means of other basic or well-defined concepts’ and in formulations about hierarchical structures. Second, the holistic view claims that concepts interrelate with each other in a web-like structure, in which no concept is more basic than another. This model is based on Wittgenstein's (1953, p. 32) idea that the cognitive structure forms a complicated network of similarities and that concepts relate to each other in many different ways. In mathematics education, Vergnaud (1988/2004, p. 85), e.g., claims that we must study conceptual fields instead of single concepts. Keywords for a holistic view are found in formulations like 'concepts are parts of various heterogeneous systems and form intricate webs' and in criticism of a molecular view.

A comment to the distinctions

Note that the views in the above distinctions should not be considered incommensurable. Jackendoff (1989/1999, pp. 305-306) argues that it is possible to have a dual view, including both a view considering concepts as abstract and a view considering them as mental. Also, this argumentation holds for the distinction between a subjective and an intersubjective view. One way of handling such a combined view is to distinguish between the student's individual concept and a given mathematical concept. Murphy (2004, p. 488), in turn, advocates a combination of a molecular and a holistic view, based on results in psychological research. As stated before, the matrix in Figure 1 together with the keywords constitute the analysing tool for my concept analysis. In the next section it is shown how this tool is used in some examples taken from various texts. The selection of examples is meant to highlight different aspects of the tool.

EXAMPLES OF ANALYSES

The following three examples aim at showing how the different views in the distinctions appear in mathematics education, and at showing the width of the analysing tool.

Example 1: A molecular view on *concept*

In Kobiela and Lehrer (2015) a molecular view on *concept* is present, as seen in the following quote:

> Defining also appears to support students' description of objects, moving them away from holistic descriptions toward more mathematical descriptions that focus on relevant parts and properties (Kobiela & Lehrer, 2015, p. 427)

An intersubjective view appears in formulations like "concepts developed by the class" (Kobiela & Lehrer, 2015, p. 431). There is neither a clear abstract nor mental view on *concept* in the text, as depicted in the left matrix in Figure 2 below.

Example 2: A holistic view on *concept*

Schacht and Hußmann (2014) explicitly use the inferential theoretical perspective, based on Wittgenstein's holistic view, and state that "[s]ince our concepts are always inferentially related by the commitments we acknowledge, this implies a holistic perspective on concepts […]" (Schacht & Hußmann, 2014, p. 99). In this text, there are both a subjective view on *concept*, sometimes addressed by the term 'individual concept', and an intersubjective view, when several students are introduced to the concept of variable (Schacht & Hußmann, 2014, p. 100). These two views are presented in the right matrix in Figure 2.

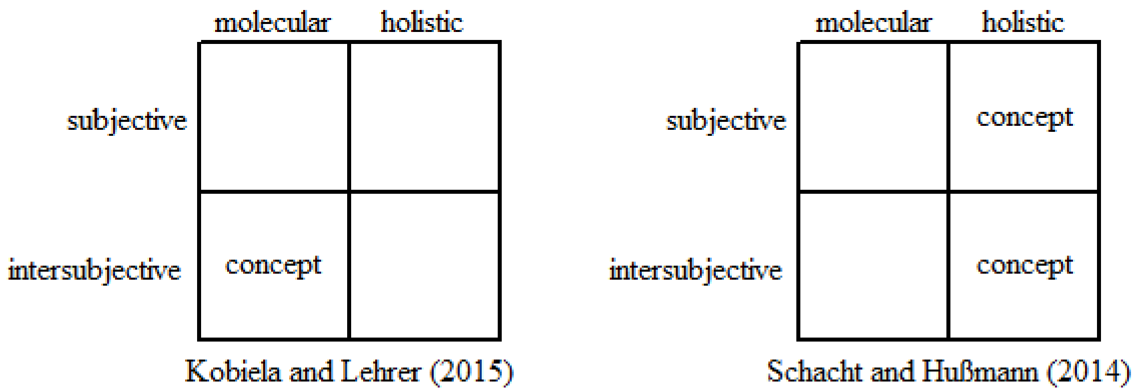

Figure 2: One molecular and one holistic view on *concept*.

Example 3: The framework of Three worlds of mathematics

In Tall (2013), concepts are sometimes seen as mental and sometimes as abstract. In the following quote, the word 'concept' refers to a mental representation (via the usage of 'schema' in a Piagetian meaning): "the duality between concept and schema is based on the same fundamental idea in which a named concept has rich internal links that reveal it to be a schema" (Tall, 2013, p. 80). Occasionally, terms like 'mental number concept' Tall (2013, pp. 6, 15) are used to address such a view. An abstract view appears in formulations like 'we think about […] the concept of number' (Tall, 2013, p. 13) and 'concepts in the calculus' (Tall, 2013, p. 7). Hence, 'mental concept' in the text refers to a mental representation, but *concept* seems, dually, to be sometimes mental and sometimes abstract.

Further, there are both an intersubjective and a subjective view on *concept* in the text. While the intersubjective view appears in formulations like "children are introduced to counting physical objects to develop the concept of number" (Tall, 2013, p. 7), the subjective view is present in formulations like '[o]ur biological brains evoke thinkable concepts by a selective binding of neural structures' (Tall, 2013, p. 24).

To summarise some results from the analysis, there are three different views on *concept* in Tall (2013), presented in Figure 3. While the first view sees concepts as abstract and intersubjective, the second view sees them as mental and intersubjective, and the third view sees them as mental and subjective. No signs indicating a subjective and abstract view have been found. Further, there is not a clear description either of a molecular view or of a holistic view in the text.

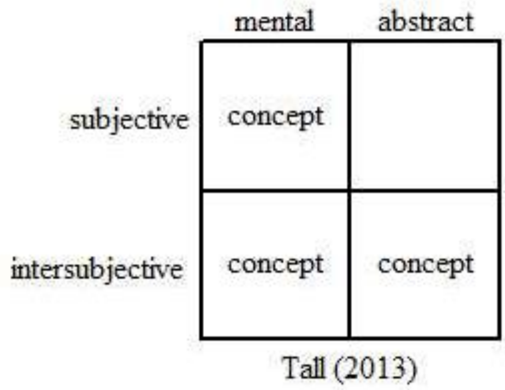

Figure 3: Views on *concept* in the framework of Three worlds of mathematics.

A summary of the concept analysis so far

These examples show that the three distinctions in the analysing tool is present in mathematics education, even if they are not present in all frameworks; one, two or three distinctions might be used in the analyses. Another conclusion is that while some frameworks have one single view on *concept*, other frameworks have several views (more or less explicit in the texts). Further, an overall conclusion and an answer to the question in the introduction is that the analysing tool can be used for comparisons between views on *concept* in different frameworks.

DISCUSSION AND CONCLUSIONS

The purpose of this paper has been to exemplify how an analysing tool may contribute to a concept analysis. From the examples shown above, the three distinctions are relevant for mathematics education and one might analyse views on *concept* in the field, with the help of this tool. However, it seems as if the three distinctions might have different roles. Concepts seen as mental and concepts seen as abstract may generally appear on two different arenas. The first one is a cognitive arena where concepts are considered mental representations. The other one is an abstract arena where concepts appear in a linguistic or mathematical context. Research in mathematics education have different purposes. Sometimes it might be to explain a psychological development of the individual, sometimes it might be to explain communication and a social context, and sometimes it is to understand the nature of mathematics. The reason for why there are several views on *concept* in Tall (2013), e.g., might be that the framework combines approaches from several theories of learning. Further, subjective and intersubjective views are found in the relation between the individual and the community, or between the student and the curriculum. The student develops his or her own subjective concept, which is evaluated against a curricular concept. The last discussion between a molecular view and a holistic view, I should say, is not about different views on what concepts are. Rather, they describe different aspects of the conceptual structure. From the limited selection of examples in this paper, a molecular view seems to be more common in geometric studies, while a holistic view seems to be more common in arithmetic studies. However, this is an assumption that needs further evaluation.

Also, the three dimensions of the tool might vary in importance, depending on the context. In some texts the difference, e.g., between mental and abstract becomes important, while in others this distinction may be less relevant. The same holds for the other dimensions. Hence, the tool should be used with care. Ideally, a clear theoretical framework should have only one view on *concept*. That would imply one cell in the matrix in Figure 1. Schacht and Hußmann (2014), however, have a dual view and Tall (2013) have three different views. There might be good reasons for using several views on *concept*. As one example, it might be appropriate to have both a subjective view and an intersubjective view. However, if different views are implicitly combined, it could be difficult to interpret what the framework actually describes, making it problematic to use as a theoretical base for educational research. It is my position that such a combined view has to be explicit and clear. One way might be to use different terms, like ‘individual concept’ and ‘concept’, when different views are addressed. This might, however, cause problems of ontological character.

Another question concerns what a methodology including an analysing tool, developed from views on *concept* in other fields, could offer. Here it might perhaps be interesting to make a comparison with the concept analysis in Yoon (2006), which includes an analysis of the notion *conceptual system* in the Models and Modeling literature. One result from that study is that there are three views on the notion in the texts (Yoon, 2006, pp. 32-36). One difference is that while Yoon (2006) analyses several notions in one single framework, and relates them to each other, I, on the other hand, analyses the single notion *concept* in several frameworks. The analysing tool seems to facilitate such a comparison between frameworks.

The analysing tool is delimited to the notion *concept* and perhaps some related notions. Also, it is developed from the perspective that it should be used for educational purposes. There are philosophical views that have not been included in the tool, since I judged that they were not relevant. Further, the tool is designed to be used for text analysis. In order to be used in other kinds of studies, there might be other aspects that have to be considered. What has yet to be done in my study is to use the analysing tool in a more systematic literature review. One result from that work will be a view on *concept* that builds a ground for a new framework for analysing concepts in mathematical problem solving. Another result will be an insight in how frameworks could be compared and combined.

References

Carey, S. (2009). *The origin of concepts*. New York: Oxford University Press.

Furth, H.G. (1969). *Piaget and knowledge: Theoretical foundations*. Englewood Cliffs, N.J.: Prentice-Hall.

Jackendoff, R. (1989/1999). What is a concept, that a person may grasp it? In E. Margolis & S. Laurence (Eds.), *Concepts: Core readings* (pp. 305-333). Cambridge, MA: MIT Press.

Jenkins, C. S. (2008). *Grounding concepts – an empirical basis for arithmetical knowledge*. New York: Oxford University Press.

Katz, J. (1972/1999). On the general character of semantic theory. In E. Margolis & S. Laurence (Eds.), *Concepts: Core readings* (pp. 125-149). Cambridge, MA: MIT Press.

Kobiela, M., & and Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. *Journal for Research in Mathematics Education*, *46*(4), 423-454.

Murphy, G. (2004). *The big book of concepts*. Cambridge, MA: MIT Press.

Nuopponen, A. (2010a). Methods of concept analysis – a comparative study. *LSP Journal*, *1*(1), 4-12.

Nuopponen, A. (2010b). Methods of concept analysis – towards systematic concept analysis. *LSP Journal*, *1*(2), 5-14.

Potter, J., & Edwards, D. (1999). Social representations and discursive psychology: From cognition to action. *Culture & Psychology*, *5*, 447-458.

Schacht, F., & Hußmann, S. (2014). Making generalizations explicit: An inferential perspective on concept-formation of variables. In P. Liljedahl, S. Oesterle, C. Nicol, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 5, pp. 97-104). Vancouver, Canada: PME.

Tall, D. (2013). How humans learn to think mathematically: Exploring the three worlds of mathematics. New York: Cambridge University Press.

Van Hiele, P.M. (1957/2004). The child's thought and geometry. In T. P. Carpenter, J. A. Dossey & J. L. Koehler (Eds.), *Classics in mathematics education research* (pp. 60-66). Reston, VA: NCTM.

Vergnaud, G. (1988/2004). Multiplicative structures. In T. P. Carpenter, J. A. Dossey & J. L. Koehler (Eds.), *Classics in mathematics education research* (pp. 84-96). Reston, VA: NCTM.

Wittgenstein, L. (1953). *Philosophical investigations*. Oxford: Blackwell Publishers.

Yoon, C. (2006). *A conceptual analysis of the models and modeling characterization of model-eliciting activities as "thought-revealing activities"*. Bloomington, IN: Indiana University.

Zalta, E. N. (2001). Fregean sense, mode of presentation and concepts. *Philosophical Perspectives*, 15, 335-359.

THE DEVELOPMENT OF THE CONCEPT OF LIMIT – ASPECTS AND BASIC MENTAL MODELS

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The concept of limit is one of, if not the basic concept of analysis. During the 20th century, various alternatives for teaching this concept, initially in a collegiate environment, were developed which considerably influenced analysis teaching at schools. Currently, it is dominated by the so-called propaedeutic or intuitive concept of limit, in which analysis teaching immediately dives into working with real functions; a formal definition of the limit is waived in favour of an intuitive approach. In the context of teaching based on understanding, this approach needs carefully developed basic mental models, if the understanding of the concept is to surpass the intuitive level and is to be advanced into a mathematically accurate understanding. In order to achieve this – and this is the central hypothesis of this article – the concept of sequences is an essential resource to develop the concepts of limit and infinity.

First we will shortly describe the main features of the *propaedeutic concept of limit.* This concept must be viewed in the context of the historic development of the concept of limit and its interdependence with the concepts of infinity and sequences. Moreover, it is now indisputable that the development of adequate perceptions – or mental models – of a mathematical concept, based on various representations and illustrations, is essential and important to teaching rooted in understanding. In terms of the development of the concept of limit, this article suggests (re-)integrating sequences and discrete ways of thinking more heavily into analysis teaching.

THE PROPAEDEUTIC CONCEPT OF LIMIT

In contemporary analysis teaching – in Germany as well as worldwide – the so-called "propaedeutic concept of limit" is the prevalent approach to the concept of limit (cf. Törner et.al. 2014). In this, analysis teaching waives a formal definition of limit in favour of an intuitive approach. This is accompanied by phrases like "x (or f(x)) approaches a value arbitrarily closely" or "x (or f(x)) differs from … arbitrarily little".

Behaviour towards infinity and near singularities

In the context of the propaedeutic concept of limit, today's students typically begin their analysis studies by analysing the behaviour of, for example, a rational function f with $f(x) = \frac{5x+1}{x+3}, x \in \mathbb{R}\backslash\{-3\}$ for "big x-values" or "towards infinity". Then, the behaviour of rational functions is analysed in the neighbourhood of singularities, where the dynamic process of approaching a singularity x_o is described with above-mentioned dictions "x tends towards x_o" or "x approaches x_o arbitrarily closely".

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 419-426). Umeå, Sweden: PME.

Access to the concept of derivation

This propaedeutic concept of limit is then transferred to determining the slope of a graph of a function at a point *P*. Coming from a function with e.g. $y = f(x) = x^2$, the slope m_{PQ} of two points *P(x, f(x))* and *Q(x+h, f(x+h))* of the graph with any point *P* on the graph and "a small number *h* with $h \neq 0$" is determined as follows:

$$m_{PQ} = \frac{(x+h)^2 - x^2}{(x+h) - x} = \frac{2xh + h^2}{h} = 2x + h.$$

If h tends towards 0, then $2x + h$ tends towards 2x. The slope of the graph with $y = x^2$ at point *P(x,y)* is then defined or determined as 2x.

This concept dates back to the two (famous) mathematicians Emil Artin's (1957) and Serge Lang's (1964) analysis lectures; they wanted to reduce formal work in their university classes and therefore returned the concept of limit to an intuitive basis. Thus, Lang establishes the concept of limit on the grounds of intuitive understanding in his definitions of continuity and differentiability of real functions; a supplemental epsilon-delta-definition of this concept is located merely in the appendix of his book. These considerations crucially influenced the development of school analysis in the context of the "intuitive limit concept" (cf. Weigand 2014).

The concept of limit in its historical development

To understand the background of this development and also to better classify problems and difficulties discovered of nowadays students it is necessary to follow the mathematical-historical developments as well as their transfer to collegiate courses and mathematics classes. Because of space reasons, we have to skip these historical considerations. We will only mention that the development of the concept of limit is to be viewed – from the beginning in its ancient times – in close relation to the concepts of infinity and sequences, and it shows how *static* and *dynamic*, *intuitive* and *formal* perceptions occur in continual interaction (cf. Greefrath et al. 2016). Moreover, there are some considerations and empirical investigations concerning the understanding of the limit concept, e.g. Monaghan 1991, Davis & Vinner 1986, Cottrill et al. 1996, Keene et al. 2014, which influenced this study.

ASPECTS ON THE CONCEPT OF LIMIT

Insightful learning and teaching of mathematical concepts requires establishing perceptions of a concept, knowing representations and properties, integrating the concept into a mind map of concepts and the ability to apply the concepts in various inner- and extra-mathematical fields. Today, the development of *Aspects* and *Basic Mental Models* of a mathematical concept is seen as an essential requirement for understanding the concept (cf. Weigand et al. 2017 and vom Hofe et al. 2005). *Basic Mental Models* are based on *Aspects* of a concept.

> An *Aspect* of a mathematical concept is a subdomain of the concept that can be used to characterize it on the basis of mathematical content.

The historic development of the concept of limit shows two aspects: The *dynamic aspect* on the possibility of an “and so forth” or a potentially infinite process, based on the (mental) successive run through the start of the natural number sequence or the gradual implementation of an act on the enactive, iconic or symbolic level, create the intuitive basis for the concept of infinity and the concept of limit.

When viewing the concept of limit from the *static aspect*, however, it is necessary to “reverse” the argumentation in terms of sequences and “stop” the dynamic aspect. Based on a fixed value, the search begins for a sequence element after which all further elements are located in a predetermined neighbourhood of the fixed value. This “inversion” of the line of thought is the basis for formal definitions of the limit.

BASIC MENTAL MODELS ON THE CONCEPT OF LIMIT

Basic Mental Models (or – in German – Grundvorstellung) give meaning to an *Aspect* of a concept.

> A *Basic Mental Model (BMM)* of a mathematical concept is a conceptual interpretation that gives meaning to it.

Interpretations with regards to content are understood inner-mathematically through various representations and illustrations, and extra-mathematically through adequate situations for application which give meaning to the concept. *BMMs* capture mathematical *Aspects* of a mathematical concept and attach sense and meaning to it. The relation *“Aspect – BMM*” of a given mathematical concept is not one-to-one. An Aspect of a mathematical concept can provide a basis for several *BMMs*. Vice versa, a specific *BMM* can be developed with respect to several *Aspects* and give them meaning. *Aspects* and *BMMs* can be considered as a specification of Tall’s and Vinner’s theoretical framework of “Concept Image – Concept Definition” (cf. Weigand et al. 2017).

Based on the dynamic and static perceptions of the limit, the following will differentiate between three basic mental models, namely the *notion of approximation*, the *notion of neighbourhoods* and the *notion of objects*.

The BMM of approximation

The intuitive idea of the *BMM of approximation* is based on the tending of sequence elements towards infinity or their approach of a fixed value. The idea of an – in principal – unlimited continuation provides the basis of the *BMM* of an infinite process. In mathematics teaching, the development of this *BMM* can be started by analysing convergent, explicitly given sequences, and continued with iteration sequences, it is then specified by the method of nested intervals or by recognizing (defining) a tangent as a limit of a family of secants. Students should develop the following knowledge and abilities or competencies in the frame of this *BMM*: They

- recognize the – generally – arbitrary continuation of the natural number series as the basis of a dynamic “process of infinity”,

- recognize the possibility of nearing the “limit”, e.g. numbers or geometrical objects (and maybe even reach it) in the unbounded continuation of “infinite processes”,
- can name examples for “infinite processes” and respective “limit objects” on the symbolic, iconic and enactive level,
- connect graphical and numerical notions with the process of “tending towards” and “approximate arbitrarily closely”,
- can – later on and at an advanced level – explain the development of the concepts of derivate and integral through the help of methods of approximation.

The BBM of neighbourhood

The dynamic process of continually running through a sequence is “stopped” by the “reversion” of the line of thought: starting from a fixed value – the limit – and a predetermined arbitrarily small neighbourhood, such that a sequence element can be found or named, after which all following sequence values are located in this neighbourhood. Ultimately, “only” a number must be found whose respective sequence value upholds a certain condition. In these terms, this notion is based on the static aspect of the limit. The *BMM of neighbourhood* is based on the idea that for any arbitrarily small neighbourhood around the limit, all further elements after a certain sequence element are located in this neighbourhood. The students

- connect graphical and numerical perceptions and representations with the “neighbourhood” of a number or a graphic object and relate it to sequence values,
- can describe limit behaviour on the verbal, numeric, graphic and ultimately on the formal level,
- can describe limits of recursively defined sequences numerically and graphically, and they know criteria for convergent behaviour in linear iteration sequences,
- can describe nested intervals and iterative methods for e.g. solving equations as a process of limit.

The BMM of objects

A limit can be a number, like a number sequence or with the determination of a circle area, it can be a geometric object like a point, a distance, or a tangent as limit of secants, a matrix with stochastic processes, or a function, as a limit of a family of functions. In the context of the *BMM of objects*, limits are viewed as mathematical objects – namely (fixed) values, matrices, or geometric objects – which are constructed or defined through a sequence of numbers, of matrices or of geometric objects. The *BMM of objects* focuses on the symbolic or formal aspect of the concept of limit. Students

- view null sequences as prototypes for convergent sequences,

- recognize the behaviour of sequence and function values for the “behaviour towards infinity” and near singularities by considering term representations – particularly with rational sequence- and function terms,
- can classify sequences and functions by considering their limit behaviour,
- can calculate limits of sequences and functions from a given term representation,
- know the constraints of calculating limits, e.g. of trigonometric or exponential sequences and functions.

