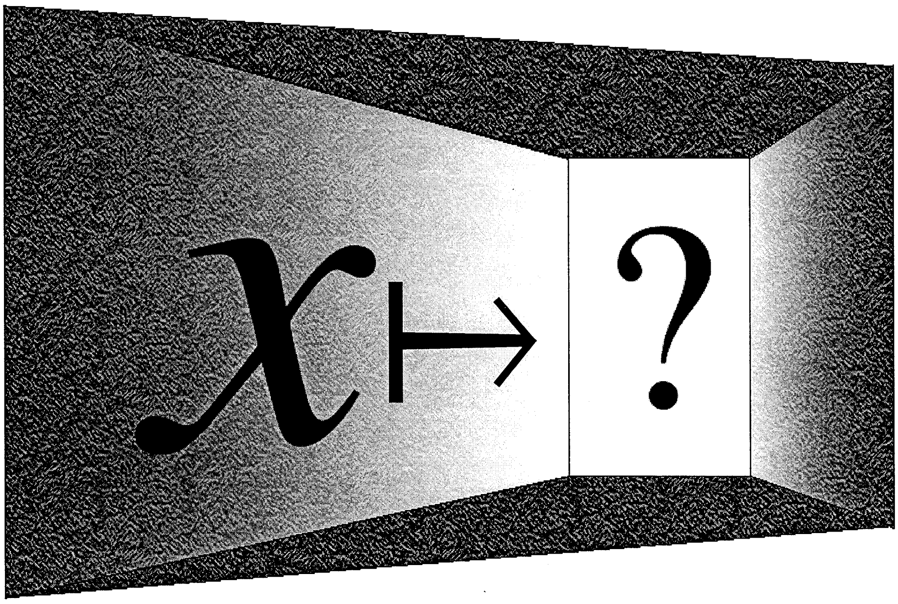
Proceedings of the

12th ICMI Study Conference

The Future of the Teaching and Learning of Algebra



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Helen Chick, Kaye Stacey, Jill Vincent & John Vincent

Volume 1

Proceedings of the 12th Study Conference of the International Commission on Mathematical Instruction

The Future of the Teaching and Learning of Algebra

Volume 1

Editors: Helen Chick, Kaye Stacey, Jill Vincent & John Vincent

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Foreword

The International Commission on Mathematical Instruction (ICMI) has, since the 1980s, conducted a series of studies into topics of particular significance to the theory and practice of contemporary mathematics education. Each Study involves an international seminar—or Study Conference—involving 50-100 participants. The Study culminates in a published volume intended to promote and assist discussion and action at the international, national, regional and institutional levels.

This, the twelfth ICMI Study, has as its focus the future of the teaching and learning of algebra. There are several reasons why it is timely to focus on this topic. The strong research base developed over recent decades enables us to take stock of what has been achieved and also to look forward to what should be done and what could be achieved. In addition, trends evident over recent years are intensifying. Those particularly affecting school mathematics are the massification of education, which is continuing in some countries whilst beginning in others and the advance of technology. Algebra is centrally affected by both of these trends. For mass education, algebra teaching highlights questions of equity and relevance. For progression to higher mathematics, students need algebra but its abstraction makes it hard to learn and hard for beginners to see a reason for learning. Simultaneously, advancing technology provides rich prospects for improving teaching. However, it also provides a challenge to the existing curriculum because so many of the routines that have been the standard diet (and where students have been most successful) are now available "at the press of a button". The result is that an algebra curriculum that serves its students well in the coming century may look very different from an ideal curriculum from some years ago. These ideas are further developed in the Discussion Document, reprinted on pages 1-6. We hope that the deliberations at the Study Conference will keep firmly in mind the goal of providing practical advice on future directions, which is imaginative and soundly based in scientific research.

The International Program Committee (IPC) first met in January 2000 to draft the Discussion Document that was then disseminated through the mathematics education community. Over 150 papers were submitted in response to the call for contributions, with many of them directly addressing the issues raised in the Discussion Document. This very strong response indicates the widespread concern with the teaching of algebra, the strong research base upon which we can go forward and an optimism that attention to this issue can result in real gains for students around the world.

The submitted papers were read by two or three members of the IPC with the assistance of additional readers. Invitations to the Study Conference were then issued to potential participants, taking into account paper submissions, areas of interest and expertise, the goals and objectives of the conference, and geographic representation. In order to retain the characteristics of previous ICMI studies, the number of participants was strictly limited, so that in many instances, not even all the co-authors of accepted papers could be offered places. In keeping the size of the Study Conference small, ICMI aims to maximise the opportunity for interaction between participants. Overall, only two-thirds of the submitted papers were accepted. The papers were submitted on the basis that they would stimulate discussion at the Conference and have not, in most cases, undergone revision prior to publication in these Proceedings. Readers may therefore expect that more developed versions of these papers will later be submitted for journal publication.

The papers are published in two sets. For most of the papers, authors have been able to give permission for the paper to be openly distributed. These are in Volumes 1 and 2. However, for a variety of reasons, other papers are reprinted only for use at the Study Conference. These are printed in Volume 3.

During the Study Conference the 100 or so invited participants will contribute to the working groups. These working groups will examine the central theme of the future of the teaching and learning of algebra from nine different viewpoints:

- Why Algebra? What Algebra?
- Technological Environments
- Computer Algebra Systems
- Early Algebra
- Tertiary Algebra
- Teachers' Knowledge for Teaching Algebra
- History of Algebra
- Symbols and Language
- Approaches to Algebra

The deliberations of the working groups will form the basis of an edited book, which will continue the ICMI Study series. Some of the publications from earlier Studies are previewed at http://www.wkap.nl/series.htm/NISS.

Finally, we wish to offer our thanks to the many people who are assisting us to make the ICMI Study a success. Members of the International Program Committee, listed on page v, have been involved in every aspect of this project, offering sound advice based on wide experience and perspectives from around the world. The Conference is held under the auspices of the Australian Sub-committee of ICMI, chaired by Desmond Fearnley-Sander. Jill and John Vincent have brought to the role of Conference Secretariat a determination to make the study a success, good teamwork, good humour, a range of organisational skills, and a concern to help the organisers and participants at the conference. Their assistance is greatly appreciated. A large group of volunteers from the Department of Science and Mathematics Education have helped with many tasks. We also are grateful for the sponsorship of DETYA (Benchmarking, Assessment and Numeracy Policy Sector); Shriro, the Australian distributors of Casio calculators, Texas Instruments (Australia); and the Rotary Club of Melbourne.

Kaye Stacey (Program Chair)

Helen Chick (Conference Secretary)

University of Melbourne

September 2001

About ICMI

The International Commission on Mathematical Instruction (ICMI) was first established at the International Congress of Mathematicians held in Rome in 1908, on the suggestion of the American mathematician and historian of mathematics, David Eugene Smith. The first President of ICMI was Felix Klein. As well as the international journal *L'Enseignment Mathématique*, ICMI also publishes a Bulletin twice a year. Starting with Bulletin 39, December 1995, the *ICMI Bulletin* may be accessed on the internet:

http://elib.zib.de/IMU/ICMI

The 12th ICMI Study

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Discussion Document for the 12th ICMI Study

The Future of the Teaching and Learning of Algebra

Introduction

This document introduces the 12th ICMI study entitled *The Future of the Teaching and Learning of Algebra*, to be held at the University of Melbourne (Australia) in December 2001. The intention is that the word 'Algebra' will be interpreted broadly to encompass the diversity of definitions around the world, extending beyond the standard curriculum in some countries. It will include, for example, algebra as a language for generalisation, abstraction and proof; algebra as a tool for problem solving through equation solving or graphing; for modelling with functions; and the way algebraic symbols and ideas are used in other parts of mathematics and other subjects. The principal interest of many participants is likely to be related to secondary school mathematics (ages 11 to 18) and algebra with real variables, but the study is also concerned with tertiary algebra (e.g. linear algebra and abstract algebra) and with algebra and its precursors for young children.

There are many reasons why it is timely to focus on the future of the teaching and learning of algebra. We are at a critical point when it is desirable to take stock of what has been achieved and to look forward to what should be done and what can be done. In many countries, increasing numbers of students are now receiving secondary education and this is causing every part of the mathematics curriculum to be scrutinised. For algebra, perhaps more than other parts of mathematics, concerns of equity and of relevance arise. As the language of higher mathematics, algebra is a gateway to future study and mathematically significant ideas, but it is often a wall that blocks the paths of many. Should algebra be made more accessible to more students by changing the amount or nature of what is taught? Many countries have already embarked on such changes, hoping to increase access and success. Alternatively, are these changes necessary: is algebra truly useful for the majority of people and, even if it is, will it be useful in the future?

An algebra curriculum that serves its students well in the coming century may look very different from an ideal curriculum from some years ago. The increased availability of computers and calculators will change what mathematics is useful as well as changing how mathematics is done. At the same time as challenging the content of what is taught, the technological revolution is also providing rich prospects for teaching and is offering students new paths to understanding. In the past two decades, a substantial body of research on the learning and teaching of many aspects of algebra has been established and there have been many experiments with adapting curricula and teaching methods. There is therefore a strong scientific basis upon which to build this study.

The study has two aims: to make a synthesis of current thinking and lessons from the past which will help set directions for future work in the field, and to suggest guidelines for advancing the teaching and learning of algebra. The rest of this Discussion Document outlines the main themes of the Study Conference.

Why algebra?

The technological future of a modern society depends in large part on the mathematical literacy of its citizens and this is reflected in the worldwide trend towards mass secondary education. For an individual, algebra is a gateway to much of higher education and therefore to many fields of employment. Educators also argue that algebra is part of cultural heritage and is needed for informed and critical citizenship. However, for many, algebra acts more like a wall than a gateway, presenting an obstacle that they find too difficult to cross. This section of the study is concerned with the significance of algebra for the broad population of secondary school students, recognising that regional and cultural differences may impact upon the answers in interesting ways. It addresses questions such as:

- Should algebra be taught to all? There has been a call for algebra for all secondary students, but what aspects of algebra are of value to all? What should comprise a minimal curriculum? How do answers to these questions relate to regional or cultural differences?
- What do we expect of an algebra-literate individual? What are the values of algebra learning for the individual, especially in view of increasingly powerful computing capabilities? Access to higher learning and employment are two values, but what are the more immediate values and how can they be achieved?
- How can we reshape the algebra curriculum so that it has more immediate value to individuals? Can we identify explicit examples in contexts meaningful to students in which algebraic ideas have clear, unambiguous value? Are there undesirable consequences of such orientations to algebra?
- How can we reshape the algebra curriculum so that specific difficult ideas are more easily accessed?

Approaches to algebra

Recent research has focused on a number of approaches for developing meaning for the objects and processes of algebra. These approaches include, but are not limited to, problem-solving approaches, functional approaches, generalisation approaches, language-based approaches, and so on. Problem-solving approaches tend to emphasise an analysis of problems in terms of equations and a view of letters as unknowns. Functional approaches support a different set of meanings for the objects of algebra; for instance, the use of expressions to represent relationships and an interpretation of letters in terms of quantities that vary. A somewhat different perspective is encouraged by generalisation approaches that stress expressions of generality to represent geometric patterns, numerical sequences, or the rules governing numerical relationships, such approaches often serving as a basis for exploring underlying numerical structure, predicting, justifying and proving. Some algebra curricula develop student algebraic thinking exclusively along the lines of one such approach throughout the several grades of secondary school; others attempt to combine facets of several approaches.

Synthesising the experience with and research on the use of various approaches in the teaching/learning of algebra leads to questions such as the following:

- What does each of these various teaching approaches mean?
- What are the algebraic meanings supported by each?

- What are the epistemological obstacles inherent in each?
- Which important aspects of algebra are favoured/neglected in each approach?
- What are the difficulties encountered by students in extending the meanings that are developed by each of these approaches to include the meanings inherent in other approaches?

Language aspects of algebra

This section considers theoretical and applied aspects of the languages and notations of algebra, in relation to teaching and learning. The evolution of algebra cannot be separated from the evolution of its language and notations. Historically the introduction of good notations has had enormous impact upon the development of algebra but a good notation for science may not be a good notation for learning. With new computer technology we are now seeing a flowering of new quasi-algebraic notations, which may offer, support or eventually enforce new notations. However, current theories of mathematics teaching and learning do not seem adequate to deal with learning about notation. It is therefore timely to focus on algebraic notations asking questions such as:

- How do theories of mathematics teaching and learning embrace the linguistic aspects of algebra and what can we propose to better take into account these aspects?
- Algebra is not a language but it has a language and the two cannot be dissociated. What does it mean to talk about algebra as a language and what are the implications of such a perspective?
- There is a wide range of theories of how mathematical concepts are learned and taught (in particular the constructivist theories) but learning a language is not just a matter of learning concepts. How do acknowledged theories of mathematics learning and teaching embrace the non-conceptual aspects of learning the language of algebra and what can we propose to better take these aspects into account?
- Would some changes of algebraic language contribute to the development of algebraic thinking, communication and understanding?
- Is it feasible and desirable to remove some of the ambiguities that are present in standard mathematical symbolism, for example in the use of the equals sign?
- Should some effort be made in the teaching of mathematics to explain and bridge differences in notation between algebra as it is taught in mathematics courses and algebra as it is used in other disciplines?
- What are the characteristics of good notation? What does mathematics education research have to say on this? Are some notational choices better for science but others better for learning?

Teaching and learning with Computer Algebra Systems

The advent of affordable computer systems and calculators that can perform symbolic calculations may lead to far-reaching changes in mathematics curricula and in mathematics teaching. This section addresses questions that arise from the increasing accessibility of computer symbolic manipulation. Answers to these questions will draw upon established research on the teaching and learning of algebra as well as reporting on recent experimental work. They may suggest new directions for research, including:

- For which students and when is it appropriate to introduce the use of a computer algebra system? When do the advantages of using such a system outweigh the effort that must be put into learning to use it? Are there activities using such systems that can be profitably undertaken by younger students?
- What algebraic insights and 'symbol sense' does the user of a computer algebra system need and what insights does the use of the systems bring?
- A strength of computer algebra systems is that they support multiple representations of mathematical concepts. How can this be used well? Might it be overused?
- What are the relationships and interactions between different approaches and philosophies of mathematics teaching with the use of computer algebra systems?
- Students using different computational tools solve problems and think about concepts differently. Teachers have more options for how they teach. What impact does this have on teaching and learning? Which types of system support which kinds of learning? Can these differences be characterised theoretically?
- What should an algebra curriculum look like in a country where computer algebra systems are freely available? What 'by hand' skills should be retained?

Technological environments

Recent research, curriculum development, and classroom practice have incorporated a number of technologies to help students develop meaning for various algebraic objects, ideas and processes. These include, but are not limited to, function graphers, spreadsheets, programming languages, one-line programming on calculators, and other specific computer software environments. [Here, we exclude computer algebra systems that are treated elsewhere.] In an attempt to characterise recent research and experience, this section will explore which aspects of specific computer/calculator environments are related to which kinds of algebra learning. This question will be explored in depth for specific examples of such technology, by addressing questions such as the following:

- For a given technological environment, what are the implicit assumptions regarding the underlying core aspects of algebra?
- Which important aspects of algebra are and are not touched upon by this environment?
- What kinds of algebra learning does this environment promote?
- What particular limitations are associated with the use of this environment and how can such limitations be dealt with?
- To what extent ought the goals of algebra education be affected by the availability of this technology?
- To which aspects of algebra learning does this particular technology make a distinctive, unique contribution?
- Are there documented long-term consequences of embedding this particular technology in an algebra curriculum, and if so, what are they?

Algebra with real data

Modelling the behaviour of real things with algebraic functions is fundamental to applications of mathematics. Using real data to teach about functions is therefore important in the curriculum, and can also be highly motivating for students. Moreover, new devices (such as data loggers) and new communications technologies (such as the internet) provide new opportunities for bringing real data into the classroom. Questions such as the following arise: What new opportunities for using real data have proved to be successful and how do they relate to research on students' learning of functions and other algebraic concepts?

- Using the history of algebra What are the strengths and weaknesses of using real data and how are these best managed in the classroom and in the curriculum?
- A commitment to using real data may lead to significant changes in curriculum content and sequence, for example by giving prominence to the exponential function over the quadratic. What changes may be required and what are their consequences?
- Interpreting real data can lead students and teachers to question why the world is as it is. What is the role of algebra education in the development of critical thinking about social issues such as economics, health and environment?

The history of algebra has been used extensively to identify epistemological obstacles in the learning of algebra and to characterise ruptures in the development of algebraic notions. Drawing on the history (or histories) of algebra from around the world, this section aims to analyse significant contributions and the value of these previous uses and also to reflect on possible avenues for research based on new areas, including:

- The history of symbolism; that is, the history of ways of representing quantities and operations in calculations;
- The history of methods for solving problems;
- The history of methods for solving equations;
- The history of the interactions of algebra with other mathematical domains (such as geometry); and
- The development of the idea of algebraic structures.

Early algebra education

This section encompasses two different readings of the title, being concerned with both the algebra education for young children - say age 6 and above - and also the initial steps in more formal algebra education, which happens in some countries when students are about 12 years old. An ongoing concern is the relationship between arithmetic and algebra. Previous research has documented ways in which students' limited arithmetical experience can constitute an obstacle to the learning of algebra, so that an earlier start might reduce the problem; approaches have been proposed to achieve that. On the other hand, a much favoured approach to initial algebra education is based on the view of school algebra as generalised arithmetic, in which case an earlier start may not be appropriate. The general point here is that different views on the relationship between arithmetic and algebra will probably result in different views on algebra education, and this most important fact is a central concern in this section. The interest in algebra education for students at an early age is recent, and so there are as yet only a few studies in this area. It is important that answers to the following questions be thoroughly research-based:

- How early is 'early algebra' and what are the advantages and disadvantages of an early start? How do the answers to these questions link to views on cognitive development and on learning, and on cultural and educational traditions?
- What aspects of algebra and algebraic thinking should be part of an early algebra education? Since the symbolic aspect of algebra is so essential, its early introduction may be beneficial, but is an awareness of algebra as a method to solve problems (for example) more important?
- What are the consequences of an early start to algebra for teachers and teacher education?

Tertiary algebra

Problems exist in the teaching and learning of tertiary algebra courses such as abstract algebra, linear algebra, and number theory. Some are similar to the problems of secondary algebra: students' difficulties with abstraction, concerns of relevance, what to do with computing technology, etc. Other problems such as proof-making or seeing the objects of calculus as algebraic objects seem particular to the tertiary level. The questions below are concerned with these issues of learning and teaching and also with the specific question of education for prospective teachers.

- What are the contributions of tertiary algebra courses to the education of prospective secondary mathematics teachers? How do secondary teachers perceive the value of their tertiary algebra courses to their teaching experience?
- Secondary algebra has been well researched, and specific obstacles have been found in making the transition from arithmetic thinking to algebraic thinking. Do tertiary level students similarly experience obstacles in making the transition from secondary-level algebraic thinking to that required for the tertiary level?
- Why are certain types of definitions difficult for students? For example, why are definitions given in terms of properties to be satisfied (for example, subspaces and group automorphisms) so difficult for students? How can this problem be addressed?
- There are specific questions about specific aspects of specific courses in algebra; for example, why do students who seem competent in R^n have difficulty with more concrete questions in R^2 and R^3? How can such questions be resolved?
- How does symbolic logic (through statements, connectives, quantifiers, qualified statements, and arguments) affect students' proof-making and their view of the value of proof-making?
- Secondary school algebra seems to lead more directly to applied mathematical modelling at the tertiary level, rather than to abstract algebra. What is going on here?
- Should secondary students learn more about algebraic structure?

Designing a National Mathematics Curriculum

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The latest version of the National Curriculum for England became effective from September 2000. This rewrite was straightforward for many subjects, but was non-trivial in mathematics. The government remit was to address concerns about pupils' attainment in mathematics yet keep change to a minimum. Strengthening the role of algebra was central to the revision. This paper focuses on algebra in the National Curriculum, with a forward glance to the post-16 mathematics curriculum. It describes what has been done and looks to the future. Algebra is the key to abstraction and generalisation. How is it best embedded in the curriculum?

Mathematics, the National Curriculum and Government Strategy

Mathematics is a core subject in the National Curriculum. All pupils study mathematics to the age of 16. There are four key stages: KS1 (years 1 and 2), KS2 (years 3-6), KS3 (years 7-9) and KS4 (years 10 and 11). Each key stage has its own Programme of Study, usually based on three strands of mathematics: Number (KS1/2) or Number and algebra (KS3/4); Shape, space and measures (KS1-4); Handling data (KS2-4). At the end of each of key stages 1,2 and 3 there are national assessments of all children. The pupils do tests in Mathematics, English and Science. Each child is assigned a national curriculum level in each subject at the end of each key stage. Most children at age 16 sit GCSE Mathematics examinations, which are set by external awarding bodies working to national criteria set by the curriculum regulatory body, the Qualifications and Curriculum Authority (QCA).

QCA has responsibility for writing the National Curriculum for Mathematics and monitoring the effectiveness of its implementation. QCA also sets, and commissions the marking of, the end of key stage tests in mathematics. It also sets the criteria for all pre-university national qualifications in mathematics (and other subjects) and looks at comparability and standards over time. More recently it has also pioneered the development of new style qualifications in mathematics for post-16 year olds.

The National Curriculum was first introduced in 1988. It was viewed as an educational entitlement for all pupils, with the idea that no matter where they came from they were presented with the same educational opportunities as pupils elsewhere in the country. National assessment of the curriculum was introduced, and the external examinations from age 16 and above were reorganised.

The curriculum was revised in 1991, 1993 and 1999 (Mathematics in the National Curriculum, 1993 and 1999). The 1999 revisions (for National Curriculum 2000, known as NC 2000) took place against a background of serious concern following England's relatively poor performance in TIMSS populations 1 and 2 (S.Harris et al, 1997) and other international comparisons, some comparisons of examination performance over the period 1975-1995. Also, there were and grave charges from universities (Howson, 1996; *Tackling the mathematics problem,* 1995) that their students seemed to be increasingly incompetent in algebra, that they could not develop or sustain multi-step chains of mathematical reasoning, that they had poor appreciation of proof. It was also the case that the students had little awareness (let alone understanding) of why mathematics was such a powerful analytical tool in the study of many other disciplines and why certain mathematical statements could be made with certainty. At the same time employers were also expressing concerns about the general mathematical competency of their work force.

Thus, the country seemed to be facing some sort of mathematical crisis. The government's response was to set up the National Numeracy Strategy (NNS)[1] as a means of injecting a greater sense of purpose into the teaching of primary mathematics in the first instance, but with a longer term view that this strategy would soon move into the secondary arena as well. The government also commissioned a wholesale review of the curriculum couched in diplomatic terms: minimum change (the teachers do not want further disruption); the attainment goal-posts are not to be shifted (because targets had been set to be achieved by 2002); address all the concerns about falling standards and lack of competence, the lack of facility with algebra, reasoning and proof, and England's poor international ratings; the curriculum and the NNS must harmonise their message. All of this posed the curriculum developers with no mean task!

The subsequent sections of this paper will look at the issues we had to consider, the resolutions that we came to for NC 2000 and how we are now looking ahead to further strengthen the intended National Curriculum for Mathematics and the wider mathematics curriculum. In what follows, the wording *mathematics curriculum* is used to refer to ages 5-19, and in particular to the 16-19 curriculum. The National Curriculum is valid to age 16. It is enshrined in an Act of Parliament and schools have to teach to the content of the National Curriculum. Compulsory schooling for all ends at age 16, and the post-16 curriculum does not carry the same legal connotations as the National Curriculum.

NC 2000

A glance at NC2000 shows that in comparison with the 1993 version of the National Curriculum there is much greater specificity of detail than before. Many more links across the subject have been stressed, and links to use of ICT and where it might prove useful to use data from other subject areas. There is much greater emphasis on Using and applying number and algebra, on Using and applying shape, space and measures and on Using and applying handling data, and in the specific details in which these are spelt out. There are also sections on the opportunities that should be presented to pupils to allow them to engage in different types of mathematical activity so as to help them make the necessary connections to develop their overall experience and exposure to mathematics and the use of mathematics. Other subjects (such as the sciences or geography), in their versions of the National Curriculum, make explicit links to particular sections of the National Curriculum for Mathematics, where this is particularly appropriate or necessary.

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[1] The name National Numeracy Strategy poses its own problems, because it suggests that politicians equate mathematics with numeracy. In practice, however, the NNS has delivered frameworks for the teaching of mathematics. Now that it has moved into Key Stage 3 it is increasingly referred to as the National Mathematics Strategy, but formally its name remains unchanged.

The overall mathematical content of NC2000 is little changed from the 1993 version, as are the level descriptors that describe pupil attainment. In this way the government requirement for minimum change without changing the attainment goal-posts has been met as far as possible. The change then is mainly presentational, through the greater specificity (much welcomed by teachers, incidentally) and a more coherent approach to progression, not only within each key stage, but going across from one key stage to another. Greater weight has been given to Number and algebra than hitherto, and this has been achieved by reducing requirements in the Handling data strand. The structure of the Programme of Study for each strand is either the same from key stage to key stage or changes in a logical way as concepts increase in sophistication through the key stages. In this way, progression in subject content can easily be tracked through.

One final major change is that the new curriculum has introduced differentiated curricula at Key Stage 4. KS4 Higher is designed for about the top 50% of the age cohort whereas KS4 Foundation is for the other half of the cohort. There is more emphasis on abstraction in the Higher Programme of Study and this is made manifest through much greater demand on the algebra and geometry requirements. Correspondingly, in the Foundation PoS there is a greater emphasis on applications to give 'relevance' to the mathematics.

The nature and role of algebra in NC2000

The importance of algebra in the overall development of the National Curriculum for Mathematics is apparent from the relative amount of space devoted to algebra.

An important influence on the content and structure of the algebra strands was the highly influential report *Teaching and Learning of Algebra pre-19*, commissioned by the Royal Society and the Joint Mathematical Council of the United Kingdom (*Teaching and Learning of Algebra*, 1997). This report looked at algebra across the entire mathematics curriculum (including post-16 academic and vocational courses) and made recommendations on the nature of school algebra, on the emphasis that ought to be placed on school algebra, on the phasing in of algebra within the curriculum, on the teaching and assessment of algebra, on new technologies in relation to supporting (but not supplanting or undermining) algebra, on supporting in-service training of teachers. The report also included some comparison of English practice with what was happening in other countries and a look at materials from textbooks, tests and examinations.

The opening paragraph of the National Curriculum for Mathematics stresses the importance of mathematics to the development of logical reasoning, problem-solving skills and the ability to think in abstract ways. It also stresses the utility of mathematics to understand and probe the world around us. At various points throughout the Curriculum, the significance of algebra for these (and other) aspects of the subject are emphasised time and again.

The algebra grows out of number. There is no specific algebra strand in the primary curriculum, but emphasis on the correct nature and use of inequality and equality signs is stressed, as is elementary work with fractions and ratio. The idea that addition and subtraction, multiplication and division are inverse processes to each other is developed, as are such concepts (but not the names) as the distributivity of multiplication over addition or the commutativity of multiplication or addition; children analyse simple sequences and they learn about the use of coordinates. Pupils learn to construct and interpret very simple formulae, first in words and then in symbols. They are also encouraged to explain their reasoning in solving problems. All this activity is considered as a precursor to the development of algebraic thinking.

Formal algebra is introduced in Key Stage 3, at the start of secondary school. The algebra strand is presented under two main headings. The first of these is *Equations, formulae and identities,* with the further sub-headings: Use of symbols; Index notation; Equations; Linear equations; Formulae; Direct proportion; Simultaneous linear equations; Inequalities; Numerical methods (to use systematic trial and improvement methods with ICT tools to find approximate solutions of equations for which there is no simple analytical solution). The second main heading is *Sequences, functions and graphs* with sub-headings Sequences; Functions; Gradients. In Key Stage 4 Higher the main headers are the same, some of the sub-headings are the same, some change, e.g. Direct proportion becomes Direct and inverse proportion, and some new ones such as Quadratic equations or Transformation of functions are introduced for the first time. Key Stage 4 Foundation includes a few new sub-headings such as Graphs of linear functions and Interpreting graphical information.

There is a deliberate to attempt to develop some symbol sense to understand how symbols are used, to use notation correctly and to use correctly words such as *expression*, *identity*, *equation* and *formula*. There is an emphasis on transforming algebraic entities through an understanding that the manipulations are simply generalisations of corresponding arithmetic manipulations. These arithmetic manipulations include detailed work on the arithmetic of combining fractions that will develop a surer foundation for eventual work (possibly at post-16 level) on rational algebraic functions. There is an emphasis on transformation of viewpoint e.g. algebraic to graphical or graphical to algebraic. There is generation of algebraic formulae. Students are encouraged to develop short chains of deductive reasoning, to explain their reasoning and to interpret key features of graphs. They learn to understand the difference between an exact algebraic solution and an approximate solution to a problem obtained by graphical or numerical methods. They apply algebra to the solution of a whole range of problems, including word problems, problems in geometry and problems in context. Notions of proof and counterexample are also introduced. Each Programme of Study includes many examples suggesting the appropriate pitch for that Programme of Study.

The national tests have now begun to reflect this strong emphasis on algebra and GCSE examinations will reflect these changes from 2003. The National Numeracy Strategy has now produced a detailed *Framework for teaching mathematics: Years 7 to 9* (Framework, 2001). This contains Key Learning Objectives for each of these three years of Key Stage 3 and numerous examples in algebra which show how the curriculum can be implemented on a year by year basis.

Where next?

We do not know if the reformulation will be effective, but we do hope that it will encourage teachers to concentrate on imparting central features of the algebra curriculum and thus help pupils to develop a much better fluency with algebra, its nature and modes of operation, and its usefulness.

We are less sure of the changes made to the geometry part of the curriculum. These changes were also introduced to strengthen notions of generality, abstract reasoning and proof. However, here we were on less firm ground as there was no report on the table comparable to the Royal Society report on algebra.

The government has allowed QCA to be pro-active with respect to future revision to the National Curriculum for Mathematics, and that if further change is still needed it might be introduced in a more gentle way than a big bang reformulation.

QCA will continue to monitor how the curriculum is working in English schools. However, thinking and planning on a broader perspective we are now engaged in collecting detailed information about the nature and role of both algebra and geometry in the mathematics curricula of different countries. We are trying to find out about: the intended curriculum and its purpose, and the actual implemented curriculum in these countries; about examination structures and differentiation; about how successful formal assessment of reasoning and proof is in reality; about how the curriculum is delivered in textbooks; about resourcing and additional training of teachers, and so on. We have an open mind where this fact finding exercise will lead to, but we will be able, at the very least, to understand what other countries are doing and why they are doing what they are doing. We will also meet with mathematics teachers, educators, researchers and curriculum planners from England and other countries so that we can share experiences and learn from each other.

We have funding for three years, and there may be a possibility of further extension. At the end of the day we may decide to produce curriculum guidance only, or we may decide that the National Curriculum needs a small amount of tweaking, or even a major rewrite (not an option that we would particularly favour at this time).

Curriculum change takes a very long time to work its way through the system, and therefore all change is to some extent a gamble. We need change. Will the present changes work? What will be the effect of introducing further change before we know whether the most recent changes are working or not? There is no simple way to puzzle one's way out of such dilemmas. A primary child who started school when the National Numeracy Strategy was first imposed will not leave school until 2012, and not leave university until at least 2015. That is too long a period in which to remain inactive, so decisions have to be made and acted upon, with the anticipation of satisfactory outcomes.

The wider mathematics curriculum

All the concerns that the National Curriculum was seeking to redress apply to the learning and understanding of mathematics studied beyond GCSE level. There is an imperative for a smooth transition to the upper reaches of school mathematics and the National Curriculum has to pave the way for this. There is also an imperative for a smooth transition from secondary mathematics to university mathematics or to university courses with a heavy mathematics component (such as courses in science, computing, engineering, economics and so on).

Changes have been made to the criteria for the GCE Advanced level Mathematics which build on the changes in NC2000. These should further strengthen understanding of algebraic reasoning, manipulation and notions of proof, and the use of algebra in context.

There are opposing views on the use of ICT. On the one hand it is seriously frowned upon because its too frequent use seems to inhibit students' algebraic development; on the other hand it can be a very useful tool for using algebra and graphs in real contexts.

QCA has now pioneered a less academic approach to post-16 mathematics by introducing new qualifications that take mathematics significantly beyond GCSE, but concentrate on developing process skills of: modelling with real data; developing and presenting reasoned arguments; learning to read and comprehend the mathematics of other people; learning mathematics across a range of realistic contexts using a range of algebraic, graphical and numerical techniques. Students must use ICT throughout and they are encouraged to apply the mathematics to some other area of their studies. They are assessed through a combination of portfolio work and written examination papers including a comprehension paper. It is hoped that this approach will provide motivation to students who might find a more formal approach not to their taste. Preliminary indicators suggest that there is much to recommend in this approach. Many students who would have great difficulty with a formal treatment of algebra can make significant progress with algebra using an applications approach motivated by seeing immediate relevance of the mathematics to other aspects of their studies and by the use of real data through the use of ICT.

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Towards a Theoretical Synthesis of Research in the Early Learning of Symbolic Algebra

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Without a framework for synthesising the vast quantity of research into students' early learning of symbolic algebra, valuable findings will remain disconnected. This paper outlines a framework based on the work of Karl Popper that may contribute towards such a theoretical synthesis. Learning is seen as occurring through trial-and-improvement of *strategic theories* in response to *concerns*, rather than through the development of context-free modes of thought. Some examples are given of how some explanatory constructs in the research literature can be reconceptualised. Examples include the CSMS letter interpretations, "procepts", "parsing errors", "cognitive obstacles", and some accounts of "algebraic thinking".

There is now a large and growing body of research on the early learning of symbolic algebra (Wagner & Kieran, 1989; Bednarz, Kieran & Lee, 1996; Royal Society / JMC Working Group, 1997). Many insights about students' perceptions, motivations, strategies, achievements and difficulties have been gained from studies as diverse in focus as CSMS (Küchemann, 1981), Sleeman (1984), Mason et al (1985), Nolder (1991), MacGregor & Stacey (1993), Radford (1995), and Linchevski & Herscovics (1996).

Yet there remains a lack of a common theoretical framework for synthesising these disparate insights. Reconciling findings between the psychological traditions of cognitive science, interpretive research, introspection or historical research can be problematic.

This paper outlines a framework based on the work of Karl Popper that may contribute towards such a theoretical synthesis. Comparisons are drawn here with other perspectives; and then some examples are given of how some explanatory constructs found in the research literature can be reconceptualised to provide additional insights and some testable hypotheses.

Outline of the Popperian psychological perspective

Aside from his defence of the idea of democracy, Karl Popper is probably best known for his philosophical critique of positivism (Popper, 1934), a critique that – with Kuhn – did much to discredit the idea that science develops by the gradual accretion of facts rather than by testing hypotheses.

Popper's critique of the "bucket theory of mind", his insistence that what can be learned is heavily dependent on the individual's prior theories "of persons, places, things, linguistic usages, social conventions, and so on" (Popper, 1963), and his view that "knowledge" in the public sense comes about through complex intersubjective processes are clearly in resonance with the work of Vygotsky.

This perspective is directly opposed to behaviourism. Rather than learning consisting in the passive, steady, repetitive accumulation of information, there are active processes of decoding and sifting, in which existing theories are modified by creative, conjectural, discontinuous trial-and-error-elimination. Campbell (1960) describes a mechanism called "Blind-Variation-and-Selective-Retention" (BVSR) for such imaginative processes, in which, by analogy with evolution by natural selection, there is "a mechanism for introducing variation", "a consistent selection process", and "a mechanism for preserving and reproducing the selected variations". Campbell also suggests that mechanisms shortcutting BVSR were themselves created by BVSR

It is important to note that a key feature of this psychological perspective is that creative theory-formation processes do not occur in isolation, but in response to the selection pressures afforded by problems of special interest to the individual – a "concern". Concerns are the fuel that power learning. Concerns would include desires, motivations, fears, and what English & Halford (1995) refer to as the "student's problem-situation model". There can also be concerns for aesthetics and social recognition (Radford, 1995).

Although there are strong similarities with Piaget's view that the learner is an active builder of knowledge rather than an empty vessel waiting to be filled, it is perhaps the crucial role of consciousness in exerting idiosyncratic selection pressures that principally distinguishes the Popperian perspective from early interpretations of the Piagetian tradition.

One subtle aspect of Popperian psychology is that action, context and theory are intertwined. Students have myriads of complicated, contextual and implicit theories (taken as constructions of reality), created from a wide variety of experiences and concerns. There is often a strategic nature to these theories in that they solve problems. For example, as will be illustrated later, the CSMS letter interpretations (Küchemann, 1981) are theories about what symbols represent in algebra; but they can also be seen as elementary strategies for dealing with letters in certain types of problems. Conversely, an action, strategy, plan, heuristic, procedure or process can be considered as a theory, in that it incorporates expectations about what is. Hence we refer to *strategic theories*. For example, suggested strategies for the student-professor problem (Clement, 1982) can be seen as theories about what mechanisms can generate a legal and valid equation from a description of a situation. By attempting to address one's concerns, new strategic theories are constructed from old (cp. the "procepts" of Gray & Tall, 1993), and these may in turn generate new concerns.

We turn now to the notion of "understanding". From the Popperian perspective, references to "sense-making", "constructing meanings", "making connections", "having sudden insight", "intuition", "building cognitive models" (e.g. English & Halford, 1995) and "having an image" (e.g. Pirie & Kieren, 1989) can be reanalysed in terms of theories. However, the Popperian perspective pragmatically links "understanding" and "problem-solving" in a way that may seem puzzling to those for whom there is a tension between understanding as individual invention and understanding as social in origin. Understanding a theory in the public domain is seen as understanding the problem-situation that the theory is intended to address, why this theory works and why other theories might fall short.

But "problems" should not be understood in any narrow or pejorative sense – indeed Popper has an almost reverential attitude to problems. It is perhaps unsurprising, then, that Lakatos – one of Popper's research students – should do so much to promote the ideas of taking students' questions seriously (since questions can be indicative of concerns) and of the teacher as co-learner in the pedagogical engagement with the problem-situation.

This Popperian notion of understanding also has resonance in Skemp's distinctions. Very often, it is not enough for a teacher that a student has grasped a particular public domain strategic theory (difficult though this may be, in itself). The teacher also wants the student to appreciate the problem that the theory is supposed to address, something of the background to the problem and why one would care about it, and the relationship between this theory and other theories. So for Popper, the activity of problem-solving has both individual and social aspects; but he also emphasises that the problem-situation includes cultural and historical aspects.

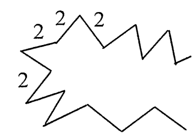
The perspective described here has some affinities with the "Triple Approach" of Drouhard & Sackur (1997). In that framework there is an actively learning subject, a social group, and a "reality" (either material or conceptual). The Popperian perspective would also count in the "conceptual reality" (called "World 3" by Popper) not just the accepted canon of past mathematicians but also the individual classroom's locally produced knowledge – written, spoken or implied (the "taken-as-shared" meanings and practices of Cobb, Yackel & Wood, 1992). It is often a conundrum for those new to Popper that "knowledge" in this theoretical realm cannot accurately be described as "justified true belief" because it does not have to be true, it is conjectural not justified, and it need not be believed by anyone!

It should be pointed out that what is presented here is very much a *psychological* rather than a *sociological* perspective. In contrast to some interpretations of Vygotsky, it is assumed that it is the individual – and this individual's interactions with the worlds of physical objects, ideas, and people, *mediated* by language, social forces, culture and history – that is the focus of study, rather than language, social forces, culture and history themselves.

So what does a Popperian reconceptualisation of findings from research into the learning of algebra look like, and can it tell us anything new? We consider some examples.

CSMS and cognitive readiness

The CSMS tasks (Küchemann, 1981) proved difficult for many children. For example, most could not give the anticipated responses to the task "Add 4 onto $3n$" or to the task "Part of this figure [on the right] is not drawn. There are n sides altogether all of length 2. What is the perimeter of the figure?"



Very few could give the anticipated responses to "If cakes cost c pence each and buns b pence each, and 4 cakes are bought and 3 buns are bought, what does $4c + 3b$ stand for?" or "Which is larger, $2n$ or $n + 2$? Explain.".

It was noticed that certain errors were made very often. Küchemann explained the popularity of the errors by cataloguing various ways in which letters were treated: "letter evaluated", "letter not used", "letter as object", "letter as specific unknown", "letter as generalized number", and "letter as variable". For example, the tendency to give a numerical answer between 32 and 42 to the perimeter question might be indicative of treating letters as having a value that must be pre-determined, or of the ignoring of letters. The answers "$3n4$" or "$7n$" (given by about half of the 12-year-olds) to "Add 4 onto $3n$" could be interpreted as the child treating letters as objects which can be collected up. Almost half of the 12-year-olds indicated that L + M + N = L + P + N is never true: this could be because they thought that L is always 12, M is always 13, and so on (letter evaluated); or because there is an M on one side but a P on the other (letter as object). Küchemann notes that treating a letter as an object (i.e. making it less "abstract") allowed many children to answer certain items successfully.

Rationalisations for the success of these letter interpretation categories in explaining the variations in students' responses to the CSMS tasks initially focussed on the possibilities of the tasks being governed by a unique hierarchy of difficulty, and of the students being under the influence of fixed maturation processes. Booth (1984) attributes some of the students' difficulties to a "cognitive readiness" factor, and cites as supporting evidence "a strong resistance… to the assimilation of the idea of letter as generalized number even within the context of a teaching programme specifically designed to address this aspect of algebra.".

Evidence was soon forthcoming that responses were dependent on the presentation and language of the tasks and on the prior curriculum of the students (O'Reilly, 1990). Yet prevailing psychological frameworks still seemed to imply that these categories indicated the presence of deep-rooted conceptions for children at different stages of cognitive development, and these conceptions dictate the way the children then respond to items.

The Popperian view – and arguably the pragmatic view of many teachers and researchers – was that these categories are simply a useful way to summarise a variety of informal strategies that students deploy in tackling such tasks. A host of studies indicates that children tend to use their own informal strategies to tackle mathematics questions rather than the standard formal methods taught in the classroom. These strategies work on easy items, but fail in fairly predictable ways on harder items. The ubiquity of certain strategies for dealing with letters can be explained by the commonality of students' prior experience.

Moreover, by linking understanding and problem-solving, the Popperian framework immediately raises the question of whether the reason that 92% of 12-year-olds could not multiply $n + 5$ by 4 was that they did not understand the question. Indeed, virtually all the CSMS items contain symbolic algebra in their formulation and therefore require familiarity with mathematical conventions to be understood. This particular task is a demand (by convention) to find a somehow simpler equivalent expression. Does a student who gives the answer "$n + 20$" really appreciate the success criteria for this task? Evidence from the errors made in the test does not allow us to decide between the hypothesis that students understood the tasks they were being asked to solve but had inadequate strategic theories for finding an answer, and the hypothesis that students did not understand the tasks and so attempted to adapt arithmetical strategies that had been successful before in what seemed like related problems.

The responses are telling us about the expectations of students as to the purposes to which symbolism is put rather than about entrenched conceptions. Consider, for example, some of the strategies identified by Kieran (1992) as characteristic of a failure to recognise "surface structure", and by Sleeman (1984) as "parsing errors" and "manipulative mal-rules": «assume implicit addition» (for example, $39x$ means $39 + x$ rather than $39 \times x$); «treat letters as digits» (for example, if $x = 3$ and $y = 2$ then $xy = 32$); and «read expressions from left to right» (for example, $a + a + a \times 2$ becomes $3a \times 2$).

The student-professor problem and reflection on meaning

Translation difficulties from English sentences into algebraic language are often exemplified using the student-professor problem, something like:

> "Write an equation, using the variables S and P to represent the following statement: 'At this university there are six times as many students as professors.' Use S for the number of students and P for the number of professors."

The anticipated equation is often reversed: 6S = P rather than S = 6P. Myriads of explanations have been produced to explain findings from studies based on variations on this problem. Since the original students for this task were undergraduates, early explanations centred on the likelihood of an impoverished set of cognitive structures rather than on the fixed maturation processes initially blamed for difficulties with CSMS. Recent research has suggested that students are insufficiently reflecting on the "meanings" of the letters (Philipp, 1992).

But in order to explain students' responses, Popperian analysis requires us to consider the students' view of the situation, and their strategic theories for dealing with it. Consider, for example, the four strategies described by Clement (1982) for constructing the equation:

> «translate the situation syntactically, phrase-by-phrase»
>
> «put the number next to the largest group»
>
> «find an operation that turns the size of one group into the size of the other»
>
> «find likely equations, choose some numbers which fit the situation described [e.g. P = 10, S = 60], substitute into the equations, and choose one which works»

What concerns have generated these strategies? Probably, students expect the equation to help with questions like "How many professors would there be if there were 120 students?". No doubt anyone who had translated "6 students for every professor" into the "erroneous" equation 6S = P would be able to think "Right, 6S = P, so 6 students for every professor. 120 students, so 20 times as many students, so 20 times as many professors.". In other words, so long as the interpretation of 6S = P remains constant, it would be possible to use ratio strategies to compute a correct solution from the reversed equation. That the resulting *form* is "wrong" stems more from convention than "understanding".

However, if the equation is to be communicated with someone else or to be subject to algebraic transformations, errors may occur. For example, one might be asked to "substitute the value S = 72 into the equation" (as opposed to "find the number of professors if there are 72 students"); or to "compare this university with another for which S = 4P + 9".

In conclusion, then, the Popperian perspective has led to the suggestion that the student-professor problem is a professor's problem, not a student's problem, in that students may lack a concern for algebraic convention. In other words, they don't know what this equation is *for*. Their expectations of the purposes to which the symbolism will be put and the strategies they use are perhaps symptomatic of a history of arithmetic tasks, rather than of insufficient reflection on meaning or of deep-rooted cognitive structures. This conclusion could be tested by simply asking students how they think the equation will be used.

We need to know more about these equation-formulation strategies so that we can work out what problems may help students to enrich their theories to cope with the algebraic domain. Clement conjectures, for example, that tasks that encourage students to seek *operations* in the first instance (rather than *relations*) might be beneficial; and this may indeed clarify why this equation might be of concern to them.

Interpretations, images, meanings and metaphors

A case can be made from the Popperian perspective that many interpretations, images, meanings and metaphors identified in the research literature are "meta-algebraic theories", generated by decontextualising strategic theories from their original problems. Meta-algebraic theories are valuable to teachers and researchers in conjecturing students' theories and concerns in a given problem-situation; but they do not provide evidence of innate cognitive structures or processes.

Furthermore, when students are faced with the challenge of representing one's theories to others, or with the need to respond to an interviewer asking what x means, what they say should not be construed as static "meanings" that underpin all algebraic thought, but as attempts-in-the-moment to capture or rationalise elusive theories or practices that have been useful in particular problem-situations. This conclusion is brought out by Booth (1984), who found that letter interpretations might bear little relation to success.

Another example of recontextualisation is interpreting expressions (Sfard & Linchevski, 1994). The theory that $3(x + 5)$ is a *computational process* would be relevant to a problem such as: "I think of a number, add 5, multiply by 3 and get 99. What was my number?". The theory that $3(x + 5)$ is a *number* might result from the problem: "Predict the x^{th} number in this sequence: 18, 21, 24, 27, 30, …". The theory that $3(x + 5)$ is a *function* might be relevant to the problem "If $f(x) = x + 5$ and $g(x) = 3x$, what is $g(f(x))$?".

Or consider interpretations of the equals sign. Kieran (1981) writes about the notions of the equals sign as a "do something signal" or "makes": "It can be argued that these notions reflect the kind of instruction that these children have received. One might then assume that later exposure to equality sentences involving the commutative and associative properties might broaden the elementary school child's notion of the equal sign. However this does not appear to be the case.". The Popperian perspective suggests that it is indeed *experience* that is the crucial factor in helping students to develop richer theories – therefore a test of this perspective is that it should be possible to improve this later "exposure" to make a difference.

A more general example is what Pirie & Kieren (1989) describe as an image-based "growth of understanding" – the transformation of a strategy that solves a problem into a theory whose properties and relationships can be examined.

Meta-algebraic theories can be valuable to students. For example, the balance metaphor for equations can help ease the learning of the strategy "What you do to one side, you do to the other.". Yet if this is the only interpretation available to students, they may miss out on the strategic theories associated with, for example, the "function machine" metaphor. So bringing interpretations, images, meanings and metaphors into a single framework can enhance the value of these meta-algebraic theories for teachers and researchers. However, whether meta-algebraic theories must be explicitly taught, or whether they are simply the useful by-products of active engagement with algebraic concerns is unclear. It could be argued that Vygotsky's description of thought development through language use and also Popper's own nominalist position as regards language would suggest that the metaphorical associations and multi-layered roles of specific language forms do *not* have to be fully appreciated before they are used.

Solving equations and the nature of cognitive obstacles

There are many informal strategies for solving equations. Kieran (1992) reports on studies showing students solving simple equations numerically using recall of number facts, counting techniques, and the "cover-up" method; and there are analytic and synthetic versions of "undoing" or "inversing". Such strategies are often fast for certain equations, but trickier or impossible for equations that have multiple instances of an unknown, negative signs, negative solutions or non-numerical coefficients. Trial-and-improvement using substitution can succeed for equations where these methods fail, but it can be time-consuming and prone to error.

There are two standard formal strategies. One method (which Filloy & Rojano 1989, attribute to Euler) is to operate on both sides of the equation with inverses; the other method (which Filloy & Rojano attribute to Viète) is to transpose terms from one side to the other. These rely on being able to identify that an equation is in one of a number of forms, and then to carry out the action appropriate to that particular form.

Many studies report that students have difficulties in solving equations by formal methods. Linchevski & Herscovics (1996) conclude:

> "... for a large number of high-school students, there are many cognitive obstacles involved in perceiving an equation as a mathematical object on which they can perform operations."

The Popperian psychological perspective would reconceptualise this finding as saying that many students have not discovered that Euler's and Viète's strategic theories can be useful in solving an equation. However, the reasons for this state of affairs would be related to the problem-situation faced by students, rather than to "cognitive obstacles". For example, they may lack a concern for solving equations in the first place; they may have informal strategies that appear more efficient or less prone to error; or they may not yet have at their disposal strategic theories on which the Euler and Viète methods depend, such as the rules governing arithmetic or the strategy of treating unknowns as if they were known.

Such aspects of the problem-situation may indeed constitute "cognitive obstacles", and if so, the impact of the Popperian perspective is just on nomenclature. However, it is possible that some people view "cognitive obstacles" in more than just this metaphorical sense – as static structures that are generic facets of someone's psychological constitution. If so, one test of the Popperian view is whether a learning environment could enable students to make the transition from informal to formal methods for themselves, without the need for superficial or rote symbol manipulation (Aczel, 1998).

Word problems and the character of algebraic thinking

Even students who are competent in standard representation and transformation tasks do not necessarily choose to use algebra to solve word problems even when other approaches appear more time-consuming and prone to error (Reed, 1984).

Lins (1992) examined students' engagement with word problems in which the use of symbolic algebra is not a prerequisite. He used a characterisation of algebraic thinking as "thinking arithmetically, thinking internally and thinking analytically" to detect when students were using algebra. Broadening the definition of algebra to encompass more than just the use of letters standing for numbers or operations was also a theme of Mason et al (1985), in which "handling the as-yet-unknown", "inverting and reversing operations" and "seeing the general in the particular" were seen as fundamental to algebraic thinking.

From the Popperian perspective, it is clear that students can have theories that can be termed "algebraic" («letters can represent numbers», for example), and an "algebraic strategy" (something like «create equations to represent relationships between unknown quantities in a situation, solve the equations, and then interpret the solutions in the context of the situation») can be contrasted with, say, a contextual "whole-part" strategy or a numerical trial-and-error strategy to particular problems. The context-specific nature of psychological processes would mean downplaying the idea of a hypothetical "mode of thought" that would be somehow independent of the problem-situation. Similarly, any "didactic cut" (Filloy & Rojano, 1989) or "cognitive gap" (Linchevski & Herscovics, 1996) would be reconceptualised in terms of strategic theories such as «treat unknowns as if they were numbers».

Lins found that students' choice of strategy is dependent on the particular numbers involved in the word problem. For example, if there were decimals involved, a whole-part strategy was used less and an algebraic strategy was used more; however, fewer students then gave the correct answer. Success also depended crucially on the *context* of the problem – a word problem about driving distances was harder than a word problem about concert tickets, even though they were soluble by the same equation. Hinsley, Hayes & Simon (1977) provide evidence that students often try to classify the problem as soon as possible in order to make use of strategies associated with standard problems.

Promoting an algebraic strategy depends on identifying the typologies of problems that students perceive, and the strategic theories they associate with these problems. If learning environments can be created in which students discover for themselves the limitations of their informal methods and so look to symbolic algebra as a natural support in exploring their ideas, they may develop a spontaneous concern to improve their representation, interpretation and transformation strategic theories.

Some concluding remarks

The examples given here have tended to illustrate the individual student's cognition and affect rather than the historical aspects or the social environment; and this is because of the emphasis in studies of the learning of algebra. But clearly the Popperian approach lends itself to a richer appreciation of learning than represented here. There has also not been space for consideration of proof or graphical examples.

Nevertheless, we have seen something of the potential for the Popperian psychology to bring together findings about algebraic learning from various research traditions, without requiring an abandonment of their fertility. The perspective is also largely neutral as regards different perceptions of algebra (e.g. algebra as a problem-solving tool versus algebra as a common language for representing, describing variation, and making generalisations). But concerns are crucial – if students do not have algebraic concerns, they will fail to cultivate the rich mix of algebraic and meta-algebraic strategic theories that may perhaps be seen as an ideal.

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TEACHING AND LEARNING ALGEBRA : APPROACHING COMPLEXITY THROUGH COMPLEMENTARY PERSPECTIVES

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Abstract : In this submission, after presenting three different research projects on algebra, we have carried out or are developing in our research team DIDIREM, at the Université Paris 7, we try to summarise what can be offered by this kind of didactic research to the ICMI Study and stress the necessity of connecting different approaches if we want to better understand the complexity of learning and teaching processes in that area and improve them in a rational way.

I. Introduction

Algebra is a crucial domain as regards the relationships students develop with mathematics. For a lot of these, and for most adults in society, algebra is the domain where, abruptly, mathematics became a non understandable world. Faced with such evident teaching and learning difficulties, didactic research has been very active during the last twenty years. It firstly tried to better understand learning processes in algebra and explain the breach mentioned above. These attempts were successful in identifying some decisive factors, such as those linked to the discontinuities existing between arithmetic and algebraic thinking modes and the specificity of algebraic semiotic practices. Didactic research also developed insightful analysis of usual teaching practices in that area, in various countries, and helped us explain their observed inefficiency. More recently, research tried to explore the potential offered by computer technologies in order to overcome the identified learning difficulties and to develop more effective teaching strategies. Books such as (Bednarz, Kieran, Lee, 1996) fairly well illustrate the richness of the research work undertaken up to now and the coherence of its results, in spite of the evident diversity of the theoretical approaches and contexts.

Our research team, DIDIREM, has been involved in didactic research in algebra for nearly ten years and we would like to rely on different pieces of research we have carried out or are carrying out, in order to contribute to the ICMI Study on Algebra. In the first part of this collective contribution, we present the way research developed, the corresponding "problématiques" and the associated theoretical frames. Then, we briefly describe three different research projects. Finally, in the last part, we discuss what can be offered by such a research work to the reflection on the future of learning and teaching algebra.

II. Development of research and of its theoretical frames

In our team, didactic research on algebra began to develop in three independent directions : the teaching and learning of linear algebra at university level, the analysis of institutional transitions in algebra and the analysis of potential offered by CAS for teaching and learning elementary algebra. This contribution mainly focuses on the work undertaken in the second direction and its subsequent developments[1]. Trough these projects, we became more and more sensitive to the complexity of this research domain and to the necessity of approaching it through different complementary but coherent approaches.

The complexity of this research domain is linked to the fact that understanding learning and teaching processes in algebra, even if one restricts to middle and high school algebra as is the case in this contribution, doesn't only require the understanding of students' learning processes seen as pure cognitive processes. This is more: this is understanding how the scientific and technological evolution influences algebraic knowledge and practices in the large today, how it changes the cultural, social and professional needs as regards algebra ; this is understanding the functioning of different complex institutional systems where algebra is taught and learnt, taking into account their respective constraints, traditions and cultures ; this is understanding teachers' culture in algebra, expectations and practices and their potential effects on students' learning ; this is understanding the way curriculum and syllabus develop, the role played by textbooks and other traditional didactic resources but also what could be offered by resources which play an increasing role such as software, CDRoms, websites...

In our research team, we try to explore and co-ordinate complementary approaches to this complexity and, in this contribution, we rely on three different projects: the initial project mentioned above about institutional transitions and its further extensions, a project on teacher professional development in algebra, and a project developing an historical perspective in order to analyse the curricular evolution and thus better understand present curricula as the result of a specific history, within a specific culture. In the following, we briefly introduce our "problématiques" and theoretical frames through the narration of the birth of the first project. At a macro-didactic level, we use the anthropological approach to didactic phenomena developed by Y. Chevallard (1992) as a global theoretical perspective to which each of our more local perspectives can be related in a coherent way.

The transition project emerged from an educational problem: in France, there exist bridges between vocational and general higher education and best students coming from vocational high schools can enter a specific one year course designed for helping the transition process. In spite of this, transition remains specially difficult and algebra plays a crucial role in its failure. Generally, such transition problems are considered as resulting from the low mathematical level of students coming from vocational high schools. B. Grugeon, who taught transition classes, hypothesised that this general interpretation was too convenient and certainly a simplistic one. She decided to approach this transition issue within the anthropological approach mentioned above. According to this approach, mathematical knowledge cannot be considered as something absolute. It strongly depends on the institutions where it has to live, to be learnt, to be taught. Mathematical objects do not exist per se but emerge from practices which are different from one institution to another one. Y. Chevallard analyse these in terms of "praxeologies", that is to say in terms of tasks, techniques used to solve these tasks, "technologies" which denote the discourse developed in order to explain and justify particular techniques, and last, "theories" that he defines as technologies for the technologies, which organise the local technological discourse in coherent structures. Within this approach, the familiar sentences : "(s)he knows that" or "(s)he doesn't know that", do not make sense, taken as such. Every institution which has to deal with some mathematical object develops an institutional relationship with this object. This relationship defines the norms and values of knowledge as regards this object, for this particular institution. Institutional relationships with

[1] The results obtained in the first direction are presented in the book edited by J.L. Dorier and published by Kluwer in 2000 : "On the teaching of linear algebra", and an independent contribution which integrates the main results obtained in the third direction is proposed by J.B. Lagrange.

one mathematical object or domain vary across institutions. Students, exposed to teaching or other socio-cultural experiences develop personal relationships to mathematical objects, which are shaped by the different institutional influences they are submitted to. This is only if their personal relationship with an object is close enough from the institutional relationship at stake that they are considered by one particular institution as "knowing this object". This theoretical frame led B. Grugeon to the following conjecture: vocational and general high schools have quite different institutional relationships with elementary algebra and the poor sensitivity of the educational system to these discrepancies could explain part of the students' difficulties in the transition process. She also supposed that being aware of such a fact, could allow the development of efficient didactic tools. This was the starting point for the research we describe in the third part of this text (Grugeon, 1995).

III. Institutional transitions

In order to test the initial hypothesis we have just mentioned, it was necessary to define a kind of external reference with respect to elementary algebra, independent from institutional values. This was achieved through the elaboration of a multidimensional structure of analysis for elementary algebraic knowledge, based on existing research in that area.

The field of algebraic knowledge can be structured around two main non-independent dimensions: the tool and object dimensions (Douady, 1984). More precisely, in its tool dimension, algebraic competence can be described through the ability to mobilise algebraic tools in order to solve different kind of problems internal or external to the mathematics field (at the elementary level considered here, problems of generalisation and proof, traditional arithmetical problems, problems where algebra appears as a modelling tool, algebraic and functional problems). In its object dimension, that is to say considering algebra as a structured set of objects with specific properties, semiotic representations, treatment modes, algebraic competence can be described through the ability to cope with algebraic objects, by taking account both their semantics and their syntax. At the level of schooling considered here, specific attention is put on the arithmetical / algebraic cut, on the ability to interpret algebraic expressions both at operational and structural level, and on the ability to flexibly adapt algebraic interpretations in order to pilot algebraic work.

The multidimensional grid of analysis

This analysis resulted in a multidimensional grid structured around six components, each of these being specified by a set of criteria with corresponding potential values, which was then used in order to analyse personal and institutional relationships to algebra. The six component are the following:

The first component has a specific role. From an institutional point of view, she allows to compare the types of problems favoured by a given institution ; from a personal point of view, it allows to evaluate the algebraic competence with respect to given institutional norms. Nine type of algebraic treatment are a priori identified, both according to the tool and object dimensions. They are only partially ordered.

The five remaining components aim at identifying and describing important characteristics, local coherences both in official syllabus and students' algebraic functioning. They are the following : (1) relationship between arithmetic and algebra, approached through the following criteria : resolution process, status of equality sign, status of letters, objects and status of these; (2) processing of algebraic expressions ; (3) connections between the algebraic symbolic register and other semiotic registers, where, according to Duval's work distinctions are made betwwen the formation of expressions and their processing, and between the nature of conversions between registers ; (4) functionalities of algebra, and (5) algebraic rationality.

The analysis of personal relationships was achieved through the elaboration of a set of 19 diagnostic tasks, we cannot describe here. The set was conceived in order to allow the identification of local coherence in students' algebraic behaviour. Thus, the value for each criteria in the grid could be assessed at least five times. The first experimentation showed that the initial values introduced for the different criteria were insufficient for covering the diversity of students' behaviour and the diagnostic potential of each task of the set. Thus local values were added and the second experimentation, one year later, showed that the new version of the grid was adequate.

Essential results

Institutional relationships were classically explored through the analysis of syllabus, textbooks, national or regional assessment tasks, but also the careful analysis of students' notebooks corresponding to their last year in vocational high school. Institutional analysis evidences differences in algebraic culture between the two institutions at stake, all the more pernicious as a quick look at the different syllabuses let the impression that the vocational syllabus is built from the general high school syllabus by cutting and pasting, then adding some specific professional uses related to finance and accountancy. They thus look very close. A more detailed analysis of the syllabuses, using the multidimensional grid, makes visible a conjunction of small differences which, taken together, characterise two very different cultures. The notebook analysis confirms this fact, while showing that the differences between syllabuses are reinforced by differences in the more global relationship the two institutions develop with respect to mathematics.

Analysis of personal relationships through the set of diagnostic tasks allowed us to built a cognitive description for each students. The micro-description made from the 19 n-uplets associated with their answers to the diagnostic could not be used as such. So, a macro-description, giving a synthetic vision, was built by identifying categories of answers, component by component. This resulted, for each student, in a quantitative description of her or his algebraic competences (first component), a qualitative identification of coherences in the algebraic functioning, and a qualitative description of his or her flexibility in connecting the symbolic system of algebra with other semiotic registers. This was expressed in terms of students' profiles. In the research process, these students' profiles were used in order to partially individualise teaching, taking into account the specific relationship each student had with algebra, a didactic strategy which resulted to be very positive for the great majority of students.

Thanks to this research work, we also became progressively aware of the diversity of possible germs for an entrance in algebraic thinking. High school teaching favours some of these at the expense of many possible other ones, and especially tends to underestimate the algebraic work with formulas which is so important in the vocational culture. Research has shown that, through adequate strategies, this entrance can be a very fruitful one if formulas are not only objects given to the student and if the technical work that formula support is not too reduced in complexity.

The subsequent developments of this research work

This research was first extended to the study of other institutional transitions, and especially the transition between junior and senior at school (grades 9 / 10). Both the initial grid and the set of diagnostic task was adapted for this purpose, but this adaptation did not rise important difficulties and it was soon achieved.

The next ambition of the group was to make the efficient tools of analysis elaborated in the framework of this research useful beyond the sole community of researchers. A first simplified version of the grid was tested with a small sample of teachers with different experience (Lenfant, 1997). The results were very disappointing. Coding students' response and then interpreting codes in terms of profiles for devising appropriate didactic actions was too complex and time consuming. For overcoming this evident obstacle, a new project was developed in collaboration with researchers in artificial intelligence. It aimed at creating a computer version of the diagnostic and at computerising coding and analysis up to the building of students' profiles. This was called the PEPITE project which resulted in the S. Jean's doctoral thesis (Jean, 2000). Transposing the paper version of the diagnostic into a computer version was not easy at all, as adequate interface and transpositions of the tasks had to be built. Successive versions were developed and tested with students. There is no doubt that working in a computer environment modifies the relationship students have with the diagnostic set, their means of work and expression, their strategies. Nevertheless, successive adaptations led to the point where these differences did not reduce the diversity of possible behaviours and did not prevent from identifying characteristics of algebraic functioning which could be also observed in standard environments. The computer version of the diagnostic (PépiTest), was then coupled with two analysis modules : PépiDiag and PépiProfil. Their elaboration was also the source of difficult problems as one can easily imagine. Successive adaptations were made, taking into account the results of comparisons between computer and hand-made codes on the same students' productions. The product is now operational and research, still in collaboration with researchers in artificial intelligence, aims at developing remedial tasks which could be proposed to students, in the same environment, after identification of their cognitive profile.

IV. The first steps of teachers' professional development in algebra

We cannot expect substantial improvements in the teaching and learning of algebra if we do not take into consideration the teacher dimension. As shown above, this issue became evident with B. Grugeon's attempts to make the tools of analysis she had elaborated in her doctoral thesis widely available and useful beyond the sole community of researchers. Two research projects were thus developed: on the one hand, the partial computerisation of the diagnostic and profile building described above, and, on the other hand, a research project aiming at improving our knowledge of teachers' professional knowledge in algebra and the ways it develops. This is the theme of the on-going doctoral thesis by A. Lenfant we briefly present now.

This research focuses on the professional development of pre-service teachers (in the following: PLC[2]) who, in France, have just passed the national competition called CAPES in order to become secondary mathematics teachers and are given one year of professional training in an IUFM (Institut Universitaire de Formation des Maîtres). During this year of professional training, they also have one class in full responsibility, 6 hours per week. So, they are moving from a student position in the educational system to a teacher position. We use algebra in order to analyse this transition process and the ways it could be more efficiently assisted through the training offered at the IUFM. Algebra is a privileged domain for such an analysis for several reasons: this domain presents evident learning difficulties which are not so easy to understand when algebraic modes of thinking and algebraic techniques have become so familiar that they are quite "naturalised", as is the case for the PLCs. Moreover, all PLCs are concerned with these difficulties as they have to teach algebra or pre-algebra in their full responsibility class.

In this research, we use the anthropological approach (Chevallard, 1997, 1999) in order to analyse the "mathematical and didactic organisations" developed by teachers with respect to algebra, the associated "mathematics and didactic praxeologies" and, more globally, the mathematical professional work of the teacher via its different professional "gestures", in and out the classroom. Professional development in algebra is seen as a complex genesis, relying on a mixture of competencies specific to this domain and of more transversal competencies which strongly intertwine in practices. We thus consider that professional competence has to be analysed in multidimensional terms. It can be modelled as a multivariate function which shapes the decisions the teacher takes in her (his) different professional gestures, the way (s)he faces unforeseen situations, the discourse and analysis (s)he develops at a more reflective level. In

[2] PLC is an acronym for high school teacher (professeur de lycée et collège, in French).

order to approach it, and inspired by B. Grugeon's research work, we have developed a multidimensional grid for professional competence in elementary algebra (MGPCA in the following) which focuses on the specific competencies and tries to describe potential underlying knowledge. We hypothesise that such knowledge influences the different professional gestures in a non-uniform way and that it cannot necessarily be made explicit. One ambition of the research is to explore the complexity of relationships between knowledge and competencies, to try to find the real influence professional algebraic knowledge plays with respect to other determinants of the PLCs' behaviour, and to evidence some regularities which could help us better understand teachers' behaviour and improve training strategies.

The MGPCA Grid

The grid is structured around three non-independent dimensions: the epistemological, the cognitive and the didactic ones. In the following, we synthesise the contents of knowledge which structure the grid according to each dimension.

The epistemological dimension :

Epistemological knowledge in the grid, is structured around, on the one hand, some important features of the historical development of algebra, on the other hand, the distinction between the "tool and object" dimensions of algebra, introduced in part III. As regards the first point, one essential epistemological characteristic is the complexity of the algebraic symbolic system and the difficulties of its historical development. Such knowledge can help to understand the difficulties met by present students. Another important point which arise from historical development is the extension and diversity of the algebraic domain. Such knowledge can support the change of epistemological views about algebra, allow to better understand the rationale for the progression in algebraic knowledge organised by the curriculum and, eventually discuss its pertinence. We conjecture that, at the beginning of the academic year, the PLCs are unaware of this complexity and tend to reduce algebra to the algebraic structures and theories they have been taught at university.

As regards the second point, the distinction between the tool and object facets of algebra, we conjecture that the PLCs, at the beginning of the academic year, mainly see algebra as a tool for solving problems which can be modelled in terms of equations and that, as teachers, they tend to over-emphasise the work on algebraic techniques.

The cognitive dimension :

This component deals with potential professional knowledge about learning processes in algebra. We have organised this part of the grid around four main points linked to resistant learning difficulties evidenced by didactic research: the relationships between arithmetic and algebra, the symbolic system of algebra, the relationships between different semiotic representations used in algebra, and the relationship to algebraic rationality. For this part of the grid, we especially rely on the categories introduced by B. Grugeon, which have been described in part III.

As regards this cognitive dimension, we conjecture that, at the beginning of the academic year, the PLCs are not aware of the diversity and resistance of learning difficulties in algebra, but that, through their practice and discussions with pairs, they soon become sensitive to most of these, even if they are not able to interpret them in coherent and analytic ways as research allows to do, and thus react efficiently. We also conjecture that, due to the general educational French culture which over-emphasises the links between rationality and geometry at the expense of any other one, integrating the role algebra has to play in the development of students' mathematics rationality results specially difficult.

The didactic dimension :

Knowledge relevant to these two first dimensions certainly influences the didactic and mathematical organisations, the teachers develop. But these are also shaped by what we will call here more specific didactic knowledge: knowledge of the curriculum, of the specific goals of algebraic teaching at a given grade, of possible progressions and activities for the teaching of algebra compatible with these and of well adapted assessment tasks, knowledge of educational resources: textbooks but also publications from the IREM (Instituts de Recherche sur l'Enseignement des Mathématiques), websites (especially as regards the use of computer tools such as spreadsheets for the teaching of algebra), etc.

The MGPCA grid is then used in order to analyse the PLCs' initial relationship with algebra, and its evolution, through different questionnaires, and through the one year long following up of a selected group of PLCs. For these, a lot of data are collected: personal note boards, representative students' copy boards, assessment tasks, videos of classroom sessions and regular interviews, and the interpretation relies on the triangulation of the different analysis carried out on this material.

