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CONCEPT IMAGES AND COMMON COGNITIVE PATHS IN THE DEVELOPMENT OF SOME SIMPLE GEOMETRICAL CONCEPTS

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Some notions are going to be defined by means of which a model for acquiring certain type of concepts, having strong visual components, will be constructed. The notion of significantly common cognitive path will be introduced. It will be used, together with the model, to analyse some data related to the development of simple geometrical concepts; however, the main purpose of the paper is just to introduce the notion and the model which seem to have certain potential to explain phenomanae in the processes of concept formation.

§ 1. Mental pictures and Mental Images of Concepts

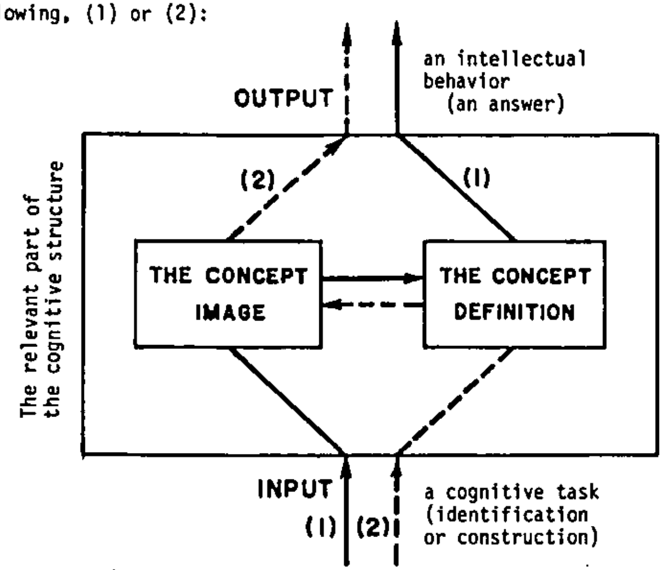
Let C denote a concept and let P denote a certain person. The P's mental picture of C is the set of all pictures that have ever been associated with C in P's mind. Besides the mental picture of a concept there might be a set of properties associated with the concept (in the mind of our person P). For instance, somebody might think (incorrectly) that an altitude in a triangle should always fall inside the triangle. He might think (correctly) that a triangle has 180^0. This set of properties together with the mental picture will be called by us the concept image (there is a more detailed discussion of the notions in Vinner, 1975 and 1980).

§ 2. Concept Definitions and their Role in Forming the Concept Images

By "Concept Definition" we mean here a verbal definition that accurately explains the concept in a non circular way. For some of our concepts we also have concept definition in addition to the concept image. For many other concepts we do not. For instance, we do not have a definition for "house", "orange" etc., although we have very clear concept images for them. They were acquired when we were children probably by means of "ostensive definitions". It is true that some concepts as these above could be introduced to us by means of verbal definitions. The word "forest" could be introduced to us by saying: "many many trees together are forest". We were supposed then to visualize many trees together and thus to form a concept image. However we claim that (1) in order to handle concepts one needs a concept image and not a concept definition. (2) Concept definitions (in case the concept was introduced by means of a definition) will remain inactive or even will be forgotten. In thinking, almost always the concept image will be evoked.

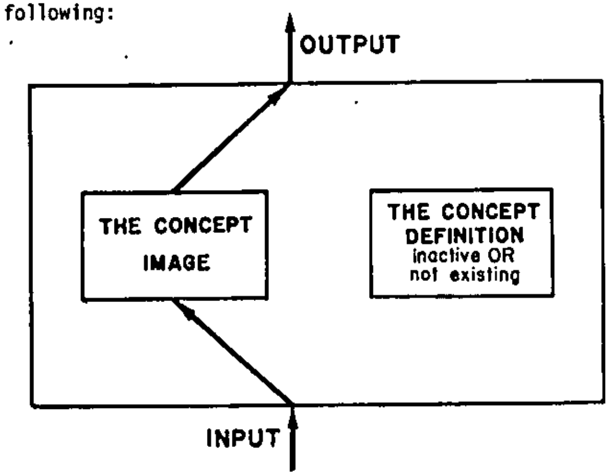
§ 3. Formal Learning, Definitions and Some Cognitive Models