The two *Aspects* and three *BMMs* are summarized in Figure 1, including the relations between the different *Aspects* and *BMMs*. The connecting lines indicate that the *Aspect* is a basis of the related *BMM* and that the *BMM* gives meaning to the *Aspect*.

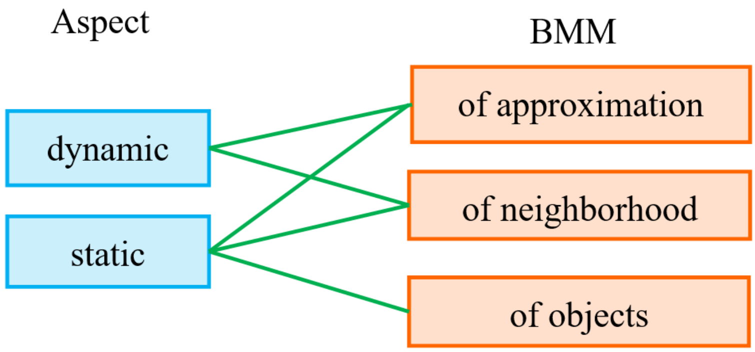

Figure 1: Aspects and BMMs of the concept of limit.

A CONCEPT FOR A DISCRETE APPROACH TO THE CONCEPT OF LIMIT

The concept of the propaedeutic limit has proven its worth to analysis teaching with the early access to the concept of derivative and thus its early integration of applications and modelling problems. However, the concept must be recognized as problematic in the context of building an adequate understanding of basic concepts of analysis, if the goal is beyond an intuitive understand and beyond training technical skills. Developing and emphasizing *BMMs* of the concept of limit in mathematics learning is one point of view if a deeper understanding of the main concepts of calculus is aspired.

A return to the formal concept of the 1960s and its extensive treatment of the concept of sequences at the beginning of the calculus class is not possible due to the current organizational constraints of mathematics teaching as well as the extent of assigned topics (e.g. stochastics, computer science, modelling). Nevertheless, there is an important difference between today’s math education and that of the 1960s and 70s: The use of digital technologies opens new possibilities for the calculation and representation of mathematical concepts. In particular, this applies to the treatment of discrete functions and sequences:

- It becomes easy to numerically or graphically display sequences “at the press of a button” from a given assignment rule;
- Limit processes can be visualized in more details by using the “zoom-tool”;

- The interactive interdependency of different representations of sequences can be analysed on the symbolic, graphic, and numerical level at the same time;
- Limit considerations with recursively defined sequences can be simplified at least on the representational level since those sequences can be numerically and graphically displayed "at the press of a button".

But, if calculations are transferred or outsourced to digital technologies and simplified for the user, *BMMs* will become even more important in learning and teaching, concerning developing understanding, for working mathematically with concepts and for anticipating solutions of problems. The following outline – in a very short form – gives the idea of a step-wise or levelled concept of a discrete *approach* to the concept of limit, based on the concept of sequences and the *BMMs* of the concept of limit.

Level 1: Approaches on the numerical and graphical level

In lower secondary classes students get experiences with limit processes in various ways under the dynamic aspect and the BMM of approximation. Examples are decimal fractions like $\frac{1}{3} = 0{,}33333\ldots = 0.\overline{3}$, the development of the concept of irrational numbers which leads to the Heron method of nesting irrational numbers like $\sqrt{2}$, or the calculation of a circle area which leads to the iterative calculation of π, following Archimedes. Geometry, in particular, provides the opportunity of visually representing limit processes, e.g. the construction of a square inside a given square (Figures with graphical representations had to be skipped because of space reasons).

Level 2: Critical considerations on the (intuitive) understanding of limits

Harmonic series are the prototype of series whose divergent behaviour cannot be developed intuitively. In his book "The Paradoxes of the Infinite" (1851), Bolzano names many examples which all show that it is an essential task of mathematics to clarify "which concept we really link to the infinite" (p. 1). These show especially the constraints of naïve or intuitive perceptions in the context of the concept of limit and they also show the limits of *Aspects* and *BMMs* and the interrelationship between these concepts.

Level 3: Explicitly defined sequences

Working with sequences in the acquisition of the concept of limit as opposed to working with (generally real) functions provides the advantage that domain and co-domain are discrete sets. Thus, (some) properties like monotonicity, boundedness and convergent behaviour can be well-visualized and more easily justified at least pre-formally in an argumentative manner than when working with continuous domains. Examples are sequences $(a_k)_N$ with $a_k = \frac{1-3\cdot k^2}{k^2}$ or $a_k = \frac{5^k}{k!}$, $k \in \mathbb{N}$.

Level 4: Recursively defined sequences – the notion of iteration

Digital technologies gain importance particularly when working with recursively defined sequences since they take over the successive calculations and representations.

Determining the limit is generally more difficult due to the lack of an explicit term representation. Yet, this is volitional and used explicitly to focus more strongly on working experimentally and heuristically. An example is the sequence $(a_k)_N$ with $a_k = -0{,}6 \cdot a_{k-1} + 3$, $a_o = 1$, $k \in \mathbb{N}$. Recursively defined sequences are well-displayed in spider web diagrams”. To work with recursively defined sequences more generally, e.g. with $a_{k+1} = A \cdot a_k$, or $a_{k+1} = A \cdot a_k + B$, a more in-depth understanding of limits is necessary (cf. Weigand 2004).

Level 5: An approach to the formal definition of limit

It is essential to acquire the *way of thinking* and the *mental model* of the “epsilon-delta-definition” or “epsilon-n_o-definition”. It is based on the *BMM of neighbourhood* for the concept of limit. A methodological resource for understanding the *dynamic-static*-interdependency in the development of the concept of limit is a dialogue between a “proponent”, trying to defend a claim or assumption, and an “opponent” who tries to refute them. Digital technologies are well-suited for visualizing this idea of “dialogical reasoning”. This “dialogue play” specifies the phrases “tends towards” or “approaches arbitrarily closely”.

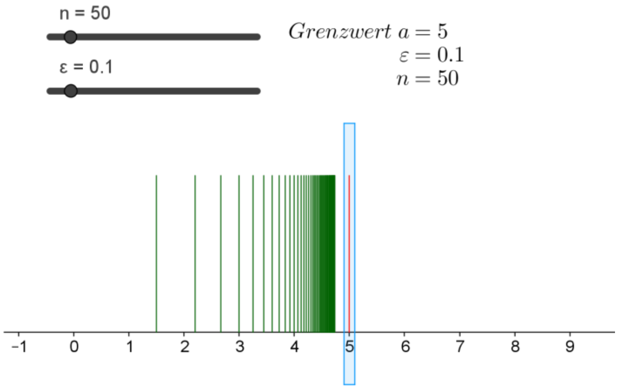

Figure 2: The first 50 elements of the sequence with $\varepsilon = 0.1$.

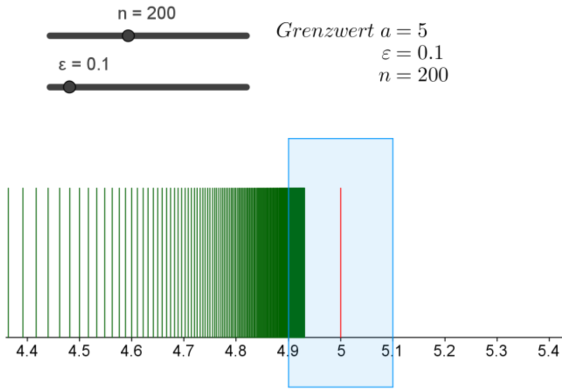

Figure 3: The first 200 elements of the sequence with $\varepsilon = 0.1$.

FINAL REMARK

The concept of the propaedeutic limit currently prevalent in analysis teaching should be enriched or complemented with mental models which build a stable base for a representation of limits coined by content and ultimately also by formality. The concept of sequences provides a helpful resource and digital technologies are important tools for calculations, for representing sequences and consequently for the developing of *Aspects* and *BMM* of limits. The possible extent or the level of such a development in calculus teaching depends on many conditions; mainly, however, it depends on the goal that students must reach at the end of the class (cf. Rasmussen et al. 2014). In teaching oriented towards understanding, there should and must be enough time available for a thorough development of the basic concepts of calculus.

At the moment a lesson study is developed and it will be tested the next months in calculus classes in Germany.

References

Artin, E. (1957). *A Freshman Honors Course in Calculus and Analytic Geometry*, Taught at Princeton University, Charlottesville, Virginia.

Bolzano, B. (1950, orig. 1851): *Paradoxes of the Infinite*. London: Routledge.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *The Journal of Mathematical Behavior, 15*(2), 167–192.

Davis, R. B., Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconceptional stages. *Journal of Mathematical Behavior, 5*(3), 281–303.

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V., Weigand, H.-G. (2016). Aspects and "Grundvorstellungen" of the Concept of Derivative and Integral. *Journal für Mathematikdidaktik, (37)* H. 1., 99-129.

Hofe, R. v., Kleine, M., Blum, W., Pekrun, R. (2005). On the Role of “Grundvorstellungen” for the Development of Mathematical Literacy – First Results of the Longitudinal Study PALMA, *Mediterranean Journal for Research in Mathematics Education, 4*, 67 – 84.

Keene, K. A., Hall, W., Duca, A. (2014). Sequence limits in calculus: using design research and building on intuition to support instruction. *ZDM 46 (4)*. 561-574.

Lang, S. (1964). *A first Course in Calculus*. Amsterdam u. a.: Springer.

Monaghan, J. (1991). Problems with the language of limits. *For the Learning of Mathematics, 11*(3), 20–24.

Rasmussen, C., Marrongelle, K., Borba, M.C. (2014). Research on calculus: what do we know and where do we need to go? *ZDM – The International Journal on Mathematics Education* 46(4). 507–515.

Törner, G., Potari, D., Zachariades, Th. (2014). Calculus in European classrooms: curriculum and teaching in different educational and cultural contexts. *ZDM – The International Journal on Mathematics Education 46*(4). 549–560.

Weigand, H.-G. (2004). Sequences - Basic Elements for Discrete Mathematics, *ZDM 36 (3)*. 91-97.

Weigand, H.-G. (2014). A discrete approach to the concept of derivative. *ZDM - The International Journal on Mathematics Education 46(4)*. 603–619.

Weigand, H.-G., Greefrath, G., Oldenburg, R., Siller, S., Ulm, V. (2017): Aspects and Basic Mental Models ("Grundvorstellungen") of basic Concepts of Calculus. To appear: *Proceedings of the PME in Singapore July 2017*.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*,12 (2), 151-169.

EXAMINING EXPLORATIVE INSTRUCTION ACCORDING TO THE REALIZATION TREE ASSESSMENT TOOL

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In this paper we present a comparison of 10 lessons based on the same task – the Hexagon task. We used the RTA (Realization Tree Assessment) tool in order to compare the implementation of this task by 10 middle school teachers undergoing professional development intended to enhance explorative instruction. We focused on three aspects in our comparison: the number of realizations, the links between the realizations, and the narratives of 'saming' algebraic expressions. Results show a wide variance between lessons in number of realizations and in the extent to which links were made between them. The quantification of these aspects enabled us to rank the lessons according to RTA "robustness" to provide a measure of explorative instruction and link it with grade level and track.

INTRODUCTION

Focusing on mathematical concepts during the lesson has been found by several major studies to be one of the most effective means for student' learning (Hiebert & Grouws, 2007). Recent years have been marked by increasing efforts to emphasize the conceptual aspects of mathematics together with an emphasis on students' agency and authority (National Council of Teachers of Mathematics, 2000). Professional development efforts have focused on helping teachers afford students opportunities to engage with cognitively demanding tasks, while clarifying important mathematical concepts and ideas (e.g. Boston & Smith, 2009). Yet this effort has been constrained by lack of sufficient tools for examining the extent to which instruction indeed affords explicit attention to concepts. In this paper, we suggest examining the conceptual aspects of mathematics instruction through the Realization Tree Assessment (RTA) tool (Weingarden, Heyd-Metzuyanim, & Nachlieli, 2017). Using the RTA, we inquire into the catalysts of explorative instruction.

THEORETICAL BACKGROUND

We define explorative mathematics instruction as instruction that supports explorative participation in mathematical learning. Explorative participation (Sfard & Lavie, 2005) is participation for the sake of producing mathematical narratives to solve problems or to describe the world. Such participation is contrasted to ritual participation, which main goal is pleasing others and which is characterized by rigid rule following and endorsement of results as “correct” according to external authority. Explorative participation is linked more broadly to the view of mathematical learning as the process by which students gradually become able to communicate about ma-

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 427-434). Umeå, Sweden: PME.

thematical objects (Sfard, 2008). These discursive objects are produced by discourse (or communication), and are made up of different "realizations" (ibid, p. 165). For example, the signifier ½, the process of dividing a pizza into two pieces, and the process of shading 3 circles out of 6, are all samed into the object "one half". Children often learn each of these realizations separately and only later come to relate to them all to one object. This is the heart of a process Sfard calls "objectification". Objectification, or talking about mathematical signifiers as "standing for" mathematical objects that "exist" in the world, is a major and necessary accomplishment for advancing in the mathematical discourse. A mathematical object can be visualized as a "realization tree" where complex objects are made of simpler ones. For example: a half is made of different realizations (1/2, 0.5, 50%, 3/6 etc.) but the whole numbers making up these realizations also have endless realizations (3 apples, 3 fingers, etc.).

Recent years have seen increasing efforts to train teachers to teach towards explorative instructional practices, but the change in teachers' practices has been found to be a complex process (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2016; Spillane & Zeuli, 1999). In particular, constructing tools for the detection of change in teachers' practices that would fit the ideas of a professional development for explorative instruction, is not a simple matter.

The Realization Tree Assessment (RTA) tool (Weingarden et al., 2017) was built in order to examine explorative instruction by assessing the extent to which students are exposed to different realizations of the mathematical object during the lesson. In our former work, we have used it mostly to visualize qualitatively differences between lessons based on an identical task. The usefulness of the tool to compare and rank the level of explorative instruction has not yet been explored. Such ranking can enable the examination of the relation between explorative instruction and other variables such as grade level or track.

In the present study we enhanced the RTA tool to provide a numerical view of explorative instruction. With this tool, we asked: how are realizations that are exposed in the classroom connected to opportunities to form narratives about mathematical objects? And how are these opportunities connected to grade level and track?

METHOD

The study reported here was performed in the context of the TEAMS (Teaching Exploratively for All Mathematics Students) project for training Israeli teachers to implement explorative instructional practices in middle school mathematics classrooms, using the "Five Practices for Orchestrating Productive Mathematics Discussions" (Smith & Stein, 2011) and "Accountable Talk®" (Resnick, Michales, & O'connor, 2010). As part of the PD, the teachers were asked to implement a task they encountered and experienced as learners in the PD session. This task is called 'the Hexagon Task' and it asks students to describe the perimeter of a general "train" in a pattern of hexagon "trains" (See Figure 1):

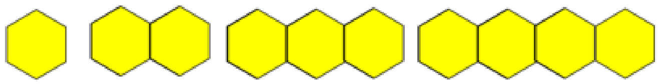

Figure 1: The Hexagons Pattern.

The task was chosen since it had previously been shown to be cognitively demanding for students, as well as productive for teachers' initial attempts to implement discussion-based instruction (Heyd-Metzuyanim et al., 2016). The Hexagon task's richness lies in its affordance to connect different algebraic expressions to a single visual mediator (the Hexagons), as there are various different algebraic expressions that express the desired perimeter. Therefore, the task also provides opportunities for "saming" the different realizations of the perimeter and opportunities for students' engagement with the mathematical concept of identical algebraic expressions.

27 teachers participated in the PD, and 23 of them implemented the Hexagon task. However, 10 lessons were excluded from the current study based on their language (Arabic) and another 3 lessons were not included due to technical reasons. Thus the analysis was performed on 10 lessons. Analysis of the RTA is preformed based on watching only the whole-classroom discussion part of the lesson. Usually, several views are required for completing a tree. However, we were able to code a video in around a ratio of 1:3 time of coding per time of video. This is much less work than preforming the analysis based on transcripts.

The RTA depicts the different realizations of a mathematical object as nodes in a “tree” (see Figures 2 and 3). We code the tree according to two criteria: (1) Coloring the realizations that were exposed to students during the lessons based on *who* articulated the realization (dark color = student; light color = teacher.) (2) Arches between the realizations are drawn where links between realizations were made during the discussion (continuous line = link made by students; dashed line = link made by the teacher). We quantify the data as follows (see Table 1): **(1) Number of realizations**: the total number of realizations that were colored. **(2) Ratio of students' realizations**: the number of dark realizations out of the total number of colored realizations. **(3) Number of horizontal links:** the total number of links that were made between algebraic expressions' and the visual mediators of the hexagons pattern. **(4) Number of vertical links:** the total number of links that were made between any other two realizations. **(5) Ratio of students' horizontal links:** the number of horizontal links that were made by students (continuous line) out of the total number of horizontal links. **(6) Students' vertical links:** the number of vertical links that were made by students (continuous line) out of the total number of vertical links. **(7) Narratives about the 'saming' of the algebraic expressions branch:** this criterion received a "1" if a narrative about the 'saming' of the mathematical branch of algebraic expressions appeared anywhere in the discussion and was offered by students and "0" if it was not. Such narratives were for example: "all those formulas are the same". We did not count under this criterion narratives of 'saming' not offered by students since practically all lessons included such a narrative authored by the teachers. In addition, for each lesson, we specify the grade level of the class and the track in the form of: track / total tracked groups in the grade.

To be able to compare RTAs one to another, we ranked each lesson on the basis of the following formula: No. of realizations/maximum realizations in the sample + ratio of students' realizations + no. of horizontal links/max horizontal links + ratio of students' horizontal links + no. of vertical links/max horizontal links + ratio of students' vertical links + saming expression. All the above were divided by 7 (number of criteria) to arrive at a ratio of 0 to 1. Thus 1 indicates the most "robust" RTA in the sample (max realizations, max links) and 0 indicates an "empty" tree (no realizations and no links).

FINDINGS

We start by describing the RTA of two lessons. This will be done both to exemplify the method and to display contrasting implementations of the task. The first lesson took place in 8th grade and was directed by Yarden. Yarden's class was the highest of 3 tracks in that grade (therefore, coded 1/3 in track column, see Table 1, line 3). In Yarden's lesson, the students were exposed to 5 different realizations, 4 of which were explained by students (see Figure 2).

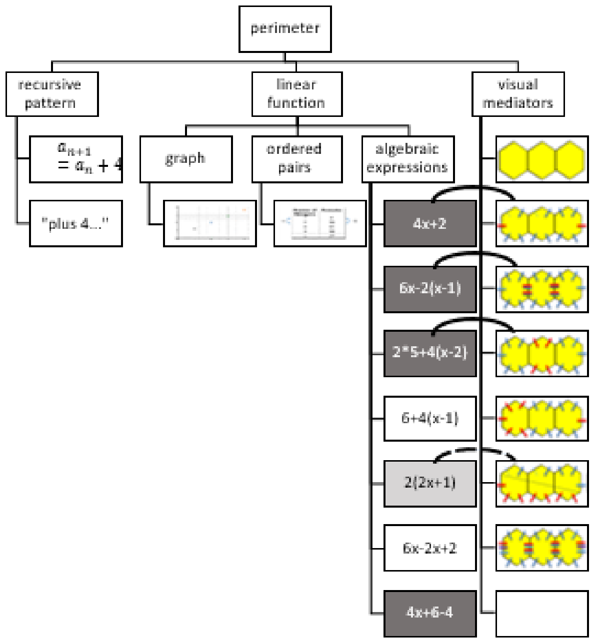

Figure 2: Yarden's RTA.

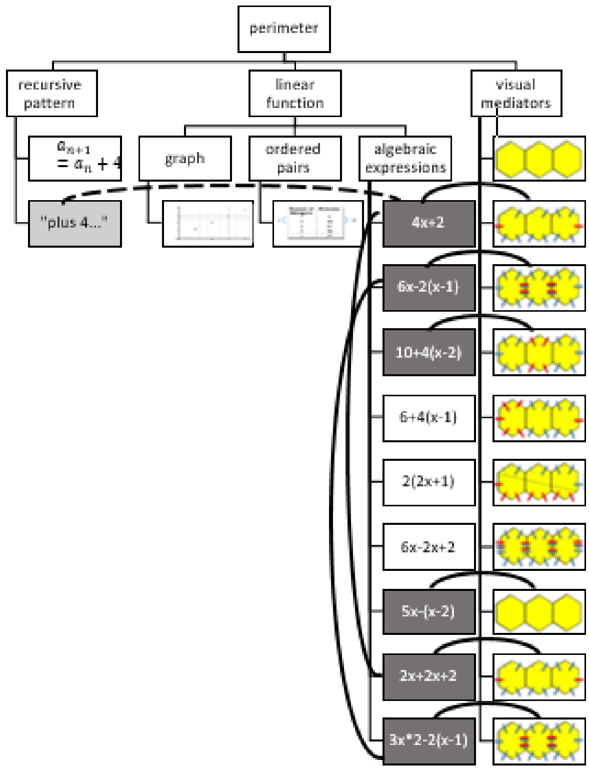

Figure 3: Tamar's RTA.