Some results

We have considered professional development as a multidimensional and complex process, involving epistemological, cognitive and didactic changes. We are perfectly aware that what can be reached through this first year of professional training is necessarily very limited. These evident limitations make all the more important to detect possible germs for priming professional development. The results we have obtained up to now are certainly very partial but they clearly tend to show that some interesting and subtle evolution take place. All of these don't directly affect the design and management of classroom situations. At a first level, they seem more able to express in a priori and a posteriori analysis of classroom sessions, and more in a collective way than in an individual way. Results also show important differences in the accessibility of the respective parts of what can be considered today as professional expertise in algebra.

For instance, as regards the epistemological dimension, data analysis shows that the training at the IUFM seems to easily destabilise the vision of algebra as a domain reduced to the field of algebraic structures and theories. For most students, boarders of algebra tend to become questionable and this helps them as expected. Nevertheless, their vision of the different tool facets of algebra remains limited and they go on over-emphasising the object dimension of algebra. Integrating a functionality of proof seems specially difficult.

As regards the cognitive dimension, training seems make them aware of the differences between the arithmetic and the algebraic processes for solving numerical problems, and through their practice, they become sensitive to students' difficulties with the symbolic system of algebra and with the conversions between semiotic registers. But, for the majority, the short training they receive does not allow internalise operational means for analysing these difficulties. As a consequence, their remedial strategies remain quite limited. Moreover, understanding that algebraic expressions have not only a syntax and an external semantics but also an internal semantics, understanding the role this semantics plays in the piloting and control of algebraic computations seems specially difficult.

As regards the didactic dimension, textbooks, and the local advisor each PLC has in the high school where (s)he teaches in full responsibility, have a predominant influence but the PLCs do not only use the textbook selected by the high school for their class. Visibly, the work they did the year before, when preparing the CAPES, which obliged them to search and select exercises on specific mathematical themes by looking at different sources has created some "habitus". Generally, the PLCs don't meet difficulties when deciding the different points they want to address, but they have more difficulties at identifying the real aims of the teaching of pre-algebra or algebra, at the grade level corresponding to their class. They also have resistant difficulties at taking into account the initial state of knowledge of their students when they come to teach algebra. Their teaching of algebra is generally not reduced to its technical component but, even at the end of the academic year, the elaboration of technological discourse seems to remain under their sole responsibility. Globally, there is a sensible evolution during the academic year: the lesson conception evolves, the relationship to assessment evolve, but, beyond the general tendencies we have just pointed out, each evolution seems to have its idiosyncrasy.

V. Curricular evolution through a case study : the case of inequalities

Understanding learning and teaching processes, reflecting on possible ways for improvement, also requires are careful analysis of the curricular evolution, of the cultural tradition it reflects through its permanencies and changes. Current curriculum is the product of an history which shapes today views, values and practices, and which conditions the viability of intended change, up to a certain point. In our team, historical analysis of curricular evolution was first developed about calculus and resulted to be specially productive in order to better understand the present state of crisis of secondary teaching in that area (Artigue, 1996). Research in algebra, within this "problématique", is more recent and benefits the conceptual tools provided by the development of research on ecology of knowledge (Artaud, 1997).

The starting point was the following: in France, the word "algebra" disappeared from the official syllabus for grades 7 to 9, the grades where traditionally algebra is introduced, more than ten years ago. Algebraic contents and practices are still present at these grades, but they are no longer grouped in a specific section of the syllabus which, from grade 6, is organised in three sections : "numerical works", "geometrical works", "data organisation and processing, and functions". Such a curricular change is far from being neutral as was pointed out by Y. Chevallard, already in 1985 (Chevallard, 1985). How does it affect the teaching and the learning of algebra in the long range[3]? In the on-going research we present here, this issue is approached through a case study: the case of inequalities at junior high school level.

The curricular evolution at junior high school level in France, as regards this particular object, can be divided into three main stages: the first one covers the first half of the 20^{th} century from the 1902 reform to the "new math" reform in 1970 ; the second one runs from the "new math" reform to the 1977 counter-reform ; the third one goes until now and its main characteristics were fixed by the 1985 reform. In the first stage, the context of equations is predominant ; in the second stage, this predominance is taken by the "functional and structural context" ; in the third stage, the situation is not so clear and we are faced with a mixture of empirical, structural and equation contexts. Let us first specify these three contexts.

The initial "equation stage"

During this first stage, inequalities are present in the algebraic part of the syllabus for grades 8 and 9, which is divided into three parts: "arithmetic", "algebra" and "geometry". For instance, in the 1902-1905 syllabus, inequalities come after operations on positive numbers, monomials, polynomials, and the solving of equations (first grade equations with one unknown, systems of two equations with two unknowns, systems of equations with more than two unknowns). The ecological "habitat" (Artaud, 1997) for inequalities is thus that of equations. Inequalities are nearly considered as equations : definitions, algebraic techniques for their solving are derived from the corresponding ones for equations, the essential objects in the algebra part of the syllabus. Inequalities are essentially used for intra-mathematical applications: in the resolution of problems modelled by equations or systems of equations, existence

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[3] As was shown by research on curricular changes in elementary analysis or calculus, analysing the effects of curricular changes requires a long term perspective : after a reform, during many years, what lives in classrooms is a subtle mixture of the old and new curriculum ; this creates an intermediary culture and long term effects, only visible when the ancient values tend to vanish, may be quite different from short term effects.

conditions for variables and parameters have to be taken into account and these lead to inequalities. This is thus their main ecological "niche".

The "functional and structural stage"

In the second stage, the context becomes structural and functional : structural because the object "inequality" is associated with the study of order structures (in 1971, middle school students are exposed to the proof that 5 is a totally ordered field) and functional since this object is depending upon the notion of function, which is introduced very early. A new organisation of the syllabus replaces the old one. For instance, the grade 8 syllabus is divided into four parts: « I. Relations, decimal numbers and approach to real numbers, II. Geometry of the straight line, III. Plane geometry », and the grade 9 syllabus into three parts: « I. Real numbers, algebraic computation and numerical functions, II. Euclidean plane, III. Euclidean plane geometry ». Inequalities are introduced at grade 8 in relationship with the fact that 5 is a totally ordered field, and, at grade 9, connection is made with the notion of interval. Equations and inequalities remain close objects but they live now together in a new "habitat", that of numerical functions, which becomes a core notion for the curricular organisation[4]. This new "habitat" reinforces the role played by the graphical register in the solving of inequalities. It also situates the work on inequalities in a new ecological chain leading to their investment in elementary analysis at more advanced levels. Simultaneously, applications for inequalities change, with both intra and extra mathematical components. From an internal point of view, inequalities become a tool for preparing calculus work as pointed out above, from an external point of view, problems linked to human, social and economic sciences, enter the field, especially linear programming problems. But these problems turn out to be too complex and they soon disappear at this grade level.

The third stage

In the third stage, as pointed out by Chevallard (Chevallard, 1985, 1989), algebra tends to be a « vanishing point » : its object is no longer the study of equations as in the classical period, neither the study of numerical structures as in the period of modern mathematics but it dilutes itself in different sections, mainly the section « Numerical Works » (inequalities are part of that section). Emblematic objects of the classic algebra, such as equations, are still present in the syllabus, and inequalities come after these, inequalities still appear linked to the relationship of order, but a new coherence for this area is still expecting to be built: algebra appears as an empirical domain deprived from structure. In the syllabus, emphasis is put on its modelling role.

In this context, inequalities appear as tools for modelling "daily-life" problems. Such choices a priori meet scientific needs (acknowledgement of the scientific importance of modelling activities in mathematical work and development), educational views (linking mathematics with concrete issues, with daily-life problems in order to evidence their usefulness and favour empirical and experimental approaches).

Further comments

These three contexts, briefly described, shape the teaching of inequalities all over the century but a more detailed analysis is necessary in order to understand their precise effects. This analysis is undertaken by T. Assude (Assude, 2000) in terms of praxeologies, looking at the way tasks, techniques and technologies evolve along the century and how this evolution can be related to the contextual characteristics described above. This analysis shows interesting moves. For instance, thirteen different types of tasks are identified and seven types of techniques, but only one task: solving an first degree inequality with one unknown, is stable all

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[4] The evolution is not so abrupt as this schematic description presents it. The functional world progressively entered the scene from the 1947 reform

along the century. The number of tasks obeys a movement of extension-reduction: four in the 1902 reform, eight in 1960, seven in 1971, five in 1977 and 1985, 2 only in 1997. The drastic reduction we observe now is specially problematic. It is not an isolated point in mathematics teaching and reflects the modes of adaptation of the French educational system to the difficulties it meets. This adaptation leads to progressive reduction in contents and to a drastic reduction of the technical ambitions of mathematics teaching. Such choices are legitimated by arguments which can appear valuable: "teach less in order to teach better", "favour conceptual understanding", but evolution tends to show that teaching less does not necessarily promote better learning and that, in mathematical activity, the technical and conceptual dimensions cannot be separated so easily: reducing drastically the possibilities for technical work can limit the field of mathematical experiences one can live and limit understanding[5]. We will come back to this point in the discussion.

Discussion and comments

In this text, we have presented three different research projects which approach learning and teaching issues in algebra from different perspectives but within a global coherent anthropological approach. What can we learn from this as regards the issues raised in the discussion document? We would like to briefly stress some points which, in our opinion, can offer a valuable contribution to the reflection.

The diversity of personal and institutional or cultural relationships with algebra. Reflecting about the future of teaching and learning algebra, even when one only considers elementary algebra requires a good awareness of this diversity and of the necessary multidimensionality of algebraic knowledge. This cannot be correctly approached through simplistic and hierarchic views, it is much more than that. In our opinion, B. Grugeon's research evidences in a convincing way the fine grained analysis which is necessary if one wants to approach the personal relationship a student has with algebra, detect in her or his functioning germs for entering algebraic thinking and practices, efficiently help her or him overcome resistant difficulties by identifying underlying coherence, taking into account the specificity of her or his culture and needs.

The complexity of competencies required from teachers if we want these make their student benefit from the advances of didactic research in algebra. A. Lenfant's thesis evidences this fact and, at the same time, shows how current epistemology of algebra in the culture but even in the educational system creates resistant obstacles to changes and professional development as it can been thought today. Elementary algebra tend to be restricted to a stereotyped and limited set of problems which do not reflect the epistemological values of this mathematical field. The role it has to play in the development of students' mathematics rationality is negated in a culture where geometry is the temple for rationality. Algebraic techniques are mostly reduced to their syntactic part, transforming algebra in a world of legal rules. This vision prevents the development of the dialectic game between syntax and semantics which is characteristic of algebraic practices as soon as one goes beyond simple routine tasks. But, at the same time, A. Lenfant's thesis shows that, if we make the effort of building adequate tools for analysing teachers' functioning, we can identify different germs for the development of a professional expertise in that area, and subtle but promising evolution in teacher-students which are still in a transition phase. Once more, even if regularities appear, there is no uniformity. Germs are different from one teacher-student to another one, according to her (his) personal sensibility and biography.

[5] For instance, limitation to only one type of inequality strongly reduces the field of situations accessible to mathematical modelling and reinforces the tendency to standardisation of modelling activities, which favour neither the understanding of modelling processes nor the understanding of algebraic concepts.

The third point, we would like to mention is evidenced by the T. Assude's research. Curricular constraints strongly shape the relationships with algebra which can live in a given institution at a given grade and the way these can evolve along time. Understanding these constraints imposes to go beyond their most apparent curricular features. On the one hand, there are syllabuses and the ambitions they express under more global curricular umbrellas, on the other hand, there are praxeologies which are potentially or effectively associated with the syllabus, in textbooks and in classrooms. The links between these two categories are far for being simple and the ideological discourse which goes with the presentation of the syllabus does not necessarily foster an adequate analysis. In France, today, for instance, there is a strong tendency to reduce technical ambitions as regards algebra, within the frame of a global discourse which opposes technical and conceptual activities in mathematics and wants to promote conceptual understanding. This is a too simplistic opposition: technical work, if not routine work, is also essential for understanding and conceptualising. Conceptual activities do not live in the air, they live supported by the manipulation of different categories of "ostensive", with the meaning given to this term in (Bosch & Chevallard, 1999), they rely on technical work. If ambitions at this level are drastically reduced, the effect can be contrary to the intention. There is a tendency to equate technical work and work without reflection and intelligence which is specially pernicious in algebra.

The last point we would like to mention deals with the relationships between research and practice. B. Grugeon's research clearly shows that transforming tools which have proved their efficiency in research into effective tools for teachers is not a trivial task. But this research also proves that transposition is possible and can be helped by technological advances. This is certainly costly ad, in our case, was only possible thanks to the collaboration with computer scientists, but we have some reasons to be optimistic. The first experiments carried out this year seem to show that the tool which exists today (see the website: http://pepie.univ-lemans.fr) is accessible to teachers and, beyond that, that it seems to be an effective tool in order to promote teachers' professional development in that area.

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From Body Motion to Algebra through Graphing

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This proposal presents an ongoing research on novices who are asked to interpret graphs on a graphic calculator. The graphs come from collected data by an on-line measurement tool. The students' performances are analysed through the lens of *embodied cognition*. Their cognitive activities, above all words and gestures, reveal crucial for the genesis of their mathematical understanding. Specifically, the so called *grounding metaphors* and *fictive motions* are cognitive pivots which trigger and support the transition from the empirical and perceptive facts to a more theoretical frame.

1. Introduction

The study of graphical representation is inserted in the mathematical curriculum at many grades, and its importance is not only related with its mathematical contents as graphs, functions, equations, but also with the modelling of phenomena in other disciplines as the scientific or economic ones.

Many researches in Mathematics Education studied the introduction of graphical representations with different aims: for example, to analyse the understanding of graphs from a point-based or a shape-based approach, e.g. within the larger perspective of process/object learning (Dubinsky & Harel, 1992). Some are interested in the study of misconceptions of students about graphs (Clement, 1989), or in the nature of their mistakes and lacks (Cobb & Bauersfeld, 1995). Others point out their relevance for supporting students in a meaningful approach to algebra (Kieran, 1994). Recently fresh trends in the research are coming, which introduce new viewpoints. For ex., Nemirovsky underlines that "learning graphing entails the enrichment of a broad range of experimental domains, involving the refinement of visual, kinesthetic and narrative resources" (Nemirovsky et al., 1998). An intriguing perspective is given by the so called *embodied cognition*, whose 'motto' is "to put the body back into the mind" (Johnson, 1987, p. xvi) and which considers the (mathematical) concepts as coming from the cognitive activities of subjects (Lakoff & Johnson, 1980; Johnson, 1987; Lakoff & Núñez, 2000). Such approaches put forward two main cognitive ingredients in students who are building the meaning of mathematical objects, namely language and gestures. More specific contributes in these directions have been given by different people. Lakoff & Núñez (2000) introduce the notion of *grounding metaphor*[1] and exploit the nature of so called *fictive motion*[2] (Talmy, 1996) with respect to some mathematical concepts. Radford (2001) points out that pupils' transition to an abstract general algebraic formula is trigged and supported by two main

[1] According to Núñez (2000), conceptual metaphors are fundamental cognitive mechanisms (technically they are inference-preserving cross-domain mappings) which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference. Among conceptual metaphors, Núñez considers grounding metaphors, which ground our understanding of mathematical ideas in terms of everyday experience.

[2] We have fictive motion when "a line is thought of in terms of motion tracing that line, as in sentences like 'The road runs through the woods'....In mathematics, this occurs when we think of two lines 'meeting at a point' or the graph of a function as 'reaching a minimun at zero'." (Lakoff & Núñez, 2000, p. 39). A fictive motion is expressed by words in natural language and may be accompanied by suitable gestures. As such it is a genuine cognitive phenomenon.

functions of language, namely the *deictic* and the *generative action* function[3]. Arzarello (2000a) studies the students' performances in terms of words and gestures in space-time. Varela (1999) points out the cognitive and neurological problems of subjects who give an order to events, which do not appear linearly ordered in perceptual experience. Arzarello & al. (2001) describe the mental dynamics of pupils who activate the past experiences and anticipate the future in problem solving.

In this proposal we present the guidelines and some preliminary results of a research program which studies the way pupils interpret graphs and use the algebraic language to investigate and represent them, from the point of view of embodied cognition. Roughly speaking, our working hypothesis is that students can do good performances in such symbolic tasks, provided that the teacher manages suitably the didactic situation, taking into account also the cognitive variables, like language, gestures and the way they can *generate* and *embody* the mathematical concepts. As a by-product, new insights on the nature of the embodied cognition machinery and concepts are achieved.

2. The teaching experiment

2.1 The research

A main problem for students who are requested to interpret graphs or numerical tables regards their static features (see Kieran, 1994; Boero et al., 1995a), which risk to block their mental dynamics, hence inhibiting a fruitful exploration (Boero et al., 1995b). In fact, to grasp cognitively the meaning of a function one needs complex dynamic activities; for ex. the so called *fictive motion* (Talmy, 1996), produced when the subject interprets in a dynamic and oriented way a graph, as if it were produced by a moving trajector. Such an activity can be observed through the words and gestures of subjects (see Lakoff & Núñez, 2000, pp. 31 and 37). From this point of view it is interesting to observe how a graph is generated on the screen of a graphic calculator which represents data on-line measured by a sensor (CBR[4]). The observer looks at a genuine oriented generation of the points in time, which is a sensibly different experience from perceiving a graph given in a holistic way. Such a dynamic graph is easier to be interpreted by subjects, if compared with a static one. This is the starting point for our first working hypothesis: suitable fields of experience (see Boero et al., 1995a) where students make experience of real and fictive motions, can support pupils while interpreting graphs. Such a field is our "Real data in real time", where pupils (aged 14-15) live some concrete experience (e.g., running); in the meanwhile some data are relieved by an on-line measurement tool and represented in real time on the screen of a graphic calculator. Successively, pupils are asked to interpret the graphs and tables on the screen, exploiting what these mean with respect to their lived experience. In the end they are asked to analyse some of their specific features (e.g. the slope) and to represent them using suitably the algebraic language. Our second working hypothesis is that body, language, and instruments mediate and support the transition of students from the perceptual facts to the symbolic representation, e.g. the algebraic one: in fact they can stimulate the production of an intense cognitive activity, which is marked by a rich language's and gesture's activity, for example with production of grounding metaphors. The purpose of our proposal is to describe the development of students' cognitive activities from bodily (e.g. perceptual, kinetic,...) to theoretical features. In such a development a crucial point is the genesis of the meaning for mathematical objects exploiting temporal explorations towards their just past experience and anticipating hypothesis and

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[3] The deictic function of language (see Radford, 2001) allows to indicate directly in the discourse some object which has not a name: words like "this", "that" are typically exploiting a deictic function. The generative action function supplies the conceptual dimension for generalising. According to Radford's analysis, the two functions start and support the genesis of concepts in algebra: language produces surrogates for (not yet existing) mathematical objects, which are grounded in the subjects' knowledge and fields of experience; metaphors are the tools by which subjects express this link and start creating that conceptual dimension, which will reveal essential for the construction of the mathematical object self.

[4] Sistema Calculator Based Ranger, Texas Instruments. For a technological description, see http://www.ti.com/calc/italia/prodotti/cbr.htm.

conjectures. Words and gestures reveal crucial in this activity; in particular language allows students a fruitful cognitive activity based on their just lived kinetic and visual experiences. This genetic process allows students: (i) producing a mathematical sense for the graphs they see on the screen and (ii) starting and supporting their transition to the algebraic register.

2.2 The educational context

The teaching experiment has involved 25 students of the 9^{th} grade attending the first class in Liceo Scientifico (a secondary school where there are 5 classes of maths and 3 of physics per week). The students at the moment of the experiment don't know the linear function and its representation, the uniform motion and the definition of velocity (maybe some of them can remember some general and qualitative information from the middle school), so the notions they have, come from their personal experience and are expressed in the natural language. The experiment is based on a functional approach to algebra (in the sense of Dubinsky & Harel (1992) and Kieran (1994)), starting from experience of collecting data, graphs, measures, analysing data and describing graphs and tables in order to look for regularities, and to express these regularities by relations, formulas, expressions, functions.

2.3 Educational and methodological choices

The teaching experiment is organised as a long term intervention of activities during the year, each activity lasting two-three (in one case four) class sessions (each class session lasts one hour). During the class sessions the students work in group of three-four pupils and they use the tools of the activity (e.g. a measure instrument or a graphic calculator or a sheet of paper, ...). In each activity they have to answer to some questions on a working proposal form, related to the construction of the meaning of a mathematical object. The researcher, who is present in the classroom during the activity (one of the author, Robutti), has the role of observer (she manages the videocamera) and guides the final discussion. In the classroom there are also a university student (who is taking her degree), two teachers of the class group (one of maths and one of physics) and a teacher-researcher (in the sense of Arzarello & Bartolini Bussi, 1998). The university student has the role of observing the group activity, teaching the students the use of the graphic calculator, or the use of a measurement tool (traditional like a meter or technological like a sensor), and so on. The teachers and the teacher-researcher observe the groups and intervene in the final discussion.

Walk or run in the corridor in order to make a uniform motion; when you arrive at the red line, come back with the same motion. The CBR will record your position with respect to time and will collect the data in a graph and in a table. The data are expressed in seconds (s) and in meters (m) respectively. Each 1/10 s a couple of data (time and position) are collected.

a) Describe the kind of motion you made in the corridor.

b) Using the graph and the table, describe how space changes with respect to time (increase, decrease, ...).

c) Analyse the graph. Is it like a line? Is it like a curve? Does that curve increase? Does that curve decrease?...

Consider the ratio: $m = \dfrac{s_2 - s_1}{t_2 - t_1}$ and use it to describe mathematically the graph of your motion (t_1 and t_2 are two subsequent time data and s_1 and s_2 are two subsequent position data).

Figure 1 The working proposal form

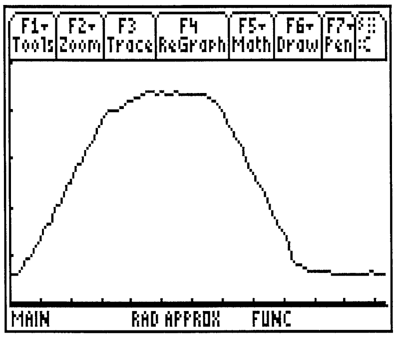

Figure 2 The graph of the motion

2.4 The experiment

The experiment (that, as a whole, lasts about 30 hours) consists in a sequence of activities so scheduled:

1. analysing a graph and answering some questions about the points and their co-ordinates;
2. measuring the length of objects with different tools (ruler, meter, ...) and finding regularities;
3. representing data in tables or graphs using a graphic calculator (a TI92, by Texas);
4. relieving data of time and distances by a sensor of position and analysing the collected data on the graph and in the table of the calculator screen (fig.1);
5. constructing models of a phenomenon, knowing the rate of change of a quantity vs time;
6. measuring data of a variable quantity vs time and modelling the phenomenon.

Each activity is divided into three parts: in the first one, the students (in small groups) explore a situation (by a proper tool or by paper and pencil); in the second one the groups answer some written questions which ask them to use/build suitable data representations (tables, graphs) to interpret the situation in a mathematical way (within pencil and paper or calculator environment); in the third and final part, the students participate in a class discussion, guided by a researcher. We shall discuss only some major points of activity number 4 (fig.1).

3. Analysis of protocols

In this chapter we describe and analyse some protocols of activity number 4 of the teaching experiment (see the list in § 2.4). In particular we shall describe some emblematic points in the dynamic genesis sketched in §2.1 looking at the performances of Fabio (M), Giulia (F), Gabriele (M) and Filippo (M). The task is given to the students with a working proposal form like in fig.1. The students make the experience running in the corridor (Fabio runs while his mates, Giulia, Filippo and Gabriele look at him and look at the screen of the graphic calculator, over which the data are represented in real time). Then the students go to the classroom, in order to analyse the result of the experience, first on the graph (fig.2) and then on the correspondent table. In what follows some excerpts from their protocols are described.

72. Fabio: *"Yes! We started..."* [he points at the starting point of the graph; he thinks]
73. Giulia: *"So..."* [she gets ready to write on the sheet]
74. Gabriele: *"So, we started from a point and we made a motion..."*
75. Fabio: *"This one"* [his finger runs below the graph drawing an imaginary line]
76. Gabriele: *"...Which should have been uniform "*
77. Filippo: *"It should be"* [Fabio runs with the finger the part of the graph which represents the motion going, that is the motion from the CBR to a red line – which was approximately considered at a six meters distance from the CBR]

The genesis of a generalising process starts in #72, where Fabio's gesture is an index towards the just past experience: in fact going back and forth from the past to the present and towards the future is essential for students' meaning production with respect to their experience. The past is rebuilt up by Gabriele in #74 through a narrative (see Nemirovsky, 1996): the process of meaning generation has a collective feature, because of the group interaction among peers. It allows pupils to connect what they have lived, observed and perceived with what they see on the screen of the calculator (fig. 2). The use of the fictive motion is supported by the bodily experience: in #75 Fabio reconstructs the shape and the orientation of the motion graph (space vs time) with the movement of his finger on the screen. In #77 this reconstruction is more conscious and interests only the first part of the graph, the linear one corresponding to the uniform motion from the CBR to the red line.

78. Gabriele: *"And then, at the end, this line"* [he moves his pen on the horizontal stretch of the graph from left to right]
79. Fabio: *"Here* [he points at the starting point of the horizontal part]*, here is when I got to the red line"*[now Gabriele and Giulia talk at the same time]
80. Giulia: *"Yes, and then you ...here"* [she indicates the starting point of the horizontal stretch]
81. Gabriele: *"Here you stopped going"* [he marks the same point with his pen; Fabio, Gabriele and Giulia indicate the same point on the graph with fingers or pens]
82. Giulia: *"...You came back* [she points at the negative slope stretch] *and then you stopped again"*
83. Gabriele: *"You stopped, you came back* [he follows the descending stretch too] *and you stopped again in front of the sensor"*[Fabio runs first the horizontal part of graph and then the descending one]
84. Giulia: *"What we are interested in is this "* [she points at the two oblique stretches with two open

The gesture of Gabriele in #78 reveals the use of the fictive motion, limited to the horizontal part of the graph, the one between the two uniform motions represented by oblique lines. The reconstruction of the just past experience with more details is pursued by Fabio's gestures (#77) and words (#79); his lived experience is shared by the other pupils who echo Fabio's utterances (#78, 80, 81, 82, 83) with their own gestures and words. The peers' interaction allows pupils to live all together Fabio's experience once more: sharing the experience culminates with the same gesture produced by Fabio, Gabriele and Giulia in #81. The episode marks a change in the way pupils approach the activity: up to #81 pupils explore the graphic on the screen and look for hints to interpret Fabio's running. Episode #81 is a *cognitive pivot*: namely the perceptual features of the first horizontal part of the graph (see fig.2) allow them to understand in a neat way the connection between signs and experience. At this point, they own this first meaning of the graph, namely the horizontal lines.

From #82 on, they use the interpretation above as a key to explain the remaining parts of the graph. This change (from an exploratory phase to an explanatory one) is important from a cognitive point of view, because it marks a starting point from an empirical approach towards a theoretical frame, which allows pupils to build up a first mathematical meaning. Giulia in #84, using the deictic function both in language and in gestures, points out the problem they must solve, namely to interpret in a meaningful and coherent way the oblique parts of the graphic. The use of fictive motion is present again in the gestures (#83, Gabriele) and helps the students to interpret and explain the shape of the graph.

85. Gabriele: *"The motion we made in the corridor is a motion..."* **...**
87. Giulia: *"Eh...uniform motion, but it's already written there* [on the worksheet], *so"***...**
91. Fabio: *"The one who ran, uhm..."***...**
94. Fabio: *"...He tried to take the steps always in the same...in the same time interval "*
95. Giulia: *"Over again"*
96. Fabio: *"In the same time interval , ehm..."***...**
106. Giulia: *"Made the motion..."* [she reads aloud what she is writing]**...**
112. Giulia: *"He (she) tried to keep the same pace there and back"*
113. Fabio: *"No, he (she) tried to keep..."*
114. Filippo: *"Eh, it's right!"*
115. Fabio: *"...During the whole..."* [he shifts his hand from left to right on his desk]
116. Gabriele: *"Walking always the same distance"*

From #85 a new subject is introduced by students, namely the motion: an interesting new generative action function starts (#94, 112, 116). The embodied experience of movement is evoked by Fabio (who lived this experience) through the "steps always ...in the same time interval" (#94), that is focusing on the regularity of motion, guaranteed by the steps' rhythm. This starts a process of discretisation of the movement, which culminates in #116, where the steps have become "always the same distance". The students understand the "rule" of the motion: "the same distance in the same time interval". However, this rule is not spelled in mathematical terms but through an embodied version, which uses natural language.

117. Fabio: *"...During the whole, during the whole time, in which I walked, I tried to, to make, to keep always the same speed"*
118. Giulia: *"So. So...the person who made the motion tried..."* [she goes back to write on the sheet]
119. Fabio: *"...To...he (she) tried to, ehm..."*
120. Filippo: *"To always keep the same speed"*
121. Giulia: *"To keep..."*
122. Fabio: *"To keep the same speed "*
123. Giulia: *"Constant speed !"* [she writes that]

A new episode starts in #117, when Fabio condenses (see Arzarello et al., 2000b, p.79) both the lived experience and its interpretations given by the group during the above discussion in the words "During the whole, during the whole time, ...always the same speed". His utterance is shared by the group (from #118 to #122) and written in an official form in #123:

"Constant speed !" So the scientific concept of constant speed has been reached through an embodied genesis, where the generative action function of language allows students the transition from a personal experience to a shared concept. It is interesting to observe the role of a marker like "always" (see also Radford, 2001). It underlines the transition from the timed experience towards non-ending processes: in fact this adverb allows the students to generalise their experience in a *de-timed* sentence (#120-123), which has the features of a scientific discourse (see Arzarello, 2000a for further discussion on this point). De-timing is produced not eliminating the time but giving a non-terminating feature to a finite and terminating process.

In the second part of the activity (from question c on, see fig.1) pupils tackle the table and the interpretation of the ratio $m= (s_2-s_1)/(t_2-t_1)$. During this session, the students pass through the same moments of the first part (the graph interpretation): the regularity of motion, the time and space intervals, the constant velocity. However a new frame is present: the numerical one.

152. Fabio: *"Here it is!* [he smiles; then he observes the table]...*So, we said: this* [he points at the first column with his finger] *is the space"* ...
155. Gabriele: *"Time increases, space also increases"* ...
266. Fabio: *"Look! if you look here* [he points a row of the table with his finger] *it has spent, more or less... about every tenth of second the sensor has checked, since there is 5.49, 5.59... "* [he goes on scrolling the table using the cursor] ...
288. Giulia: *"Time increases always"* ...
299. Giulia: *"Then space begins to decrease, while time continues to increase"*...
302. Gabriele: *"And we put: up to 4 m they grow together, in an irregular way"*...
318. Giulia: *"...'cause time goes always on of a tic, tic; instead.."*
319. Filippo: *"How?"*
320. Giulia: *"Yes of a tic, tic. You know: the watch makes tic tic"*...
324. Gabriele: *"We must write that up to 4 m both have grown up...At least I think so..."* ...
329. Giulia: *"Up to the stop they have grown up together"* [she checks numbers on the screen jumping from one column to the other]
330. Fabio:*"Together, but not in a regular way"* ...
340. Gabriele: *"After the first part they have started decreasing, the space has started decreasing and the time has gone on increasing"*
341. Giulia: *"Ehm...backward? But it does not work backward, that is..."*...
343. Gabriele: *"Arrived?... uhm... ...but not in a regular way; coming back it has happened the same...time has gone on increasing and the space diminishing"*...
345. Gabriele: *"...Even if not in a regular way"*

Numbers put forward immediately two new ingredients: (i) an unbalance between the (supposed) regularity found before (#120-123) and the regularity after which time increases (#266, 288, #318-320) and the irregularity of numbers on the table (#330, 340, 343, 345); (ii) the exploitation of a functional dependence of space versus time (#152, 155, 299, 329, 340). Pupils connect the two frames using metaphors and gestures, in order to manage the unbalance.

389. Fabio: *"Wouldn't it have been so if...?"* [he puts together his forefingers' tips and forms with them a triangle without a base]. ...
403. Fabio: *"...Had you done it more or less right..."* [he moves his open hand from left to right regularly]
404. Filippo: *"A straight line"*...
406. Gabriele: *"I should have...*[he looks at the graph]...*We should check if it was a...as you said* [he makes again the triangle with fingers] *or a straight line that makes...*[he draws a curve in the air]*"*..
430. Giulia: *"In the sense that you had done so...then, who knows?... you start running, you know, ...the stuff grows up, then it goes up again and turns, isn't it? On the contrary you have gone straight* [with the pencil she draws an imaginary line], *in short. You have gone away, hence the space has increased* [she moves both her hands on the left]...*the time has increased together* [again both her hands on the left], *however more or less, altogether, they have gone straight* [with the pencil she draws a straight line]...*Ok ...."*...
441. Gabriele: *"Because if you take a straight line...*[he raises his pencil in a vertical position]...*if time increases* [he raises more his pencil]...*if time increases also the space increases* [again his pencil up]...*we cannot get a curve* [with his pencil he traces a curve]...*for me it is a straight line"*...
469. Fabio: *"..It should have been a straight line like so...but since, while I was walking,..I have not been very precise, .."* [he moves his fingers on the desk simulating his walk]...

The complex dynamic illustrated by the episodes above shows the way pupils build up an *ideal* regular motion which solves the unbalance regularity/irregularity; this fictitious movement has four ingredients: the fictive motion (#389, 403, 406), the idealised regular motion of the body (#430), the mathematical language (#404), the ideal linear dependence (#441). They live together while the students make new ideal experiments (#430, 469). This gives a new sense to the calculations asked by question c, namely puts forward the necessity of approximating values in order to get the linear dependence:

564. Giulia: *"Is 0.20 divided for 0.20"* ...
567. Filippo: *"Second case"* ...
589. Giulia: *"0. 18 divided for 0.20"*
590. Filippo: *"Ok, more or less 0 point?"* ...
593. Filippo: *"Well, 0.9"* ...
597. Giulia: *"Let's write 1 more or less"* ...
601. Giulia: *"The rate is the same. Yes, more or less."* ...
606. Giulia: *"Then 1 is the constant"*

The numerical data are used by the students to calculate the ideal slope; then the obtained values of the slope are used to interpret the 'ideal' graph. Fictive motion has created the fictive 'real' model of mathematics.

4. Some partial conclusions

As the written protocols of all the students reveal:

(i) 22 students have performed the activity understanding the meaning of the graph, namely they have done the parts a, b, c of the task;

(ii) 16 students have understood the meaning of the formula (final part of the task).

Most students show a good linguistic production and a flexible co-ordination among different registers: verbal, graphical, algebraic. Moreover there is an interesting genesis of the mathematical concepts through metaphors, fictive motion and managing of the inner times (Varela, 1999; Arzarello et al., 2001). We can observe this intense cognitive activity through their gestures and linguistic productions, as pointed out in §3.

It is interesting to observe that their cognitive activity passes through a complex evolution, which starts in their bodily experience (namely, running in the corridor); goes on with the evocation of the just lived experience through gestures and words; continues connecting it with the data representation and culminates with the use of algebraic language to write down the relationships between the quantities involved in the experiment. The recalling process has a double nature: from the one side words and gestures start the generative action function towards a suitable representation of what they have done (i.e. with tables, graphs, functions,..); from the other side, it allows a meaningful interiorisation of their experience. In fact, there is a dialectic between mathematical concepts (for ex., a function) and their representations (for ex., its graph), which grows up through the generative action function supported by language and gestures.

Some didactical conclusions can be drawn from our experience and possibly be confirmed by the research, which is going on in the meanwhile. a) The approach to functions in the school often inhibits or curtails experiences that encourage the productions of fictive motions schema. For ex., the graphs in books and exercises generally have a static and holistic aspect. But new technology allows teachers to design experiences where graphs can be presented in a dynamical and genetic way. b) Using grounding metaphors seems to facilitate such functions as the generative and generalising ones, which can support students in the transition to a meaningful managing of algebraic language. In fact metaphors are based on usual cognitive activities that all people can do. However, grounding metaphors may be not always appreciated in the class of mathematics, since they have not a rigorous flavour. On the contrary, encouraging their production by students can facilitate the understanding of formal aspects of mathematics. As a by-product, our findings suggest that a genetic structure appears in the way metaphors are produced, which intertwines deeply with inner times of pupils. Their cognitive activity shows a continuous dynamic movement from the present to the past (their lived experience) and to the future (the hypothesis or the de-timed sentences). In a different context, such phenomena have been studied by the authors (Arzarello et al., 2001) and by Guala & Boero (1999). The analysis of connections between inner times, rhythms and metaphors reveals as a promising field of investigations in Mathematics Education, from the point of view of research (genesis of mathematical object) as well as the one of practice (which cognitive activities can the teacher encourage to facilitate pupils' understanding of mathematics?).

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ON TRANSFORMATIONS OF BASIC FUNCTIONS

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Using a genetic decomposition for the concept of transformation, interviews with college students in a pre-calculus course were analyzed. The course was based on transformations of functions and included writing as a process and the use of graphing calculators. Results show that students need a stronger understanding of function to fully understand the concept of transformation and depend on the graphing calculator to be able to analyze the properties of functions. The calculator didn't help students construct the concept of transformation. More of the writing students developed the transformation concept at a process level and recognized transformations on complex functions.

The function concept is considered a basic pre-requisite for the understanding of other concepts included in most undergraduate mathematics courses. The courses in which students are to strengthen their knowledge of functions are either College Algebra or Pre-calculus courses. In some universities, these courses have focused on the teaching of some basic functions, typically $f(x) = x, f(x) = x^2, f(x) = a^x, f(x) = \log(x), f(x) = \frac{1}{x}$, and their properties. From the study of these functions the student is introduced to more general functions by means of transformations: translations, rotations, and stretches of the basic (benchmark) functions. Many of these courses use calculators as tools to help students in visualizing the transformations on functions and some of them use process writing as a tool to foster students' abilities to construct the necessary abstractions in order to develop a better understanding of the concepts of the course.

A lot of research has been conducted on students' difficulties and understanding of the concept of function, with some studies including the use of calculators and the use of writing in mathematical courses. There is not as much knowledge from research about transformations. This information is necessary if courses involving the study of transformations are to be used to help students deepen their understanding of functions. Then one may be able to assess "if" and "how" pre-calculus courses based on transformations of functions can be successful.

Previous studies about students' understanding of transformations (Eisenberg and Dreyfus, 1994: Block, 1998) indicate that this is a difficult concept for students. They suggest that students do not easily recognize transformations on functions and that even if they have fewer difficulties when working with transformations in simpler functions such as linear and quadratic, they cannot handle them with ease. They suggest that courses based on transformations don't get the expected results because the learning of transformation requires a deeper understanding of function, maybe at an object level. Additionally, special care has to be taken when teaching these courses to be sure that students have acquired a sufficiently sophisticated knowledge of functions before proceeding to the teaching of transformations.

The act of writing has been described as a problem solving process (Hayes, 1989; Flower, 1985). As such, the focus of the writing is the thinking and the writing process rather than the product, and the process is recursive, meaning that it does not necessarily proceed on a linear path from step to step. The writing assignments establish a line of communication between the student and the instructor. Although not each student can recite in class each time, every student does complete the writing assignments. Through feedback from the instructor, each student interacts with the concepts and receives comments on the correctness of her or his thinking. The instructor also receives feedback on the understanding of the students. One of the most popular types of writing assignments used in collegiate classes is explanatory writing in class for assessment of student understanding (Keith & Keith, 1985; LeGere, 1991, Meyer, 1991). This type of assignment is useful in enabling the instructor to have an ongoing dialogue with each student in the class and to attempt to redirect mathematically incorrect student thinking.

A number of studies have shown that visualization and mathematical understanding can be improved with the use of hand-held graphing devices (Demana & Waits, 1990; Schoaf-Grubbs, 1992, Adams, 1997). There is the belief that the use of calculators in function courses could be useful for students because whereas hand sketching of graphs is slow, calculator graphing is fast and accurate. When errors occur in the manual computation of coordinates, the resulting graphs will be incorrect and may lead to incorrect generalizations. The ability to generate many graphs on the same coordinate axes is widely considered to help students generalize the relationship between the algebraic representation and the graphical representation of functions.

The present study has several purposes:

Acquiring a better understanding of students' conception of transformations;

Analysis of the success of a pre-calculus course based on the teaching of transformations of benchmark functions;

Analysis of whether the use of graphing calculators aid in the construction of function concepts and whether writing as a tool helps students make the required abstractions and transfer their understanding to other graph properties such as domain and range.

In particular this study intends to analyze the following research questions:

(1) How do students use graphing calculators in learning transformations and properties of functions? Is the use of graphing calculators fostering the constructions of concepts?

(2) What effect does writing have on learning transformations and properties of functions?

(3) Does the approach of such courses contribute to students' understanding of the properties of functions especially domain and range?

Theoretical framework

The theoretical framework used as a basis of this study is APOS (Action, Process, Object, Schema) theory (Asiala et al, 1996). A genetic decomposition of the concept of transformations was developed:

At an Action level, students are able perform transformations on functions in an analytical context by substituting values in them one by one and by drawing the graph of the function based on the evaluation of independent points. These students depend on concrete visualization to be able to deduce properties of transformed functions and they have a static conception of transformation at best.

At a Process level, students are able to coordinate the actions of evaluating the function at different points and the change in the properties that follows from the application of the transformation in the case of simple functions, either in a graphical or analytical context. These students have developed a more dynamic conception of transformation and can think of intermediate steps between the initial and the final states based on the properties of the basic functions, but only when these are very simple.

Students at an Object level have encapsulated the process of applying a transformation to any function and can tell in advance the properties of the transformed function. They are able as well to identify a complex function as the result of applying transformations to a basic one independently of the representation of the given information.

A genetic decomposition of domain and range was also used in the analysis. To recognize the domain of a given function, a student needs to look at the rule or the graph of the function and identify those numbers for which the rule does not apply or those numbers for which there is no corresponding point on the graph. To do this, students only need to do the action of identifying those numbers from the rule or the graph of the function.

Algebraically, to be able to recognize the range of the function, students need to apply the rule of the function to each point in the domain, then interiorize these actions and coordinate all their results to be able to construct the set of points that correspond to the domain points under the function. This constitutes a process.

To graphically find the range of the function, students need to recognize that each point on the y-axis is the result of the action of applying the function to each point in the domain of the function, and to coordinate each of the vertical values of the curve representing the function with their corresponding values on the vertical axis and to interiorize these actions in order to collect them into a set, which is the range. Once the student is aware of this, then he is able to think graphically of the range as the set of y values that correspond to the image of the curve under the y-axis.

Methodology

This study was conducted at a medium sized mid-western university during a single semester using four sections of Pre-Calculus of about sixty students each. The courses involved two instructors, each of whom taught both a writing section and a non-writing section. A common set of class lecture notes was used, and all sections used graphing calculators extensively during class, for homework assignments, and during quizzes and exams.

The writing students wrote in-class responses to questions or responses to situations that had been discussed in class on previous days. These students received written feddback on their responses the following day. The non-writing students completed additional examples of problems instead of writing in class. Graphing calculators were incorporated into the course as a tool for exploration and verification. The instructors presented the curriculum topics using algebraic, graphical and numeric approaches and comparing what information one got most easily from each representation. In the study of function transformations, the students used the graphing calculators to examine many graphs to see the change in shape and location when a particular transformation was applied and eventually when a combination of several transformations were applied to a given basic function.

After the course was over, audiotaped interviews were completed with 24 students. The students were stratified into three groups according to their incoming skill level on a placement test. Two students from each section at each skill level were chosen at random to be interviewed. The results presented here are derived from these 24 students' data, including their interviews, tests, and for the students in the writing group, their writing assignments. Pseudonyms are used in identifying student responses.

The interview questions analyzed here are:

- Sketch a graph of $f(x) = 2(x - 5)^2 + 3$. Explain your graph.
- From your study of algebra, what differences do you expect to see between the graph you drew above and the graph of $g(x) = \dfrac{2}{x-5} + 3$?
- What similarities do you expect to see between the graph you drew above and the graph of $g(x) = \dfrac{2}{x-5} + 3$?
- Determine the domain and range for each of the following:

  $f(x) = \sqrt{x+2} + 1$ $\qquad$ $g(x) = |2x - 3|$

Results

The overall approach including calculators and writing with emphasis on transformations of benchmark functions was not as successful as expected. Results of this study suggest that students need a stronger background in algebra to be able to benefit from this kind of approach. When working with functions, actions are performed on variables, while when working with transformations the students need to conceive the function itself as the variable and perform actions on it (Goldenberg, 1988). It has been shown that many college students don't have a rich understanding of the different uses of variable within algebraic contexts (Ursini & Trigueros, 1997). If students have difficulties relating algebraic variables, it is not surprising that they are not able to generalize the concept of variable to include functions as variables.

Students used the graphing calculators in a variety of ways. In addition to using them as visualization tools, the students depended heavily on the graphing calculator to be able to analyze the properties of functions, even in the simpler cases. Additionally, even when they could see the graph of a function either on the calculator screen or on paper, many of these students were not able to recognize the properties of the given function, such as domain and range.

All students used the calculator to examine the graphs of the functions they were analyzing. For students who were proficient with families of functions this was only a confirmation of their understanding, but for other students looking at the graph constituted an obstacle. These students used the calculator to perform actions on functions, but they didn't reflect on their actions or generalize the effect of these actions. It seems that the use of the calculator itself is not fostering students' constructions of concepts related to functions. These teaching approaches need to be complemented by the design of activities with the calculator that promote students' reflections on their actions. For example, Gwen was unable to say much about the original transformed parabola until she was allowed to graph it on her calculator, but then was unable to identify the vertex. When asked about differences, she first graphed the hyperbola, then said "it would be an asymptote type graph because, um, I'm not exactly sure why, but I graphed it and this is what it looked like." Still looking at the graph, when prompted further, she said "well instead of a parabola it goes in a straight line and then goes up and down and then goes in a straight line again and it has an asymptote at ...um...*x* equals five."

Some students developed an understanding of transformations at a process level. These students benefited more from the use of the calculator and were capable of recognizing transformations on linear and quadratic functions. They appeared to possess an object understanding of such functions which forms the basis of their understanding of how a transformation on a function changes its graph and its properties. For example, Rose writes down the list of three transformations to the benchmark parabola and draws a correct sketch without her calculator. She asked to verify her work with her calculator. For the hyperbola graph, she recognizes there will be an asymptote at $x = 5$ but is unable to identify the other transformations or the rational function benchmark.

Only a few students have an object understanding of transformation. These students can identify transformations on any of the given functions and are able to describe the effects of the transformation on the graph and properties of the functions.

We found that more students who used writing activities as part of the course were classified as having a process level concept of transformation of functions. Seven out of twelve writing students were classified at this level, as opposed to four students from the non-writing group.

Even though writing students performance on transformations was at a higher level, this understanding was not used when they explained domain and range of functions and they showed many difficulties when working with these properties. On the domain and range question about the radical function, Irene said: "I think both the domain and range would be all the reals." On the same question, regarding domain, Gwen states "you can't have a zero under ...or a negative under the square root so it'd have to be ... negative two to infinity and negative infinity to negative two." After examining the graph of the radical function on the calculator, Harry indicates that the range is $(-\infty, \infty)$.

The explanatory writing with feedback may have helped students interiorize their actions on functions. Writing allowed another mode of exploration and expression which helped students make the necessary abstractions on their actions to interiorize them as processes and also to coordinate some of those processes. Doug, in his writing assignments, was able to explain clearly what the effect of *a, h,* and *k* are in the equations $y = a(x-h)^2 + k$ and $y = \dfrac{a}{x-h} + k$, but struggled in the interview over which transformation caused which action on the graph. He did recognize that the second function would be a different benchmark with asymptotes but having the same transformations.

Students' ability to write about their understanding increased with practice during the semester, so a longer study may have shown more dramatic results. The students who had been away from a mathematics class for a year or more appeared to benefit the most from the writing assignments (Baker, 1994).

Most students in the study could only work with domain and range when a graphical representation of function was presented to them. Students showed more difficulties recognizing the range of functions, even in the graphical context. Identifying range is thus a higher level activity for students than identifying domain and therefore more difficult. Results of this study clearly show this is the case. Doug, looking at the graph of the radical function, first states that "the range would be all positive real numbers." The student is so focused on the radical, he completely ignores the *+1* after the radical. When asked by the interviewer about why the graph lies only above the *x*-axis, the student says if the *y*-value were zero, the whole thing would be undefined. He actually substitutes in $y = 0$ algebraically and completely confuses himself about whether it is *x* or *y* that is -1 in that case. Finally he simply states that "all I know is it can't be in the negatives for the range."

Given that the design of the course was based on the study of transformations on benchmark functions, we expected that some students would use them to find domain and range of functions. None of the students in this study identified domain and range of functions using transformations. There is no evidence in our data that could point to students relating transformations on benchmark functions to finding domain and range of transformed functions.

In a previous study, Eisenberg and Dreyfus (1994) suggested that an object conception of function might be a prerequisite to the effective understanding of transformations although they didn't have empirical support for this assertion. We found in this study that this seems to be the case. Students who don't have an object conception of function are not able to recognize transformations on functions even after being taught to use them as a starting point for the analysis of the properties of general functions.

Students that have a process or object understanding of functions are capable of understanding the transformations of functions in a dynamic way. By this we mean that the student can think abut the curve as an object and think about how it changes when transformations are applied to it. The student can think about one or more actions to the curve to change its shape or position in the plane. When a student can think about the changes without actually performing the movements, it shows a process conception of transformation. We found some students that showed this dynamic conception of transformation. For example, Olivia, after reading the problem in analytical form stated "this is obviously a parabola, and there's ... a vertical stretch factor of two and a horizontal shift to the right five and vertical shift up three and it's relatively narrow" (to the benchmark).

One striking result found in this study was that many students considered simple rational functions as linear functions, even after having studied the hyperbola as a benchmark function. We think this occurs because the function expression contains a variable with exponent one, which is strongly associated to linear functions, although in this case it appears in the denominator of the rational function. It has also been shown that students who have a weak algebra background rely strongly on symbols to make inferences about algebraic expressions. (Ursini & Trigueros, 1998).

As found by other researchers, this study shows that students have less difficulties when the basic functions are linear or quadratic and that vertical transformations seem to be easier for students that horizontal ones. This result can be explained in terms of the theoretical framework used in this study: vertical transformations are actions performed directly on the basic functions while horizontal transformations consist of actions that are performed on the independent variable and a further action is needed on the object resulting from the first action to get the result of the transformation.

More research is needed on students' understanding of transformations of functions. Eisenberg and Dreyfus proposed that transformations could be ordered in terms of their difficulty for students and left as an open question whether it was possible that strategies for teaching functions based on transformations could succeed. The results found in this study suggest that, in the case of vertical and horizontal transformations, pre-calculus courses based on transformations cannot succeed in their goal of helping students develop a deeper understanding of the concept of function unless more activities are designed to foster students' reflection upon actions and to help them to construct an object conception of the basic functions.

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Solving equations: Will a more general approach be possible with CAS?

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Increased access to CAS will impact on the type of learning that will occur in mathematics classrooms in the future. In considering a course where CAS calculators are available at all times the approach to solving equations will most likely change. One outcome is that students will have the opportunity to deal with general functions on a more regular basis. To be able to solve equations, students will need to develop the ability to use algebraic manipulation to rewrite equations in a form where they can use standard techniques to find a solution.

Solving equations with CAS

Access to CAS allows for a reconsideration of approaches to solving equations with students. When using CAS to solve equations there are some questions where the answer can be obtained in one step using the solve feature however there are others where this is not the case. Students need to develop general strategies to solve equations including skills to deal with equations that CAS may not be able to solve immediately. Equations that cannot be solved in one step require the selection of an appropriate mathematical technique, which is often algebraic, to find a solution for a problem. These general approaches are useful for students understanding of equation solving, even for problems where CAS provides immediate answers. It is probable that the use of CAS may enable students to develop a more algebraic approach to solving equations.

CAS use requires good algebraic understanding

In the CAS-CAT research project (http://www.edfac.unimelb.edu.au/DSME/CAS-CAT) we have considered approaches for solving equations when students have access to CAS calculators for all mathematical work, including assessment. Development of teaching material for solving equations involved consideration of the skills students require when CAS performs routine calculations formerly done by hand. Reflection on the skills and mathematical understanding needed to solve equations suggests that what is important may be different when CAS is readily available. The possibilities for teaching and learning are certainly different and a more generalised development of an approach to solving equations is possible.

In attempting a range of problems using CAS it has become increasingly obvious that good algebraic understanding is essential to solve equations both without, but especially with, access to CAS. The extent to which CAS assists in solving equations depends largely on the equation. For simple linear equations, the CAS calculator is always able to solve the equation quickly, but so will most students using pen and paper. Beyond simple linear equations a CAS does not always give a result that is easily recognisable by students, as it will often switch into complex mode or the particular mode setting may result in an unexpected output. Another possibility is that a CAS is not able to find a solution in one step. In these cases, students need considerable algebraic understanding to transform equations into a form where the CAS can assist in finding the answer.

Consider the equation $\frac{pw}{\sqrt{p^2+100}}-1=0$ that formed part of the solution to a question on a year twelve examination (Board of Studies, 1998) in Victoria. To find the solution to this equation, students had to solve for p. Initially students should attempt to solve this equation for p using the solve feature on a CAS. It is worth noting that not all CAS can solve this equation directly for p. If the CAS is not able to produce a solution immediately, students will need to find a number of equivalent equations in order to be able to rewrite in a form where p is the subject. When solving this equation, students could start by rewriting the equation as $pw=\sqrt{p^2+100}$. A good first step here is to write the equation without fractions. This is a useful technique for students, particularly in attempting to rewrite in a form where solutions can be produced using standard techniques. Students should then recognise that the resultant equation is easier to deal with by squaring both sides and considering powers of two rather than non-integer powers. This gives $p^2w^2-p^2=100$ and then $p^2=\frac{100}{w^2-1}$ where $w^2>1$, resulting in $p=\pm\sqrt{\frac{100}{w^2-1}}$. Students then had to interpret the result in the context of the original question. This question required students to have a good sense of solving equations. After obtaining $pw=\sqrt{p^2+100}$ an approach was required that enabled isolation of a power of p, in this case p^2. This example was not straightforward for all CAS. With some CAS students almost had to carry out the same procedure as if using a 'by-hand' method. For some CAS, the steps stages in solving this equation were:

1. Try to use the inbuilt solve feature of the CAS. If this works, write down the answer and interpret this in the context of the original problem.
2. If the inbuilt solve feature does not give an answer immediately, then students will need to rewrite the equation to isolate p^2. At each stage of working, try to use the solve feature of the CAS. Rewriting could involve factorisation, simplification and use of inverse relations.

To solve this question students require a level of algebraic understanding, even when CAS is readily available. The example shown above demonstrates the importance of basic algebraic manipulation in solving some questions where CAS may be unable to produce an answer. A unit on solving equations needs to help students continue to develop this basic algebraic manipulation and the ability to produce a range of equivalent equations for a given equation.

The form of an equation is a key element in solving equations

Students need to develop an understanding of approaches for solving equations that are applicable in a variety of contexts and for a range of different functions. The following examples will attempt to illustrate some algebraic skills that are useful for developing approaches to solve a variety of equations. A benefit of using CAS is the opportunity to explore more functions with students and to overcome simple errors made by students when using by hand techniques.

A key element in being able to solve equations seems to be the ability to recognise the form of an equation (see for example, Pierce & Stacey, 2001). Students should to be able to recognise an equation as linear, quadratic, cubic or as another standard type of equation. Following identification of the type of equation, algebraic manipulation could be used to rewrite the equation in a form where known techniques can then be used to solve the equation.

An essential skill for students is the ability to recognise equivalent equations and use algebraic manipulation to produce a range of equivalent equations. For example,

$4\sin x + 6 + 2\sin^2 x = 4$ is a quadratic equation in sin x and an equivalent equation is $\sin^2 x + 2\sin x + 1 = 0$. This ability to produce equivalent equations enables students to rewrite equations in a form where they can recognise the type of equation they are solving and hence the nature of the solution expected. This is particularly useful when the number of solutions may not be apparent to students from the initial equation that they are dealing with. Whenever possible students need to understand what they are expecting for an answer particularly when different CAS present answers in different ways.

The next example shows how we can help students develop an understanding of the form of an equation. This illustrates a general way of considering equations and provides an approach for solving equations that will be useful if students are solving equations that are unable to be solved directly using a CAS. The ability to recognise the form of an equation will be useful in developing a general understanding of solving equations.

In the following examples, $f(x)$ will be taken to represent basic power functions x^n and basic transcendental functions such as e^x, *sin x, cos x, tan x* and $log_a\, x$.

Consider the equation $f(x) = \dfrac{4}{f(x) - 3}$ where students are required to solve for x. In dealing with equations that involve fractions, the first step would be to rewrite this equation in a standard form as $f(x)(f(x) - 3) = 4$ and then $[f(x)]^2 - 3f(x) = 4$. Students should recognise that $[f(x)]^2 - 3f(x) = 4$ is a quadratic equation in $f(x)$ and that this can be rewritten as $(f(x) - 4)(f(x) + 1) = 0$. This results in $f(x) - 4 = 0$ and $f(x) + 1 = 0$ giving $f(x) = 4$ and $f(x) = -1$. These equations are solvable for x if the inverse is available, possibly over a restricted domain. For example if $f(x) = e^x$ then $f^{-1}(x)$ will give the solution, whereas, if $f(x) = \sin x$ then the family of solutions can be found using arsin x. Depending on the basic function, students will need to state any restrictions on the values that arise through rewriting equations. They will also need to interpret the solution obtained in the context of the original problem. The emphasis here is on recognising the form of the equation, in this case as a quadratic in $f(x)$, and then standard techniques enable the solution to be determined. For the problem just considered, essentially there are six stages in solving the equation:

1. Rewrite the equation without fractions.
2. Recognise the form of the equation $[f(x)]^2 - 3f(x) - 4 = 0$ as a quadratic.
3. Use a standard technique, in this case factorisation, to solve the equation for $f(x)$: $(f(x) - 4)(f(x) + 1) = 0$
4. Use the inverse relation to write down the solution, $x = f^{-1}(4)$ and $x = f^{-1}(-1)$ if the inverse exists or use special techniques for trigonometric functions.
5. Write down a solution for the problem where the inverse is not uniquely defined. For example, $x^2 = 4$ so $x = \pm\sqrt{4}$. If $f(x) = \sin x$ then $f(x) = -1$ has a family of solutions.
6. Check the answer in the context of the original problem.

Students could use this approach for any basic function $f(x)$, so the techniques used to solve this equation are applicable for a range of problems. The approach where students identify the form of an equation is useful in the solution of many equations. Clearly, when students have CAS they can try to use the inbuilt solve feature whenever possible however an understanding about solving equations is important.

A general approach to solving equations

An approach to help students develop their skills at solving equations could be to consider equations of a general form and then move to specific examples, rather than consider many specific examples and then generalise the results obtained. For example, if students start with an equation such as $a = \dfrac{b}{f(x)} + d$ they could find $f(x)$ first before writing down an equation to solve for x. This result could be used to find $f(x)$ for a range of different values of a, b and c and for a variety of basic functions. This type of simple algebraic manipulation will be important even when CAS is available. Similarly, for an equation of the form $a\,f(x) + b = c$ students can write $f(x)$ in terms of a, b and c, namely $f(x) = \dfrac{c-b}{a}$. They could then consider what the solution will be for a range of different basic functions such as $f(x)= x^2$ or $f(x) = \sin x$. $f(x)$ can be stated for each case using the general result. Students need to recognise that any equation of the standard form $a\,f(x) + b = c$ can be rewritten to give $f(x)$. This approach can be used to solve equations of the form $4f(x) + 5 = 7$ where a, b and c have been allocated specific values. Students could consider examples such as $4x^4 + 5 = 7$, $4x^{0.5} + 5 = 7$ or $4\cos x + 5 = 7$ and write equations for $f(x)$ in each case. Solving any equation of this form gives $f(x) = 0.5$ and hence $x = f^{-1}(0.5)$ where the inverse exists. A similar approach could be used to solve equations of the form $f(ax+b) = c$ for x.

There needs to be an emphasis on how there may be restrictions on the values that x may take. For example, when solving and either squaring or taking the square root this may affect the solutions that are possible. The solutions that students provide for problems should reflect this understanding of the restrictions on possible answers. Although CAS will solve most equations, it may still be important for students to have some understanding of a general approach for solving.

For students in their final years of secondary schooling there will be a number of algebraic skills requiring familiarity. In order to be able to get equations into a form where the solve function of a CAS can be used students may need to be able to employ some techniques that they would use to solve equations by hand. Being able to recognise equivalent equations will be useful for approaching questions where CAS may not provide an answer immediately and in this situation students may also need to complete part of the solution using a method similar to a by-hand method.

What would we expect with CAS? – some initial thoughts

We should be encouraging students with access to CAS to attempt to solve most given equations using the inbuilt solve feature. Exceptions would be cases where it may be more efficient for a student to simply write down an answer rather than enter an equation into a CAS, or where an answer may be immediately apparent. Examples could include the solution of a linear equation or an equation such as $e^x = 2$ for x.

Students should be aware that it is not always possible to obtain an answer in one step using the solve feature of a CAS calculator and that at this stage there are limitations to the equations that CAS will solve. To use CAS students may still need to rewrite an equation so that it has a more standard form and this will require algebraic skills. Students may need to carry out intermediate steps in the process of finding the answer for an equation when the CAS cannot immediately produce answers. If a CAS cannot solve an equation in one step then this ability to rewrite an equation may assist students in producing a standard form that the CAS can solve. In a unit on solving equations there would be an expectation that students would develop a general sense of the connections between various algebraic expressions or equations and how to produce equivalent forms.

Once students have obtained an answer using CAS they will need to write down the answer to the given problem in the context of the original problem. Many problems will require students to interpret results after obtaining a solution. For example, students may be required to substitute values for a parameter to find a solution within a restricted domain and then comment on the results in the context of the problem. An example is in stating the solution for the equation $2\,sin^2 x - sin\,x - 1 = 0$ where $0 \le x \le 2\pi$. After an output has been produced using CAS, students will need to find the answer in the restricted domain by substituting values for the parameter.

The implication of this more general approach to solving equations being possible is that the type of question we can ask of students can change. There is the scope to ask more questions that probe student understanding and that require interpretation of results. This opportunity to ask general questions enables us to explore students' algebraic understanding. For example, a question may require students to write down an equation with a specified number of solutions if $af(x) + b = 0$, rather than give a specific question with stated values for a, b and c. This type of question would require students to choose a solution to an equation that satisfies particular requirements. To do this requires a good understanding of equations and the type of solution that can result from different functions, $f(x)$. CAS allows students to explore more functions, tackle more interesting questions and use a more general approach to solving equations.

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Revealing and Promoting the Students' Potential in Algebra: A Case Study Concerning Inequalities

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This paper presents a study concerning novices' cognitive apprenticeship in the field of inequalities. A specific educational context was designed with the purpose of revealing and enhancing the students' potential in dealing with inequalities according to a functional approach. A preliminary analysis of students' solutions is provided. Then, the nature and functions of some "grounding metaphors" (surfacing in students' solutions) are discussed, as well as the possibility of enhancing their use by students.

1. Introduction

In many mathematics domains, mathematics education research must face widespread, strong difficulties in teaching and learning specific subjects. Difficulties met by teachers and students frequently bring to postpone those subjects and/or reduce their teaching to procedural aspects. In some cases epistemological, didactical and cognitive analyses can help planning teaching experiments which allow researchers to better understand the reasons for these difficulties and reveal students' potential in dealing with those subjects. Innovative educational choices are an expected reasonable outcome.

In this paper we present the guidelines and some preliminary results of a research program conceived according to the above perspective and concerning the approach to inequalities in 8^{th}-grade. Our working hypothesis is that a functional approach to inequalities (i.e. an approach based on the comparison of functions), when suitably managed by the teacher, can reveal (from the research point of view) and allow to exploit (from the curriculum design point of view) a students' potential in Algebra which goes far beyond the mathematics content involved (inequalities). Our preliminary results support this hypothesis and enlighten some conditions concerning the educational setting which students' success seem to depend on.

During the analysis of students' protocols, an important aspect surfaced in students' behaviour: the use of many metaphors in their solutions. The relevance of body-related metaphors in mathematical thinking has been clearly shown by Nunez et al (1999); in particular they provide an interesting example concerning continuity of functions. Nunez (2000) describes *conceptual metaphors* as follows: "*Conceptual metaphors are fundamental cognitive mechanisms (technically, they are inference-preserving cross-domain mappings) which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference*". Amongst *conceptual metaphors*, Nunez considers *grounding metaphors* (i. e. conceptual metaphors which "*ground our understanding of mathematical ideas in terms of everyday experience)*. Concerning metaphors, this paper has three aims: to show how different kinds of *grounding metaphors* can intervene (as crucial tools of thinking) in novices' approach to inequalities (and functions); to discuss possible refinements of the idea of a *grounding metaphor* deriving from the analysis of students' behaviour and related to the cultural variety of possible everyday life source domains; and to investigate how *grounding metaphors* can become a legitimate tool of thinking for students in the field of Algebra.

2. Inequalities: A Challenge for Teaching and Research

In most countries, inequalities are taught in secondary school as a subordinate subject (in relationship with equations), dealt with in a purely algorithmic manner that avoids, in particular, the difficulties inherent in the concept of function. For instance, in Italy and some other countries students are taught to deal with second order inequalitities depending on a parameter (e.g. $x^2+Kx+1>0$) in a very rigid, prescriptive way: they must solve the equation $x^2+Kx+1=0$ (distinguishing between the three cases: no real solution, two coincident real solutions, two distinct real solutions); then they must build up a table where "concordances" and "variations" of signs, in relationship with the values of the parameter K, provide the solutions for the given inequality. As a consequence of this approach, students are unable to manage inequalities which do not fit the learned schemas. For instance, according to different independent studies (cf Boero, 2000; Malara, 2000), at the entrance of the university mathematics courses in Italy most students fail in solving easy inequalities like $x^2-1/x>0$. This task was proposed to a sample of 58 students entering the Faculties of Science of Genoa and Pisa Universities: less than 60% engaged in solving the inequality; most of them performed the following transformations: from $x^2-1/x>0$ to $x^2>1/x$ to $x^3>1$. Few students took care of the case $x=0$; less than 10% made a distinction between the case $x>0$ and the case $x<0$. Graphic heuristics were not exploited and algebraic transformations were performed without taking care of the fact that the > sign does not behave like the = sign (for a comparison with other countries, see Tsamir et al., 1998; Assude, 2000).

We may ask ourselves what are the reasons of this situation. One reason could be the fact that equations (and inequalities) are considered (in most of European countries, including Italy) as a typical content of school Algebra; this subject matter is distinguished from Analytic Geometry and does not include functions. This might explain why inequalities (and equations) are not dealt with in those countries from a functional point of view. But even in countries where functions (and Analytic Geometry) belong to school Algebra (cfr. NCTM Standards, 1989) the procedural, algebraic approach prevails in many curricula and even in innovative proposals (cf Dobbs and Petersen, 1991). Another possible reason is that inequalities are a very complex and demanding subject; dealing with few and well codified cases in an algorithmic way appears as a consequence of these intrinsic difficulties. In order to support this interpretation we may consider mathematicians' work when they solve equations with approximation methods, deal with the concept of limit or treat applied mathematical problems involving asymptotic stability: the functional aspect of inequalities plays a crucial role. This fact is often neglected in traditional teaching (see above). Under the same perspective we can make the hypothesis that an alternative approach to inequalities based on the concept of function could provide an opportunity to promote the learning process of the difficult concepts involved (cf Harel and Dubinsky, 1992) and the development of the inherent skills. It could also ensure an high level of control of the solution processes of equations and inequalities (Sackur and Maurel, 2000).

3. Method

We have planned a teaching experiment in two VIII-grade classes with two aims: investigating the feasibility of an early functional approach to inequalities; and revealing students' potential and difficulties in dealing with this subject as a special case of comparison of functions. We choose to guide VIII-grade students in a cooperative, gradual enrichment of tools and skills inherent in the functional treatment of inequalities. Then we have analysed how (in relatively complex tasks) they were able to use their knowledge and increase their experience in an autonomous way.

3.1. The educational context

36 VIII-grade students (divided into two classes) were involved. A rather common routine of classroom work consisted in individual production of written solutions for a given task (if necessary, supported by the teacher with 1-1 interventions), then the teacher guided comparison and discussion of students' products; possibly, the adoption of other

students' solutions in similar tasks followed. The didactic contract included the exhaustive written wording of doubts, discoveries, heuristics.

3.2. Specific content and educational choices

As concerns the **content,** the concepts of function and variable have been approached through activities involving tables, graphs and formulas. At the beginning the geometrical context (area, perimeter, etc.) was prevailing, then it has been progressively left aside. At the beginning the function was presented as a machine trasforming x-values into y-values (*machine view* in Slavit, 1997), then classroom activities focused on the variation of y as depending on the variation of x (*covariance view*, ibidem). By this way a dynamic idea of function gradually prevailed on the static consideration of a set of corresponding pairs (*correspondence view*, ibidem). As a consequence, a peculiar aspect of the concept of variable was put into evidence (a variable as a movement on a set of numbers represented on a straight line: cf Ursini and Trigueros, 1997). The point-by-point construction of graphs was discouraged. As a consequence, the ordinary table of x, y values was frequently exploited as a tool to analyse how y changed when x changed (column-vertical analysis) and not as a tool to read the line-horizontal point-by-point correspondence between x-values and y-values. The algebraic and graphical settings were strictly related (formulas were read in terms of shapes in the (x,y) plane, while graphs evoked formulas). Finally, the approach to inequalities was realised by comparing functions. Students had to compare functions by making hypotheses based on the analysis of their formulas.

As concerns the **educational choices**, the following points were considered crucial:

- classroom discussions about "*what do we loose and what do we earn*" when a function is represented through formulas or graphs or tables or common language;

- different ways of representing given functions have been encouraged and compared (cf Duval, 1995). Even the *metaphors* used by students to describe the role of different pieces of the same formula have been encouraged and discussed.

3.3. Individual task

"*Compare the following formulas from the algebraic and graphic points of view. Make hypotheses about their graphs and motivate them carefully, finally draw a sketch of their graphs.* *A)* $y=x^2-4x+4$; *B)* $y=-x^2+4$"

The above task was significantly more difficult and complex than the previous ones, which concerned the comparison of functions like $y=x^2$, $y=-x^2+4$.

All individual solutions were collected. We collected also some recorded interviews performed after the end of the activity and concerning the strategies produced by students and their general ideas about inequalities.

4. Preliminary Results

4.1. Some quantitative data

• 5 students out of 36 did not succeed in tackling the problem (they did not understand it, or were stuck);

• 7 students (out of 31 who were able to tackle the problem) arrived to incorrect conclusions (sometimes due to trivial mistakes in calculations).

• 3 students (out of 31) build up graphs point by point, while the others compare functions in a dynamic and global way with different strategies.

4.2. Some qualitative aspects

From a qualitative point of view collected data are interesting for the following reasons:

a) 28 students (out of 31) show good skills in coordinating different linguistic registers (formula, graph, verbal language, etc.)

b) a plurality of strategies of comparison between the two functions, frequently consisting in a personal blend of strategies and moves compared and discussed in previous classroom activities. In particular we can find:

b1. Strategies based on the relationships between formula and shape of the graph (6 out of 28 who keep the didactical contract).