What we said in the previous section is true mainly about informal learning of concepts. In formal learning the situation might be different. Here the concept definition might become a "part of the game". Students are required to give definitions. (In J.L. Austin and A.G. Howson, 1979 it is claimed that "in England the tradition is to rely on the formal definition in higher education and informal or ostensive definitions in primary or secondary schools" (p. 181). However, it is not so clear whether this is a tendency or a practice.) But the (cognitive) reason for asking somebody to give definitions should be the assumption that definitions help to form concept images and also that they are useful in carrying out some cognitive tasks. Thus, if we assume two cells (not biological) in the cognitive structure, one for the concept image and one for the concept definition, our impression is that the model, implicitly assumed by many teachers, for some cognitive tasks is similar to the following, (1) or (2):



These models do not reflect the practice. There is no way to force a cognitive structure to use definitions, neither in order to form concept images or in order to handle a cognitive task. Some definitions are too complicated to deal with. They do not help in creating mental pictures in the students' mind. Hence, they are useless. On the other hand, there are some definitions that do make sense but the moment some specific examples are given by the teacher or by the book, they form the concept

image and because of (1) and (2) of the previous section the definitions might become inactive or even be forgotten. Thus, the model for the above cognitive task is more similar to the following:



§ 4. Revealing the Concept Image and Teaching

Sometimes we test students in order to see whether they have the knowledge of some very simple geometrical concepts as right angle, isosceles triangle, an altitude in a triangle and so forth. We do it by means of a multiple choice questionnaire where they are asked to identify the concepts or to construct specific examples. Very often we discover that our students do not know the above concepts and they have wrong concept images. These images might be a result of the specific set of examples given to the students. Probably, there is an (implicit) assumption that the students are supposed to use the concept definitions when a cognitive task is imposed on them and therefore there is no need to give them numerous different examples, but, as we claimed before, such an assumption has no ground.

Here are three examples: 1. In a textbook for the second grade (in Hebrew) the author defines an isosceles triangle as a triangle which has two equal sides. All the isosceles triangles which are drawn in the text have a horizontal basis. Let us draw now a set of triangles some of which are isosceles triangles but only one of these has a horizontal basis. Will it be a surprise if only this triangle will be identified as an isosceles triangle by our students?

2. In a geometry textbook (Geometry with Coordinates, SMSG, part I, pp. 143-144) an angle is considered as "the union of two concurrent rays". A straight angle is "the union of two opposite rays". There is only one drawing of a straight angle in the book, of course, a horizontal one. Again, will it be a surprise if a student will identify only straight angles which are horizontal?

3. In another book by SMSG (Mathematics for Junior High School, Vol. 2, part I, pp. 194-195) right triangles are discussed. In all of them one side of the right angle is horizontal (there are ten right triangles altogether). It seems unnecessary to ask our question once again.

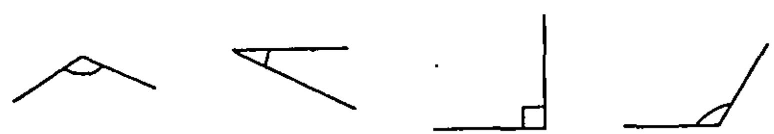
Thus, revealing the concept images of our students become very important for teaching. Not only that it might give us better understanding of our students (knowing what caused them to act as they acted) but also it might suggest some improvements to our teaching which formed such wrong concept images.

§ 5. The Sample, the Questionnaire and the Results

We tested 550 students in grades 7, 8 and 9. Their schools were officially declared as culturally deprived (namely, over 60% of the students are culturally deprived). The students were supposed to learn several geometrical concepts in their elementary schools, concepts as ray, angle, triangle etc. In their textbooks, these concepts were introduced by means of examples and formal definitions ("two concurrent rays are an angle" and so forth). The students did not have any official geometry lessons before the test but they met the above geometrical concepts in their mathematical activities from time to time.

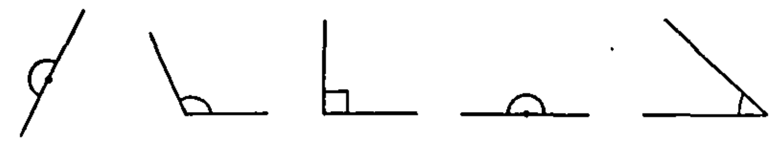
Out of the four questions in the test (which are concerned with this paper) three had an identification task and one had a construction task. Each question item will be denoted by a capital letter after which, in parentheses, will be written the percentage of right answers.

QUESTION 1: CIRCLE EVERY OBTUSE ANGLE.