For example, one of Yarden's students, who presented the 2*5+4(x-2) realization, wrote this algebraic expression while relating to each one of the terms in the expression: "the 5 represents the 5 sides in each sequence (points to the 5 'external' sides in the rightmost hexagon). The 2 is to multiple it for the other side (points to the 5 'external' sides in the leftmost hexagon)". The (x-2) term and the multiplication by 4 were explained by pointing to the internal (connecting) sides of the hexagons. Although there was a substantial number of students' horizontal links (3/4), there were no vertical links at all. This means that although the students were exposed to different realizations and to the links to the visual mediator of each realization, there was no public 'saming' of those different realizations. The discussion thus had a "show and tell" feeling, where each student presented his or her solution but links between solutions were not made. Not surprisingly, 'saming' narratives were not found during Yarden's whole classroom discussion.

In contrast to Yarden's lesson, Tamar's students (see Figure 3, and line no. 9 in Table 1) were exposed to a greater number of realizations (7) and they explained most of the realizations (6/7) themselves. Horizontal links between algebraic expressions' and the visual mediator of the hexagon pattern were made consistently and always by the students (6/6). In addition, three vertical links between realizations were made during the discussion. In particular, the students linked between two algebraic expressions: (1) 2x+2x+2 and 4x+2, (2) 3x*2-2(x-1) and 6x-2(x-1), and the teacher linked between the 4x+2 and the 'plus 4' realizations (explaining that each hexagon added to the train contributes 4 sides to the general perimeter).

Some of the vertical links were not declared explicitly but rather implicitly. For example: after one student explained the 4x+2 realization, another student presented the 2x+2x+2 realization. The student started explaining this expression but another student stopped her and said, while laughing: "It's cheating, 2x + 2x is like 4x… I also have one [laughs] 4x + 1 + 1". Those implicit links mark the final part of the 'saming' process, where students have already 'samed' the realizations and have come to talk about them as being equivalent. In Tamar's lesson, where there were multiple vertical links, students also authored narratives about the saming of the general "branch" of algebraic expressions. For example, students concluded that "all expressions lead to 4x + 2". Such narratives were not observed in Yarden's lesson.

As a whole, the RTAs of Tamar's lesson thus show a deeper engagement with the concept of equivalent expressions as compared to Yarden's lesson. This, although at surface level, Tamar's lesson included quite a few realizations authored by students. Yet the main difference between the lessons could be seen in the number of links between realizations, and especially the vertical links, which signal the "saming" of different algebraic expressions. These differences led to the robustness of Tamar's lesson, which was quantified as 0.87, compared to 0.42 in Yarden's lesson.

A similar analysis preformed on the other 8 lessons elicited several points of comparison as will be elaborated next (see Table 1).

Relation between realizations, links and 'saming' narratives: as a general trend, the greater the number of realizations presented during the whole classroom discussion, the more links (horizontal and vertical) can be seen in the tree. Though this may seem self-evident, this relation does not always exist. Some lessons (such as Yarden's) *do* include multiple realizations, yet links (horizontal or vertical) do not appear in them.

Thus, the presence of realizations in the classroom public sphere is not a *sufficient* condition for the opportunities to objectify. However, our small sample does hint that such presence is a *necessary* condition. Thus we see that in lessons where only few realizations were presented, hardly any links were made (though they could have been made even between few realizations) and even less narratives were formed about the "sameness" of the algebraic expressions branch. We conclude from this that the number of realizations that students are exposed to during the lesson is one essential catalyst for creating links and objectification.

Lesson no.	Grade	Track	Ratio of students' reali-zations	Ratio of Students' horizontal links	Ratio of Students' vertical links	Saming ex-pressions	RTA Robust-ness
1	9	4/6	1/4	0/2	0/2	0	0.22
2	8	3/3	0/6	0/5	0/3	0	0.33
3	**8**	**1/3**	**4/5**	**3/4**	**0/0**	**0**	**0.42**
4	9	3/3	4/7	3/3	0/5	0	0.58
5	7	1/2	2/7	1/2	2/5	1	0.65
6	7	1/4	4/5	2/3	3/3	1	0.75
7	8	No track	4/4	3/3	4/4	1	0.84
8	8	No track	5/5	1/1	5/5	1	0.84
9	**9**	**1/4**	**6/7**	**6/6**	**2/3**	**1**	**0.87**
10	9	1/2	7/7	6/6	2/2	1	0.91

Table 1: Results of the RTA's coding of the 10 lessons.

Relation of grade level and robustness of the RTA: One would expect an increase in the robustness of the tree as the grade of the classroom advances. This, since students in the 9th grade are expected to be more familiar with mathematical ideas related to the equivalence of algebraic expressions. However, as Table 1 shows, the connection between grade level and robustness of the RTA was weak, if existing at all. Thus, there were lessons in 9th grade which were low in robustness (e.g. lines 1 & 4) and there were 7th grade lessons which were relatively high in it (e.g. lines 5, 6).

Track: Unlike grade level, the track of the classroom seems to have a closer connection with the robustness of the RTA. Low tracks (e.g. track 3 out of 3) figure prominently at the bottom part of the table (ranks 1, 2 & 4) while the upper part contains only high tracks (track no. 1 out of 2 or 4) or classrooms that were not tracked. We interpret this finding as indicating that students sitting in low-achieving tracks had less opportunities for objectification than their peers in high-achieving tracks. Of course, low-track students also authored less realizations, forming a vicious cycle that may perpetuate ritual participation in these tracks.

Types of lessons: We interpret the table as consisting of three types of lessons. The first are those that have sparse RTAs, and hardly any links. These lessons (such as 1 and 2 in our table) are characterized by low attention to concepts and low student authority as can be seen in the small number of realizations present, and the fact that any links, if made, are authored by the teacher. The second type of lessons are the middle-scoring RTAs. These (lessons ranked 3-6 in out table) often have multiple realizations presented and even multiple links. However, the relatively low ratio of links made by students shows that the teacher was "pulling" the classroom towards new realizations and new links. This may show that the classroom is learning something new and that the teacher is trying to insert new ideas. The final type are lessons that are characterized by high attention to concepts and high students' authority (ranking 7-10 in our table). These lessons have very consistent and high ratios of realizations and links, and students author almost all of them. These lessons may be very productive and show high levels of exploration. RTA robustness may also indicate that the students have become quite familiar with the mathematical object and that 'saming' had already previously occurred in that classroom.

CONCLUSION AND DISCUSSION

Our main goal in this paper was to examine the opportunities for students' explorative participation during lessons. These opportunities include exposure to different realizations, encouraging students to create links ("saming") between realizations, and improving students' mathematical learning through the creation of narratives about the mathematical objects. The analysis of the 10 hexagon lessons using RTA afforded us the opportunity to better understand what catalyses explorative instruction: the exposure of students to broad numbers of realizations and to links between realizations. This exposure seems to be most productive when narratives and links are made by the students, not solely by the teacher.

Our findings indicate that students' grade and their level of familiarity with algebraic content has no relation to their explorative participation. Robust RTAs from 7th grades show that even when students are not yet very familiar with algebraic expressions, they can offer multiple realizations and form links between them. The situation is less encouraging in low-level tracks, where we see much less student authority, less realizations and less links between them. Our worry is that students in such tracks receive less exposure to different mathematical objects, even when explorative tasks are offered to them. This findings continues previous studies showing the negative effects of tracking on students' explorative participation (Boaler & Staples, 2008). Particular illuminating, in this respect, are the two heterogeneous classrooms in our sample, figuring high in RTA robustness. These show that it is possible, and perhaps even more fruitful, to implement explorative tasks in such classrooms.

The use of the RTA tool in this study continues our previous research (Weingarden et al., 2017) where we used it to qualitatively examine and visually represent different levels of explorative instruction. Here, we have shown its utility to compare numerically between a relatively big numbers of identical lessons. Of course, the possibility to examine lessons based on an identical task is quite rare. We intend to pursue the usefulness of the RTA to compare between lessons that are based on different tasks in future studies.

References

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *The Teachers College Record*, *110*(3), 608–645.

Boston, M. D., & Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education*, *40*(2), 119–156.

Heyd-Metzuyanim, E., Smith, M. S., Bill, V. L., & Resnick, L. B. (2016). Change in Teachers' Practices Towards Explorative Instruction. In *In Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (pp. 393–400).

Hiebert, J., & Grouws, D. (2007). The effects of classroom mathematics teaching on students' learning. *Second Handbook of Research on Mathematics Teaching and Learning* –371 ,*1* , 404.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA.

Resnick, L. B., Michales, S., & O'connor, M. C. (2010). How (well structured) talk builds the mind. In R. Stermberg & D. Preiss (Eds.), *Innovations in Educational Psychology: Perspectives on Learning, Teaching, and Human Development* (pp. 163–194). New York: Springer.

Sfard, A. (2008). *Thinking as communicating*. New York: Cambridge University Press.

Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? — Early numerical thinking revisited. *Cognition and Instruction*, *23*(2), 237–309.

Smith, M. S., & Stein, M. K. (2011). *5 Practices for Orchestrating Productive Mathematics Discussions*. National Council of Teachers of Mathematics. Reston, VA: National Council of Teachers of Mathematic.

Spillane, J. P., & Zeuli, J. S. (1999). Reform and Teaching: Exploring Patterns of Practice in the Context of National and State Mathematics Reforms. *Educational Evaluation and Policy Analysis*, *21*(1), 1–27. https://doi.org/10.3102/01623737021001001

Weingarden, M., Heyd-Metzuyanim, E., & Nachlieli, T. (2017). The Realization Tree Assessment tool: Assessing the exposure to mathematical objects during a lesson. In *In Proceedings of the tenth Congress of the European Society for Research in Mathematics Education*.

ACTION STRATEGIES IN SPATIAL GEOMETRY PROBLEM SOLVING SUPPORTED BY DYNAMIC GEOMETRY SOFTWARE

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This study is aimed at characterization of action strategies in spatial geometry problem solving supported by Dynamic Geometry Software (DGS), by means of a measure allowing dynamic monitoring of visual difficulty during problem-solving moves. Twenty-one high-school students were engaged in DGS-supported solving of spatial geometry cube-related problems, in individual work-sessions. Data analysis consisted of identification of changes in the visual difficulty of the sketches undertaken by the students on the computer screen and characterization of their problem-solving moves. The results suggest that the students used DGS to reduce visual difficulty in a non-linear process, influenced by their spatial abilities, the initial visual difficulty of the problems and the solution-stage at which the DGS is employed.

THEORETICAL FRAMEWORK

Mathematics curricula tend to emphasize the importance of studying spatial geometry as a means of developing spatial aptitude. Yet, spatial geometry presents a serious challenge for high school teachers and students (Bakó, 2003). Past research has shown that the challenge stems, at least partially, from the need to visualize 3-D objects from their 2-D representations: many learners find it difficult to “see in space” and are often unaware of the loss of information in transit between a 3-D object and its 2-D sketch (Parzysz, 1988). Additionally, the learners tend to rely on visual aspects and ignore theoretical inferences (Bakó, 2003). Thus, the well-known in planar geometry conflict between figural and conceptual aspects of sketches, involves in 3-D geometry also a conflict with visual perception (Ferrara & Mammana, 2014).

While many studies indicate the effectiveness of DGS in teaching geometry, DGS-assisted learning processes are different in spatial and planar geometries. Some of these differences appear to be related to individual spatial ability (Dan & Reiner, 2014). Widder and Gorsky (2012) observed that students characterized by low levels of spatial abilities tend to compensate for their difficulty "to see in space" by extensive use of DGS when it is available. These learners perform many measurements and fewer rotations. In contrast, students endowed with high-level spatial abilities use the software relatively less, and perform mainly rotations. High spatial ability students seem apt to rotate 3-D objects in their minds and tend to use DGS mainly for self-validation and for abbreviation of mental processes. Widder and Gorsky (2012) attempted to explain these findings by means of conceptual tools offered by the Theory of Cognitive Load (Sweller, 1988). However, these explanations must be considered as

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not particularly comprehensive. This is for two reasons: first, methods for measuring cognitive load are still not readily available, and second, cognitive load can account only for part of possible mechanisms behind individual differences in students' strategies of using DGS.

Other researchers explored students' strategies in using DGS by attempting to link the types of actions performed by means of DGS with their cognitive outcomes. Arzarello et al. (2002), and later Olivero and Robutti (2007), drew upon the concession that geometric sketches play a dual role in geometry problem solving. On the one hand, they are related to theoretical geometric objects, and on the other hand they visualize graphical spatial properties meant to stimulate perceptual activities. Accordingly, learning with DGS can be described as an interaction between bottom-up cognitive processes (from the sketch to the theory) involved in conjecturing geometric regularities in the sketch, and top-down cognitive processes (from the theory to the sketch), involved in verification or refutation of the conjectures. By suggesting a hierarchical classification of dragging and measuring modalities in DGS, according to the different approaches and goals of learners, these studies established a theoretical framework for investigating learners' actions when solving planar geometry problems with DGS. Or (2008) expanded this framework to include the "glass-ball" perspective dragging modality in 3-D DGS. Leung (2008) suggested an additional approach, by relating the changes that occur on the computer screen to the Theory of Variations (Marton & Booth, 1997). This theory postulates that, to reach profound understanding of a phenomenon, learners must be simultaneously aware of its many aspects. DGS provides a sort of virtual reality in which changes can be viewed. Leung (2008) suggested a classification of DGS dragging and measuring modalities, based on the learners' intentions to create variation that leads to a simultaneous distinction between the various object properties, and thereby to understanding and learning.

Either way, although providing useful theoretical tools for describing the learning processes with DGS, these classifications of the dragging and measuring modalities do not address the issue of explaining the differences between the action strategies undertaken by learners with different spatial abilities in DGS-supported 3-D geometry problem solving. Moreover, previous studies do not provide comprehensive explanations of how exactly DGS helps students to overcome visual difficulties.

In our study, we sought to gain a deeper understanding of the variety of strategies that students endowed with different spatial abilities employ in DGS-supported 3-D geometry problem solving. The study relies on the premise that spatial perception relates both to the visual discernment of geometrical features in the sketches, and to the ability to manipulate mental images of 3-D objects depicted by these sketches (Ferrara & Mammana, 2014). Accordingly, we hypothesized that learners work with DGS to alleviate the visual difficulty embedded in the sketches, and that differences in the strategies that they employ when using DGS can be related to the differences in their spatial abilities. The goal of this study was to examine this hypothesis and to answer two questions:

(1) Do learners work with DGS to alleviate the visual difficulty embedded in 2-D sketches depicting 3-D geometry objects?

(2) What characterizes the action strategies employed by students endowed with different spatial ability (SA), when solving spatial geometry problems of different visual difficulty levels with the help of DGS?

METHODOLOGY

To answer the research questions, we have constructed and validated a visual difficulty measure (VDM) of 2-D sketches depicting cubes with auxiliary constructions (Widder, Berman & Koichu, 2014). VDM is based on calculating the ratio between two types of information embedded in the sketch: *Potentially Helpful Information* (PHI) that may elicit visualization and support deductive reasoning, and *Potentially Misleading Information*, that may hinder perception. Visual difficulty is higher as the sketch contains more PMI and less PHI, i.e. as VDM is lower (see examples in Figure 1).

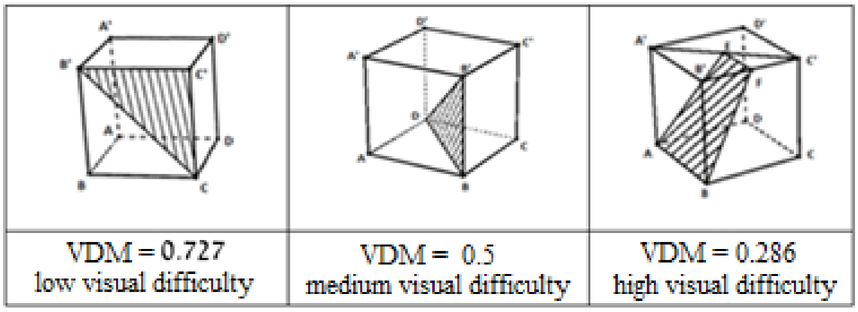

Figure 1: Different visual difficulty levels of cube-related sketches (VDM).

The study included twenty-one 12th grade students (17 years old), studying mathematics at the highest stream-level: seven of low, seven of medium and seven of high SA. SA was determined using the standard test PSVT-ROT (Guay, 1976). All participants engaged in DGS-supported solving of seven cube-related problems of different visual difficulty levels, as determined a-priori by VDM, during semi-structured interviews. Figure 2 exemplifies a high visual difficulty problem presented during the interviews. Students were given free choice regarding whether and when to use the Cabri 3D software. Interviews were recorded, transcribed and summarized, with the purpose of allowing reflective interpretation of data.

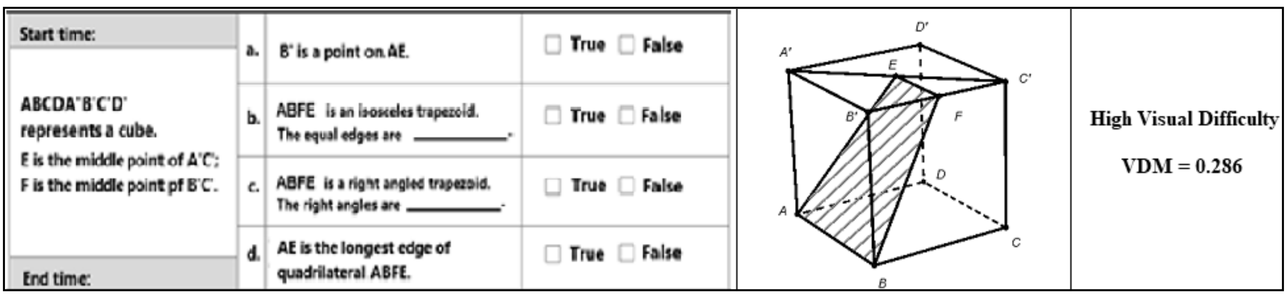

Figure 2: An example of a high visual difficulty problem.

Data analysis included:

(1) monitoring of changes occurring in visual difficulty of sketches on the computer screen by calculating VDM for sketches at *critical screens*, defined as screens where students performed measurements of segments or angles, or stopped their rotation activity with DGS for at least two seconds. The VDM values of the sketches at critical screens were used for constructing strategy-graphs, for each student and for each problem, describing the type of actions performed (rotations and measurements) (see Figure 3). By using these graphs, we sought to examine the hypothesis that learners are working with DGS to alleviate visual difficulty;

(2) counting the total number of types of actions (rotation and measurement at critical screens) performed by students with different SA, to solve problems of different visual difficulty with DGS. We hoped this counting reveal characteristics of the strategies employed by learners.

RESULTS

A close look at the strategy-graphs (see examples in Figure 3) confirmed our hypothesis that learners operate with DGS to alleviate visual difficulty. However, the alleviation processes were non-linear as a rule, and differed for students endowed with different spatial abilities. We illustrate this finding by presenting cases of three students (Pseudonyms: Sarah, Rachel and Ben).

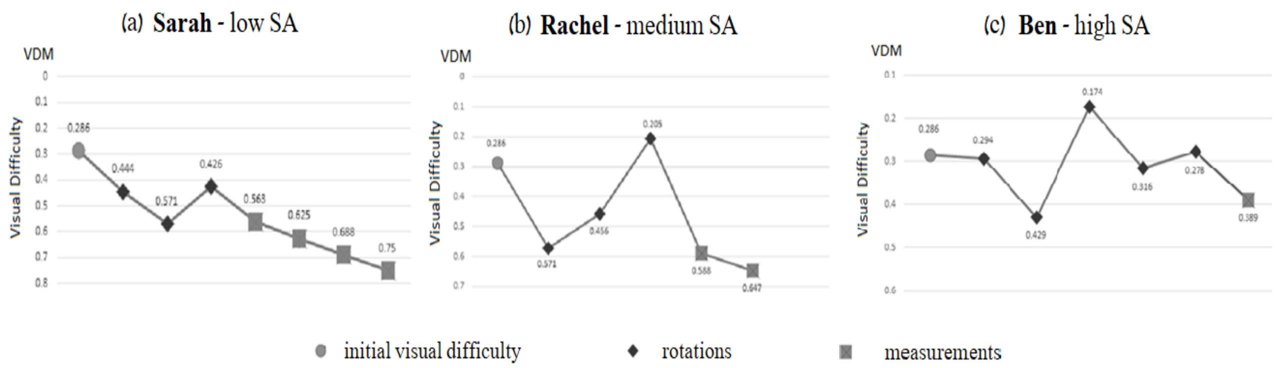

Figure 3: Strategy-graphs for solving the question in Figure 2.

The case of Sarah (low SA)

Sarah began using the software at the outset for testing the hypothesis that trapezoid ABFE was not equilateral, but right angled (see Figure 3(a)). By performing two rotations, Sarah created a sketch where the trapezoid looked right-angled. Still not entirely sure, Sarah performed a third rotation, which completely changed her mind, because the trapezoid on the screen suddenly looked equilateral. Apparently being heavily reliant on visual aspects, and less on theoretical reasoning, Sarah corrected her answer accordingly. She was indecisive even when she tried to think theoretically: "…a trapezoid cannot be both equilateral and right-angled, right? ...". Finally, Sarah reached the correct answer performing a series of measurements on the screen, thus adding to the potentially helpful information in the sketch and facilitating visual difficulty (see Figure 3(a)). When asked how she arrived at her correct initial guess, she said, "... this is something I learned at school - always refer to the angles around the vertex of the cube as right ...". Thus, it seems that Sarah succeeded in answering the question correctly, without reaching a deep understanding of the spatial geometric situation.