Formulas, or some of its parts, are associated with a peculiar shape of graph (Ex. 1, Davide). In the case of Davide x^2 evokes the shape "parabola" and the other elements of the given formula are interpreted in terms of transformations of the prototypical graph: for instance the presence of a negative sign before x^2 suggests the "reversed U shape". The dynamical aspect of the solution consists in the interpretation of the different parts of the given formula as transformations of the prototypical formula. Davide seems to ask himself questions like "*What does it mean +4 in the graph? And what about the sign - on the left of* x^2 *? What does it mean –4x in the graph?*".The most interesting fact is that the graph seems to be "an object", a shape which can be moved as a whole.

Ex. 1, Davide

In the graph x^2 and - x^2 are two parabolas: the former is over 0 and looks like **U**, the latter looks like a reversed **U***. In both functions there is +4, thus we can say that both parabolas stand over the x-axis of +4

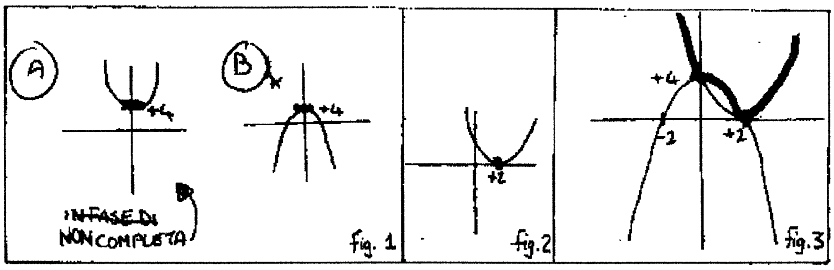
(Fig. 1: he sketches the graphs of y= x^2+4 *and y= -* x^2+4*)*

<u>In function A there is an operation (-4x) which moves and transforms* this parabola.</u>. When x=2, y=0, thus we can say that , when y=0, the parabola stands in x=2.

(Fig. 2: a parabola with the vertex in the point (2, 0))

This parabola does not go under zero when x is negative. <u>This parabola starts slowly, because -4x decreases its speed*</u> ,but successively goes up quickly.

(Fig. 3: superposition of parabolas A and B) [...].



Figures 1, 2, 3 - Excerpts from Davide's protocol.

b2. Strategies based on the relationship between formula and increase or decrease of y when x increases (14 out of 28).

In this case students analyse the behaviour of the formula according to $x>0$ or $x<0$ (Ex. 2, Lorenzo). We can imagine that Lorenzo asks himself questions that are very different from those considered by Davide: "*What does it happen if x<0? And if x>0? Does it increase? Does it decrease?*". In this case the only dynamical aspect concerns the analysis of how *y* changes in relationship with *x*. Even in this case the approach is global: it is the whole function that decreases or increases when *x* changes. For Lorenzo the graph represents a synthesis and the visual validation of his previous analysis.

Ex. 2, Lorenzo

• In both formulas you have always found a +4. In the former x^2 you always have a positive result, while in the latter you never find it, because there is a minus before.

• If $x<0$: The former increases more because there is x^2 , always positive, and the sign changes also in -4x, which turns out $+4x$. On the contrary, the latter dicreases only because I take out an x^2 and at a certain point the $+4$ cannot keep the - x^2 above the zero. There is no meeting point*.

• If $x>0$: The former decreases for a certain period (from 0 to +2) because the +4 and the x^2 don't keep * the – 4x, while after that *[the function]* increases. The latter for a certain period stays*above zero (from 0 to +2) because the +4 supports the - x^2 while after that it decrease below zero. Meeting points x=0, y=4 and x=2, y=0

[He draws the superposed graphs with a comment]

The latter is bigger than the former from 0 to +2

b3. Strategies based on the search for "remarkable points" (8 out of 28).

These students through the discovery of remarkable points (*x=0 y=..., y=0 x=...*) perform an analytic-inductive analysis of the functions under comparison and get an idea of the shape of the graph. In particular a student finds some points of the first function, then declares:"*It does not touch the origin and y is never negative!!*". This behaviour can be considered (in the case of the two functions) as intermediate between the preceding two. Indeed neither the use of the graph is prevailing (like with Davide), nor the analysis of how *y* changes in relationship with *x* (like with Lorenzo), but these two aspects are intermingled.

c) Plurality of meanings of pieces of formulas

In the task of comparing the two functions $y= x^2- 4x+4$ and $y= - x^2+4$ a crucial difficulty concerns the role of "*-4x*" in the first fuction. We can observe how, according to the peculiar strategy of each student, "*-4x*" takes diferent meanings. For instance, in the case of Davide (Ex.1, underlined parts): "*-4x*" is responsible for "*moving and transforming the parabola*"; later on "*-4x*" takes another meaning: it "*diminishes the speed of increase*". For Lorenzo (Ex. 2, underlined part) "*-4x*" is "*responsible for the decrease of the function between 0 and 2*", in relationship with the other parts of the formula. Other students declare that "*if -4x did not exist the two parabolas would be equal: the former oriented upwards and the second oriented downwards*". This diversity of meanings attributed to the same piece of the formula enriches the interpretation and representation of the whole formula..

5.Grounding Metaphors

In students' protocols some metaphors were indicated with *. What functions did they fulfil in students' solutions? In some cases, the communication function was relevant. We want to detect other, possible functions. The challenge coming from an *embodied cognition* perspective (Lakoff and Nunez, 2000; Nunez, 2000) is to ascertain at what extent *grounding metaphors* can support the learning of inequalities and overcome some of the students' difficulties by entering their problem solving processes as crucial *thinking tools*. Another problem concerns the necessity of establishing whether novices' *grounding metaphors* in the field of inequalities do substitute more advanced thinking tools; in this case their importance should be only temporary in the students' career. For this reason in this part of our experimental study we considered both novices (VIII-graders) and university students (four mathematics Ph.D. students) engaged in structurally similar tasks.

5.1. Grounding Metaphors and Inequalities

In the Annexe 1 some excerpts of an VIII-grade student's solution are reported. Also a solution from a Ph. D. is reported (Annexe 2), to show impressive similarities between the novices' strategies in dealing with open problems concerning inequalities and the efforts of an expert young mathematician in dealing with a similar, more difficult task (not covered by learned procedures). For space restrictions we will consider only one crucial step of the solution: the search for *pivot points* around which the direction of the inequality changes. In many protocols we find one (or more) of the following metaphors:

- dynamical reference to increasing and decreasing values, and the necessity of a meeting point supported by consecutive dynamical gestures of one hand (firstly indicating increase, then decrease, or vice-versa); words are coherent with this body dynamic representation: students speak about *going up* and *going down* of the two functions, and thus they must *"meet in one point"* (*meeting metaphor*): *"one graph goes up steeper and steeper from below and must meet the other which increases and then goes down"* (see Annexe 2. See also Lorenzo's protocol).

- reference to the imagined (or drawn) shapes of the two graphs, and the necessity of a meeting point supported by static crossing of the two arms; again words are coherent with this static body representation: the two graphs *"must have one point in common"* (*intersection metaphor*);

- *balance metaphor*: in this case the idea of a possible equilibrium between the values of the two functions drives the student's attention towards values of x which are near to satisfy the equation. A reference to physical trials performed in order to reach the equilibrium point is evident (see Annexe 1 for an example).

5.2. Discussion about Grounding Metaphors in Students' Protocols

In our opinion, the reported excerpts and the examples of the metaphors surfacing in students' attempts to find the 'pivot points' raise three relevant questions:

Can we speak of grounding metaphors?

The communication function does not seem to be the most important function in the students' protocols: their metaphors *"project the inferential structure of a source domain onto a target domain"*, according to Nunez's description (see Section 1). The relationships established in the different source domains (for instance: balance equilibrium) serve as crucial references to infer conclusions in the target domain (functions and inequalities). In particular, the necessity of a point belonging to both graphs derives from the necessity that can be experienced in the source domain (see later for a detailed analysis).

Can we consider the grounding metaphors used by students as spontaneous, or may we identify their origin in classroom activities?

The knowledge of students' background brings to the hypothesis that words and gestures (strongly encouraged by the teacher during the previous classroom activities on functions: see Section 3) allowed different kinds of grounding metaphors concerning functions and variables to become legitimate and spread in the classroom. *Legitimacy* means that students were allowed to overtly reason through those kinds of grounding metaphors. This is not frequent in mathematics teaching (even in lower grades): abstract reasonings are privileged. *Spreading* means that overt, legitimate gestures and words were freely adopted by the schoolmates, according to their personal needs; by this way different grounding metaphors became accessible as thinking tools for dealing with variables and functions. For instance, the balance metaphor was produced by the author of the first protocol (Annexe 1) in a previous situation; the teacher promoted a discussion about it; then it spread in the classroom. If this perspective is appropriate to describe what happened in the two classrooms, the teacher's role seems to be crucial in order to provide students with the opportunity of accessing powerful grounding metaphors. Potentially every student can use them; but this potential does not translate into an effective, appropriate use if this use is not legitimate and supported by appropriate signs (particularly words and gestures).

In the case of the search for pivot points how can we distinguish between the three kinds of metaphors described above?

The very nature of the source domains show important differences between the different metaphors. All of them fit the Nunez's description of *grounding metaphor* (see Section 1), but the nature of the evoked *everyday experience* is not the same in the different cases: again considering the three grounding metaphors surfacing in the research for the pivot points, in the first case we can recognize a reference to an everyday experience concerning crossing of movements. This physical experience supports the necessity of a *meeting point*. In the second case, a familiar situation of a necessary crossing of two continuous lines is evoked: the activity of drawing lines on plane surfaces provides the support for the visual necessity of a *common point*. In the third case, an everyday life situation is evoked: a technological tool (the balance) provides the physical necessity of an equilibrium point (in fact, the *pivot point*). These remarks suggest the consideration of different kinds of *everyday experience*, with different relationships between *culture* and *body*: an immediate relationship in the first case, a visual culture-mediated relationship in the second case, a technology-mediated relationship in the third case. In other words, we could speak of different culture-mediated necessities for a *pivot point* in the three cases.

6. Conclusion

As a consequence of epistemological analyses (which reveal the importance of the functional aspects of inequalities in mathematicians' work), and didactical analyses (which show how the current approach to inequalities based on purely algebraic procedures produces very limited learning results), and keeping into account cognitive studies about functions, a teaching experiment aimed at exploring the feasibility of an early functional approach to inequalities with WIII-grade students was planned and analysed. Collected data seem to support our didactical and educational choices; in particular, inequalities appear an interesting issue to study the students' construction of different aspects of the concepts of function and variable, and a promising learning context for these concepts.

A relevant aspect of our study concerns the educational choices related to the aim of a cooperative enrichment of tools and ways of reasoning useful to deal with inequalities. Here we may say that the approach to inequalities based on a global, dynamic approach to functions seem to fit very well with this aim: graphic representations, gestures, *grounding metaphors* concerning functions were very easy to share in the classroom (once the teacher decided to encourage their use by students).

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Annexe 1: some excerpts from the first part of an VIII-grade student's solution to the problem presented in Subsection 3.3:

"Due to the fact that x^2, that is x multiplied by itself, is there, we should get two parabolas. The presence of +4 makes them start from +4 when x=0

The first curve will meet twice the second: at +4 (on the y axis) and in another point to be found. The first curve will go down, in the first quadrant, below 4 when the weight of 4x will be greater than x^2 and will go up again when this situation will be over.

When does the first curve go below +4? Moving from 0 on the right, the first curve will go below +4 till when -4x will balance x^2, that is they will become equal

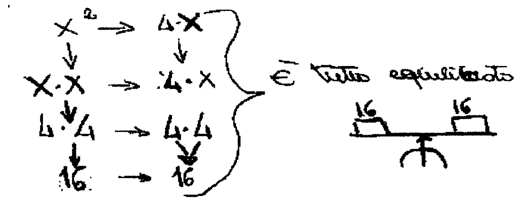

Figure 4 - An excerpt from the original protocol

The second parabola is a very usual parabola, but it is translated "upwards" by +4 and made negative as an effect of $-x^2$ which makes everything negative. It will remain over the level 0 for a while, until +4 will remain greater than $-x^2$. But when will x^2 equilibrate +4? [drawing: again a schematic representation of a balance]. *When $-x^2$ will give a number that will annihilate +4, that is -4. When? -2·2+4=-4+4=0 [...]"*

Annexe 2: a Ph. D. student's solution for the following problem: "*To find where xsinx>x^2-1*". The student is invited to tell aloud what he thinks.

(from audiorecording and notes taken by the interviewer)

"It is evident that the parabole overcomes the other function when x grows in absolute value, because xsinx cannot be bigger than the absolute value of x. It is a parabole compared with two straight lines outcoming from the origin and moving upwards [he makes gestures showing the two curves, then he makes a sketch on paper]. *The problem is what happens near to zero. Indeed if x=0 I see that the parabole is below the other function. But... xsinx... it is a pair function, a symmetric function... Well, I can consider only the positive side of the x axis. Here I imagine... x^2-1 is like x^2 lowered by one* [gestures in the air: a parabole then a lower parabole]. *OK, the other function goes up and down, but definitively it will remain below the parabole... I have already said it* [he points to the drawing]. *Now I must coordinate what happens near to zero and what happens at large* [he carefully draws the parabole y=x^2-1, the y=-x and the y=x straight lines]. *I must be more precise, and see where xsinx meets the x axis* [he makes a sign for 1, 2, 3, 4, 5, 6, then he makes a sketch of the graph of xsinx for x>0, saying: "*it goes up and down between these two straight lines*"]. *It looks fine. Oh, oh, this sketch is not precise enough - I must find where xsinx meets x... OK, sinx=1, it is here* [he makes a sign on the straight line by going up from the value of approx. 1.5 on the x axis, then he draws a more precise graph of y=xsinx between 0 and 3]. *xsinx=x^2-1 ... no precise solution, but a solution do exist, I see here, one graph goes up steeper and steeper from below and must meet the other which increases and then goes down. But I should get no more than one solution... Let us see: if x=1.5, x^2-1 makes 2.25-1, that is 1.25... bigger than one, but not so bigger... It means that the meeting point is near to 1.5 on the left... OK, my drawing was OK! I can see that x^2-1 overcomes xsinx out of this interval* [he rapidly completes the drawing by symmetry on the left, and makes symmetric gestures with the two hands to indicate the two symmetric parts of the x axis, out of the central interval]".

Research Based Instruction: Widening Students' Perspective When Dealing with Inequalities

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Abstract

A joint research study involving Israeli and Italian students is described here. This paper aims to suggest a possible research based approach to encourage introducing algebraic situations from a general and global point of view. Inequalities were chosen as one possible illustration. The students' ways of reacting to standard and non-standard tasks that underlie a similar mathematical idea are investigated, in view of drawing implications for teaching.

Introduction

In the last decades much progress has been made in the teaching of algebra. Various approaches have been suggested and reports regarding students performance under these changes have been published. However, when relating to high school curriculum in both Italy and Israel, a non-negligible number of students still study in the traditional manner, i.e., the teacher presents a sequence of topics, theorems and rules, which are demonstrated in suitable sets of examples. Home assignments are usually a repetition of tasks similar to those experienced in class.

The discussions are usually dedicated to local, specific mathematical situations.

For example, the teacher declares the topic, such as "quadratic inequalities" and then presents a sequence of tasks going from "easy" to "difficult". It should also be mentioned that many high school teachers complain about time restrictions due to a very loaded curriculum and they find it to be their own responsibility to provide students with "efficient" approaches. Moreover, in many cases even teachers who are willing to integrate computer based activities face practical problems.

This paper aims to suggest a possible research based approach to encourage introducing algebraic situations from a general and global point of view. Inequalities were chosen as one possible illustration. Inequalities play an important role in mathematics. They are part of various mathematical topics including algebra, trigonometry, linear planning and the investigation of functions (e.g. Chakrabarti & Hamsapriye, 1997; Mahmood & Edwards, 1999). They also provide a complementary perspective to equations. Accordingly, the American Standards documents specify that all students in Grades 9-12 should learn to represent situations that involve... equations, inequalities and matrices (NCTM, 1989). They further recommend that students should "understand the meaning of equivalent forms of expressions, equations, inequalities and systems of equations and solve them with fluency" (NCTM, 2000, p. 269). To implement these NCTM recommendations, it is crucial to consider students' ways of thinking about inequalities.

However, so far, research in mathematics education has paid only little attention to students' conceptions of inequalities (e.g., Dreyfus & Eisenberg, 1985; Linchevski & Sfard, 1991; Tsamir & Almog, 1999; Tsamir, Tirosh, & Almog, 1998). Most of the related articles provided teachers' and researchers' suggestions for instructional approaches, usually with no research support. They recommended, for instance, the sign-chart method (e.g., Dobbs & Peterson, 1991), the number-line method (e.g., McLaurin, 1985; Parish, 1992), and various versions of the graphic method (e.g., Dreyfus & Eisenberg, 1985; Parish, 1992; Vandyk, 1990).

The small number of published studies has tended to describe students' reactions to a few inequalities of the types commonly presented in class, and usually reported only one or two difficulties. For instance, studies pointed to students' tendency to make invalid connections between the solution of a quadratic equation and its related inequality (e.g., Linchevski & Sfard, 1991; Tsamir, Tirosh, & Almog, 1998). They related to students' tendency to regard transformable inequalities as being equivalent (e.g. Bazzini, 1997). They further identified the need to use logical connectives (Parish, 1992), and found solving inequalities with "R" or "ϕ" was extremely difficult (Tsamir & Almog, 1999).

The present study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities. During discussions of Working Group 2 at PME22 (1998), and of Project Group 1 at PME23 (1999), it was found that in both Italy and Israel, algebraic inequalities receive relatively little attention and are usually discussed only with mathematics majors in the upper grades of secondary school. Discussions are usually limited, emphasising the "practical" algorithmic perspective of algebraic manipulations. Generally, attention is paid mainly to "How to solve?" instead of "Why to solve it this way?" or "How can I be sure that the solution I reached is the correct solution?" Moreover, in both countries, the two researchers witnessed students' and teachers' frustration with the difficulties encountered when dealing with inequalities. Consequently, the two researchers assumed that students in the two countries might also encounter similar difficulties in solving inequalities, and decided to put a mutual emphasis on investigating related issues.

A collaborative study was designed to investigate students' ways of solving standard and non-standard tasks with similar, underlying mathematical ideas. As discussed in the following, our suggestion, which comes from the results of this study, is not very demanding with respect to the time restrictions mentioned by the teachers and is not bound to technological or non-technological environment. It just offers a trend for discussion making connections between mathematical topics and dealing with each topic from a global perspective

Methodology

Aim of the study

The main purpose of our study is to investigate the students' ways of reacting to standard and non standard tasks that underlie a similar mathematical idea. More specifically, we gave the students tasks that examined their ability to solve inequalities presented in the way they were used to in their classes, and tasks involving inequalities presented in a different manner.

Procedure

In both countries, the classes' mathematics teachers distributed the questionnaires, during mathematics lessons. The students in each of the countries were given approximately one hour to complete their solutions, which usually was enough time.

The students were given six tasks, presented in the manner to which they were used to in their classes, i.e., "solve" tasks, which were regarded as standard tasks. They were also given nine tasks, related to the same mathematical issues, which were presented in a non-customary manner. These were regarded as non-standard tasks. The researchers analysed, categorised and summarised the different solutions.

Here, we chose to focus on 4 of the 15 tasks that were given to students.

Participants

One-hundred-and-seventy Italian high school students and 148 Israeli high school students participated in this study. Both the Italian and the Israeli participants were 16-17 year old mathematics majors. That is, in both countries we examined students who were aiming to take final mathematics examinations in high school. Success in these examinations is a condition for acceptance to academic institutions, such as universities.

In their previous algebra studies, the participating students had studied the topic of algebraic inequalities, including linear, quadratic, rational and absolute value inequalities. In both countries, the participating students were taught this topic in a traditional way, being presented with different methods for solving the different types of inequalities. For example, parabolas or the number line to solve quadratic inequalities, and "multiplying by the square of the denominator" for the solutions of rational inequalities.

Tools

A 15-task questionnaire was administered in both countries. Italian and Hebrew versions were given to the Italian and Israeli students respectively. Two types of tasks were the following

- Going from a possible solution to create a suitable inequality (task 1)
- Using parametric inequalities in order to investigate a wide range of possibilities (tasks 6,7,12)

The tasks analysed here are the following

Task 1

Consider the set S= {x∈R: x=3} and check the following statement:
S can be the solution of both an equation and an inequality.
Explain your answer.

Research findings indicated that when relating, for instance, to quadratic inequalities, students encounter difficulties in the cases where the solution is of the type: x=a, R or the empty set.(e.g. Tsamir, Tirosh & Almog, 1998; Tsamir & Almog, 1999) It was also reported in several documents that students encounter difficulties when given tasks presented in a way different from what they are used to (Tsamir & Tirosh, 1999).)

Taking into account these data, we constructed a task asking the students whether x=3 can be the solution of an inequality. This type of task, going from the solution to the problem is not customarily discussed in class. It compels students to take into account the variety of topics he/she studied, in order to be able to come up with a suitable example.

It has a wider mathematical issue of considering the statement and understanding that "not finding a suitable example" does not prove that there is no such example.

We expected the first part to be easy, but the second part is usually not discussed in class and we expected it to be problematic.

All the students, however, had the needed knowledge to correctly answer both parts. For the second part they could provide examples derived either from quadratic equations or from absolute vales. (e.g. $x^2 \leq 3$, $| x^2-3 | \leq 0$)

Task 6

Check the following implication:

$$ax < 5 \Rightarrow x < \frac{5}{a} \qquad \forall a \in R,\ a \neq 0$$

Task 7

Check the following implication:

$$ax < 5 \Rightarrow x < \frac{5}{a} \qquad \forall a \in R$$

Task 12

Solve the inequality $(a-5)x > 2a-1$ *, x being the variable and a the parameter.*

Research findings indicate that students tend to multiply both sides of an inequality by a negative number without changing the direction of the inequality. In a wider sense, several researches suggested that students apply procedures that are valid in the context of equations for solving inequalities even though they are not necessarily applicable (e.g., Tsamir, Almog & Tirosh, 1998)

There is also reported evidence that students are inclined to reduce algebraic rational expressions without checking the limiting zero cases. They also tend to multiply by the denominator in both equations and inequalities without taking into consideration the zero cases or the negative cases.

All these tendencies have been taken into account when designing tasks 6, 7, and 12.

Tasks 6 and 7 may provide us information regarding students' performance with inequalities of the type ax<b. In task 6 we provided the limitation $a \neq 0$, which is sufficient in the case of equations. Our aim was to see whether students regarded this limitation as sufficient for inequality as well.

In both tasks 6 and 7 students were asked to determine the equivalency and to justify their claims.

Task 12 dealt with the same issue in a different manner, asking the students to solve a similar given parametric inequality.

Results

The results will be presented in the following order. First, an analysis of Italian and Israeli students' responses to Task 1, then their responses to Tasks 6, 7, 12.

Students' Reactions to Task 1

Table 1: Frequencies of students' solutions and justifications to Task 1 (in %)

		ISRAEL N=147	ITALY N=150
TRUE*		**51.4**	**48.3**
	Valid explanation	5.4	2.0
	A system of inequalities	15.5	0.7
	X=3 belongs to the solution	3.3	3.0
	Other**	27.2	42.5
FALSE		**48.6**	**51.7**
	A solution of inequality is an inequality	19.5	22.0
	Other**	29.1	29.7

* Correct response
** Irrelevant or missing justifications

In both countries, none of the students had any problems in correctly responding that x = 3 can be the solution of an equation. Most of them accompanied their responses by an example, usually of a first-degree equation, such as 2x-6=0. This, however, was not the case, in both Israel and Italy, with the participants' responses to the question of whether x = 3 can be the solution of an inequality.

Table 1 shows that in both countries, only about 50% of the students who responded to this task, correctly claimed that x = 3 can be the solution of an inequality. Still, most of them did not accompany their claims by any justification and only a few students, Israeli or Italian, gave valid explanations. These latter explanations were usually the presentation of the following example of the quadratic inequality $(x-3)^2 \leq 0$. More prevalently in Israel, but also in a few Italian cases, they explained that the claim "x = 3 can be the solution of an inequality" is true, because x = 3 can be the solution of a system of inequalities.

Another type of interesting justification, given by a small number of Israeli and a small number of Italian participants, was that "the claim is true, because x = 3 can belong to the set of solutions of an inequality." This justification was accompanied by illustrations, such as, $5x-10>0$, further explaining that "the truth set (or solution) of this inequality is $\{x: x>2\}$, and 3 is one of the values that satisfies this condition, and therefore x = 3 belongs to the truth set of $5x - 10 > 0$."

Students' reaction to Tasks 6, 7, and 12

Table 2: Frequencies of students' solutions and justifications to Task 6 (in %)

ISRAEL n = 125	ITALY n=147	JUDGMENT JUSTIFICATION
%	%	
41.6	**73.6**	**TRUE**
9.6	20.9	No justification
28	46.8	a ≠ 0 it is possible to divide (like an equation) Since
3.2	2.9	Substitution of a number for 'a'
0.8	3	Others
59.4	**24.3**	**FALSE***
2.4	5.1	No justification
52.8	13.8	If a>0 it is OK to divide (relating to the sign of 'a')
1.6	1.8	Valid Example
1.6	5.6	Others (relating to an equation)

Table 2 shows that about 60% of the Israeli participants and about a quarter of the Italian ones gave the correct answer: "False". In their justifications, about half of the Israeli students and about 15% in Italy provided a justification, with reference to the sign of 'a'. It should be mentioned, however, that typical justifications of about 30% of the Israeli students were merely that "this claim is correct only for positive values of 'a'". While correctly limiting the range for 'a' they said nothing about zero or negative 'a's. The other participants who justified these "False' responses explained, for instance, that "for negative 'a's the direction of the inequality should change, for a=0 it is impossible to divide by 'a', but for positive values the given implication is correct." .

The answers relating to equations are mainly concerned with a ≠ 0, like in the case of equations.

Table 3: Frequencies of students' solutions and justifications to Task 7 (in %)

ISRAEL n = 123	ITALY n = 133	JUDGMENT JUSTIFICATION
%	%	**TRUE**
8.9	**9.5**	
3.3	3.5	No justification
1.6	3.0	Dividing by a (like equation)
4	3	Other
91.1	**91.4**	**FALSE***
6.5	10.5	No Justification
23.6	9.4	$a>0$, $a<0$, $a\neq 0$ Relating to all cases
44.7	66.4	$a\neq 0$ Relating only to
0.8	0.8	Substituting a number for 'a'
15.5	3.5	Other

Most Italian and Israeli participants correctly responded to Task 7 and the percentages of the correct answers given in the two countries are similar (91.1 and 91.4 respectively). However, at a closer insight, there is evidence that most students gave the correct answer with wrong justification. In particular, most prevalent were justifications, such as, "the answer is 'False', because 'a' might be zero and it is impossible to divide by zero", or "there is no restriction about the need to have non-zero 'a'". The high percentage of those who related only to $a \neq 0$ confirms the persistence of the "ghost of equation".

Table 4: Frequencies of students' solutions and justifications to Task 12 (in %)

The complete solution

a-5>0	a>5	$x > \frac{2a-1}{a-5}$
a-5<0	a<5	$x < \frac{2a-1}{a-5}$
a-5=0	a=5	No solution

ISRAEL n = 99	ITALY n=139	JUDGMENT JUSTIFICATION
%	%	
13.1	**15.6**	**FULL SOLUTION***
86.9	**85.4**	**INCORRECT SOLUTION**
40.4	44.6	x>(2a-1)/(a-5)
15.2	24.4	Relating only to $a \neq 5$, relating to the equation
2	6.5	Solving a as f(x)
29.3	8.9	Other

Task 12 was solved correctly by a low percentage of students (13.1 of Israeli students and 15.6 of Italian students). The analysis of incorrect solutions gives evidence that in most cases (40%-45% of both Italian and Israeli) students did not consider the range of value of the parameter 'a' at all. A non-negligible number excluded only non-zero values, claiming, for instance, that "for $a \neq 5$ $x>(2a-1)/(a-5)$". They solved the inequality with routine procedures, the same as they adopted for equations.

Discussion and implications

The data we obtained shows that students who have studied inequalities in the traditional way face difficulties when presented with "non traditional" tasks.

The data analysis suggests a closer investigation of the origin of such difficulties.

Our main assumption now is that previous instruction plays a major role. A traditional approach makes students able to solve standard tasks (for example that of solving equations) but they often feel lost when required to handle symbols and meaning. We address the traditional approach trying to claim that with minor changes much can be done.

Instead of "teacher gives task" and "student solves", we suggest fostering a higher level of investigation and discussion about the language they use. As far as algebraic expressions are concerned, students should be requested to be more reflective on the tasks they face.

As in task 1, going from possible (and likely reported as problematic) solutions to create suitable tasks, students are allowed to approach algebra from a holistic point of view.

As in task 6, 7 and 12, parameters play a major role in deepening understanding, since the survey of all possibilities is required.

Doing algebra is not just formal manipulation, but rather a competence, which deeply involves understanding. In this perspective, it is important to provide students with different ways to approach the same mathematical ideas. For example, like here, asking them not only to solve, step by step, different tasks given by the teacher, but also to create tasks of their own, according to specified guidelines.

The traditional way of teaching gives students tools to algorithmically solve inequalities. We strongly suggest that discussion should also be theoretically oriented. Furthermore, we suggest that from the very beginning algebraic topics should be faced from a wider standpoint (and not only going step by step through local examples).

Finally, we mention that mathematics is not a set of disconnected pieces of knowledge, but an interrelated network. In this perspective teaching algebra must no longer consists of separate issues, but must be reconsidered in its whole complexity and underlying connection.

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A Problem-Solving Approach to Algebra: Accounting for the Reasonings and Notations Developed by Students

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Even though an abundant corpus of research exists on the subject, the teaching of algebra, particularly during the initial stages when students construct a certain relationship to this new knowledge, remains a central issue in mathematics teaching. Within this problematic, the question of the transition from arithmetic to algebra is of central concern, particularly in the case of problem-solving. The research reported in this article represents an attempt to shed light on this transition, by a survey of previous work and on the possible avenues of approach that lend to developing algebraic reasonings among students in a context of problem-solving.

Introduction

Research into the learning of algebra conducted by a team of fellow researchers over the last several years (Bednarz et al. 1994, 1995, 1996; Schmidt, Bednarz, 1997) has been aimed at clarifying the conditions under which students construct algebraic procedures in situations conducive to the emergence and development of such reasoning activity. In the past, this work has centred on one of the major heuristic functions of algebra—namely, its use as a problem-solving tool, which historically has played a role in the development (Charbonneau, Lefebvre, 1996; Radford, 1992) and teaching (Chevallard, 1990) of this field of mathematics. The studies that we conducted among students at different academic levels (Bednarz et al., 1994, 1995, 1996; Bednarz, Gunzman, in press) and among university students in preservice education (Schmidt, Bednarz, 1997) has enabled us to shed light on the difficulties encountered in connection with the transition between two fundamental learnings—arithmetic and algebra—by showing how such difficulties stem both from the specific nature of the problems presented in each field and from the nature of the arithmetical and algebraic types of procedures used to solve these problems (Bednarz, Janvier, 1996; Schmidt, Bednarz, 1997). In addition, these studies have enabled us to highlight the complexity of the algebra problems generally presented to students (Bednarz, Janvier, 1994) and, in terms of current teaching practices, the discontinuities characterizing the transition from one academic level to another (Marchand, Bednarz, 1999; in press).

This research prompted us to use these findings as a basis for further investigating the teaching situations and intervention strategies that can serve to stimulate the emergence and development of algebraic procedures among students in a problem-solving context. We will present a synthesis of previous research and a possible avenue illustrated by an intervention that was conducted among a group of Secondary 2 students (ages 13 to 14) during their introduction to algebra (Landry, 1999). These two parts will highlight different components of a problem solving approach to algebra.

The transition from arithmetic to algebra in the context of problem-solving

Even though an abundant corpus of research already exists on the subject, the teaching of algebra, particularly during the initial stages when students construct a certain relationship to this new knowledge, remains a central issue in mathematics teaching. Within this problematic, the question of the transition from arithmetic to algebra is of central concern. For one, students have already acquired a long experience of arithmetic by the time they first take up algebra. In this acquisition process, they have developed a certain number of ideas about problems, ways of approaching problems, familiarity with notations that constitute a basis of reference, etc. that they will then draw on to interpret the new situations put to them.

Research conducted among students who completed instruction in algebra provides evidence of several sources of resistance during the transition from arithmetic to algebra (Schmidt, 1994, Schmidt, Bednarz, 1997). Among other causes, this resistance stems from: the very status accorded by these students to symbolism (a designatory as opposed to operational status); the nature of the reasoning procedures deployed in each field (algebraic problem-solving procedures are from the outset, based on states via a symbolic substitute, whereas arithmetical problem-solving procedures are endowed with potential for evolution in the form of an elaboration articulated on relationships and transformations, without consideration for states); and the nature of the validation performed in either field (Margolinas, 1991, Pycior, 1984). Arithmetical problem-solving procedures works off of known quantities and contextual meanings, whereas algebraic problem-solving procedures must resort to other criteria for determining the validity of the reasoning implemented.

Our research also shed light on this transition from arithmetic to algebra. By improving the characterization of problems generally encountered in arithmetics and algebra, the analytical framework developed by the team thus lent itself more fully to differentiating between both types of problems in terms of the nature of the relationships between quantities and their interlinkage. Thus, one facet of the transition between arithmetic and algebra was made to stand out more clearly (Bednarz, Janvier, 1996): in arithmetics, the problems presented to students are generally said to be "hooked up, connected"—that is, a relationship can be easily established between two known quantities, thus laying the groundwork for a potential "arithmetic" type of reasoning procedure (in which the known quantities of a problem are used to arrive at the unknown quantity). In algebra, on the other hand, the problems generally presented to students are "disconnected"—that is, at the outset, no link or connection can be established directly between the known quantities.

Further insight into this transition between arithmetics and algebra was made possible by means of a closer analysis of students' reasoning procedures. A series of experiments was thus conducted among Secondary 1 students (132 students, ages 12 to 13) prior to any introduction to algebra. It served to identify the methods that they adopted to resolve various types of problems traditionally presented in algebra; by the same token, these experiments served to make out various stable profiles of arithmetical reasoning procedures (Bednarz, Janvier, 1996). Individual interviews with several students selected on the basis of these various reasoning-type profiles were then used to confront these reasoning procedures with the type of reasoning habitually expected in algebra. This confrontation served to highlight the difficulties characterizing the transition from one type of handling (arithmetical) to another (algebraic): the refusal to operate on the unknown quantity; difficulty perceiving a generator (essential to the equation-building phase and to choosing one or more unknown quantities); and difficulties associated with either the symbolization of relationships between quantities or with the type of substitution required by the transition to a single unknown quantity (Bednarz, Janvier, 1996). This analysis of the conceptual changes characterizing the arithmetic-algebra transition, a shift that many students fail to negotiate, thus brings forward the question of potential interventions in this field.

Introduction to algebra: foundations of an intervention

Several studies have shown the contribution to introductory algebra represented by the informal problem-solving strategies developed by students prior to any initiation in this subject (Filloy, Rubio, 1991, Sutherland, Rojano, 1991; Kieran et Chalough, 1993). Rojano (1996), in that perspective, has argued that:

> Trial and error, together with other strategies considered informal and which are found in students beginning the study of algebra, are indeed a real foundation upon which the methods or strategies of algebraic thought are constructed (p. 137).

According to the above-mentioned studies, these informal strategies are implemented in a computer environment (logo and spreadsheets) that helps students to symbolize their problem-solving procedures. Other interventions have proceeded similarly. The didactic intervention model developed by Filloy, Rubio (1991) used explorations with numbers to lead off the analysis of the problem with students. These various results show the plausibility of introducing students to algebra by means of incorporating problem-solving procedures previously constructed in arithmetic. The intervention that we developed in the introduction of algebra is in line with this same perspective, taking account of reasonings previously built by students in arithmetic (Landry, 1999; Bednarz, Landry, 2001)

A certain conception of the arithmetic-algebra transition underlying the intervention

Brown et al. (1998) have critiqued research on introductory algebra, challenging the very notion of a transition from one field of knowledge (arithmetic) to another (algebra), at least in the way that this notion is often expounded in promoting the idea of a "didactic break" or discontinuity between the subjects of arithmetic and algebra (and the teaching procedures specific to each) in effect amounts to favouring a certain model that defines both a current status of the learner (functional in arithmetic) and a desired future status (functional in both arithmetic and algebra). The educator's initial overture to students, which appears to open up a range of various potentialities (informal strategies used by students in arithmetic), instead gradually narrows in on a single, expected response (in this case, standard algebraic-type reasoning), thus failing to encourage the development of shared, reflexive understanding of the situations at hand in a way that does justice to this complex, multifaceted learning process.

At this juncture, socio-constructivist bases served to problematize the way that we viewed this transition, in line with the reflections and conclusions of studies by Brown (Bednarz, Garnier, 1989; Garnier, Bednarz, Ulanovskaya, 1991; Larochelle, Bednarz, Garrison, 1998): in this transition, the capacity for generalization in the field of algebra depends on a meaningful interpretation of multiple experiences (Steffe and Kieren, 1994).

> Constructivism is better grounded in an epistemology where a person's understanding of any content is based on complex connectedness among it emerging through multiple and varied experiences of the learner.

A constructivist perspective draws on knowledge of students' arithmetical reasoning procedures

The above-mentioned perspective stresses the complexification of reasoning procedures rather than a break between arithmetic and algebra. Our knowledge of the arithmetical reasoning procedures developed by students (prior even to any introductory course in algebra) for solving the problems habitually presented in algebra would then serve as a basis for structuring the intervention. In effect, the experiments conducted in Secondary 1 brought out the various problem-solving procedures used by students, and highlighted major differences in the way students picture quantities and the relationships between the quantities of a problem (Bednarz, Janvier, 1996): trials and errors, false position, in which students operated on states, "structure-reckoning" in which students directly operated on the relationships and transformations at hand, are some procedures previously identified.

This research also highlighted the distance separating a number of these procedures from algebraic-type reasoning procedures. Such is the case, notably, of trials and errors, students were prompted to engage in reasoning via a series of actions using intermediate states, and did not work a priori toward an overall account-taking of relationships between quantities. On the other hand, false position, in which students use a kind of fictional quantity, and structure-reckoning procedures, in which students take account of the relationships and their linkage, could prove useful in a transition to algebra. At that point, a number of questions cropped up in connection with the choice of situations to be implemented in order to foster the complexification of students' reasoning procedures.

What components should enter into the process of selecting situations?

A number of major components emerged from the analysis of algebra research. For one, the difficulties associated with the meaning accorded by students to algebra symbolism and notations (Booth,1984, Bednarz et al, 1992, Clement, 1982, Kieran, 1981, Lee, Wheeler, 1989, Lochead, Mestre, 1988, Matz, 1980) argued for an intervention offering students opportunities for seeing the necessity and relevance of a transition to this symbolism. In the framework of this intervention, situations were constructed with a view to incorporating and extending what Kieran (1996) has called "generational activities"—i.e., situations that encouraged students to investigate number patterns in context as a starting point for constructing and employing symbolic notations. From this perspective, algebra appeared as a tool of generalization serving to endow symbolism and symbol use with meaning. Accordingly, problem-solving was dealt with as a particular extension of these situations. We thus delved deeper into work at this specific level, drawing on the previously described problem-solving procedures as part of an attempt at prompting students to complexify them.

Weighing choices of 'didactical' variable problems

Previous research by the team led it to develop a analytical framework for analyzing algebra problems in terms of relational calculations (Vergnaud, 1976, 1982). This framework could be used to account for the relative complexity of these problems and, as well, to grasp and anticipate the possible lines of approach to a given problem, coupled with the difficulties observed among students (Bednarz, Janvier, 1994). The general structure of a problem brought into play both known and unknown quantities, in addition to the relationships between these quantities. These relationships were either provided explicitly in the problem statement or left implicit, and then had to be reconstructed by the student.

According to our team's previous analysis, three basic classes of problems are generally taken up in algebra: 1) unequal sharing, a type of problem generally presented during introductory algebra and which involves the relationships of comparison between unknown quantities (example: 380 students are registered in sports activities for the season. Basketball has 76 more students than skating and swimming has 114 more than basketball. How many students are there in each of the activities?); 2) problems involving transformations of unknown quantities over time (example: Luke has 3,50$ less than Michael. Luc double his amount of money whereas Michael increases his amount by 1,10$. Now Luke has 0,40$ less than Michael. How much did Luke and Michael each have to begin with?); 3) and problems involving relationships between non homogeneous quantities by means of a rate (example: 588 passengers must travel from one city to another. Two trains are available. One train disposes only of 12 passengers cars while another train disposes of 16 passengers cars. Supposing that the train with 16 passengers cars will have 8 cars more than the other train, how many cars must be attached to the locomotive of each train?).

The cognitive complexity of the task for students was brought out by the analytical framework developed by the team on the basis of the relational calculations involved in the representation and resolution of such problems (the nature of relationships between known or unknown quantities; composing (or not composing) these relations; the number of relations to be managed; the linking of these relations; formulation of the relationship, etc.). This framework developed out of a systematic analysis of the various types of problems presented in the algebra sections of textbooks, and was tested among several groups of students in grades Secondary 1 to 5 from several different schools. The results of these various experiments confirmed the influence of the factors of complexity that had been initially identified during the analysis of problems: the impact of the nature of relations for the same structure (for example, additive versus multiplicative relations of comparison; the composition of two heterogeneous relations, one additive and the other multiplicative); the number of relationships (implying a more or less complex handling of data); the interlinking of relationships (reference to the same generator; composition of two relationships; or presence of indirect relationships referring to different generators, etc.), and so on. This framework constituted an essential basis for designing the classroom intervention, for these prior analyses guided the selection of problems and their gradated introduction over the course of the intervention.

The situation design phase thus proceeded on the basis of the previously described theoretical background and research. The steps in this phase could be summarized as follows: 1) an analysis of students' reasoning procedures and conceptions. At this point, one of the essential points resided in the prior, close analysis of the main reasoning procedures used by students in arithmetic (in connection with the greater or lesser distance separating these procedures from algebraic reasoning procedures), in order to design teaching situations that were capable of triggering an evolution in these reasoning procedures. Another, equally important basis was to be found in an analysis of students' conceptions of the symbolism and main notations used in algebra—an indispensable foundation for algebraic reasoning activity—again, in order to design teaching situations that would trigger further development of these conceptions (the intervention had to contribute to constructing a meaning for the symbolism used in algebra); 2) an analysis related to the characteristics of of algebraic problems: complexity of problems; didactic variables that were likely to influence the involvement of students.

Context of intervention and methodology

A teaching experiment was conducted among a group of 24 Secondary 2 students who, at the outset, presented learning difficulties in mathematics. The students selected for inclusion in the group had been identified as being "weak" in terms of learning mathematics; these students had passed the Secondary 1 mathematics courses with an average grade between 60% and 75% or had failed their Secondary 2 mathematics course (they thus presented a wide spread of ages, from 13 to 16). The challenge with respect to problem-solving and learning algebra seemed all the more formidable as a result.[1] This study was thus directly concerned with reaching out to the broadest population possible, and not merely making do with including students from regular or strong classes.

The experiment was enacted over a four-month period (September to December). Three teaching sequences were developed with a view to prompting students to construct a meaning for the symbolism presented and developing skills in algebra problem-solving. Throughout this intervention, we attempted to document: the evolution of procedures deployed by students; the gradual restructuring of these procedures; and the conditions which, during this intervention, contributed to the emergence and development of algebraic reasoning procedures.[2] Toward that end, a written examination that included the problem situations previously tested by the team (Bednarz, Janvier, 1996) was administered to all students prior to any experimental intervention, for the purpose of identifying the arithmetical reasoning procedures spontaneously developed by students in response to various types of problems habitually proposed in algebra. A written test was also given to students at the end of the experiment project for the purpose of identifying the reasoning procedures deployed during the intervention. Individual interviews were conducted among 13 students who, on the basis of their results on the written test, were chosen according to their most noticeable difficulties in problem-solving. These interviews were conducted in two phases: once, prior to any intervention, for the purpose of identifying, from the outset: the arithmetical reasoning procedures implemented by students during the solving of problems habitually presented in algebra, students' difficulties, and the conditions under which these reasoning procedures were produced; and again, at the end of the teaching sequences, so as to see how these students evolved in terms of the reasoning procedures deployed (restructuring of procedures; the kinds of support they drew on). These thirteen students, who from the outset presented different profiles of problem-solving procedures, were the subject of particular monitoring throughout the intervention. The answers, comments and reasoning strategies that they put forward during group discussions in class were analyzed for the purpose of understanding how, in connection with the more specific situations put to students, the transition from arithmetic to algebra proceeded in a problem-solving context.

The teaching sequences

In this section, I will briefly outline the three teaching sequences that were developed for the intervention. The first sequence was designed to induce an initial transition to algebra in a context of generalization. Algebra research has provided evidence of the difficulties that students have using and understanding the symbolism and notational conventions current in algebra (Bednarz, Dufour-Janvier, 1992, Booth, 1984). This sequence was intended to overcome some of the difficulties habitually encountered when algebra is first taken up in the classroom. The main goal was to ensure that students grasped the relevance of a shift to algebra in a situation of generalization and that they constructed a certain meaning for the symbolism. The main phases of this sequence could be described as

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1 Research by Perrin-Glorian (1993) has highlighted the persistent difficulties experienced by these students, particularly with respect to problem-solving (the notion that they form of a mathematics problem; the search for algorithms; little validation of the problem-solving process; difficulties transfering knowledge into another field, etc.)

2 This work was performed within the framework of a doctoral thesis. For further details concerning this experiment and an analysis of its results, see Landry (1999).

follows: 1) the identification of number patterns presented in context, with the support of an illustration; 2) the construction of a verbal message that accounted for this regularity and could be used to locate a certain term quickly; 3) a group discussion in class concerning the various messages devised, followed by validation of these messages in context by students; 4) the resourceful use of the equivalence of a number of these messages; 5) a shift to symbolism, class discussion concerning the symbolic messages produced, plus validation; 6) the transition to problem-solving using the constructed formulas.

The second sequence constituted the beginning phase of solving a particular class of arithmetic problems, namely, the class of problems involving relationships of comparison. An analysis of students' specific difficulties in terms of solving algebra problems provided evidence, among other things, of the major obstacles that students experience apprehending relationships of comparison and their interlinkage (Mayer, 1982, Bednarz et al, 1996). Thus, the second sequence centred the intervention on the development of skills related to handling these relationships in a context of solving arithmetic problems (formulation of relationships of comparison based on a context; illustration of these relationships; students' formulating and solving more or less complex arithmetic problems involving relationships of comparison; etc.).

The third sequence involved a shift to algebra in a context of problem-solving founded on the learnings of sequence 1 (context of generalization serving to establish a general formula) and sequence 2 (skills of representing and handling relationships of comparison). This third sequence was designed to foster the transition to algebraic reasoning procedures, and took into account: 1) the prior analyses, particularly with respect to the previously developed problem situations and analytical framework (Bednarz, Janvier, 1994); and 2) students' reasoning procedures, with emphasis being placed on work designed to foster an integration of overall relationships between quantities (Bednarz, Janvier, 1996). The main phases of this sequence could be described as follows: 1) a starting situation (a context of stocktaking which made the relationships between various quantities explicit, only the relationships) for the purpose of forcing students to reflect on the choice of a generator (is it possible to find the number of items using only the information provided and identify the overall quantity of items inside the warehouse?); 2) the construction of a general verbal message accounting for a way of calculating the quantity of articles in the warehouse; 3) modifications made to one of the quantities and reflection over the impact of this modification—so as to grasp, on the one hand, the applicability of this message to a whole class of situations, and on the other, the status of the numbers involved in this situation; 4) the transition to problem-solving using the message thus constructed; and 5) complexification of the relations of comparison proposed and the extension to other problems.

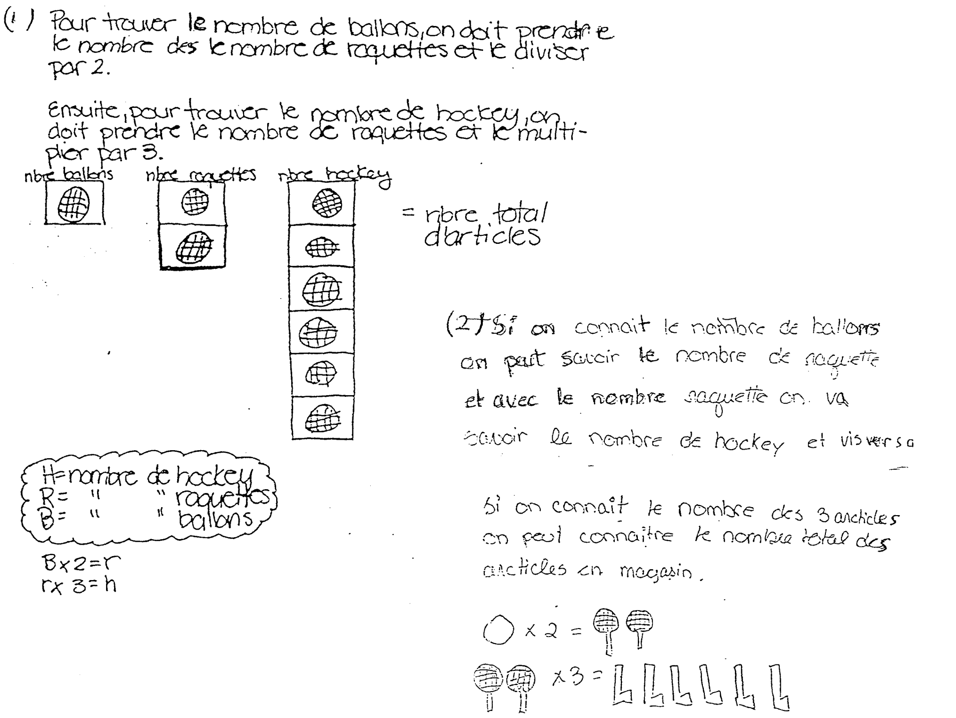
Analysis of results: Some further avenues of reflection opened by this intervention

Close analysis of the comments and responses of students during the intervention (more particularly the last teaching sequence) provided evidence of their receptiveness to a possible choice of various generators (opening on to various choices of unknown quantities), as we can see in this situation:

(The starting point situation): a son (hired by his father to do an inventory) let him the following message: three types of articles were counted. There are 2 times more rackets than balls, and 3 times more hockey sticks than rackets. The students were then asked the following question: Do you think the father can get by with this message his son left him in order to find what is in the warehouse? Some answers given by the students
No, because it is necessary to know the number of an article
It is necessary that he gave at least the number of rackets to find the other objects. You did 2 times more and 3 times more
No, because it will be necessary to have the same number of rackets
No, it is impossible because it is necessary to know the number of balls
You have to count all the balls and after that you did the calculations
No, there is no numbers then we can't know how much rackets, balls and hockey there are. But if we want to know how articles there are, it is necessary to know how much balls?

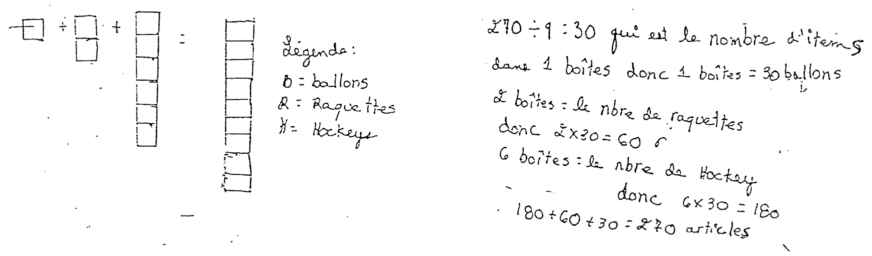
The analysis of their productions also highlight the simultaneous use of various modes of representation: a verbal message accounting for the overall quantity; an illustration accounting for the relationship between quantities; and the use of a intermediary symbolic form of expression—as is shown in the examples below:

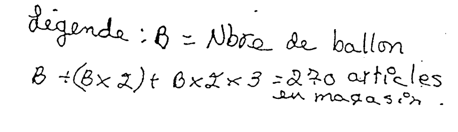
The situation: there are 2 times more rackets than balls and 3 mores hockey sticks than rackets....(the construction of a general verbal message accounting for a way of calculating the quantity of articles in the warehouse)



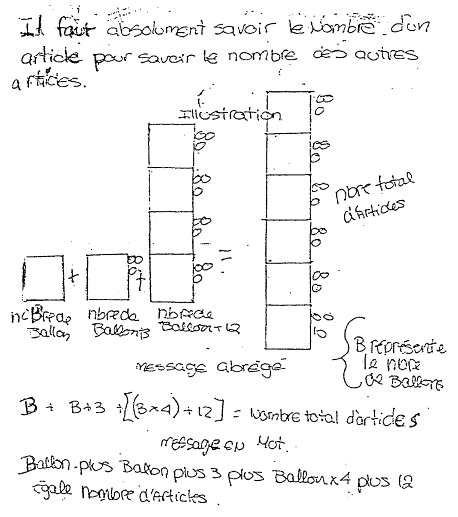
These examples give clear illustration of the richness of students' non-standard representations, which served (as we can see below) as a support in the transition to algebraic reasoning procedures in problem solving; in this process, students operated on the unknown quantity to arrive at the result using the expression that they had previously formulated.

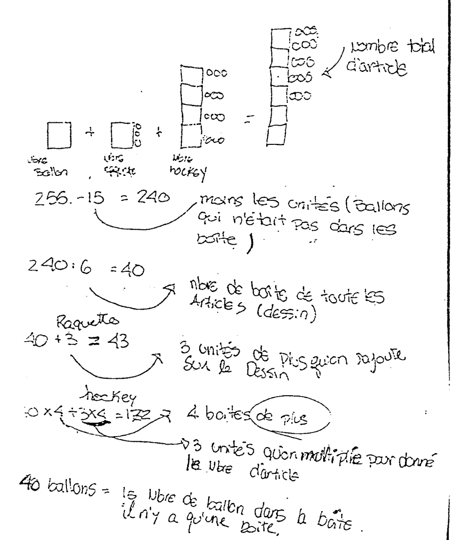
The problem: There are 2 times more rackets than balls and 3 times more hockey than rackets. If there are 270 articles in the warehouse, can you find the number of balls, rackets and hockey sticks?





An other more complex problem: There are 3 rackets more than balls and 4 times more hockey sticks than rackets. If there are 255 articles in the warehouse, how many balls, rackets and hockeys?





These examples also highlight the role that apparently is played by the intermediate notations developed by students, and the possible interplay occurring between the formulation in natural language (the verbal message)—wherein the illustration accounts for both the relationships of comparison between the various quantities—and the symbolic notations constructed by students. These various representations stand out as fundamental components of the transition to an algebraic reasoning and of the construction of a meaning surrounding algebraic symbolism and notations. In this instance, classroom discussion and validation played an essential role in explicating writing conventions elaborated by students (teaching sequences 1 and 3)and reasoning procedures, thereby offering students a means of validating their methods. Finally, from the point of view of the teaching and learning of algebra, this intervention shed light on the possible transition from an approach based on generalization (developed in sequence 1 and at the beginning of sequence 3) to problem-solving (sequence 3). Accordingly, various meanings of the letter were made use of in each approach: generalized number in the formulas elaborated by students to account for a regularity and locate a certain term quickly ; unknown quantity when they solve problems. Various components of algebraic thought were encouraged: the construction of a formula describing a general mode of calculating a certain term, the construction of a general formula to describe relationships between quantities and their linking, capable of being applied to an entire class of problems; in the solving process, analytical reasoning in which students operated on the unknown quantity. Finally, the reflection conducted by students on comparison relations and the influence of the modification of a quantity on the other ones opens on a possible transition to another component of algebra, the introduction of functions, by forcing them to reflect on the status of numbers (generator, parameter).

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Tertiary Algebra and Secondary Classroom Practices in Number and Algebra: Closing the Gap

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Prospective secondary Mathematics teachers seem to consider it a difficult task to establish connections between tertiary algebra and their future classroom practices. We argue that this process is essential in their training, and teacher trainers should dedicate time and effort to help them in this task. In our case, a discipline with such aim was included in the curriculum. In view of the scarcity of reading materials for this discipline, we were lead to write one ourselves. Here we discuss reflections on our practices and on the research results that provided guidelines for the main features of the text.

The Secondary Mathematics teacher-training course offered by our University is the oldest established in Brazil. One of the basic principles underlining the present version of the course is that the sum of solid mathematical formation with basic pedagogical and didactical knowledge is essential in training teachers, but it cannot be considered enough. Teacher training courses should also include disciplines especially planned to allow the students to reflect upon the importance for their future classroom practices of establishing connections between tertiary and secondary Mathematics.

Specific disciplines aiming at establishing these connections have been included in our teacher-training curriculum a few years ago. They are still considered as an experiment and are constantly being reassessed by students, teachers and administration. Lecturers in charge of these disciplines report several difficulties, the lack of appropriate reading materials being considered the most serious. Relying on our practice and on the field research developed while ministering these courses, we have developed a textbook, which aims at establishing the necessary connections between tertiary and secondary levels in numbers and algebra. In this article, we discuss some of the research results that were used to orient the choice of contents to be included in the text.

Background

There were many reasons leading us to conclude that it was necessary to include materials specially designed to highlight the connections between the undergraduate level mathematical disciplines and the subject matters to be taught at secondary school. There are indicators, coming from research, that a solid subject content matter may be essential for a successful teacher (for examples, see Ball and Feiman-Nemser, 1988; Grossman et al., 1989; Stein et al., 1990; Ball, 1991; Leinhardt et al., 1991; Belfort S. Moren, 2000). Nevertheless, Ball and McDiarmid had already taken one step further: they began to investigate the special qualities prospective teachers' actual subject content knowledge seemed to lack in order to foster good classroom practices. Ball (1988) stated that "...[secondary teacher candidates' additional studies] do not seem to afford them substantial advantage in explaining and connecting underlying concepts, principles, and meanings" (pg. 24), while McDiarmid argued that "elementary and secondary teachers frequently lack connected, conceptual understandings of the subject matters they are expected to teach" (pg. 1). Researched texts in this direction are not frequent in the literature. This may be due to the fact that " [to] identify current practices ... is a difficult task because there is usually a gap between what is said about teaching practice and what is really done in classrooms" (Laborde, 1996, pg. 506).

In her book, based on research done for her PhD thesis, Ma (1999) discusses the importance of the "*profound understanding of fundamental mathematics* (PUFM)" demonstrated by some elementary teachers. To explain her concept of PUFM, Ma (1999) needs three preliminary definitions. She defines *understanding a topic with depth* as "connecting it with more conceptually powerful ideas of the subject. The closer an idea is to the structure of the discipline, the more powerful it will be..." (pg. 121). She also conceptualises *understanding a topic with breadth*, meaning "to connect it with those [other mathematical topics] of similar or less conceptual power..." (pg. 121), and she considers that "depth and breadth, however, depend on *thoroughness* - the capability to 'pass through' all parts of the field - to weave them together" (pg. 121). Based on these three concepts, she defines profound understanding of fundamental mathematics as "an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough" (pg. 120).

As a consequence of PUFM, some elementary teachers "do not invent connections between and among mathematical ideas, but reveal and represent them in terms of mathematics teaching and learning" (Ma, 1999, pg. 122). According to her, the work of these teachers displays the following characteristics: connectedness, multiple perspectives, awareness of the basic ideas, and longitudinal coherence.

Our practice teaching subject matter in undergraduate and graduate courses directed at *secondary mathematics* teachers suggests that the difficulties our students face with subject content are of a similar nature (at the very least in our geographical area - see, for instance, Belfort and Guimarães, 1998 and Belfort et al., 2001). Nevertheless, if we add our experiences to other researchers' results, we may at least guess that this may be a widespread problem. At least part of the reason for the poor performance displayed by secondary students in international tests of mathematics (Robitaille et al., 1993) might be related to difficulties some teachers themselves have in understanding in depth the mathematical content they teach.

If we attempt to transpose Ma's ideas to the situation of the study of numbers and algebra at secondary level, we are faced with the twofold connection of this concepts, to the previous knowledge of elementary arithmetic the students would have and, in the other hand, to the algebraic structures studied in tertiary algebra (as, for instance, in the initial chapters in Birkhoff and Mac Lane, 1953). Explanations for and justifications of some of the mathematical procedures studied at secondary school may be provided only if one resorts to knowledge of algebraic structures and properties (a simple example: several of the divisibility criteria often taught in the final years of primary school depend for their justification on the properties of the integer numbers as an integrity domain, and of the general properties of congruence in this set).

So it seems that, if the teaching of secondary algebra is to display characteristics of connectedness; multiple perspectives, awareness of the basic ideas and longitudinal coherence, the teacher would have not only to know tertiary algebra, but also to be able to reflect on its connections with the arithmetic and algebra of fundamental school. Such were the premises guiding the writing of the text for the discipline. But first, it was necessary to assert whether, after sitting for the first discipline on tertiary algebra of the undergraduate curriculum, which deals precisely with the algebraic structures displayed by the numerical sets N, Z and Q, and the generalisation to more abstract structures, the students still needed help or were able to easily establish the connections themselves.

Field Research and Analysis of the Test Results

Looking for experimental confirmation for the degree of difficulty prospective teachers might have to establish the connections between secondary and tertiary algebra by themselves, we tested the students at the beginning of the experimental discipline were this connections would be explored. All the students enrolled had been through at least one tertiary algebra discipline before. The test consisted of two parts. The first had problems that could be solved with elementary integer arithmetic, and for which the students were required to provide more than one solution. In the second part, the questions asked the students to decide whether a set of propositions about the natural numbers was true, always justifying their answers.

Linking the Pre-test Results with Tertiary Algebra Discipline Results

We tabulated the results of the 45 students who took the test, and compared their results against their performance in the first tertiary algebra discipline. Let us consider initially the seven students who were able to present more than one solution for at least three out of the four problems in the first part of the test (that is to say, the fraction of the sample showing the greatest flexibility in dealing with numbers, arithmetic and secondary school algebra). We can point out the following similarities among them:

- They were all able to correctly justify their options in at least four out of five questions in the second (true or false) part of the test; and
- They were all in the top 20% in the discipline of tertiary algebra.

Contrast this with the ten students of the sample who just made the grade in that discipline (seven of them had to sit through it twice):

- *Not one* of them was able to present arithmetic solutions for any of the problems in the first part of the test;
- They have all failed to distinguish the correct alternative in at least two out of five questions in the second part of the test; and
- None justified correctly more than one true false option in the second part of the test.

Such a strong relationship supports the common sense notion that solid and flexible number and algebra models at secondary school are important to furnish a firm basis on which the student can rely to apprehend the more abstract and general structures to be studied at undergraduate level.

Analysis of Students' Answers to Selected Questions of the Test

Question 1, Part I

The statement of the first question of the test was taken from an old arithmetic textbook for secondary schools (Roxo, 1928), used in the most important state schools in Rio de Janeiro for many years:

Find the pair of numbers such that their product is 736 and, when four units are added to one of them, the product of the new pair of numbers is 864.

We requested the students to present more than one solution to this problem.

In table 1, we present a summary of the analysis of the solutions to this first question. Notice that every one of the students chose the algebraic first order system resolution as one of their solutions (one of them made a mistake when modelling the system).

number of solutions presented ➔ types of solutions ➔	1 - incorrect	1	2	total
first order equation system	1	28	16	45
arithmetic, using only natural numbers	0	0	7	7
solutions using integers and/or rational numbers	0	0	3	3
trial	0	0	5	5
other (trial variation)	0	0	1	1

Table 1: students' answers to the first question - part I

On the other hand, 29 students were not able to elaborate any other solution to such an elementary problem. Considering the second solution offered by the remaining 16 students, four different types of solutions were found:

- Arithmetic: seven students solved the problem by arithmetic reasoning. They all justified their answers in the exact same way: "if we add four units to one of the factors, the product will increase by four times the other factor, so the difference (864 - 736) is four times a number (a). The other factor (b) is 736 divided by a".
- Solutions involving operations with Z and Q: these three students justified, step by step, the system resolution they presented as their first solution. In order to do so, they used integers and/or rational numbers. One example: "we know that the product is 763, so we can say that the symmetrical of the product is -736. If we add this number to 864, we obtain 128, and this number is four times one of the factors. If we multiply 128 by one fourth, we discover one of the factors... and so on ";
- Trial: these five students tried out different pairs of numbers whose product was 736. They added four to one of them, and multiplied again. All of them eventually got the correct answer. One of them did a quite organised set of experiments, finding all the divisors of 736 and pairing them. He also estimated the values of the products $[a \times (b + 4)]$ to reduce the number of verifications he had to do.
- Finally, a student presented a unique solution, which can be considered of a variation of the trial solutions: by factoring 736 in prime factors, he found out that 23 was a divisor of 736. By factoring 864 in prime factors, he found out that $3^3 = 27$ was a divisor of 864. As $23 + 4 = 27$, he verified that the pair (23,32) is the solution.

Comments and Reflections (test, part I)

It seems natural and positive that prospective secondary teachers should choose the linear equation as their first option when solving an elementary problem. On the other hand, prospective secondary teachers should reflect about mathematics and its teaching and learning. Considering the research-based reflexions mentioned above, the ability to solve a problem in different ways should be developed in prospective teachers as a way to encourage flexible and consistent mathematical reasoning. The small group of students who seems to be more 'flexible' (Gray and Tall, 1991) would still benefit from the opportunity of analysing different solutions. As for the other students, they do not seem to be aware of the need for flexibility: they know an optimised way to solve the problem and don't bother with alternative solutions. Perhaps we should consider as part as our job to convince them otherwise.

Question 1, Part II

This question asked the students to decide whether the affirmative bellow was true or false, justifying their answers:

If two given natural numbers c and d are divisors of the product $a \times b$ (of two natural numbers), we can conclude that one of the given numbers divides a and the other divides b.

option	total	analysis of the justifications	total
False	37	counter-example	16
		false counter-example	8
		attempted general proof	7
		no justification	6
True	8	failed attempt to prove the result	2
		"proof" by examples	4
		no justification	2

Table 2: students' answers to the first question - part II

In table 2 above, we present a summary of the analysis of this question's solutions. Although 37 students decided that the affirmative was false, only 16 of them were able to present a correct counter-example to justify their answer. All the others had trouble, either in deciding on the veracity of a basic division result or in justifying their reasoning.

The difficulty in justifying suggests that these students either had problems during the previous tertiary algebra discipline (we mentioned above evidence that this is true for a number of them) or are not transferring the acquired knowledge to different situations (suggesting that their knowledge did not become flexible enough).

Some examples of solutions offered by the students show misconceptions in basic mathematical concepts and/or difficulties in mathematical reasoning when justifying a result:

- We classified a justification for the false option as a 'false counter-example' if it went something like this: "we know that $3 | 60$ and $5 | 60$; and we also know that $60 = 10 \times 6$. But $3 \nmid 10$ and $5 \nmid 6$ ";
- Students who made an attempt to prove the falsity of the statement used two different types of justifications: they either made an attempt to use a general reasoning (7 cases), or used blatantly false results (1). One example of the first case: "it is false because it is not possible to say that $c \mid a$ knowing that $c \mid a \times b$. It may happen that $c = m \times n$, with m being a divisor of a and n a divisor of b";
- Among the eight students who who thought the statement was true, four presented an example as means of justification: "we know that $3 \mid 60$ and $5 \mid 60$; and we also know that $60 = 6 \times 10$. We can observe that $3 | 6$ and $5 | 10$".

Comments and Reflections (test, part II)

Different levels of difficulties in dealing with basic number results were observed. This suggests that most students are not applying their recently acquired knowledge in tertiary algebra to think again about the elementary results that they will teach. On the other hand, if these students did not have 'concrete' number models to support their learning of tertiary algebra, the discipline could have revealed to be too abstract for them. There might be that a lack of sufficient knowledge in basic arithmetic was interfering with their learning process during tertiary algebra. If this is so, it is little wonder that many students consider algebra to be one of the most difficult and abstract disciplines in their teacher-training course. Students' opinions after taking the new discipline (which we will exemplify later in this article) seem to support this hypothesis.

When we consider the task of justifying their answers, they demonstrated different levels of mathematical reasoning. Inductive reasoning, incomplete and/or misleading arguments were found among their justifications. On the other hand, if they had had the opportunity to reflect on the very sample of justifications they produced, they would probably understand what is wrong with them - and these reflections might lead to better justifications next time. Again, it seems that there is a job here to be done by teacher trainers.

Closing the Gap*

We defined the main guidelines of a textbook (Belfort & Guimarães, 2000) intended for a discipline aimed at highlighting the connections among secondary number and algebra and tertiary algebra. Special features of the text reflect these guidelines, and it is expected that they will help our students in the intended process. We used a few techniques commonly applied when writing distance education materials to ensure that, in the textbook, no mathematical work is done without a previous discussion about why it is necessary and how it can be related to secondary algebra and number topics. Some features of the text are:

- we offer the student diverse opportunities to work with (and reflect upon) the basic properties of numbers, arithmetic and secondary school algebra contents;
- we use the history of mathematics not only to motivate the students to read the text but also as a source of interesting algebraic and numerical problems;
- we present in the text frequent comments establishing the connections between the topic under study and examples of actual practice in the secondary school classroom;
- every mathematical concept is correctly defined in the text, and possible difficulties in its construction by secondary students are often discussed;
- we punctuate the text with commentaries on mathematical reasoning and justification, always associated to examples taken from the text: what it means to show that a result is false (use of counter-examples) and different methods of proof;
- we present justifications for every result studied, most times including a mathematical proof, and we do not present any formal proof without a previous discussion of the need for it and of the strategy in searching for it;
- we furnish ample opportunity for the critical analysis of texts and questions found in Brazilian textbooks;
- we present several examples of wrong and/or incomplete problem solutions to be analysed by the student; and
- we include sequences of harder supplementary problems to keep the interest of those students who seem to establish the connections we desire with more ease.

The bibliography consulted for the contents of the text is very extensive. It includes undergraduate algebra textbooks, texts on the history of number theory and of mathematics, books on mathematical recreations, research articles on mathematical education, ancient and recent school textbooks, and many others. Even though the text is not yet in its final form, it has

* The notion of 'gaps' between different learning stages can be found in Hart (1987), where she discusses the use of concrete materials in elementary education.

been tested with undergraduate students and also with practising schoolteachers, and we have initial data indicating a positive influence on teachers' subsequent classroom practice.