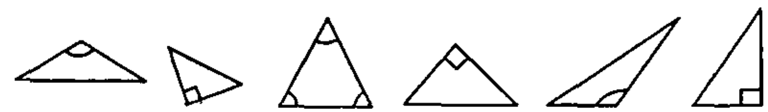
A (62.7) B (82.4) C (87.4) D (76.7)

QUESTION 2: CIRCLE EVERY STRAIGHT ANGLE.



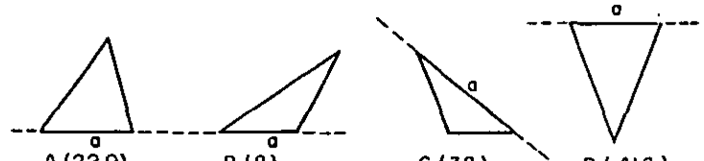
A (79.6) B (96.5) C (96.7) D (92) E (95.4)

QUESTION 3: CIRCLE EVERY RIGHT TRIANGLE.



A (86.4) B (67.6) C (76) D (41.3) E (92.2) F (75.6)

QUESTION 4: DRAW THE ALTITUDE TO THE SIDE DENOTED BY a IN EACH OF THE FOLLOWING TRIANGLES:



A (22.9) B (8) C (38) D (41.6)

The following can be said about the concept images:

1. The Obtuse Angle. Obtuse angles which have a horizontal ray are identified more easily than others. Some concept images contain only obtuse angles with horizontal rays. As we know teachers and textbooks, we can say that in both there is a tendency to draw obtuse angles with a horizontal ray. Probably the "gravitational factor" has also some role in forming the above concept image. An angle is "stable" ("can stand") only if it has one horizontal ray (the other one being ascending rather than descending). This might cause people to draw angles with a horizontal ray. As a result of this a concept image that contains only obtuse angles with a horizontal ray can be formed.

2. The Straight Angle. The concept image of many students for this notion contains only horizontal straight angles. Everything that was said about the obtuse angle can be said about the straight angle as well.

3. The Right Triangle. 76% of the students have a concept image for this notion that contains a right triangle with a vertical side and a horizontal side. Only 68% of the students have concept images that contain right triangle as in 3B. This triangle is obtained by means of a small rotation from a right triangle having a horizontal side and a vertical side. 59% of the concept images do not contain right triangles as in 3D. The gravitational factor plays a major role also here.

4. The Altitude in a Triangle. Note first that less than half of the students succeeded even in the easy items of the question. The failure in 4B can be a result of a (sometimes implicit) common belief that an altitude should always fall inside the triangle. The fact that the success percentages in 4C and 4D are so close to each other is, perhaps, due to the fact that both triangles are (almost) isosceles triangles. Our guess here is that the concept of an altitude to the basis in an isosceles triangle took the place of a general concept of altitude. Note that here the "gravitational factor" is not dominant anymore.

§ 6. The Notion of Common Cognitive Path

There are some approaches in cognitive theories which distinguish between cognitive levels and implicitly or explicitly suggest what might be called "cognitive paths" in learning. Gagné, for instance, speaks about hierarchies in systems of concepts (Gagné, 1970). Bloom and Avital distinguish between level of thinking (Bloom, 1956, Avital, 1968). Piaget points at stages in the intellectual development. In this paper we deal with simple isolated concepts that have several aspects or components. These aspects are not necessarily acquired simultaneously. Let us denote by A, B, C, D, four different aspects (components) that partially form a certain concept (or concept image). Assume that a certain learner acquired these aspects in the order they were written above. Then $A \to B \to C \to D$ is a cognitive path of the above concept for the above learner. Now assume that there is a group of people such that $A \to B \to C \to D$ is a cognitive path for everybody in the group. Then we will say that it is a common cognitive path for this group. In such a group we will not find people who know D without knowing also A, B and C or people who know C without knowing also A and B and so on. Suppose we had a question that examines aspects A, B, C and D of our concept and we administered it to a group of people (not necessarily the same one as above). Denote by a, b, c, d respectively the subgroups of people that answered correctly the items that test aspects A, B, C, D. Suppose, finally, that it was found that $a \supsetneq b \supsetneq c \supsetneq d$. We may claim then that $A \to B \to C \to D$ is a common cognitive path for this group (in the sense that nobody in the group can know D without knowing also A,

B, C and so on). If x is a set denote by m(x) the number of elements in x. It is easy to see that $a \supsetneq b \supsetneq c \supsetneq d$ is equivalent to

(1) $m(a) > m(b) > m(c) > m(d)$

(2) $\frac{m(a \cap b \cap c \cap d)}{m(d)} = \frac{m(a \cap b \cap c)}{m(c)} = \frac{m(a \cap b)}{m(b)} = 1.$

(It has some similarity to Guttman scales, see for instance SPSS, 1975). Certainly it is not realistic to expect (1) and (2). It seems that in a more realistic situation it will be reasonable to expect in (2) a ratio smaller than 1 but quite close to it, say 0.8. However, a simple analysis (that will be given elsewhere) shows that this is not the case. As a result of this analysis we suggest the following definition.