The case of Rachel (medium SA)

Rachel began solving the problem in Figure 2 without using the computer. It was evident that she was using theoretical reasoning:

> "... ABFE is not an equilateral trapezoid because A'C' is a diagonal of the base of the cube, whereas B'C' is a cube's edge…the diagonal is bigger than the cube's edge... hence the AA'E and BB'F triangles do not overlap, and AE cannot be equal to BF ...I think it could be right-angled… "

Rachel began using the software only after she had made an initial conjecture, because it was difficult for her to decide whether ABF was the right angle. To resolve this doubt, she began rotating the model (see Figure 3(b)): "... I have no idea about the right angle, but the trapezoid seems to me equilateral ...". Contradicted by what she saw on the screen, Rachel felt confused and found it hard to accept her own reasoning as proof. Finally, she decided to measure AE and BF. As the two segments were unequal, she decided to measure angle ABF. Additional PHI received by measuring, lowered the visual difficulty in the sketch, and helped Rachel confirm her initial conjecture. Rachel succeeded in solving the problem, but claimed that she had been confused by the visual details on the screen.

The case of Ben (high SA)

Ben started solving the problem in Figure 2 without DGS, articulating theoretical explanations for most of the geometrical aspects of the problem. Ben then used the software to validate his theoretical arguments: "…I use the software because it is available...not because I cannot think by myself...it's just nice to get instant feedback and it helps seeing things on the screen instead of just imagining them...". Ben performed several rotations (see Figure 3(c)). Observing the changing screens, Ben discovered geometrical features for angles he was not asked about in the problem formulation. Although very confident, Ben was surprised when asked why the ABFE was a trapezoid. He tried to answer a more general question: "... are two lines on parallel planes necessarily parallel? ...". To our understanding, Ben's preoccupation with a self-imposed general question indicates a deep understanding that is beyond the given geometric situation. Unable to find an answer on his own, he measured angles ABF and EFB, and asserted that the angles complemented each other to $180°$. Thus, he empirically showed that EF and AB were parallel. Immediately afterwards, Ben managed to prove his empirical finding using the midline theorem for triangle A'B'C'. Observing the changes in visual difficulty during Ben's activity with DGS, can be interpreted as using DGS not only for solving the problem, but for creating a space of variations for the search of additional properties.

Counting rotation and measurement actions

The counting of rotation and measurement actions at critical screens shows that as SA increases, students tend to perform fewer actions with DGS (see Figure 4), and to use more rotation and less measurements, compared to low and medium SA learners (see Figure 5). These findings are in line with the findings of Widder and Gorsky (2012). Furthermore, as seen in Figure 4, students with medium to high SA, performed an increasing number of actions as the problems' visual difficulty increased. However, for learners of medium SA, the significant increase in the average number of actions was in the transition from easy to medium problems, whereas among learners of high SA, the significant increase was in the transition from medium to hard problems. This may indicate that learners of medium SA significantly increase their efforts when solving medium visual difficulty spatial problems, while learners of high SA significantly increase efforts only to solve high visual difficulty spatial problems. Interestingly, for learners of low SA, the change in visual difficulty did not result in a significant change in the average number of actions, nor in the relative number of rotations and measurements (see Figure 4). This finding may indicate that learners with low SA are "blind" to spatial geometrical situations, i.e. do not distinguish between different visual difficulty levels of spatial problems and use the same strategy to solve all problems.

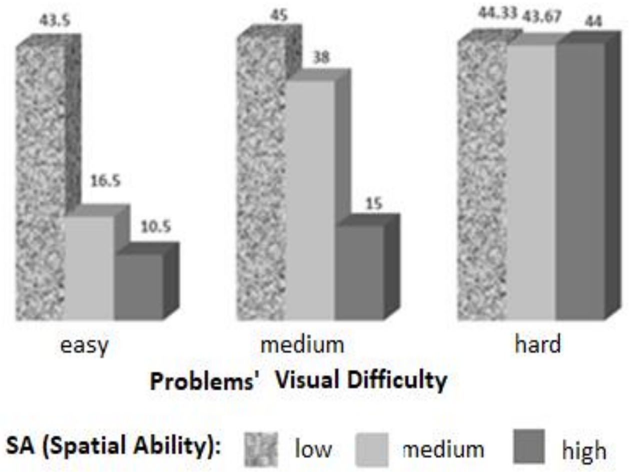

Figure 4: Average number of performed actions.

Surprisingly, for the hard questions, the average number of actions performed by students with different spatial abilities did not significantly differ (see Figure 4 above), although the types of actions performed were distinct: as their spatial ability increased, students performed more rotations and fewer measurements (see Figure 5). This is in line with the findings of Chase and Simon (1973) on performance differences between novices and expert chess players, who performed similarly when presented with unfamiliar chessboards, but used different thinking strategies. These findings indicate that less familiar geometric spatial situations, of high visual difficulty, may be equally challenging for all learners. However, learners endowed with different SA use different heuristic strategies for solving spatial geometry problems with DGS.

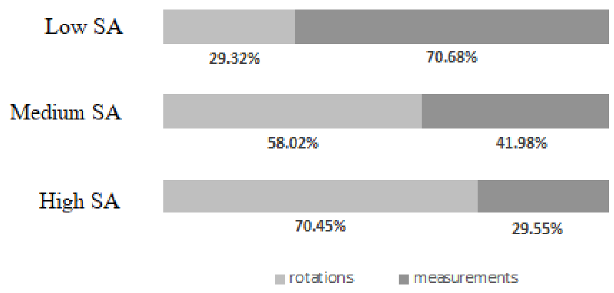

Figure 5: Percentage of performed types of actions for hard questions.

DISCUSSION

Previous studies have struggled to understand the process of learning with DGS, offering descriptive classifications of DGS dragging and measuring modalities, and possible theoretical explanations for spatial problem-solving processes through DGS. In this study, the use of VDM enabled us to empirically test a hypothesis about the use of DGS for alleviating visual difficulty of 2-D sketches depicting 3-D objects on the computer screen, while attending to individual differences in SA.

As exemplified by the three strategy-graphs of Sarah, Rachel and Ben, findings indicate that learners do work with DGS to alleviate visual difficulty, and the patterns of visual difficulty change are closely related to the patterns of actions undertaken to solve problems in spatial geometry with DGS. The process of lowering visual difficulty is a nonlinear process, influenced by the learners' individual SA, as well as by the initial visual difficulty of the problem and the solution-stage at which DGS is employed. All three students started their exploration with DGS after making a conjecture, but differed in the ways of employing DGS in relation to the conjecture. Ben integrated the software only for testing his solution, while Sarah and Rachel began to use the software during the early stages of solving. Both Sarah and Rachel were caught up in visual confusions on the computer screen and assumed incorrect assumptions at different stages of problem solving. These findings are consistent with our expectations that the visual information on the computer screen will guide students' strategies in solving geometric spatial problems with DGS.

Interestingly, strategy-graphs and findings concerning the number and type of performed actions partially support explanations in terms of cognitive load (Sweller, 1988), but also in terms of variation (Leung, 2008), and in terms of moves between the graphical field of the sketch and the theoretical field (Arzarello, et. al., 2002; Olivero & Robutti, 2007). For example, similar cognitive load levels could explain why low SA learners employ similar strategies for different visual difficulty levels of spatial problems, and why the effort of students with different spatial abilities did not significantly differ when approaching hard questions. However, the use of DGS for verification and testing, by high SA learners like Ben, can be better explained as a search for variation, while fluctuations between the sketch and the theory can describe Sarah's and Rachel's interaction with DGS. Therefore, we conclude that alleviating visual difficulty with DGS is not an isolated factor in characterizing action strategies of learners endowed with different SA. Further research is needed for establishing the roles played by visual difficulty, cognitive load, the need to connect visual stimuli to theory, and the search for variation.

References

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A Cognitive Analysis of Dragging Practices in Cabri Environments. *ZDM*, 34(3), 66-72.

Bakó, M. (2003). Different projecting methods in teaching spatial geometry. Electronic Proceedings of the third Conference of the European Society for Research in Mathematics Education.

Chase, W., & Simon, H. (1973). Perception in chess. *Cognitive Psychology*, 4, 55-81.

Dan, A., & Reiner, M. (2014). A comparison in mental load, using EEG in two different learning environments. *Meeting of SAN (Society of Applied Neuroscience) 2014.* Utrecht, Netherlands.

Ferrara, F., & Mammana, M.F. (2014). Seeing in space is difficult: an approach to 3d geometry through a dge. Research Report in the Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education. Vancouver, Canada.

Guay, R. B. (1976). *Purdue spatial visualization tests*. Purdue Research Foundation.

Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning*, 13, 135-157.

Marton, F. & Booth, S. (1997). *Learning and Awareness*. New Jersey: Lawrence Erlbaum Associates, INC, Publishers.

Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. International Journal of Computers for Mathematical Learning, 12, 135–156.

Or, A. (2008). Designing a Teaching Unit in Cabri 3D Environment for Concepts of 3D Figures in Hong Kong Secondary Mathematics Curriculum. In *Proceedings of Topic Study Group 22: New Technologies in the Teaching and Learning of Mathematics, 11th International Congress on Mathematical Education.* Monterrey, Mexico, Academic Press.

Parzysz, B. (1988). 'Knowing' vs. 'seeing': Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics, 19*, 79-92.

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science, 12, 257–285.*

Widder, M., & Gorsky, P. (2012). How students solve problems in spatial geometry while using a software application for visualizing 3D geometric objects, *Journal of Computers in Mathematics and Science Teaching*, 32(1), 557-588.

Widder, M., Berman, A., & Koichu, B. (2014). Dismantling Visual Obstacles to Comprehension of 2-D Sketches Depicting 3-D Objects. *Proceedings of PME 38/ PME-NA 36 Conference: Mathematics Education at the Edge*. Vancouver, Canada. Jul 15-20, 2014.

MATHEMATICS ACHIEVEMENT AND THE ROLE OF WORKING MEMORY AND ATTENTION - EVIDENCE FROM A LARGE-SCALE STUDY WITH FIRST GRADERS

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Besides pure intelligence and subject-specific skills, there are other determinants which substantially affect learning outcomes in mathematics. Working memory capacity is one of the key determinants and as recent studies show that it can be trained, it´s highly relevant for improving mathematical learning processes. In this study, we investigate the relationship between mathematics achievement, and three dimensions of working memory as well as attention and inhibition control. We used data from highly standardized computer-based tests from more than 500 first grade students. As a second perspective we also considered teacher ratings for the same students. Our findings confirm a strong and highly significant correlation between math achievement and different working memory and attention subtests.

INTRODUCTION

Not just pure intelligence and subject-related skills, but also other cognitive skills and executive functions substantially affect important individual life outcomes such as health, educational attainment, life satisfaction, and labor market outcomes. One key determinant for various skills and thus for life outcomes is working memory (WM) capacity—the ability to mentally store and process information (Baddeley & Hitch 1974).

First, WM capacity is highly correlated with analytical problem solving and general fluid intelligence (Engel de Abreu et al. 2010). Second, WM capacity has been found to be related to children’s performance in math and language tasks (Raghubar et al. 2010, Menon 2010). Third, WM capacity has been found to be strongly related to self-control (Schmeichel et al. 2008). This view has been supported by evidence from neuroscience based on neuroimaging methods indicating an overlap of the brain regions that are involved in working memory tasks and time-discounting or self-control tasks (Wesley & Bickel 2014). Another strand of literature report WM capacity to be correlated with attention control (Unsworth & Spillers 2010), which is often considered as relevant for all school subjects. In turn, there is evidence that children with ADHD often have WM impairments (Alderson et al. 2010). Finally, there is a growing body of literature from different fields indicating that WM can be trained: In neuroimaging studies, WM training generally led to increase in brain activity in WM relevant areas of the brain (Dahlin et al. 2008, Olesen et al. 2004). What makes

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working memory so relevant for mathematics education is the fact that it is strongly related to mathematical achievement on the one hand and trainable on the other hand. There are several recent studies which confirm these findings for different contexts (see further down). But as Raghubar et al. note in their conclusions: "*Also missing from the literature is a discussion of the overlap between attention and working memory in relation to mathematics.*" (2010, p. 119)

The aim of this paper is to examine this open question from the literature for the case of primary school students. To do so, we use data from a large-scale study (Schunk et al. 2017) that has measured the three components WM, attention and mathematics together with a series of control variables.

THEORETICAL BACKGROUND

Working Memory Capacity and Related Constructs

Working Memory Capacity (WMC), as defined by Alan Baddeley, is a brain system that "*provides temporary storage and manipulation of the information necessary for such complex cognitive tasks as language comprehension, learning, and reasoning*" (Baddeley 1992, p. 556). The large general interest in research on WMC is based on the finding that WMC is central to a wide range of cognitive abilities as well as life outcomes, attention, self-control, and even the regulation of emotions (Moffitt et al. 2011).

Reasoning Ability: WMC is highly correlated to analytical problem solving and general fluid intelligence (Süß et al. 2002). Also for children, WMC can predict general fluid intelligence (while short-term memory does not do that, see Engel de Abreu et al. 2010).

Maths and Reading Abilities: WMC is related to educational outcomes like math and language abilities (Gathercole & Pickering 2000). Studies for different age groups as well as for participants with and without learning difficulties reveal a strong relationship between WMC and abilities in arithmetic (Raghubar et al. 2010; Dumontheil & Klingberg 2011). Remarkably, WMC might be a stronger predictor of academic success than IQ (Alloway & Alloway 2010). Low WMC seems to constrain the acquisition of skills in reading and mathematics (Gathercole et al. 2006) and WM training is included successfully in interventions for underperforming students (e. g. Yates & Lockwood 2014).

Attention and Concentration: There is also a strong link between WMC and attention or attentional control (Unsworth & Spillers 2010). Ignoring distracting information or inhibiting unwanted responses to distracting stimuli was found to be improved for individuals with higher WMC (Colflesh and Conway 2007). Overall, working memory capacity seems to be closely linked to performance in attention tasks, especially when distracting stimuli have to be ignored or automatic reactions have to be suppressed.

To sum up, there is evidence for working memory capacity being linked to various factors relevant for learning. Particularly, WMC seems to be crucial for (i) general reasoning abilities, (ii) academic achievements –e.g., in math – and (iii) concentration abilities such as attentional control and inhibition of unwanted responses.

Measurement Issues

WMC is not a single, easy-to-measure construct. According to Baddeley's definition (see above), WMC involves aspects of storage and processing (or manipulation) of information.

The storage component is usually identified with short-term memory, which describes an amount of information that is temporarily in a very accessible state (Cowan 2017). This memory has proven to be further separable into a verbal and a visuo-spatial component (confirming Baddeley and Hitch's (1974) model of the "phonological loop" and the "visuo-spatial sketchpad"). Thus, WMC might be best described as (i) the ability to hold or manipulate task-relevant information in STM and (ii) the general ability to control attention (Cowan 2017).

Therefore, measurements of WMC should account for this rather complex structure of the construct and should employ multiple tasks (cf. Shipstead, Redick, and Engle 2012, p. 6). Short-term memory is usually measured using so-called simple span tasks, while WMC is best reflected in complex span tasks (Engle et al. 1999). Simple span tasks consist of recalling sequences of information and rely solely on storage. Complex span tasks involve some component of both storage and (parallel) processing of information. For our case of mathematics achievement in primary schools all types of tasks are of interest.

RESEARCH DESIGN

Participants and Data Collection

In our study, we concentrate on first grade primary school children (6-7 years of age). We conduct our study with more than 500 students from 31 classes in 12 public primary schools of a German city. While the goal of the main study was an intervention study, the present paper only focuses on data from the first evaluation wave which was conducted before the start of any intervention. All data was collected by a professional data collection service provider that was hired by us.

Students completed highly standardized computer-based tests including tasks in working memory (phonological and visuo-spatial), maths performance, attention and inhibition control and IQ. All the tasks were computer-based, using touchscreen, and headphones. Furthermore, teachers completed an online questionnaire on student´s skills as well as on some background characteristics of teachers, classes and individual students.

Main Variables

Mathematics performance (Outcome variable): Basic numeracy was assessed by three different subtests. In the first subtest, the children were asked to simultaneously detect the number of balls on a two by ten grid. The other two subtests were about addition and subtraction, one about mental arithmetic with orally presented problems and one with written problems.

For geometry, we used a test in which children had to guess how many triangles or rectangles fit into a larger geometrical shape. In all math tests, children entered their answer by using a calculator-like grid with the digits 0 through 9 which appeared on the touch screen.

In addition to the test measures for maths performance we asked the teachers to rate each student on a scale from 1 to 7. The question which corresponds to the measure of the three arithmetic subtests was: “How good is the child in adding and subtracting numbers?”.

Working Memory: To account for the complex structure of WMC, we adopted three different tasks to measure WMC. Hence, we conducted a simple span task (Digit Span) in the area of verbal short-term memory, and a complex span task in each area of visuo-spatial (Location Span) and verbal (Object Span) WMC.

The *Digit Span task* (variable “WM: Digit Span”) was a simple forward span short term memory test. In this test, the child listened to different sequences of one-digit numbers in the range from 1–9. After each sequence, the child was asked to indicate the numbers heard in the correct order.

The *Location Span task* (variable “WM: Location Span”) was a complex span task measuring visuo-spatial WMC. For every item in this test, the child had to detect the “odd” shape out of three shapes. After the entire sequence of items, the positions of the identified shapes had to be recalled in the correct order.

In the *Object Span task (variable “*WM: Object Span”), the child listened to a sequence of words he / she had to remember. Since this was a complex span task, after each word the child had to decide whether it represented an animal or not by pushing a button “Animal” or “No animal”.

Attention control: To measure concentration and inhibition abilities we employed a GoNoGo task (adapted from Gawrilow & Gollwitzer 2008). The child has to push a red button on the touch screen every time one of four different animals appeared on the screen. However, when a fifth animal appeared, they must not push the red button. We used the sum of commission errors (pushed, although “NoGo”-item) as a measure for inhibition control (variable “Attention: Inhibition Errors”) and the sum of omission errors (not pushed, although “Go”-item) as a proxy for attentional control (variable “Attention: Attention Errors”).

As a third proxy for attention abilities (variable “Attention: bp-Test Errors”) we choose the “bp task” (drawn from Esser et al. 2008). In this test, the child was presented several pages full of similar looking letters and the child was asked to tap each ”b” and ”p” without marking any other letters.

Methods

In order to examine the overlap between attention and working memory in relation to mathematics, we use the above-mentioned three outcome variables: the performance in the geometry test, the mean performance in the arithmetic subtests and additionally the teacher ratings for arithmetic. We regress each of these outcome variables on our three measures for WMC as well as our three measures for attention. As there is a high correlation between IQ and WMC as well as between IQ and math achievement, we also control for children’s IQ based on a subset of Raven’s Matrices IQ test (Bulheller & Häcker 2002) that has been conducted in our sample. Importantly, all these variables are z-standardized with a mean of 0 and a standard deviation of 1. Moreover, we added gender and age to the regression as further standard control variables. Finally, we added school fixed effects to our regression to control for unobserved differences between the schools.

PRELIMINARY RESULTS

First results of the fixed effects regression model are shown in the following table.

	Test Performance in Geometry		Test performance in Arithmetic		Teacher Rating for Arithmetic	
	Coef. (SE)	p	Coef. (SE)	p	Coef. (SE)	p
WM: Digit Span	.11 (.05)	.010	.24 (.04)	.000	.39 (.07)	.000
WM: Location Span	.16 (.04)	.000	.23 (.04)	.000	.27 (.06)	.000
WM: Object Span	.13 (.04)	.004	.09 (.04)	.036	.06 (.08)	.442
Attention: bp-Test Errors	.03 (.04)	.422	-.14 (.04)	.000	-.21 (.07)	.003
Attention: Inhibition Errors	.01 (.04)	.757	-.10 (.03)	.003	-.01 (.07)	.851
Attention: Attention Errors	.04 (.04)	.328	-.05 (.03)	.089	.01 (.05)	.872
IQ (control measure)	.33 (.04)	.000	.16 (.04)	.000	.29 (.07)	.000
(further control measures including gender, age and school fixed effects are not listed here)						
	n = 556		n = 544		n = 540	
	R^2 = .32		R^2 = .44		R^2 = .35	

Table 1: Results from Linear regression of WMC and Attention on Mathematics achievement (M=0, SD=1), robust standard errors.

Working Memory

Although we control for IQ, the different WMC measures all are significantly related both to the arithmetic as well as the geometry test outcomes. The effect sizes are between 0.09 and 0.24. The strongest association exists between the digit span and the location span measures with arithmetic test performance. For arithmetic these effect sizes are even higher than the effect size of intelligence. Although smaller in magnitude, even the object span measure has a significantly positive effect on arithmetic test performance. This is remarkable, because this measure has no obvious relationship to mathematics like numbers or positions. Focusing on the teacher´s perception of student´s arithmetic skills, the standard errors are a bit larger but the effect sizes for digit and location span as well. For object span we do not find any association to the teacher ratings. Importantly, the teacher was blind to the testing results and was just asked to rate the student´s abilities to add or substract numbers. Hence, it seems that the teacher´s assessment of student´s arithmetic ability highly correlates with the ability needed for the digit span and the location span task but not the verbal object span task.