Conclusions

A critical analysis of the responses of the prospective teachers to this work is still in progress. We intend to analyse their test results and their answers in questionnaires in which they could give their opinion after each of the three thirds of the text (and the discipline). Although the analysis is not yet finished, some issues were brought to our attention by the students' comments on the questionnaire to evaluate the first third part of the course (natural numbers):

- A little over a half of the students said the discipline was well placed in the curriculum (third year). They declared that they had by then become mature enough to understand the importance of connecting tertiary and secondary algebra. On the other hand, the other half said that this discipline would be better placed at the beginning of the course, because it would have helped them in developing models they could have applied during their first tertiary algebra course. They also declared that reading the text and working through the proposed exercises helped them to perfect their understanding of proofs and justifications.
- Five students said (in 'free comments', as this question was not in the questionnaire) that the work with the text had been important to them because it helped them in reading mathematics. Using one student' s words: "whenever I read a theorem now, I ask myself: what is the need for this result in the theory? What arguments can be used to prove it? - And I understand better what I am reading. Consequently, I am also expressing myself better mathematically "
- Students who declared they were already teaching acknowledged that the course had a great influence in their practices. Using one student' s words: "I changed the way I look for activities to be used in classroom. I used to choose activities that I believed my pupils would enjoy, without thinking about its objectives. Now, I decide what I want them to learn, then I choose appropriate activities".
- All students perceived the discipline as needed in the course. One of them said: "At the beginning, I thought it would be an easy discipline, and that I already knew the contents well. I never thought it would be such hard work to understand other people's solutions to problems I know how to solve. I realise now that if I do not make an effort to understand what's going on inside pupils' minds, I can dismiss as wrong perfectly good solutions."

We are well aware that it is an impossible job to close all the 'gaps' for the prospective teachers we have been training, and that was never our objective. Different students will have different backgrounds. Consequently, they will have different 'gaps' to be closed. We can say that our main goal was to convince these prospective teachers that subject content matter knowledge is at the very root of their jobs - and that it is also the most powerful of all tools they can apply as mathematics teachers. To use it effectively, it is necessary to develop a deep understanding of the mathematics they will teach, making it more flexible and establishing connections among different mathematical topics. We hope to have made them aware that this is a never-ending job.

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Algebrafying the Elementary Mathematics Experience
Part II: Transforming Practice on a District-Wide Scale

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We illustrate what it means for teachers to algebrafy their practice by describing a teacher's classroom work and how she gradually transformed her practice to achieve the kinds of classroom practice described. She is among a set of teacher leaders now providing regular support for their peers in as part of a school district-wide initiative to improve mathematics learning. This initiative also involves seminars for 30 school principals co-led by the authors and the district superintendent, and is integrated into the larger district improvement plan mandated by the state of Massachusetts due to the district's poor performance on state examinations.

Introduction

In this companion paper to the content-focused paper submitted by Kaput & Blanton, we first provide some concrete illustrations of the kinds of classroom practice that represent the goals of our work with the teachers who are implementing the "algebrafication" strategy described in the companion paper. Of particular importance is the need to render the algebrafication of teachers' practice *generative and self-sustaining,* and not dependent on continuing engagement with teacher professional development activity or pre-constructed resource materials that teachers "implement." Our goal is that teachers become independent developers of algebraic reasoning in their classes using whatever material resources that they can find, adapt or generate.

We begin by sharing excerpts from one of our 3rd grade teachers' reflective writings ("Jan," a pseudonym) that detail a selected set of algebra-rich problems and the classroom scenarios in which they occurred. Not only do these vignettes illustrate how she is algebrafying mathematical tasks, they also point to the emerging instructional strategies that she uses and, as a result, the development of a classroom culture that supports the processes of justification and deliberate argumentation that accompany acts of generalizing and formalizing underlying the development of algebraic reasoning in its several forms. We will follow these excerpts with discussion and interpretation in terms of our three dimensions of algebrafication.

The last part of the paper is a brief description of how we are implementing the algebrafication strategy on a district-wide basis, involving 30 schools in ways that ensure that it is *systemic* and hence will become a permanent feature of elementary school practice.

Illustrations of Classroom Practice in the Process of Being "Algebrafied"

Following are two excerpts of a 3rd grade teacher's reflections on her classroom work, roughly 5 months apart, as she is coming to terms with the strategy, relatively early in her development. This illustrates not only what we mean by our algebrafying strategy, but also how it begins to emerge in a teacher's practice. Space limitations prevent a fuller inclusion of the student talk and artifacts that lie behind these reflections.

Jan's Reflective Writings—An Excerpt (Spring, 1999)

Multiplication Tables. I was teaching the 5 table. The children of course find this table very easy, so I figured this would be a good place to start to find missing factors. I gave the class the problem '5 x a = 20'. John immediately raised his hand and gave the answer "4". I asked him how he knew the answer so quickly. He answered that he knew that 5 x 4 = 20, so the answer had to be 4. I then asked if you did not know this, how could you find the answer? Jeff responded with 'count by 5's until you get to 20'. I asked for any other ways. Dan said that he could find the answer by subtracting. I asked him to explain his thinking. He said take 20 – 5, 15 – 5, 10 – 5, and 5 – 5. Then he said to count the number of times you had to subtract 5. There were 4 times that you subtract, so the missing number had to be 4. I was impressed!

Missing Numbers. After doing the problem 5 + 8 = __ + 9 (adapted from Carpenter, personal communication—see the accompanying Carpenter/Franke submission), I decided to make up more of these kinds of problems for them to do:

$\Delta + \Delta = 6$

$9 + \Delta = 12$

$\Delta =$ ___

This activity started out with just about every student asking how to do this. I wanted to see what strategies they would use to figure out the problems, so I only told them that the triangles had to have the same number. Some of the children did guess and try. They did not understand that the numbers had to be the same. Then all of a sudden a few of them noticed that if you did the doubles first, it was much easier to get the remaining number. I asked if their answers made sense. They didn't like the idea at first to check it on their own, but after a while they felt good about knowing their answer was correct.

Introducing "Letters". I wanted to get the class started using letters to represent a number or numbers. So we talked about what we could use to replace a number or an empty box. After some discussion, someone came up with the idea to use a letter. I put 1 x b on the board. John finally figured out that the answer was b. I then asked what other number sentence could I use and another student answered b x 1 = b. So then we started to replace the b with all different numbers and multiplied the number by 1. They used a calculator to check these problems. I then asked them to make up a rule that they thought they could always follow. They had to write the rule. Every one of the students was able to say in their own way that any number times 1 always equals the other number.

I tried the same thing using 0. I wrote d x 0 on the board. Because they had done the problem with the 1, they knew that d x 0 = 0. Again, they were able to see that 0 x d = 0. We once again replaced the letter with any numbers. The numbers got bigger. (This brought in them reading large numbers for a review.) Again, they used the calculator to check. The first time they put in their large number, Dan announced loudly, "Hey, that '0' took away my number!" So then we tried adding and subtracting numbers on the calculator before multiplying by 0 (e.g., 569 + 222 – 123 + 256 x 0 = 0). They were quite surprised to see that everything they did became a '0' when they multiplied by 0. So we took it a step further and replaced the (nonzero) numbers with letters: A + B = C, C – D = E, E + F = G, G x 0 = ?. They all knew the answer would be '0'. They are having a lot of fun with numbers!

The 100th Day of School Challenge. On the 100th day of school, I challenged my 3rd grade and Julie's 5th grade to come up with as many number facts as possible that would equal 100. This had a few of my students very excited. Many of the students started to randomly come up with the facts that equaled 100 in no particular fashion. However, 2 boys who do not appreciate homework went home and started making up problems (without being assigned to do so) to put on the chart. This was rather interesting. One of the boys started making problems this way:

1+99 =100; 2 + 98 = 100; 3 + 97 = 100.

He continued this pattern all the way to 50 + 50 = 100. Another boy, without knowing what the other student had done, had a similar response:

200 – 100 = 100; 199 – 99 = 100; 198 – 98 = 100.

I thought it was interesting that the boys discovered a pattern to follow, but did so in different directions.

In order for the children to put their facts on the chart, they had to check to see if the problem was on the chart already. They found the approach used by the 2 boys easy to check their facts against. Adding competition with the 5th grade class really motivated my 3rd grade class!

A Second Excerpt of Reflections 5 Months Later (Fall, 1999)

Using a Balance Scale to Interpret Missing Addend Problems. As an introduction to finding missing addends, we used a balance scale with 5-gram weights. On one side of the balance there were 9 weights. The other side of the balance had 5 weights. The children were asked if there were the same number of weights on each side. They were able to see right away that there was a difference because the scales were not "even". When asked how to make them even one student said to take away some of the weights from the side that had 9. The student kept removing weights until the balance was even. Another student then saw that you could add to the side with fewer weights until the scales were even. They then counted how many more were needed to make 9. The class was then asked to come up with a number sentence to model the scale. They needed a little help with this. We continued using the balance having the children make up their own problems. Doing this, Jordan came up with the idea of what if we didn't know how many weights we had. To him, it wasn't important except to make both sides equal. In thinking about his approach, it made me think that he is already thinking algebraically. He was working with the unknown and it didn't faze him.

They understood what the '=' sign means. When writing a number sentence such as 9=?+5, they were able to arrive at an answer. What I think is so significant about this is not one of the students said that the problem was written incorrectly. In the past, if the answer were written first, many of the children would be confused. In this case, the number sentence was written to mirror the balance.

I am now finding more algebra in what I have taught for many years, but have not realized it. Now, however, I feel that I am getting better at bringing out more with the class by the types of questions that I ask. I find myself being very fascinated at the math language that my third graders use. Listening to them think, talk, and explain to others has also been a help to the way I now try to teach. The children are so excited about math. I think sometimes the best lessons in math are not necessarily taught, but investigated. Kids need the chance to explore and not always be shown what to do. Try to let them find out on their own. It has more meaning for them.

<u>In-Out Charts</u>. I gave the class an In-Out chart. First we started by having them make their own charts. We played "Guess My Rule". We discussed if the 'in' number was greater than the 'out' number. Then we had to subtract to get the answer. If the 'in' number was less than the 'out' number, we had to add. These charts went right along with finding the missing addends. It was interesting to hear how they arrived at the their answers. Stephanie said that she added 3 to 10 to get 13, then she thought she might have subtracted 3 from 13. Then she realized it didn't matter. Ashley added that she counted from 10 to 13 and then said that that is really subtracting 10 from 13. The more these children are talking and explaining how they are getting their answers, the more they are willing to share their answers. This is good because it lets us know how they are thinking.

<u>The Handshake Problem.</u> Today we tried the Handshake activity. To make this more manageable, we decided to break the class into groups of 3's to start. We just explained that they had to shake hands with each person in the group only one time. The children had difficulty understanding this. After a few minutes, we decided to have a group of 3 students model this activity. While the group was shaking hands, the others were getting restless, so we had another group of students record responses on the board. I found it necessary to label the handshakes on the board. This really helped. They looked up to see who had already shaken their hand. In the first group, I had Kevin, Shawn and Jordan. Kevin shook with Shawn, so we called that KS. Kevin shook with Jordan, so we called that KJ. Shawn shook with Jordan and we called that SJ. The recorders wrote 3 people: 2 + 1 = 3 shakes. We then added another person to the group and continued recording in the same way and got 4 people: 3+2+1 = 6 handshakes.

We continued adding one at a time to the group. It was important for them to see the pattern that was forming. I also wanted to know why there was always 1 less than the number of people shaking. I was surprised that they were able to say that you cannot shake hands with yourself.

One of my students, Shawn, then asked us what if there were 20 people in the group. This surprised me because it was a question that usually I would have asked. He then went to the board and wrote that the number of handshakes would be 19+18+17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1.

We were impressed! In the meantime, Andrew was off with another group. His group was working nicely and seemed to be able to figure this out. Although they didn't record their responses in the way we did, they saw very early on that when you added another person to the group, you only needed to count the number of handshakes for that person with the other group members. Then, you could just add this amount to the previous figure. It was good to see that they were counting on, which was an issue that had come up recently when we worked on patterns in triangular numbers.

Reflections on Jan's Reflections

The Matter of Content—The Five Aspects of Algebraic Reasoning

First, regarding content, it is apparent that each aspect of algebraic reasoning outlined in the companion paper except #2 (syntactical manipulation) appears in the classroom episodes, and further, that generalization and the explicit expression of that generality is at the heart of all of them. Note that most episodes combine different aspects of algebraic reasoning, which reflects the nature of algebraic reasoning more broadly—that its aspects are most fruitfully used in combination. Hence, in the last episode regarding the Handshake Problem, we see modeling coupled with an examination of the structure of an arithmetic sum when one student generalized not the numeric pattern, but rather the forms of the unexecuted sums to write the answer for a group of 20 people. Not recounted here is an extension of this activity that involved regrouping these sums into pairs: (1+19)+(2+18)+... In effect, the class was able to work with these arithmetic sums in an algebraic way—basically engaging in the syntactic aspect of algebraic reasoning as well.

Not recounted above is another episode where a student focuses on computing 5+12 to determine if the resultant sum is even or odd. What Jan did next (spontaneously) with the student was inherently algebraic. By using numbers (45675 and 23675) that made it impossible for the child to perform the computation, the teacher forced the child to focus the properties of evenness and oddness and to implicitly treat the numbers as placeholders—again, an example both of using numbers in an algebraic way as well as dealing with the structure of the number system—aspect # 3.

The Algebrafication Strategy Reflected in Jan's Practice

We now attempt to identify a few characteristics of Jan's practice from these writings that show how she is coming to terms with the algebrafication strategy (the fuller paper will include much more data on which such reflections and conclusions must be based). A critical characteristic of Jan's practice is her seamless extension of arithmetic conversations to algebraic conversations, reflected in almost all the examples she described. Rather than being bound by a prescribed "algebra activity," she is able to "algebrafy" what could otherwise be an ordinary arithmetic conversation. We have observed this on numerous occasions in Jan's class and, based on teachers' written and oral reports in the after-school seminars for the teachers, we suspect that the same occurs in the classes of many of the other teachers as well). In effect, she is able to algebrafy tasks and activities whose origins are in arithmetic. We also see an element of spontaneity, a somewhat "on the spot" algebrafication—to us, an indication of the developing of a teacher's "algebra eyes and ears."

Another characteristic of teachers' practice is reflected in the ways that teachers such as Jan are able to *generalize an activity*, what we refer to as "activity engineering." That is, the teacher doesn't depend solely on activities used in our sessions with teachers (note however, that Jan was familiar with the Handshake Problem because it had been used in one of our after-school seminars for teachers). But there is an explicit ability to cull other resources and adapt them to their particular grade level that expand on a common theme, such as a particular number pattern or algebraic processes involving missing number sentences since none of the problems discussed was in her textbook in the form that it was used in her class.. It turns out that another teacher ("Julie," the 5th grade teacher mentioned by Jan) instigated a significant mathematical experience for her seminar peers when she brought in a geometrically-based pattern problem (counting the number of embedded equilateral triangles in an equilateral triangle made of unit-triangles described in Kaput & Blanton, 1999). This turned out to be an important benchmark in prompting teachers to think about how to expand a given activity. In fact, one of the implicit goals of our work (and for us an indication of generative teacher change) is to move teachers beyond direct reliance on tasks we provide.

We also see this characteristic of generativity manifested in Jan's classroom as the spiraling of algebraic themes over significant periods of time (this has been on ongoing phenomenon throughout our observations although not well-reflected in the excerpts above due to space limitations). In our preliminary analysis, we have found some of these themes to include conversations about even and odd numbers (such as the one mentioned above), the development of students' symbol sense, and the use of missing number sentences to model a variety of mathematical situations.

Another form of 'integration' that we find noteworthy in terms of illustrating how a teacher's practice becomes algebrafied in a sustaining way is the integration of aspects of algebraic reasoning. In other words, while doing what might be considered a stand-alone algebraic activity, the teacher pulls into this another algebraic process in a very natural way that alters the complexity of the original task and, in essence, pushes the "algebra envelope". This occurred in a class where she asked students to use Base-10 blocks to solve missing number sentences, a task that is itself algebraic in nature. After a discussion with students in which they shared the different strategies they used to solve the problem, she focused on the sentence '14=6+n' and expanded this problem in the following way. First, she asked students to solve '140=60+n', then '1400=600+n'. After another discussion about how students had arrived at their solutions, Jan turned solving this family of problems into a pattern finding activity, thus superimposing on an open number sentence activity an additional layer of pattern expression, indicating an ability to freely combine aspects of algebraic reasoning in her activities. We take this to be an indication that she is internalizing the aspects of algebraic reasoning into her practice in a deep way.

Finally, while it is very difficult to capture in a few excerpts of teacher reflections the nature of the classroom culture that her practice fosters, it should be apparent from the illustrations that active inquiry and generalization were deep aspects of her classroom practice reaching beyond, yet intimately linked to, algebra-rich content and curriculum activities. Video excerpts will be used to illustrate this characteristic of her algebrafied practice more fully.

Taking Algebrafication to an Entire City School District

We will describe our work with teachers in after-school seminars, the process of building teacher-leaders who then run their own seminars, and then the Leadership Academy for school principals and the connection between this initiative and the broader district improvement activities.

The Nature of Teacher Seminars

The three dimensions of algebrafication are explicitly addressed in year-long after-school cross-grade seminars in the kinds of activities (and additional collaborative work as described in the district context below). The seminars include the following kinds of activities.

- Teachers examine their own texts and instructional materials for problems and activities that can be the basis for developing, discussing, and justifying generalizations. We begin by offering the teachers a series of increasingly complex examples that they themselves work on in small groups and then share their solution strategies. They are then asked to examine their texts for such beginning during the seminar and then for "homework." Since some of these are designed into their text and identified as such, they are readily able to collect these for discussion in the next seminar. Repeatedly, and from the outset, teachers are asked to try the problems with their own students, not as an enrichment, or pull-out activity, but as a mainstream instructional activity. (Most teachers for whom this represents a very different kind of practice understandably treat it as separate from their "real" teaching.) We examine in detail what the students might learn from these problems, what number sense they might develop, and what expressive skills they might be developing. An important feature of this subsequent discussion is comparing how a given problem or problem type takes different forms at different grade levels, and in turn, how students' skills and understandings as well as developmental levels progress across the grades. This grounding of reflection and action in the teachers' existing instructional materials base is continued through the entire seminar and has become a critical component in raising teachers' sensitivity to how actions in their own classrooms are significant for other grades.
- Teachers are asked to "algebrafy" existing problems—basically, to transform arithmetic problems into activities that support the development of algebraic reasoning skills, both by creating patterns and by relaxing constraints of arithmetic problems (see the accompanying paper). Again, they are scaffolded by some initial examples done in the seminar, asked to modify them for their own grade level, and try them in their own classes. They are asked to report on the resulting classroom activity in the next seminar.
- Teachers read and discuss excerpts from teacher-writings and similar writings by Schifter, Ball and others—as another way of reflecting upon what they are doing, both in class and in the seminar. An interesting activity for the teachers is to reflect on how what we are doing is different from what is reported in the accounts that they are reading (mainly the explicit algebrafying and the contextualization within the development of algebraic reasoning) and how

it is similar (active roles for students, centrality of communication, classroom culture of inquiry, teacher attention to student thinking and how it develops).

- After the first 4 seminar meetings, which begin to build confidence, a community and common understandings, we enter the heart of the seminar, which involves the teachers in working on carefully chosen and mathematically challenging mathematical generalization and formalization tasks. The intent of these problems is to immerse the teachers in a substantive mathematical learning experience that can then serve as the foundation for deeper, experientially based discussion and understanding of the processes of generalization, expression of generality: What different forms can it take, ways a specific instance embodies a generality, how the experience of expressing a conjecture or communicating it to another affects one's understanding of it, especially through the use of different representations or language, how one's computational skills were engaged or improved, how we use or see arithmetic statements in new ways, and how different peoples' conceptions of the problem differ and interact during group work. These kinds of questions are explicitly addressed in the second key step: follow-up discussion in the seminar as solution strategies and actual solutions are discussed, compared and contrasted. The third key step is the teachers' assignment to reformulate the problem for use in their own classes. The fourth key step is to report in the following seminar on student work on the problems based on student artifacts and to discuss the differences in the problems across the grades, the different strategies within a classroom, how these approaches were similar or different from the teachers' approaches, and how we might alter the problem, if at all, *for use by other teachers.*
- This 4-step cycle is repeated for new challenging problems for much of the remainder of the seminar, deliberately linking the teachers' mathematical development to their regular practice, with a side-product being a rich source of materials, from their textbooks, and from other sources that they can use in subsequent years.
- During the last third of the seminar, attention is increasingly focused upon the teachers' classrooms as a whole—the 3rd dimension of algebrafication. In particular, is a culture of inquiry developing, do students question each other and expect justification of mathematical statements as a standard way of operating, are they learning communication skills, both oral and written, are there differences across classrooms or (when relevant) schools, and how do the teachers themselves feel about the evolution of their practice?

Rendering the Innovation Systemic—The Leadership Academy for Principals

Following upon the pattern of activities and building on the professional development resources constructed over the past two years, our aim over the next three years is to engage every district K-5 teacher (approximately 350 in all) in a series of after-school seminars running approximately every other week for a year. These are run by us and the teacher leaders from prior work and by additional teacher leaders from the pool of seminar participants given additional training in the coming two years. They use a set of resources developed in the initial seminars, but these are treated very explicitly and deliberately as a work in progress to be added to and improved for use by others, *not* a closed resource to be implemented. Most, but not all, seminars are school-based and led by a teacher from that school. In some cases, for practical reasons seminars are offered outside the particular school in which its participants teach.

Importantly, since this entire effort is part of an overall District Improvement Plan (sanctioned by the State Department of Education due to low student performance), the seminars are coordinated with a Leadership Academy for all K-5 principals and curriculum coordinators co-directed by the authors and the district superintendent which meets once per month through the school year during the school day. One goal of the Academy, beyond helping principals understand and support the algebrafication strategy in evaluation and hiring as well as day-to-day supervision, is to assist them in redesign of the school day to create time for teachers to collaborate and extend professional development activities during the school day on a permanent basis. Another is to enable principals to promote "Math Literacy" on a school-wide basis, patterned on a district-wide literacy program and to integrate it with other district initiatives and imperatives so that it is truly a systemic change in the district.

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School algebra:
Primarily manipulations of empty symbols on a piece of paper?

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The paper is based on students' written responses to selected items in a large-scale assessment project in Norway. Findings related to students' knowledge of the symbolic notations in algebra are discussed. The findings are related to the Norwegian tradition to focus extensively on executing rules for symbolic manipulations. A general interpretation of the data collected is that the main outcome of school algebra for a large proportion of student is that they have tried to learn rules for manipulation of symbols.

This paper is based on a large-scale assessment project, KIM, initiated and funded by the Norwegian Ministry of Education. It involves mathematics from grade 5 to 11, where assessment is used as a basis to aid conceptual development. Two of the objectives of the project are to develop:

- integrated test- and in-service training packages that can be used by teachers as part of their assessment practice
- a collection of test-instruments of diagnostic character, which can be used as a staring point for teaching practice within various parts of the subject matter.

KIM has developed sets of diagnostic test items for various parts of the mathematics curriculum, and thus intended to cover most of the key concepts of school mathematics when the project is finished. Written materials for teachers are produced for each set of diagnostic test items. The main focus of these materials is on reporting and discussing the extent of misconceptions and conceptual obstacles identified by the diagnostic test items. The items are developed to enquire into different aspects of the particular concept in question. Teaching activities are designed to help teachers address specific difficulties encountered.

In selecting diagnostic items for the KIM project the main intention was to inform teachers about several well-known conceptual obstacles in students' algebra learning. Several of these items have been used in previous studies, for example the CSMS study. Küchemann (1981). The research perspective had a second priority. The data reported in this paper ar based on written responses to diagnostic items from respectively 1805 grade 6 students from 100 classrooms, 1953 grade 8 students from 91 classrooms and 1957 grade 10 students from 90 classrooms. The average age was respectively 11.5, 13.5 and 15.5.

Introduction

There is a large body of research literature about the teaching and learning of algebra. It documents students' difficulties in grasping fundamental aspects of the notations used. This involves how to write simple expressions and equations containing variables, numerals, operation signs and brackets. (e.g. Booth, 1984; Herscovics, & Linchevski, 1994; Janvier, 1996; Kieran, 1981; Küchemann, 1981; MacGregor & Stacey, 1996; Stacey & MacGregor, 1996 and 1999.). One aim of the KIM project was to make teachers aware of learning outcomes of an extensive emphasis on symbolic manipulation in the teaching of algebra. A national survey of Norwegian students' understanding of different aspect of algebra was needed to promote a change in teaching. The teaching activities linked to the national survey are designed to help students to close the cognitive gap between arithmetic and algebra (Herscovics, & Linchevski, 1994).

Issues in the transition from arithmetic to algebra

The priority of multiplication and division

The intention of three of the items of problem 1: *Write the correct number to go in each of the boxes:* 1b: □ · 2 + 4 = 12, 1c: 3 + 2 · □ = 15, and 1d: 25 – 2 · □ = 17 is to focus on students awareness of the priority of multiplication compared to addition or subtraction. Respectively 61, 77 and 88 percent of the students answered item 1b correctly. The most common wrong answer was 2. These answers are probably a result of first adding 2 and 4, and next multiply 6 by 2. Table 1 illustrates this strategy is much more prevailing in 1c. One additional reason for the large increase of incorrect answers could be the tendency to perform calculations from left to right. The answer 5 indicates that these students add 3 and 2 and write the result in the empty box.

Item 1c	G. 6	G. 8	G. 10
No answer	4	4	2
6 correct	13	17	33
3	67	73	63
5	8	3	1

Table 1. Percentages of correct, and most common incorrect responses. Item 1c.

The idea of priority amongst operations is even less transparent in an algebraic setting than in numerical calculations. It is the author's opinion that this has to be specifically dealt with in arithmetic to be applied in an algebraic setting. That students do not manage this convention for numerical operation is presumably because it is difficult to understand, but rather that this is usually not specifically dealt with in arithmetic teaching.

Students' use of symbols

Lack of closure and evaluate a specific unknown

Item 4b of the KIM was taken from Küchemann (1981): *If e+f = 8 then e+f+g =* .... This item can be solved by matching. Table 2 demonstrates that operating with *g* as a specific unknown caused problems for the majority of our students. Basically the students tried to solve this problem of symbolizing in two different ways. One was to join the number 8 and the specific unknown *g* and the other were to evaluate *g* in different ways.

Item 4b	G. 6	G. 8	G. 10	CSMS
No answer	27	23	13	-
8 + g or g + 8 correct	4	8	38	41
8g, 8·g or similar (joining)	4	11	27	3
9	21	16	3	6
12	18	16	7	26
Other numbers evaluated	21	22	8	-

Table 2. Percentages of correct and most common incorrect responses. Item 4b.

The CSMS study involved fourteen year olds, and would on average be half a year older than our eight grade students. It should also be noticed that the formalistic teaching of algebra starts in grade 8. Note that CSMS students performed better than our grade 10 students. In our sample there is a large, and increasing, proportion of "joiners" as student grow older. We also notice that there are different "popular" ways of evaluating g between the two samples. Apparently, near the end of compulsory school, grade 10, students stop evaluating variables, but the part of students who have problem with the lack of closure of algebraic expressions are growing. This is an opposite change by age compared to CSMS results shown in table 2. Similar lines of thinking illustrated by the responses to item 4b are also observed for other items used in of the KIM project, for example items 3b: "*Add 2 and n+5*" and 3c: "*Add 4 and 3n*". Respectively 4, 9 and 37 percent of the students answered item 3b correctly. The corresponding figures for item 3c were 4, 7 and 46. Two types of joining, avoiding lack of closure, were observed in item 3b: $2n + 5$, joining 2 to n, and $7n$ or $n7$, adding the numbers and join the variable. The percentages of joiners on item 3c were respectively 63, 69 and 39, compared to 31% in the CSMS study.

Simplifying expressions

The test contained eleven expressions to be simplified for grade 8 and 10. We will first consider the following items, 8b: $a + 4 + a - 4$; 8c: $x + y - x + y$ and 8d: $(a + b) + (a - b)$. The facility level of these items was respectively 33, 12, and 8 percent for grade 8, raising till respectively 79, 57, and 40 percent for grade 10. A relatively small numbers of "joiners" were observed for all of these items, just below 10% for grade 8 and around 4% for grade 10. Notice that the careful professional progression from item 8b to item 8d has a large effect on the responses. The most common incorrect answers to item c was 0, $0x + 0y$, $0xy$ or similar, respectively 21% and 16%. They probably treat $x + y$ as *one* object. It is more difficult to explain the reason for the response 0 or $0a$ (12, and 5 percent). Two thirds of the grade eight students who answered 0 on item c also gave the same response to item d.

The power notation is also brought into play in grade 10 students' responses to the items above as well as to item 8e: $3a - (b + a)$, presumably because this content is currently taught at this grade level. In addition the conjugate equality in algebra pops up in many students' responses to item 8d, as illustrated in figure 1 below.

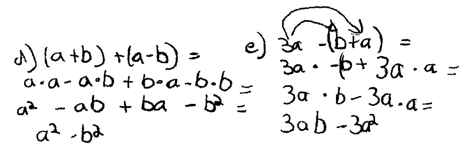

Figure 1. Example of wrong application of the conjugate equality.

The power notation is employed by between 6 and 9% of the grade 10 students on items 8b to 8e. In addition the conjugate equality is employed by 15% on item 8d.

The most difficult item to simplify was item 8j: $\frac{4x+2}{8x}$. We found more than 40 structurally different responses to this item, for example the difference between numerator and denominator, or only between the parts of the numerator that contained the unknown. Only 2% of the tenth grade students gave a correct response. Some students answered 0.75 (or equivalent), dropping x and cancel, others $0.75x$, dropping x, cancel and introducing x again. Others operated as illustrated in figure 2

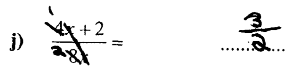

Figure 2. Example of simplifying of expressions.

Other students cancel $4x$ by $8x$ to get 0.5 and then add 2 to give the solution 2.5, while others again introduce the x again and write $2.5x$. Another alternative is illustrated by figure 3:

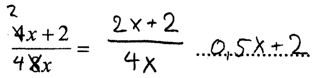

Figure 3. Example of simplifying of expressions.

$4x + 2$, $-4\ x + 2$ and $\frac{2}{4x}$ are all examples of subtractions between nominator and denominator. Two percent of the students managed to cancel "everything" and answered 0, and seven percent, nearly four times as many managed to find a correct solution, cancelled as illustrated in figure 4.

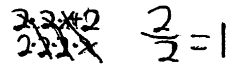

Figure 4. Example of simplifying of expressions.

Letter used as a generalised number.

Problem 13 below is used to discuss students' comprehension of how letters are used to represent a generalised number. The format is probably unfamiliar to many students, but still gives valuable information of typical interpretations.

Problem 13: Tick the box for the correct answer:

a. $a + b + c = c + a + b$

☐It is always true ☐It is never true ☐It is sometimes true

Explain the reason for your answer.

The same questions were asked to the following identities:

b. $4 + x = 4 + y$

c. $2a + 3 = 2a - 3$

d. $1 + m + n = 1 + p + n$

Table 3 show the distribution of the ticks (correct ticks are indicated by *).

	Non response	Always true	Never true	Sometimes true
a. G. 8	12	*39	12	33
a. G. 10	7	*65	9	18
b. G. 8	14	11	23	*31
b. G. 10	7	4	54	*34
c. G. 8	15	9	*60	14
c. G. 10	9	4	*72	14
d. G. 10	12	5	49	*33

Table 3. Percentage distribution of students' responses to problem 13.

The qualities of students' explanations are not considered in table 3. Students who ticked but not explained were also registered in table 3. Grade 8 students gave fewer explanations than the older students. Among those who ticked for the correct alternative it was respectively 31%, 58% and 43% who did not give any reason for their choice for the three items in grade 8. The corresponding percentages for the four items in grade 10 were 18%, 23%, 29% and 25%. The percentages of none explanations amongst those students who ticked for a wrong alternative are much higher than those above. We notice also that the two items (b and d) where the identity *could* be correct under certain conditions are more difficult for both grades. It is conspicuous that the alternative *never true* is more attractive than the correct alternative in these cases. One reasonable explanation could be that students have much less experience with = as a symbol of equivalence than as a symbol which separate a calculation task from its solution. A further discussion of this will follow below when analysing types of student explanations.

The most common correct explanation to item a was that the order of addition is indifferent. Two third of the students who made a correct tick gave such reasons. But there were also explanations that indicate diffuse ideas. For example: *"It is the same, but usually they comes in* *alphabetical order. When we add it is the same which factor comes first".* (All explanations are direct translations). Some use one or two numerical examples, implying that these are general, such as: "*The answer is always the same because if you add 1 + 2 + 3 = 6, and turn it around you get 3 + 2 + 1 = 6."* Others refer to that the letters are the same, which could indicate that they interpret the letters as objects.

The most common incorrect explanations are based on that the order of the variables is changed, for example: It is always true because "*It is correct that the letters should be separated, but they have to come in alphabetical order a + b + c".* Other students made connections between variables and reading, it is never true because *"a + b + c is not the same as. Example: bil(car) is not the same as hus(house)."* Others again conceive letters as objects. Figure 5 shows an example (in Norwegian). It is never true: "*because a+b+c = abc and c+b+c = cba these are totally different things."*

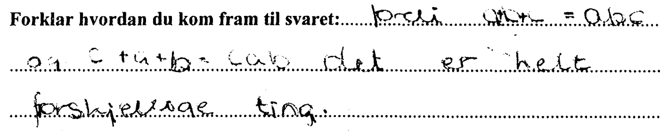
Forklar hvordan du kom fram til svaret: fordi a+b+c = abc og c+b+c = cba det er helt forskjellige ting.

Figure 5. Example of student's response to item 13a. (In Norwegian)

For all the items of problem 13 we find that many students conceive the equal sign as an operator, a signal for a numerical operation or to perform a procedure. The "task" is to the left of the equal sign, and the solution to the right. It is never true because *"When it says a + b + c the answer is abc.* Amongst the students who ticked for *sometimes true*, respectively two third and 80% in grade 10 and 8 gave no reason for their choices. The majority of students who gave reasons refer to the order of the letters involved as in Figure 1.

Item b turned out to be much more difficult. We notice that a considerable larger part of the grade 10 students believe that this equivalence can never be true, than those who understand that it can be fulfilled under certain conditions. The most common correct type of explanations refer to x and y as general expressions for numbers that *may* have the same value. Explanations such as "*It depends of the values of x and y", "If they have the same value it is true, if not it is never true.*" are typical correct explanations. One student wrote: *"x and y have usually different values, but they may also be the same by chance in an equation".* It was respectively 9% and 19% of the students who gave these types of explanations. More than two thirds of the eight graders and thee quarter of the tenth graders who ticked the correct box *and* wrote an explanation apply this type of reasoning. A few student, 1%, based their reasoning on one or more numerical examples that fulfilled the equivalence.

The most common reasons among those who ticked for *never true* are related to that x and y **have to** represent different numerical values **because** they are symbolised by different letters. Answers such as "*x and y are different in any case*", "*4 + x and 4 + y cannot be the same because x and y never stand for the same number"* and *the values of x and y are not equal in the <u>same</u> task".* Students' who consider variables to represent objects arrive at the same conclusion, for example *"x is one word/thing and y is another word/thing".* Respectively 12% and 29% of all the students of the two gave reasons of such types.

As for item a, we find several examples of explanations that reflect the interpretation that the identity represents a calculation task to the left and the solution to the task to the right of the equal sign. For example: *"The solution to the task cannot be true because 4 + x becomes 4x and <u>not</u> 4 + y", "Because one has to have a y in the task to get a y in the solution"* and *"Where does this y come from? One cannot suddenly bring it in when it wasn't mentioned in the task".* On the other hand, the explanation *"Because it is an equation, and equations are always equal on both sides,"* indicates a correct interpretation of the equal sign, but the reasoning is associated with the symbolisation rather than with numerical values of these symbols.

Table 3 shows that 11% and 4% ticked for *Always true*. Around half of these students explained their reasoning. Most of these students argue along the same line: since it is indifferent which letter one chose to represent the variable, the identity will always be valid.

It is obviously easier to establish a correct content knowledge to item c than to the other items of problem 13. For grade eight, the differences between c and the other items of this problem are especially large. One reason could be that one can apply other strategies that were not applicable in the previous items since they involved more than one variable. It is here possible to apply some of the lines of reasoning that were not possible in the previous items. All the same it is a large proportion of the students that just tick one of the alternatives. Respectively 43 and 29 percent of those who ticked for *never true* did not explain their reasons. The following line of reasoning is linked to the outcomes of an addition and a subtraction *"2a is the same number on both sides and cannot add a something on one side and subtract on the other side to keep it the same".* Some students, respectively 26% and 41% of all students who wrote an explanation, focused only the signs of the equality for example, "*+ and – are not the same".* In a few cases, 1% and 4%, based their argumentation on one or more examples. Few students who ticked a wrong box gave an explanation for their choice. The only error type that could be traced to a systematic line of reasoning can be illustrated by the following response: *"When carry the numbers across = you have change sign"*, referring to a procedure for solving equations. This line of reasoning was used both by those who ticked *always true* and those who ticked *sometimes true.*

Item d was given only to tenth graders. It has the same structure as item b. The identity has three variables on each side of the equal sign, and two of the variables occur on both sides. Even though it is possible to use the same argumentation as in item b, one would expect that this item should be more complicated, both because of the number of variables and that the identity does not contain an explicit number, but table 3 shows that roughly the same number of students tick for the correct option. Likewise it is about same number of students who tick for *never true*, 54% and 49%. Comparing the individual responses to these items we find that 83% of those who ticked *never true* alternative for item d did the same for item b and equivalent, 77% of those who ticked *sometimes true*, did the same for item b. It should be noted that many of these students do not explain their choices, but respectively 85% and 86% of those students who explained their ticks for either *never true* and *sometimes true* used similar reasoning on both items, which indicates consequent thinking.

Conclusion

A general conclusion from the analysis of our data is that the majority of students apply arbitrary, but to some degree consequent, procedures in transforming algebraic expressions. These are usually linked to bits of partial understanding of arithmetic, but also procedures in other fields of mathematics. Their procedures become in this way isolated and therefore difficult to apply in unfamiliar situations. It is my opinion that the best way to overcome this is to strengthen the links between arithmetic and algebra.

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LINEAR EQUATIONS AND INTRODUCTORY ALGEBRA

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Although a functional approach involving patterning is now commonly used to introduce algebra in schools, research studies have indicated a range of difficulties encountered by students with such an approach. An alternative is a problem-solving approach that involves students solving linear equations. While this approach reflects the historical development of algebra, it has traditionally posed difficulties for many students particularly when the unknown occurs on each side of the equation. This paper illustrates an approach to introductory algebra that involves students solving linear equations in which the unknown occurs on each side of the equation. There is evidence that with this approach, students do not experience the difficulties that often result from traditional approaches.

Deciding how to introduce beginning algebra is a difficult task for any teacher. They may choose a functional approach that emphasizes the input/output metaphor of a computer and in which letters as variables can represent a range of values; a problem-solving approach that involves formulating and solving equations; a modelling approach that emphasizes reality, representation, and relevance; or an approach that emphasizes processes in making mathematical generalizations. It is often argued that since equations involve algebraic expressions and algebraic expressions involve variables, the sequence of presentation should be first the idea of variable, then algebraic expressions, and finally equations. Whatever sequence is chosen, there is a considerable body of literature that can help us understand some of the cognitive obstacles involved in each approach (see for example, Kuchemann, 1981; Chalouh and Herscovics, 1984; Booth, 1988; Herscovics, 1989, and Sfard and Linchevski, 1994).

One way to help make a sensible decision may be to first examine the historical development of algebra. The word algebra comes from the Arab word al-jabr translated by Schwartzman (1994) as the reunion of broken parts illustrated by rewriting $2x = 10 - 3x$ as $5x = 10$ (Lesser, 2000). Al-jabr appears in the title *Hisab al-jabr w'al-muqabalah* (al-Khwarizmi, 825 A.D.) which was the first practical guide to solving linear and quadratic equations (Baumgart, 1989). However, although the birth of algebra can be attributed to al-Khwarizmi, it was not until Viète in the late 16th century that letters were used as unknowns in equations, as generalised unknowns in expressions and also as variables in functions. It was the use of letters as variables in functions that led to the development of The Calculus.

Because the idea of function is the cornerstone of The Calculus, it is sometimes argued that algebra should be introduced through a functional approach where for example, the expression $2x + 3$ is the function that maps every number x onto another number y. Such an approach however denies the preceding history of algebra led by al-Khwarizmi and later by Viète who used letters such as x in equations to represent an as yet unknown value. It also pays little attention to the growing body of research evidence that raises a number of concerns about this approach (Booth, 1984; Kuchemann, 1981; MacGregor and Stacey, 1993a, 1994, 1995a; Orton and Orton. 1994; Warren, 1997; Yerushalmy and Shterenberg, 1994).

The historical development of algebra indicates that solving equations was what algebra was initially all about. The use of letters as generalised unknowns in expressions and then as variables in functions came much later. Harper (1987) offers some support for the view that students pass through stages in development of algebra understanding which reflect this historical development. There are however aspects of the pedagogical approaches that are traditionally used to develop students' equation solving skills that need careful consideration (MacGregor and Stacey, 1993b, 1995b). Problems that can be reasonably easily formed into linear equations are typically of the type, *"I am thinking of a number. I multiply it by 4 and add 3 to the result to get 27. What number am I thinking of?"* If x is the starting number, then the equation for the problem is $4x + 3 = 27$. Filloy and Rojano (1984) note that problems which can be represented algebraically as $x + a = b$, $ax = b$, or $ax + b = c$, can be solved by arithmetic methods. Figure 1 shows such a method known as "backtracking" for solving $4x + 3 = 27$.

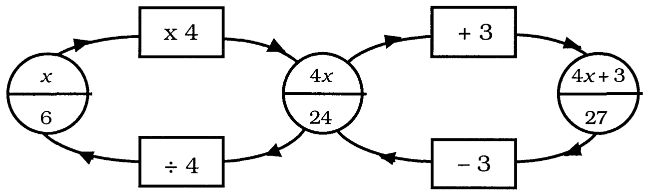

Figure 1. Flowchart showing "Backtracking" to solve equations of the type $ax + b = c$.

The upper part of the flowchart is used to form the equation $4x + 3 = 27$ and the lower part shows the reverse or "backtrack" steps that produce $x = 6$ as the solution. Filloy and Rojano however claim that a "didactic cut" occurs with equations such as $ax + b = cx + d$ in which the unknown occurs on each side of the equation. Such equations cannot generally be solved by arithmetic methods such as that shown in Figure 1. Equations of this type can be formulated from, *"I am thinking of a number. 4 times the number plus 3 is the same as 3 times the number plus 9. What number am I thinking of?"* This can be represented by the equation $4x + 3 = 3x + 9$. Pirie and Martin (1997) argue however that the "didactic cut" proposed by Filloy and Rojano is merely a consequence of "approaching the solution of equations through appeal to arithmetic parallel thinking" p. 161. They followed an approach used by a teacher with mathematically less able students where the supposed stumbling block did not occur. Rather than easing students into linear equations through considering equations such as $2x = 6$, $x + 5 = 11$, and $3x + 2 = 14$, this teacher began his lesson by immediately focussing on equations that were not drawn from "real-life' contexts and had as yet unknowns on each side of the equal sign. Within a two-week period, the students were all successfully solving equations of the form $ax + b = cx + d$. Pirie and Martin showed how the teacher sent his students down a path of growth which they analysed and accounted for in terms of the levels: Primitive Knowing, Image Making, Image Having, Property Noticing, and then Formalising, of the model of growth of mathematical understanding proposed by Pirie and Kieren (1989).

The approach reported on by Pirie and Martin forms the basis of an introduction to algebra in a new text series for students aged 12 to 14, see Britt, Hughes, and Souviney (2002). As with the students in the Pirie-Martin study, students here are sent down pathways where they continually meet new mathematical challenges. They begin algebra by focussing on equations with unknowns on each side of the equal sign and which have no specific links to real-life contexts. Figure 2 shows examples of the tasks and how they can be analysed in terms of the Pirie-Kieren model.

Level of Task Understanding	Mathematical Tasks	Purpose of Task
Primitive Knowing	Find numbers that go in the boxes. □ + □ + 19 = □ + 62 The total value of the numbers to the left of the equal sign is the same as the total value of the numbers to the right of the equal sign.	This open task with an infinite number of solutions uses students' knowledge of part-whole relationships and focuses on the meaning of the equal sign in equations.
Image Making/Image having	For each equation the *same* number goes in each box. Find the number. □ + □ + 15 = □ + 47 □ + 27 + □ = □ + 92 19 + □ = □ + □ + 256 As they work on tasks similar to these, students look for a short cut strategy for finding the value of the unknown number.	The ideas established in the previous tasks form the basis for 'Guess and Check' strategies to find the box number. Students shift back and forth recursively between the levels Image Making and Image Having in the Pirie/Kieren model for mathematical understanding.
Image Having/Property Noticing	Students explain their own short cut strategies and examine those of others. □ + □ + 27 = □ + 42 For example, a box on each side of the equal sign can be covered. ■ + □ + 27 = ■ + 42 giving □ + 27 = 42. Since the total value of the numbers to the left of the equal sign is the same as the total value to the right of the equal sign, the value of the unknown can be found from 42 – 27 = 15.	Students reflect on their mental actions as they search for patterns in order to generalise their actions to produce a short cut strategy that works consistently. At no stage are the students applying given rules – they devise their own short cut strategies or explain those of others.

Figure 2. Developing strategies for solving "Box" equations with the unknown on each side of the equal sign.

The 'Big Idea' that is the focus in this work with equations is that the total value of the numbers to the left of the equal sign is the same as the total value of the numbers to the right of the equal sign. Students compare the component values on the left with those on the right to invent their own short cut strategy for solving box equations. Mathematics for these students is about experimenting with ideas, those that work as well as those that don't. Reflecting on and explaining ideas that don't work is as valuable as developing successful short cuts. Mathematics is about understanding ideas not simply applying rules and procedures.

Figure 3 shows the transition from box equations to equations of the form $ax + b = cx + d$. Students are encouraged to propose and develop sensible strategies for solving such equations before formalizing them as efficient algorithms for solving linear equations.

Level of Task Understanding	**Mathematical Tasks**	**Purpose of Task**
Primitive Knowing/Image Making	Rewrite each equation using x for the unknown. So $\square + \square + 19 = \square + 62$ becomes $x + x + 19 = x + 62$ giving $x + 19 = 62$ so that $x = 62 - 19 = 43$.	This is the first time that students encounter the idea of a letter standing for a particular but as yet unknown number.
Image Making/Image Having	$x + x + x + 82 = x + x + x + x + x + 28$ can be written more simply as $3x + 82 = 3x + 2x + 28$. Find the value for x. Then find the value for x in $2x + 83 = 4x + 27$ and in $5x - 14 = 3x + 46$	Students are introduced to $x + x + x$ as being 3 times x or $3x$. When they compare the left side of the equation with the right side they see that $82 = (82 - 28) + 28$ so that $2x = 82 - 28 = 54$ giving $x = 27$.
Image Having/Property Noticing	Students look for short cut strategies for solving equations and investigate short cuts of others. For example, suppose Isaac just looks at the equation $9x + 27 = 12x + 3$ and then writes $24 = 3x$ so $x = 8$. Explain his reasoning.	Students reflect on their mental actions to see if they can invent a simple strategy for solving equations of this type.
Property Noticing/ Formalising	Students use and extend their short cuts to solve a variety of equations such as $5x + 4 = 46 + 2x$ and $128 + 3x = 5x + 32$ Then they see if they can find a strategy for solving equations such as $5x + 4 = 46 - 2x$ and $128 - 3x = 5x + 32$ Finally they use any of the strategies that is appropriate for the given situation to solve a variety of linear equations of the type $ax + b = cx + d$.	Students apply their short cut strategies to a variety of equations that involve the unknown on both sides and only addition. Then they see if they can extend their strategies to solve equations that have unknowns on each side and involve subtraction. The intention is for students to develop robust procedures for solving linear equations.

Figure 3. Making the transition from "Box" equations to linear equations of the form $ax + b = cx + d$.

The approach shown in Figure 2 and Figure 3 has similarities with that used by Linchevski and Herscovics (1996) in a teaching experiment that involved cancellation of numerical terms of equal value and also terms in the unknown on either side of the equal sign. They found that their cancellation method for equations involving only addition remained stable for students even after a period of seven months. Difficulties with their cancellation method emerged however with equations such as $5x + 4 = 46 - 2x$ where the unknown occurs on both sides of the equal sign and at least one of the unknowns is subtracted rather than added. While there are similarities between the approach I have illustrated and that used by Linchevski and Herscovics, there is an important difference. Linchevski and Herscovics focussed on a particular method involving cancellation in which students learned that equations remained balanced following a correct cancellation process. By contrast, in the approach that I have illustrated, students carefully reflect on their mental actions with the goal of developing one or more mental strategies or short cuts for solving linear equations. For example, for the equation $5x + 4 = 46 + 2x$, one such strategy is to first notice that $5x$ is $3x$ more than $2x$, or the difference between $5x$ and $2x$ is $3x$, so that $3x + 4 = 46$. Then $3x = 42$ since $42 + 4 = 46$. A second strategy is to set the difference between the unknowns equal to the difference between the numerical terms. So $5x - 2x = 46 - 4$ or $3x = 42$ and $x = 14$.

Applying either of these strategies to $5x + 4 = 46 - 2x$ on the other hand presents a new challenge. First, $5x + 4 = 46 - 2x$ can be expressed as $5x + 4 = 46 + {}^{-}2x$. Then, using the first strategy above, $5x$ is $7x$ more than ${}^{-}2x$, (the difference between $5x$ and ${}^{-}2x$ is $7x$), giving $7x + 4 = 46$. So $7x = 42$ and $x = 6$. Applying the second strategy gives $5x - {}^{-}2x = 5x + 2x = 46 - 4$ and $7x = 42$, $x = 6$. It may also be helpful to encourage students to see if they can develop a new strategy by exploring the effect of adding equal values or identical algebraic expressions to each side of a variety of linear equations. For example, adding zero to each side of $5x + 4 = 46 - 2x$ gives $5x + 4 + 0 = 46 - 2x + 0$ which is identical to $5x + 4 + (2x - 2x) = 46 - 2x + 0$ since $2x - 2x = 0$. This in turn gives $7x - 2x + 4 = 46 - 2x$ leading to $7x + 4 = 46$. Or just adding $2x$ to each side gives $5x + 2x + 4 = 46 - 2x + 2x$ which leads to $7x + 4 = 46$ as before. So, instead of developing a single algorithm for solving equations as Linchevski and Herscovics did in their teaching experiment, this approach continually advances students' mathematical thinking towards a flexible and robust approach to solving linear equations.

The strategies arising from students' mathematical thinking can lead to the traditional algorithms for solving linear equations. We saw for example, that $5x + 4 = 46 + 2x$ can lead to $5x - 2x = 46 - 4$ and then that $5x + 4 = 46 - 2x$ can lead to $5x + 2x = 46 - 4$. In the first equation, $+ 4$ on the left side of the equation became $- 4$ on the right side of the equation. Also $+ 2x$ on the right became $- 2x$ on the left. Then in the second equation, $+ 4$ on the left became $- 4$ on the right and $- 2x$ on the right became $+ 2x$ on the left. So addition changes to subtraction and vice versa for numerical terms and algebraic expressions when they change sides of an equation. Earlier in Figure 3, we also saw that $3x + 82 = 3x + 2x + 28$ becomes $82 = 2x + 28$ by ignoring $3x$ on each side (this can be construed as subtracting $3x$ from each side). Then finally we saw that $5x + 4 = 46 - 2x$ becomes $5x + 2x + 4 = 46 - 2x + 2x$ when $2x$ is added to both sides. So equal values or identical algebraic expressions can be added to or subtracted from both sides of equations.

The development and subsequent efficient use of both of these algorithms is one of the possible outcomes of the approach proposed in this paper. First ideas of algebra are introduced through the teaching of equation solving in a way that reflects the historical development of algebra and is consistent with recent research findings (Pirie and Martin 1997; Linchevski and Herscovics 1996). On the one hand, Pirie and Martin claimed that the so-called "didactic cut" did not occur for the students in their study. On the other hand, Linchevski and Herscovics argued that, while their cancellation procedure led to successful outcomes for students solving equations where the unknown occurs on both sides and which involve addition only, a cognitive obstacle emerged later for equations that involved subtraction of the unknown. The approach proposed here recognizes both of these views. A cognitive obstacle does exist but can be overcome if a teaching approach is used that pushes students towards developing their own sensible strategies arising from individual and collective sense making.

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Additive Relations and Function Tables[1]

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We present work with a second grade classroom where we carried out a teaching experiment that attempted to bring out the algebraic character of arithmetic. In this paper, we specifically illustrate our work with the second graders on additive relations, through the children's work with function tables. We explore the different ways in which the children represented the information of a problem in the form of a self-designed function table. We argue that the choices children make about the kind of information to represent or not, as well as the way in which they constructed their tables, highlight some of the issues that children may find relevant in their construction of function tables. This open-ended format pointed to how they were understanding and appropriating tables into their thinking about additive relations.

Although tables are an integral part of the mathematics curriculum, we know surprisingly little about how this tool lends itself to the understanding of functions, particularly of additive functions. The same cannot be said of other tools related to functions, such as graphs, where there has been considerable research describing children s understanding and construction of them (e. g., diSessa, Hammer, Sherin, & Kolpakowski, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Nemirovsky, Tierney, & Wright, 1998; Tierney & Nemirovsky, 1995), or other types of diagrams (e. g., Sellke, Behr, & Voelker, 1991; Simon & Stimpson, 1988). What is the nature of children s understanding of tables? How does their understanding evolve? What do children learn about functions when working with tables? And what issues might educators and curriculum developers need to take into account when designing learning activities that employ tables?

Our work stands in contrast with research carried out up to this moment. Sellke and his colleagues view the use of linking tables or diagrams as "overrid[ing] the influence of the intuitive models and the numerical constraints associated with them, overrid[ing] the inaccurate conceptions many students have of the two operations, reflect[ing] multiplicative relationships between the quantities, remain[ing] true to the semantic meaning of the problem, and [being] effective regardless of the numerical characteristics" (Sellke, Behr, & Voelker, 1991, p. 31). Simon suggests that concrete diagrams are helpful for children to refer to when "confusions arise with the abstractions" of the algebraic problems they are confronted with (Simon & Stimpson, 1988, p. 140). Streefland has found that "the ratio table is a permanent recording of proportion as an equivalence relation, and in this way contributes to acquiring the correct

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concept" and "serves the progression in schematising" (1985, p. 91). These researchers have focused on the use of tables to enhance or guide students' understanding of functions. Our focus has been that of uncovering the understandings already present by analyzing the original self-designed tables constructed by young children. This psychogenetic approach to children's representations is similar to that carried out in the area of literacy by Ferreiro (e. g., Ferreiro, 1988; Ferreiro & Teberosky, 1979) and in the area of musical notations by Bamberger (e. g., 1988).

In our classroom research, we have noticed that children tend to construct tables that differ significantly from the conventional tables they are shown by us, their teachers. This has suggested to us that children's re-construction of tables can inform us about how they are working tabular representations into their thinking—their thinking in general and about functions in particular. A case in point is that of a child named Joey. When asked to construct a table to display the data available we had been working with in a word problem, he began to peek at the printed table on the last page of his handout. As he re-constructed the table, he intermittently flipped to the end page to verify his work and move to the next step. Figure 1 shows the table he had used as a model.

	Jessica	**Daniel**	**Leslie**
Day 1	7	4	0
Day 2	9	6	2
Day 3	12	9	5
Day 4	14	11	7
Day 7	20		
Day 10		20	
Day 16			20
Any day		x	

Figure 1. The table that Joey had been peeking at

Figure 2 shows Joey's representation of the table. At first glance, he simply transposed the table, organizing the columns by days (even though it was organized by children's names in the original table) and the rows by children. However, he did not place the children's names only once at the beginning of each row. Instead, Joey seemed to find it necessary to write the child's initials in each one of its respective cells. In other words, in Joey's re-construction, Jessica's initial recurs in each cell of row 1; Daniel's initial recurs in each cell of row 2; and Leslie's initial recurs in each cell of row 3.

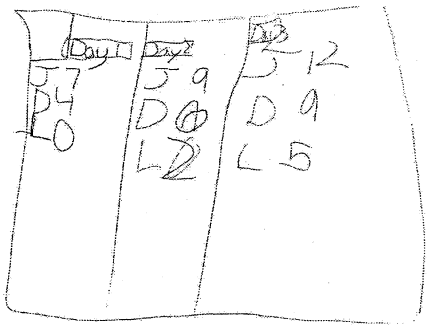

Figure 2. Joey's table, constructed while peeking at the table in Figure 1

In the present paper, we explore children's self-designed function tables with the goal of: a) learning about what they consider relevant in the construction of a function table; b) learning more about children's understanding of additive functions, as reflected in their self-designed tables.

The study and its context

The work reported in this paper is part of the work we have been doing during the last three years with children between 8 and 10 years of age. In our work, we have been exploring some ideas about how to bring out the algebraic character of arithmetic (see Brizuela, Schliemann, & Carraher, 2000; Carraher, Brizuela, & Schliemann, 2000; Carraher, Schliemann, & Brizuela, 1999, 2000; Schliemann, Carraher, & Brizuela, 1999). Most efforts to "algebrafy" arithmetic (Kaput, 1995) have begun with multiplicative reasoning and linear functions. We believe that there is no need to put off the algebrafication of arithmetic until multiplicative structures. Moreover, children's initial intuitions about order, change, and equality do arise in additive situations. During this time, central to our work has been trying to understand how young children work algebraic representations into their thinking. In this paper, we extend these views into an exploration of how children understand, construct, and think about tables in additive situations.

The specific examples that we use in this paper are taken from our work with the children in four second-grade classrooms in a public elementary school from a multi-cultural community in greater Boston. Joey, mentioned above, is one of these second graders. These examples are part of our longitudinal work with students as they move from their second through fourth years of schooling. During the second grade, we met with the students once a week during a six-week period. Our curriculum that semester was focused on an exploration of additive structures. Midway through the semester, we interviewed the children in pairs. The purpose of the interviews was two-fold: to learn about the students understandings of the concepts, and to hold more individualized interactions with the students, as learning opportunities. We interviewed a total of 39 children. During that interview, we presented the students with the following problem which is the same that Joey, above, had been presented with. A model table was included at the end of the children s handouts for a different task during the interview, but was not shown to the children during the part of the interview that we will be focusing on here.

Day 1:

Jessica, Daniel, and Leslie each has a piggy bank where they keep the money they receive from their grandmother. One day, they counted the money they had and found that Jessica had 7 dollars, Daniel had 4 dollars, and Leslie had none. They then decided not to spend any more money and keep all the money their Grandma would give them in their piggy banks.

Show how much money each one has.

Day 2

On the second day Grandma came to visit and gave 2 dollars to each one of the children. They put the money in their piggy banks.

Show what happened.

Day 3

On the third day Grandma came to visit and gave 3 dollars to each one of the children.

Show what happened.

How much money does each one have now?

Show in a table what happened from day 1 to day 3.

Looking at children's self-designed tables

After going through the problem, we asked each child to show in a table what had happened in the problem, from day 1 through day 3. We did not give the children a model to follow in building their tables, although they had already worked with tables in their classes with us.

The range of representations

The children's responses spanned the range from the very idiosyncratic, like Jennifer's (see Figure 3) to the more conventional, like Joseph's (see Figure 4). Interestingly, Jennifer and Joseph had actually worked together on this problem even though their responses were radically different.

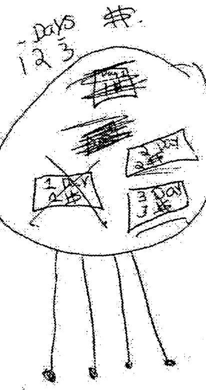

Figure 3. Jennifer's table

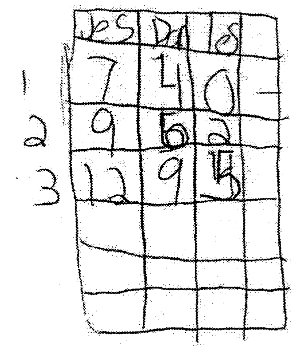

Figure 4. Joseph's table

Jennifer explained that she drew a table (literally!), with its four legs. She described the table she made in the following way:

> Jennifer: See, I wrote days up here (pointing to the word "days"), and then wrote, there was three days, right (pointing to the numbers)? And that is including money (pointing to the dollar bill). And then I drew squares and it says,...on day two they got two dollars and on day three they got three dollars.

Joseph, on the other hand, explained that he did a "regular table...it's just got days and names" (maybe feeling slightly intimidated by Jennifer's creation!).

Following a temporal order

Of the thirty-nine children interviewed, over half of them (22 of the 39) followed a temporal order in their construction of the tables. By this we mean that they placed the names of the characters as headers for the columns and the days of the week as headers for the rows. Thus, going down a column also reflects the passing of time. This is usually the conventional way to construct tables. Joseph's table is an example of this kind of construction (see Figure 4, above). In contrast, an example of a table that does not follow this order is that of Joey, in Figure 2, above.

Relevant and irrelevant information

An analysis of the children's tables also indicates what kind of information they considered to be relevant and irrelevant. In their tables, children were more likely to consider the names of the characters in the problem (e. g. Daniel, Jessica, and Leslie) as relevant information, that could not be obliterated. Thus, only 3 of the 39 children (8%) left out the characters' names in their tables. An example of a table that left out this information is that of Jessie (see Figure 5, below).

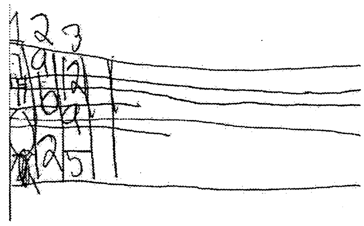

Figure 5. Jessie's table, which does not show the characters' names

In Jessie's table, it is hard to discern who each of the amounts refer to. The fact that there are so few children who obliterated this information may indicate that it is very important for quantities to have some kind of referent or, in this case, "owner" (see Schwartz, 1988, 1996). Along the same lines, 14 of the 39 children (36%) repeat the names of the characters in their tables. Raymond's table, for example, seems to indicate that he felt compelled to include the names of the characters of the problem—the "owners" of each amount—in each one of the cells in his table (see Figure 6, below).

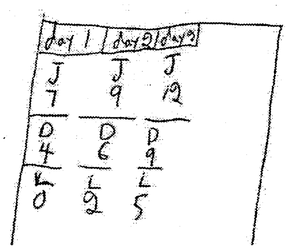

Figure 6. Raymond's table, which repeats the names of the characters in each cell

In contrast, the second graders we worked with did not consider the day numbers as relevant as the names of the characters. Thirteen of the 39 children (33%) did not write the names of the days in their tables (compared to the 3 children who did not write the names of the characters). Briana is an example of a child who did not explicitly register the day numbers in her table (see Figure 7, below). This may indicate that in her view, the inclusion of this information would have been redundant in her table.

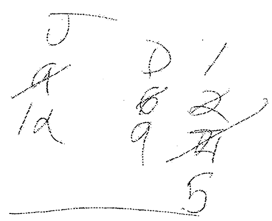

Figure 7. Briana's table, which does not show the day numbers for the problem

It seems that the day numbers are information that is considered more superfluous than the names of the characters. Going along the cells (be it down the columns, as would be done conventionally, or across the rows, as was done by some children like Jessie and Raymond) shows the passing of time. Some children may think that therefore it would be redundant to indicate the day numbers. Additionally, very few children (2 of the 39 children, or 5%) actually repeat the day numbers in their tables (compared to the 13 children who repeated the characters' names in their tables). Adam, for example, repeated the information about the day numbers in his table (see Figure 8, below).

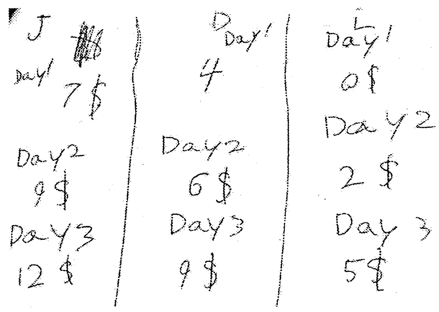

Figure 8. Adam's table, where he repeats the day numbers in each cell

Moreover, when providing a verbal explanation of his representation during the interview, Adam felt compelled to repeat not only the day numbers but also each character's name, each day:

> Adam: Day one, Jessica had seven dollars. Day one, Daniel had, uhm, four dollars, and day one Leslie had zero dollars. Day two, Jessica had nine dollars. Day two, Daniel had six dollars. Day two, Leslie had two dollars. Day three, Jessica had twelve dollars. Day three, uhm, Daniel had three dollars. Day three, Leslie had five dollars.

The choices children make about the kind of information to repeat and to obliterate from their tables highlight some of the issues that these children may find relevant in their construction and re-construction of function tables. The referents for the amounts are important to them, while an indication of the temporal order of the events is not as important. In children's minds, they may find that there are other ways in which tables can indicate temporal order, but not the referents to the amounts included in the table.

Looking into additive relations

Of the thirty-nine children interviewed, most children (36 of the 39, or 92%) showed in their tables the cumulative amount of money that each child had on each one of the three days—that is, how much money each child *had* and not how much they *got*. Tables such as Joseph's (Figure 4), Briana's (Figure 7), and Adam's (Figure 8) are examples of tables in which only the cumulative amounts are depicted in the tables. On the other hand, the other three children showed in their tables the amount of money gained each day instead of the total amount of money—that is, how much they *got* and not how much they *had*. Jennifer's table (Figure 3) is an example of this kind of representation. Another example of this kind of representation is that of Mariah, who shows the amounts gained by drawing a circle around them. The circled parts at the bottom of the representation are the amounts the children *got* from their grandmother on the second and third days (see Figure 9, below).

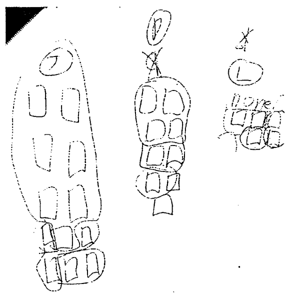

Figure 9. Mariah's representation, showing how much the children *got* on each day

Thus, in thinking about additive relations they were representing in their tables, most of the children focused on total amounts; that is, on how much each child had on each one of the three days. A variation of this focus can be seen in representations such as Briana's (Figure 7, above) and Jessie's (Figure 5, above). In their tables, they crossed out the amount of money the children had on the previous day once they noted how much they had on the following day. Thus, when they moved on to day two, they crossed out the amount on day one. In their representations, they seem to need to clarify that the amounts represented in each row should not be added to the amounts in the previous rows, but that each new row makes the previous ones invalid. A few children, however, focused on the *differences* between the amounts that the children had on each one of the days, or how much the children *got* on each one of the days. This is the case of Jennifer and Mariah described above.

Concluding remarks

We take seriously and at heart a question posed now ten years ago by Kaput (1991): How do material notations and mental constructions interact to produce new constructions? (p. 55). While we do not believe that we could answer this question conclusively based on the data here presented, we can indicate that there is a great need to look further and deeper into children s construction of function tables. By developing better understandings both of how their representation of tables evolves and of the understandings about additive relations that is reflected in their representations, we can better serve them as teachers and as curriculum developers. We have begun to unearth some of children s understandings about additive relations reflected in their representations. Still to be further explored is the interaction *both* of their constructions and of conventional tables on the development of their understandings about additive relations.

Even though their experiences with function tables are counted, the second grade children we worked with have developed sophisticated tabular representations of the data presented in a problem focusing on the progression of money gained over time. They construct function tables that make sense and that reveal the workings of their logic about the problem at hand, and they adequately organize and represent the problem situation. Interestingly, children incorporate into their representations conventional features of function tables. Over half of the children we interviewed followed a temporal order in their tables, in terms of rows (for characters) and columns (for the passing of time). Additionally, information such as showing the characters names seems to be relevant to children when constructing these tables, which is consistent with Schwartz s (1988,1996) suggestion regarding the importance of having referents for quantities. Additionally, data tables seem to help uncover children s conceptions and understandings about additive relations.

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Natural Algebraic Activity

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How is a classroom culture established where students develop a need for algebra? What does the teacher do? How is algebraic activity recognised? We share the practice from a research project[1] in the U.K. where algebraic activity, in requiring a shift in level of abstraction of communication, supports the learning of mathematics. Algebra is not seen as a separate or separable curriculum strand, but as a support for talking about abstract concepts. Rather than students hitting a wall or barrier in its learning, the mixed ability 11-12 year old students on the project, in exploring their own questions within complex problems, seem to express naturally structural concepts such as inverse, infinity, limit and distributivity.

Introduction

What does it mean to talk about establishing a classroom culture where students develop a need for algebra? Before teasing out what we mean by algebra and algebraic activity and telling some of the story of the journey that has led to us asking this question as part of a current research project, it seems worth stressing our strong perspective about working with algebra. This view is that discussions about what should and should not be included in an algebra curriculum are subordinate to considerations about the environment in which the teaching and learning of mathematics exists. If students meet a barrier in their learning of mathematics when introduced to algebra we can make decisions to reduce the content of the curriculum but that will not effect the students' learning. In the U.K., observations of difficulties in students learning algebra led, over time, to the gradual reduction of the amount of algebra included in the pre-16 curriculum (RS/JMC, 1997) until there were effects on the movement of students to advanced courses (Winter et al., 1997).

This paper is the story of our (a teacher, Alf Coles and a teacher-educator, Laurinda Brown at the start who now see themselves as co-teacher-researchers) journey to try to teach mathematics in such a way as to work contingently with what students bring with them to the mathematics classroom. We teach skills in meaningful contexts, creating a 'community of inquirers' (adapted from Lave and Wenger, 1991, Schoenfeld, 1996) where the students and teacher are doing mathematics. To do mathematics they need to be able to talk about abstract concepts, often supported by use of notation and some formal algebraic expression and manipulation. What is striking is that many skills are used without being taught explicitly. We have, increasingly, noticed the naturalness of students use of mathematical concepts such as limit, inverse, infinity and distributivity. These students are offered sufficient complexity with which to work and are not falsely limited by a syllabus or scheme of work or by being presented with something they do not understand.

What is algebra?

In asking ourselves the question: what is algebra? we wanted to find a definition with which we could work to try to understand what students actually did which could be described as algebraic activity when they were engaged in doing mathematics. In reviewing current research on algebra, strands emerged to do with context, meaning-making, complexity and control which we found useful in our thinking. Introducing and using algebra in a context is talked about from a view which we support that:

> Traditionally, algebra in schools has been dealt with at a syntactical level; the students have no 'meta-control'; they know that they are allowed to do some things and not others, and obviously they sometimes make mistakes ... to improve the situation one can call to mind an algebra which is always linked to a context; not necessarily to the (often unreal) 'real world problems', but to the properties of numbers, or to the manipulation of functions, in all cases where it is necessary to interpret the result (Menghini 1994, p.13).

We see the important task as making symbol representation meaningful in contrast to one person who made a submission to the Cockcroft report (1982) who represented the view that: Mathematics lessons are very often not about anything. You collect like terms, or learn the laws of indices, with no perception of why anyone needs to do such things (para 462). Similarly:

> ... it was the lack of this (linking symbols to the situations they represent) that led to failures in the past teaching of algebra: the children who failed thought of x and y as meaningless marks that had to be played with by peculiar rules (Sawyer, quoted in Anderson 1978, p.20).

One argument is that meaning might be achieved through working with students on thinking mathematically, where algebra is one component:

> One major part of the effort to reform secondary school mathematics is the project of changing the goal of studying school algebra from mastery of symbolic manipulations to the ability to reason mathematically (Yerushalmy 1997, p.431).