Definition: Let A, B, a, b be as above and assume $m(a) > m(b)$. Denote by N the total number of people in our group. Define now two numbers by the following:

$r(a, b) = \frac{m(b) \cdot m(a)}{N}$, $\Delta(a, b) = m(a \cap b)$ (It is easy to see that r(a, b) is approximately the number of members of b who are also in a, provided b was randomly chosen.) Assume $\Delta(a, b) > r(a, b)$. We will say that $\Delta(a, b)$ is α- significantly greater than r(a, b) if the probability of χ^2 comparing r(a, b) (as an expected value) and $\Delta(a, b)$ (as an observed value) in b is less than α. Finally, we will say that $A \to B$ is an α-significantly common path, if $\Delta(a, b)$ is α - significantly greater than r(a, b). If $\Delta(a, b)$ is α-significantly smaller than r(a, b) we will say that $A \to B$ is definitely not a common cognitive path. In all other cases we will say that there is no indication on a significantly common cognitive path.

Thus we have defined our notion for two step path but it is obvious how this definition can be generalized for any number of steps. For instance, if we have aspects A, B, C and we have $m(a) > m(b) > m(c)$ then in addition to r(a, b), $\Delta(a, b)$ that we should compare according to the above definition we should also compare r(a , b, c) and $\Delta(a, b, c)$.

We will apply now the above notion to the questions of our test. In questions 1 - 3 we had an identification task. Such a task generally include two kinds of items. Those which should be identified as the concept and those which should be identified as the non-concept. Call identification of items of the first kind by C-identification (concept identification) and identification of items of the second time by NC-identification). Thus in questions 1-3 we will look for two different common cognitive paths, one for the C-identification and the other for the NC-identification. In question 4 we will look only for one cognitive path. The results are the following:

Question 1: D—A is an α-significantly common cognitive path, $\alpha = 0.005$, for the C-identification. There is no indication on significantly common path for NC-identification.

Question 2: D—A is an α-significantly common cognitive path for C-identification, $\alpha = 0.005$. There is no indication on significantly common cognitive path for NC-identification.

Question 3: F—B—D is an α-significantly common cognitive path for C-identification, $\alpha < 0.005$. E—A—C is an α-significantly common cognitive path for NC-identification, $\alpha = 0.005$.

Question 4: D—C—A—B is an α-significantly common cognitive path for the construction task, $\alpha = 0.005$.

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The effects of previous knowledge on learning geometry

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As part of a general research program[1] about the utilisation of cognitive models for teaching and learning processes in school a research group at the University of Osnabrück investigates changes of states of cognitive development during the mathematics' instruction of 5th grade students (about 10 years old). For two instructional units, "axial reflection" and "nondecimal number systems", teaching programmes have been developed. Additionally, diagnostic tests were constructed which aim specifically at previous knowledge and the content dependent mechanisms of cognitive processing. The influence of these tested learning-prerequisites on the actual processes of learning and thinking were studied by individual observation of each student during his work on the teaching programme. The investigation aimes at a comparison of content specific and content independent properties of thinking processes in both areas "axial reflection" and "nondecimal number systems" to clarify the relation of more area specific and more general modes of thinking. This report is restricted to the area of axial reflection and exemplifies our procedures and gives some first results.

1. Theoretical foundations

The visual objects of school-geometry develop the human ability of thinking in a specific way: geometrical relationships are perceived by direct visual contact and thinking about these relationships can proceed within this direct visual contact. Many teachers assume that the pupil perhaps produces spontaneously semantic interpretations while his cognitive system is concretely interlocked with the object of thinking. But adverse to the assumption of many teachers exatly this the pupil is unable to do. But only later, after having attributed meaning to some relationships (e.g. to the position of two lines as "vertical") he will be able to make further explorations. Concerning the area of axial reflection, the objects of thinking are lines and points in the plane, and just with these the 10 years old pupil has scarcely accumulated every-day experience. But in spite of this he relativly quickly achieves to organize the perceptual field into parts, unities and subunities. (Koffka, 1935).

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