Attention

Two of our three measures for attention also significantly predict the children´s test performance in arithmetic. The more errors in the bp-test and the more errors with pushing the button at "NoGo"-items, the lower is childs performance in arithmetic. Apparently, despite the inclusion of three working memory measures and one IQ-measures – which both also implicitly capture children´s attention – our specific attention measures still have considerable explanatory power and are important determinants of arithmetic performance. In contrast, for the teacher ratings for arithmetic we only find an association with the bp-task and for the geometry test, we do not find any association with attention. As there are less measures related to the teacher ratings of arithmetics.

SUMMARY AND OUTLOOK

With the regressions of the data from more than 500 first graders and their teachers, we found promising results concerning our question about the overlap between attention and working memory in relation to math abilities. In line with the cited literature we find that all three dimensions of working memory and two of three measures for attention are significantly linked to arithmetic test achievement, although we control for children´s IQ. The effect seems stronger for working memory capacity than for our attention measures and it is more clear on arithmetic than on geometry. Despite these obvious effects the teacher seems to take only a part of the underlying skills of these not-easy-to-measure constructs into account when he/she rates the student´s arithmetic skills based on his/her holistic observations from every day lessons. It appears that WMC and attentional capacities are differentially related to different mathematical skills – arithmetic and geometry – as well as teacher´s assessment of student´s skills.

Our regression we also checked for similar variables and got similar results. Nevertheless we still have to do further robustness checks and a more detailed look at moderation and mediation effects of WMC and attention to get a more precise answer to our question. This is planned for the time before our presentation of the results on the Conference in July. A more detailed look on the overlap between WMC and attention will also allow us to provide a more concise link to the current literature as well as to school practice.

As a limitation we know that although our measures have been collected with highly standardized procedures they of course suffer from the usual measurement error. Moreover, since we have neither exogenously manipulated working memory capacity nor attention our regression results cannot be interpreted causally, but they rather shed light on the link between working memory, attention and mathematics.

References

Alderson, R. M., Rapport, M. D., Hudec, K. L., Sarver, D. E., & Kofler, M. J. (2010). Competing core processes in attention-deficit/hyperactivity disorder (ADHD): do working memory deficiencies underlie behavioral inhibition deficits? *Journal of abnormal child psychology*, *38*(4), 497–507.

Alloway, T. P., & Alloway, R. G. (2010). Investigating the predictive roles of working memory and IQ in academic attainment. *Journal of experimental child psychology*, *106*(1), 20–29.

Baddeley, A. D., & Hitch, G. (1974). Working memory. *Psychology of learning and motivation*, *8*, 47–89.

Baddeley, Alan. 1992. "Working memory." *Science* 255 (5044): 556–559.

Bulheller, S., & Häcker, H. O. (2002). *Coloured Progressive Matrices* (CPM). Deutsche Bearbeitung und Normierung nach JC Raven.

Colflesh, G. J., & Conway, A. R. (2007). Individual differences in working memory capacity and divided attention in dichotic listening. *Psychonomic Bulletin & Review*, *14*(4), 699–703.

Cowan, N. (2017). The many faces of working memory and short-term storage. *Psychonomic bulletin & review*, *24*(4), 1158–1170.

Dahlin, E., Neely, A. S., Larsson, A., Bäckman, L., & Nyberg, L. (2008). Transfer of learning after updating training mediated by the striatum. *Science*, *320*(5882), 1510–1512.

Dumontheil, I., & Klingberg, T. (2011). Brain activity during a visuospatial working memory task predicts arithmetical performance 2 years later. *Cerebral Cortex*, *22*(5), 1078–1085.

Esser, G., Wyschkon, A., & Ballaschk, K. (2008). *BUEGA: Basisdiagnostik Umschriebener Entwicklungsstörungen im Grundschulalter*. Göttingen: Hogrefe.

Engel de Abreu, P. M., Conway, A. R., & Gathercole, S. E. (2010). Working memory and fluid intelligence in young children. *Intelligence*, *38*(6), 552–561.

Engle, R. W., Tuholski, S. W., Laughlin, J. E., & Conway, A. R. (1999). Working memory, short-term memory, and general fluid intelligence: a latent-variable approach. *Journal of experimental psychology*: General, *128*(3), 309.

Gathercole, S. E., & Pickering, S. J. (2000). Working memory deficits in children with low achievements in the national curriculum at 7 years of age. *British Journal of Educational Psychology*, 70(2), 177-194.

Gathercole, S. E., Alloway, T. P., Willis, C., & Adams, A. M. (2006). Working memory in children with reading disabilities. *Journal of experimental child psychology*, 93(3), 265-281.

Gawrilow, C., & Gollwitzer, P. M. (2008). Implementation intentions facilitate response inhibition in children with ADHD. *Cognitive Therapy and Research*, 32(2), 261-280.

Hattie, J. (2008). *Visible learning: A synthesis of over 800 meta-analyses relating to achievement.* Routledge: New York.

Menon, V. (2010). Developmental cognitive neuroscience of arithmetic: implications for learning and education. *ZDM*, 42(6), 515-525.

Moffitt, T. E., Arseneault, L., Belsky, D., Dickson, N., Hancox, R. J., Harrington, H., ... & Sears, M. R. (2011). A gradient of childhood self-control predicts health, wealth, and public safety. *Proceedings of the National Academy of Sciences*, 108(7), 2693-2698.

Olesen, P. J., Westerberg, H., & Klingberg, T. (2004). Increased prefrontal and parietal activity after training of working memory. *Nature neuroscience*, 7(1), 75-79.

Schmeichel, B. J., Volokhov, R. N., & Demaree, H. A. (2008). Working memory capacity and the self-regulation of emotional expression and experience. *Journal of personality and social psychology*, 95(6), 1526.

Schunk, D., Berger, E., Fehr, E., Mueller, H. & Winkel, K. (2017). The KIDS-WIN-Study: Design of a Working Memory Training Intervention in Primary Schools. Working paper.

Shipstead, Z., Redick, T. S., & Engle, R. W. (2012). Is working memory training effective?. *Psychological bulletin*, 138(4), 628.

Süß, H. M., Oberauer, K., Wittmann, W. W., Wilhelm, O., & Schulze, R. (2002). Working-memory capacity explains reasoning ability—and a little bit more. *Intelligence*, 30(3), 261-288.

Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and individual differences*, 20(2), 110-122.

Unsworth, N., & Spillers, G. J. (2010). Working memory capacity: Attention control, secondary memory, or both? A direct test of the dual-component model. *Journal of Memory and Language*, 62(4), 392-406.

Yates, S. M. & Lockwood, M (2014). Intervention for middle school students with poor achievement in mathematics. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*, Vol. 5, 393-400. Vancouver, Canada: PME.

Wesley, M. J., & Bickel, W. K. (2014). Remember the future II: meta-analyses and functional overlap of working memory and delay discounting. *Biological psychiatry*, 75(6), 435-448.

ELEMENTARY AND SECONDARY STUDENTS' FUNCTIONAL THINKING WITH TABLES AND DIAGRAMS

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Tables are widely used for supporting functional thinking in the early grades. This research explored whether diagrams might also be adequate for this endeavour. 322 students in late elementary and early secondary grades resolved a multiple-choice test that required the identification of unknown instances of a dependent variable. Items of varying difficulty were presented in tabular and diagrammatic formats. Diagrams facilitated better performances in easy and difficult items amongst students in elementary grades and in the lowest secondary grade – who had little or no formal algebraic instruction – but not in the highest secondary grade. Responses denoting non-functional thinking were in general selected at chance-levels. The potential of diagrams for supporting the early development of functional thinking is discussed.

INTRODUCTION

There is much interest for supporting the development of algebraic thinking in the early grades. A key approach to early algebraization consists of introducing ideas such as variable, covariation, generalization, and symbolism, with tasks designed under the framework of the function. Functional thinking (FT) is an umbrella term coined to describe the body of knowledge and abilities required for making sense of these tasks and ideas. The usage of multiple representations is thought to play an important role in the development of FT, since the construction and generalization of patterns and relationships require mastering diverse linguistic and representation tools (Blanton and Kaput, 2011). According to Smith (2008), FT is a form of representational thinking that focuses on the relationship between two or more varying quantities, progressing from specific relationships involving individual instances to the generalization of that relationship across instances.

There are various ways in which functional relationships can be represented, including for example coordinate graphs, natural language, idiosyncratic symbols and drawings, formal algebraic notation, figural patterns, tables, real-life scenarios, "function machines", etc. And yet, most studies addressing FT tend to rely on tables. This seems adequate considering the properties of this representation. Tables are thought to make functional relationships transparent, thereby facilitating the process of generalization. It has been suggested that tables can help to spread the cognitive load in a way that allows students in second grade and beyond to focus on more complex tasks such as symbolizing (Blanton & Kaput, 2011). Tables are also helpful to identify functional

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relationships independently of the situation that originates them. For example, second grades can recontextualize variables (number of tables and number of people) from the physical context of a chair assignation problem, to the mathematical context of tables, in order to analyze their underlying functional relation (Cañadas, Brizuela, & Blanton, 2016).

Although tables are powerful tools for supporting FT, there are reasons to believe that it would be worth finding representations to complement their use. Functional ideas are relevant in situations that can be represented in ways other than tables. Moreover, research addressing diverse representations might result in the enrichment of the toolbox employed by teachers to deliver algebraic ideas in the classroom.

Here we present an exploratory study, the objective of which was to assess the potential that physics diagrams might have for facilitating FT. We compared the performance displayed by Mexican students in later elementary education and early secondary education whilst resolving a task requiring the identification of a missing instance of the dependent variable, presented analogously in tables and shadow-cast diagrams. The research questions were: RQ1: How do diagrams compare to tables for facilitating the solution of tasks involving functional relationships? RQ2: Do children address functional relationship differently in diagrams and tables?

REPRESENTING FUNCTIONAL RELATIONSHIPS WITH DIAGRAMS

There is much potential in exploring diverse representations for supporting FT. The coordination of diverse representations of functional situations can help learners to develop the capacity to analyze, describe, and symbolize different types of patterns and relationships. Representations might have distinctive properties that can be used for supporting FT. For example, natural language possess familiarity, pictorial representations of patterns can bridge the gap between natural language and symbols, and graphs can help to link FT with geometrical ideas. Below we outline some ways in which the properties of physics diagrams might support the understanding of functional relationships.

The potential of diagrams for supporting the learning of algebra in secondary and higher education is well documented. For example, diagrams can successfully aid the solution of complex equations amongst students in pre-algebra courses (Chu, Rittle-Johnson, & Fyfe, 2017). And yet, diagrams have been seldom studied in the early algebra literature (Smith, 2008). However, there are reasons to argue in favour of its usage for supporting FT. First, children are likely to be familiar with diagrams. Diagrams representing a range of phenomena are pervasive in schools. In fact, diagrams are commonly used for introducing children to a range of physical phenomena, including complex topics such as general relativity (Pitts, Venville, Blair, & Zadnik, 2014). Although there might be different levels of sophistication for interpreting diagrams, children become familiar with reading diagrams from early grades. Second, diagrams can display visually the qualities of the relationship between quantities. For example, in the case of shadow-cast diagrams like the ones used in this study (Figure

1), the qualities of the covariation between the magnitudes involved are visually represented. The length of a shadow is visually linked to the height of its corresponding object, in relation to the light source. This sort of visual representation of the relationship between quantities is missing in function tables. Third, diagrams can represent continuity. This is relevant considering that functions often involve continuous data that needs to be discretised in order to be represented in a table. Fourth, the contextual cues offered by diagrams might elicit learners' useful intuitions and informal knowledge. One example relevant to the current study, is that even 5-year-olds take into account the relationship between object size and light-object distance for estimating shadow lengths (Ebersbach & Resing, 2007). This suggests an intuitive tendency to establish relationships between variables in order to produce an output. Perhaps this kind of intuition might support the identification of functional relationships.

METHOD

Participants

A total of 332 children in five grades across elementary and secondary school, from Mexico, took part in the study. In Mexico the elementary and secondary systems are composed by 6 and 3 grades respectively. Therefore, our sample is useful for comparing across years and across educational stages. The most important aspect of comparing educational stages is that the Mexican elementary curriculum does not address algebraic concepts, whereas formal algebra instruction begins in secondary school. Consequently we were able to begin to explore the possible effects of formal algebraic instruction on FT across representations.

The study sample was taken from one elementary school and one secondary school, both were government-funded schools located in a mid-size city in south-central Mexico. These schools are adjacent to each other, so share a socioeconomic context, with a medium-level of marginalization according to national statistics (SEP, 2017). The mathematics performance of the elementary school is around the national average, whilst the secondary school has a slight overrepresentation of students in the lower level of performance, but is nevertheless still close to the national average. After removing 10 individuals who returned more than 75% of missing responses, the final sample was taken from elementary grades 4 (n = 59), 5 (n = 45), and 6 (n = 73) and from secondary grades 1 (n = 68) and 2 (n = 77), and was drawn from two classes in each grade.

Questionnaire

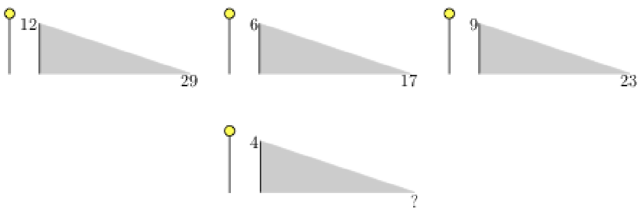
We administered a questionnaire designed to compare the effects of tables and diagrams over children's functional thinking. The items in the questionnaire required the identification of an unknown instance of the dependent variable. We consider this to be an indicator of functional thinking because in order to find an unknown value of y given a particular value of x, an individual needs to resolve and generalize the rule governing the relationship between the numbers involved in the two variables.

The tabular format presented pairs of numbers ordered in two adjacent columns. The diagrammatic format also presented pairs of numbers, but these were contextualized in a figure designed to represent the relationship between the height of a pole and the length of its shadow. Diagrams were designed to accurately represent different shadow cast situations. Examples are shown in Figure 1. The questionnaire contained one section for tables and one section for diagrams, each of 12 items, six of which were ‘easy’ and six ‘difficult’. In the easier items the functional rule involved either an addition or a subtraction, whereas in the difficult items the rule involved either a multiplication or the combination of a multiplication with an addition or a subtraction.

Children were asked to respond each item by choosing one of four options. One option was congruent with functional thinking. Another two options were congruent with two ways of responding to similar exercises with function tables that we identified in a large dataset (Xolocotzin & Rojano, 2015) and the literature (e.g., Tanışlı, 2011), namely recursive and local. The recursive thinking option implied that the subject resolved the unknown number by looking at a pattern formed by the numbers in the dependent variable, neglecting the relationship with the independent variable. Local thinking consisted of relating the unknown with its corresponding pair in the independent variable, in a way that does not apply to the rest of the pairs. The fourth option was a distractor, which consisted of a number that was ostensibly unrelated to the presented data. For example, if the to-be-identified unknown was smaller than its pair in the independent variable, the distractor option was a number larger than its corresponding pair in the independent variable.

a	*b*
4	9
2	5
6	13
5	?

(a) 4
(b) 17
(c) 15
(d) 11



(a) 3
(b) 35
(c) 12
(d) 13

Figure 1. Examples of difficult tabular (left) and diagrammatic (right) items. Respectively, the options congruent with functional thinking are (d), resulting from $b = 2a+1$ and (c) resulting from $b = 2a +5$. Options (b) and (b) are consistent with recursive thinking. Options (c) and (d) are local thinking, and options (a) and (a) are distractors.

Procedure

Individuals were tested at the start of the academic year. The representation order was counterbalanced within grades, and items were presented in two different randomized orders. Questionnaires were administered in groups in one-hour sessions, allocating approximately 15 minutes for each format. Subjects were invited to participate in voluntary basis, with clarification about the confidentiality of their responses and after obtaining permission from school authorities and parents.

RESULTS

Functional thinking responses

The first analysis assessed functional thinking. The number of responses consistent with functional thinking was the dependent variable in a mixed ANOVA including the within-participants factors Representation (table/diagram), and Difficulty (easy/difficult), and the between-participants factor Grade (E4/E5/E6/S1/S2). The results revealed significant main effects for all the factors. These were mediated by significant two-way interactions, which sources were identified with post-hoc Bonferroni comparisons. The interaction representation x grade [F (4, 317) = 3.199, p = .013, η_p^2 = .039] indicated that children performed better in the diagrammatic format in grades E4 to S1, but not in S2. The interaction difficulty x grade [F (4, 317) = 6.375, p < .001, η_p^2 = .074] indicated that children across grades performed significantly better in the easy items than in difficult items across grades, but this effect was significantly smaller in S2. These results are illustrated in Figure 2.

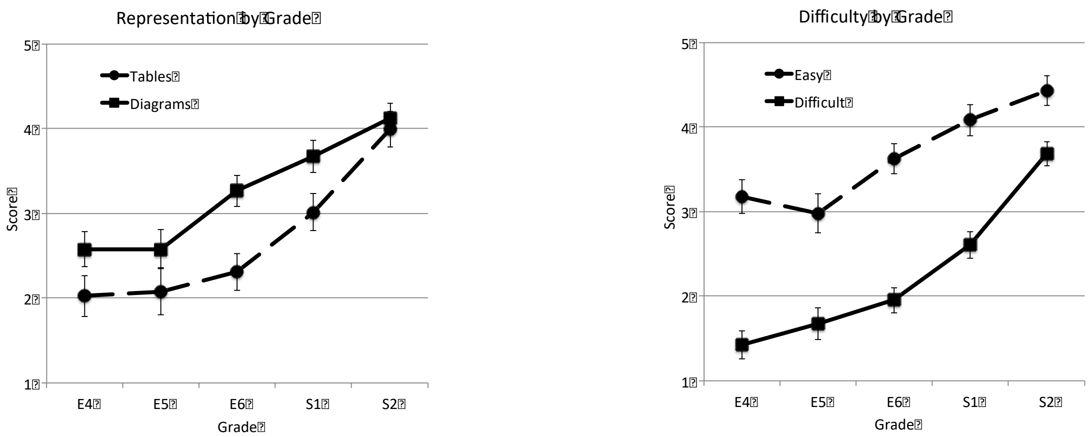


Figure 2. Plots showing marginal means of performance by grade for each representation type (left) and each operation type (right).
Error bars show ±1 SE of the mean.

The interaction representation x difficulty [F (1, 317) = 21.425, p < .001, η_p^2 = .063] indicated better performances in the diagrams in both easy and difficult items, but this effect was significantly larger in the easy items. The interaction representation x difficulty x grade was not significant. These results are shown in Figure 3.

Responses denoting non-functional thinking

A second analysis addressed the rates of responses denoting non-functional thinking, with a breakdown by grade, representation and difficulty. The number of responses consistent with recursive and local ways of thinking, as well as the number of distractor selections and missing responses, were all below the expected by chance (1.5). The lowest selection rate was observed in the distractor responses of S2 children to the difficult diagram items (*M*= .155, *SD* = .365), whereas the highest selection rate was observed in the local thinking responses of S1 children to difficult diagram items (*M*=1.44, *SD* = 1.29). Given the low rates of these response types, no further analyses were carried out.

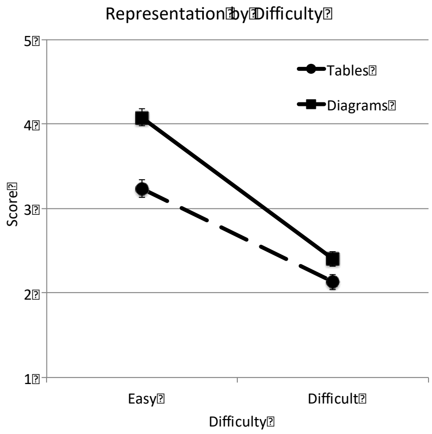


Figure 3. A plot showing the marginal means of performance for each representation by problem difficulty, collapsed across grades. Error bars show ±1 SE of the mean.

DISCUSSION

The current study is an initial step to explore the potential that diagrams might have for complementing other representations, such as tables, in the development of FT. Below we discuss the way in which the results responded to the research questions.

How do diagrams compare to tables for facilitating the solution of tasks involving functional relationships?

Students in all elementary grades and in the lower secondary grade performed better in the diagrams. In contrast, students in the higher secondary grade tested (S2) showed similar performance in tables and diagrams. We think that this interaction between representation and grade is interesting considering the differences in algebraic instruction received across grades.

The Mexican elementary curriculum does not include algebraic content, and considering that participants were tested at the start of the academic year, we can say that students in grade S1 had an incipient algebraic formation when they responded to the questionnaire. What this suggests is that diagrams might have certain properties that facilitate the identification and generalization of functional relationships amongst children without formal algebraic knowledge. Such properties might become less important after receiving algebraic instruction.