Although students do need some fluency in symbolic manipulation:

> The manipulation of symbols is only a small part of what algebra is really about, the traces that are left behind after mathematical thinking has taken place (Mason 1992, p.5).

One implication of this is that:

> symbolic manipulation should be taught in rich contexts which provide opportunities to learn when and how to use those manipulations (Arcavi 1994, p.32).

In other words algebra should arise from complex situations:

> Algebraic symbolism should be introduced from the very beginning in situations in which students can appreciate how empowering symbols can be in expressing generalities and justifications of arithmetical phenomena ... in tasks of this nature, manipulations are at the service of structure and meanings (Arcavi 1994, p.33).

There is never an end-point in this conception of learning mathematics. If I am learning to reason mathematically to structure my thinking about problems, then what I learn is in an ongoing state of complexification and enrichment.

Here we had our link to the challenge (Sutherland, 1991) of creating a school algebra culture in which students find a need for algebraic symbolism. The need we envisage here is for expression of awarenesses within complex situations. This clearly places onus on us as teachers to create a classroom culture in which there is the possibility, from the very beginning, for students to work at and attempt to express what they are aware of. What we are prepared to notice and are able to perceive is to a large extent dependent on the culture around us, and the language available to us.

We view the developing culture and ethos of our classroom as a 'community of practice' (Lave and Wenger, 1991), where the practice is mathematical 'inquiry' (Schoenfeld, 1996). The learning of algebraic thinking is part of learning mathematics and is situated in the classroom interactions. The students are not entering a community of mathematicians but they and their teacher can become a 'community of inquirers'[1].

A unifying strand through all these quotes is the sense of algebra as an evolving language that can emerge from situations and contexts that are already laden with meaning. Algebra can be used to express and offer insights into those situations. It is in this emergent expression and consequent empowerment that students can discover a need for algebra.

We chose to use the following components of algebraic activity in the project:

> (i) Generational activities which involve: generating expressions and equations which are the objects of algebra, for example, equations which represent quantitative problem situations; expressions of generality from geometric patterns or numerical sequences; and expressions of the rules governing numerical relationships.
>
> (ii) Transformational rule-based activities, for example, factorising, manipulating and simplifying algebraic expressions and solving equations. These activities are predominantly concerned with equivalence, form and the preservation of essence.
>
> (iii) Global, meta-level activities, for example, awareness of mathematical structure, awareness of constraints of problem situations, justifying, proving and predicting, and problem-solving. These activities are not exclusive to algebra (Kieran, 1996).

Within the discussions of a national working group (RS/JMC, 1997) this definition was the one which covered sufficient of the member's interpretations of algebra that it could be accepted by all. In our case, such a broad definition allows us, as teachers, to work on our recognition of what algebra is in what the students do and does not arbitrarily limit what we see.

Background

In a pilot project two pairs of high achieving students, one pair of 15 year olds and one pair of 18 year olds were interviewed as they worked on a problem set by Alf. This problem could be tackled algebraically. The major difference between the two pairs of students was the control with which the older ones first explored the problem numerically, until they had some sense of what was going on, and then moved effectively to an algebraic representation and solution showing evidence of all three of the components of algebraic activity. The 15 year olds, on the other hand, reached for the symbolism quickly but became bogged down in the transformational work.

This experience led to our asking the question: would it be possible work with 11 year old students so that when they themselves were aged 15 they would be operating as the 18 year olds did? In September, 1998 we started a project, funded by the Teacher Training Agency (TTA), to investigate the question: Can we develop a school algebra culture in which students find a need for algebraic symbolism to express and explore their mathematical ideas? (Sutherland, 1991, p.46).

During that year we worked on ways to characterise students' 'needing to use algebra' and saw this as linked to them being able to ask and answer their own questions related to contexts. We reported in one case study about a student using algebra in response to a numerical problem fifteen weeks after the start of the project:

> Alex clearly shows evidence of insight into the structure of the problem ... The difference that strikes us here ... is that the algebra has arisen from a question of Alex's ... As he worked through the general case the structure of the problem was illuminated ... we would argue that Alex's need for algebra came through the posing of his own question: why? and that this came out of a pattern spotted ... after the process of doing a few examples (Brown & Coles, 1999, p.159).

Methodology and methods

We were successful in a bid[1] to extend the TTA project into 3 more classrooms in the U.K. during the academic year 1999/2000. The aims of this ESRC project are:

1) To create year 7 mathematics classroom cultures which provoke a need for algebra.

2) To investigate the similarities and differences developed in each of the teacher's classrooms.

3) To investigate the nature and extent of the support needed from the collaborative group of teachers to plan their classroom activities starting from the students' powers of discrimination.

4) To develop theories and methodologies to describe the complex process of teaching and learning.

We believe that all individuals have powers of discrimination which allow them to learn, to make sense of the world they perceive, through an awareness of difference. What we experience through our actions is an interpretation based on all of our past. Therefore two people cannot see the same thing nor share the same awareness. We can communicate, however, because we can talk about the details of common experiences, exploring differences, and in doing so the gap between interpretations can be reduced. We were still investigating students' 'needing to use algebra' and considered that teaching and research strategies would develop throughout the project. Learning through the process of the doing of the research places us as enactivist (Reid, 1996, Brown and Coles, 1997, 1999). In what follows we describe how the ESRC project was constructed to give insights into the aims above, then link the four principles of our enactivist theoretical frame to methodology and methods.

Over one academic year, September, 1999 to July, 2000 which is split into 3 terms we (3 teachers, 1 teacher-researcher and 3 researchers) are investigating the samenesses and differences in the developing algebraic activity in the classroom cultures of the 4 teachers through:

- working in a collaborative group, meeting once every half-term for a full day and corresponding through e-mail

- videotaping each teacher for one lesson in every half-term and researchers observing teachers in the classroom at most once a fortnight in teacher/researcher pairs

- every half term interviewing a) each teacher and b) 6 of each teachers' students in pairs, selected to give a range of achievement within the class

- encouraging students to write a) in doing mathematics and b) at the end of an activity, about 'what have I learnt?': photocopies of all the 'what have I learnt?'s are collected from each teacher as well as all the written work of the 6 students interviewed

- each researcher being responsible for viewing the data collected through one or more strands; teacher strategies, student perspectives, algebraic activity, samenesses and differences in the classroom cultures, teachers use of same/different in planning to teach through students using their powers of discrimination.

The classroom cultures were developed through the teachers sharing with their students that the year was about them 'becoming mathematicians'. This overall structure is to support our looking at what students and teachers do in these classrooms. Schemes of work and organisational structures within the schools are different and it is not our intention to change these. The content of the lessons would still be decided by the teachers within those structures but during the day meetings there would be time to plan together, given those constraints, to allow students to use their powers of discrimination.

A first principle of enactivism is the recognition that we cannot take in the details of everything that is happening around us. We are naturally selective since there is a limited capacity to what we can attend to. What we notice and the connections we make guide our actions, often implicitly. It is in this sense that cognition is placed as being 'perceptually guided action' or 'embodied actions' (Varela, 1999, p.12, p.17).

We are therefore giving space within the classrooms for the students to work at making connections and for them also to communicate these to the whole class. The teachers make their decisions contingently upon what they perceive through their awareness of samenesses and differences within what the students are sharing. The teacher cannot be in control of the content nor hear and respond to everything that is happening in such classroom interaction. We try to set up the possibility of the students making connections through common boards used for sharing questions, conjectures, homework etc. (see Sutherland's (2001) paper, written for the ICMI Study Group, for the illustration of these ideas through the detailed description and analysis of one video-taped lesson).

The second principle of enactivism (adapted from Varela, 1999, p.10) is the belief that we are what we do. It is our actions and perceptions that make us who we are and these are dependent on the whole of our past experiences. Consequently the data collection on the project is done over time and we are working with students who have just started at a new school to establish new behaviours in their mathematics lessons related to 'becoming mathematicians'.

The third principle we adopt is that we take multiple views of a wide range of data:

> The aim here is not to come to some sort of 'average' interpretation that somehow captures the common essence of disparate situations, but rather to see the sense in a range of occurrences, and the sphere of possibilities involved (Reid, 1996 p.207).

Multiple views of the data are captured through the different strands that the researchers are investigating. We take one incident and interpret it through these different strands and also tell stories of the changes that are happening over time. One powerful way of working with multiple views is through the use of the videotapes, using short extracts and talking through the details of what we see. What seems important is that overlapping themes emerge over time.

As a fourth principle, from these overlapping and interconnecting themes, theories which are 'good enough for' (Reid, 1996) a purpose emerge:

> theories and models ... are not models of ... they do not purport to be representations of an existing reality. Rather they are theories for; they have a purpose, clarifying our understanding of the learning of mathematics for example, and it is their usefulness in terms of that purpose which determines their value (Reid, 1996 p.208).

There is no sense of there being a 'best' theory for our work. Our theories are 'good-enough for' our actions and the ideas that we continue to think about and use as teachers and researchers are those that inform our practice. We recognise what is useful for the practice of our teaching by what we are doing in the classrooms. What is not useful does not happen. We see our research about learning as a form of learning (Reid, 1996 p.208) where our learning is gaining a more and more interconnected set of awarenesses about our teaching of mathematics. These are not the explicit products of this research, however, what is crucial here is the process of developing such theories and actions in the classrooms.

Meta-commenting

At the start of the project the group of teachers and researchers explored what they meant by thinking mathematically. The members of the group all had different ideas but the teachers went into their first lessons giving to their students the purpose for the year of becoming a mathematician and intending to comment about the mathematical behaviours of students publicly to the whole group as they observed them (we call this process metacommenting, which is also a strand researched by Laurinda). The question 'why?' seemed important for all the teachers and was metacommented on in the early lessons of the project. What did these metacomments look like at the start of the year? For example, observing in Alf's classroom on 21/9/99 there is the following exchange:

> Alf: What question were most people working on for their homework?
> Student: Why does it always come to 1089?
> Alf: There's a question before that (smiling).
> Student: How does it work?
> Student: I know, does it always work!
> Alf: Two people have ones which didn't.

Through stressing the asking of questions and setting homework based on working on their own questions and using the students' homework to develop discussions at the start of the lesson it is perhaps unsurprising that the students have already begun to ask the question 'Why?' for themselves. The teacher is running the discussion but about a process even though the class and the teacher are actually working on a mathematical problem.

In an interview during the evening of the day when the first lesson of the year with his new year 7 class had taken place, Alf reported saying to the class:

> ... about becoming a mathematician ... that what I mean by that is if you're thinking mathematically then it's about noticing things about what's around you and it's about writing things down about what you notice and often what you'll be writing down will be a question about something which you've noticed - maybe you've seen a pattern and a question that mathematicians often ask is 'why?' so you might spot a pattern and think about 'why does that pattern work?'. Make a prediction maybe based on that pattern. Say why you think that pattern will continue (Alf, First lesson interview, 9/9/99).

From the start of the year Alf metacommented on these behaviours in the classroom, for instance, in the following extract he comments on the 'very mathematical question':

> Alf: What were you doing last lesson?
> Student: Carrying on from before, 4 digit numbers.
> Alf: Were you working on a question?
> Student: How many different answers?
> Alf: That's a very mathematical question. Any others?
> Student: I've been looking at whether I can know which column any number goes in.
> Student: I've been seeing whether the conjectures worked.
> Alf: Try to be organised about how you try (DO2, 29/9/99).

What the students picked up on in their writing about 'what have I learnt?', however, was'getting organised' (here Alf suggests that students 'try to be organised') which itself became able to be commented upon and which the students themselves used to describe what they were doing:

> Alf: Can someone explain what we were doing last time?
> Student: Trying to get 3 down.
> Alf: Can you expand on that?
> Student: We were trying to be organised doing 6, 5, 4 ... down (DO2, 13/10/99).

Although Alf is initially commenting on the asking of questions (the student's question 'How many different answers?' is described as a 'very mathematical' one) this question 'How many different answers?' is itself an example of one where the students have to 'get organised' or work systematically to answer it.

Alf does not respond within the mathematics in these extracts to validate or correct statements, but comments about the ways of working. Conversations often begin at the start of lessons across the project with a discussion about what happened last time or a sharing of homework where the students set the agenda. The classroom cultures were set up and metacomments became less frequent as students worked on problems over time (typically three weeks for the exploration of a starting problem in Alf's classroom).

Natural algebraic activity

One particularly rich activity in Alf's classroom was the following:

> Functions and graphs: A game was played where a rule or function (of the form N goes to 2N +1) was guessed and students described how they had found the rule in a range of different ways on the common board. The invitation was to plot such rules as graphs and explore.

The students' written work where they are encouraged as normal to write to explore what they were thinking, gives evidence for a range of algebraic activities and procedures from Kieran's (1996) classification (taken from the exercise books of each of the 6 interviewees:

> Student 1: I have looked at various rules that got the same answers and I worked out they are in fact the same rules, just written differently. We found that (2N + 5) x 2 is the same rule as 4N + 10. So, if we had (3N + 4) x 5 it is the same as 15N + 20. But is $(N + 3)^2$ the same as $N^2 + 3^2$?
>
> Student 2: On the graphs there are two types of lines curved and straight. There are different rules to make either line. I have worked out both types of rules. Here are two conjectures: 'If you have N x N in the rule, it will be curved' and 'If it is straight then it doesn't have N x N in the rule'.
>
> Student 3: If we try different rules will the graph still work? Can we be organised in finding all the rules?
>
> Student 4: A question I am going to do next is 'Is it always a curved line with two n's?'
>
> Student 5: From what we were talking about I have learnt that not all the graphs are straight and not all the graphs are curved.
>
> Student 6: When it goes up in curved lines, why isn't it going up in the tables?

Alf had been using metacomments for some years to introduce working on process with his classes and it became apparent during the TTA research that there was space in this classroom for a new level of awareness. Now that the students were getting organised and asking and answering their own questions and being mathematical, listening and hearing, it was as if the algebraic activities of problem solving and classifying and using notation within meaningful contexts were leading to a natural use of ideas of mathematical structure such as infinity and inverse (see Figure 1). Alf now finds himself metacommenting on the students applying a limiting process or wondering 'what would happen the other way' because that is the way the students are working and thinking, not because he has introduced those ideas to them.

This is not a teaching prescription and these are not experimental classrooms. For everyone involved in the project, teachers, researchers and learners we are developing our awarenesses or learning i.e. we are engaged in a greater level of abstraction through talking 'about' the teaching and learning of mathematics, the developing cultures in the classrooms or mathematics itself in the case of the students. These awarenesses are rooted in developing practice. The function of algebra in these mathematics classrooms and we now believe more generally, is to allow this 'talking about' of ever more abstract processes. For this reason, rather than prescribing an algebra curriculum (the differences in the uses of n - functionality, generalised number etc. have become talked about within the class) or deciding to leave such a difficult topic until the later years of the school curriculum, we would advocate the full range of algebraic activity to be present in all the process of teaching and learning of mathematics.

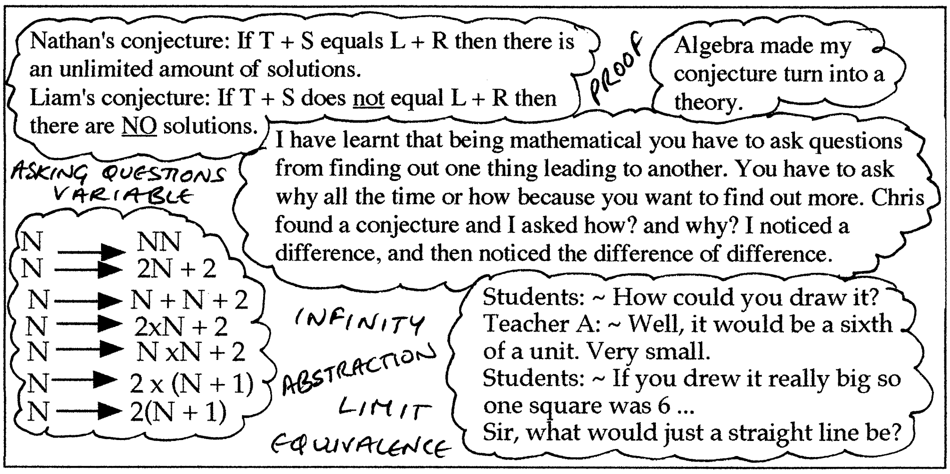

Figure 1. A collage of students' comments and written work to illustrate the mathematical interaction

Notes

1. 'Developing algebraic activity in a 'community of inquirers'' Economic and Social Research Council (ESRC) project reference R000223044, Laurinda Brown, Rosamund Sutherland, Jan Winter, Alf Coles. Further information at www.regard.ac.uk.

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What algebra is required in 'high stakes' system wide assessment? A comparison of three systems.

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This is a report of a preliminary comparative study across three examination systems into the types of algebraic experiences that are being assessed in high stakes assessment at the end high school, where access to hand held technology is assumed. The study has found that manipulative algebra is still a requirement of examinations particularly the skills of transforming and substitution.

Introduction and background

There are a variety of approaches taken to develop a curriculum for school algebra, according to (Bednarz, 1996) these include, the rules for transforming and solving equations, the solving of specific problems, the generalization of laws governing numbers, variable and function and algebraic structures. The author also recognises that with the increasing availability of technology there is likely to be an increasing focus on the concepts of function and variable.

While (Heid, Choate, Sheets, & Zbiek, 1995) have provided advice on how assessment should look when they state "More emphasis will be placed on conceptual understanding than on the execution of algorithms, and more attention will be given to the processes by which mathematical truths are obtained than to the outcomes of those processes. The changing emphasis, as well as the changing content, suggest a need for assessment instruments that differ both in content and in form." (p.37).

Curriculum innovations and changes in assessment often go hand in hand and it is evident that for successful change in curriculum it must be supported by changes in assessment (Morgan, 2000). (Barnes, Clarke, & Stephens, 2000) make this point forcefully when they state, "It is our contention that assessment should be recognised, not as a neutral element in the curriculum, but as a powerful mechanism for the social construction of competence. The imperative is to realize and exploit the significant role that assessment plays in the process." (p.625). The influence that external assessment has on what happens in the classroom should not be underestimated. Therefore, if there is going to be a move away from manipulative algebra to one of developing conceptual understanding and symbol sense, then this need to be reflected in the assessment.

The acceptance of hand held technology into the end of high school "high stakes" assessment is a relatively recent phenomena. Gradually examining boards have moved from graphing calculator neutral examinations to examinations where questions are set assuming the students have a graphing calculator. It is our intention to look at the types of algebraic skills assessed by three examining boards in different countries and to determine the impact that the availability of this technology has had on the algebraic requirements in the examinations.

Introduction to International Baccalaureate

The International Baccalaureate Diploma programme caters for over 1000 schools in 100 countries. There are four mathematics subjects all of which include topics found in most upper high school curricula worldwide (Brown, 1999; Brown & Davies, 2000). In 1995 students were allowed to take graphic calculators into all mathematics examinations, although it was stated that the questions would be written in such a way that students without graphic calculators would not be disadvantaged. In 1997, it was announced that from 2000, examination questions in three of the four mathematical subjects would be written assuming that the students had access to a graphic display calculator.

Introduction to Victorian Certificate of Education, Australia.

The board of studies approved the use of graphic calculators from 1997. In (VBOS, 1998) it was stated, "In developing 1999 examinations in Mathematics Methods and Specialist Mathematics, the Board of Studies will direct the setting panels to assume that all students have access to approved graphics calculators." (p.7).

Introduction to Denmark Upper Secondary Education

The Danish upper secondary education is divided into the general upper secondary (Gymnasium and Higher Preparatory Exam) and the vocational upper secondary education (Business and Technical Colleges). All four qualify for higher and further education entrance, and offer mathematics at A-level, which is the highest level of mathematics offered, and though the syllabi vary they are considered equivalent. The A-level courses in the vocational upper secondary are more business/technical oriented than the general upper secondary courses. Consequently the four A-level courses also differ in how much emphasis is placed on "pure" mathematics and applied mathematics.

In the Gymnasium the final examination in mathematics A, consists of an oral and two written examinations. The written examinations are: A four-hour examination with aids, where the graphing calculator is assumed (Danish Ministry of Education, 1999) and the student may use all texts and other aids but excludes programs and computer algebra calculators, and a two-hour examination without any aids whatsoever. The final mark for a candidate is given, based on the answers to specific questions as well as a general assessment of the two sets of answers, as one overall mark, (Danish Ministry of Education, 2000b).

Methodology

Examinations papers were collected from the three above mentioned examination boards. To ensure some consistency between the content of examination papers, it was necessary to choose those examinations whose main focus was on functions and calculus. As a consequence the following examinations were chosen.

Victorian Certificate of Education, Mathematical Methods, Examination 1 and Examination 2, 2000. (VBOS, 2000a, 2000b)

Studentereksamen Maj-Juni 2000, (Danish Ministry of Education, 2000a)

International Baccalaureate Organisation Mathematical Methods (SL), paper 1 2000. (IBO, 2000)

A scheme of analysis to investigate the use of algebra in examinations was developed from the work of (Senk, Beckmann, & Thompson, 1997) and (Forster & Mueller, 2000) resulting in the following table, Table 1.

Category	Response	Description
Reasoning	Yes	Item requires justification, explanation or proof.
	No	No justification, explanation or proof is required.
Interpretation of Question	Yes	Students need to make connections between the information given and appropriate mathematical structures.
	No	Equations already provided or instructions given
Solving Equations	Linear	Uses linear equations only
	Polynomial	Uses polynomial equations, includes square and cubic roots
	Trigonometric	Uses trigonometric equations
	Exponential, Logarithmic	Uses exponential and logarithmic equations
	Parametric	Use parametric equations
	Other	
Algebraic Manipulation	Transforming	The changing of the form of an equation from one form to another
	Expansion	Removal of brackets
	Factorisation	Students are required to factorise the expression to gain marks.
Function	Substitution	Substitution into the given functions is all that is required
	General	Function provided in general form only. E.g. $f(x) = g'(x)$
Role of Technology	Processing	Used only to calculate or could be used for trial and error solutions.
	Pedagogical	Forms an integral part of the question, by expecting students to experiment in the question with different values.
Technology Required	Required	Question requires technology and cannot be done without it.
	Neutral	Question can be done with or without technology.
	Excluded	Question specifically excludes the use of technology.
Graph Given	Yes	Graph provided as part of the question.
	No	

Table 1. Categories for analysing Examination Questions.

Comparing Assessment

Overview of the curriculum

Looking at the curriculum of the examining boards provides an indication of the composition of the mathematics subject being assessed. In all cases functions and calculus are included. The VCE curriculum specifically includes Algebra and Probability; the IB includes Probability and Statistics, Vector Geometry as well as Number and Algebra, and; the Danish curriculum includes Numbers, Geometry and Vectors, and Statistics and Probability. Though there are differences between each of the curricula the general impression is that the subjects under consideration are roughly equivalent in content.

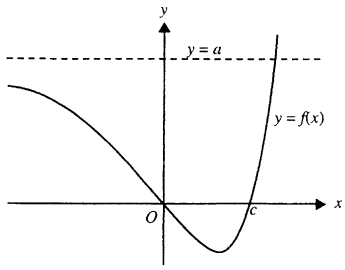
Looking at questions

Each question was considered, including taking account of the different solution paths that are available when using a graphing calculator. Once the question solution has been determined, we used the categories to analyse the question taking account of these different solution paths. As expected the questions at the beginning of the paper tended to be easier both in terms of the solution method and the level of conceptual understanding required to complete the questions. For example Question 1, Examination 2 ,Victorian Certificate of Education.

The graph of the function $f: R \rightarrow R, f(x) = e^{2x} - 2ke^{x} + 3$ is shown. The graph of f has a horizontal asymptote y = a. The graph of f passes through the origin and the point (c, 0)

a) Write down the exact value of a.

b) Show that k = 2.



c) Use calculus to find the exact values of the coordinates of the turning point.

d) i. Find the exact value of c.

ii. Hence use calculus to find the exact value of the area of the region bounded by the graphs of f and the x-axis.

e) Let g be the function whose graph is the reflection of the graph of f in the y-axis.

i. Sketch the graph of g on the axes given .

ii. Write down the rule for g. (VBOS, 2000b)

The instructions on the front page of the examination booklet include the following statement "Where an exact answer is required to a question, appropriate working must be shown and calculus must be used to evaluate derivatives and definite integrals."(VBOS, 2000b) It is for this reason that all parts, except *e)*, of the question were considered to exclude the use of the graphing calculator. Parts *a)* to *d)* inclusive were considered to be instrumental in nature where the solution, required the use of standard rules and procedures. There was only a minimal amount of algebraic manipulation in parts *a)* and *b)* to find the value of a and to show that $k = 2$. The use of function notation was evident but the students were only expected to substitute into these functions at certain points to find the solution. Part *e)* required students to reflect the curve about the y-axis and then find the rule. An understanding of functions and their reflection was required and a graphing calculator could have been used for trial and error but otherwise would have been of minimal use for the entire question aside from the checking of answers..

Questions which appeared later in the examinations required more steps and often reasoning and interpretation on the part of the student, as indicated in the following example:

Opgave 7a. *When a certain patient is given a glucose injection into the blood, the concentration of glucose in the blood (measured in milligram per 100 millilitre) is a function g of time t (measured in minutes). Given that g(0) = 60, and that g is a solution to the differential equation* $\frac{dy}{dt} = 20 - 0.22y$

Define a rule for g. Find how much time passes from t = 0 to a point in time where the concentration of glucose in the blood reaches 80 milligram per 100 millilitre.
(Danish Ministry of Education, 2000a)

Though this question was much more challenging than the first example once again the steps required to complete the problem are of a standard classroom type. Though, in this case, the students were required to interpret the question to obtain information that was relevant in the solution process. This question was written with a graphing calculator in mind. It could have been equally solved without one. The question required students to perform the algebraic tasks of transformation, expansion as well as the ability to substitute into the exponential functions found in the solution.

A question that specifically required the use of the graphing calculator was:

Q4d. Using $x = -2\cos(2\pi\ t) + \sin(8\pi\ t) + 3$.

Find the first time after t = 0, correct to the nearest one-hundredth of a minute, when this model predicts that the hammer will be at its least distance from the platform. Find this least distance, correct to the nearest millimetre. (VBOS, 2000b)

The complex nature of the function would ensure that students use the graphing calculator. Apart from the use of substitution into the function to find the minimum distance, there was no other algebraic skills involved.

Discussion

Overall there were 64 questions analysed from the different examination papers, these questions were categorised using the system as outlined previously. The results of the categorisation are listed in Table 2 below. In some categories there were multiple responses for some questions while for other questions the category was not applicable.

Category	Response	Description
Reasoning	Yes	19%
	No	81%
Interpretation of Question	Yes	47%
	No	53%
Solving Equations	Linear	17%
	Polynomial	17%
	Trigonometric	16%
	Exponential, Logarithmic	16%
	Parametric	3%
	Other	11%
Algebraic Manipulation	Transforming	33%
	Expansion	9%
	Factorisation	5%
Function	Substitution	63%
	General	6%
Role of Technology	Processing	77%
	Pedagogical	3%
Technology Required	Required	13%
	Neutral	41%
	Excluded	30%
Graph Given	Yes	19%
	No	81%

Table 2. Occurrence of various categories as a percentage.

We will confine our discussion to the categories that specifically relate to algebra:

Reasoning and interpretation of question

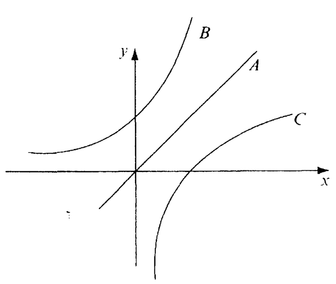
It is evident from Table 2 that most questions were set in such a way that there was no reasoning required, that is the students were only expected to find the answer without justifying their solution or the method used. The questions were equally split between those that required some interpretation and those that did not. Many of the questions that required interpretation were those from the topics of probability and statistics in which no equations were given and often students had to determine which probability distribution was required. An example of a question from functions requiring reasoning and interpretation is given below.

14. The diagram shows three graphs. A is part of the graph of y = x.: B is part of the graph of $y=2^x$.

C is the reflection of graph B in line A. Write down:

(a) *the equation of C in the form y = f(x) ;*

(b) *the coordinates of the point where C cuts the x-axis.* (IBO, 2000)



The students were required to interpret the graphical and the algebraic information and make connections between them. They then needed to deduce that they were required to find the inverse function from the given information. These skills though not algebraic in nature are important if they are to develop skills in problem solving.

Equations and Algebraic Operations

There was a majority questions involving exponential equations in the VCE, and almost no linear equations. The other two examining boards had an even distribution of the different types of equations. In most cases students were only required to solve one or two-step equations that could be solved either algebraically or numerically depending on the requirements of the question. The following examples are intended to demonstrate the level of difficulty found in all examinations.

Question 3. The students are given the picture of a time capsule made up of a right circular cylinder of height h cm, and radius r cm, with hemispherical caps of radius r cm. They are also told that the volume is V cm^3. *They are then required to express V in terms of r and h. (The formulas for the volume of a cylinder and a sphere are provided in the accompanying formula sheet). Part b Given that V = 8000, show that* $h=\frac{8000}{\pi r^2}-\frac{4r}{3}$. (VBOS, 2000b)

The solution to this manipulation can be completed in three steps and is indicative of the maximum level of transformation required by all three examining boards. A question that required a combination of transformation and expansion is given below.

Opgave 4. In a 3D coordinate system, a line is given by the parametric equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ *and a sphere is K is given by the equation* $(x-4)^2+(y+2)^2+(z-5)^2=11$. *The sphere and the line have a point P in common. Find the coordinates of P.* (Danish Ministry of Education, 2000a).

The students need to use the two equations to find the point of intersection; the complexity of the algebra lies in the 3 D coordinate system and the parametric equation. Once the parameter has been removed then the students only needed to substitute for x and y into the equation for the sphere, solve for z and then solve for the other two coordinates. The difficulty in this question lies not in the algebra but in the insight required to recognise the solution path.

Functions and graphs

Where functional notation was used in almost all cases students only needed to substitute values into the function to determine a solution. Example of such a question includes

Question 15. Let $f(x)=x^3$.

(a) Evaluate $\frac{f(5+h)-f(5)}{h}$ *for h = 0.1.* (IBO, 2000)

Where the general form of the function was used eg. $f(x)$ it was always coupled with a question where the use of a graphing calculator was either of no use or excluded all together. For example

Question 6.

On the set of axes provided, sketch a continuous curve with equation $y=f(x)$ *having the following properties*

$f(0)=0 \quad f'(0)=0$

$f(4)=0 \quad f'(3)=0$

$f'(x)<0 \text{ for } \{x:x>3\}$ (VBOS, 2000a)

$f'(x)>0 \text{ for } \{x:x<3\}\setminus\{0\}$

The question uses function notation but it is not possible to use technology at all, the students need to interpret and relate the information to sketch the correct shape.

Conclusion

As we stated previously this is a report on a preliminary study which forms part of a larger report to be made available at a later date. (Heid et al., 1995) recognise the impact that technology is likely to have on the study of algebra when they note, "the study of algebra is bound to change dramatically with the infusion of currently available and emerging technology. What was once the inviolable domain of paper and pencil manipulative algebra, is now within easy reach of school-level computing technology." (p.1).

The evidence to date is that examination questions still expect some algebraic manipulation as part of the assessment, contrary to what (Heid et al., 1995) have suggested. It would appear that some questions are being written specifically to insist on the use of manipulative algebra and exclude the use of technology. This perhaps reflects the uncertainty in the examining boards minds of what is important in mathematics education and what is not.

The use of function notation appears to have the following roles

- to test transformation of functions
- to test understanding of rates of change
- to test summation as a key concept of calculus.

The type of questions involving functions are often, at best, technology neutral but more likely to be technology excluded.

Extending this survey to other examinations particularly those sat by students who intend to follow tertiary courses whose main focus is the mathematical sciences would provide further insight the applications and types of algebra expected in assessment.

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Algebra in an Age of Numerical Mathematics

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In this paper we demonstrate that algebraic manipulation, for the vast majority of people, could be replaced by the availability of fast numerical computation and dynamic visualisation. Algebra can then focus on understanding and describing general patterns that arise.

Part of the justification for teaching algebra in school is that it will be needed by those students who continue on to tertiary studies in engineering, the sciences, finance and economics, psychology, and the many other areas which rely heavily on mathematics and statistics. However, the use of algebra as a manipulative skill is of little importance in these areas. Instead the focus is on modelling and interpreting solutions, with the intermediate solving step typically being of no interest to the user.

There is increasing focus, particularly with recent advances in graphics calculators, on Computer Algebra Systems (CAS). These effectively give the user help in the solving step with their algebra. However, in this paper we will suggest that it may be more appropriate to leave out the algebra entirely and focus on numerical methods. A user will ultimately have particular numbers which describe their problem and will want numbers to describe the solution. Algebra may be needed to describe the model to others or to a computer, but manipulative skill is not necessary to obtain the numerical solution.

Functions

The logarithm tables that were universal just twenty years ago served a simple purpose. For a collection of standard functions, they gave output values for a range of input values. This range of application could be extended by knowing the important algebraic rules for logarithms and trigonometric functions, used in combination with the tables. This required a range of algebraic skills in knowing how to use these identities for specific values of interest.

The pocket calculator has replaced the tables but the role is still the same. We give an input value and the calculator gives an output value. Indeed the algorithms used are likely to be very similar to those used to generate the paper-based tables. The advantage of the calculator is that it is determining the values in "real time", rather than having to pre-generate them, allowing us to give any input value we like. The algebraic skills are no longer needed to obtain function values.

It is interesting to note that the use of tables has not fully died out. In statistics, tables of distributions such as the Standard Normal and Binomial are still present in most textbooks. This is particularly interesting for the Binomial distributions, since a straightforward, though rather daunting, formula exists for calculating the values in the table, whereas the Normal values must be found by numerical approximation. There is an algebra here too for working out values not given immediately by the tables, such as interval probabilities. These tables and their algebra of probability calculations can now be replaced by the functionality of statistical software and many graphics calculators. As these tools become more widespread then perhaps the statistical tables will disappear as the logarithm tables have in the past.

As users we take it for granted that "sin" and "Normal" have been specified and that the tables or software or calculators will give us correct answers for our queries. As technology improves, the algebraic skill required to obtain the correct answers diminishes. Our aim in this paper is to show how this could be extended to a range of areas which are typically seen as important applications of algebra. Once the function has been specified then algebraic effort in finding values is replaced by numerical methods of computation. These methods then lead to visualisation and exploration, and thus algebra which is more focussed on generalisation rather than manipulation.

Calculus

We will focus on some examples from calculus, since this is often where algebra leads for students proceeding to tertiary study.

One area where the ability to carry out algebraic manipulation has been important in the past is indefinite integration. In particular, the use of trigonometric identities is vital for many algebraic integration problems and techniques such as integration by parts and by change of variable. A first-year mathematics course will often involve the same large quantity of practise integrations that students experienced when learning algebra. For example,

$$\int \sin^2(x)dx = \tfrac{1}{2}\int (1-\cos(2x))dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

Yet why do we want to know this integral? A standard application is to find the area under a curve. For example, if the velocity of an object at time t was $\sin(t)$ then the integral,

$$s(t) = \int_0^t \sin(u)du,$$

will give the position of the object at time t. This is a complete specification of the function $s(t)$. A computer can evaluate $s(1.9)$ using a numerical method. Mathematica's NIntegrate gives $s(1.9) = 1.32329$ (to five places). It so happens that $1 - \cos(1.9) = 1.32329$, and indeed $s(t) = 1 - \cos(t)$ for all t, but it is not necessary to know this in order to make a graph of $s(t)$, as in Figure 1.

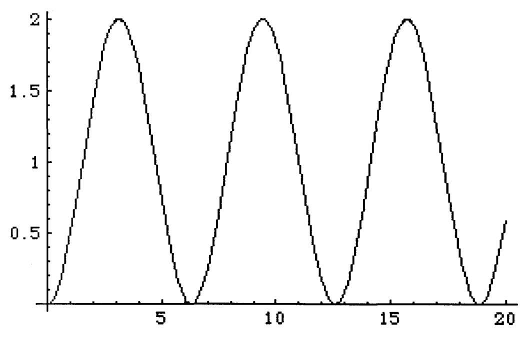

Figure 1. Graph of $s(t)$ for t in [0,20]

This makes philosophical sense as well since carrying out the algebra to determine that $s(t) = 1 - \cos(t)$ does not really help us find the solution. Unless we are lucky enough to want the position at time $t = \pi/6$ or the like, we will still need to use our calculator (or tables!) to find the value of $s(t)$. At the moment this is more convenient since calculators usually have a "cos" button but do not have a "numerical integration" button. Yet many graphics calculators can do numerical integration, and were doing so before they started to acquire algebraic capabilities.

The need for algebra now comes in exploring and interpreting the behaviour of the function. We can alter parameters in the model and see how the solution varies numerically or visually.

As a second example, we can specify a function by giving a differential equation that it satisfies. A simple example is to define $f(x)$ by

$$\frac{df}{dx} = 2x, f(0) = 0.$$

We do not need to solve this initial value problem algebraically to be able to make a graph of $f(x)$. Figure 2 uses Mathematica's NDSolve to plot values from –2 to 2.

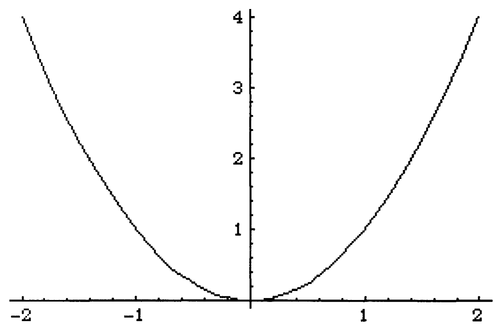

Figure 2. Graph of $f(x)$ for x in [-2,2]

Again it so happens that $f(x)$ is the same function as the function x^2. As a further example, consider the more interesting initial value problem

$$\frac{dy}{dx} = -ky, \; y(0) = 1.$$

Figure 3 shows graphs of $y(x)$ for $k = 1$ and $k = 2$. In practice, the value of k and any other parameter could be controlled dynamically to explore its effect on the solution. We don't need to know about the exponential function or any of its algebra to describe the pattern of decay we see from this graph.

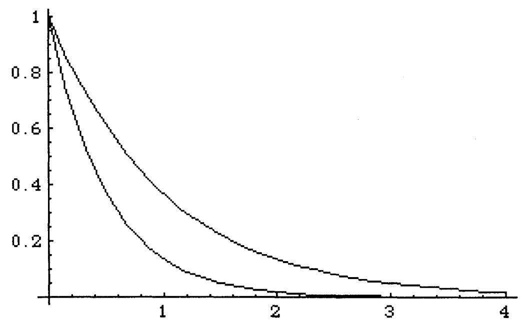

Figure 3. Graph of $y(x)$ for x in [0,4] for k=1 and k=2

Finally, consider the famous integral

$$\int e^{-x^2} dx .$$

This integral is important because of its role in the Normal distribution, yet we cannot evaluate it algebraically. As seen in many differential equations, algebraic manipulation does not always allow us to find an answer and so we must resort in such cases to numerical methods. This begs the question as to why we don't use the numerical methods from the very beginning, since they can work on a wider range of problems.

Computer Algebra Systems

In the first three examples above, a Computer Algebra System, such as Mathematica or Maple or one of those now available on a graphics calculator, could have been used to carry out the manipulative algebra required. However, this manipulate algebra is often simply the "solving" step in a practical problem, and the computer algebra system is helping us find the solution. This is analogous to using a calculator to do long multiplication by working out all the intermediate numbers and adding them up.

We usually begin with a description of a problem involving numbers and want a solution involving numbers. It may be unnecessary to use algebraic manipulation, whether by hand or with CAS, to carry out the steps in between. Instead we can focus on interpreting the result graphically, looking at sensitivity to parameters with dynamic visualisation, and so on. Students should be able to use algebraic language and symbols to describe the problem, but can then go straight to exploring the results of their model without the algebraic work.

Conclusions

We have shown examples where numerical methods could replace algebraic manipulation. This could be taken to the extreme, replacing all algebraic work by numerical methods, though in practice it may be that students still learn manipulative skills but are made more aware from an early age that they can find solutions using numerical methods. This is the same as teaching students how to do long multiplication and division; in practice they will use a calculator to do these things. However, the availability of numerical methods and visualisation may allow the focus of algebra to move more easily towards understanding and describing generality.

Algebra for All:
What does it mean? How are we doing?

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This paper will describe an approach with the following elements:

- a rough taxonomy of aspects of algebraic performance;
- a broad assessment instrument for looking at these aspects, used separately and together
- some exemplar reviews of the levels of performance that have been achieved by a few treatments;
- discussion of a benchmarking tool that can help support practical progress in this important area.

While the focus is on practical progress in typical classrooms, the paper raises issues of principle for the research and development communities, including that of "treatments v principles" (or "How much do the details matter?")

This outline is presented mainly in note form, representing work in progress. Insofar as it contributes a different and useful view, it may be of interest to the IPC. If so, it will be developed further in time for the Conference, including further contributions from Alan Bell, Malcolm Swan, the Shell Centre team and collaborating projects. The work outlined will continue.

Background

There has been a lot of fine work on the learning of algebra, probing conceptual understanding and misunderstanding, and more will be reported at this ICMI meeting. Attention is focussed on matters of principle in the learning and teaching of algebra; these are clearly important.

However, these principles, even if fully understood, are far from enough in themselves to determine the design of a curriculum. It is possible, even likely, that the details of design and development are at least equally important in determining the learning outcomes. Thus it seems worth looking at, and probing in some detail, the outcomes from actual curricula. So this paper takes an approach, focussed on classrooms rather than research environments. We look at two separate aspects which, in research terms, may be called "treatments" and "instruments".

The treatments in question are curricula designed for, and sometimes delivered by, typical teachers. The emphasis is on performance, which is in some sense directly observable, rather than on understanding. It also reflects a major focus of our current work on balanced assessment in MARS – the Mathematics

Assessment Resource Service. The instruments are at the moment an opportunistic mixture of research instruments and assessment tools which have been used variously to probe performances related to "algebraic thinking".

Standard tools, such as TIMSS, APU or the CSMS task set, have strengths – each task is focussed on a specific element of performance, linked to a specific concept. This is also a complementary defect – though it makes them relatively 'clean' probes, the chains of reasoning are short.

On the other hand, most significant problems where mathematics gives extra power involve longer chains of reasoning, where several concepts and skills have to be selected and effectively deployed in the context of the problem. It seems that a balanced assessment of mathematical performance should thus include a substantial weight of the latter. (In a driving tests, you might want to specifically test steering, gear changing and looking in the rear view mirror but driving on the road (or, as with aircraft) at least on a good simulator does seem important) MARS work with a range of client systems, some of them particularly concerned with "algebraic thinking" has developed a range of tasks which assess performance in more holistic way, involving substantial chains of reasoning on worthwhile problems. In benchmarking, these and the more detailed probes complement and illuminate each other.

(This complementarity also reflects an issue of appropriate 'grain size'. It is not profitable to do biology or engineering by using quantum mechanics; though it is indeed the underlying dynamics in these fields, it is too detailed and cumbersome for practical use in understanding those complex domains. So, if there are 84 misconceptions which students show in the subtraction of two 3-digit whole numbers, it may be more profitable to shift the focus and analyse and handle the learning and teaching processes in a more "clumped" way – for example, by focussing on students' learning to detect and debug their own misconceptions.)

"Doing algebra" – what does it mean?

"Doing algebra" has a very wide range of meaning, even in the school curriculum, from substituting numbers in a given formula or extending a simple pattern to constructing a formal proof – for example, that the powers of 2 cannot be written as the sum of consecutive natural numbers, and beyond. Each aspect of performance has value as a contribution to the mathematical power of those who have it. There is some good evidence that nearly all children can achieve the former kinds of performance, but the last kind is achieved by few.

Here we attempt a provisional partly-research-based taxonomy of algebraic performance, with a rough order of difficulty. (The taxonomy will be developed and the caveats will be made clear) In compiling it, we have leant on several studies. As always, the empirical difficulty levels they report are dependent on the implemented curriculum in the system studied. (Everyone has their favorite example: In the UK National Curriculum, multiplying two decimals comes *several years before* knowing 0.29 <0.5, called "decimal inequalities" – because historically it gets little attention in the classroom, probably because teachers think it is obvious)

With this caveat, the order below is probably one of increasing difficulty. As with all general statements, the level of difficulty depends on the complexity and familiarity of the specific problem; the order here is for first achievements of this kind.

P1 extending number and geometric patterns, stepwise
(You may choose whether to call such unsymbolised generalisation algebra or pre-algebra)

P2 formulating verbal rules for stepwise pattern extensions

S1 substituting numbers in formulas, evaluated with calculators

C1 following simple algebraic programming commands, let A=B+C
understanding simple formulas in spreadsheets

F1 formulating verbal rules for functional relationships, or explanations for general results
(eg in patterns, proportional situations)

C2 formulating simple algebraic programming commands, let A=B+C
formulating simple formulas in spreadsheets

C3 formulating simple programs with branching structures
constructing simple spreadsheets, copying formulas

S2 formulating well-understood functional relationships symbolically

S3 rearranging linear algebraic expressions, solving simple equations

S4 rearranging algebraic expressions, solving simple equations

R1 following, critiquing and modifying chains of symbolic reasoning
following, critiquing and modifying spreadsheet structures

C3 formulating procedural programs
constructing linked spreadsheets

S5 inverting functional relationships, solving equations

R2 constructing deductive chains of symbolic reasoning

R3 constructing general symbolic proofs

Looking at algebraic performance – an instrument

How might we assess the levels of performance in these various aspects of algebra? Two elements seem important (though in many studies only the first is used)

- Short tasks which probe one aspect of performance, and of understanding. This "clean" approach has obvious attractions. However, in doing algebra as in any other complex activity, the whole is more than the sum of the parts – achieving the elements separately does not guarantee the ability to integrate them into a coherent and reliable attack on a substantial problem. (Even conversely, some "weaker" students show higher levels of performance on complex tasks than on the elements within them, tested separately – probably because the task is more meaningful) Thus we need:
- Longer more complex tasks, closer to those in which algebra is actually useful, where the focus of attention is broader so as to assess in a more holistic way the student's ability to integrate different aspects – of algebra, of other mathematics and of understanding of the context of the problem.

Correlations between performance on whole tasks and the 'subskills' are likely to provide non-obvious results and some interesting insights.

We are building such an instrument, starting from two main sources:

- for specific focus, the CSMS algebra set (Kuchemann, Hart et al), which have been widely used in systems around the world
- for holistic assessment, the range of "algebraic thinking" tasks developed by the MARS projects for Grades 4 to 10.

In consultation with others, we add other tasks that seem to offer additional dimensions of probing.

The development of a 'benchmarking tool' for the various aspects of algebraic performance is a major goal of this work. Illustration of its various elements will be an important part of any presentation.

What has been achieved, by and with whom?

Here we look at a few selected treatments to exemplify our approach. For each:

A The nature and provenance of the treatment will be briefly described.

B The scale and the circumstances of its implementation will be outlined, with attention to the typicality of the teacher and student populations involved.

C The evidence of the performance levels achieved in different aspects of algebraic performance will be reviewed and the gaps noted.

(In this outline, there are very brief notes on three projects) It will be no surprise that the full stories, too, are incomplete, even fragmentary. How we might provide a fuller more reliable picture is the concern of this paper.

School Mathematics Project
We choose this treatment for two complementary reasons. These materials were very widely used in English secondary mathematics classrooms (over 60%) during the 1970's. This was the environment in which the CSMS study, mentioned in the last section, was carried out. In that sense it provides a favorable case for study, rough though the evidence remains.

A SMP produced two sets of textbooks for students aged 11 to 16. The "number books", 1–5, were designed for and developed with students in selective schools by a writing team of teacher-authors, mainly based in such schools. The explanatory approach might be called "closely guided discovery". They were designed for the top 25% or so of performers, and the GCE)-level examination they took. The materials were popular and successful but proved challenging, even for this group. There was a demand for "SMP for all" so the "letter books" A-H were developed. They covered much the same conceptual ground but with much less demanding tasks. It was these materials that were most widely used.

B These materials were dominant in UK classrooms in the 1970's and early 80's. Professional development was provided for teachers new to the materials, and fairly widely taken up.

C The picture is qualitatively clear, and a familiar one. The careful and thorough CSMS study (and of the APU) focussed on individual concepts. A familiar pattern emerged, with fewer than 25% of students showing broad understanding. Some of the easier aspects, above, were widely accessible but, for example, the ability to manipulate algebraic expressions reliably was rare. This was reflected in the form and results of the more holistic O-level and CSE examinations. The uses of algebra in tackling practical problems were neither probed nor taught.

Computer Assisted Mathematics Program
We choose this treatment for two very different reasons – a new approach with well-researched successes.

A CAMP was developed by David Johnson and his collaborators at the University of Minnesota in the 1960's, again for students aged 11 to 16. Developed with students in a university school, using a link to the University mainframe computer, the goal was to see how far computer programming in BASIC, as a regular element of the curriculum, could forward the learning of mathematics. The materials are in the form of textbooks which, of course, require computer support.

B For obvious reasons of technological provision and organisation, this was inevitably a relatively small-scale development, with good support for the teachers involved.

C The research back-up for this funded project was substantial, with carefully developed instruments and control groups. In the study of algebra, Tom Kieren showed that about 50% of CAMP students aquired fluency in algebra, about twice the (normal) proportion in the control group. This was ascribed partly to their learning symbolisation in the computer programming, a "semi-concrete bridge" to the more flexible, and thus slippery, meanings of symbolic expressions in algebra. The details are interesting.

It is interesting and ironic that, though the technology has become much more widely available, there has been no direct successor to this work – indeed, programming as such has disappeared from contention as a curriculum element. The work on spreadsheets is, perhaps, a indirect descendent.

Connected Mathematics Project

We choose to feature this treatment because, among the 12 curriculum materials projects funded by the National Science Foundation, it is both one of the more widely-used ones and one of the most studied. It takes the beginnings of algebra very seriously, and at an earlier age than is customary in the US.

A CMP developed materials for the Middle School Grades 6 to 8. The NSF-funded design team is based at Michigan State University, with development in classrooms across the US. Each year's materials are published as a series of about 10 units on different areas of mathematics. The focus is on meaningful learning of concepts and skills through problem solving,

B The materials are widely used in school districts which are committed to reforming their mathematics curriculum along the lines set out the the Standards developed by the National Council of Teachers of Mathematics. Since materials are usually adopted district-wide, the sample of teachers and students is reasonably representative.

C There is a good deal of independent evidence on the performance of students in a group of CMP schools, compared to a comparison group. For example, CMP students performed well ahead of the others in tackling substantial problems; on the short tasks of ITBS, which are closer to traditional skill-focussed curricula, the CMP students moved over three years of work from rough parity to significant gain. Of course, only early algebra is involved, even at Grade 8.

It would be of interest to compare performance levels on the different aspects of algebra in strong-implementation classrooms using materials from the other parallel NSF curriculum projects, particularly those in the High School, where most of algebra learning takes place. Resrach evidence on these is now emerging. What patterns of strengths and weaknesses would we find in each case? (Another intriguing middle school example is 'Mathematics in Context', in which the Netherlands group from the Freudenthal Institute played a major role in the design and development)

Where do we go from here?

To summarise, the goals of this work are:

- to develop a useful array of performance assessment instruments which, along with evaluation of the strength of implementation in specific classrooms, can be used to evaluate in some detail the impact of specific curricula on student performance in different aspects of algebra, both microscopically and holistically.
- to collect results on a variety of curricula, and relate them to a qualitative assessment of the design features of the materials concerned.

We see this global treatment-focussed approach as a complement to the many micro-studies of the understanding, learning and teaching of algebra.

Further work in this direction is in progress and will continue.

Number Theory and the Transition from Arithmetic to Algebra: Connecting History and Psychology

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Number theory is typically viewed as a generalisation of whole number arithmetic or as an specialised adjunct to algebra. This paper suggests that number theory may have a more central role to play in helping learners make the transition between arithmetic and algebra. Research in teaching and learning number theory, however, remains in a relatively insipient state. Phenomenological considerations toward the cognitive history of mathematical thinking can serve to inform research in the psychology of mathematics education in this regard.

Number Theory, the study of the properties of the integers, as a subject in and of itself, at times seems all but banished from the schools. Essentially exiled to institutes of higher learning, some basic topics of the subject are typically found scattered about preK-12 and preservice teacher mathematics curricula (e.g., NCTM, 2000). Perhaps this accounts, at least in part, for why teaching and learning number theory, despite the substantial amount of work that has been done studying the teaching and learning of arithmetic and algebra, has only recently become a focus for research in mathematics education, and the focus of most of this work has been oriented toward the subject and pedagogical content knowledge of prospective teachers (e.g., Zazkis & Campbell, 1996a, 1996b; Campbell, 1999; Campbell & Zazkis, forthcoming a). At this point, very little is known about the cognitive, instructional, and curricular implications of number theory. Should number theory be taught and learned in a more systematic manner, and can it be more deliberately and cohesively situated within the K-12 curriculum? Overall, preK-12 mathematics educators and curriculum designers have had little guidance from research regarding the cognitive and instructional utility of number theory *per se*, or of more central concern here, its potential utility in expediting the transition between arithmetic and algebra.

In some respects, number theory is already well situated between arithmetic and algebra, insofar as it is often thought of as both a generalisation of whole number arithmetic and as a specialised "subset" of algebra. It has long been recognised, on the one hand, that introductory topics of number theory, such as multiples, divisors, factors, divisibility, prime factorisation, the division and Euclidean algorithms, etc., contribute toward a conceptual understanding of whole number arithmetic. The central theme of Standard #6 for the middle grades in the NCTM *Curriculum and Evaluation Standards for School Mathematics* (entitled "number systems and number theory") was concerned with facilitating students' understanding of the underlying structure of mathematics and of arithmetic in particular (1989, p. 91). On the other hand, number theory can also be thought of as a subset of algebra dealing solely with the integers or whole numbers. What is more often referred to by 'algebra' in the K-12 curriculum, however, involves working with (lower order) polynomial equations and functions over the domain of real or rational numbers. Various problems from number theory are often studied under the guise of algebra (e.g., Healy & Hoyles, 2000). Presumably, this is because problems from number theory typically involve some degree of algebraic manipulation and reasoning with variables. In these ways it at least seems natural to think of number theory in algebraic terms, if not as a *bona fide* subset of algebra.

Overall, there are a wide variety of ways to think about number theory with respect to arithmetic and algebra: In terms of the relation between the concrete and the abstract, for instance; or with regard to whole and integer numbers in relation to rational and real numbers; or with regard to various coefficients, parameters, variables, and functions over these numerical domains; or with regard to reasoning and proof, etc, not to mention matters of instruction and pedagogy associated with the teaching and learning of all of these things. Despite its utility for helping students develop better understandings of the abstract conceptual structure of whole numbers and integers, and despite its algebraic characteristics with respect to variables and mathematical reasoning, number theory itself is often viewed by mathematics educators as being more tangential than central to the mainstream preK-12+ arithmetic-algebraic continuum. Indeed, this adjunct status of number theory to arithmetic and algebra appears to be so prevalent within mathematics education that it will be referred here to as *the standard view*.

One reason in support of the standard view is that number theory, as the study of the properties of the integers, constitutes a well-defined subject content area. As such, number theory can be taught and learned separately from arithmetic and algebra. Perhaps the main reason why number theory is considered tangential to the transition between arithmetic and algebra is that there is, after all, much more to both arithmetic and algebra than the domains of whole numbers or integers. Arithmetic and algebra, as taught in the K-12 curriculum, typically treat integer (and whole) numbers as subsets of real (and rational) numbers. Pedagogically, this subsumption of numerical domains is usually taken as given (e.g., Freudenthal, 1983, p. 103). Accordingly, the integers, for all intents and purposes, are taken to be *conceptually* identical to the subset of the reals known as the integers. This identification, however, is a relatively recent formal development in the history of mathematics. An apparently related perception underlying the standard view is that it is more important for students to gain proficiency with rational numbers, and that in any case they will gain sufficient experience with whole numbers working in various problem-solving contexts (NCTM, October 1998, p. 206).

Even with the adjunct status of number theory in the preK-12 curriculum, there are good reasons for considering its role with respect to teaching and learning algebra. The ICMI discussion document (February 2000) indicates, "... the word 'algebra' will be interpreted broadly" (p. 1), and as we have seen, there are ways in which number theory would certainly fall within that category. Indeed, the standard view of number theory constitutes an important "microcosm," or "conceptual field" (Campbell & Zazkis, forthcoming b) for study and analysis in addressing cognitive gaps (e.g., Herscovics & Linchevski, 1994), and the overall transition between arithmetic and algebraic thinking. The main question of concern here, however, is whether number theory can and should play a *more central* and *less adjunct* role in helping students to make that transition. In other words, just as algebra is considered the gateway to higher mathematics, is there a sense in which study of properties, variables, and functions over whole numbers can be considered as the gateway to algebra? With the need for more research in this area, the historical development of mathematical cognition can provide some insight into the kinds of questions that need to be investigated in order to determine the extent to which number theory can contribute to the psychological development of young learners, and the extent to which it may help expedite the transition from arithmetic to algebra.

Consider, for example, the closely related set of conceptual, procedural, and semantic differences between indivisible and divisible units respectively: counting and measuring; discrete quantities and continuous magnitudes; and more generally, between whole number arithmetic and rational number arithmetic. There is evidence to suggest that some learners are hard pressed in their understanding of differences and relations between whole number and rational number division (Campbell, forthcoming a), which in turn, are also grounded on the difference between indivisible and divisible units. It may be an unanticipated consequence of the standard view toward number theory that researchers, teachers, and learners are not typically aware of the manifest importance of these differences in the cognitive history of number.

A Brief Cognitive History of Number

Given that a full, accurate, and essentially unabridged story could be told about the cognitive history of the development of mathematical thinking, and given the theoretical and conceptual development of number served as a gateway on the road to algebraic understanding in the cognitive history of Western culture, why should it follow that such a developmental path serve in the same or even a similar sense with regard to the conceptual history of mathematics in other cultures--or more central to this paper (as I cannot consider the question regarding cultural relativity here), with regard to psychological development of learners today? Various aspects of the historical development of mathematical cognition may serve to inform, inspire, and enlighten teachers and students alike, but one should not thereby assume that psychological development must always be constrained or enacted in similar ways. Determining necessary (or even identifying conducive) conditions for improving the psychological development of algebraic thinking is a much more difficult question, and it may ultimately be the question upon which determining the manner and extent to which number theory may serve as a gateway from arithmetic to algebra must reside. Having said that, however, there may very well be important ways in which key shifts the historical development of Western thought actually do limit or enable the psychological development of learners' understandings of mathematics.

From an historical point of view it is evident that the early origins of ancient Greek mathematics were predominantly concerned with the properties of and relations between whole numbers. Different structural forms of whole numbers are evident in Pythagorean figures such as "square," "oblong," and "triangular" numbers (which I suspect were quite possibly early precursors of the Platonic forms). The observation that two sequential triangular numbers form a square number provides an exemplary case of seeing the general in the particular (Mason & Pimm, 1984). Exploring why the expressions "n^2," "n(n+1)," and "n(n+1)/2" provide general algebraic forms for square, oblong and triangular numbers respectively provides a preliminary indication of the potential role of number theory in helping learners develop more intuitively grounded meanings of variables, functions, and proofs ranging over whole numbers--thus helping them make a more meaningful conceptual transition from arithmetic to algebraic understanding. It is also evidently a part of Western cultural heritage that number theory, at least in the sense of the theoretical or conceptual (vs. the applied or procedural) study of whole numbers, preceded the development of modern algebra (Klein, 1992/1968). The study of Pythagorean figures provides a important clue as to the *conceptual* origins of numerical forms, which have through the course of time come to be represented as general algebraic formulae. There is clearly much more to the cognitive history of mathematics than this, however, and mathematics educators and researchers alike would benefit from knowing more about it than we typically do. At the very least, such a history should provide an "existence proof" as to how a conceptual transition from arithmetic to algebra is possible. As tenuous, winding, and dark as the roads travelled over the ages might be, there are important signposts of which to take note.

Elsewhere, I have attempted to provide some insight into key metaphysical (*qua* cognitive) shifts involved in the developmental history of number theory that need not, for the most part, be revisited here (e.g., Campbell, 1996; 1998; 1999; 2000). For the sake of the purpose at hand, however, I will expand upon an issue that calls the standard view into question: a couple of key developments in the cognitive history of number involving the numerical unit. One crucial qualitative shift in the notion of the numerical unit was from the Pythagorean proto-atomic, spatially extended, geometrical notion of unit (Kirk & Raven, 1966/1957) to the Platonic idea of an *indivisible unit* with no concrete attributes whatsoever (a notion that appears to be related to Parmenides' notion of "the One," see Plato, 1945/~388-378 B.C.E.; and for more in this regard, Campbell, 1998). A second qualitative shift of paramount importance was from the Platonic to the Aristotelian concept of a numerical unit as a *divisible unit* of measure (Klein, 1992/1968). To what extent might these shifts in the cognitive history of Western thought have occurred separately or independently of each other? Is there a sense in which each provides necessary or sufficient conditions for the emergence of the other? Must psychological development recapitulate historical development? More specifically, in what manner and to what extent is the *cognitive* history of mathematics of educational relevance?

It is helpful to recall that psychology was in the not too distant past still considered as a branch of metaphysics (Preece, 1967). If a plausible connection between the historical and psychological development of mathematical thinking is to be found, it may be evident in the historical development, or what I have been referring to as the *cognitive history* of mathematical thinking in Western culture. If so, then one might ask a question such as: Could the Aristotelian concept of an arithmetic unit as a continuous and divisible unit of measure have been so readily developed without the benefit of the Platonic idea of a discrete and indivisible unit? Similarly, could Platonic ideas regarding number have been developed independently of Pythagorean proto-atomic notions about the geometrical nature of number? Whether such questions can be answered though a comprehensive study of the cognitive history of mathematical thinking in the affirmative or not, simply formulating such questions can serve well to inform and guide research into the psychology of mathematics education. It will be helpful first, though, to inquire into what possible grounds there might be for relating or connecting developments in the cognitive history of Western culture with the psychological development of learners today.

Phenomenological Considerations

Phenomenology provides one way of discerning and establishing meaningful connections between historical and psychological developments in mathematical thinking (c.f., Klein, pp. 117-125; Campbell, 2001). The basic objects of experience provide a natural point of departure for discerning different notions of a numerical unit. Given the ancient Greeks' perceptual experience of objects was not substantially or qualitatively different from our own, one may reasonably infer that the phenomenological structure of their perceptual experience of objects was not substantially or qualitatively different from ours as well. We learn from experience that some objects tend to present themselves as having a sense of integrity to them, such as trees, stars, and computers for instance, and there are other kinds of objects that can be more readily separated or broken apart without altering their essential nature, such as wood, water, and sand. One can break a piece of wood, for instance, into two pieces of wood, but one should usually avoid trying to break a tree into two trees. Observation of and reflection on these different kinds of objects provide phenomenological grounds for distinguishing between things that can be more readily or rightfully divided than others--thus providing a pedagogical basis for identifying things that cannot be broken into pieces without compromising their "integrity" (that is, without inducing substantive changes in what they are), versus those that can.

Given there are phenomenological grounds for distinguishing between indivisible and divisible objects in a child's day to day lived experience, what can we possibly gain from that? A classic question of Piagetian constructivism concerns the extent to which children are capable of reflecting on and abstracting from the operational structures of their own experience, and at what point in their lives are they typically capable of doing so. More specifically of concern here is to what extent are purely conceptual abstractions--such as indivisible and divisible units, whole and rational numbers, variables, operations, functions, and reasoning over these conceptually distinct numerical domains--within the reach of young children? Would a more phenomenologically informed instructional focus on these concepts from a number theoretical view prove more effective than conflating these conceptually distinct arithmetic domains and subsuming number theory into algebra over the rational numbers? This would seem to be a lynch-pin question regarding any determination as to whether number theory should play a more central and less tangential role in the transition between arithmetic and algebra from the early into the middle grades. Phenomenological considerations regarding the cognitive history of mathematics may suggest or lend support to certain novel and innovative pedagogical practices but determining the degree of their success is a task for research.

One approach to investigating connections between historical and psychological developments of mathematical thinking--and, more specifically, the role of number theory in developing mathematically meaningful understandings of the transition between arithmetic and algebra--is to investigate the quality of learners' understanding when the pedagogical approach that is used places historical and psychological developmental trajectories in conflict, as is, I would suggest, most commonly the case with the standard view. Children's difficulties in understanding the relations between whole numbers, fractions, and decimals are well known and have been well documented (e.g., Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989; Markovits & Sowder, 1991; Mack, 1995). These kinds of difficulties can become quite deeply entrenched, manifesting themselves well into adulthood. Consequently, they appear to be difficulties that can not be assumed to have been alleviated with subsequent exposure to algebra. Research into prospective teachers' content knowledge (a population absolutely crucial to any systemic aspirations for future improvements in mathematics education) indicates much of their understanding of basic arithmetic remains sparse, fragile, and disconnected (e.g., Ball, 1990; Campbell, forthcoming a; Simon, 1993). Such difficulties could be taken as an indication, if not conclusive evidence, that *not* clearly distinguishing between whole number arithmetic and rational number arithmetic, between whole number and rational number division, between indivisible and divisible units, and the procedural, conceptual, and semantic relations between them, presents a general problematic in understanding basic arithmetic for learners of all ages.

The fact that so many learners have difficulties understanding differences between various representations of whole, integer, rational, and real numbers, and the fact that so many learners experience difficulties with the transition between arithmetic and algebra may be associated with a wide variety of factors. The age, dispositions, attitudes, beliefs, socio-economic status, cultural background, gender of the learner, characteristics of teachers, parents, and peers, for instance, may all be contributing factors to one extent or another. I am suggesting, however, that perhaps the main factor involved in making the transition between arithmetic and algebra is the *conflation factor*: the procedural, conceptual, and discursive conflation of whole number and rational number arithmetic, and the corresponding extension of this conflation implicitly residing in the standard view toward number theory with regard to algebra. Furthermore, I am suggesting that the root of this conflation factor is grounded in a relatively recent development in the history of mathematics that has *logically* subsumed whole (and integer) numbers as a formal subset of rational (and real) numbers. This development appears to have motivated and encouraged some serious pedagogical mismatches between the historical, psychological, and formal development of mathematical understanding (see Campbell, 2001).

A possible justification for living with the conflation factor, and the standard view toward number theory which appears intimately related to it, is to claim that young children are simply not developed or experienced enough to grasp the various abstract distinctions and relations to be made between whole number and rational number arithmetic. Furthermore, no one is at all likely, under any circumstances, to recommend forsaking the latter for the sake of the former (what teacher would willingly forestall using the number line as a pedagogical tool?). Any suggestion that instruction in rational number arithmetic be suspended, say, until the middle grades, for instance, could hardly be taken seriously, and it is important to be clear that nothing of the sort is being suggested or implied here. Rather, from a cognitive point of view, it is more the commonly held assumption that young children are incapable of abstract thinking that is being challenged as being mistaken and misguided, both on theoretical and empirical grounds (e.g., Egan, 1997; Carraher, Schliemann, & Brizuela, 2001). Ironically, it may very well be the case that the cognitive difficulties in children's understanding of basic arithmetic is a result of selling short their cognitive abilities. It may also prove to be the case that their cognitive difficulties are in large part a manifestation of pedagogical mismatches between the historical and psychological development of mathematical thinking in Western culture.

It is also important to be clear at this point that it is not being suggested or implied that the psychological development of learners should be an exact recapitulation of the historical development of mathematics, even if such a thing were possible. Such a view is clearly absurd. Rather, what is being suggested here is that the cognitive history of mathematical thinking may be constituted of a few key moments that collectively constitute a developmentally significant sequence of critical shifts in understanding, and furthermore, that a phenomenologically informed consideration of these key moments can potentially lead to significant insights regarding the nature of mathematical cognition and, thus, have significant implications for curriculum and instruction. I have offered, as a possible case in point, two key moments in the history of the development of mathematical thinking regarding the numerical unit.

Even if there are no detrimental mismatches regarding key moments in historical versus psychological development in contemporary mathematics education, how might research help to sort out the reciprocal relations and interdependencies between cognition and instruction with regard to determining the proper role of number theory in the contemporary curricula? One possible framework for coordinating such a task is the notion of a *conceptual field*. Vergnaud (1982) has described a conceptual field as "... a set of situations, the mastering of which requires a variety of concepts, procedures and symbolic representations tightly connected to one another" (p. 36). Although much work has been done over the past two decades in studying additive and multiplicative conceptual fields with regard to arithmetic and algebra, and although learners' understanding of some basic topics from elementary number theory have recently been studied (e.g., Zazkis & Campbell, 1996a; 1996b), a more comprehensive and systematic program of research in this area seems warranted but has yet to fully materialise--either with regard to number theory as conceptual field unto itself, in relation to other conceptual fields, or in relation to the historical contexts from which mathematical thinking has emerged from in the first place. Some recent attempts, however, have been made to help contribute to and encourage the further development of such a program of research in a variety of areas concerning cognition and instruction (e.g., Campbell, 1999; Campbell & Zazkis, forthcoming a).

Conclusion

If the new National Council of Teachers of Mathematics *Principles and Standards of School Mathematics* (NCTM, 2000) is any indication, the prevailing standard view of mathematics educators regarding the tangential role of number theory with regard to arithmetic and algebra implies that teachers should prioritise rational number arithmetic and that students will basically learn to discern between whole number and rational number arithmetic through the contexts in which they are used. Indeed, working with and recognising "equivalent" numerical representations is now being stressed. Research indicates, however, that learners have notorious difficulties in understanding conceptually conflated connections between whole, fractional, and decimal representations of rational numbers. Unfortunately, for many learners, the main commonality between arithmetic and algebraic understanding is procedural in orientation. Although learners can readily intuit whole number and rational number arithmetic in terms of discrete objects or continuous magnitudes respectively, learners often fail to reify the many procedural, conceptual, and linguistic distinctions that follow from those observations. Moreover, many students seem to be readily defeated when it comes to transferring the mathematical knowledge and understanding they do have beyond the specific contexts in which it was learned. On the bright side, the new *Principles and Standards* does promote the study of introductory topics of elementary number theory in the middle school curriculum, and the utility of number theory in reasoning and proof is becoming more widely recognised as well.

Historical considerations regarding the early development of mathematical cognition in Western culture suggests that number theory may have been given short shrift with regard to the importance of its role the transition from arithmetic to algebraic thinking. It is certainly the case that the standard view tends to conflate fundamental procedural, conceptual, and linguistic distinctions between whole numbers comprised of indivisible units and rational numbers comprised of divisible units. Being conscious and clear about these distinctions, and by drawing on number theory to study the structure of whole numbers; by relating arithmetic forms to algebraic formulae (such as triangular numbers with "$n(n+1)/2$," respectively); and by reasoning with and proving equational relationships between those forms and formulae (such as "two consecutive triangular numbers form a square number" and $n(n+1)/2 + (n+1)(n+2)/2 = (n+1)^2$), etc., may provide an important psychological gateway for learners from arithmetic to algebra.