The information offered by diagrams is richer in quantity and quality than the information offered by tables. Tables showed only numeric data, whereas the diagrams used in this study presented both numeric and schematic data. As in the case of tables, the diagrams showed numeric representations of the quantities involved. However, diagrams also showed, albeit in a largely abstract manner, the natural mechanisms that linked the variables. Children had more information available to make conjectures directed to identify the unknown. For example, the diagrams presented the position of the light source, and also presented analogue visual and numeric data regarding the height of the pole and the length of the shadow. Larger or smaller numbers accompanied larger or smaller icons of poles and shadows. Students might have used this information to make sense of the qualities of the covariation between the pole height and the shadow length. For example, to notice when the numbers in the dependent variable were larger or smaller than the numbers in the independent variable and apply the corresponding additive, subtractive or multiplicative operations to figure out the unknown.

The properties of diagrams outlined above might have become irrelevant for students in S2. Perhaps they devaluated non-numerical information in the analysis of the mathematical situation presented, and concentrated on figuring out the relationship between variables by purely numerical reasoning, which is a kind of reasoning prompted by tables (Cañadas et al., 2016). Moreover, this result may reflect that the Mexican secondary curriculum overemphasises the operation of "formal" symbolic representations in the algebraic domain, which is thought to hamper the understanding of functional relationships.

Finally, it is worth mentioning that across grades, difficult items were harder than easy ones, and although diagrams facilitated better performances than tables in both kinds of items, this effect was larger in easy ones. These results suggest that the benefits of diagrams are mediated by arithmetic proficiency. Even if the qualities of the covariation between quantities portrayed by diagrams facilitates conjectures about their functional relationship, arithmetic proficiency is still required to figure out the specific rule governing such relationship, which is required to find the unknown.

Do children address functional relationship differently in diagrams and tables?

One explanation for the low rates of non-functional responses could be that children applied the same sort of knowledge to resolve tables and diagrams. The properties of diagrams mentioned above do not seem to prompt non-functional strategies to a different measurable extent than tables do. Based on this, we argue the higher performances observed in the diagrams are due to its specific properties mentioned above.

Concluding remarks

To summarize, these data make it possible to argue that diagrams can be an option for complementing tables to support FT in the early grades. Future studies will be made for understanding the cognitive mechanisms that underlie the interpretation of functional relationships represented in diagrams. This will be useful to develop instructional approaches directed to provide teachers with more tools for introducing algebraic ideas in the early grades.

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References

Blanton, M., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5–23). Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-17735-4_2

Cañadas, M. C., Brizuela, B. M., & Blanton, M. (2016). Second graders articulating ideas about linear functional relationships. *The Journal of Mathematical Behavior, 41,* 87–103. https://doi.org/10.1016/j.jmathb.2015.10.004

Chu, J., Rittle-Johnson, B., & Fyfe, E. R. (2017). Diagrams benefit symbolic problem-solving. *British Journal of Educational Psychology*, *87*(2), 273–287.

Ebersbach, M., & Resing, W. C. M. (2007). Shedding new light on an old problem: The estimation of shadow sizes in children and adults. *Journal of Experimental Child Psychology*, *97*(4), 265–285. https://doi.org/https://doi.org/10.1016/j.jecp.2007.02.002

Pitts, M., Venville, G., Blair, D., & Zadnik, M. (2014). An exploratory study to investigate the impact of an enrichment program on aspects of einsteinian physics on Year 6 students. *Research in Science Education*, *44*(3), 363–388. https://doi.org/10.1007/s11165-013-9386-6

SEP. (2017). *Planea. Plan Nacional para la Evaluación de los Aprendizajes. Resultados 2016*. Secretaría de Educación Pública. Retrieved from http://planea.sep.gob.mx/ba/

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 133–159). New York.

Tanışlı, D. (2011). Functional thinking ways in relation to linear function tables of elementary school students. *The Journal of Mathematical Behavior*, *30*(3), 206–223. https://doi.org/http://dx.doi.org/10.1016/j.jmathb.2011.08.001

Xolocotzin, U., & Rojano, T. (2015). The development and arithmetic foundations of early functional thinking. In *9th Congress of European Research In Mathematics Education*. Prague, Czech Republic: ERME.

USING THREE FIELDS OF EDUCATION RESEARCH TO FRAME THE DEVELOPMENT OF DIGITAL GAMES

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In this article, we explore three theoretical perspectives that inform the development of high-quality, research-based, digital instructional materials. In our team's efforts to develop a game-based learning applet for an existing inquiry-oriented curriculum, we have sought to theoretically frame our approach so that we can draw on the corpus of researcher knowledge from multiple disciplines. Accordingly, we will discuss three bodies of literature – realistic mathematics education's approach to curriculum development, inquiry-oriented instruction and inquiry-based learning, and game-based learning – and draw on parallels across the three in order to form a coherent approach to developing digital games that draws on expertise in each field.

INTRODUCTION

Researchers in undergraduate mathematics education have developed curricula that draw on the curriculum design principles of Realistic Mathematics Education (RME) and are intended to be implemented using inquiry-oriented (IO) methods (e.g., Larson, Johnson, & Bartlo, 2013 (abstract algebra); Rasmussen & Kwon, 2007 (differential equations); Wawro et al., 2013 (linear algebra)). IO curricula fall within the broader spectrum of Inquiry-Based Learning (IBL) approaches that focus on student centered learning through exploration and engagement (Ernst, Hodge, & Yoshinobu, 2017) facilitated by an instructor's interest in and use of student thinking (Rasmussen, Marrongelle, Kwon, & Hodge, in press). In this paper we give examples from one IO curriculum, but also use quotes and references from the more general IBL literature.

In our current project we are exploring the extent to which technology can help mathematics educators extend inquiry-oriented (IO) curricula into learning contexts that are less conducive to inquiry-oriented approaches. Game Based Learning (GBL) provides a reasonable approach to addressing the constraints that large class sizes or non-co-located learning place on instructors' implementation of IO curricula. GBL studies show a clear relation between games and learning as games provide a meaningful platform for large numbers of students to engage, participate, and guide their learning with proper and timely feedback (e.g., Barab, Gresalfi, & Ingram-Goble, 2010; Juul, 2009). Despite advances in technology and policy initiatives that support development of active learning, few digital games exist at the undergraduate level that explicitly incorporate a research-based curriculum. In this paper, we explore the three theoretical perspectives of RME, IO/IBL instruction, and GBL in order to identify the ways in which the three perspectives align and might contribute to the development of digital media that incorporate knowledge and practices gained from each perspective.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 459-466). Umeå, Sweden: PME.

We begin with a brief discussion of each of the three theoretical framings illustrated with specific examples. For the first two framings we describe a task sequence and strategies for implementing that task sequence that come out of the Inquiry Oriented Linear Algebra (IOLA) curriculum. For the third framing, we provide a brief outline of a mathematics game, Rolly's Adventure, developed by the third author, who drew on GBL principles in her game design. We then draw on each of these examples to demonstrate how aspects of RME, IO/IBL instruction and GBL align with each other and to point out a few ways that RME and IO/IBL might be used to inform design of future games, especially as we, the authors, move towards the development of a new digital game rooted in the existing IOLA curricular materials.

RME and IOLA

RME is a curriculum design theory rooted in the perspective that mathematics is a human activity. RME-based curricula focus on engaging students in activity that lends itself to the development of more formal mathematics. Researchers rely on several design heuristics to guide the development of RME-based curricula (Gravemeijer, 1999; Zandieh & Rasmussen, 2010). In this paper, we focus on Gravemeijer's (1999) four levels of activity to show how curricula might reflect the design theory. *Situational* activity involves students' work on mathematical goals in experientially real settings. *Referential* activity involves models-of that refer to physical and mental activity in the original setting. *General* activity involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting. *Formal* activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and no longer require prior models-for activity.

The IOLA curriculum (http://iola.math.vt.edu) draws on RME instructional design heuristics to leverage students' informal, intuitive knowledge into more general and formal mathematics (Wawro, Rasmussen, Zandieh, & Larson, 2013). The first unit of the curriculum, referred to as the Magic Carpet Ride (MCR) sequence, serves as our example of RME instructional design (Wawro et al., 2013). The first task of the MCR sequence is *situational* activity in that it asks students to investigate whether it is possible to reach a specific location with two modes of transportation: a magic carpet that, when ridden forward for a single hour, results in a displacement of 1 mile East and 2 miles North of its starting location (along the vector <1, 2>) and a hoverboard, defined similarly along the vector <3, 1>. As students work through this task and share solutions with classmates, they develop notation for linear combinations of vectors and connections between vector equations and systems of equations, providing support for representing the notion of linear combinations geometrically and algebraically.

The second task in the sequence supports *referential* activity. In this task, students are asked to determine whether there is any location where Old Man Gauss can hide from them if they were to use the same two modes of transportation from the first problem. As students work on this task, they begin to conceptualize movement in the plane using combinations of vectors and also reason about the consequences of travel without actually calculating the results of linear combinations. This allows students to form conceptions of how vectors interact in linear combination without having to know the specific values. The goal of the problem is to help students develop the notion of span in a two-dimensional setting before formalizing the concept with a definition. As with the first task, students are able to build arguments about the span of the given vectors and rely on both algebraic and geometric representations to support their arguments.

As students transition from the second task of MCR, they have experience reasoning about linear combinations of vectors and systems of equations in terms of modes of transportation in two dimensions. In the third problem, students are asked to determine if, using three given vectors that represent transportation modes in a three-dimensional world, they can take a journey that starts and ends at home (i.e., the origin). They are also given the restriction that the modes of transportation could each only be used once for a fixed amount of time (represented by scalars c_1, c_2, and c_3). The purpose of the problem is for students to develop geometric imagery for linear dependence and linear independence that can be leveraged through students' continued referential activity toward the development of the formal definitions of these concepts.

In the fourth task, students have the opportunity to engage in general activity. In this task, students are asked to create their own sets of vectors for ten different conditions – two sets (one linearly independent and one linearly dependent) meeting each of the five criteria: two vectors in $\mathbb{R}^2$, three vectors in $\mathbb{R}^2$, two vectors in $\mathbb{R}^3$, three vectors in $\mathbb{R}^3$, and four vectors in $\mathbb{R}^3$. Students also create conjectures about properties of sets of vectors with respect to linear independence and linear dependence. Students work with vectors without referring back explicitly to the MCR scenario as they explore properties of the linear in/dependence of sets of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$; furthermore, students often extend their conjectures to $\mathbb{R}^n$. Finally, students engage in formal activity as they use the definitions of span and linear independence in service of other arguments without having to re-unpack the definitions' meanings. This does not tend to occur during the MCR sequence but rather during the remainder of the semester as students work on tasks unrelated to the MCR sequence.

Effectiveness and Challenges of Inquiry-Oriented Instruction

Effectively implemented inquiry-oriented instructional approaches have been related to improved levels of conceptual understanding and equivalent levels of computational performance in areas ranging from K-12 mathematics, to undergraduate mathematics, physics, and chemistry (e.g., Kwon, Rasmussen, & Allen, 2005). To enact an RME curriculum, a classroom must engage students in inquiry into the mathematics of the problems posed. These classrooms are problem-based and student-centered, characteristics that overlap with other Inquiry Based Learning (IBL) and active learning classrooms (Laursen, Hassi, Kogan, & Weston, 2014). Consistent with others in the field (e.g., Kuster et al, 2017), in this work, we consider inquiry-oriented instruction to fall under the broader category of inquiry-based instruction. Research has shown that students who engage in cognitively demanding mathematical tasks have shown greater learning gains than those who do not (Stein & Lane, 1996). Furthermore, Stein and Lane (1996) found that those gains were greater in classrooms where students were encouraged to use multiple representations, multiple solution paths, and where multiple explanations were considered.

Implementation of the MCR task sequence described above is dependent on an inquiry-oriented classroom environment. Rasmussen and Kwon (2007) describe inquiry both as student inquiry into the mathematics through engagement in novel and challenging problems and instructor inquiry into students' mathematics to provide feedback to advance the mathematical agenda of the classroom. The MCR sequence is comprised of tasks that allow for multiple strategies and representations. Since the tasks are non-trivial, students are challenged with debating their answers and explaining their arguments. In addition, Tasks 2 and 3 each allow students to engage in mathematical activity that can be leveraged by the instructor to introduce formal definitions (span in Task 2, linear independence in Task 3). In both cases the instructor serves the role of broker between the classroom community and the mathematical community (Rasmussen, Zandieh & Wawro, 2009; Wenger, 1998) by taking student ideas and connecting them with the formal mathematical definitions. This brokering move of "interpreting between communities facilitates the students' sense of ownership of ideas and belief that mathematics is something that can be reinvented and figured out" (Zandieh, Wawro, & Rasmussen, 2017).

Game-Based Learning

Game Based Learning (GBL) is the use of digital games with educational objectives to significantly improve learning outcomes. Games are designed to be enjoyable and fun where students overcome challenges and goals (including educational goals) by gaining mastery of the rules within a constrained environment (Dickey, 2005). Research in game-based learning has emphasized the importance of incorporating thoughtful learning theories into the design of games (e.g., Williams-Pierce, 2016; Gresalfi, 2015), particularly by engaging students in activities in a problem-solving scenario so that students have opportunities to build on their understanding through reflective abstraction on their prior activity towards more advanced ways of thinking. We illustrate GBL with examples from *Rolly's Adventure* (RA), a videogame developed by the second author to support student learning about fractions.

RA begins with Rolly in the top left of the screen (see Figure 1). Rolly needs to roll past the obstacle (the gap) in the middle of the screen. The player's avatar is below Rolly in the purple hat. The player can choose from three options to press at the bottom of the screen. If the player chooses incorrectly the area explodes in fire and the golden bricks in the center show the result of the choice (see Figure 1c.) In Figure 1, the player chose the single black circle and this did not change the size or shape of the golden brick. They then received feedback that their answer was incorrect (the fire that sends their avatar back to start over), and what the direct result of their action was (one black circle results in a single golden brick). Such instantaneous feedback and failure are considered crucial aspects of supporting learning during gameplay (e.g., Juul, 2009). If the player chooses the two black circles, the size of the initial brick doubles to fill the space and Rolly (and thus the player) is able to move past the obstacle, thus receiving positive feedback as to the accuracy of their choice.

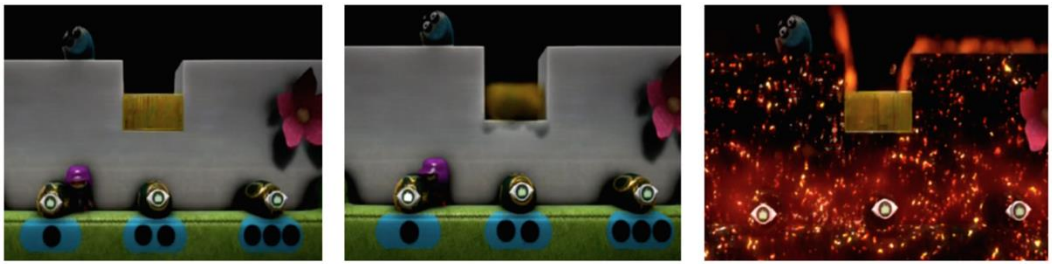

Figure 1: (a) The player (shown here in a purple helmet) enters the puzzle; (b) the player activates the first button; (c) the puzzle catches on fire.

As the player progresses through the challenges the brick or bricks in the obstacle will change in relationship to the space, and the way that the choices are indicated will also change. For example, the golden brick in Figure 1 represents one-half of the hole (the obstacle), and the next puzzle (not shown) has a block that represents one-fourth of the hole, following recommendations that halving a half is a natural next step in the learning of fractions (e.g., Smith, 2002). RA was designed specifically to begin with simpler puzzles and become more complex as players move through the trajectory, such that as players develop generalizations about the game, new puzzles emerge that continue to challenge and nuance these generalizations. Mathematical notation becomes introduced that supports the player in being more precise and accurate just as they begin to struggle, as a way of developing a sense of "intellectual need" (Harel, 2013) so that players find the notation immediately useful. The game also becomes more complex by, for example, presenting bricks that are not an integer multiple of the size of the hole or are larger than the hole.

RA was designed specifically with GBL principles to support players in mathematizing their own gaming experience, and engaging in mathematical play (Williams-Pierce, 2016, 2017). In this fashion, RA serves as a proxy for the role of the instructor in the brokering process (Rasmussen, Zandieh & Wawro, 2009; Wenger, 1998), in that the game requires players to act as producers (Gee, 2003) in reinventing the mathematics underlying RA. In other words, an intentionally designed mathematics game can serve as a responsive digital context that mediates interactions between the player, the game, and the mathematical community.

CONNECTING GBL, RME, AND IO INSTRUCTION

The game design principles outlined above and illustrated with *Rolly's Adventure* align well with the nature of inquiry-oriented instruction using an RME-based curriculum. In Figure 2, we draw heavily on Gee's (2003) notion that good game design is good learning design to show parallels between principles of game design, RME curriculum design, and inquiry instruction and learning. Statements in the boxes of Figure 2 are all quotes or close paraphrases of various authors as indicated.

	Theoretical Framing		
	GBL	RME	IO/IBL
Structure of task sequence	Good games confront players in the initial game levels with problems that are specifically designed to allow players to form good generalizations about what will work well later when they face more complex problems. [1]	Lessons should have experientially real starting points and engage in situational, referential, and general activity. [4]	----
Nature of the tasks	Good games operate at the outer and growing edge of a player's competence, remaining challenging, but do-able ... [therefore] they are often also pleasantly frustrating, which is a very motivating state for human beings [1]	Challenging tasks, often situated in realistic situations, serve as the starting point for students' mathematical inquiry. [3]	IBL methods invite students to work out ill-structured but meaningful problems. [2] Students should solve novel problems [3]
Teachers' role	Good games give information "on demand" and "just in time," not out of the contexts of actual use or apart from people's purposes and goals… [1]	Teachers to build on students' thinking by posing new questions and tasks. [3]	Students present and discuss solutions; instructors guide and monitor this process. [2] Empower learners to see mathematics as a human activity. [3]
Students' role	Games allow players to be producers and not just consumers. [1]	Empower learners to see themselves as capable of reinventing mathematics [3]	Students construct, analyze, and critique mathematical arguments. Their ideas and explanations define and drive progress through the curriculum. [2]

Figure 2: Aligning three areas of our team's expertise that inform game design.
[1]Gee, 2003; [2]Laursen et al, 2014; [3]Rasmussen & Kwon, 2007; [4]Gravemeijer, 1999

Looking across the rows in Figure 2 we see that both digital games and RME curricula place importance on the structure of the task sequence. The sequence should start with an activity in which students can immediately engage, but that has the potential to be generalized to a more sophisticated understanding that will help in solving more complex problems. We see this both in the increasing complexity of the tasks in *Rolly's Adventure* (RA) and in the magic carpet ride (MCR) tasks.

In considering the nature of the tasks we see that GBL, RME and IO/IBL all place emphasis on tasks that are novel and ill-structured allowing for a challenging but do-able problem-solving experience. The RA game (Williams-Pierce, 2017) and the MCR tasks (Wawro et al., 2013) have both been shown to be challenging, but manageable. A digital game based on MCR would share this novel approach.

The teacher's role in inquiry classrooms is particularly important (Rasmussen & Kwon, 2007; Rasmussen et al., in press). Games can take on some of these roles. A well-designed game can intervene at desired junctures and provide real-time guidance or feedback based on the situation that the player is facing. A game can take on the role of the broker between the player (student) and the larger mathematics community by game play being mathematically consistent and by students being gradually introduced to accepted mathematical notation and terminology.

Ultimately the first three categories are aimed at creating an optimal environment for student learning. The students' roles include producing ideas and explanations that allow for their guided reinvention of the mathematics. In RA players create increasingly nuanced generalizations as more complex situations are presented. Student creation of generalizations also occurs in the MCR sequence (Rasmussen, Wawro, & Zandieh, 2015). Our goals as we work toward creating a digital game based on the MCR sequence will be for players of this game to construct, analyze and critique mathematical arguments in the game scenario. For this to happen students need to both (1) experience the mathematical principles/structures through the feedback from gameplay and (2) reflect on their experiences and codify them in some way. In addition to having aspects of the game serve in the teacher role, the game may also need to have aspects that serve in the role of other students in the classroom with whom a student would collaborate in an IO or IBL setting (Ernst et al., 2017).

In conclusion, we believe that these overlapping aspects of GBL, RME and IO/IBL provide a solid starting point for creating a digital game based on the existing IOLA curriculum. As development progresses we will be able to explore affordances and constraints of the digital environment in comparison with the in-person IO classroom.

References

Barab, S., Gresalfi, M., & Ingram-Goble, A. (2010). Transformational Play: Using Games to Position Person, Content, and Context. *Educational Researcher*, *39*(7), 525–536.

Dickey, M. D. (2005). Engaging by design: How engagement strategies in popular computer and video games can inform instructional design. *Educational Technology Research and Development*, *53*(2), 67–83.

Ernst, D. C., Hodge, A., & Yoshinobu, S. (2017). Inquiry-based learning. *Notices of the AMS*, *64*(6).

Gee, J.P. (2003). *What video games have to teach us about learning and literacy*. New York: Palgrave MacMillan.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical thinking and learning*, *1*(2), 155–177.

Gresalfi, M. S. (2015). Designing to support critical engagement with statistics. *ZDM*, *47*(6), 933–946.

Harel, G. (2013). Intellectual need. In K. R. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 119–151). New York, NY: Springer.

Hamari, J., Shernoff, D. J., Rowe, E., Coller, B., Asbell-Clarke, J., & Edwards, T. (2016). Challenging games help students learn: An empirical study on engagement, flow and immersion in game-based learning. *Computers in Human Behavior*, *54*, 170–179.

Juul, J. (2009). Fear of failing? the many meanings of difficulty in video games. *The video game theory reader*, *2*, 237-252.

Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (in press). Inquiry-oriented instruction: A conceptualization of the instructional components and practices. *PRIMUS*.

Kwon, O. N., Rasmussen, C., & Allen, K. (2005). Students' retention of mathematical knowledge and skills in differential equations. *School Science and Mathematics*, *105*(5), 227–239.

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. *The Journal of Mathematical Behavior, 32*(4), 693–711.

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry- based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education, 45*(4), 406–418.

Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior*, 26(3), 189–194.

Rasmussen, C., Marrongelle, K., Kwon, O.N., & Hodge, A. (in press). Four goals for instructors using inquiry-based learning. *Notices of the American Mathematical Society*.

Rasmussen, C., Wawro, M. & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Education Studies in Mathematics, 88*(2), 259–281.

Rasmussen, C., Zandieh, M., & Wawro, M. (2009). How do you know which way the arrows go? The emergence and brokering of a classroom mathematics practice. In Roth, W-M. (Ed.), *Mathematical Representations at the Interface of the Body and Culture* (pp. 171–218). Charlotte, North Carolina: Information Age Publishing.

Rosenheck, L., Gordon-Messer, S., Clarke-Midura, J., & Klopfer, E. (2016). Design and Implementation of an MMO: approaches to support inquiry learning with games. In *Handbook of Research on Gaming Trends in P-12 Education* (pp. 33–54). IGI Global.

Smith, J.P. (2002). The development of students' knowledge of fractions and ratios. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (p. 3–17). Reston, VA: National Council of Teachers of Mathematics.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*, 50–80.

Wawro, M., Rasmussen, C., Zandieh, M., & Larson, C. (2013). Design research within undergraduate mathematics education: An example from introductory linear algebra. *Educational design research—Part B: Illustrative cases*, 905–925.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* Cambridge university press.

Williams-Pierce, C. (2016). Rolly's Adventure: Designing a fractions game. In M. Wood, E. Turner, M. Civil, & J. Eli (Eds.), *Proceedings of the Psychology of Mathematics Education – North American chapter conference* (pp. 1690–1697). Tucson, AZ.

Williams-Pierce, C. (2017). *Fractions, mental operations, and a unique digital context.* In E. Galindo & J. Newton (Eds.), *Proceedings of the Psychology of Mathematics Education – North American chapter conference* (pp. 1349–1352). Indianapolis, IN.

Zandieh, M., Wawro, M., & Rasmussen, C. (2017). An example of inquiry in linear algebra: The roles of symbolizing and brokering, *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 27:1, 96–124.

Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, *29*, 57–75.

UNFOLDING AND *COMPACTING* WHEN CONNECTING REPRESENTATIONS OF FUNCTIONS

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Connecting representations of functions is both, means for developing a deeper understanding of functions and an activity students need to learn. Hence, it is necessary to specify in a more detailed way what learners need to do for adequately connecting representations of functions. This paper empirically identifies two important sub-processes of connecting representations: unfolding and compacting of comprehension elements of functions.

THEORETICAL BACKGROUND

Connecting representations as an important activity when dealing with functions

Connecting representations is an important instrument for learners to develop conceptual understanding (Duval, 2006). This fact applies especially for the function concept (Leinhardt, Zaslavsky, & Stein, 1990). However, many learners struggle with this activity (Niss, 2014). Niss explains this phenomenon by pointing to the common core remaining implicit:

> "One important issue that arises in this context is the fact that functions can be given several different representations (e.g., verbal, formal, symbolic (including algebraic), diagrammatic, graphic, tabular), each of which captures certain, but usually not all, aspects of the concept. This may obscure the underlying commonality – the core – of the concept across its different representations, especially as translating from one representation to another may imply loss of information. If, as often happens in teaching, learners equate the concept of function with just one or two of its representations (e.g., a graph or a formula), they miss fundamental features of the concept itself." (Niss, 2014, p. 240)

Hence, adequately connecting a function's representations requires conceptual understanding of the function concept, especially of the common "core" of the function concept in different representations. However, it needs further research to specify these conceptual demands in more detail. This paper focuses on the research question:

Which sub-processes are necessary when connecting representations of functions?

Before dealing with this research question empirically, it is necessary to identify relevant *comprehension elements* that are crucial for learners when connecting representations. Such a specification of the common "core" of a function allows identifying and describing necessary sub-processes of connecting representations in more detail.

Cognitive psychological background: the idea of a "comprehension model"

In order to specify the "core" of the function concept, this paper draws upon the cog-

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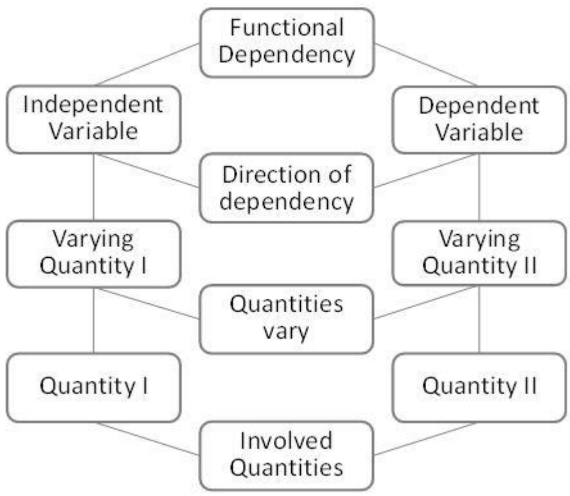
nitive psychological approach of “comprehension elements” (Drollinger-Vetter, 2011). According to Drollinger-Vetter (2011), a whole “comprehension model” has three levels: (1) the level of the concept including its connections to other concepts, (2) the level of representations and their connection, and (3) the level of “comprehension elements” that are the various aspects of the notion one needs to know in order to understand the whole notion itself. This approach has been applied to the notion of function (cf. Prediger & Zindel, 2017): Based on the conceptualization of understanding a concept as being part of a mental network of concepts (Hiebert & Carpenter, 1992), level (1) contains the notion of function and the relation to other notions as formulas, variables or magnitudes for example. Level (2) contains the representations of functions (verbal, numeric, graphic, and symbolic, e.g. Niss, 2014; Leinhardt, Zaslavsky, & Stein, 1990) and their connections (Niss, 2014; Duval, 2006). The most important level for this paper is level (3). In order to explain learners’ difficulties, “researchers have introduced a number of terms and distinctions” (Niss, 2014, p. 239). Hence, identifying “comprehension elements” on level (3) depends on how one conceptualizes *understanding* the function concept. This paper conceptualizes understanding as adequately addressing the “core” of the function concept when connecting representations.

Specifying necessary comprehension elements for connecting representations of functions

Therefore, the following comprehension elements were specified as relevant in the first cycles of the overarching project (Figure 1 left; cf. Prediger & Zindel, 2017).



FlatWatch1.0

The Streaming-provider FlatWatch1.0 offers renting an unlimited number of films for a fixed-price per month. Additionally, you can buy films permanently. By the functional equation f(x) = 4.5 x + 9.99, the price in one month is assigned to each number of films bought.

a) How many films did Sina buy when she has to pay 45.99 Euro?

b) What does Sina pay in one month when she has bought 5 films?

Figure 1: (Left) Comprehension elements of the core of the function concept (Right) FlatWatch1.0 task (Zindel, in preparation)

The following example (FlatWatch1.0; Figure 1 right) illustrates the relevance of the different comprehension elements. In order to solve the task it is necessary to identify the (two) involved quantities, here “number of films bought” and “price in one month”.

Identifying the ||quantity I|| and ||quantity II|| (these signs ||…|| mark comprehension elements from the model) is a potential obstacle in learners' solving processes. Additionally, one needs to know that these quantities can vary (Janvier, 1998). This comprehension element is necessary for understanding that there is a relationship between the two involved quantities and that both quantities can have different values. Solving the FlatWatch1.0 task requires identifying ||x|| and ||f(x)|| in the symbolic representation and ||number of films bought|| and ||price in one month|| in the verbal representation not only as ||involved quantities||, but also as ||varying quantity I|| and ||varying quantity II||. This realization allows evaluating values in the function equation. However, in order to be able to find the right place for evaluating the given value, one needs to identify the ||direction of dependency||. In FlatWatch1.0, the price in one month depends on the number of films bought. Learners need to interpret this information with regard to the roles of the variables as ||independent variable|| and ||dependent variable||: here the number of films bought appears in the role as ||independent variable|| and the price in one month as ||dependent variable||. Connecting this information with the symbolic representation allows solving the first part of the word problem by evaluating the given price of 45.99 Euro for f(x) and calculating x representing the number of films bought. However, learners have often difficulties to identify these roles (Moschkovich, 1998).

Within this model, students are called to have acquired conceptual understanding when they are able to address comprehension elements of the core of the function concept in a flexible way. Adequately connecting representations requires adequately addressing the same comprehension elements in both representations as well as their adequate connection. This specification of comprehension elements for connecting representations of functions allows the researchers to provide in-depth descriptions of learners' processes when connecting representations and thereby to identify empirically relevant sub-processes.

METHODOLOGICAL BACKGROUND

The methodological background of the overarching project (see Zindel, in preparation) is the research programme of Design Research with a focus on learning processes (Prediger, Gravemeijer, & Confrey, 2015) that intertwines the research activities of specifying the demands of dealing with functional relationships, designing and refining a teaching-learning arrangement and deepening the empirical insights into students' learning processes. In total, 16 design experiments in laboratory setting (with pairs of two learners) and 3 design experiments in classroom settings were conducted and videotaped. Thereby, approximately 1890 minutes of video material was collected. For the research question of this paper, learning processes of eight students were qualitatively analysed with regard to learners' theorems- and concepts-in-action (cf. Vergnaud, 1996). A theorem-in-action (marked with <…> in the following text) is a "proposition that is held to be true by the individual subject for a certain range of situation variables" (ibid., p. 225). Concepts-in-action (marked with ||…||) are "categories (objects, properties, relationships, transformations, processes, etc.) that enable the subject to cut the real world into distinct elements and aspects […] according to the situation and scheme involved" (ibid., p. 225). Here, the underlying concepts-in-action can be comprehension elements of the model or individual comprehension elements.

EMPIRICAL INSIGHTS

Due to limitations in length, this paper presents only a short insight in one of the case studies: the case of Fynn and Svenja (both 15 years old). This case illustrates which sub-processes learners need when connecting verbal and symbolic representations of functions and how learners deepen their conceptual understanding within these sub-processes. Therefore, this paper identifies the underlying theorems- and concepts-in-action und analyses their development by comparing a scene at the beginning and a scene in the middle of the design experiment.

Crucial point in learning processes: sufficiently *unfolding* and *compacting* comprehension elements of the core of the function concept

At the beginning of the design experiment, Fynn and Svenja do not unfold the given function sufficiently when working on the FlatWatch1.0 task (cf. Figure 2 for Fynn's and Svenja's first notations literally translated from German).

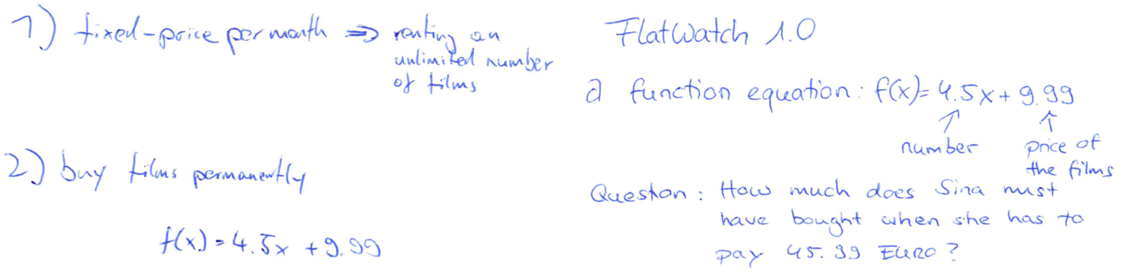


Figure 2: (left) Fynn's and (right) Svenja's notations on the task "FlatWatch1.0"

The transcript starts when Fynn and Svenja begin to explain their notations.

Fynn	I've simply written down the main things.
Svenja	Well, I've written down the functional equation and the task that I – I've also written down for example here the number *[points to the word "number" in her notes]* and the price *[points to the word "price" in her notes]* of the films. So that I know what is what. Then maybe how one could calculate (…) – here *[points to the "9.99" in the functional equation in her notes]* that is the price of the films, I think, and the number *[points to the first summand in the functional equation in her notes]*.

Fynn's utterance is presumably based on a theorem-in-action such as <In order to solve a function word problem, one needs to write down the main information>. This strategy is not wrong in general, but he cannot adopt this strategy adequately for the concrete case, which becomes visible in his notation. Fynn identifies the information ||monthly basic price|| and ||buy films permanently|| as the "main things" (Figure 2). However, it would have been necessary to identify the ||number of films bought|| as ||independent variable|| and the ||price in one month|| as ||dependent variable|| in order to be able to solve the task. To conclude, Fynn does not unfold the verbal and symbolic representations sufficiently here.

Svenja's utterance could be based on a theorem-in-action such as <In order to solve a function word problem, one needs to identify the meaning of both summands>. This theorem-in-action unfolds the functional relationship regarding the two summands. Therefore, Svenja unfolds the verbal representation with regard to two quantities, ||number of films|| and ||price of the films||. Besides, she unfolds the symbolic representation in ||summand 1|| and ||summand 2||, by writing down the meaning of both summands in her notation without separating the variable x and the constant in front of x. With the ||number of films|| she addresses one of the ||involved quantities||. However, she is not able to connect this information adequately with the symbolic representation because she does not distinguish between the variable x and the constant in front of x. With the ||price of the films||, she addresses one of the constants instead of the variable f(x). Indeed, this connection is adequate. Nevertheless, Svenja does not address the meaning of the variables in both representations adequately. Although Svenja unfolds the given representations of the function more than Fynn, she does not unfold them sufficient to be able to connect the representations adequately.

This case study illustrates a general problem of many learners. Not only Fynn and Svenja, but also many other learners did not unfold the given representations of the function sufficiently in order to solve the FlatWatch1.0 task.

Supporting learning processes: eliciting processes of *unfolding* and *compacting* comprehension elements of the function concept

Hence, a teaching learning arrangement has been developed to elicit such processes of unfolding. One task in the teaching learning arrangement is to explain the congruence and incongruence of varied verbal representations (Prediger & Zindel, 2017). In this second part of the design experiment learners set up function equations to streaming offers. Afterwards, they argue the match or non-match of varied descriptions (verbal representations) to their function equations (Figure 3). In the following, Svenja's learning process illustrates that this task can support a more far-reaching process of unfolding the representations.

At first, Svenja argues that phrase D ("With the equation, I can – in dependency of the number of bought films – calculate the price in one month") does not fit to the DreamStream offer:

Svenja	The first one does not fit at all *[points to phrase D]* because one does not pay any films extra. You can rent as many films as you like and you only pay this 19.99 Euro plus this registration fee.

In her explanation, she addresses the ||number of films bought|| as quantity that is not one of the ||involved quantities|| in the function equation of the DreamStream offer. The underlying theorem-in-action is presumably <Verbal and symbolic representations are incongruent when the ||involved quantities|| differ>. Thereby, she unfolds both, the verbal and the symbolic representation, with regard to the ||involved quantities||. By this step of unfolding, she is able to explain the incongruence of the representations adequately.

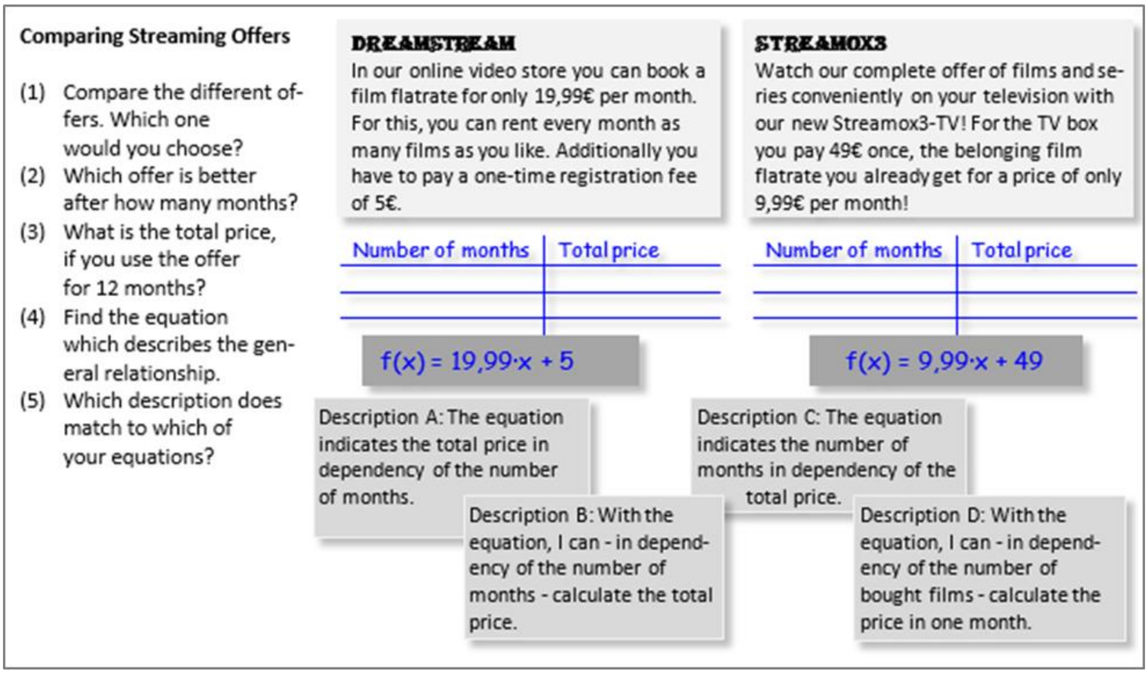


Figure 3: Excerpt from the teaching-learning-arrangement (Descriptions A-D literally translated from German) (similar in Prediger & Zindel, 2017).

Next, Fynn and Svenja argue that phrase C (“The equation indicates the number of months in dependency of the total price”) does not fit to the DreamStream offer either.

Fynn *[3 sec]* Erm – it deals with the months.

Svenja Yes, here it deals with the number of months and there *[points to the DreamStream offer]* we want to calculate the price.

Fynn’s utterance is presumably based on a theorem-in-action like <Verbal and symbolic representations are incongruent when the ||direction of dependency|| differs>. Admittedly, he only addresses the ||number of months|| as one of the quantities. However, due to the context he presumably addresses this comprehension element in the role as ||dependent variable||. Svenja agrees and states more precisely the differing ||dependent variable||. By this unfolding with regard to the ||direction of dependency||, they are able to explain the incongruence of the representations adequately.

Later in this session, Svenja states that phrase B (“With the equation, I can – in dependency of the number of months – calculate the total price”) fits to the DreamStream offer. The following utterance is her first attempt to reason a congruence of representations.

Svenja Erm – in dependency of the months here *[points to phrase B]*. So that this *[points to the function equation of DreamStream]* – let’s say *[points to phrase B]* dependency are five months […] that one can calculate the price at all – the total price one has to pay after five months *[points to the function equation of DreamStream]*

Svenja uses an example and explains the congruence by referring to the way of calculation. Her utterance is presumably based on a theorem-in-action such as <Verbal and symbolic representations are congruent when one can calculate the same quantity by the same other quantity>. Thereby, she addresses implicitly the meaning of the variables. Explaining an exemplary way of calculation offers her to compact the comprehension elements and describe the functional relationship in a compacted way.

In contrast to the scene at the beginning, Fynn and Svenja unfold sufficiently here: Both learners are able to unfold verbal representations with regard to the involved quantities and the direction of dependency. This sub-process of unfolding allows them to explain the incongruence of verbal and symbolic representations adequately. Furthermore, in contrast to many other learners in the overarching project, Svenja found a way to describe the function in a compacted way for explaining the congruence of representations, thus her process ends with a sub-process of compacting. This development is summarized and visualized by the model (Figure 4).

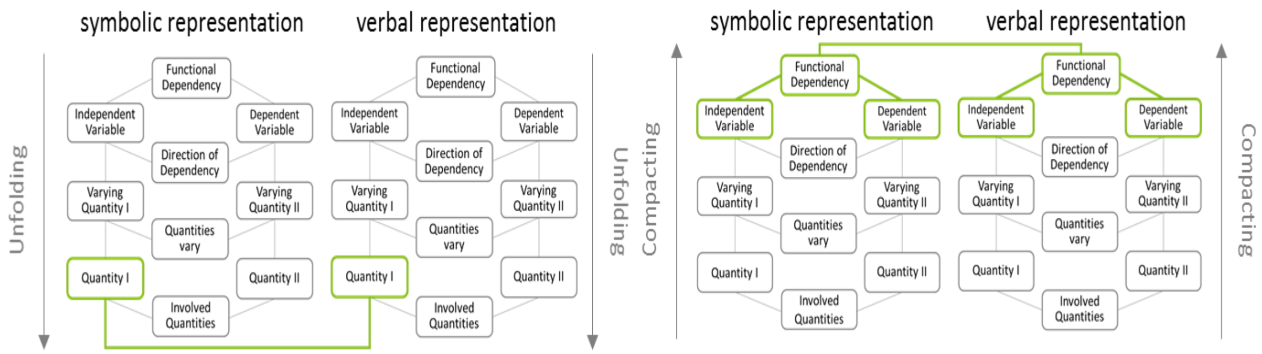


Figure 4: Svenja’s activated comprehension elements (*unfolding* for explaining incongruence (left) and *compacting* for explaining congruence (right)).