It is quite possible that the historical tradition of Western thought could have unfolded quite differently than it has, and quite certain that it has unfolded in a much more complex manner than could ever possibly be portrayed. The question is, to what extent can historical developments in mathematical thinking, such as the shift from the Platonic idea of number (i.e., as a discrete quantity comprised of indivisible units), to the Aristotelian concept of number (i.e., as a continuous magnitude comprised of divisible units), inform educational theory and practice. I have no ready answer for this question. I would suggest, however, that whatever answers one provides will inevitably have some bearing on the question as to whether or not elementary number theory constitutes either a central or tangential matter in resolving cognitive gaps and expediting the transition between arithmetic and algebraic thinking. An important question to consider in this regard is whether one can develop a purely *conceptual* understanding of continuous measure prior to, or perhaps even simultaneously with a *conceptual* understanding of discrete quantities. If so, the standard view toward the role of number theory might be vindicated. Whatever the case may be, I would suggest that if mathematics educators depart in their ways of teaching from the historical ontogenesis of mathematical thinking, in the sense of epic cognitive shifts such as those noted above, then there should be good phenomenological, psychological, pedagogical, and certainly *not* just logical grounds for doing so.

Plato warned long ago that coming out of the cave of perceptual appearances into the dazzling and potentially blinding light of intellectual understanding too quickly could be counterproductive. The intellectual subject matter of algebra typically involves functions, equations, coefficients and variables ranging over rational and real numbers. A clear focus on understanding properties and structures of whole numbers may provide learners with invaluable, if not necessary scaffolding for overcoming cognitive gaps in these areas, thereby expediting the transition from the more concrete intuitive realms of arithmetic into the more lofty abstract realms of algebra. However, in order to help determine the overall effectiveness of recapitulating key developmental moments in the historical ontogenesis of mathematical thinking--in the sense of adhering to possible interdependencies of major metaphysical (*qua* cognitive) shifts such as those noted herein--it is will be necessary to conduct longitudinal studies using pedagogical methods specifically designed to evoke those kinds of realisations.

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Developing Algebraic Reasoning in the Elementary School: Generalization and Proof

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We characterize the development of elementary school children's algebraic reasoning as reflected in their ability to generate and justify generalizations about fundamental properties of arithmetic. Building on the work of Robert Davis, we focus on four primary themes: (1) equality, (2) making generalizations explicit, (3) representing generalizations in natural language and using algebraic notation, and (4) levels of justification and proof. We concurrently consider the development of children's algebraic reasoning and the nature of classroom interactions that support that development.

A growing consensus has emerged that it is necessary to reconceptualize the nature of algebra and algebraic reasoning and to provide students opportunity to engage in algebraic reasoning earlier in their education (Kaput, 1998; National Council of Teachers of Mathematics, 1997, 1998). The artificial separation of arithmetic and algebra traditional in school mathematics curricula deprives students of powerful schemes for thinking about mathematics in the early grades and makes it more difficult for them to learn algebra in the later grades (Kieran, 1992; Matz, 1982).

For the last four years, we have studied the development of students' algebraic reasoning in the elementary grades (Carpenter & Levi, 2000). In this research we have worked intensively with a group of teachers to construct instructional contexts that support the development of algebraic reasoning. The research has documented that children are capable of learning to reason algebraically earlier than commonly assumed and that learning to reason algebraically not only provides a basis for smoothing the transition to algebra in the later grades, but also deepens understanding of basic arithmetic.

From Arithmetic to Algebraic Reasoning

Algebraic reasoning can manifest itself in a number of forms. Kaput (1998) has identified five interrelated forms of algebraic reasoning:

- Algebra as generalizing and formalizing patterns and regularities, in particular algebra as generalized arithmetic.
- Algebra as syntactically guided manipulations of symbols.
- Algebra as the study of structure and systems abstracted from computations and relations.
- Algebra as the study of functions, relations, and joint variation.
- Algebra as modeling.

In our research, we have focused on the first and third forms of reasoning identified by Kaput. In particular we focus on two central themes: (a) making, justifying, and using generalizations about arithmetic operations and relations; and (b) using symbols to represent mathematical ideas and to represent and solve problems.

Previous research suggested that students in traditional classrooms are not aware of the underlying structure and properties of arithmetic operations (Chailkin & Lesgold, 1984; Collis, 1975; Kieran, 1989). In our research we have found that young children are capable of making such generalizations, constructing ways of representing them, and justifying them if they are provided appropriate opportunity (Carpenter & Levi, 1999, Carpenter et al., 1999;). These findings are consistent with other recent studies (Bastable & Schifter, 1998; Davis, 1964; Kaput, 1999; Schifter, 1999; Tierney & Monk, 1998).

Drawing on Robert Davis's (1964) seminal work on the Madison Project, we have used the solution and discussion of true, false, and open number as a primary context for helping teachers understand and focus on student thinking. These number sentences provide a context to initiate conversations that can lead to generalization and to introduce discussion of notation that might be used to express those generalizations. This context has proved particularly fruitful in engaging students in algebraic thinking and making that thinking visible to teachers. Specific ideas that we have addressed include: (a) equality as a relation, (b) generalization about number and properties of operations, (c) representation of generalizations, and (d) the progression of forms of argument that students use to justify generalizations. The following discussion is based on work with 15 elementary school teachers teaching and their students in grades 1 through 6 (ages approximately 6 to 12), including in-depth case studies in three classes.

Equality

Kieran (1992) characterizes the distinction between arithmetic thinking and algebraic reasoning as a shift from a procedural perspective of operations and relations to a structural perspective. One of the hallmarks of this transition is a shift from a procedural view to a relational view of equality, and developing a relational understanding of the meaning of the equal sign underlies the ability to make and represent generalizations. Behr, Erlwanger, and Nichols (1980), Erlwanger and Berlanger (1983), Kieran (1981), and Saenz-Ludlow and Walgamuth (1998) have documented, however, that children in the elementary grades generally consider that the equal sign means to carry out the calculation that precedes it, and this is one of the major stumbling blocks when moving from arithmetic to algebra (Kieran, 1981; Matz, 1982). Our research demonstrates, however, that children can learn to think of equality in relational terms with appropriate instruction (Carpenter & Levi, 2000; Falkner, Levi, & Carpenter, 1999; see also Kieran, 1981; and Saenz-Ludlow & Walgamuth, 1998).

At the beginning of the study, fewer than 10% of the students in any classes from Grade 1 to 6 demonstrated any evidence of relational understanding of the meaning of the equal sign. Given a problem like 8 + 4 = + 5, the majority of students at every grade responded that 12 should go in the box to make the number sentence true. The second most popular answer was 17. Teachers encouraged children to make explicit their conceptions and address the alternative conceptions held by different students in the classes. To provide a context for this discussion, teachers asked students to consider a range of true and false number sentences that challenged different conceptions of the meaning of the equal sign. For example, students would be asked to consider whether the following number sentences were true or false: 5 + 3 = 8, 8 = 5 + 3, 8 = 8, 5 + 3 = 5 + 3, 5 + 3 = 3 + 5. Virtually all students agreed that the first number sentence was true. They were not so sure about the others, but as the unfamiliar forms of number sentences were introduced and compared to 5 + 3 = 8, students were put in a position to examine and make explicit their implicit ideas about how the equal sign was used. In making their assumptions and conceptions explicit and discussing them with other students, some students began to question and change their conceptions of the meaning of the equal sign. The changes were not immediate or easy, but at the end of the year, the percent of students in our study demonstrating a relational understanding of equality ranged from 66% in Grades 1 and 2, to 84% in Grade 6.

Generalization about Number and Properties of Operations

When students make generalizations about properties of numbers or operations, they make explicit their mathematical thinking. Generalizations provide a class with fundamental mathematical propositions for examination as well as opening up students' thinking for analysis and discussion. Students have a great deal of implicit knowledge of about properties of arithmetic operations, but they generally have not examined generalizations about properties of numbers and operations explicitly or thought systematically about them. The trick is to find problems that provide a context to make their implicit knowledge explicit. Discussion of appropriately selected true and false number sentences provides such a context.

The following example, is taken directly from a case study of a group of second-grade students (Carpenter & Levi, 2000). This example not only illustrates how the students made their implicit knowledge explicit, it also shows how they refine the language used to articulate the generalization. Students were asked whether the following number sentence was true or false: 78 - 49 = 78. The following interaction occurred:

Children: False! No, no false! No way!
Teacher: Why is that false.
Jenny: Because it is the same number as in the beginning, and you already took away some, so it would have to be lower than the number you started with.
Mike: Unless it was 78 - 0 = 78. That would be right.
Teacher: Is that true? Why is that true? We took something away.
Steve: But that something is, there is like nothing. Zero is nothing.
Teacher: Is that always going to work?
Lynn: If you want to start with a number and end with a number, and you do a number sentence, you should always put a zero. Since you wrote 78 - 49 = 78, you have to change a 49 to a zero to equal 78, because if you want the same answer as the first number and the last number, you have to make a zero in between.
Teacher: So do you think that will always work with Zero?

Mike, interpreted the question as whether it was necessary to change the 49 to a zero.

Mike: Oh no. Unless you 78 minus umm 49, plus something.
Ellen: Plus 49
Mike: Yeah. 49. 78 - 49 + 49 = 78.
Teacher: Wow. Do you all think that is true? [All but one child answered yes.]
Jenny: I do because you took the 49 away, and it's just like getting it back.

Essentially the children generated another generalization (a + b - b = a), although they had not yet articulated it as a general rule. It is a somewhat more difficult generalization to articulate than the zero properties for addition and subtraction, so after some discussion of the specific example, the teacher returned to sums and differences involving zero with the following example: 789,564 - 0 = 789,564.

Children: That's true.
Teacher: How do you know that is true? Have you ever done that? Ann?
Ann: I will tell you. All those numbers take away zero, you won't take away anything, so it would be the same number.

After another example in which the children immediately respond that 0 + 5869 = 0 was true, the following discussion ensued:

Teacher: So we kind of have a rule here don't we. What's the rule?
Ann: Anything with a zero can be the right answer.
Mike: No. Because if it was 100 + 100 that's 200.
Jenny: That's not that we are talking about. It doesn't have just plain zero.
Ann: I said, umm,, if you have a zero in it, it can't be like 100, because you want just plain zero like 0 + 7 = 7.

After some additional discussion to clarify that the children were talking about the number zero not zero in numbers like 20 or 500, the children were challenged to state a rule that they could share with the rest of the class .

Ellen: When you put zero with one other number, just one zero with the other number, it equals the other number.
Steve: Not true.

Teacher: Wait. Let me make sure I got it. You said, "If you have a plain zero with another number." With another number? Like just sitting next to the number?

Ellen: No, added with another number, or minus from another number; it equals that number.

The group collectively came up with the rule: "Zero added with another number equals that other number." They also came up with the generalizations: "Zero subtracted from another number equals that number," and "any number minus the same number equals zero." One student, Steve, came up with several generalizations about multiplication. The comments came up in response to Ellen's initial generalization about zero, which did not specify an operation.

Steve: I wasn't thinking about the zero stuff plus another number equals that same number that we added to the zero, but umm I was thinking about if you were [inaudible] a number times zero would be zero. . . . 7 X 0 = 0. That is what I am trying to say, because 7 zeros or 0 sevens would be zero, and if you just add 7 zeros you would just get zero. . . .Even a high number times zero would be zero. . . . Even the highest number you can think of times zero would be zero.

Teacher: How do you know that, Steve?

Steve: . . .Like 256 times zero, 256 zeros and any amount of zeros would be zero.

Steve had initially objected to Ellen's overly general statement of a zero rule. Having clarified the rules, we questioned Steve about whether his rule was an exception to the rules for addition and subtraction. He responded that those rules were like 27 X 1 = 27. This comment suggested that Steve had at least some level of understanding of the parallels between additive and multiplicative identities.

In these examples, children readily applied generalizations about zero to determine the truth value of number sentences. They not only applied them to solve given problems involving zeros; they came up with number sentences that embodied additional principles (78 - 49 + 49 = 78), and one student, Steve, spontaneously came up with generalizations that did not evolve out of solutions of specific number sentences. Children often tried to state their generalization using a specific example (It's like 7 + 0 = 7), and they often used specific simple cases to validate their generalization. They all were confident, however, that their generalizations held for all numbers. The justification of their assertions generally referred to zero representing "nothing," but they did say that zero was a number. Ellen's initial statement of the zero principle was overly general and not accurate, but collectively the students identified the limitations and constructed more focused, valid generalizations. At this point the generalizations were stated in natural language, which could be awkward and imprecise.

Another feature of the discussion was that the students seemed to demonstrate a good conception of the appropriate use of counter examples to challenge other students' claims or generalizations. For example, Mike challenged Ann's generalization that "Anything with a zero can be the right answer" with the counter example 100 + 100 = 200. That forced Ann to revise her assertion to make it more precise. Steve also challenged Ellen's generalization about operations with zero by bringing in multiplication. The students had a little more difficulty articulating reasons that their assertions were true, but they generally seemed to recognize that a single case or several cases did not prove a statement was always true, and they attempted to use arguments that would apply to all numbers. "All those numbers take away zero, you won't take away anything, so it would be the same number." "Any amount of zeros would be zero."

Representation of Generalizations with Symbols

One of the outcomes of making the use of true, false, and open number sentences an integral feature of classroom activity is that it provides students access to a notation for expressing generalizations precisely using variables (Carpenter & Levi, 2000; Davis, 1964).

Students who worked with open number sentences in flexible ways quite readily adapted them to represent generalizations. There were several ways that teachers encouraged students to do this. One was to ask directly whether there was a way that a specific conjecture could be expressed using an open number sentence. Students initially proposed number sentences, such as $7 + 0 = 7$, that did not express generality, but with a little scaffolding they generally arrived at appropriate number sentences. In two case studies of fourth and sixth grade classrooms, 80% of the students used variables to express generalizations such as $a - a = 0$.

Progression of Forms of Argument Students Use to Justify Generalizations

We have identified different forms of argument that children use to justify their conjectures and key tasks that elicit more advanced forms of argument. In the primary grades, children tend to justify propositions by example, but as they move into the third and fourth grades they begin to see the limits of arguing by example and begin to employ more sophisticated forms of argument. The following example, which is drawn from a case study of a sixth grade class (Valentine & Carpenter, 2000), illustrates the progression of forms of argument that students use to justify commutativity of multiplication.

In early November the teacher, Ms. V, asked the students to determine what number could replace n to make the following number sentence true: $325 \times 6 = n \times 325$. Only one student, Daniel, explained the array model in his written response. Thirteen students wrote that the numbers must be the same because the equal sign means "the same as." These students explained in a variety of ways that both numbers should be represented on both sides of the equal sign. One proposed that "If you take 325 from both sides you only have 6 and n. So n equals 6." Three students explained that the products would be the same if n was 6. One student thought that n should be the product of 325×6. She still interpreted the equal sign to mean, "the answer comes next." Because of the large numbers involved, none of the students actually carried out the calculation.

In the class discussion that followed, most students provided ambiguous explanations about why $n = 6$. A number of students gave responses similar to Abby, who responded: "N equals 6, because you have one 325 on each side of the equal sign and since both sides are equal, the other number on each side would be equal to each other." These students never mentioned switching the order of the numbers or the operation of multiplication.

For many of these students, the sameness of the two sides of the equation was the prominent feature of the number sentence that they attended to. This is clearly illustrated in Karl's response: "I think it is 6 because the two 325's are the same and so that means that the 6, the n has to be 6 if they are going to be the same thing. When asked directly whether the order of numbers could be switched for all operations, virtually all students did acknowledge that commutativity only applied to addition and multiplication, but at this point many of their explanations and justifications were not clearly focused.

Daniel, the one student who had provided a principled explanation for his response, explained that with an array with 325 dots in 6 rows and an array with 6 dots in 325 columns you are counting the same number of dots. Ms. V ask the class to look at 3×5 and 5×3 arrays. By the end of the session, about five students had demonstrated some understanding of Daniel's array model, but it was questionable whether they could apply it to larger numbers or more general cases. Furthermore, the argument was only applied to a specific case, and even Daniel had not publicly argued for a general proof of the commutative property of multiplication.

In early December Ms. V gave assignments to small groups of students to write on one of three different tasks. Each group had a large sheet of grid paper to present their solution to the entire class. One group was asked to decide whether the statement 124 x 396 = 396 x 124 was true or false, to write a generalization for equations like this, and justify that the conjecture was true for all numbers.

The group wrote, "The conjecture for multiplication and addition is that it does not matter which order the numbers are in. The answer will always equal the same thing." This group then drew two arrays of fifteen dots, one 3×5 and the other 5×3.

Jordan: Here it is 3 x 5 (pointing to the array) and here it is 5 x 3.

Daniel: It does prove that 3 x 5 equals 15 and so does 5 x 3 equal 15. But does it prove for every number you can do that?

Jordan: Yes.

Arial: It was just an example. It could have been 2 x 4 and it would have been 8.

Jordan: I could give another example. See this [pointing to the dot arrangement] is six. Here this is 3 x 5 and this is 5 x 3.

Ms. V: Jordan, what do you think Daniel's question is?

Jordan: How does this prove that it would work for any problem?

Ms. V: OK that's the problem.

Arial: I can say now why it proved it. Because that's 5 and that's 3. So it shows that if you switch the numbers around you would have the same answer, Even though you could have different like numbers like 4: what I mean by different numbers is not like 3 and 6 and 3 and 5. But I mean like 2 and 6 and 6 and 2.... But...well...it's just showing that you can flip them around.

Daniel: But then again, I can see why it works. I know it works. Actually, I am not sure why it works. I don't see why another one up there would help. I can see why...I just don't see any proof.

Arial: There are a billion possibilities. Like 10 and 20 and 20 and 10 numbers. It is the picture that changes. Well, there would be more dots.

Karl: With any number, you can just turn it. Like if you make a dot array and you just turn it. It's *always* gonna have the same number of dots, no matter what.

Arial: Yeah.

Daniel: That's a little better. It's just that I didn't want more and more examples.

Abby: Well, Daniel, you can spend a lot of time trying a lot of examples and then you think you know it would work for any numbers.

Daniel: I know. But then again there are an infinite amount of numbers so it shouldn't matter. You need to show that it always works.

Ms. V: Did you accept Karl's explanation?

Daniel: I accepted Karl's explanation.

Jordan: Take any group of dots...and flip it on its side. It will be the same number of dots!

Jordan and his partners used an array model to demonstrate that the order of numbers could be switched, but their model only illustrated a specific case. They did not take the further step of arguing that the process they had demonstrated could be applied to any number (at least any whole number). Daniel insisted on a new level of justification. He stated explicitly that numerous examples would not yield a proof. Jordan's suggestion that the array could be turned and not change the number of dots didn't satisfy Daniel. He probed until Karl explicitly stated that the process could be applied to any number. Jordan's final statement also acknowledged the necessity of making a general argument that applied beyond the specific case being illustrated. Arial, on the other hand, continued to talk about switching the numbers around, and in the next episode it became clear that she still had a superficial conception of commutativity.

In this episode, the students had assumed almost complete responsibility for class discussion. Ms. V only interjects to clarify that the group has actually reached some consensus about making an argument about the generality of the proof. Although Ms. V did not guide the discussion, it remained focused on establishing norms for justification. Daniel played a key role in this discussion, actually assuming the role that one might expect of the teacher. He understood the limitations in the initial argument and pressed the other students to show that the argument they were using generalized to all whole numbers. Rather than simply stating the generalization himself, he asked a question of the other students to get them to articulate the principle.

In this class episode and the one that precedes it, we see how students negotiated norms for articulating and justifying generalizations. True, false, and open number sentences represented artifacts that provided a focus for discussion, and provide a context that makes it possible for students to assume substantial responsibility for deciding what counts as a legitimate statement of a generalization and what counts as justification.

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The Reification of Additive Differences in Early Algebra: Viva La Diférence!

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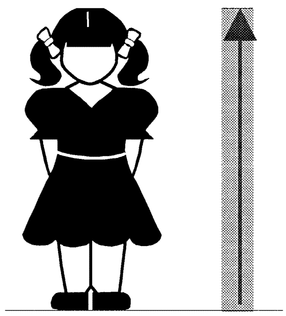
We look at the emergence of 9-year old students' concept of *additive difference*. The concept entails a tension between *process* and *object*. But even more strikingly, reifying the concept requires that children adopt analogies across diverse representational contexts. We will look at examples of students' reasoning about children's heights in contexts associated with number lines, counting, line segment diagrams, and arithmetic-algebraic notation. The examples show that *subtraction* comprises a small yet essential part of the concept of *difference*. We consider implications for research and curriculum development in early algebra and early arithmetic education.

Several years ago we began to appreciate the importance of the concept of *difference* when we designed the following problem for a class of 9-year old students (Carraher, Brizuela, & Schliemann, 2000):

> Tom is 4 inches taller than Maria.
>
> Maria is 6 inches shorter than Leslie.
>
> Draw Tom's height, Maria's height, and Leslie's height.
>
> Show what the numbers 4 and 6 refer to.



Maria Maria's Height

The problem talks about the *differences* in heights among three characters without revealing their actual heights. This problem seemed appropriate for introducing students to additive functions. The heights could be thought to vary insofar as they could take on a set of possible values. Of course that was *our* view. The point of researching the issue was to see what sense the *students* made of such a problem. We will review our original findings and then describe what we have learned more recently through teaching experiments with other third grade students[i]. Then we will relate our findings to additive structures and early algebra.

Initially, some students interpreted the numbers as referring to the heights of the characters in the story, while others suggested adding the numbers 4 and 6 to obtain the height of the third child. Eventually the students began to talk about differences in heights and we invited three volunteers to the front of the class to represent the characters in the story. When we asked the students to point to the differences between the heights of pairs of children—that is, to show what the numbers 4 and 6 referred to—they responded in surprising ways. Typically, they would indicate the difference by placing their hand first on top of the head of one child and afterwards on the head of the other. Sometimes they would simply place their hand on the top of the taller child's head. Alternatively, they would point to where, on the taller child's body, the shorter child's top of head would reach. When we asked them whether they were referring to a single point on the taller child's body, expecting that they would sweep out the region from that point to the top of the respective child's head, they typically insisted on the single point idea. Another strategy was to place their hand on the top of one head and then move diagonally to the top of the other child's head. Each of these strategies conveyed some understanding of the difference in heights yet differed in significant ways from the convention of expressing a difference through the distance by which one quantity surpasses or falls short of the other.

We realized that our concept of difference (based on the fixed distance by which one child's head towered above or fell below that of the other along a vertical dimension) relied on conventions not yet adopted by the children. Some children seemed to represent the difference in heights through the very act of comparing (note in particular the case of sequential head-tapping). These children were more inclined to view the difference as a *process* rather than an *object*. This distinction between *process* and *object* was further highlighted when Anne Goodrow, a member of our research team, noted that children who were puzzled about where to locate a difference in heights of two students solved the problem without hesitation when it was framed by a question such as, "how much does Martha have to grow to be the same height as Paul?" (Their upward, never downward, gesticulations conveyed the process of growth.)

The present discussion evokes Sfard and Linchevski's (1994) observations on the reification of mathematical concepts. But we use *reification* in a somewhat broader sense here. We think of reification in terms of a widening of a concept across multiple contexts. We speak not merely of the increasing number of representations. An additive difference behaves and expresses its properties diversely in different representational mini-systems. In conventional representations such as those we will consider here—number lines, written algebraic notation, line segment diagrams, the subtraction and addition of sets—students must learn the conventions for expressing properties of differences and how they interrelate. The reification we refer has to do with children's travelling through the different contexts and representations.

Differences Revisited

What's the Difference?: Location vs. Distance

Two years later we took up a discussion of heights with another class of third grade students. We initially focused on enacting height differences with diverse children in the class, and in exploring the relationships between two pairs of heights through line segments and diagrams. Bárbara, who was teaching that class, asks Jennifer to show the difference between the heights of Jeffrey and Adriana (Jeffrey is a full 10 inches taller than Adriana, but no measurements have yet been taken.) Jennifer expresses the difference by pointing to Jeffrey's shoulder, which is the highest point that Adriana reaches:

Bárbara [pointing to Jeffrey's shoulder]: That is the difference?

Jennifer: No. Like, up, um...

Bárbara: Where does [the difference] start and where does it end?

Jennifer: Mm, it goes up.

Bárbara: It goes up to where?... It *starts* here. [Bárbara indicates Jeffrey's shoulder]. Let's pretend it starts here. Where would it end?

Jennifer: At his head.

Bárbara [her hands spread between Jeffrey's shoulder and the top of his head]: At his head. Could you say something like this?

Jennifer: Yeah.

Bárbara: That would be the...

Jennifer [holding her hands apart as Bárbara had]: So that would be the difference.

Jennifer faced similar difficulties to those faced by the third graders we had worked with two years before: how to represent the difference as a distance as we wanted instead of as a location, as she felt naturally inclined to do.

Reversibility in Comparisons: Taller Implies Shorter

Bárbara then introduces a ruler and with it the idea that one can measure the children's difference in heights without measuring the heights of either child. She guides Jennifer to place the ruler atop Adriana's head and to read off the number 10, which lies at the same height as the top of Jeffrey's head. The class appears comfortable calling that difference 10 inches. Jennifer explains that Adriana is 10 inches shorter than Jeffrey, and Nathan volunteers that Jeffrey is 10 inches taller than Adriana. Although this reversibility may seem obvious to us as adults, it is not necessarily obvious to many nine-year-old children. Max also understands that Adriana would have to grow 10 inches to be the same height as Jeffrey, and Risa laughs as she answers, in response to Bárbara's query, that Jeffrey would have to shrink 10 inches to be as tall as Adriana.

At this point Bárbara presents the heights problem to the children precisely as it was given two years earlier to our original class. This time the students seem to quickly understand that the numbers relate to numerical differences instead of total heights. (Although this may not be surprising, given that we had worked with the students with differences in a broad range of contexts during the previous school year and in the classes leading up to the present one.)

Expressing in Notation the Differences Between Unknown Heights

When Bárbara asks, "Tom is 4 inches taller than Maria; does it say how tall Tom is?" the class issues an emphatic "No!". The students respond likewise for the cases of Maria and Leslie. Bárbara suggests that because the characters' heights are unknown, the class could call Maria's height N. (These students were already familiar with the convention of using N to represent an unknown.)

Bárbara: Now if Maria's height was N, what would Tom's height be?

Students: N plus 4.

Bárbara: Why?

Students: Because he would be 4 inches taller.

Bárbara: Mm, hmm. And what would Leslie's height be?

Nathan and students: N plus 6.

Nathan: Because Leslie is 6 inches taller.

A Difference as a Subtraction

Bárbara writes the expression "Tom – Maria" on an acetate overhead and asks the class what the expression refers to. A student explains that it is the difference between Tom's and Maria's heights. Bárbara pursues the idea of subtraction.

Bárbara: So if you subtract the height of Tom's, the height of Maria from the height of Tom, what would you get?

Students: Four minus four.

Bárbara [hearing only the final 'four']: Four, right? ...

Student [insisting that both fours are to be included]: **Four minus four**.

Bárbara [thinking she heard incorrectly]: Oh. Four?

Students [continuing to insist]: **...minus** four.

Bárbara [having finally written "4-4" on the overhead]: Why four minus four?

Student: Because...

Bárbara: This is the difference, the result of that four? Isn't that four inches? The height of Tom, the difference between Tom and Maria, isn't that four?

Jennifer: Mm hmm.

Bárbara: Just plain Four. Four inches.

The students seem to have been thinking that a subtraction requires a taking away of one number from another. Bárbara is actually thinking of the difference as the *result* of the subtraction and hence as a single value. This momentary misunderstanding mirrors the tension between process and object. The students are thinking in terms of the former, the teacher in terms of the latter.

Inferring the Difference of Two Differences

Shortly thereafter, Bárbara draws attention to the fact that Leslie and Tom's difference is still unknown.

Bárbara: Leslie and Tom. We have to figure out at some point what the difference between Leslie and Tom is.

Nathan: Two!

Bárbara: Why two?

Nathan: Because you...six minus four equals two.

One might think that Nathan has merely made a lucky guess. After all, given the fact that two numbers had been provided and the students were accustomed to working with addition and subtraction problems, only three distinct possibilities existed for binary operations (6-4, 4-6, and 6+4). Our past experience had shown us, however, that children were more likely to add the numbers four and six assuming they stood for total heights, to arrive at a new total heights of 10 inches for the third character involved.

Inferring the Order of the Characters in the Story

Bárbara then calls once more upon Jeffrey and Adriana to represent two of the protagonists in the story. They quickly understand that Jeffrey should represent Leslie and Adriana should represent Maria. The discussion moves to the issue of finding a student to be Tom.

Nathan [who is somewhere between Adriana and Jeffrey in height]: Can I come up?

Bárbara: Why does Tom, someone said that Tom has to be medium.

Nathan: I'm medium!

Bárbara: Why? ... Well, why does Tom need to be medium, Risa?

Risa: Because he can't be taller than uh Leslie because Leslie's the tallest.

Nathan [looking for an acting role]: I'm the perfect size!

Bárbara: Leslie's the... so how come Tom can't, how come Tom can't be shorter than Maria?

Risa: 'cause Maria's the shortest, and that's saying that Tom's, like, the second shortest...

Risa's explanation may be questionable, but she is quite correct in her conclusion about the order of heights.

Making Diagrams of Unknown Heights

Bárbara then asks the students to make their own drawings of the three characters, showing what they know about the problem. In Figure 1, Jeffrey represents the heights as vertical line segments, the darkened regions of which correspond to the known differences.

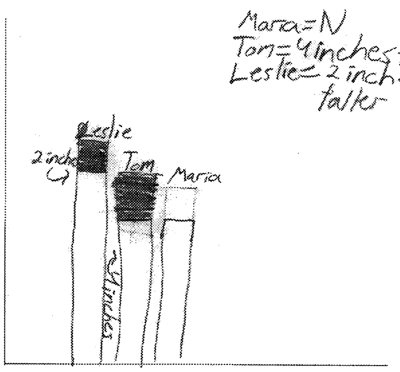


Figure 1. Jeffrey's drawing and notation for the height's problem

Ramon's (Figure 2) line segment drawing does not explicitly indicate the differences as parts of line segments. He seems to set Leslie's height at 6" and Tom's height at 4". The difference of 2" is consistent with the problem, but the numbers, 6 and 4 should correspond to differences. Here they do not. Note also that Maria's height, if indeed 6" less than Leslie's, would have to be zero inches.

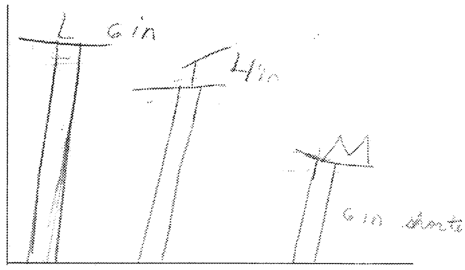


Figure 2. Ramon's drawing of the heights of the three characters

Expressing the Differences on a Variable Number Line

Jennifer notes on her own that it is possible to express the heights of the three characters in the story as positions on a variable number line, referred to in prior classes as the n-number line as opposed to the regular number line. Jennifer's diagram is shown below (see Figure 3):

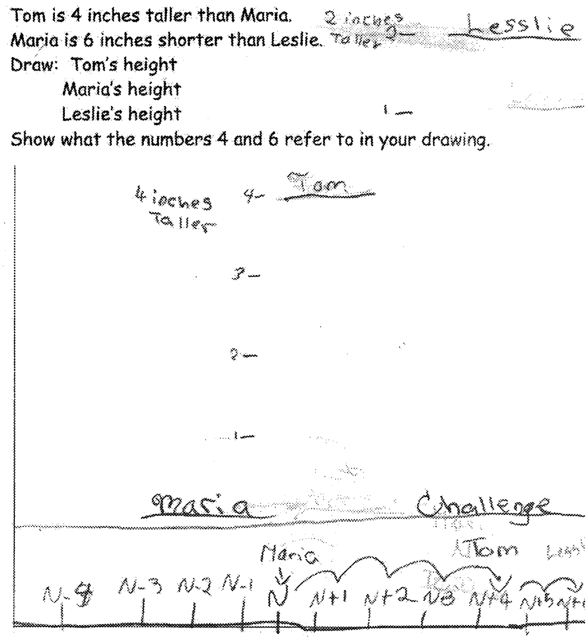


Figure 3. Jennifer's drawing (notches) showing differences but no origin. She also makes use of a variable number that forms the basis of subsequent discussion

Bárbara adopts Jennifer's number line as a basis for a full-class discussion of the relations among the heights. She further adopts Jennifer's assumption that Maria is located at N on the variable number line. See Figure 4 (middle number line) below.

Bárbara: … Now if Maria's height was *N*, what would Tom's height be?

Students: *N* plus four.

Bárbara: Why?

Students: Because he would be four inches taller.

Bárbara: Mm, hmm. And what would Leslie's height be?

Nathan and students: *N* plus six.

Nathan: Because Leslie is six inches taller.

It is remarkable that Jennifer realizes that a representational tool introduced in earlier classes would help clarify the problem at hand. It is equally impressive that the remaining students appear comfortable with the idea and easily infer the values of Tom and Leslie from Maria's value.

Bárbara wonders to herself whether the students realize that the decision to call Maria's height *N* was arbitrary. So she asks the students to assume instead that Leslie's height was *N*. The students immediately argue that Tom would be at *N*-2 on the number line and Maria would be at *N*-6. See Figure 4 (bottom number line) below.

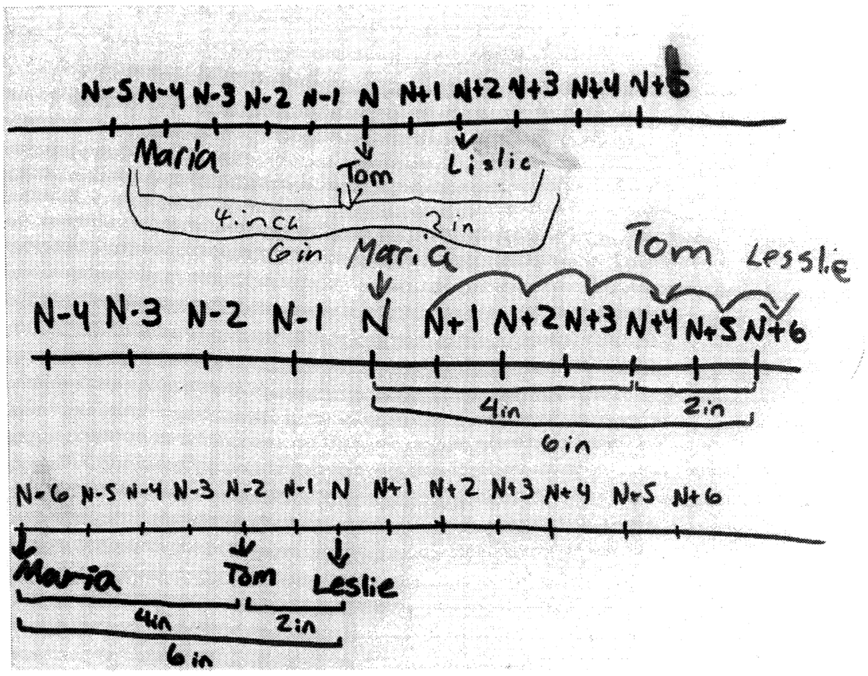


Figure 4. Three variable number line representations (on overhead) used by students and teacher to discuss the cases where [middle] Maria is attributed the height of *N*; [bottom] Leslie is assigned the height of *N*; and [top] Tom is assigned a height of *N*

Numbers as Fence Posts vs. Numbers as Intervals or Distances

Bárbara then asks the children to assume that Tom had been assigned the value of *N*. Max goes to the front of the class and places Leslie at *N*+3. See Figure 4 (top number line) below; there is an erasure under *N*+3 where Max had first incorrectly put Leslie's name. Why does he do this, even when questioned by Bárbara? Max realizes that the difference between Tom and Leslie is two, but nonetheless places her three units to the right of Tom. This is an example of the "fence post" issue. Students are well accustomed to the idea that a number refers to the count of elements in a set, that is, a set's cardinality. However, the issue before children often is: what should I count? On a number line two sorts of elements suggest themselves. One can count the number of "fence posts" or notches: each numeral is located at a notch along the number line. In Max's case he seems to have counted the number of numerals lying between *N* and *N*+3, the delimiters. Mathematical convention dictates that one count the number of *unit intervals* separating the delimiters. In the present case, one counts the two intervals, [*N*, *N*+1] and [*N*+1, *N*+2]. This may seem like a minor issue, but if a student is thinking of the number of integral points on the line rather than the number of unit-distances or intervals, misunderstandings are likely to arise in a wide variety of situations.

But if we focus too much on such momentary adjustments that students like Max may be required to make, we may fail to see the larger picture; namely, that by the end of the lesson the students are relating the given numerical differences to a number of symbolic representations: algebraic notation, line segment diagrams, number lines (including variable number lines), subtraction, counting, and natural language descriptions. The concept of additive difference does not reside in any one of these representations. And we cannot observe the students' concepts directly. But the fluidity with which students move from one representational context to another assures us that their understanding of *additive difference* is robust and flexible.

Concluding Remarks

An *additive difference* is a rich mathematical concept that manifests itself in a wide variety of contexts and through a diversity of representational forms. Although it is related to subtraction, it is not reducible to subtraction. Although it is related to displacements on a number line, it does not reduce to the idea of a number line displacement. An additive difference plays a key role in the emergence of early algebraic understanding in that it is central to the concepts of addition and subtraction and their conceptualisation as functions. But it does not end there. Additive structures lie at the heart of many advanced ideas in mathematics, including fractions, measurement, and even statistical concepts such as the analysis of variance. It is important to begin to nurture their development from very early mathematics education. Along the way, the term, *difference,* may begin to serve an integrative function, a linguistic handle that unites otherwise disparate situations. (Learning the concept entails far more than learning to use the term; nonetheless the linguistic representations of the concept are very important symbolic vehicles that should not be underestimated in mathematics education.)

Students may initially treat differences as the results of processes or actions (of growing, moving etc.) that change as an entity moves from one state to another. This is a legitimate way of thinking about differences, which never needs to be abandoned. But students also need to view differences as bona fide quantities that can be represented much in the same way as states can. This process-object tension is useful for us to gain an initial appreciation of the diversity of meanings the concept may take on. However, in order to develop well thought-out theories of learning and curriculum development we need to go far beyond this general characterization. We need to look at the particular ways in which notation can help students represent and work through specific mathematical issues (Brizuela, Carraher, & Schliemann, 2000). When students move from one mini-representation, such as from a set approach to a number line representation, particular kinds of issues arise that cannot be anticipated by a general theory of reification. We used the fence post issue as a case in point. We could highlight others, such as learning to distinguish between first and second-order differences (differences of differences) or learning to move between instantiation and generalization (for example, using N both to represent a particular value and the set of all possible values).

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[i] The National Science Foundation has provided support for this work under the grant, "Bringing Out the Algebraic Character of Arithmetic" (#9909591). We wish to thank Judah Schwartz and Pat Thompson for their discussions with us about additive differences.

Learning algebra by using it: A promising approach to using calculators in the classroom.

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This paper discusses research results on using graphic calculators to help students make the transition from arithmetic to algebra. In this study school algebra is seen both, as a language to express generalizations and as a tool to negotiate problem solutions. Selected results of three interrelated studies carried out in Mexican middle schools are discussed with the aim of putting forward some possible answers to questions proposed for the ICMI Study on The Future of the Teaching and Learning of Algebra: What kinds of algebra learning does this environment promote? To which aspects of algebra learning does this particular technology make a distinctive, unique contribution? To what extent ought the goals of algebra education be affected by the availability of this technology? Are there activities using graphic calculators that can be profitably undertaken by younger students?

Background

This paper encapsulates results of three interrelated studies that were carried out with 11-to 12-year-old students who had not received algebra instruction. The students' age was chosen in order to closely study the potential of a teaching environment specially designed to use graphic calculators as cognitive tools before students are exposed to traditional algebra instruction.

The lack of connection between arithmetic and algebra has been identified as an influencing factor that might prevent students from successfully using algebra as a tool for expressing and justifying generalizations and solving algebra problems (Lee and Wheeler, 1989; Sutherland and Rojano, 1993). The major aim of the present study was to investigate the extent to which students' arithmetic notions can be exploited to help them assign meanings to algebraic expressions and develop strategies to confront algebra problems, supported by the graphic calculator. In order to create a learning environment, where the cognitive facts to be studied take place, a block of teaching activities was designed for each of the following settings: (i) algebraic contexts to recreate numerical notions (working with the yet unknown); (ii) number-based contexts to develop meanings for algebraic expressions (finding and algebraically expressing the rule governing linear number patterns); (iii) visual contexts to model algebra problems (graphs of linear and quadratic functions). A study was implemented to have the students' work observed within each of these contexts.

The structure and content of the teaching activities were influenced by research on natural language acquisition. The principle "the use of language determines its meanings" was borrowed from Bruner's work (1980, 1982, 1983). A condition derived from this principle is that the teaching activities must not require the students to previously know mathematical rules and definitions. Underlying this principle is the hypothesis that students can assign meanings to arithmetic and algebra codes by using them (Cedillo, 2001).

The fieldwork was carried out during two 50-minute classroom sessions per week throughout 32 weeks. The research method used was an approach based on qualitative analysis (Miles and Huberman, 1984). Six of the students in the experimental class were chosen to have their work observed using the case-study technique. The main sources of data were the following: (i) case-study students' written work throughout the fieldwork; (ii) individual interviews which were carried out with the case-study students; (iii) notes addressing relevant children's interventions taken by the researcher after each classroom session during the fieldwork. Each of the case-study students was interviewed three times, twice during the study, and once at the end. Every interview was video recorded and transcribed. The same questions were asked of each student; however, the interviews also included questions related to specific characteristics of each of the case-study subjects in accordance with their written work and the notes taken by the researcher at the end of every classroom session.

The three studies were carried out in two stages: a first approach with a small group of students, and a replica with a wider population. The first stage was done with a group of twenty-five 11-to 12-year-old students who were at Year 1 in a private school; each student in the group worked with their own calculator and the researcher played the role of the teacher during the whole school year. The students were allowed to work alone or in small groups at their choice. The second stage was a replica of the first stage aimed at obtaining further empirical evidence by implementing the study with 'real full time teachers' attending average size classes in public Mexican schools (40-45 students per class). In the second stage the researcher limited to prepare the teachers and to closely observe and analyze the work done in the classroom; the replica also enriched the first study in that it was carried out under more standard conditions in terms of school's environments. Three out of twenty teachers who were willing to participate were chosen. The main criteria for choosing them were their willingness to participate and the school's socioeconomic environment. The teachers were selected as follows: one teacher who worked in an upper middle class school in the country side, one teacher who worked in a typical middle class public school in Mexico City, and one teacher of a working class school in Mexico City. Each teacher carried out the teaching experiment with two groups of 11 to 12-year-old students. A total of 250 students took part in the study.

Despite the strong differences among school environments and teachers, the results drawn from each school that took part in the replica did not significantly differ among them or from the outcomes of the first study (Cedillo, 1996). One of the most stable results of these studies is that the students extended the trial and refining strategies they developed while negotiating solutions within arithmetic-based environments to face algebra problems using algebraic tools. This result directly relates to the "learning by doing" principle used to shape the learning environment.

In the next sections each of the aforesaid studies is briefly described. Due to space constraints the results of the study were summarize by means of vignettes taken textually from students' answers in an attempt to encapsulate the salient features suggested by the data. All questions and activities referred to in the vignettes were administered to the case-study students. Finally, a section is devoted to a more general discussion of the results through the light of the questions mentioned in the abstract of the paper.

Algebraic contexts to recreate numerical notions

The activities designed for this part of the study were intended to make the students work by inspection and exploration with the still unknown, within a number-based context. A series of 60 worksheets were designed pursuing this purpose, each addressing a new problem situation. The selected examples for this section are the following:

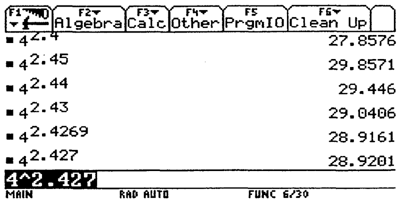
- Can you find the missing number in $4^x=29$?
- Can you compute 438+725 without using the 'plus key'?

Students' responses to $4^x=29$

First approach: "There are no numbers that make this possible because $4^2=16$ y $4^3=64$."

A second approach: "I'm getting closer … I tried with $4^{2.5}$, it gives 32 …"

Subsequent approaches: The figure on the right hand shows a calculator's screen with some students' responses. This screen was amplified by a video projector and displayed at the front of the classroom. A student could come to the front and type his/her answer to be displayed only if it improved the previous one.



Discussion

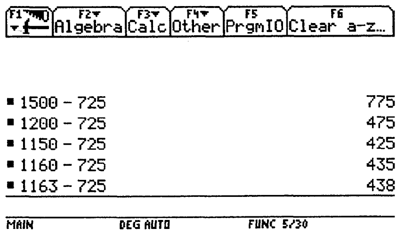
The first students' approach seems to be quite sensible. Why did they have to think of fractional exponents? The teacher invited them to abandon the task as a way of challenging them. Most of the students did not accept such a teacher's invitation and keep on trying to work out the task exploring with the calculator which eventually led them to try with fractional exponents. This encouraged the group to go ahead in that direction and found they were able to get closer and closer. Eventually, they started to suspect they will never find an exact solution, which gave a good opportunity for the teacher to talk about irrational numbers and fractional exponents.

Though $4^x=29$ was the first equation the students met in the course, the aim of this activity was not that they learn about exponential equations. The aim was to put students in a good position to further elaborate on some of the notions they already had about numbers and arithmetic operations. The data suggest that these students were learning about the following topics: (i) the use of the calculator as a tool for exploring and refining their conjectures; (ii) order with decimal numbers; (iii) notions of approximation by defect and excess ("$4^{2.4}$ is below 29, $4^{2.5}$ is above … the solution has to be between $4^{2.4}$ and $4^{2.5}$".); (iv) The notion of an equation as "an operation where a number is missing"; (v) how to read algebraic expressions as $4^x=29$. After this experience, equation solving with any kind of equations was for them a kind of playful situation where they just have to find the "missing number".

Students' responses to "compute 438+725 without using the plus key"

This activity was one of the most difficult for students to complete. Nevertheless, they did not accept the teacher's help to show them a way of solving it. After considerable struggle the students were able to produce answers as the ones shown below.

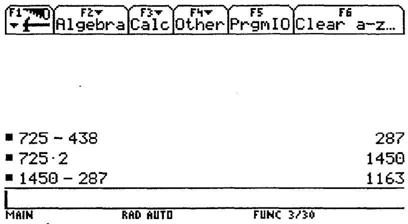
Trial and refining approaches. This was the students' most frequent response. The following figure illustrates this strategy taken from a student's approach.



Interview excerpts

"I began with 1500 because it is bigger than 725 and 438. Then I took away 725 from 1500, it gave 775 … 775 is too far from 438, then I chose 1200 and took away 725 from it, it gave 475, is close to 438 … Well, I keep on doing like that till I got 1163, 1163-725 gives 438 … This is my method".

Systematic approaches. There were students who developed approaches that actually are algorithms "to add without adding". The figure on the right hand shows one of these algorithms (725+438=1163)



Discussion

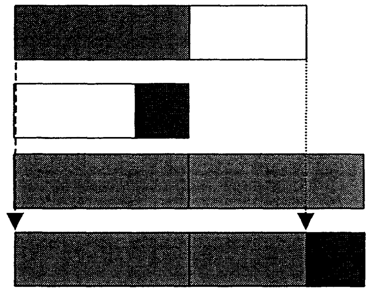
The high frequency (92%) of the students' responses based on trial and refining approaches strongly suggest that this strategy better fits their ways of reasoning. Their reactions during the fieldwork also suggest that the experience they had confronting the activities, no matter how elementary or correct their approaches were, put them in a good position to take somebody else's view. The following extract provides evidence for this. Atahualpa, the student who developed the algorithm shown above was not one of the best in his class. The group of students who were the best did not want to accept his finding and asked him to explain why his "method" works. Atahualpa was not able to offer a clear explanation, he only was able to describe his approach. Then, the best students engaged themselves in making evident that he was wrong. When they showed their work to the group, they accepted that what they found was an argument that shows that Atahualpa's method was fine "no matter which numbers you want to add up". They sowed their argument by using pieces of cardboard which is illustrated in the following diagram.

"The grey color rectangle is the bigger number, say '**A**'. The white color one is the smaller, say '**B**' ... Together it gives **A+B**"

If we take away **B** from **A** the black rectangle is left.

Now we double **A**.

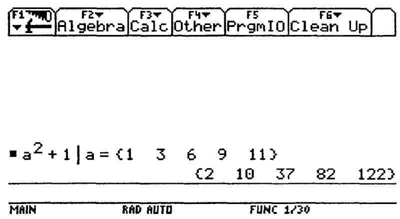
Let's take away the black color rectangle from the double of. **A+B** is left!



These students' explanation also shows that they were going back and forth from the particular to the general. In fact their argument is a diagrammatic representation of the algebraic identity $a+b=2a-(a-b)$, which is the method found by Atahualpa. Again, this intuitive students' approach gave the teacher a good opportunity to review topics about negative numbers.

Number-based contexts to develop meanings for algebraic expressions

Working with the calculator's "home screen" the student can assign a number value to a letter; then, in terms of the letter "variable", define (and edit as needed) an algebraic expression, and request the calculation of the number named by the expression.



It should be stressed that no instruction was given as to nomenclature or algebraic concepts. As far as the students were concerned, the algebraic expressions were merely programs that allowed the calculator to "understand" what they wanted it to do. This calculator's facility was used to make students produce algebraic expressions and get immediate feedback from the machine. The only rule used here is factual: they get or do not get the desired number pattern. Sixty worksheets were designed for students to complete, each worksheet included new mathematical elements. The following examples show the type of tasks the students were asked to complete.

a) The table below was made by using the calculator.

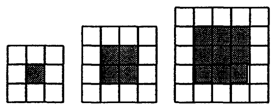
Input	1	4	6	9
Output	1	7	11	17

Can you program your calculator to produce the same table?

b) Program your calculator so that it produces the same table as the one below. After you have done this, make a program such that, if you enter the output value, the calculator gives you the input value.

Input	2	5	7	10
Output	8	17	23	32

3. Look at the following pictures and draw the next two pictures.



a) How many squares are needed to make the frame of the black square in picture number 27?

b) How many squares are needed to make the frame of the black box in picture number 100?

c) Can you program your calculator so that it helps you answer any questions as the ones before?

Meanings assigned by students to letters and algebraic expressions

While sorting out a task in individual interviews concerning the use of the calculator's language, the students were asked the following questions: (i) "What does the letter you use when you make a program in your calculator mean to you"; (ii) "What does a program like the ones you have made in your calculator mean to you?" The following vignettes illustrate the meanings that students gave to letter and algebraic expressions.

"The letter I use in a program is the name of a memory in the calculator, but I would rather think that a letter personifies a number, any number...see...you input a letter in the program and you can give it different values (he types in the program 3×A–2 and runs it with different values) ... see ... the program understands that it has to calculate a new result for each number you input...you don't have to change the letter ... only the value" (Diego, average boy student)

"A program is used to do something...to complete a table or to solve a problem ... I mean ... it serves to tell the calculator how to do what I have in my head to solve a problem " (Jenny, above average girl student).

Students' responses to geometrical patterns

"I noticed that the number of squares changed from one figure to the next like the multiples of four ... That is, there are 8 squares in the frame of the first figure, 12 squares in the second, 16 in the third, and so on ... Then I thought that multiplying by 4 was a good idea ... Then I made the program a×4... I knew that program would not work, it is jus a way of beginning to think ... Then I thought that a×4+8 would work, but if a=1, a×4+8 gives 12 and I wanted 8 ... That is why I took away 4 and made the program a×4+8–4, or the same, a×4+4 ... This program works well for any other figure in the sequence ... I mean, with this program is easy to answer how many squares should the figure number 100 have, if a=100 ... it gives 404 squares (Jimena, average girl student).

Discussion

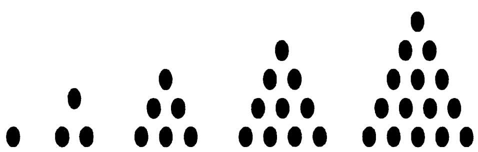
Similarly to what was observed within arithmetic-based environments, trial and refining strategies was the most frequent type of students' approaches (94%). The data gathered suggest that a key aspect to the development of these strategies and meanings was that the calculator allowed them to use a programming code, not just as a means of editing, but also as a language for negotiating problem solutions: "it serves to tell the calculator how to do what I have in my head to solve a problem". The data indicate that availability of calculators makes it possible for the activities concerning recognition of numerical patterns to go beyond encoding what is being represented, as is the case when working with pencil and paper. Jimena's response to this question clearly describes how her experience with trial and refining strategies led her to effectively use algebraic code to face the problem situation: This suggests that in the calculator-based environment, algebraic symbolization is rather the result of interaction between what is known (arithmetic), and the process of attaining a goal (making the calculator reproduce a table of given values). For example, the program 3×B-1 generates the numerical pattern 2, 8, 17, 23, when B=1, 3, 6, 8. If this was the pattern they wanted to generate, they knew that the program they had written was correct, if it was not, they could reanalyze the pattern and try again.

Visual contexts to model algebra problems

Using computational resources and exploit the visual context it offers to approach the study of functions has been a topic widely studied during the last few years (Hector, 1992; Ruthven, 1992; Hitt, 1996). The didactical approach used in the present study aimed at putting students at the position of learning about function graphs by using them. In accordance to this the students were taught only how to produce a graph by entering an equation into the calculator's graph editor, how to use the "TRACE" tool to "walk through the graph", and how to produce scatter graphs and display the table of values corresponding to a given equation. The activities consisted on (i) making the students observe the change in the graph when they changed one of the parameters of a given equation; (ii) producing graphs of linear and quadratic functions satisfying given conditions, as producing parallel graphs to a given linear graph, or producing graphs so that all them have the same intersection point; model problem situations in order to ask questions about the situation they were modelling.

The example that will be discussed is the following:

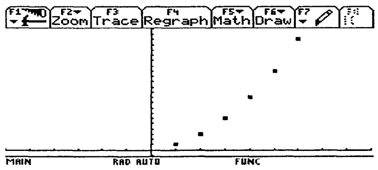
Look at the following diagram. These figures are the five first triangular numbers. Imagine you continue drawing the sequence of figures without making any mistake. How many points will have the figure that appears in the place number 729?

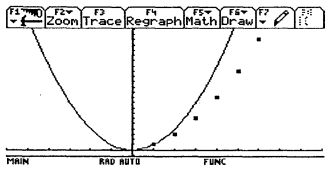


Students' responses to the" triangular numbers question"

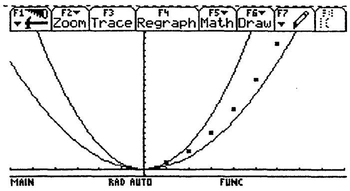
First approach: The students made a graph using the data provided by the diagram. They noticed "it looks like a parabola"

Second approach: "I tried with x^2 ... but it is too small" (sic).





Subsequent approaches: I tried with $0.5x^2$ ...See, the solution has to be between x^2 and $0.5x^2$



Discussion

Some of the students (20%) eventually got $y=0.5x^2+0.5x$ as a model to confront the problem situation. Others (60%) obtained good approximations and by extrapolation were able to correctly answer the question. This kind of approach was attempted by 85% of the students, which seems to confirm that trial and refining was an strategy that better fitted their ways of reasoning. It is clear that such students' unconventional strategies closely relate to their experience using the calculator as an aiding tool to confirm/refute their conjectures. The same question was asked to 30 mathematics teachers within a workshop. It seems worth mentioning that 90% of them answered that this question relates to "adding up the first natural numbers", that "this problem can be solved by a formula", and so on, but none of them was able to remember that formula nor to somehow deduce it. Most of them were really surprised when they were told how to face the question using graphs and the equivalence between $y=0.5x^2+0.5x$ and $S_n = \frac{1}{2}n(n+1)$. They explained they have never thought of a relationship between that formula and a function graph.

Final remarks

What kinds of algebra learning does this environment promote?

The data gathered suggests that graphic calculators can be used to promote a students' self-pace way of learning that allows them to confront the study of algebra following their own ways of reasoning. This approach to algebra seems to help pupils overcome the constraints imposed by a formal approach, where particular ways of teaching may lead them to see algebra as an enormous bag of rules, formulas and tricks to have in mind. Furthermore, the approach of algebra here reported seems to democratize the learning of this discipline, 89% of the students taking part in this study were able to pass the course, both in accordance with the assessment criteria used in the study and using the standard goals required by the Mexican Ministry of Education to every school in the country. According to the tests results administered by the School Board, the low and average attainment students were the most favored by this approach; the high attainment students reached the results they were expected to obtain.

To which aspects of algebra learning does this particular technology make a distinctive, unique contribution?

The study here reported suggests that a graphic calculator based-environment can be profitably used to help middle school students develop unconventional algebraic strategies and meanings. These notions seem to be a crucial background on which students, who will follow a scientific or technological studies at university level, might successfully confront a more formal approach to algebra as the language of higher mathematics.

To what extent ought the goals of algebra education be affected by the availability of this technology?

According to the Mexican Curriculum for Middle Schools, the activities used in this experimental study overtly go beyond the instruction goals proposed for Year 1. The kind of problem situations used in this study in the last two sections are usually introduced during Years 2 and 3 in Mexican Middle Schools. The results of the present study strongly suggest that a new curricula is needed if technology based-approaches are introduced in the classroom. The results here reported also suggest that, before the Ministry of Education introduces the use of technology in schools, it is necessary to intensively work with teachers in order to update their professional standards.

Are there activities using graphic calculators that can be profitably undertaken by younger students?

This study indicates that activities that are left to the last school years in secondary schools can be administered to students who have just finished the elementary school. The data gathered suggest the need to implement a study with younger pupils in order to investigate the feasibility of using the activities based on arithmetic contexts with students in the last grades of primary school.

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L'Algebrista: a microworld for symbolic manipulation

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Within the theoretical framework of Vygotsky's theory, the paper presents a new algebra software. The main features of the software are described and their potentialities are discussed. According to the Vygotskian theory, expressions and commands may be thought as external signs of the Algebraic theory, and as such, they may become instruments of semiotic mediation: In other terms they can be used by the teacher, in the concrete realisation of classroom activity, according to the motive of introducing pupils to syntactic manipulation as a theoretical activity.

Introduction

As clearly shown by previous research Studies the evolution of algebraic symbolism can be described in terms of "procedural-structural" terms, and, in particular, according to the psychological model described by Sfard (1991) this evolution requires a long period of transition. Procedural character of pupils' conceptions related to literal terms and expressions tends to persist; at the same time, although symbolic manipulation of literal expressions is largely present in school practice, the absence of "structural conceptions" appears evident (Kieran, 1992, p. 397).

Limits related to a procedural approach to symbolic manipulation have been often pointed out, so as the need of a "structural-relational" approach in order to master symbolic manipulation in a productive way (Arzarello, 1991). Poor strategic decisions has been described, made by students with extensive algebra experience, but unable to identify the right transformation to be accomplished: when the task does not explicitly indicate what transformation has to be performed, pupils are unable to take a decision, "go around in circles" (Kieran, 1992, p. 397) carrying out transformations without any clear gaol.

A key point of structural approach is the notion of "equivalence relation" between expressions, which can be defined as follows: two algebraic expressions may be interpreted as computation procedures and it may happen that, whatever values are attributed to the letters involved, such procedures will to the same numerical result. According to this definition of equivalence one may assume the classic properties of sum and multiplication as axioms of a theory, within which the equivalence between expressions can be proved.

Thus, symbolic manipulation makes sense within a theoretic system; it is characterised by activities of transformation of expressions using the rules given by the assumed axioms. Certainly this perspective is not very common in school practice (at least in Italy), yet it is exactly the perspective we assumed.

A previous study project, concerning pupils introduction to geometry theory (Mariotti et al., 2000), clearly showed how a computer environment may offer a support to overcome the well known difficulties related to theoretical perspective. In particular Mariotti analysed the semiotic mediation that can be accomplished by the teacher using specific instruments offered by the Cabri environment (Mariotti, in press) .

In the same stream a research project, still in progress, has been set up; a computer microworld, L'Algebrista (Cerulli 1999, Cerulli&Mariotti, 2000), was designed incorporating the axioms defining the algebraic equivalence relation. A prototype was realised and experimented in ninth grade classes. In the following sections the software will be described together with a short account of a teaching experiment.

L'Algebrista: an overview

The software is a microworld incorporating the basic theory of algebraic expressions. Activities in the microworld consist in transforming expressions, a chain of such transformations correspond to the proof of the equivalence of expressions.

What follows is a list of the main ideas underling L'Algebrista:

- A **symbolic manipulator** which is totally under the user's control. It is going to be a microworld of algebraic expressions where the user can transform expressions on the basis of the fundamental properties of operations, which stand for the axioms of the local theory.
- Axioms are represented by the "buttons of the properties of the operations" which must not have any implicit behaviour; the buttons must realise only transformations which are **directly** implied by the axiom they represent. Furthermore, a button must not apply recursively an axiom, but only once.
- Buttons that represent equivalence relationships must be reversible and must include the inverse functionality as well. This is required to make explicit the meaning of equivalence between expressions, and to associate the correct meaning of equivalence to the "equal" sign ("=").
- Some buttons will represent or recall conventions of the mathematical community, while other will represent or recall conventions, negotiated within the classroom community. Thus there will be a chance to make explicit some conventional aspects of the activities, in order to make explicit the conventionality of Mathematics.
- The interaction is based on direct manipulation, using the mouse to select expressions and to click buttons. Thus the user does not have to learn any coding language in order to interact with the system. This feature differentiate L'Algebrista from other computer environments quite popular in school practice such us LOGO and PASCAL, where the user must learn a peculiar language to interact with the computer.
- L'Algebrista is not able to do any transformation if it is not guided explicitly by the user using the above mentioned buttons. Differently from what happens with other symbolic manipulators, the user has the total control on the transformation activity.
- When proved, a new theorem may be represented as a new button and added to the system of axioms and theorems. Thus the evolution of the microworld will go on in parallel with the evolution of the theoretical system.

Brief description of the software

After the start up sequence L'Algebrista offers the user to chose between four different menus (Figure 1): ***Base, Meta, Aiuto*** and ***Info***. As one might imagine, the menus *Info* and

Figure 1: The *Base* menu and the *Meta* menu

Aiuto give information concerning L'Algebrista and how to use its microworlds and environments. The *Info* menu contains the common explanations concerning the copyright and licence; the *Aiuto* menu contains information about the main features of the different menus and an "on-line help".

The Base menu

This menu introduces the user to the main working environment of L'Algebrista; here the user can chose between several *Teorie* (Theories), i.e. microworlds of algebraic expressions. In figure. 2 it is possible to see the three basic theories we are using in a classroom experimentation which is going on by now.

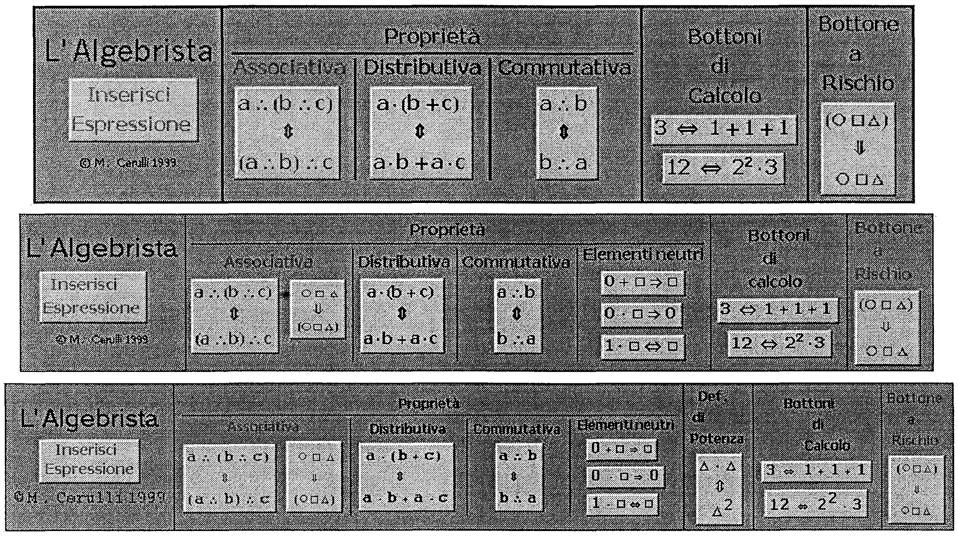

Figure 2: Teoria 0, Teoria 1 and Teoria 2

Each *theory* is made of palettes (windows containing buttons) and notebooks (working environments). To start the activity the user has to write an expression in a notebook and insert it the microworld, then the manipulation of the expression will be carried out by selecting sub-expressions and clicking on the buttons available in the palettes. Every button (except one, as discussed below), always produces expressions which are equivalent to the expressions it is transforming.

The Meta menu

The word *meta* in this case stands for "meta theory", in fact this menu offers two instruments to be used to create new theories.

The first instrument is called ***Il Teorematore***, it lets the user create new buttons to represent new transformation rules that can be included in the palettes and used to manipulate expressions in L'Algebrista.

The second instrument is called *Personalizza Palette* (Palette personalisation) and is just a notebook containing a collection of ready made buttons and instructions concerning how to create a palette using those buttons and the buttons created with *Il Teorematore*.

With this two instruments teachers and pupils can actually build palettes including the axioms, definitions and theorems they prefer.

Description of the interaction with the basic microworld offered by L'Algebrista

Let's now describe the main commands of the theories presented in the *Base* menu analysing some peculiar aspects of the computer-user interaction.

Figure 3 shows a palette of L'Algebrista; more precisely it represents the first theory we used in our teaching experiments and corresponds to *Teoria 0* in the *Base* menu. This palette is divided in four sectors corresponding to: the button *Inserisci Espressione (Insert expression)*; the buttons of the properties of sum and multiplication; the computations' buttons; the *risky button* ("*Bottone a Rischio*"). This partition is coherent with the distinction between the roles played by each button in classroom activities. In particular, the bottoms representing the properties of the operations were separated from the buttons that execute computations, in order to distinguish the activities of transformation based of the axioms, from those based on numerical computations.

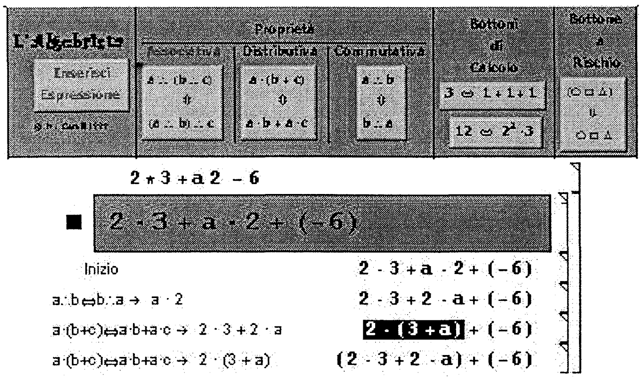

Figure 3: In a notebook the user writes the expression to work with («2 * 3 + a 2 - 6» in the example), then after selecting it the button Inserisci Espressione is clicked, thus L'Algebrista creates a new working area where the buttons are active.

An example of interaction is reproduced in Figure 3. The user writes on a notebook the expression he/she wants to work with ("2*3+a 2 - 6" in our example), then he/she selects the expression and clicks *Inserisci Espressione* ("Insert Expression"), thus L'Algebrista creates a new working environment where the original expression is marked on its left with the label *Inizio* ("start"). The operation of *inserting* the expression is fundamental because it proclaims the entrance into the microworld where it is possible to act only using the buttons offered by L'Algebrista.

We observe that when an expression is *inserted*, its new instance comes out with some changes: every multiplication is represented with a dot ("•"), so either stars ("2*3") or spaces ("a 2") are substituted with a dot ("2•3+a •2"); every subtraction is transformed into sum and every division is transformed into multiplication. L'Algebrista does not know subtraction, and division: this follows from a precise didactical choice because we wanted pupils to work in a "commutative environment".

Interaction always happens by selecting a part of an expression and clicking on a button. The selection was designed so that it is not possible to select parts of expressions which are not sub-expressions from an algebraic point of view. For instance, given the expression **a•b+c** it is not possible to select **b+c**, if one tries to do it the software will automatically extend the selection to **a•b+c**; on the other hand one can select **a•b** or **c** or **a** etc. This feature corresponds to fact that the expressions of this microworld incorporate a fundamental algebraic characteristic of mathematical expressions: their tree structure.

Going back to the previous example, the expression can be now transformed by selecting the term **a•2** and clicking the 'commutative property button'; a new expression is produced (written just below), the term **2•a** is substituted by the term **a•2**, while on the left a label indicates the button used and the sub-expression it was applied on. Going forward we transformed one part of the expression using the distributive property, and in the following step, using the same button[1] we inverted the previous transformation. Coherently with our didactical hypothesis, the buttons incorporate all the functions of the properties of operations without

[1] The button checks the structure of the expression and then decides how to transform it; in case no structure is recognised then the expression is left as it is.

advantaging any peculiar direction. Note that most of the symbolic manipulator use another, different, command in order to invert a specific command.

L'Algebrista's buttons always produce a correct expression, that is equivalent to the original expressions to which they had been applied; the only exception is the *Risky Button* which is used to delete parenthesis: for instance it can transform **a+(b+c)** into **a+b+c** but it can also transform **a•(b+c)** into **a•b+c**. This button has been put aside and highlighted, so that the user can distinguish it from the others and use it with particular attention. Its meaning and its use is to be negotiated in the class in order to make clear the conventional use of parenthesis and its relation to algebraic axioms.

We conclude this section with a couple of observations on the notations used: the commutative and the associative properties have been represented using the symbol "∴", instead of "+" and "•", this is certainly related to a matter of economy, but also it is intended to familiarise students with generalisation of structures.

Il Teorematore

Il Teorematore (Figura 4) is a peculiar environment which allows the user to create new buttons. The use of *Il Teorematore* is very simple, one just has to write the new transformation rule, to select it, and finally to click on the button *Teorema*. In our opinion it is fundamental that the user does not have to learn any coding language to create new buttons, but he/she just has to use mathematical symbols.

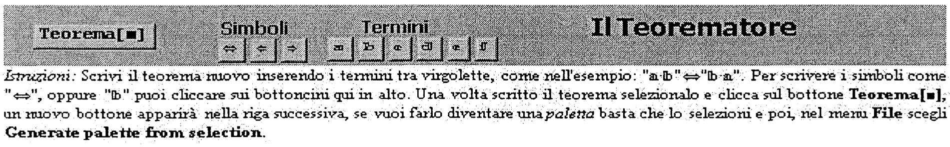

Fig.4: Il Teorematore

Thanks to *Il Teorematore* the theory incorporated in L'Algebrista can grow together with the user's mathematical knowledge. In other words, the user can create as many buttons as he/she wants, and can then use them in his/her future interactions with L'Algebrista.