CONCLUSION

Connecting representations is both, a learning medium for deepening conceptual understanding of functions (e.g. Duval, 2006) and a learning content (e.g. Niss, 2014). This paper has identified two important sub-processes learners need to fulfill for successfully connecting representations: *unfolding* and *compacting* of comprehension elements of the core of the function concept.

The first part of Fynn and Svenja’s case study illustrates that appropriately unfolding comprehension elements of the core is crucial for dealing with function word problems. They were not able to solve the task because they could not connect the given verbal and symbolic representation regarding the meaning of the ||independent variable|| and the ||dependent variable||. The second part of Fynn and Svenja’s case of illustrates how task design can elicit processes of unfolding and compacting and thereby deepen conceptual understanding. Explaining incongruence of representations can elicit processes of *unfolding* with regard to the incongruent elements like the ||involved quantities|| or the ||direction of dependency||. Explaining congruence of representations can elicit processes of (re-) *compacting* of comprehension elements to the denser comprehension element ||functional dependency||. These phenomena appeared in other learning processes, too. In the overarching project, the design element of dealing with varied verbal representations supports learners' processes of connecting representations by eliciting processes of unfolding and compacting (Prediger & Zindel, 2017; Zindel, in preparation).

Drawing the conclusion, this paper has shown that unfolding and compacting of comprehension elements of the core of the function can be important activities for successfully connecting representations. Managing these activities is a demand that students need to learn. How these processes can deepen learners' conceptual understanding by raising their awareness for the comprehension elements of the core of functions long-term needs further research and a larger sample.

References

Drollinger-Vetter, B. (2011). *Verstehenselemente und strukturelle Klarheit.* Münster: Waxmann.

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational Studies in Mathematics, 61*(1-2), 103–131.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York, NY: Macmillan.

Janvier, C. (1998). The notion of chronicle as an epistemological obstacle to the concept of function. *The Journal of Mathematical Behavior*, *17*(1), 79–103.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, Graphs, and Graphing: Tasks, Learning, and Teaching. *Review of Educational Research*, *60*(1), 1–64.

Moschkovich, J. N. (1998). Resources for Refining Mathematical Conceptions: Case Studies in Learning about Linear Functions. *The Journal of the Learning Sciences*, *7*(2), 209–237.

Niss, M. A. (2014). Functions Learning and Teaching. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 238–241). Dordrecht: Springer Netherlands.

Prediger, S., Gravemeijer, K. & Confrey, J. (2015). Design research with a focus on learning processes. *ZDM Mathematics Education*, *47*(6), 877–891.

Prediger, S. & Zindel, C. (2017). School academic language demands for understanding functional relationships. *EURASIA Journ. Math., Science & Techn. Ed., 13*(7b), 4157–4188.

Vergnaud, G. (1996). The Theory of Conceptual Fields. In L. P. Steffe, P. Nesher, P. Cobb, B. Sriraman, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 219–239). Mahwah, NJ: Erlbaum.

Zindel, C. (in preparation). Den Funktionsbegriff im Kern verstehen – *Entwicklungsforschungsstudie zu Bearbeitungs- und Lernwegen in einem fach- und sprachintegrierten Lehr-Lern-Arrangement.* (PhD Thesis in preparation). TU Dortmund University, Germany.

THE ROLE OF HORIZON CONTENT KNOWLEDGE IN TEACHERS' RECOGNITION AND INTERPRETATION OF STUDENTS' MATHEMATICAL MISCONCEPTIONS

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Mathematics Knowledge for Teaching (MKT) remains a research challenge as far as its content, structure and role in teaching the subject matter are concerned. The paper examines the recognition and interpretations of 106 secondary mathematics teachers of hypothetical students' misconceptions related to the dual decimal representation of a rational number. The analysis of these interpretations aims to provide an insight into the role of Horizon Content Knowledge, one of the components of the MKT which has attracted less the interest of the researchers so far.

INTRODUCTION

Teachers' knowledge of the subject matter has attracted considerable attention in the recent years. This attention, strengthened by educational evidence suggesting that teachers' intellectual resources affect students' learning (e.g., Ball, Thames & Phelps, 2008), was fuelled by Shulman's work in the 1980s, who distinguished three categories of teachers' content knowledge: (a) Subject Matter Content Knowledge (SMK), (b) Pedagogical Content Knowledge (PCK) and (c) curricular knowledge. Since then, models of studying and developing Mathematical Knowledge for/in Teaching (MKT) have been suggested and explored based on data obtained primarily from primary education (e.g. Ball, et al., 2008; Rowland, Huckstep & Thwaites, 2005).

One of the central questions of the relevant research concerns the nature and the content of MKT leading unavoidably to examining the mathematical demands of classroom teaching. Fundamental mathematical ideas, like function, real number, limit, are taught informally at school, even at the upper high school, appealing mainly to intuitive knowledge rather than to formal definitions and theoretical foundations. This is often seen as a 'transitional' stage of accessing powerful mathematical ideas like those identified in their historical development, where cornerstone mathematical concepts and symbols were initially used loosely, before defined rigorously. For the above-mentioned ideas, MKT is the knowledge related to the special requirements of this 'transitional' stage for mathematically legitimate knowledge to be built.

In this article, we focus on a such a case, studying secondary school teachers' knowledge related to the teaching of periodic decimal numbers with infinite decimals of period 9 via their interpretations of students related misconceptions.

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THEORETICAL BACKGROUND

Many researchers have attempted to identify a typology of SMK adopting a variety of approaches. Ball, Thames and Phelps (2008) subdivided Subject Matter Knowledge (SMK) into: (a) Common Content Knowledge (CCK), ("the mathematical knowledge and skill used in settings other than teaching"), (b) Specialized Content Knowledge (SCK) ("the mathematical knowledge and skill unique to teaching") and (c) Horizon Content Knowledge (HCK) ("an awareness of the mathematical topics that are related over the span of mathematics included in the curriculum"). They also distinguished Pedagogical Content Knowledge (PCK) into: (a) Knowledge of Content and Students (KCS), ("knowledge that combines knowing about students and knowing about mathematics"), (b) Knowledge of Content and Teaching (KCT) and (c) Knowledge of Content and Curriculum (KCC) ("combines knowing about teaching and knowing about mathematics") (Ball et al., 2008, p. 309 - 403).

At a later stage Ball and Bass (2009, p.6) described HCK as having constitutive elements: (i) a sense of the mathematical environment surrounding the current "location" in instruction; (ii) major disciplinary ideas and structures, (iii) key mathematical practices and (iv) core mathematical values and sensibilities. They described HCK as "a kind of mathematical 'peripheral vision', a view of the mathematical landscape, that teaching requires" (p. 1), while Jakobsen, Thames and Ribeiro (2013) argue that "HCK is neither common nor specialized, and it is not about curriculum progression, but more about having a sense of the larger mathematical environment of the being taught" (p. 3128).

HCK is often associated with Advanced Mathematical Knowledge. Ball and Bass (2009) describe HCK as "a kind of elementary perspective on advanced knowledge that equips teachers with a broader and also more particular vision and orientation for their work" (p. 10). However, Zazkis and Mamolo (2011) "consider application of advanced mathematical knowledge (AMK) in a teaching situation as an instantiation of teachers' knowledge at mathematical horizon" (p. 9).

Although Ball et al. (2008) link HCK with KCC, Jakobsen et al. (2013) think that HCK "... it is not about a curriculum progression ..." (p. 3128). During their teaching, teachers are invited to respond to mathematical problems, which are primarily meaningful within the school environment. That is, to establish a framework of accepted propositions or approaches that can be used by the students to prove a proposition adopting alternative approaches or excluding those containing elements of the proposition to be proven. Elements of this knowledge, which are not included in the curriculum, also constitute part of HCK as a special kind of knowledge necessary for teaching mathematics.

Summarizing, HCK is mathematical knowledge for the needs of teaching which refers to issues of the wider environment of the mathematical issues under negotiation in the teaching context, unlike the SCK which is directly associated with the issues being taught. It contributes to the development of students' broader mathematical concerns but also to the understanding of the causes of their possible misconceptions. It relates to deeper and broader knowledge of the mathematical ideas being taught, playing a specific and distinct role in didactic approaches at every level of education. It includes knowledge that does not necessarily belong to the curriculum, but it is associated with ideas that are part of the curriculum.

Mathematical knowledge for teaching: the case of decimal numbers.

Tall and Scwarzenberger (1978) found that first year university mathematics students tended to believe that the representation 0.999… was a number less than 1. Li and Tall (1993) found students who agreed that $1/9 = 0.1+0.01+...$, yet they did not accept the equation written in reverse order as $0.1+0.01+... =1/9$. Identifying epistemological projections into students' understanding of notations like the above, Vinner and Kidron (1985) argued that the infinite decimal is perceived as one of its finite approximations, or as a dynamic creature which is in an unending process. Fischbein, Tirosh and Hess (1979) referred to a student who argued that $1+1/2+1/4+1/8+...$ is 2−1/¥, indicating that s/he viewed the limit object as having the same properties as the objects tending to the limit. Based on this interpretation, Tall (1986) argued that students who consider that 0.999... is less than 1 believe that, since any of the terms of the sequence 0.9, 0.99, 0.999, ... is less than 1, so is also its limit. He called such a limit "generic", that is, conceived as having the properties that are common to all the terms in the sequence.

Dubinsky, Weller, Mc Donald and Brown (2005b) offer two explanations for this issue. The first one is that the confusion was due to some students who perceive 0.999... as a process while 1 as an object. Particularly, they note:

> The difference between the two conceptions is that a process is conceived by the individual as something one does, while an object is conceived as something that is and on which one acts. (p.12)

The second explanation is that students "actually conceive of 0.999... as consisting of a string of 9s that is finite but of indeterminate length". So, they accept that conceptions such as the difference from 0.999... to 1 exists and is infinitely small.

The above suggest that the understanding of the representation of decimal numbers with infinite digits is closely relate to epistemological issues. This is line with the historical development of the relevant concepts and notations, indicating that the difficulties encountered by students are not due to lack of knowledge but have deeper causes. Zoitsakos, Zachariades and Sakonidis (2013) found that even mathematics teachers conceptualize the decimal numbers with infinite digits mainly as processes. In this paper, we focus on the role of HCK in teachers' conceptualizations of this representation.

METHODOLOGY

Adopting the idea that teacher knowledge is potentially better explored and developed in situation-specific contexts (e.g., Biza, Nardi & Zachariades, 2007), secondary mathematics teachers were invited to recognize and interpret four fictional students' responses reflecting subtle misconceptions related to the representation 0.3999… as follows:

> A final year secondary school teacher gave the following question to his students: "What is the meaning of the representation 0.3999... (infinite number of 9)"? Four students gave the following answers:
>
> Student A: The representation 0.3999... means a process that tends to 0.4,
>
> Student B: 0.3999... is a number that tends to 0.4,
>
> Student C: 0.3999... is the number just before 0.4, and
>
> Student D: The representation 0.3999... is the sum of 0.3+0.09+0.009+... but, as it continuously increases, it cannot be equal to a number.
>
> (a) What could be the teacher's goal in asking this question? (b) Comment on each student's answer according to the thought process articulated, the positive and the negative points imprinted in his/her view and his possible misconceptions (if there are any), and (c) If you were a teacher in this class, how would you help these students overcome the misconceptions you identified?

Fictional students' answers are based on the findings of the relevant research: students A and B statements focus on the difficulty of discriminating between number 0.3999 ... (a different representation of number 0.4) and the sequence 0.3, 0.39, 0399, ... (a sequence with limit 0.4); student C's statement seeks to shed light on how teachers will deal with the property of density; and with student's D statement we aim to explore how teachers negotiate an infinite sum which comes up as the expansion of a periodic decimal number with infinite digits.

The questions raised in the scenario were answered by 106 practicing secondary school mathematics teachers in writing (36 males and 70 females, with teaching experience ranging from 0 to 20 years). The answers were provided in the context of an entry examination paper for a Mathematics Education postgraduate program. Teachers were invited to think about the scenario questions within a rather complex environment of issues related to concepts, such as the dual nature of mathematical objects, as processes and as concepts (Sfard, 1991; Gray, & Tall, 1994) and the dual nature of infinity, as actual and potential (Dubinsky et al., 2005a). Moreover, important mathematical practices such as the proof of the equivalence of different representations ensuring the accuracy and consistency in mathematical language and symbolism are enacted. These are features that, as mentioned before, constitute the HCK in relation to the mathematical idea at hand. In this study, which is part of a larger project examining HCK's role in teaching mathematics, we focus on this role in teachers' identification and interpretations of students' misconceptions.

To analyse the data, we read carefully each response and initially categorized them in terms of their mathematical correctness. We then analysed the responses in each resulted category, trying to identify features of HCK in the justifications employed in teachers' interpretations of students' errors.

ANALYSIS AND FINDINGS

Our analysis suggested that in trying to recognize and interpret students' misconceptions, teachers' statements draw on a wide range of ideas and employ arguments that might be mathematically right, wrong or even vague. The quantitative distribution of this range of teachers' answers is reflected on the categories presented in the table below, followed by the results of a qualitative analysis (content analysis) of the responses in each category. a) Correct recognition & interpretation (CR&I), b) Correct recognition and wrong interpretation (CR&WI), c) Correct recognition and vague interpretation (CR&VI), d) No evidenced recognition (NER), e) Wrong recognition in agreement with the student (WRAS), f) Ambiguity about recognizing (AR).

	CR&I	CR&WI	CR&VI	NER	WRAS	AR
St. A	19	13	15	29	14	16
St. B	21	18	9	27	14	17
St. C	25	17	25	18	3	18
St. D	27	8	20	20	18	13

Table 1: Teachers' recognitions and interpretations of students' misconceptions.

From table 1 we can see that 19% to 25% of the teachers recognize and interpret correctly the misconception in students' statements. The picture is not very different for the teachers whose response offers no evidence of recognition of students' misconceptions, whereas the teachers agreeing with one of the fictional students' statements are overall a little less (not exceeding 18%). Teachers tend to agree more with student D (18%) and less with student C (3%). Other teachers' responses either recognize misconceptions but based on mistaken interpretation (never exceeding 18%) or recognize misconceptions, offering, however, vague interpretation (up to 25%).

The first column concerns mathematically correct responses (right recognition of misconception and interpretation with no mathematical contradictions). Students' A and B misconceptions appear to have provoked slightly less mathematically correct interpretations than those of students' C and D.

For example, (teacher) T39's response below is ambiguous concerning the recognition of student A's misconception, but acknowledges rightly the misconceptions in the remaining students' statements and thus is included in the first column for students B, C, D but not for student A.

> Student A uses the meaning of function and limit in his reply. But it is not clear if, when asked, he can show why he tends to 0.4. [...] Student B has not understood that a number is something fixed and cannot tend to somewhere else. [... Student C] confuses that there is a next number and a previous one as in natural numbers. [... Student D] makes a big mistake since he reports that the number keeps increasing to finally say that it is not equal to 0.4.

The quote indicates that T39, while having a satisfactory understanding of the properties and nature of numbers, he does not consider 0.3999… as a number. He might think that the three dots at the end of the representation signify something special which he relates to the sequence $\alpha_n = 0.39\ldots9$ (n times 9), n = 1, 2, ….

Sometimes, the recognition of a student's misconception by a teacher is not accompanied by a mathematically correct justification (second column in table 1). This appears a little more often for student's D statement. For example, teacher T21 recognizes a misconception in student C, but provides a wrong interpretation. In particular, he reports:

> For learner C the misconceptions are several. [...] This number is continuously approaching 0.4, it is not just its previous one. It is a periodic number that will continually approach 0.4.

T21, knowing that 0.3999…is a number, he interprets the three dots as indicating a continuously changing number.

The third column in table 1 concerns the teachers who recognize the existence of a misconception in a student's statement, but provide unclear interpetation. This appears to be the case least often for student's B answer. For example, T92 reports:

> Student A made the mistake that number 0.3999 ... can constantly increase in the same way, but it will never become number 0.4, because the number changes very little, it is just that its decimal digits increase.

The fourth column of table 1 reports on the teachers who do not express their opinion as to whether the respective student's statement has some misconception, this being the case a little less frequently for students' C and D statements.

The fifth column represents the teachers who agree with the opinion of the respective student, appearing rarely when commenting on student's C answer. An example is T48's statement: "Correctly the number is tending to (approaching) 0.4, although the limit is not displayed. I believe that student B thought about this way. I think his answer is right".

The sixth column refers to the teachers whose statements are unclear about whether a misconception is recognized or not for a student, although they sometimes might know the correct answer. For example, teacher T16, who gives a proof of the equality 0.3999 ... = 0.4, states: "Students A and B have obviously learned the meaning of limit, (like the other two students) and they have seen it in an intuitive manner".

Overall, only 31 out of the 106 referred to the equality 0.3999… = 0.4 and 22 of them gave at least a proof for this. Nevertheless, many of these 31 teachers did not recognize satisfactorily at least one of the students' misconceptions (e.g. T16). Only the responses of 19 teachers were mathematically correct (that is, they recognized all students' misconceptions and interpreted them correctly). However, there is a small number among these teachers who recognized students' misconceptions without mentioning the corresponding equality.

DISCUSSION AND CONCLUSIONS

Ball et. al. (2008) argued that simply recognizing a wrong answer is classified as CCK, but "sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about meaning in ways that are distinctive of specialized content knowledge (SCK)" (p. 402).

The findings of this study indicate that HCK is also crucial for the recognition and insightful interpretation of students' misconceptions, especially when the teaching object is related to advanced mathematical issues such as the double representation of the rational numbers with period 9. To recognize and make sense of students' misconceptions about decimal representation 0.3999…, a teacher appears to tend to draw on ideas such as limit of sequence, infinite series and the density of rational numbers that are part of the wider mathematical environment of the topic being taught, that is, to HCK. However, these attempted associations are not always correct. For example, some teachers tend to believe that the symbol 0.3999… does not signify a specific number but an infinite process, confusing the representation 0.3999…, which is the limit of the sequence $\alpha_n = 0{,}39\ldots9$ (n times 9), n = 1,2,… with the sequence itself. This confusion gives rise to problems in teachers recognizing and interpreting students' misconceptions.

Thus, it seems that for the recognition and the interpretation of some misconceptions of students concern issues associated with advanced mathematics related not only to SCK but also to HCK and the correct connection between them are crucial.

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References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Ball, D., & Bass, H. (2009). With an eye on the mathematical horizon: knowing mathematics for teaching to learners mathematical futures Paper presented at the 43rd Jahrestagung der Gesellschaft fiir Didaktik der Mathematik, Oldenburg, Germany Retrieved 15 May 2011 from www.mathematik uni-dortmund de/ieem/BzMU/ BzMU2009/ BzMU2009-Inhalt-fuer-Homepage htm

Biza, I., Nardi, E., & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education*, *10*, 301–309.

Dubinsky, E., Weller, K., McDonald, M. A., & Brown, A. (2005a). Some historical issues and paradoxes regarding the concept of infinity: An APOS-based analysis: Part 1, *Educational Studies in Mathematics, 58*, 335–359.

Dubinsky, E., Weller, K., McDonald, M. A., & Brown, A. (2005b). Some historical issues and paradoxes regarding the concept of infinity: An APOS-based analysis: Part 2, *Educational Studies in Mathematics, 60*, 253–266.

Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics, 10*, 3–40.

Gray, E., & Tall, D. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *The Journal for Research in Mathematics Education*, *26* (2), 115–141.

Jakobsen, A., Thames, M. H., & Ribeiro, C. M. (2013). Delineating issues related to Horizon Content Knowledge for mathematics teaching. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of CERME 8, the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 3125–3134). Antalya: Middle East Technical University, Ankara.

Li, L., & Tall, D. (1993). Constructing Different Concept Images of Sequences and Limits by Programming, In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. Lin (Eds.), *Proceedings of the 17th Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 41–48). Tsukuba, Japan.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255–281.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on process and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36.

Tall, D., & Schwarzenberger, L. (1978). Conflicts in the learning of real numbers and limits, *Published in Mathematics teaching*, *82,* 44–49.

Tall, D. O. (1986). *Building and testing a cognitive approach to the calculus using interactive computer graphics*. Ph.D. Thesis in Education, University of Warwick.

Vinner, S., & Kidron, I. (1985). The concept of repeating and non repeating decimals at the senior high level. L. Streefland (Ed.) *Proceedings of the 9th International Group for the Psychology of Mathematics Education Conference*, (Vol. 1, pp. 357–361). Noorwijkerhout, The Netherlands.

Zazkis, R., & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics*, *31*(2), 8–13.

Zoitsakos, S., Zachariades, Th., & Sakonidis C. (2013). Secondary mathematics teachers' understanding of the infinite decimal representation of a rational number. In A. M. Lindmeier & A. Heinze (Eds.) *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 441–448). Kiel, Germany: PME.

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