Coherently with its basic principles, L'Algebrista, thanks to Il Teorematore, lets the user create new buttons corresponding to bi-directional transformations, that is buttons which incorporate all the functionality of equivalence relationships. This feature differentiate strongly L'Algebrista from most popular symbolic manipulators. In particular DERIVE does not allow the user to create any new command, while other didactical softwares (such as MILO and Theorist) let the user create only one direction commands that can be inverted only by using other commands.

Finally we observe that Il Teorematore **does not** check mathematical correctness of a new transformation rule. This is a consequence of a specific didactical choice: we want the pupil to be responsible for the validation of a new theorem or transformation rule. Thus it is the student who will have to control his/her set of axioms and theorems, i.e. the theory he/she is building in L'Algebrista.

Outlines of the classroom experimentation

The experimentation started during the school year 1999/2000 in a 9th grade class, and it is still in progress. Here we are not going to give a detailed description of the two experiments, but we will indicate the outlines of the sequence of activities and the basic ideas inspiring it. First of all we recall that our educational goal concerns:

- to introduce pupils to symbolic manipulation;
- to introduce pupils to a theoretical perspective.

According to our hypothesis, the concept of *equivalence relation* is the basic principle underling symbolic manipulation, thus it represents the starting point of pupils activities.

We introduce the problem of comparing expressions, taking into account the fact that, at this school level, pupils consider numerical expressions as equivalent when they give the same number as result. Thus it is not difficult to negotiate the interpretation of numerical expressions as computation schemes, which will be equivalent if they give the same result.

The idea of interpreting expressions as computation schemes allows one to introduce the properties of sum and multiplication as principles (theory axioms) that determine a priori whether two computation schemes lead to the same result: if two expressions are equivalent on the base of such properties then the computation of the two expressions must lead to the same number. Thus a new equivalence relationship between expressions is introduced:

if one expression can be transformed into another using the properties of sum and multiplication, then the two expressions are equivalent.

In the microworld L'Algebrista this corresponds to:

two expressions are equivalent if it is possible to transform one into the other using the given buttons.

Once this equivalence relation is accepted, pupils are asked to compare expressions. A new terminology is introduced: one says that the equivalence of two expressions is **proved** if one expression is transformed into the other using the axioms; vice versa one says that the equivalence is **verified** if the calculation of both the expressions leads to the same result.

With literal expressions the difference between *proof* and *verification* becomes even more definite: the use of axioms becomes the only way to state the equivalence between two expressions, whilst numerical verification (substituting the letters with numbers and computing the expressions) becomes the main way to prove that two expressions are not equivalent.

Il Teorematore can then be used to add a selected choice of proven equivalencies to the set of buttons to be used for new proofs.

Semiotic Mediation

Within the Vygotskyan framework of semiotic mediation theory, a central role is played by the signs used to mediate mathematical meanings. Thus when analysing, and/or designing, an environment, its fundamental to study the nature of the signs to be used to mediate mathematical meanings. In our opinion, analysing didactic softwares should take into account such differences of representation so as their implications in terms of meaning construction and their functioning as instruments of semiotic mediation (Mariotti, in press).

A full discussion on this topic is beyond the scopes of this paper, thus we will limit to discuss on some aspects strictly related to L'Algebrista and more generally to symbolic manipulators.

Let us take the case of algebraic expressions as represented in L'Algebrista and algebraic expressions as represented in the paper and pencil environment. If we consider an expression, written on a sheet of paper, no interaction with the specific environment will make explicit its structure; in that case the structure of the expression can be implicitly imposed by the expert, but might be totally absent for novices.

The representation of an expression in L'Algebrista incorporates its mathematical tree structure[2], and this structure becomes explicit, "tangible", when the user interacts with the environment[3]. This feature actually differentiates L'Algebrista also from other symbolic manipulators, such as DERIVE, where, in order to transform a sub-expression of a given expression, the user has to write it again in the buffer; in this case the user needs to know a priori (independently from the software environment) the structure of the expression, in order to chose the right sub-expression to rewrite. On the other hand, in the case of L'Algebrista, part of the theoretical control is incorporated in the computational object so that actions on the expression are submitted to that control. The specific *selection function*, which embeds such

[2] Actually this feature is inherited from *Mathematica*, and is called *structured selection*.

[3] See the description of the software above.

control, constitutes an external sign (Vygotsky, 1978) of the theoretical control, and as such it can function as a semiotic mediator. An example of how pupils use the selection sign as an external control referring to the theoretical properties, during the manipulation of an expression, is provided by the following protocol in fig.5.

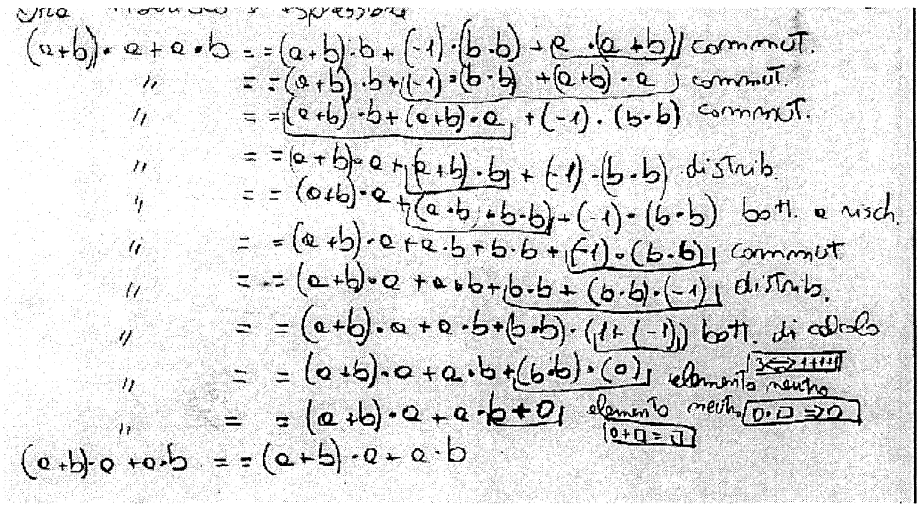

Figure 5: In the case of a comparison task, performed in paper and pencil environment, the protocol shows that pupils use signs clearly derived from L'Algebrista, in particular the *selectin function*, or the iconography of the buttons.

Expressions are not the only things that can be represented in a symbolic manipulator environment: actually any command represents some specific procedure or theorem. Again one can compare various softwares discussing, for example, how the distributive property (an axiom) is represented. Most symbolic manipulators (ex. *Mathematica*, DERIVE, MILO, Theorist) split the distributive property into two distinct commands, usually called **Factor** ("Multiply out" in MILO) and **Expand**. Thus, if the distributive property is expressed by the equality **a•(b+c)=a•b+a•c**, the command **Expand** corresponds to "going from left to right" and **Factor** corresponds to "going from right to left". **Factor** and **Expand** maybe considered one the inverse of the other, and correspond to the two functions of the distributive property. In other terms Derive introduces a procedural interpretation of the sign "=", and may be an obstacle to the correct interpretation in relational terms.

In L'Algebrista the distributive property is represented by a unique button, called "button of distributive property", and represented as **a•(b+c)=a•b+a•c**. This button performs two transformations, one inverse of the other, so it actually *works* as an equivalence relationship: it allows the user to transform an expression into another and vice versa.

Furthermore the used iconography expresses the effect of the button on the algebraic structure of the expressions, whilst in the case of "Factor" and "Expand", this words do not say anything a priori about the relation between the structures of the two expressions involved.

In conclusion, the "button of distributive property" expresses the character of equivalence relationship of the distributive property, and incorporates both its functions. On the other hand, the commands Factor and Expand, represent the functions of the distributive property but cannot express the fact that it is an equivalence relationship.

Finally it is worth to observe how the L'Algebrista environment can offer instruments of semiotic mediation related to the theoretic aspects of algebra:

- expressions in L'Algebrista environments are signs of algebraic expressions;
- given buttons are signs of axioms and definitions;
- new buttons, built using Il Teorematore, are signs of theorems;

- transforming an expression into an other using the buttons corresponds to proving that the two expressions are equivalent, the produced chain of justified steps corresponds to a proof.

The aim of the discussion, concerning the above mentioned examples, is to show that a very wide range of mathematical concepts can find a counterpart in computer environments. Furthermore, a single mathematical concept can be represented using very different signs that make explicit different aspects of the same concept. But the crucial point remains the effectiveness of such signs as instruments of semiotic mediation. The experimentation in progress aims at analysing the process of semiotic mediation as it is accomplished in the classroom practice.

Conclusions

The development of information technologies leads to many discussions, one of those concerns the revision of school curricula taking into account the changes brought by this development. The ideas we presented in this paper give an example of a new way to approach symbolic manipulation (Ita. "calcolo letterale"). Our proposal is to be considered as concerning questions relative to the introduction of pupils to theoretic thinking. Thus symbolic manipulation has been interpreted taking a theoretic perspective and the particular software environment has been designed as embedding Algebra Theory.

The axioms incorporated in the buttons of L'Algebrista become tools that pupils can learn to use to transform expressions in order to attain activities' goals. The distinction between buttons representing axioms, and buttons for computations, helps distinguishing the terms "proof" and "verification"; and may contribute to build the meaning of *proof* as well as the idea of *theory*. Furthermore the possibility of creating new theorems and making them usable, offered by "Il Teorematore", lets the student participate in the activity of *theory* evolution.

A research study has been set up with the aim of analysing the process of semiotic mediation in the case of L'Algebrista. The research project has been set up taking advantage of the previous study concerning the use of Cabri, the experimentation, still in progress, is devoted to analyse the joint use of L'Algebrista and Cabri: the main hypothesis concern the synergy provided by the two environments in rapport to the objective of developing a general idea of theory.

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New insight in mathematics by live CAS documents.

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Abstract

Traditional education in mathematics is mostly a matter of hindsight, and many mathematics texts offer little opportunity for students and learners to gain insight. Here we show how experiments in CAS in tertiary education and in the final years of secondary education can lead to new ways of handling problems, new conjectures, new visualisations, new proofs, new correspondences between theories and sometimes even new definitions. Experimentation will also have effect on the teaching of algebra which now takes place in parallel with the introduction of Computer Algebra systems and symbolic programming. Many topics from this presentation involve simulations by CAS, and some are taken from abstract mathematics where resistance to the introduction of CAS has been heaviest. For each document the implication on the teaching of algebra is commented. The live documents are offered in classroom, lecturing with data projector, and also reside on servers as a template for project work by freshmen in tertiary education.
All documents are written in Mathematica. Some of these were written under the "Exploot" project (experimental learning environment using education technology), funded by the Education Department of the Flemish regional Government in Belgium.

Introduction

Most modern mathematics texts are well structured, nicely formatted and polished, but offer little leeway both for the teacher and the learner. Results are presented only when authors feel they cannot immediately be improved, and the subsequent authors using these results or teachers mentioning them will present results in a simplified form or with alterations to accommodate the user groups they are addressing. It is a filtering process that in each step reduces the content, trivialises the theory, and which moreover hides the creative effort from the initial authors. Important applications, which motivated the development, are often left out and the historic build-up is forgotten.
The final user is not aware that the build-up of mathematics has always been a creative process which most of the time involves lots of experimentation. We do not realise how much work has been done by great mathematicians. Students tend to think mathematical discovery only comes by stroke of genius. Students are convinced they will never have such luck, and this adds to the general uneasiness about mathematics and its teaching.
Many educators therefore want to replace the traditional paradigm using the sequence

definition -> theorem -> proof -> corollary (-> application)

by an approach which is more historic using the discovery chain

problem -> experiment -> conjecture and idea of proof.

Nowadays introduction of a CAS allows lots of experimentation by the students, thus helping to find reasonable conjectures. It can offer new insight in how to prove these conjectures and

may point to new mainstream developments in mathematics, and may even point to some topics that are now obsolete. Some mathematicians and teachers may regret that certain topics in the curriculum no longer deserve the attention they used to receive, but this is the way science evolves.
Introduction of a CAS has a profound impact on the traditional algebraic skills that students should learn. While some skills are not useful any more, the student should retain an ability to control the output by the computer in simple cases. He has to gain more insight in the mathematics behind a problem to get his computer input organised correctly. Inputting is a writing assignment that improves his skills and gives him a deeper understanding of his solution.
The active documents below are written in Mathematica, a general-purpose CAS that allows efficient symbolic programming closely following mathematical ideas [1]. Some topics were developed with funding by the Flemish government [2]. Part of this work can be ported to other CAS environments as well. While it is a drawback that a good CAS does not come for free, it will involve much larger expenditures in manpower if every teacher has to start writing from scratch his own programs, rather than adopting a widely circulated commercial product. Moreover, in some of the examples efficient programming is included in the build-up of mathematical ideas. In the last presentation we show how list processing, rather than procedural or vector processing, should be included in algebraic reasoning.

1. The burial of trigonometry.

Cosine and sine functions are merely projections of a point moving along the unit circle, called harmonic movement. It is therefore not surprising that all trigonometric formulas can be derived from a geometric feature in the complex number plane, and rote learning of formulas is no longer useful. A rapid introduction to the complex number system is possible using a naïve description of polar co-ordinates in the plane, complex multiplication by adding polar angles and the theorem of Thales. A straightforward representation of complex polynomials allows the user to work out problems involving roots of unity [2]. It is unfortunate that algebra curricula forbid multiplication in the plane, and that therefore rotations cannot be represented in a simple way. Many problems concerning physical movement in the plane can be modelled by parametric plots. Powerful applications, such as encryption techniques, complex eigenvalues for two by two matrices and the discrete Fourier transform, can all be offered at an early stage after this introduction of complex numbers.

2. Snowflakes, dragon curves and other strange sets in the plane.

Using complex numbers again, together with a turtle graphics description of the path, one obtains elegant descriptions of these fractals, and experimentation leads to insight in some sets of small or zero measure in the plane. We approximate the Koch snowflake from inside, and from outside by iterating the design of the Mitsubishi logo. The effect is more visual in the plane than it is with fractals in the real number system such as the Cantor ternary set, the simplest algebraic infinite product. Programming fractals in the plane is simple. Moreover it is linked to non-commutative word problems on a finite alphabet in algebra.
The reason why many fractal curves are not differentiable also becomes obvious from such constructions.

3. Limits: guessing the accuracy of approximations.
The well-known Stirling approximation of n! using logarithms is coarse for small values of n, but its relative error decreases as n increases. The question is to have an idea how good this approximation is, and if more precise forms are possible. Plotting values does not help much because of the huge scale of such plots. The relative error looks like some negative power of n, and we can ask which exponent is involved. Having some idea of the speed of approximation allows the user to decide with what precision the computer has to compute. An approximation has to be more precise if it is used in an elaborate computation later on.
In many problems the main question is not whether there is a limit, but rather how fast this limit is approached. This allows approximate computation, and a framework for approximate algebra has been developed in the past decades. The rule of L'Hopital is no longer sufficient, but has to be replaced by some careful analysis of the orders of growth involved. This approach uses formal series and follows ideas due to Landau and Hardy. Learning the effect of a composition of orders of magnitude makes it is possible to do efficient approximate algebraic reasoning.

4. The exponential function as a transition between discrete and continuous processes.
Many problems in calculus or real analysis have their counterparts in a discrete setting: discrete functions are transformed into functions with continuous variables to be able to use differential or integration techniques, differential equation models are converted to recurrences by a stepwise approximation, dynamic systems can be described by systems of differential equations or by iterating matrices, and the eigensystem descriptions are running in parallel. Such duality should be learned from the very beginning. It is important to shift from discrete to continuous phenomena and conversely whenever this is useful for the solution of a problem. Discrete solutions are given by powers of variables while the continuous solution involves the exponential function.
It is therefore unfortunate that curricula tend to partition mathematics into subheadings that allow only one of these approaches, either the discrete one (in linear algebra) or the continuous one (in calculus and analysis).

5. Perfect shuffle and combinatorial identities.
The exponential also arises in other problems related to probability theory such as the race horse derangement involving all permutations in a finite set. It is surprising that the solution of this problem is independent from the size of the problem. Here the approximation of combinatorial expressions by expressions involving exponential functions allows simplification and fast computation of solutions.
Using exponential functions is also the idea behind the theorem of large numbers and more generally in the use of probability densities.

6. Making quotient groups without checking normality of subgroups.
Building quotient group is perhaps the most difficult topic in abstract algebra. Programming the construction of cosets closely following the usual definitions we arrive at colour diagrams that exhibit the structure of a quotient group without checking the normality of the subgroup beforehand for non-commutative groups [3]. This text offers templates for programming other results in group theory and other abstract constructions. The colour diagrams allow the illustration of properties and structure of classical groups in the classroom, facilitating

understanding by the students [4]. Isomorphic groups and certain decompositions are discovered at a glance. A (solitary or) two player game concerning square matrices with entries in a finite cyclic group of rainbow hues allows students to practice their understanding.

7. Continuous objects, and generalised functions.

The ε–δ condition is not understood by many students, and difficult to check point by point. By looking at graphs as a whole, it is easy to see it is merely a property from dynamic geometry, translating figures in the plane. This holistic approach allows definition of continuous objects that are no longer functions, such as inverses of functions, other multiple-valued relations, and even geometric figures and densities. The formulation of classic theorems, such as the fundamental theorem of calculus and the behaviour of the solutions of a differential equation, is much easier in this setting.

The transition to mathematical transforms, defined as functions acting on other functions, becomes straightforward.

8. Looking at Fourier series from a distance.

Using the representation of polynomials one can introduce complex power series visually. By projecting the plots on horizontal and vertical planes in space, one obtains the cosine and sine Fourier series with the given sequence of coefficients.

Delicate results concerning harmonic conjugates and a simple proof of Gibbs' phenomenon arise naturally.

9. Symbolic programming

Algebra is about replacing numbers by symbols, so that results are obtained for a large set of numbers, or all possible solutions at once. Symbolic programming follows this mathematical reasoning closely. Instead of handling results for specific numbers as is still done on elementary pocket calculators, it is much more interesting to write down in a CAS general expressions depending on parameters and delay the replacement by the actual value as long as possible, thus handling all possible cases at once.

A useful example is the dynamic system behaviour of models in economy and in other domains. In many applications it is the transition from one type of behaviour to another that is the most interesting part, and this can be shown by animations running over values of the parameter.

10. Using list spaces.

Important mathematical objects such as polynomials or fractal geometric objects arising in the examples above are difficult to describe by vectors of fixed size. This is a problem in procedural programming where the size of an object has to be decided well in advance. The advent of list processing allows a concise and more flexible approach, together with a description of the objects that corresponds to mathematical intuition. The algebra word problems are a good example where list processing is essential. Stability of algebraic operations and characterisation of structures by fixed points from some construction are another good example. The potentially unbounded size of lists allows handling of infinite sets and functions defined on infinite sets. These ideas are best learned as early as possible.

Conclusion.

The circulation of live documents, in which experimentation by the student is allowed and encouraged, will have a tremendous impact on the teaching of algebra and on the future curricula.

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Developing Algebraic Thinking: Past, Present, and Future

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This paper presents a definition for algebraic thinking to analyze research and curriculum reform efforts in the United States from the past decade. Based upon these discussions, the author recommends that algebraic thinking be developed through multiple representations in curricula and pedagogy for students ages 9-12. Examples are given to demonstrate a method for teaching of percents through proportional reasoning for students age 10-11 and for teaching algebraic thinking through real-world applications.

Traditional instruction in algebra emphasized direct instruction of routine procedures with abstract symbols with little emphasis upon conceptual understanding and applications. In the past decade in the United States, the National Council of Teachers of Mathematics, in the *Curriculum and Evaluation Standards for School Mathematics* and *Principals and Standard for School Mathematics*, proposed that instructional methods in algebra include investigation, multiple representations of concepts, and real-world applications (NCTM, 1989, 2000). In addition, the study of algebra has been purported as a course necessary for all students and not only for college intending students. These reform recommendations have raised questions: how do we define algebraic thinking? How can we structure content to develop students who think algebraically and not merely procedurally?

For this discussion, three broad indicators are defined as essential to algebraic thinking:

1. Ability to think in symbolic language, to understand algebra as generalized arithmetic and to understand algebra as study of mathematical structures.
2. Ability to understand equality and equations of algebra and to apply these within real world problem solving settings.
3. Ability to understand relationships of quantities through patterns, defining functions, and applying mathematical modeling.

In order to analyze where curriculum, pedagogy, and research should evolve as we move into the twenty first century, major influences from past research and current curriculum changes need to be examined.

A Quick Review from the Past

Research relating to the definition of algebraic thinking given above can be categorized into three main topics: the concept of variable, the concept of equation, and the concept function. Studies have continued to document the difficulty students demonstrate in the understanding of the abstract symbol representations in these three areas. First, the research in algebra completed by Kuchman (1981) assessed student thinking related to six interpretations of a variable (originally developed by Collins, 1975) that students may confront in algebra. A variable in an equation, such as $2x - 7 = 11$, represents a numerical value while the variables in a function, $y = 2x - 1$, represent a relationship between quantities. Kuchman found that the majority of 13-15 year olds in his study were only able to use variables at the two lowest levels, solving for an unknown in a simple equation and simplifying variables in addition and subtraction expressions ($2x + 5x =$). They were unable to analyze a generalization such as which is larger, $2n$ or $n + 2$?

Research findings on the understanding of equation have also revealed students misunderstanding of the symbolic forms of algebra. Wagoner (1981) found that only 38% of sample of 29 students could answer the question correctly, "For the equations, 7 x *W* + 22 = 109 and 7 x *N* + 22 = 109, which would be larger, *W* or *N*? Kieran (1989) noted that arithmetic and algebra share the same = symbol and students often interpret = as a "here's the answer" symbol rather that as an equivalence statement. Many students in algebra learn to solve an equation procedurally yet do not understand the concept of the equivalence.

The concept of function is also complex and involves levels of understanding. Sfard (1987) found that the majority of students in his study of 16 and 18 year olds viewed a function at the operational level, as an algorithm for computing rather than as a structural concept, a correspondence between two sets. Boers-van Oosterum (1990) also found that students' conceptual understanding lagged far behind their procedural knowledge. Other studies have detailed the obstacles students encounter in learning algebra as a symbolic system (Lee & Wheeler, 1989; Herscovics, 1989)

A question arises concerning algebraic thinking. Can a student develop algebraic thinking without the ability to perform symbolic procedural algebra manipulations? In a case study of an algebra student with a learning disability, Crawford and Sinecrope (1996) found that the student could "solve" an equation such as $3(x + 5) + 6 = 27$ mentally, used basic arithmetic and reasoning to solve algebraic word problems, and could determine algebraic rules for patterns in function tables. One of the student's greatest strengths was his visual ability with reasoning from graphs. Yet, the student was unable to write the procedural steps to solve a simple equation such as $3x - 8 = 5$. The student could think in a symbolic language, understand equality and relationships of quantities through patterns. These are all basic characteristics defined above as "algebraic thinking".

These notations from past research are merely a sampling of the research depicting the need for an algebra curriculum that supports the construction of concepts through multiple representations rather than predominately focusing on symbol manipulation. The findings suggest that a 'new" algebra should develop the meaning of variable, equality, and function. This curriculum should build the foundations of algebraic thinking that would continue throughout further mathematics coursework.

Curriculum Reform

In 1991, in the USA, the state of North Carolina adopted a policy requiring all high school graduates to complete Algebra I. Concurrently, the state mathematics supervisors and mathematics educators redefined the state algebra curriculum to be aligned with the NCTM *Curriculum and Evaluation Standards* (NCTM, 1989). A statewide inservice program to retrain algebra teachers to implement the new curriculum and pedagogy was conducted during the summers 1992 and 1993 (Crawford & Shotsberger, 1995; Crawford, Chamblee, & Rowlett 1998). Today, teachers across the state are successfully using graphing calculators and applications to teach a curriculum more suited for all students. (Department of Public Instruction, 1999)

In this curriculum, teachers are recommended to develop the concept of slope as a rate of change and with multiple representations. Slope traditionally has been thought of as merely a memorized formula. However, students can investigate the slope from a table, a graph or an equation. Consider the following problem and solution in figure 1. In this example, the slope is interpreted from a graph. However, another student might write an equation from the table or merely use the table data to write the slope and y-intercept. A teacher can have students justify, clarify, and communicate their findings to compare the different representations (Crawford & Scott, 2000).

Discuss the meaning of the slope and y-intercept for the data.

Basketball tickets sold	10	20	30	40
Money in the cash box	75	115	155	195

Student response.
The slope m = $4 which is the cost per ticket. The y intercept is $35 which is the amount of money that the person began with in the cash box.

Figure 1. Application for Interpreting Slope and Y-intercept

Before the new curriculum changes, the primary goals in algebra were 1) simplify, 2) solve and 3) graph. With the new algebra curriculum, these goals are reversed: 1) graph, 2) solve then 3) simplify (Kysh, 1991). Functions can be introduced much earlier in the course and applied throughout. With the use of graphing calculators, students can analyze problem situtations, graph the data then calculate the line of best fit. Quadratic applications with maximum and minimum values and exponential applications can be graphed. First year algebra students can investigate compound interest, inflation, depreciation, consumption rates, and population growth. (see figure 2)

Date	Customers without power
Sept. 6	1,159,000
Sept. 7	804,000
Sept. 8	515,000
Sept. 9	340,500
Sept. 10	195,200
Sept. 11	77,000
Sept. 13	37,600

1. Make a scatter plot of the data and discuss any patterns or trends. Estimate when all power would be restored.

2. Based upon the initial customers without power, what percent of those customers had their power restored after one day? After two days? After three days?

3. Determine the equation for the model of best-fit. Use the equation to predict when all of the customers would have their power restored.

Figure 2. Hurricane Fran Data, September, 1996
(from *Algebra Resources,* Department of Public Instruction (1999)

In the North Carolina Mathematics Curriculum, middle grades (ages 11-14) and elementary teachers also prepare their students with prerequisite learnings in patterns and parealgebra. Students age 10 to 11 describe relationships within sets of ordered pairs in the form of tables or graphs. At age 11-12, students describe, extend, analyze and create a wide variety of patterns to investigate relationships and solve problems. They apply informal and formal methods to solve simple one and two step equations. At age 13-14, they graph linear equations and investigate patterns with both linear and nonlinear data without symbolic equations.

Teachers in North Carolina have implemented content changes in algebra through traditional textbooks. Results from state written achievement tests given at the end of the algebra course reveal that the new content is indeed being taught. However, these content changes do not assure that students have developed the ability 1) to think in a symbolic language, 2) understand equality and 3) understand the relationships within mathematical modeling as defined as "algebraic thinking". Teachers use graphing calculators for modeling, however, it is not known *how* the teachers use the technology. Are graphing calculators merely a procedural tool or a tool for students to construct representations and understanding? No data has been collected to determine the extend that instructional methods within the classroom have changed.

In the United States, data from several reform curricula have provided rich information about the development of algebraic thinking. Students in the Computer Intensive Algebra (CAI) curriculum (developed by James Fey and M. Kathleen Heid), have outperformed control group students in mathematical modeling, interpreting and linking representations, and in problem solving abilities (O'Callaghan, 1998). They have a richer understanding of the concept of variable (Boers-van Osterum, 1990). These studies have also shown that students have attained an equal level of skill development as the students in the control groups. Data from the Core-Plus Mathematics Curriculum (CPMP) indicate that this reform curriculum is more effective than traditional curricula in developing problem solving in algebra within real-world context and in applying algebra within multiple representations (Huntley et al., 2000). However, the traditional curriculum was more effective in developing skills in symbolic manipulation without context and without the use of graphing calculators. The CPMP curriculum structures content to develop an ability to apply equations within real-world settings and understand the relationships of functions which are essential to algebraic thinking (defined above). Students think in a symbolic language within the problem contexts. Students may not have the opportunity to understand algebra as generalized arithmetic which is defined above as an indicator for algebraic reasoning. However, these data provide insight into curricula that can richly enhance algebraic thinking. As mathematics educators analyze further the results from such reform curricula and begin to plan for the future, a definition of algebraic thinking may need to be refined.

Towards the Future

In international comparisons, the US algebra curricula continue to fall short in terms of student achievement (Harmon, 1997). Results of the TMSS study indicate that the typical US middle grades and algebra curricula attempt to cover too many topics to adequately provide depth in development of concepts. Teachers tend to expect that students must master arithmetic before beginning to learn algebra. Although the algebra curriculum may have changed, many mathematics teachers continue to teach with direct instruction approach with little discussion of concepts. With state accountability programs built upon scores on multiple choice tests, teachers have begun to narrow curriculum back to the learning of skills in expense of using problem solving, reasoning, and communication to develop algebra concepts.

It is quite reasonable to expect that the basics of algebraic reasoning can be developed fully at ages 9-12 which could then be followed by the content of a first year algebra course at ages 13-14 or earlier. Rather than assume that arithmetic must be mastered before algebraic thinking in developed, the two can be developed simultaneously. The learning of rational number can be enhanced with more emphasis on proportional reasoning. For example, ratio tables can be used when teaching students ages 10 -11 mental computation with percents (Lamon, 1999). The tables in figure 3 may not be the most efficient method for solving each problem. However, each table reflects a mental process that applies proportional reasoning. Students can be encouraged to build their own table for a percent problem then discuss the advantage and disadvantage of each table. Basic arithmetic computation with decimals and fractions can be applied while students work with rational numbers within their tables.

Solve with a ratio table.

a. ____ is 24% Of 42

Part	Whole	Mental Math
15	40	
7.5	20	Divide by 2
3.25	10	Divide by 2
32.50	100	Multiply by 10

b. 30% of ____ is 24.

Part	Whole	Mental Math (step)
30	100	
3	10	Divide by 10 (1)
6	20	Multiply by 2 (2)
1.2	4	Divide by 5 (3)
7.2	24	Add (3) and (4)

c. ____ is 24% of 42

Part	Whole	Mental Math
24	100	
2.4	10	Divide by 10 (1)
9.6	40	Multiply by 4 (2)
.48	2	Divide (1) by 5 (3)
10.08	42	Add (2) and (3)

Figure 3. Ratio tables for solving percent problems, ages 10-11.

As students move from tables to graphs of proportions and solving proportional equations, their proportional reasoning is then transferred as the beginning of algebraic thinking. Students ages 10-11 can also be given real-world problems to investigate through multiple representations, a table, graph and then an equation (figure 4).

Tina's dad can drive approximately 176 km in 2 hours.

1. Make a table to show how far he can drive in 2, 4, 6, and 8 hours.
2. Draw a graph of your data.
3. At this rate, approximately how far can Tina's dad travel in 5.5 hours?
4. How long would it take for a trip that is a distance of 300 km?
5. Write an equation to use to calculate the distance based upon the number of hours traveled.

Figure 4. Multiple Representation, Age 10-11.

With a strong emphasis to develop the concept of variable, equation, and function at ages 9-12, the foundations for algebraic thinking can be grounded in both curriculum and pedagogy. This can be followed by an algebra program that can indeed develop 1) an ability to understand algebra as generalized arithmetic 2) an ability to apply equations and problem solving in real-world settings and 3) understand the relationship of quantities through functions and modeling.

Closing Remarks

The past decade has enveloped great changes in the first year algebra curriculum in the United States. Less emphasis is placed upon symbolic manipulation with more emphasis on applications. Now the goal that students understanding algebra must be attained. For the next decade, we need to refine the definition of algebraic thinking for ages 5-8, 9-12 and 13-college. These definitions should further guide research, pedagogy, and curriculum development.

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The author, a former algebra teacher, directed the NC MATH Algebra Project that provided inservice for approximately 750 teachers across North Carolina during the summers 1992, 1993 focusing on problem solving, technology, and new applications in algebra. She was one the writers of the *Algebra Resources,* published by the NC Department of Public Instruction that was distributed to teachers across the state.

Learning Algebra with Technology: The Affordances and Constraints of Two Environments

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In this paper we discuss the use of two technology environments (the graphing calculator with motion detectors as a data collection device and a computer microworld) for the learning of algebra at the upper secondary level. Through students' investigations about periodic phenomena, we find the co-emergence of students' symbolizing activity and sense-making activity within each environment. Evidence of the developing use of algebra for generating powerful conjectures, providing explanatory arguments, and describing physical phenomena symbolically is discussed. The affordances and constraints offered by each environment are contrasted.

The role of algebra in school mathematics has been described by National Council of Teachers of Mathematics (NCTM) as a major component of the school curriculum and as a unifying theme for that curriculum. The recently issued *Principles and Standards for School Mathematics* states quite simply: "All students should learn algebra" (NCTM, 2000, p. 37). The view of algebra at the upper secondary level includes the understanding of patterns, of structure, of quantitative relationships, and of the analysis of change in various contexts. Clearly the availability of computational media (such as graphing calculators, spreadsheets, data collection devices or computer microworlds) impacts the teaching and learning environment for all of these aspects of algebra in significant ways. In this paper, I will present a brief overview of a modeling approach to the design of an instructional unit that uses both graphing calculators and a computer microworld. I will then describe the results of the investigations into the notion of periodicity by upper level secondary students. The nature of the algebraic thinking in each technology environment will be described. Finally, I will discuss the affordances and constraints of each environment on the students' learning.

Theoretical Perspective

A modeling approach to the teaching and learning of algebra focuses on the mathematization of situations that are meaningful to the learner. This brings to the foreground two important shifts in the teaching and learning environment: (1) the use of contexts that will elicit the creation of useful systems (or models) by the learner that in turn can be used to describe or explain the context and (2) the cyclic development and refinement of such models in ways that are shareable among students and increasingly generalizable. In modeling tasks, the student's goal is to make sense of the context so that s/he can mathematize it in ways that are meaningful to her/him.

In the case of algebra, the sense-making activity of the learner is closely linked to the learner's symbolizing activity. Gravemeijer and colleagues have described this link as "the reflexive relation between symbolizing and sense-making, and the dynamic character of this relation" (Gravemeijer, Cobb, Bowers & Whitenack, 2000, p. 235). These researchers emphasize that symbolizations emerge and develop meaning while the learner is engaged in the activity of symbolizing. The reflexivity in the relationship between sense-making and symbolizing suggests, on the one hand, the value of student initiative in interpreting situations and phenomena. On the other hand, students also need to be able to use and reason with the conventional symbolizations which are used by the community of mathematicians and, in turn, by the designers of graphing calculators, spreadsheets, and computer microworlds.

In earlier research, I have used the distinction made by Bliss and Ogborn (1989) between expressive modeling and exploratory modeling to investigate and describe this difference between the expression of student sense-making and student exploration of conventional symbol systems (Doerr, 1996, 1997). The expression of students' ideas as they strive to make sense of an experienced situation can be supported by a number of technological tools, including spreadsheets, multi-representational function software, and graphing calculators. However, the technological tool itself is not seen as inherently expressive or exploratory, rather it is the nature of the students' activities with the tool in an instructional setting with particular goals that is expressive activity or exploratory activity. A spreadsheet, for example, can be used to support the expression of students' ideas about relationships or it can be a pre-built model of relationships (usually reflecting an expert's model or way of thinking about the relationships) that the student is to explore through parameter variations or alternations of the underlying relationships. The students' activity and the instructional setting mediate the use of the tool and the meanings that are constructed by the students (Meira, 1995; Matos &Carriera, 1997).

Several researchers have investigated the use of tools such a spreadsheets (Matos, 1995; Sutherland & Rojano, 1993), multi-representational function software (Borba & Confrey, 1996; Confrey & Doerr, 1994, 1996; Schwartz & Yerushalmy, 1995), and graphing calculators (Doerr & Zangor 2000; Guin & Trouche, 1998; Ruthven, 1990) to support the expression of students' ways of symbolizing experienced phenomena. Other researchers have used computer microworlds to engage students in an exploratory process of creating meaning for representations by investigating both the representations and the phenomena that they symbolize (Kaput & Roschelle, 1997; Nemirovsky, 1994; Niedderer, Schecker, & Bethge, 1991). The research of Noss, Healy and Hoyles (1997) is particularly interesting as an example of a microworld intended to support students' understanding of algebraic symbols. These researchers point out that in many word problems intending to evoke pattern recognition schema in ways that lead to generalizations that there is a disconnect between the learner's actions and the algebraic expression. In discussing the well-known matchsticks problem, they observe that "with real matches the actions on them necessitate no push towards generality, no need for mathematical expression" (p. 206). The algebra is an end in itself, an appendage to the problem. The algebra is disconnected from the actions on the matchsticks. Furthermore, the algebra "neither illuminates the problem nor provides a means for validating its solution." (p. 205). That is, the algebra is its own endpoint, not a tool for understanding, solving or validating the problem situation.

These researchers go on to argue that computational media can provide a dynamic, two-way linkage that connects the learner's actions on objects (for example, matchsticks) with a trail of documentation (for example, a series of programming statements) that can be manipulated to produce actions on the objects. In this way, the media establish a linkage between actions on objects (matchsticks) and actions on symbols (Logo statements) and the learner can act in either mode. Such programmable results provide a trace or record of the learner's thinking that are valuable for researchers seeking to understand how it is that learners make sense of symbols. Central to the notion of engaging in a computational microworld is that the meanings which the learners constructed became "elaborated, refined and enhanced, rather than one in which one set of meanings ("the concrete") was supplanted by another ("the abstract")." (p. 225). In this way, these researchers argue that algebra becomes a tool or a way of thinking about relationships, rather than as an end in itself.

In this paper, I will report on how two technological environments were used to support both the sense-making and symbolization activity of students as they engaged in learning about the phenomena of periodicity. The students explored conventional representational systems in a computer microworld environment and expressed their own ideas as they made sense of periodic phenomena while using a graphing calculator and a motion detector. I will describe each of the technological environments, including some of the implicit assumptions regarding the underlying core aspects of algebra. I will conclude the paper with a discussion of the kinds of algebra learning that took place and the important aspects that were supported and constrained by each environment.

Data Sources and Analysis

This study is part of a larger research program designed to understand student learning through mathematical modeling in technology enhanced environments.[1] This classroom-based, observational case study took place with 31 students, ages 15 to 17, in two classes of pre-calculus in a suburban setting, taught by the same teacher. The classes were observed over a seven week unit of study on trigonometric functions. The classes met for 270 minutes per week. All of the students had either TI-82 or TI-83 graphing calculators. In general, the students had used their calculators for well over a year before taking this course and were quite familiar with its functions. The students used motion detectors to collect position vs. time data that was displayed as a graph on their calculator and stored in a data table.

The students had access in their classroom to the MathWorlds software, that had been installed on the computers. This software environment (Kaput & Roschelle, 1997) is a microworld for exploring one-dimensional motion in which any combination of three graphs (position vs. time, velocity vs. time, and acceleration vs. time) can be linked to an animated simulation and to each other. The central focus of the MathWorlds software is the exploration of how the same phenomena of one-dimensional motion can be represented with different graphs (i. e., position, velocity, and acceleration). The MathWorlds software includes an animation of a character (or a set of characters) moving in a horizontal or vertical direction. This motion is controlled by the piecewise graphical manipulations of its position, velocity and acceleration graphs which are also bi-directionally linked to each other. This link enables learners to change a character's motion by acting on any one of the graphical representations. Unlike function graphing software, this environment does not include bi-directionally linked tables of values, graphs, and algebraic equations to represent the same data set, rather the central linkage is between the graph and the motion of a character.

Classroom instructional activities regularly alternated between modeling problems investigated by the students within a small group and whole class discussion for sharing progress, discussing solution methods and extending results. The instructional tasks were designed so that the students would create quantitative systems which could be used to describe and explain the patterns and structures in an experienced situation and to make predictions about the behaviour of the situation. The students were asked to interpret data, to find meaningful representations of the data (typically tables, graphs and equations), and to generalize relationships beyond the particular situation at hand. The concept of the rate of change of a function and the transformations of a trigonometric functions were addressed throughout the problem situations.

All class sessions were observed by two or more members of the research team. Extensive field notes, transcriptions of audio-taped group work, students' written work and transcriptions of video-taped whole class discussion constituted the data corpus for this study. In an observational study such as this, there are necessarily some difficulties in observing students' use of the graphing calculator and the computer software. To address this, the mathematical investigations were designed to be completed by small groups of students and then later shared in a whole class discussion. This format generated discussion within the small groups where students explained to each other what they did on their calculator or on the computer; the discussion in the whole class setting revealed the emerging meanings that were forming among the students and with their teacher. The results of the analysis of two particular episodes are reported so as to illuminate particular aspects of algebra which emerged from each environment: the computer microworld and the graphing calculator with motion detectors.

Results

The first episode occurred as the students explored periodic position and velocity graphs by controlling the periodic motion of a character through graphical manipulation. In an earlier unit, the students had explored the linked relationship between the position and velocity graph using both the microworld and the graphing calculator with motion detectors. So in this instance, it was the periodicity of the graphs which was the focus of their attention. The graphs shown in Figure 1 illustrate the link between the velocity and the position graph. Each of the piecewise segments in either graph could be transformed by graphical actions on the screen using the mouse. Following this exploration, the students were investigating the question of how changes in the velocity graph effected the linked position graph. The students had been discussing the effect of changes in the period, amplitude, horizontal shift and vertical shift of the velocity graph on its corresponding position graph. The discussion was characterized by arguments from related pairs of graphs which the students had created with paper and pencil.

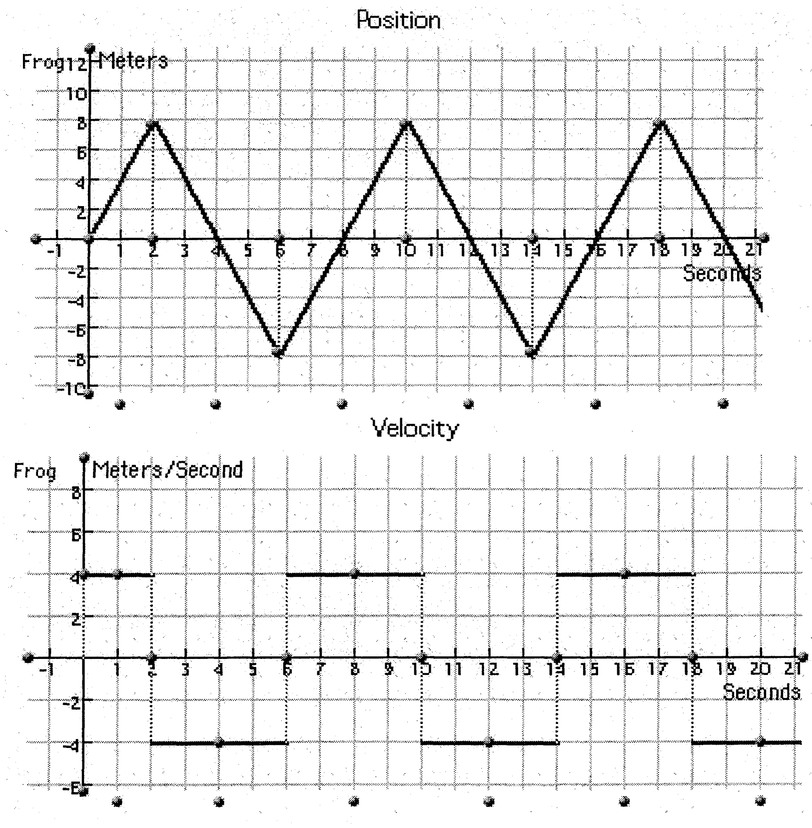

Figure1. Linked, manipulatable, piecewise position and velocity graphs.

Two students were presenting the findings of their group's investigation about the effect of a vertical translation of a periodic velocity graph on its associated position graph. The manipulatable graph segments had become the basis for arguments about the relationship of the changes. At this point in the unit, the students were using paper and pencil graphs rather than the micorworld graphs. One student was presenting her group's argument that a vertical shift in a periodic velocity graph resulted in a non-periodic position graph:

> Jennie: the reason it's [the position graph] not periodic any more is because we did have a balance and we had the same number of positives, areas. Where velocity is negative, eventually we lost negatives and gained positives, so therefore it [the position graph] didn't go down as much as it went up and that's why it's not periodic any more!

In this argument, Jennie has compared the negative areas and the positive areas between the velocity graph and the x-axis and argued that they must balance in order for the position graph to remain periodic. She reasons that the "negative areas" and "positive areas" were initially equal and in balance. When these areas are no longer in "balance," she argues that the loss of "negative area" and the gain of "positive area" results in the position graph going up more than it goes down. This lack of balance in area is the reason why the position graph is no longer periodic.

This is followed by a discussion of how this result depended on the initial periodicity of the position graph. That is, Jennie had argued that by shifting the velocity graph vertically, its linked position graph changed from being periodic to non-periodic. Another student engaged in a conversation with the teacher that the argument holds since the resulting non-periodic position graph could have been the initial position graph. Other students seem to think that more evidence is needed. At that point, Jessica elaborated on the effects of the initial condition to show that the position graph can be changed from non-periodic to periodic:

> Jessica: This is to show that you can take a non-periodic [position graph] and with a similar shift you can turn it into a periodic graph. Like, this is periodic velocity. Like, because the velocities do not... they don't cancel out, they're.... They don't like counteract each other at all. They just keep adding on top of each other because they're all positive. Um, you're gonna be, like, keep going up in the position graph for that. But if you shift it down, like, go down, you know, 2 and a half to negative 2. Jennie, did you <unclear> They cancel out, so that it's periodic.

Immediately following this argument, Jennie made the following powerful conjecture:

> Jennie: So now here's a question. Is there any [periodic] velocity graph that no matter how vertically you shift it, that it will not, that there's no way you can make it periodic?

This conjecture shows a shift in reasoning from what *could* happen to the position graph, i.e., specific instances of how the position graph can change, to a generalized statement about what might be true for all velocity graphs. This conjecture subsequently became the focus of much discussion, including an attempted (and rejected) counter-example of a horizontal line and later resolution by an argument stating that it is always possible to find a "balance line" for the positive and negative areas. In fact, this was a graphical argument for the Mean Value Theorem, supported by the students' notions of the relationships between the linked graphs. In this particular episode, the salient features of algebra which emerge are in its power for generating conjectures and supporting explanations. The symbols for these arguments and for the students' reasoning are the graphical symbols which they could manipulate in the microworld.

The second episode resulted from the students' investigation of one-dimensional harmonic motion using their graphing calculators and a motion detector. In this episode, the students are controlling the motion of a spring to produce a graph through the motion detector. This is precisely an inversion of their previous actions that were on the graphs to produce animated motion in the microworld. The students were given the task of finding an expression that could be used to describe the motion of the spring. There were fundamentally two approaches that students took in approaching this task. The first approach is illustrated by a student, basing his strategy on the recognition of the class of trigonometric functions and his knowledge about the symbolic representations of transformations, started with an equation of the type y=Asin(B(x+C))+D. This student systematically varied the parameters A, B, C, and D in his equation in order to obtain a visual fit as he checked the "fit" between the plot of the data set and the graph of the analytic function.

Other students did not use this fitting approach, but instead approached the problem by analyzing the patterns in the data. For example, one student determined the amplitude, both the horizontal and vertical shift, and the frequency by tracing the coordinates of consecutive maximum and minimum points in the data set. This approach enabled her to find a meaningful equation based on her knowledge of trigonometric functions and their transformations. In this case, the check between her conjectured equation and the data set was confirmed as she observed that the graph and the data were visually aligned along critical features such as local maxima and minima. For this student, the parameters came from her analysis of the data, rather than from fitting to the data.

For both of these students, as well as others in the class, the activity with the harmonic motion of a spring was accompanied by descriptive language about the spring's motion: how far above the ground it was held, how fast it was moving during various points in the cycle, why the velocity had to be zero at the maxima and minima of the position graph, why the position graph did (or did not) start with the natural beginning of a cycle on the y-axis, how the actual graph had compared with their earlier sketches prior to using the motion detector, and how their actions with the spring effected the amplitude of the graph. Later many of these issues were linked with the values of the parameters in the equations which the students' formed. In this sense, the symbols of the algebra emerged for descriptive power about experienced phenomena. Unlike with the microworld with its animated motion, in this case, the students' descriptive language included the physical motion of the phenomena which they controlled.

The students were also asked to find a cosine function that could be used to describe the data (since in all cases the function that they found was a sine function). Nearly all students had difficulty with this task. Since this part of the task was given as homework, there is no observational data to illuminate why this presented such difficulty. There was no evidence that would suggest that the in most cases the students even used their graphing calculator to check the validity of the transformed equation. It did not appear that the students were able to use symbolic notation to help in transforming their equation from a sine function to a cosine function.

Discussion and Conclusions

In examining these two episodes, I found that two important but different aspects of algebra became salient in each environment. In the first episode, the bi-directionally linked and manipulatable graph segments became the basis for the students' algebraic reasoning about the relationship of the behavior of the position and velocity graph. In this sense, as in the work of Noss, Healy and Hoyles (1997) described earlier, for these students, the actions and objects in the microworld were the supportive of the actions and objects in the algebraic world. I am, of course including graphical objects as algebraic objects or "symbols" that had taken on particular meaning as a result of acting on them in a graphical environment. The particularly powerful use of algebra (in the inclusive sense of graphical objects) that emerged was the development of powerful generalizations through the posing of conjectures and the development of explanations that used ideas of areas, accumulation and balance. A significant shift in the students' algebraic reasoning occurred as they moved from arguing from specific examples to arguing about conjectures that might always be true. The microworld thus provided an opportunity for the students to explore the designers' understanding of the relationship between the position graph and the velocity graph (i. e. the conventional mathematics of that relationship), but the approach was through the use of graphical objects rather than the language of equations. Through their exploration, the students were able to express their own generalizations about consequences of that relationship in their language and using paper and pencil arguments about the areas between the velocity graph and the x-axis..

Although the microworld did not provide any linkages to the language of tables or equations, the graphing calculator and motion detector provided access to both these representational forms as well as access to the control of the graph through the physical motion of the spring. Although the microworld had links from the graphs to the motion of a character, the characteristics of the motion were only faintly present in the students' arguments. They had shifted to a language of graphs. In working with the motion detector and springs, certain characteristics of the physical motion remained salient throughout their work. The two strategies for finding an equation (described above as a visual fitting strategy and a analytic strategy) used the underlying notion of the generalized symbolic form of a trigonometric equation. In this case, the algebraic symbols provided descriptive power for the physical phenomena. The students were able to describe the meaning of the parameters of the function in terms of the physical motion of the spring and confirm that with their collected data. While they appeared to have clear understandings of the role of the parameters in terms of a particular function, they were not able to easily shift between the co-functions.

The computer microworld, with its linked position and velocity graphs, afforded the students an environment in which to explore the well-established mathematical relationship between these two graphs. This exploration was based on graphical objects and the simultaneously bi-directionally linked position and velocity graphs. It was not argued from the usual equations which relate a function (such as position vs. time) and its derivative (velocity vs. time). In this case, the changes in the function and in its derivative were examined as they interacted with each other. While the motion of the character itself seemed no longer significant, it is likely that this occurred once the students understood the relationship of the position graph to the motion. The students' language had shifted to the language of the linked graph segments, which they acted on with paper and pencil. The powerful idea which emerged from their explorations was a generalized statement about the Mean Value Theorem from calculus, but deeply situated in the context of the possibility of always being able to create a periodic position graph by shifting a periodic velocity graph. The calculator environment, on the other hand, supported the students in their connections between the physical phenomena and the meaning of parameters in the expression of a function that can be used to describe that phenomena. Neither environment supported the shifting between algebraic symbol expressions. It should be noted that in earlier work with these students (Doerr & Zangor, 2000), the graphing calculator was supportive of the symbolic shifts for exponential functions, thus suggesting that the need for further investigation of this issue with trigonometric functions. Together, the two environments provided opportunities for student to generate conjectures, to provide explanatory arguments, and to describe phenomena symbolically.

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Access to Algebra: A Process Approach

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Algebra I is often a gatekeeper, preventing students from taking other mathematics and, in some cases, science courses. This paper describes a program that has shown success with heterogeneous groups of diverse students in developing algebraic concepts and skills through a problem-solving approach. The use of problem-solving processes in task construction and associated instructional approaches allows students access to the mathematics while they develop strong high-order thinking skills.

Algebra I—a stumbling block for many students. More and more schools are requiring that ALL students must take algebra I. While subsequent mathematics courses are dependent upon a student's success in algebra, the ability to make generalisations and apply them is a skill needed and valued in other areas and careers. Thus it is imperative that algebra be structured in a way that maintains the rigor of the mathematics but allows access to the algebraic content for a broader range of students.

The Curriculum Research & Development Group (CRDG) of the University of Hawaii has developed an algebra curriculum that meets the needs of a diverse group of students (Rachlin, Matsumoto, Wada, & Dougherty, 2001). The curriculum infuses worthwhile tasks constructed specifically to invoke discussion, through instructional techniques that push the content development beyond rote memorisation for all students.

Algebraic Tasks

A basic premise used in the development of the algebra curriculum by the Hawaii Algebra Learning Project (HALP) is that algebra should go beyond formulas and memorised or rote procedures. In fact, the algebraic content should be couched in problem-solving contexts that motivate higher-order thinking. Recognized problem-solving processes developed by Krutetskii (1976) are used as the foundation for the task construction. These processes include generalisation, reversibility, and flexibility.

Generalisation is the ability to 1) subsume a particular case under a known general concept and 2) deduce the general from particular cases to form a concept. In the first case, students are given tasks that are carefully ordered so they can determine commonalities across the solutions and their methods. For example, in problem set 7-2, students are asked to: Find three pairs of algebraic expressions, each pair having a product of a^5. Rather than the teacher simply scoring their solutions for correctness, students share many solutions in class discussion. Since the pairs are not restricted to monomials, students include many types of algebraic expressions. A typical discussion involves the observation that 1 is a monomial and its product with a^5 is a^5. And, some students notice that following the pattern or generalisation that they have made about the behaviour of exponents in multiplication, a^0 times a^5 is also a^5. Thus they begin their exploration of zero and negative exponents with a conjecture that a^0 must represent 1. Note that at this initial stage of development, students have not considered whether or not all values for a would be appropriate.

In the second case, students may be asked to find something specific, based on a generalisation. For example, in problem set 7–9, students are asked to: 2.a. Find two binomials whose product is a binomial when simplified; 2.b. Find two binomials whose product is a trinomial when simplified; 2.c. Find two binomials whose product has four terms when simplified; and 2.d. Find two binomials whose product has five terms when simplified. As students again share their solutions, the teacher will ask them to explain the generalisations they used to solve the problems. These generalisations form the broader concepts that drive the fluent skill development.

Notice that in either case, students may be shifting between the two types of generalisations. They may have already seen the generalisation in the first case, and use it to create the specific example. The opposite would hold true in the second case. Regardless, these tasks force students to see the "big picture" rather than only applying a procedure.

Reversibility is the "ability to restructure the direction of a mental process from a direct to a reverse train of thought" (Krutetskii, 1976). Students typically learn to do a problem in one way, with certain conditions given. When those conditions are changed, students often believe that they are seeing an entirely different problem. However, students who have developed the process of reversibility view the given problem as one in a class of problems, rather than an isolated case. In problem set 10–6, students are given this problem: 4.d. Find two fractions with unequal denominators whose difference is 3/8. The problem gives students an "answer" and they must supply the "problem."

Flexibility is the ability to switch from one level of thinking about a problem to another. It may be shown within or across problem. Within-problem flexibility refers to the student's ability to solve a problem in multiple ways. The development of flexibility is encouraged through problems like this one in problem set 4–7: 1.a. Solve for j: $5j + 6 = 3j + 15$. 1.b. Solve the same equation by another method. As students move through the chapter on solving equations, they develop the skill to solve equations using guess-and-test, tables, graphing, equivalent equations, working backwards, diagrams, manipulatives, and algebraic manipulations.

Across-problem flexibility refers to a student's ability to use the solution from one problem to solve another problem. That is, students see connections between and among problems. For example, in problem set 4–10, students are given this problem: 1. Graph the solution set for each inequality: a. $5t - 3 > 17$ b. $5(t + 1) - 3 > 17$ c. $5(2t + 2) - 3 > 17$. Many students would solve each problem independently rather than noticing the connections across the problems.

The tasks developed in the HALP materials reflect each of the three processes. In some cases, the problems may reflect all three processes. In the examples given above, it is clear that these problems could be solved in multiple ways.

The open-endedness of the tasks allows students from diverse backgrounds and experiences to attempt the problems. In traditional tasks where there appears to be only one correct answer and, in some cases, one solution method, students may not engage in the problem at any level. Without engagement, students cannot gain access to the mathematics. To get such engagement, the structure of the problem must represent to the student a chance for success.

Rather than focusing on isolated and fragmented skills, the problems must build from previous knowledge. By introducing each new concept with a problem-solving task, students are forced to use their previous knowledge and experiences to solve it. Thus they connect the new topics with previously developed ones. This creates a solid foundation for the development of concepts and skills. Additionally, since the teacher does not introduce the new topics with a lecture, students have the opportunity to introduce it in class the next day through the class discussion.

Assigning problems as homework *before* discussing the topic gives students an opportunity to think about the concept and to take ownership of the ideas. But, not all problems on one problem set will be introductory problems. On a given day, students may encounter five nonroutine problems, each on a different topic and at a different stage of concept development. They may also deal with two or more multipart problems reinforcing skills from earlier topics. Rather than focus on a single topic in isolation for forty-five minutes in one day, students discuss each topic and its relation to other topics for more than forty-five minutes over a five-day span as they work on the sequence of five nonroutine problems per concept. Each concept is given time to mature before it becomes routine.

Finally, tasks must lend themselves to rich discussion. It is through the discussions that students negotiate the meaning of the concepts and validate or refute their understandings. The well-designed tasks in HALP support this knowledge development and form the basis for the associated instructional approach.

Instructional Approach

When there is a shift in the way concepts and skills are developed, there must be an associated shift in the instructional method. However, any shifts should support student learning, rather than representing a current instructional fad.

The instructional techniques developed in association with the tasks represent a research-based approach. A shadowing technique (Rachlin, Matsumoto & Wada, 1987) was used so that an analysis could be made of the effects of particular strategies on student learning. The findings from that research indicate that a primary consideration must be to create a classroom environment conducive to exploration and risk taking (Dougherty, Slovin & Matsumoto, 2000).

The classroom environment "forms a hidden curriculum with messages about what counts in learning and doing mathematics" (NCTM, 1991, p. 56). It is a place where teacher and student beliefs come together to create a common vision about mathematical knowledge and what constitutes reasonable mathematics activity (Lubinski, 1994).

While good tasks are necessary, they are not sufficient to create this environment. It requires a shift in the teacher's role as well—a shift from the lecturer to the questioner. That is, in order to improve student learning, the teacher has to understand how students are making sense of the algebraic content. This is accomplished by giving students the opportunity to discuss their solutions and associated generalisations or observed patterns. From the discussion, the teacher continues to push the algebraic understandings by posing generalisation, flexibility or reversibility questions.

To provide an appropriate environment for this discussion, students need an opportunity to comment on each other's problem solutions, ask questions, and suggest alternative ways of thinking about the mathematics. This process is best accomplished through a collaborative group approach.

In each collaborative group, students share their solutions to a specific problem from the homework set. Their responsibility is to negotiate the mathematics content by validating the solution methods used or the answers found. Within a specified time period (usually seven to eight minutes), students prepare a short presentation about their problem.

This group use differs from a cooperative group setting where students must work together to create something—a solution or a product. In a collaborative group, students are engaged in an atmosphere that focuses on an intellectual discussion of their understandings of the mathematics. As they share solutions, other group members must listen critically and respond to the ideas presented. These collaborative group discussions are then shared publicly before the whole class. A broader audience can now respond to the mathematical ideas.

The communication of ideas through discussion is central to developing student understanding. However, when combined with specific writing tasks, the learning becomes even deeper.

Writing occurs throughout the instructional process. Solutions to problems do not only reflect the mathematical processes used to solve a problem, but they also include a textual description of the student thought processes. The textual description often provides the support or justification for selecting a particular solution method or conveys questions the student may have had about the problem or its solution.

Additionally, our research indicates that another technique that is effective in promoting deeper understanding is the inclusion of structured journal writing (Dougherty, 1996). The journal prompts are selected in three categories: content, process (or metacognitive) and affective/attitudinal. The prompts are aligned with concept and skill development so that students confront them at an appropriate stage in their learning experiences.

One cannot neglect to include multiple representations as part of the communication activities. Since the problems promote multiple solution methods, the learning is enhanced when students are encouraged, and often explicitly asked, to create multiple representations of the mathematical content. These representations include, but are not limited to, diagrams, graphs, manipulatives, and algebraic symbols.

The communication features define the instructional process. Through the exchange of ideas and support for those ideas with strong evidence and justification, students confront and negotiate contradictions and validations of their own understandings. In the process, they develop a common language for continued discussions.

Evidence of Learning

Without evidence that rich tasks and an appropriate instructional approach impact student learning, little can be gained. The HALP conducted an evaluation using a pretest-posttest norm-referenced design to determine the value added for students during an academic year of algebra I. An open-ended, constructed-response test (Harcourt-Brace's Goals™) was selected as the best match for instructional design and appropriate mathematical content. The results of the evaluation are shown in the following table.

Site	Pre-percentile	Post-percentile	Percentile Gain
1	37.5	54	16.5 ($p<0.05$)
2	50.5	71	20.5 ($p<0.001$)
3	71.5	86.5	15 ($p<0.001$)

Table 1. Evaluation data for *Algebra I: A Process Approach*

At all sites, large gains beyond normative expectation were found (normative gains would have resulted in no changes in percentiles). Young et al (1998) noted that a somewhat remarkable finding was that, even though there were large differences in pre-test means at the three sites, the gains shown at each site were very similar in magnitude (between 15 and 21 percentile points) indicating a significant value-added component.

Final Thoughts

It is not a simple task to bring together all of the factors that create a curriculum in algebra I that is accessible by a broad group of students. That curriculum must include 1) rich tasks, 2) time for development of understanding, 3) opportunities to negotiate content and 4) a classroom environment conducive to discussion. However, to orchestrate the implementation of such a curriculum in the classroom, the teacher is central to bringing the algebra to life.

The HALP research on teacher change has formed a cornerstone of the curriculum development process. It has shown that teachers must have an intensive professional development experience linking the materials to the background research and classroom practice to enhance student achievement. Consequently, the National Staff Development

Council recognized the project's professional development institute as one that creates change in teacher actions and student achievement (Killion, 1999). But, the professional development institute is not enough. Follow-up support is needed for a minimum of one year so that support is given when obstacles occur.

This paper presents components of a successful curriculum development project that has shown to enhance student achievement in algebra. There is no one component that can be highlighted as the panacea for making algebra accessible for all students. A successful program requires a cohesive blending of many factors.

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The Emergence of Algebraic Structure

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We claim that despite differences in the algebraic topics, the settings, and the students' ages, there are fundamental parallels in algebra learning experiences at levels ranging from junior high school to university. These parallels are due to the very nature of algebra, which creates the need for students to structure and restructure their knowledge at every stage of the learning process, and to reach ever higher levels of generalisation as well as of abstraction. In this paper we support the above claim by describing, within the same theoretical framework, three processes of learning different algebra topics. The framework we use is a model for observing and describing processes of abstraction in context, called the dynamically nested RBC model of abstraction.

The dynamically nested RBC model of abstraction

Mathematics educators have commonly regarded abstraction as consisting of focusing on some distinguished properties and relationships of a set of objects rather than on the objects themselves. Abstraction is thus a process of decontextualization. On the other hand, Davydov (1972/1990) views abstraction as starting from an initial, undeveloped form and ending with a consistent and elaborate final form. Similarly, Ohlsson and Lehtinen (1997) see the cognitive mechanism of abstraction as the assembly of existing ideas into more complex ones. Noss and Hoyles (1996) go even further. They situate abstraction in relation to the conceptual resources students have at their disposal and see it as attuning practices from previous contexts to new ones. Hence, in their view, students do not detach from concrete referents at all. Leaning on ideas of these and other authors, Hershkowitz, Schwarz & Dreyfus (2001; referred to below as HSD) define abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. The use of the term activity in this definition is borrowed from Activity Theory (Leont'ev, 1981) and emphasises that actions occur in a social and historical context. The reorganisation of knowledge is achieved by means of actions on mental (or material) objects. Such reorganisation is called vertical, if new connections are established, thus integrating the knowledge and rendering it more profound.

According to this definition, abstraction is not an objective, universal process but depends strongly on context, on the history of the participants, on their interactions, and on artefacts available to them. As abstraction is an activity consisting of actions, the research by HSD included the identification of actions involved in abstraction. They focussed on epistemic actions, that is actions relating to the acquisition of knowledge (Pontecorvo & Girardet, 1993). The three epistemic actions of abstraction they identified are Recognising, Building-With and Constructing, or RBC. In many social contexts, such as small group problem solving, participants' verbalisations may attest to these epistemic actions thus making abstraction observable.

Recognising a familiar mathematical structure occurs when a student realises that the structure is inherent in a given mathematical situation. The process of recognising involves appeal to an outcome of a previous action and expressing that it is similar (by analogy), or that it fits (by specialisation). *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. *Constructing* is the central step of abstraction. It consists of assembling knowledge artefacts to produce a new structure to which the participants become acquainted. The same task may thus lead to building-with by one student but to constructing by another, depending on the student's personal history, and more specifically on whether or not the required artefacts are at the student's disposal. Another important difference between constructing and building-with lies in the relationship of the action to the motive driving the activity: In building-with actions, the goal is attained by using knowledge that was previously acquired or constructed. In constructing, the process itself, namely the construction or restructuring of knowledge is often the goal of the activity; and even if it is not, it is indispensable for attaining the goal. The goals students have (or are given) thus strongly influence whether they build-with or construct.

The three epistemic actions are the elements of a model, called the dynamically nested RBC model of abstraction (or shorter, the RBC model). According to this model, the genesis of an abstraction passes through three stages: (a) a need for a new structure; (b) the construction of a new abstract entity; (c) the consolidation of the new abstract entity. Stage (b) is characterised by a constructing action within which recognising and building-with actions are nested. A constructing action incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions. Furthermore, constructing actions may be nested in other, higher level constructing actions. Stage (c) is characterised by repeated recognition of the new structure and building-with it in further activities with increasing ease. This could occur, for example, in problem solving. But a particularly interesting case is building-with the abstract entity E constructed in stage (b) in order to construct an additional new abstract entity F. In this case, stage (c) for E forms part of stage (b) for F, and the building-with E actions are nested in the constructing F action.

HSD have argued that this model fits the genesis of abstract scientific concepts acquired in activities designed for the special purpose of learning. In such activities the participants create a new structure that gives a different perspective on previous knowledge. The model is operational: It allows the researchers to identify processes of abstraction by observing the epistemic actions and the manner in which they are nested within each other.

In this paper, we present three very different algebra activities and point out how each of them fits into the general framework of the model. A thorough analysis is not feasible within the available space and would require a full-length paper for each activity. The order in which the activities are presented has been chosen according to the features of the model illustrated by each activity rather than according to the mathematical content.

Activity 1: The linearity of a transformation of a vector space

This activity is taken from a conceptual linear algebra course, which was designed on the basis of an epistemological analysis of elementary linear algebra (Sierpinska, Dreyfus & Hillel, 1999). The aims of the course included the generation of a thorough conceptual understanding of the notions of vector, linear combination and linearity of a transformation. The students were MA students in mathematics education; they had taken a computational, matrix based linear algebra course in college, between seven and eighteen years earlier. At the beginning of the present course, a synthetic-geometric view of vector spaces is taken, for which the two-dimensional vector plane serves as a representation. The introduction of coordinates, n-tuples and matrices is deferred until students have had the opportunity to become acquainted with the basic notions vector, linear combination and linear transformation. Prior to the activity to be discussed here, students had acquired experience with various transformations, some linear and some non-linear ones. For example, in the activity immediately preceding activity 1, they were presented with five pairs of transformations and had to judge, for each pair, whether or not the two transformations were equal. These transformations were presented on a computer screen by means of a vector v and its image T(v). Students moved v and observed the effect on T(v). Linearity had not been introduced to them. The aim of activity (designed by Anna Sierpinska and presented in Figure 1) is to lead up to the introduction of the notion of linearity of a transformation.

This report is based on the written answers of twelve students who worked in pairs on the activity. All pairs answered (i) as expected, that is A2 $\rightarrow$ M2, A4 $\rightarrow$ M4, C2 $\rightarrow$ K2, and C3 $\rightarrow$ K3. We observed that despite their experience with linear combinations in earlier activities and with transforming vectors in the preceding activity, most students did not think in those terms. Rather than use vectors and their linear combinations to represent the given points and find their images under the transformation T, they mostly thought of the task in terms of deforming one grid into the other one. However, their answers to (ii) and (iii) were more varied and can be classified into four groups:

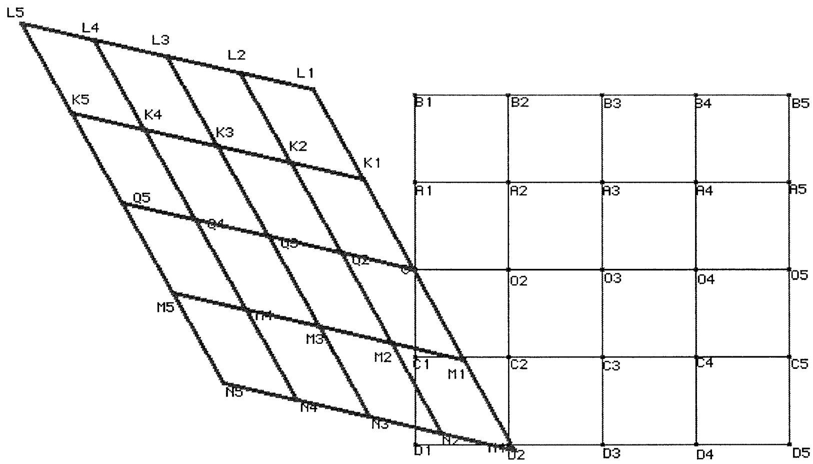
(a) Two pairs applied linearity and explicitly mentioned the linearity conditions [$T(ku)=kT(u)$ and $T(u+v)=T(u)+T(v)$] under (c).
(b) One pair applied linearity without explicitly mentioning it.
(c) Two pairs constructed (linear) functions f: $x \rightarrow -x + 0.2x + 0.5y$ and g: $y \rightarrow -y+0.2x$, using the term 'function' but not the term 'linear'.
(d) One pair could not make sense out of the question; they stated that M1 and Q2 were not on the thin grid but rather on the thick one.

Using the grid representation of a transformation

Work in pairs. This activity is to be done with paper and pencil.

(i) Look at the Figure below. The point A1 was transformed by a transformation T onto the point M1, and the point O2 was transformed onto the point Q2. Where did the points A2, A4, C2, C3 go?

(ii) Assume that the coordinates of the point M1 in the thin grid are (.5, -1) and the coordinates of the point Q2 in the thin grid are (-.8, .2). Let X be a point with coordinates (65,-42) in the thin grid (not visible in the figure; you have to imagine it). Would you be able to tell what would be the coordinates in the thin grid of this point X after the transformation?



(iii) Identify any assumptions about the transformation that you may have made when answering (a) and (b) in this activity.

Figure 1. Activity preparing the introduction of linearity

The task was designed to provide an experience with the need for linearity before the algebraic formulation of this notion, and thus to create an explicit stage of transition to the algebraic formulation in the abstraction stage. Parts (ii) and (iii) of the task were intended to guide the students toward reflecting about the concept, its scope, its formulation by means of algebraic language, and thus its general and abstract version. The students' awareness of the fact that they made some assumptions is an essential aspect of the need for the notion of linearity. The example thus illustrates stage (a) of the genesis of abstraction according to the RBC model, which consists of the need for a new structure.

Activity 2: Projection in an inner product space

This activity is taken from a regular second semester university linear algebra course. In this course, several students chose to work on some selected assignments in a Maple lab, and under supervision. The particular activity involved the following task: "Approximate x^3 with a polynomial in $\mathbf{P}_2$. Use the inner product defined by the definite integral of the product of functions from 0 to 1". Prior to this activity, there were four lectures devoted to Inner Product

Spaces (IPS). The least-square approximations were discussed in the context of finding the projection of a vector on a subspace in a general IPS. The instructor also discussed the solution of a similar problem involving cos(x), instead of x^3 (see Dreyfus and Hillel, 1998, for full details).

This report is based on the observation of four students who worked on the task during a single lab session lasting more than an hour. We note, first of all, that while the problem that was worked on in class referred explicitly to a vector space (C[0,1] - the space of continuous functions on the interval [0,1]), and a subspace $\mathbf{P}_2$ (the vector space of polynomials of degree ≤ 2), the actual task only makes a reference to an inner product. The students initially attended only to the concrete aspect of the task, namely, that of "approximating a function" . They already possessed a prototype solution of how the coefficients of the approximating quadratic function are found, and they were simply considering what modifications had to be made in order to accommodate the new function. In other words, for the students, the task was initially totally disconnected from its theoretical underpinning and they could operate at the procedural level, while using Maple to evaluate simple integrals.

To mesh all the relevant aspects of the theory (vector spaces, subspaces, inner products, norms, orthogonality, and projection) into a coherent structure takes a lot of mathematical maturity. We didn't expect that this one task would enable the students to reach such a level of abstraction. However, the activity nicely illustrates the process of abstraction, as the students try to restructure their knowledge by constructing a more powerful notion of projection. Initially, they spoke most confidently about those items, which were explicitly mentioned in the task, namely, "approximations", "inner product" and "$\mathbf{P}_2$". They recognized these items from earlier activities but since the modifications they had to make in order to build-with them in the present task were not completely transparent, they were led to allude to other aspects of the theory, such as "least-square", "norm", and "orthogonal basis". The crucial notion of a projection was only recalled when the observer asked them about the connection between least-square methods and IPS. It was clear from their ensuing discussion that their "concept image" of projection was still strongly tied to the geometric context of projecting a vector in R^3 on a plane, reinforced by the very use of the word "projection" with its geometric connotation. They first said "...you can do the <u>image</u> or something like that", then a vague "the <u>projection</u> of it on that", and "projection on the <u>plane</u>". But the task imposed the need to think in terms of a new structure, since functions were involved. A bit later in the session, the students spoke in terms of "we are <u>given x^3</u> and we have to approximate it on the <u>plane</u>" and then of the "projection of it [x^3 in this case] on the <u>plane of P_2</u>". Thus, while the idea that one needs to project "something" on a plane remains, we see that they have started add some new levels of generality and structure, so that in the first instance, the "something" is a function rather than a geometric vector, and in the second instance the "plane" itself is $\mathbf{P}_2$ which they recognize as consisting of polynomial functions. While there is clearly some mixing up of geometric with the more general aspects of vector spaces, there is an initial attempt here to reorganise some notions into a new structure. Later on in the session they refer to the "canonical" geometric diagram of a projection which they find in their class notes: "The projection of v on W - the plane W". Though the reference here is again geometrical, the use of the symbols v and W suggests the potential for generality. Only late in the session, projection is spoken of in general terms: "You know your vector, you take your projection - it is the closest one on that subspace to that [vector]".

While this process of abstraction in context did not necessarily lead the students to completely reorganise their previous notions of vectors so as to include classes of functions, a need for reorganising their knowledge arose from the requirements of the task and the students did construct a deeper notion of projection by tying together concepts that were, for them, initially disparate. The example thus serves to illustrate stage (b) of the genesis of abstraction according to the RBC model, namely, the construction of a new abstract entity.

Activity 3: Towards explicit generalisation

We focus here on the work of two Grade 7 students who participated in an activity named "*Dots and what more?*". This activity was chosen from an introductory algebra course in the Compu-Math project (Hershkowitz et al., in press), which consists of a sequence of activities organized around problem situations. These were designed to facilitate students' construction of structures of mathematical concepts and models, and the use of various mathematical processes (hypothesising, making generalisations, testing hypotheses, interpreting representational information, solving and justifying). All activities were open - no guidance for solution was provided, nor instruction whether to make use of a spreadsheet program (Excel) which was at their disposal.

This activity took place during the last month of the school year. Thus students were already quite familiar with the spreadsheet, and were accustomed to work with peers. The tasks in the activity were of increasing difficulty. In the various tasks students were asked to discover regularities in the number of dots in a sequence of dot patterns, and to express these algebraically. The activity was carried out in two class periods, 2 days apart. In the first lesson and at home students worked on several sequences of dot patterns. The second lesson took place in the computer lab and was devoted to the Rhombus task (see Figure 2).

Note: In this kind of task, students may obtain a sequence of numbers, and describe a given pattern in a spreadsheet, by using one of the following generalisation methods:

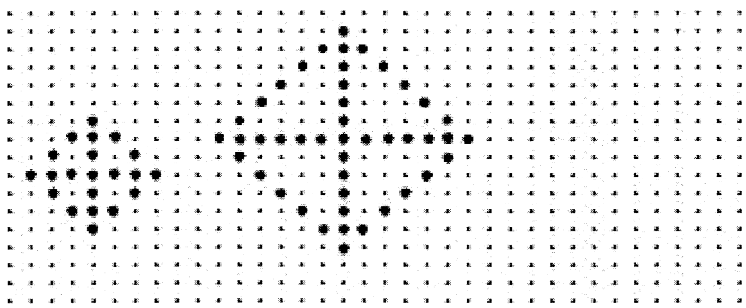
(a) Relating recursively to the previous number in the sequence (usually appearing in the previous cell of the same spreadsheet column)

(b) An explicit generalisation - using the position number N (usually appearing in the same row in an adjacent column).

Whenever possible, most students tend to use the recursive method, in which they consider locally the difference between two consecutive numbers of the sequence. With the spreadsheet tool, the recursive strategy, which is primarily local, may become global by the action of dragging, thus leading to a recursive generalisation. The tasks in this activity were designed to promote generalising of the dot pattern into an explicit algebraic expression, using the position number method. The design of the task supports the connection between a specific *counting method* of the number of dots in a given dot pattern sequence and the corresponding algebraic expression.

The Rhombus task

The following drawings are the third and sixth elements in a dot pattern sequence:



(i) How many dots are in the fifth element?

(ii) How many dots are in the seventh element?

(iii) How many dots are in element number 20?

(iv) Is it possible that the number of dots for a certain element will be even? Explain!

(v) Is it possible that the number of dots for a certain element will be 3557?

For each of the above try to calculate first and than check with the Excel.

Figure 2. The Rhombus Task

In most of the tasks of the first lesson and homework "our" two students already succeeded in making explicit generalisation using N.

In the Rhombus Task, however, they faced a higher level of difficulty and struggled to generate a proper counting method, which would lead to a correct algebraic generalisation. After each attempt to come up with a general expression, they inserted it in Excel and then dragged it and compared its value with the counted number of dots in a particular cell. On this basis, they rejected their first generalisation, (N+1)*4 +(N-1)*4 +1, and the subsequent one, (N+1)*2+(N-2)*2+(N-2)*4 +1. However, when they investigated (N-1)*9, it gave a correct number of dots for the sixth element in the sequence, so they accepted on it for a while.

Many interesting issues emerge from their discourse like: The question whether it is possible that an algebraic generalisation "will work" for only a few elements of the sequence and not for others. Or the question if it is possible that the algebraic generalisation is correct, while the computer (Excel) fails to generate from it the correct results.

On the whole, one can say that the students were aware of the fact that they had to create an explicit algebraic generalisation that has "to work" for each N. The students constructed this kind of algebraic generalisation in the tasks of the first lesson of the activity - "*Dots and what more*?". It was then available to them when they struggled to solve the Rhombus task. This shows how previously *constructed* knowledge – generalising to an explicit algebraic formulation in terms of N – is becoming *building-with* in the Rhombus Task. It thus illustrates stage (c) in the genesis of abstraction according to the RBC model: the consolidation of abstracted knowledge.

The integration between the students' cognitive paths and their peer interaction path is very rich and deserves a special analysis and interpretation. It forms part of the context in which the process of abstraction takes place. The role of interaction in processes of abstraction according to the RBC model has been discussed more extensively elsewhere (Dreyfus, Hershkowitz & Schwarz, in press).

Conclusion

We have presented a brief overview of an operational model for describing and analysing processes of abstraction in context. According to the model, abstraction proceeds in three stages: need, construction and consolidation. In this paper, we have used examples from different topics (beginning algebra to inner product spaces) and shown that despite differences in the algebraic topics, the settings, and the students' ages, the model applies equally in all cases because there are fundamental parallels in algebra learning experiences at levels ranging from junior high school to university. These parallels are due to the very nature of algebra, which creates the need for students to structure and restructure their knowledge at every stage of the learning process in order to reach ever higher levels of generalisation as well as of abstraction.

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THE CONCEPT OF PARAMETER IN A COMPUTER ALGEBRA ENVIRONMENT

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The study that is presented here concerns the learning of algebra in a computer algebra environment and, more specific, the learning of the concept of parameter. Students of 14 – 15 years old used a TI-89 symbolic calculator during a five-week period. They studied the parameter in different roles such as placeholder, changing quantity and generalizer. The results indicate that using parameters in the computer algebra environment requires a clear view on the roles of the different letters. Also, the reification of a formula seems to be important for an appropriate instrumentation.

Introduction

Algebra and computer algebra are two issues that are subject of many discussions among mathematics educators nowadays. Of course, these topics are related to each other: as the availability of (hand held) computer algebra devices is increasing, the question is raised what aspects of algebra lose relevance and which algebraic concepts or techniques are indispensable for mathematical thinking. The idea that the computer algebra tool just takes over the algebraic work has shown to be too simplistic. Rather, there seems to be a subtle interplay between algebraic considerations of the user and the procedures that can be left over to the technological tool. Therefore, a computer algebra environment might even provoke the development of algebraic concepts, techniques and strategies.

In the study that is presented here, this idea of using computer algebra for the learning of algebra is explored for the case of the learning of the notion of parameters in algebraic relations and solutions. First, the aim of the research is explained. Then we present the theoretical framework, followed by the didactical scenario. After some information on the research method and the experiment, two episodes of student behavior are described, that concern the solution of systems of equations that contain parameters. A conclusion finishes the paper.

The research project

The study presented here is part of an ongoing research project called 'The learning of algebra in a computer algebra environment'. The general research question of this project is:

> How can the use of computer algebra promote the insight in algebraic operations and concepts?

This question is specified in two sub-questions:

1. How can the use of a computer algebra system contribute to a higher-level understanding of parameters as they appear in algebraic expressions and functions?
2. How does instrumentation of computer algebra take place and what is the role of the relation between machine technique and mathematical conception?

Why parameters?

A reconsideration of algebra education is currently taking place in the Netherlands as well as in many other countries. For a better understanding of the background of this paper we need to explain the approach of algebra in The Netherlands. The introduction in algebra during the first years of secondary education (students of 12 – 15 years old) takes place in a careful way. Much attention is paid to the exploration of realistic situations, to the process of mathematization and to the development of informal problem solving strategies. Variables, for example, are often words that have a direct relationship with the context where they come from. The translation of the problem situation into mathematics is an important point, whereas the formalization is delayed. Although this is a good pedagogical policy in general, as a consequence the algebraic skills such as formal manipulation are developed relatively late and to a limited extent. This may cause some difficulties at upper secondary level, especially for students at the highest and most theoretical level. Therefore, a reconsideration of algebra education is taking place, and the role of computer algebra is a part of that.

The conception of the parameter can be a suitable topic to try to bridge this gap. Variables and parameters are in the heart of algebra. The parameter is an 'extra' variable in an algebraic expression or function that generalizes over a class of expressions, over a family of functions, over a sheaf of graphs. The parameter can be considered as a meta-variable: the a in $y = a.x + b$ can play the same roles as an 'ordinary' variable, such as placeholder, unknown or changing quantity, but it acts on a higher level than is the case for a variable. For example, a change of the parameter value does not affect one single point locally, but the complete graph globally. The different roles of the variable are resurfaced, but now at a higher level, and the generic function becomes the object of study. The concept of parameter, therefore, is adequate for enhancing the abstraction of concrete situations, so that the more formal and general algebraic representation can become a natural part of the students' mathematical world.

Why computer algebra?

Taking into account the affordances of technology in general, and the algebraic capacities of computer algebra in particular, it seems obvious to use a computer algebra system (CAS) for the purpose mentioned above. It can serve as a powerful and open algebra environment that allows students to concentrate on the concepts and the problem solving strategy. We conjecture that performing procedures in the computer algebra environment contributes to the development of insight in algebraic operations and concepts such as substitution and the distinction of the different roles of letters. Using the machine, the students don't have to worry about the calculations and this may enhance a more global conception of the problem solving procedures.

On the other hand, however, computer algebra can be demanding in its use. Guin and Trouche (1999, p. 205) pointed out that an adequate use of computer algebra tools requires making explicit the different roles of the letters to a further extent than is the case for paper and pencil work. This kind of explicitness that computer algebra demands is a burden but in the mean time it can stimulate the student to handle the operations more consciously.

Furthermore, from the perspective of Realistic Mathematics Education the integration of computer algebra is not a trivial matter. In Drijvers (2000) the issue is raised whether the development of informal strategies and the process of vertical mathematization, that are so important in this educational theory (e. g. see Gravemeijer, 1994), are stimulated in a computer algebra environment. As far as the informal strategies are concerned, it seems that the computer algebra environment does not support them. For vertical mathematization the CAS seems more appropriate.

Previous research and theoretical framework

Previous research

Much research has been done into the learning of algebra in general and, more specifically, on the notions of variables and parameters (e.g. see Küchemann 1981, Usiskin 1988, Wagner 1981, Warren 1999). Tall and Thomas (1991) provide a short overview of difficulties with algebra in general and stress the relevance of algebra:

> ... there is a stage in the curriculum when the introduction of algebra may make simple things hard, but not teaching algebra will soon render it impossible to make hard things simple. (Tall and Thomas, 1991, p. 128)

Concerning variables, one can find classifications of the different roles of variables such as unknown, indefinite, generalized number, dynamical variable and parameter in many publications (e.g. Usiskin, 1988). Küchemann (1981) distinguishes between the variable that is changing dynamically and the variable that is representing a set. This distinction can also be applied to the parameter that can be considered as a 'sliding parameter' or as a 'family parameter'.

Not so many studies have been published on the learning of the parameter. Bloedy-Vinner (1994) stresses the hierarchy of substitution and the implicit quantifier structure that is often involved while using parameters. These 'hidden quantifiers' also are described by Furinghetti and Paola (1994), who state that parameters are conceptually more difficult than variables.

Several studies have been devoted to the use of the computer for the improvement of the learning of algebra in general, and that of the concept of the variable in particular. Tall and Thomas (1991) use a computer environment as a generic organizer of examples and non-examples. They claim that the understanding of algebraic concepts had significantly improved. Graham and Thomas (2000) successfully used the graphing calculator to stress the placeholder-role of the variable. Boers-Van Oosterum (1990) also claimed to improve the conception of the variable by using different software packages. Brown (1998) used a computer algebra environment for generalization of patterns, for solving equations step by step and for solving number problems. None of these studies used information and communications technology tools (ICT-tools) for the learning of the concept of the parameter, as is done in the project described here.

Theoretical framework

An important part of the theoretical framework of this study is the theory of the *instrumentation* of ICT-tools. Following French mathematics educationalists (Artigue 1997, Lagrange 1999abc, Guin & Trouche 1999, Trouche 2000), we consider the development of instrumentation schemes as a crucial and non-trivial step in the appropriation of an ICT-tool. In the acquisition of such schemes, technical skills and mathematical conceptions are interwoven. Examples of this complex relationship can be found in Drijvers & Van Herwaarden (2001).

A second part of the theoretical framework concerns the dual character of mathematical concepts that have both a procedural and a structural aspect. Sfard (1991) uses the word *reification* for the gradual development of a process becoming an object. In the function concept, for example, the process of calculating function values may develop into the image of a function as an object that is represented by a formula or a graph. Sfard and Linchevski (1994) elaborated this for the case of algebra. Close to the idea of reification is the encapsulation that is described by Dubinsky (1991). Dubinsky states that encapsulation of processes into objects is an important step in reflective abstraction. He suggests that performing processes using a computer may stimulate its encapsulation. As a third means of representing the bilateral nature of mathematical entities we mention the procept that has been developed by Gray and Tall (1994). 'Procept' is a contamination of process and concept. The authors stress the flexibility that learners of mathematics need in order to be able to deal with the ambiguity of mathematical notations. In 3+5, the + may be an invitation to perform the process of addition, whereas in $a+b$ the + is only a symbol that defines the object 'the sum of a and b'.

It is our conviction that the theories of reification, encapsulation and procept are very relevant to the learning of algebra and to the instrumentation of computer algebra tools. For the students, the mathematical entities in such an environment may tend to have a structural character, whereas the processes are more distant to the objects than is the case with work using the traditional paper-and-pencil.

Methodology and didactical scenario

Research design and methodology

As a research paradigm, the developmental research method was used (see Gravemeijer, 1994). According to this methodology, the researcher tries to develop (local) instruction theories by means of constructing and developing thought experiments and educational experiments in the classroom situation. This involves a cumulative process of consideration and testing. Developmental research design typically makes use of qualitative, close-to-the-students, observations.

Therefore, the research method was mainly qualitative. The most important data consisted of audio recordings and field notes of classroom observations, audio recordings of mini-interviews with students, and written work of the students. The data were analyzed by coding the classroom incidents and solution methods according to previously defined categories. While doing so the most dominant categories came up clearly.

Development of the didactical scenario

A conceptual analysis of the phenomenon parameter led to the identification of three essential steps in the learning trajectory: the parameter as a placeholder, as a changing quantity and as a generalizer. The role of a parameter as an unknown is not mentioned explicitly, as this role tends to change the hierarchy between parameter and variable. We thought it would be confusing for students to focus on this role; implicitly, however, the parameter in some situations acts as an unknown.

Table 1 summarizes the three phases. Also, it indicates by means of what kind of activities the students are supposed to pass to the next phase and how computer algebra supports these activities.

parameter role	*a in $y = ax+b$*	*graphic model*
placeholder	*a* contains specific values, one by one	one graph, that can be replaced by another
changing quan-tity, 'sliding' parameter	*a* walks through a set dynamically	'comic' of the dynamic graph
generalizer, 'family' parameter	*a* represents a set, generalizes over situations	a sheaf of graphs

student activity	*CAS function*
systematic variation of parameter values	solve equations substitute animate graphs
generalization of situations and solutions	graph sheafs solve parametric equations

Table 1. Outline of the learning trajectory

Classroom experiment: aim and situation

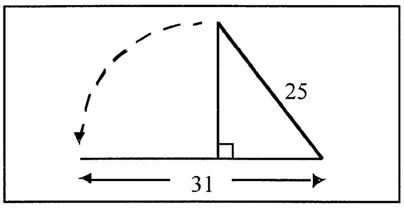
The aims of the classroom experiment were to investigate if the students' conception of parameter would develop according to the didactical scenario and if computer algebra serves as an aid for this. Also, we were interested in the process of instrumentation of the computer algebra tool with respect to the different roles of the letters involved.

The classroom experiment took place during a five-week period in the spring of the year 2000. The subjects were 50 students of 14 – 15 years old, divided into two classes. The students were high achieving in general but not specifically in mathematics. As computer algebra tool the TI-89 symbolic calculator was used at school as well as at home. Each class had four mathematics lessons of 45 minutes each week. Because the students had no previous experience with technology such as graphing calculators, using this type of handheld technology was really new to them. Their knowledge of formal algebra was quite limited. For example, the general solution of a quadratic equation had not been taught so far, so the help of the symbolic calculator was needed in case such an equation was encountered.

Episodes of student behavior

Classroom observations indicate that it was important that students are aware of the different roles of the letters, especially if there are parameters in the equations. We saw that some students found it natural to use parameters to generalize a relation or procedure, whereas others seemed to be confused by several letters in one expression, each having a different role. The differences between the students were considerably, as is shown in the following episodes, that concern the phase when students use the computer algebra environment to solve (systems of) parametric equations.

John and Rob work at the following assignment:



The two right-angle edges of a rectangular triangle together have a length of 31 units.
The hypotenuse is 25 units long.
***a**. How long is each of the right-angle edges?*
***b**. Solve this problem in case the total length of the two edges is 35 instead of 31.*
c. Solve the problem in general, that is without the given values of 31 and 25.

At question ***c*** John and Rob wrote down in their notebooks:

$$a^2 + o^2 = p^2$$
$$o + a = s$$
$$o = s - a$$
$$a = s - o$$

Then they entered into the TI-89:

solve($o^2 + a^2 = p^2 \mid o = s - a, o$).

The wrong letter at the end. The response of the machine was:

$0 = -s^2 + 2.a.s - 2.a^2 + p^2$

The boys corrected this by solving this a second time, now with respect to a:

solve($0 = -s^2 + 2.a.s - 2.a^2 + p^2, a$)

This time John explained the choice for the letter a at the end:

"You want to know the a".

This way they found the solution for a expressed in the parameters s and p.

John and Rob introduced the parameters themselves and generalized the problem solving strategy of the concrete cases of questions ***a*** and ***b*** without difficulties. Their solution schema did not seem to be confused by the presence of the parameters. However, there were some instrumentation problems at the start that may be related to a limited consciousness of the roles of the different letters.

For others, however, the presence of parameters was an extra complication, as illustrates the following observation of Sandra. The assignment was this time to calculate the dimensions of a rectangle with given perimeter and area. Using the viewscreen Sandra tried to demonstrate to the class how the corresponding system of equations could be solved:

$$b + h = s$$
$$b * h = p.$$

First Sandra entered a re-written version of the first equation in itself:

solve($b + h = s \mid b = s - h, b$)

The machine replied: true.

Rob commented: "She did not use the p, she did not use the second equation."

Sandra changed the command into:

$$\text{solve}(b + h = 20 \mid b = s - h, h)$$

Before the generalization the value of the parameter s had been 20, and apparently she felt the need to return to the concrete case. The result, $s = 20$, is logical but Sandra does not notice that. Then she realizes that Rob was right, but she entered $b + h = p$ instead of $b * h = p$:

$$\text{solve}(b + h = p \mid b = s - h, h)$$

The machine replied s = p, which is also reasonable. At the end Sandra entered

$$\text{solve}(b + h = s \mid b = p/h, h)$$

and that gave the right answer.

Sandra's behavior gives the impression that the parameters were an extra, complicating factor in the problem solving process. In earlier situations she had shown that she was able to apply this solution scheme correctly in concrete cases without parameters. It is not clear whether she really perceived a formula such as $b = s - h$ as an object that can be substituted.

Conclusion

The conclusion of the above exemplary episodes is that the didactical scenario to use parameters for generalization was confirmed by some of the students such as John and Rob. Computer algebra was helpful to clarify the problem solving strategy. For others, the use of parameters complicates the situation to a greater extent. In order to be able to solve parametric equations, it seems to be important that students are able to perceive formulas as objects. If a formula invites to a calculation process in the eyes of the student, s/he will find a general solution containing parameters hardly satisfying. Some students were able to distinguish the roles of the different letters, whereas others seemed to be confused by the amount of variables.

The conjecture that performing procedures in the computer algebra environment would enhance the understanding of the global mathematical conceptions behind the procedures was confirmed only to a limited extent. Some students did not overcome the difficulties with the instrumentation scheme for solving systems of parametric equations. The equilibrium between paper-and-pencil work and machine work during the instrumentation process may be quite delicate, and we probably did not pay enough attention to the 'traditional' approach. Meanwhile, the teachers reported that they could benefit from the students' machine experience while treating the solution of quadratic equations after the experiment.

As a consequence for teaching, we would conjecture that the development of both the technical and the conceptual side of the instrumentation schemes deserve explicit attention. This can be done by means of student interaction, classroom discussions and demonstrations, so that the instrumentation process gets a more social character.

Note

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This paper is based on Drijvers & Van Herwaarden (2001) and has much resemblance to the author's research report that is submitted to the PME25 conference.

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ICMI Discussion Paper

The trial of Algebraic Calculators in Senior Mathematics by Distance Education

David Driver

Background

The 1992 Queensland syllabus for the year 11 and 12 subject Mathematics B, for internally assessed students states: "it is appropriate that students are confident and competent in using a calculator." The parallel syllabus for externally assessed students requires that "students should be confident and competent in using a graphing calculator." There are no restrictions on either the model of calculator used or its use in tests or examinations. (Mathematics B is the pre-tertiary courses used as a prerequisite for tertiary study in mathematics based faculties such as science and engineering)

In late 1999, it was anticipated that the next revision of the syllabus for internally assessed students, due for release in late 2000 for implementation in 2002, would increase both the potential and requirement for students to demonstrate their competence in the use of either graphing calculators or computers.

Graphing calculators have been readily available for over ten years internationally, and widely used in Queensland to varying degrees for six or seven years since the implementation of the 1992 syllabuses. Their price has decreased considerably in this time.

Brisbane School of Distance Education (B.S.D.E.) has been a leading state school in the extensive use of graphing calculators in Mathematics B, preparing students for both internal and external assessment. Its student population is extremely diverse, as indicated by the student enrolment categories:

Distance – school age students unable to attend a mainstream school due to geographical isolation

Overseas – school age students studying for a Queensland pre-tertiary qualification while living overseas

Travelling – school age students moving around Australia

Medical – school age students who are unable to attend a mainstream school due to a medical condition

Approved – school age students given official exemption from attendance at regular schools (often following exclusion from a mainstream school)

Home School – school age students who pay a fee for the use of distance education materials under the supervision of a parent at home

School Based – school age students attending either a state or non-state school who study one or two subjects in distance mode (in most cases, a subject which is not available in the regular school)

Re-entry – students who have been absent from school for a minimum of 6 months and are over 18 years of age.

The enrolment characteristics of the students who participated in the trial of algebraic calculators in 2000 is included in Appendix 1.

In the distance education mode, it is considerably easier, cheaper and more equitable to use calculators which can be issued to students on hire, than to require students to have daily access to a computer. Calculators have the additional advantage over computers that students can have ready, daily access to them for both learning and assessment. Since the introduction of the 1992 syllabus at B.S.D.E., students have been able to hire either a Sharp EL9300 or Casio fx 7400G from the school for a nominal fee. All students enrolled in the subject, whether for internal or external assessment, have been required to have access to a graphing calculator at all times.

Self-paced learning materials developed for use in distance education are written to a design brief, based in the relevant syllabus, by curriculum writers on contract with Access Ed – a government instrumentality distinct from B.S.D.E. The materials can be used with minimal alteration in mainstream schools as an alternative to both conventional teaching and graphing calculators. They are available for purchase by teachers in mainstream schools through Access Ed.

Rationale

Algebraic calculators are pre-programmed with a Computer Algebra System (CAS) making them capable of symbolic manipulation. They have been available for several years. A greater range of models is now available and their price has reached the point where they are affordable. (The current GST exempt price of algebraic calculators is comparable to the price of graphing calculators five years ago.)

Given the impending syllabus changes, it was decided that this was an ideal time to phase algebraic calculators into senior secondary mathematics education at B.S.D.E..

Through the project, students would have the option of acquiring and using new mathematical techniques (or knowledge and procedures) by either conventional pen-on-paper techniques, using a graphing calculator (calculators with CAS capability all have the full functionality of a graphing calculator) or using the algebraic calculator, in conjunction with existing distance education print materials or new materials developed as part of the project. A list of the identified knowledge and procedures in the curriculum materials where a CAS calculator may have an impact were identified and are included in Appendix 2.

The project aims to maintain B.S.D.E.'s position as a leading school in the utilisation of calculator technology in Senior Mathematics.

Several state schools were approached with a view to their participation in this project. Two of these expressed a strong interest in being involved in the trial. Unfortunately cost restrictions precluded their involvement, as funding for the trial was only available from within the budget of B.S.D.E. The materials developed under the project, however, are available to other schools if they choose to make use of them, in much the same way as they can use other curriculum materials developed for distance education, to supplement other resources or to use as self paced instructional materials.

During the planning phase of the trial, an evaluation was made of the Texas Instruments TI-89, Casio FX 2.0 and Hewlett Packard 48G was made and it was decided to use the TI-89 due to its functionality, user friendliness and guaranteed avalability.

The Project

The project involved the development of teaching resources which take advantage of algebraic calculator technology and trialing their use in both year 11 and year 12 Mathematics B. The new materials compliment existing distance education print materials, text resources and limited direct teacher contact. They provide students with alternative learning approaches and allow students to choose a learning approach that suits their individually preferred learning style. These materials form a supplement to the teaching materials known as a *Mathematician's Toolchest*. The existing teaching materials already include two versions of the *Mathematician's Toolchest* - one for the Sharp EL9300 and another for the Casio fx 7400G. The contents of the TI-89 *Mathematician's Toolchest* is included as Appendix 3.

At the commencement of the 2000 school year, students who had been awarded either a minimum grade of C+ in year 10 Junior Mathematics (Extension) or High Achievement (B) in year 11 Mathematics B in 1999 were given the option of being involved in the trial of an algebraic calculator. Year 12 students had a year's experience using a graphing calculator. Year 11 students had only about a month's experience. The restriction on which students were invited to participate was due to concerns that students who were weak at algebra may either: be unable to benefit from the use of an algebraic calculator; or become over-reliant on it and not develop the necessary knowledge and procedures required by the course. Approximately 80% of students who were given the option of being involved in the trial chose to participate. No students who wanted to be involved in the trial were excluded.

Most students were provided with an introductory tutorial on the use of the new calculator and provided with an initial set of instructional materials on its use. Students who were unable to attend the tutorial were given limited instructions by phone or at a residential school in March. Throughout the year, as additional modules for the *Mathematician's Toalchest* were developed, these were sent to students.

Most of the students participating in the trial were transferred to a single class taught by the author. All students involved in the trial were able to use their calculator for class work and during both formative and summative assessment All summative tasks were vetted to ensure that students with access to an algebraic calculator did not have an unfair advantage over other students.

The objectives of the project were to:

- develop teaching resources which take full advantage of the latest developments in calculator technology to supplement existing learning materials and programs;
- free students of the need for repetitive manipulation of symbols and allow them to investigate real-life contexts and problems;
- provide students with learning resources which will enable them to achieve success in Mathematics at the highest level possible; and in the process
- maintain BSDE's position as a leading school in the use of calculator technology; and
- make copies of the teaching resources available to other teachers and schools through various avenues such as workshops at the annual conference of the Queensland Association of Mathematics Teachers.

The intended outcomes of the project were:

- the production of teaching resources intended initially for use in distance education but suitable for use in regular schools;
- enhanced learning opportunities for senior secondary students who are currently under some disadvantage due to their limited access to computer technology;
- the promotion of conceptual understanding and higher levels of achievement in Mathematics of participating students;
- a greater appreciation by students of the power of mathematical modelling;
- the increased ability of students to develop, analyse and apply mathematical models;
- an improved ability of students to solve unfamiliar real-life and purely mathematical applications (solve problems) using familiar mathematical techniques (knowledge and procedures); and
- to establish a basis for introducing the new technology to Mathematics C and / or lower secondary mathematics.

The methodology employed was as follows -

1. Students were provided with continuous access to a state-of-the-art graphics calculator with built-in computer algebra system, as a substitute for the current graphics calculator.
2. An introductory tutorial was conducted for students to introduce them to the capabilities of the new calculator.

3. Teaching resources, across a range of syllabus topics, which employ learning approaches alternative to existing print materials, were developed by the author, published and made available to students who nominate and are selected to participate in the trial.
4. Students have the choice of using the additional features of the new technology both as a learning tool, in regular 'classroom' activities and for all formative and summative assessment tasks.

The performance measures to be applied are:

A survey of trial participants to determine their attitudes towards the new technology and its use both at the end of the first year of the trial and on completion of the project; and

A comparison of the levels of achievement and performance on selected assessment tasks of trial participants with the performance of similar students in previous years or in the same cohort who elect not to use the new technology, but retained full access to a graphing calculator.

Interim Results

At the completion of the first year of the trial, all students involved in the trial (including one student, 83 years of age, who withdrew because he found the new calculator too difficult to use) were sent a survey. A copy of the survey is attached as an Appendix 4.

The overwhelming view of the students is that the trial should be continued. (The only respondent to disagree was the elderly student mentioned earlier.) Of the 22 respondents (out of 25 participants) 18 replied that the trial should be continued in its present form, and 3 expressed the opinion that the trial should be modified. Their comments mainly related to the need for additional *Toolchest* modules, and this concern has been addressed for both continuing trial participants and participants joining the trial in 2001. All participants were sent a complete set of modules at the commencement of the 2001 school year.

A significant number of students found it more difficult to learn how to use the algebraic calculator compared to their previous experience using a graphing calculator. (This was more apparent with the students who had previously used a Sharp calculator). However, 78% of students reported that the algebraic calculator was easier to use once they had learnt how to use it.

Only three students reported that they felt that their overall results had declined since using the algebraic calculator, whereas seventeen were of the opinion that their overall results had improved. Of greater significance, only one student reported that their algebra skills had declined since using the algebraic calculator but 14 students believed that their algebra skills had improved.

These student perceptions are not well supported by other data, however. Of the six internally assessed year 12 students, when comparing exit results with their year 11 results at certification, the achievement level of three students had declined slightly, two students had improved slightly and one student improved considerably. Of the five internally assessed year 11 students, two achieved at a significantly higher level than expected and the other three achieved at a level close to that expected, based on their year 10 results.

Exact comparisons are difficult since trial participants were not randomly selected and all were achieving at an average standard or better prior to being invited to participate. A direct comparison of the achievement of the trial participants and a control group is difficult for a number of reasons. The limited size of the trial and diverse nature of the sample means that only small subgroups can be directly compared.

Having said that, the evidence available to date suggests that, students who have used the algebraic calculator have achieved a higher level of achievement than would otherwise be expected. Figure 1 shows the academic results of the internally assessed year 11 students and their corresponding year 10 results. The two linear trend lines are almost parallel. The vertical distance between them, of about half a level of achievement, indicates that the students with algebraic calculators achieved a level of achievement about half a level higher than would be otherwise expected on the basis of their year 10 results.

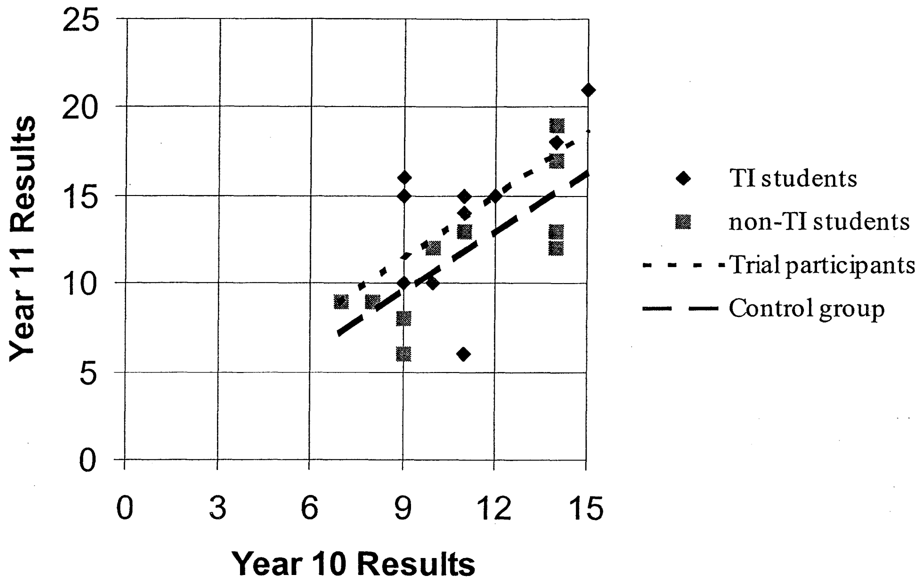

Fig 1 Comparison of Year 10 and Year 11 Results

Although the sample is quite small, Figure 2 shows that a similar trend exists for the year 12 students in the trial. The linear trend line of their results (based on their result for year 11) is approximately parallel to and about half an achievement level above that of the control group.

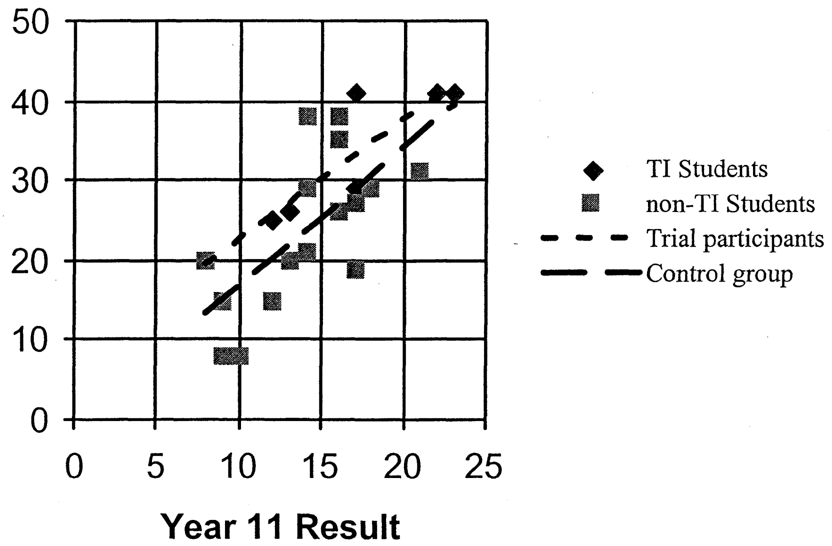

Fig. 2 Comparison of Year 10 and Year 11 Results

Conclusions

Based on the interim results, the school will continue the trial in 2001 with almost all of the year 11 students (both internally and externally assessed) continuing and an increased number of year 11 students joining the trial.

Further evaluation, with an increased sample size, should allow more reliable conclusions to be drawn about the impact of the use of the algebraic calculator on student performance and their attitudes to both the technology and mathematics.

If the algebraic technology is made available to all internally assessed students, then the curriculum implications for both junior mathematics (years 8 – 10) and senior mathematics will need to be reviewed. In particular, the question of how much algebra should be taught in years 8 – 10, and how this should be taught to best equip students to take full advantage of the algebraic technology in year 11 will need to be carefully considered. One option would be to increase the use of graphing calculators in year 10.

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Appendix 1

Trial Participants' Characteristics and Achievement

Student	Year (2000)	Age	Category	Assessment Mode	Year 10 Result	Year 11 Result	Year 12 Result
SA	11	Secondary	Travelling	External	C-		
RK	11	Adult	Re-Entrant	External			
RC	11	Adult	Re-Entrant	External			
CC	11	Adult	Re-Entrant	External			
WM	11	Secondary	Approved	Internal	A	H3	
HS	11	Secondary	Medical	Internal	A+	VH1	
CL	11	Secondary	Approved	Internal	B	S5	
BJ	11	Secondary	Approved	Internal	B-	L5	
ST	11	Secondary	Approved	Internal	B+	S5	
FD	11	Secondary	Approved	Internal	C+	L5	
KE	11	Secondary	Approved	Internal	C+	H1	
TJ	11	Secondary	Approved	Internal	C+	S5	
CJ	11	Secondary	Distance	Internal	B	L1	
DD	11	Secondary	School Based	Internal		VL5	
GK	11	Secondary	Approved	Internal	B	S4	
NR	11	Secondary	Medical	Internal	B	S3	
QJ	11	Secondary	Medical	Internal		S2	
UA	11	Secondary	Medical	Internal		VL5	
VC	12	Adult	Re-Entrant	External			S
BT	12	Secondary	Approved	Internal		H2	S9
HD	12	Secondary	Medical	Internal		VH2	VH1
PD	12	Secondary	Approved	Internal		S3	S6
RK	12	Secondary	Medical	Internal		H2	VH1
WB	12	Secondary	Distance	Internal		VH3	VH1
WC	12	Secondary	Approved	Internal		S2	S5

Year 10 results are on a 15-point scale E^{-} - A^{+}
Year 11 Internally assessed results are on a 25-point scale VL1 – HV5
Year 12 Internally assessed results are on a 50-point scale VL1 – HV10
Year 12 Externally assessed results are on a 5-point scale VL – HV
(Very Limited – Limited – Sound – High – Very High)

Appendix 2

Knowledge and procedures in Mathematics B impacted by the use of algebraic calculators

Syllabus Content	Approach
Solving equations	Substitute then solve
Changing the subject of an implicit equations	solve w.r.t. the new subject
Trig ratios	solve w.r.t. unknown
Sine rule	solve w.r.t. unknown
Solving quadratic equations	solve w.r.t. variable
Solving higher order polynomial equations	solve w.r.t. variable
Index laws	Factor
Exponential equations	nSolve
Gradient of secants	factor
Algebraic derivatives	Differentiate
Using the derivative (stationary points)	Solve the derivative
Extrema	fMin
Writing and solving equations	Solve
Simultaneous equations (2)	Solve
The length of a curve	arcLen
Logarithmic functions revisited	nSolve
Applying exp and log functions	nSolve
Derivative applications	Differentiate
Product rule	Differentiate
Chain rule	Differentiate
Quotient rule	Differentiate
Combining the rules	Differentiate
Return to differentiation	Integrate
Antidifferentiation	Integrate
Antidifferntiation and motion	Integrate
The indefinite integral	Integrate
Derivative of e^x	Differentiate
Derivative of ln x	Differentiate
Applications of exponential and logarithmic functions	nSolve
Differential equations	nSolve
Applications of d.e.'s	nSolve
Binomial distribution	Expand
Integral features	Integrate
Integrating exponential, reciprocal and periodic functions	Integrate
Area applications	Integrate
Integrals of f(ax + b)	Integrate
Integrals, areas and applications	Integrate
Differential equations	Integrate
Geometric Progressions	nSolve
Finite sums	List, seq, sum
Interest review	nSolve
A compound interest model	nSolve

Appendix 3
Mathematics Toolchest Modules

Basic features	T 01
Editing	T 02
Named variables	T 03
Data in lists	T 04
Sorting lists	T 05
Central tendency	T 06
Frequency graphs	T 07
Box plots	T 08
Five number summary	T 09
Standard deviation	T 10
Trig ratios	T 20
Inverse trig ratios	T 21
Sexagesimal angles	T 22
Converting degrees and radians	T 25
Storing equations	T 30
Graphing	T 31
Tracing a graph	T 32
Locating x intercepts	T 33
Graphing vertical lines	T 34
Gradient of tangents	T 36
Parametric graphs	T 51
Basic algebra	T 60
Solving simultaneous equations	T 62
Stationary points	T 63
The nature of stationary points	T 64
Limits	T 65
Solving trigonometric equations	T 66
Fitting regression lines	T 67
Solving equations step-by-step	T 68
Transposing formulae	T 69

Appendix 4

Brisbane School of Distance Education

Mathematics Department

Survey of TI-89 users

As a participant in the trial of the Texas Instrument TI-89 algebraic calculator, it would be appreciated if you could answer the following questions. As a school, we need to establish whether or not the additional cost of this calculator compared to the alternatives is sufficiently beneficial to students to warrant expansion of the program.

Name (Optional) ____________________

1 What year level were you this year? Year 11 / year 12 (please circle)

2 Under which assessment mode were you studying? Internal / external

3 Which subjects were you studying in 2000? Mathematics B / Mathematics C

4 How long (in months) have you been using the TI-89? ______________

5 Had you used a graphical calculator prior to this? Yes / No (If no, proceed to question 7)

6 a Which calculator (make and model) had you used previously? ____________

  b Compared to the other calculator, how would you compare the TI-89 for ease of:

    i learning how to use the facilities? easier / no different / harder to learn

    ii use once you learnt how to use it? easier / no different / harder to learn

7 Compared to what you expected, how has the TI-89 affected your

  a overall result for mathematics improved / been unchanged / declined

  b algebra skills improved / been unchanged / declined

8 Did you find the Toolchest notes for the TI-89

  a easy to use yes / sometimes / no

  b covered all of the skills you needed? No / yes (If yes, proceed to question 10)

9 For which skills would you like additional Toolchest notes? ____________________

10 Do you think that the TI-89 trial should be continued? Yes (in its present form) / modified / no

11 What is the main reason for your answer to question 9? ____________________

____

12 How do you feel about your level of preparation for using an algebraic calculator?

Not very well prepared 1 2 3 4 5 Very well prepared

13 How confident do you feel about your use of an algebraic calculator when sitting for exams?

Not very confident 1 2 3 4 5 Very confident

14 Please add any other comments which you think would make future student's use of the TI-89 more effective.

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Thank you for your time. Please use the reply paid envelope to return your completed survey to David Driver at B.S.D.E., by mid December if at all possible.

Researches in Language Aspects of Algebra: a Turning Point?

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I have been working in the domain of the learning and teaching of algebra for more than fifteen years, first as a Ph. D. student then as a researcher in "Didactics of Mathematics" (the French name for the research in Mathematics Education). Coming from an educational background of mathematics, logics and linguistics I was interested from the very beginning on a linguistic point of view on algebra. My Ph. D. Dissertation was a description, in terms of a generative grammar, of the symbolic expressions of elementary algebra. At this moment I felt that my work belonged to an emerging stream of research in mathematics education. This "stream" appeared to be a confluence of the ever-increasing studies on the role of the language in mathematics education and the researches on algebraic learning and teaching, which began to appear as an autonomous research field. Little by little, researchers with this common interest (the language aspects of algebraic learning) met in various special interest groups1, workshops2, seminars3, and Internet forums4 or on occasion of editing books5 in this topic. They came from very diverse theoretical backgrounds, many of them had more than this unique centre of interest, and they had very different vocabulary or research paradigm. However, all these researchers have had in common the conviction that algebra is not just a matter of "contents" but is also characterised by a particular language (or "system of representations", or "semiotic system" etc. according to their particular vocabulary). Therefore students, from this point of view, have to learn how to read, write, understand, speak, and above all how to use this particular language in order to solve problems and to "think algebraically".

Amongst the researchers interested on the language aspects of algebra, one could distinguish two main centres of interest6. The characteristics of the language7 of algebra and its pedagogical consequences interest a first group. The main concern of the second group is the use of computers in algebra education; then, they are interested on the question of language and representations of algebraic objects since computerised environments

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1 like the various *PME Working Groups*, whose organizers or co-organizers were Rosamund Sutherland, Teresa Rojano, Luis Radford, Alan Bell, Jean-Philippe Drouhard, Sonia Ursini...

2 like *WALT*, the *Workshop on Algebra Learning in Turin*, organized by Ferdinando Arzarello, the *Working Conference of the ESRC Seminar Group* (London) organized by Rosamund Sutherland...

3 like SFIDA, the French-Italian Seminar on Didactics of Algebra, co-organized by Ferdinando Arzarello, Jean-Philippe Drouhard and Giampaolo Chiappini ...

4 like the Algebra Working Group organized by JimKaput...

5 like *Research Issues in the Learning and Teaching of Algebra*, S. Wagner & C. Kieran (Eds.), *Approaches to Algebra*, N. Bednarz, C. Kieran & L. Lee (Eds), or *Algebraic processes and structure*, Rosamund Sutherland & Teresa Rojano (Eds.).

6 Of course these scientists may have more than one centre of interest, this is not a rigid classification.

7 or "system of representations",, etc.

involve necessarily modules of production and interpretation (or representation) of algebraic expressions.

Gradually the researchers on linguistic perspective in algebra developed common ideas (if not a common language). At present a presentation of a recent research or an ongoing study in a specialised seminar or colloquium (as the "SFIDA" seminar) does not require justifying the need to focus on the linguistics aspects of the studied phenomenon. To adopt such a perspective have become quite obvious and the emphasis is now put on the results - of what is *seen* from this point of view. Gradually too, most researchers began to explore new areas (reasoning, situated cognition, social construction of knowledge, computer environments, history of algebra...), as if the main ideas in language aspects of algebra have been already well known and established and do not deserve more investigation.

The situation of the impact of the language approach in the studies on learning and teaching algebra however, is contrasted. On the one hand, most computer environment take into account, more or less, this language approach, for the reasons cited above. On the other hand, this approach seems to have little impact on the other studies on learning and teaching algebra; and the more applied are the studies, the less they seem related to the language. Some curricula or national school programs seem written as if nobody had ever written anything about these questions of language8.

Two main obstacles could be identified to a full acknowledgment of the role of the language in algebra education. The language of algebra is a very complex language that looks apparently simple (mainly in comparison with natural languages), and its mastering is difficult although it looks apparently easy. Therefore the importance of this language in learning and teaching is often underestimated. This is the first obstacle. It is not easy to overcome it since this complexity (and this difficulty) is evident for every researcher who try to describe it, but remain barely intuitive for the others. Moreover, a research study is generally expected to find simplicity in complex phenomenon, not to show that an (apparently) simple phenomenon is actually complex. In the latter case, the researcher must proof that the complexity holds within the phenomenon itself and not within his/her description; which is not easy by far.

A second obstacle is due to the nature of the student's linguistic knowledge. Most theoretical frameworks for the learning of mathematics9 concern obviously mathematical knowledge. The point is that linguistic knowledge is not, strictly speaking, mathematical (see for example Kirshner, 1989). The grammatical structure of an algebraic expression, or its numerical (or logical) value (its "denotation") is a linguistic knowledge10. A student has to know this, and to know *how* a given expression is compound, but this knowledge has little to do with a conceptual knowledge. Hence theories of mathematical learning might be poorly convenient for such linguistic knowledge. Can this knowledge be "constructed" as other conceptual knowledge in a constructivist perspective? There is no obvious answer; however, we could make the reasonable hypothesis that not every kind of knowledge is constructed in the same way. In short, the nature of linguistic knowledge is an obstacle to the impact of the linguistic perspective since the learning of such linguistic knowledge might pose thorny problems to the "mainstream" theories of mathematical learning.

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8 to be honest I must say that some curricula or national programs seem written as if nobody had never written anything about research in mathematics education

9 see for instance *Theories of Mathematical Learning*, Edited by Leslie P. Steffe, Pearla Nesher, Paul Cobb, Gerald A. Goldin & Brian Greer, 1996, Lawrence Erlbaum.

10 although it plays a role for mathematical comprehension: we speak of the nature of a knowledge, not of the role it plays. Obviously non-mathematical knowledge does play a role in learning of mathematical knowledge.

That is why, at present, the linguistic perspective is at a turning point. It will play a prominent role, provided that the researchers could get over the present obstacles to its diffusion in the teaching and learning of algebra.

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Algebra Worlds

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This paper addresses the future of school algebra. It foresees the learning of algebra in environments — games or virtual worlds — which engage and engross the learner. It offers a glimpse of some of the features that such worlds may have, and it briefly considers learning in virtual worlds in relation to current perspectives on the learning of algebra and the nature of mathematics itself.

In [1], Carolyn Kieran concludes a comprehensive survey of research in the teaching and learning of algebra with an analysis in terms of procedural and structural aspects. In brief, her conclusion is that students cope better with procedural than with structural ideas, and that many fail to achieve any real sense of the structural aspects of algebra. Further evidence for this may be seen in the problem that is now bedevilling university mathematics departments around the world: the shift of students from mathematics to computing. Whereas physics and other sciences depend upon structural aspects of mathematics, you can now make good money and have a career working in the purely procedural side; and to many of those who program, the activity is almost hypnotically entrancing.

Kieran argues "for extending the content of algebra by adding more emphasis to activities that promote the development of procedural interpretations and make explicit the transition from procedural to structural conceptions". Evidently, the increasing use in the curriculum of the computer as a tool has a bearing on this issue. The main purpose of the present paper is to argue that the computer offers opportunities for a truly radical change in the teaching and learning of algebra, a change that can greatly facilitate the acquisition of both structural and procedural ways of thinking.

What is advocated here is the development of mathematical worlds in which the learning of algebra is accomplished almost as a side effect accompanying creative construction, exploration and investigation. With the aid of a sufficiently powerful and well-designed computer algebra system, algebra can be learned, potentially, as easily as natural language or games are learned. This claim is made not as fact but as something that future research may establish; appropriately for this Study, it is a claim about the future of algebra learning. Before the claim can be tested it is necessary to build such worlds. The system used by the author to develop prototype worlds of the kind considered below is Mathematica.

Less radically and more immediately, use of a computer algebra environment such as Mathematica can fruitfully be integrated with standard approaches to the teaching of algebra. A secondary purpose of this paper is to consider ways in which such use can facilitate, here and now, the transition from arithmetic to algebra that many find so difficult; in particular, a specific example from the Kieran paper is discussed from this perspective. The suggestion is that good use of a well-designed computer algebra environment supports incorporation of procedural elements in the learning process. Mathematica provides a uniform language for interacting with the machine, in conversational fashion, with language close to standard mathematical language, and in some respects superior to it. Moreover, as the material presented shows, Mathematica can serve to bridge the gap between natural language and mathematics — the familiar source of difficulty in so-called "word problems".

Introduction

The focus of research in mathematics education is, naturally enough, upon what makes mathematics difficult: the idea of a variable, the transition from procedural to structural thinking, equality, word problems, ... Let us ask instead: what is it that makes other kinds of learning easy? Certain activities of the mind come easily to kids: learning to talk, playing games, making things. These activities, in themselves, may be quite difficult. Mastery may be rare. No matter. Most of us, and especially kids, can learn to play a new card game or to make a tower with Lego, and can enjoy the learning. Algebra should be like that. The thesis of this paper is that right use of the computer can make this happen. What is needed is not trivial: it is the development of algebra worlds.

At the heart of the problem with learning algebra lies the fact that the role of the student is too often passive. In total contrast, doing algebra with a computer algebra system like (raw) Mathematica is an activity in which the user-player, confronted by a blank sheet, must be completely active — it is the computer that is the passive player. The problem that must be addressed is how to bring together these two extremes: to design worlds in which the student takes an active role, motivated by the desire to make things, to explore or to win; uses and acquires algebra as a means to this end; and hands to the computer those tedious but necessary tasks that the computer does best. Learning becomes a three-way dialogue between teacher, student(s) and computer. Bringing in the computer opens the possibility of greater autonomy for the student. (For the dialogue to flower, the computer needs greater autonomy too.) The participants are to be seen as equals, each bringing its own capabilities to the dialogue: the innate disposition of the student to learn, the symbol-processing power of the computer, the wisdom of the teacher as facilitator of the learning process.

Virtual worlds are not new. Every game is a virtual world. All games, to a greater or lesser extent, call upon mathematical abilities. The computer, and especially computer algebra systems, offer the possibility of creating games or worlds in which mathematical learning, and especially algebra learning is an implicit objective. Of the mathematical worlds that have been developed, the best known and most pedagogically successful type, so far, is the dynamic geometry world of Cabri Geometre, Geometer's Sketchpad and Cinderella's Cafe. Although the hidden mechanism of dynamic geometry is algebra, this is not really and algebra world. For algebra, something closer to Monopoly or the popular computer game Sim City is needed. Development of such worlds is not trivial. It may require many person-years. But it must be done before we can start to work within such worlds in the classroom, and before we can begin to adequately test the thesis of this paper: that algebra learning and attitudes to algebra will immensely benefit from such developments.

A world

Imagine this. A kid plays a game at a computer. The game is creative rather than competitive. Tools are available for the making of buildings and villages. What you make will be visible on the screen. But you must give your instructions to the computer in language the computer speaks, mathematical language. A prototype world of this kind, implemented in Mathematica, was presented in Fearnley-Sander (1994). Here I will try to convey the flavour of this world, and indicate extensions that will exercise additional mathematical skills. The purpose is not to advertise this very rudimentary particular world, but rather to suggest the possibilities that will be opened up by the (inevitably demanding) development of better algebra worlds.

The Village World is an environment for the creation of 2-dimensional scenes, using basic geometry, transformations, iteration and chance. A scene is made up of objects, and the most basic objects are particular rectangles, triangles and other geometric shapes. For example, here is the way to specify some features of an object called **window**:

```
isaRectangle[window] := True;
```

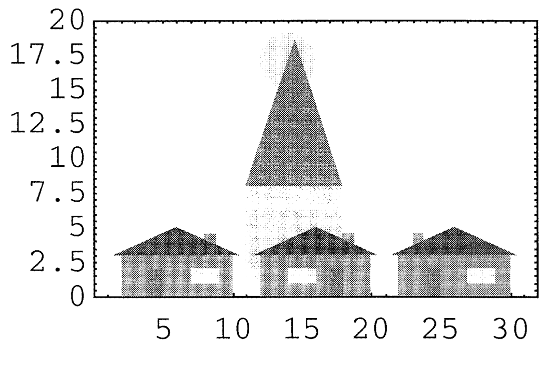
```
base[window] := 2;
height[window] := 1;
colour[window] := Yellow;
```

Other objects such as **chimney** and **roof** may be specified similarly. A building is made up of components like this. The following gives some of the features of a building called **myHouse**:

```
isaBuilding[myHouse] := True;
parts[myHouse] := {wall,chimney,door,window,roof};
locationIn[wall,myHouse] := {0,0};
locationIn[chimney,myHouse] := {6,0};
...
isaScene[scene1] := True;
parts[scene1] := {myHouse,JacksHouse,JillsHouse,
                  theChurch,theSun};
locationIn[scene1,myHouse] := {0,2};
...
isaRectangle[scene1] := True;
base[scene1] := 32;
height[scene1] := 20;
```

The created scene may now be viewed:

```
draw[scene1]
```



The mere construction of **scene1**, well guided by a teacher, will exercise algebraic skills in an enjoyable way. This is only a beginning. Scenes may be elaborated and modified by making changes to the code; objects may be given additional attributes; and algebraic reasoning may be applied to investigation of the properties of objects and to the solution of problems.

Let us immediately confront some objections that might be made to working with the Village World in a classroom.

• *Objection: it's not algebra.* On the contrary, an algebra in the technical sense is precisely what has been constructed. Moreover it is a setting for motivated algebra problems, in the kind of way sketched below.

• *Objection: too much typing.* Although creation of **scene1** needs many lines of code, copy and paste reduces actual typing to a fraction of that. Typing, copying and pasting are skills that kids need, anyhow, and acquire more easily than adults. Moreover there are many things to do with **scene1** that require very little typing — changing some colours, for example, or moving some objects.

• *Objection: too much irrelevant syntax.* On the contrary, the syntax is almost the standard one for functions. **base** is a function that takes rectangles to numbers; and **base[window]:=2** would standardly be written "Let base(window)=2." Use of square brackets (enforced in Mathematica) and **:=** eliminates ambiguities in the standard use of round brackets and the equals sign. Such ambiguities are a part of what makes algebra difficult. They should be eliminated.

• *Objection: too limited.* Construction of scenes like this one will help to achieve, transparently and almost as a side effect, two steps in the learning of algebra: grasping coordinates (and, indeed, relative coordinates) and understanding the role of functions. These are not trivial steps. Understanding is acquired largely as a result of visual and other feedback from the computer. Moreover the village world, as described so far, is a mere beginning. Like other simulated worlds, it is open to expansion and refinement in infinitely many ways.

• *Objection: Mathematica is too powerful and complex for this purpose.* Mathematica is a state-of-the art working environment for doing mathematics. No-one learns to use its full capabilities. No matter. You use it for what you want to do. Its power is useful, even for doing simple things. And it makes the doing of conceptually simple but computationally complex tasks, such as linear programming, available for possible use in high school. You don't have to know how a tool works to use it and use it well.

• *Objection: Mathematica is too expensive.* Granted. Currently student copies in Australia cost about $300 — too much, but not outrageously too much.

How might we refine the Village World? We could make it a game requiring the solving of various problems. One kind of game will require the building of villages with limited resources. A more sophisticated version will introduce competition between players for resources. Elements of chance will be included, disaster and good fortune, insurance and the borrowing and lending of money, rents and interest. These are algebraic activites, requiring algebraic language and algebraic problem solving. Real life, not to mention Monopoly, tells us that playing a money game is absorbing.

Let me sketch an easy problem that might arise in playing a Village game. In reading the discussion you should be aware that **bold type** is input by the student user while `non-bold type in this font` is output from the Mathematica engine. Also I have purposely used **:=** in some places where the straight = would also be acceptable Mathematica code; this is to avoid considering computational subtleties that are not relevant to the task at hand.

An easy problem

To figure out your profit when you sell a house that you have built to another player, you need to work out the cost of building it. The house has four large windows 3.2 metres wide and 1.2 metres high, and two small .5 metres wide and 1.2 metres high. You buy glass at $25 per square metre. How much does the glass cost you?

```
windowCost[myHouse] := 4*3.22*25 + 4*.5*25;
windowCost[myHouse]
372.
```

Let us specify some other costs, and compute the total cost.

```
doorCost[myHouse] := 8*97;
```

```
wallCost[myHouse] := 3250;

roofCost[myHouse] := 2100;

totalCost[myHouse] := windowCost[myHouse]+
  doorCost[myHouse]+wallCost[myHouse]+roofCost[myHouse]

totalCost[myHouse]

6498.
```

What if we reduce the number of doors to 6?

```
doorCost[myHouse] := 6*97;

totalCost[myHouse]

6304.
```

We see that changing bits and pieces of **myHouse** induces change in **totalCost[myHouse]**.

Examine this approach to the simple problem of specifying and computing a cost. It looks procedural. You have told your computer how to work out some things and you have asked it to work out some things. Now change perspective to a structural one. What you are dealing with is nothing but an algebra in the technical sense defined in advanced mathematics courses. No need to tell that to your class of twelve year olds, of course! Our algebra of house costs has two sorts: the sort of houses, and the sort of dollar amounts; the first has no operations, while the second has the familiar operations of arithmetic. One sort has objects like **myHouse** as the elements of its carrier, the other has objects like **3.22**. There is no barrier between the procedural and the structural. The algebraist may view this algebra as quite trivial, but from the point of view of the algebra teacher, that is no criticism. It is praise. Of course beginning speakers of mathematics should start with the easiest structures.

Take another look. Take a programming perspective. To convey the facts about a world he is working/playing with to the computer you have written an object-oriented program. Perhaps, as a beginner, you simply filled in a template. No matter. **myHouse** is an object with a number of attributes such as its **roofCost**. Another object **yourHouse** of the same sort might have a different **roofCost**. Where, you might ask, is the object-oriented programming language? Where is Eiffel or Java? Well, Mathematica is such a language. But I prefer to say: mathematics is an object-oriented programming language. Among other things, of course. One of the many things you do when you do mathematics is object-oriented programming.

Problems like this are standard fare in the mathematics classroom. What is added here is this: it is more fun to solve problems that are part of a game. There are many computer games that offer challenges of the kind described here. What I am advocating, though, is the use of a symbol-processing environment such as Mathematica as the interface to the simulation engine. The language of Mathematica is, pretty much, the language of mathematics. Some of the small ways in which Mathematica differs are annoying but some actually offer improvements over what we are used to. In particular, procedural equality is distinguished notationally from structural equality.

Turning once again to our easy problem, let us see how it can motivate a hard step in the process of acquiring algebraic thinking: coming to grips with variables. The code for **totalCost[myHouse]** is fine, but what if we want **totalCost[JacksHouse]** and **totalCost[JillsHouse]**? One possibility is to duplicate the code and make appropriate replacements. But the following does the job in one hit for all the houses that we may ever build this way:

```
totalCost[aHouse_] := windowCost[aHouse]+
  doorCost[aHouse]+wallCost[aHouse]+roofCost[aHouse]
```

The underbar marker on **aHouse_** tells Mathematica that **aHouse** is a variable. It is explicit and simple. It is not something that, in some mysterious way, must be inferred from context about a letter.

The objects of algebra

What are the objects of school algebra? My suggestion is that in the initial years of school algebra the objects are, and should be the same as those of arithmetic: numbers and entities to which numbers pertain, such as people and polygons. It is quite non-standard to view objects like **myHouse** or **Yellow** as mathematical, but this is what I am advocating.

The move to algebra, in its early stages, is a move to richer language in which to talk about numbers and number properties of entities like people and polygons. Language is enriched by the introduction of variables, equations, functions and so on. While bare arithmetic is a language of commands (Evaluate `76*53`), algebra also embraces assertions (x^2`-1=(x-1)(x+1)`), definitions (x^2`=x*x`), questions (If $x^2=x$, what is x?) and other types of sentence.

In accordance with the Vygotskian view cogently expressed in Filloy and Sutherland (1996), objects other than numbers are constructed by the learner, more or less slowly, out of the engaged use of this enriched language. While the objects that have been the focus of university algebra are structures (such as groups and fields), the main objects that emerge from school algebra are functions and equations. In some ways these are more difficult to internalize than structures. While functions are well understood as mathematical objects, the same cannot be said of equations. (Fearnley-Sander and Stokes (1997) and (2000) reflect an ongoing effort to theorize a general notion of equation as object.) It is the author's view that some of the conceptual difficulties of school algebra spring from inadequacies in our current formalization of mathematics itself, inadequacies that are in process of being dealt with as mathematics comes to grips with computation.

Computer algebra in today's classroom

World's of the kind envisaged will take time to develop, and more time to reach the classroom, even experimentally. But Mathematica is here now, and, I want to argue, can be used with benefit now. Indeed, Mathematica itself is a world, a world in which algebra can be done, though not a world designed for the learning of algebra. Let us consider, in the Mathematica environment, a problem discussed in Kieran (1994) in relation to research on algebra learning.

A problem from Kieran (1994), p. 393:

The Westmount Video Shop offers two rental plans. The first plan costs $22.50 per year plus $2.00 per video rented. The second plan offers free membership for one year but charges $3.25 per video rented. For what number of rented videos will these two plans cost exactly the same?

```
plan1Cost[num_] := 22.50 + num*2;
plan2Cost[num_] := num*3.25 ;
```

This formulation reflects almost exactly the statement of the problem. It is just put in the formal way required by the Mathematica engine. Each of the two lines is a Mathematica program, and you can immediately run the programs to see what they do, and check that they match what is expected.

```
plan1Cost[5]
32.5
plan2Cost[5]
16.25
```

We can run these programs with many different inputs, and compare the results. Below is code for doing that in one hit. Template code of this complexity would of course need to be introduced carefully by the teacher; but it will then be reusable by students for many different purposes.

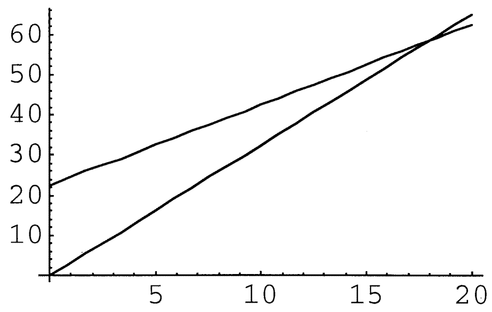
```
Table[{num,
  plan1Cost[num],
  plan2Cost[num],
  plan1Cost[num] ≥ plan2Cost[num]},
   {num, 0, 20}] // TableForm
```

0	22.5	0	True
1	24.5	3.25	True
2	26.5	6.5	True
3	28.5	9.75	True
4	30.5	13.	True
5	32.5	16.25	True
6	34.5	19.5	True
7	36.5	22.75	True
8	38.5	26.	True
9	40.5	29.25	True
10	42.5	32.5	True
11	44.5	35.75	True
12	46.5	39.	True
13	48.5	42.25	True
14	50.5	45.5	True
15	52.5	48.75	True
16	54.5	52.	True
17	56.5	55.25	True
18	58.5	58.5	True
19	60.5	61.75	False
20	62.5	65.	False

Hand calculation of this table would be laborious and very prone to error; it is precisely the kind of task that should be done by a computer. People, even school kids, have better things to do with their minds.

The essential information in the table is easily accessible graphically:

```
Plot[{plan1Cost[num],
  plan2Cost[num]},
  {num,0,20}];
```



We seek the point of transition from one plan being best to the other. The answer is plainly visible in both the table and the graph. But algebra will work in far more complex cases of the same underlying problem. The following asks the Mathematica engine to evaluate an equation. Only the most basic simplification is done.

```
plan1Cost[num] == plan2Cost[num]

22.5 + 2 num == 3.25 num
```

Mathematica's **Reduce** command fully simplifies the equation:

```
Reduce[plan1Cost[num] == plan2Cost[num]]

num == 18.
```

Of course it is not being advocated here that such a simple equation should be handed over to a computer algebra system. Two things are suggested: that the notations used to formalize the problem for the computer (such as **plan1Cost[num]**) can facilitate understanding; and that understanding may be cemented by use of the computer algebra system for solving similar but more computationally intensive and realistic problems. Moreover the computer algebra system allows early access to conceptually simple but computationally demanding areas such as constrained optimization problems.

There are many occasions on which the traditional use of letters in algebra, and especially that universal kill-joy **x** is unnecessary — kids easily learn to type fast and accurately (and this is a very worthwhile skill to acquire), and using full words or more elaborate expressions as above eases the burden on memory and leaves more brain space available for dealing with the substantive aspects of a problem. Although this may seem to add to the labour of doing algebra, copy and paste (plus Mathematica's error detection) makes it manageable, especially by school students. The skills required are skills that should be learned, skills that can enable each of us to harness the computer as an extension to the brain.

Conclusion

This paper takes up Kieran's observations concerning the use of programming as an aid to making the transition from arithmetic to algebra, and advocates the use of a computer algebra system such as Mathematica for this purpose and beyond this purpose for the development of worlds that engage the algebra learner. In Mathematica "programming" is more appropriately viewed as having a conversation with the (Mathematica engine in) the computer. You tell the computer certain things, and you ask the computer questions. A program is what emerges as you progress from mere arithmetic questions to making assertions using a language enriched by the use of equality in its various distinguished roles, and by the use of pronouns.

Not to be confused with what is advocated in this paper is the vogue in mathematics teaching for real world applications. The idea is that knowing that a piece of mathematics is used in real work will motivate the student. This view is mistaken, and mistaken in a simple way. It is, of course, true that if you want to accomplish a task, and that requires learning something, then you will be strongly motivated to do the learning. But this does not carry over to work that someone else might want to do some time. For most of us, the knowledge that elaborate calculations of, say, compound interest are needed by some people in the workplace is no motivation for learning to do those calculations — let someone else do them. Motivation comes with engagement: compound interest is interesting to the accountant, the bank manager, people who are involved in the game of money.

The perspectives of university algebra, familiar to most mathematics teachers, are undergoing change. The impetus for change is coming from an unexpected quarter: computer scientists wanting to make more sense of the activity of computing than the mere following of rules have been inventing and re-inventing mathematics. The computational ways of thinking that dominated mathematics in centuries prior to the twentieth are making a come-back. Mathematics is being seen, again, as a way of talking about computation. New insights from computing are requiring the creation of new mathematics. It is a process that promises to invigorate the learning of algebra.

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Developing a Curriculum of Beginning Algebra in a Spreadsheet Environment

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We present here several considerations that influenced the design process of an investigative activity in algebra. The factors that influenced the design process were related to the students' thinking processes in their use of algebraic models and of spreadsheets as a technological tool. The designers' main concerns related to the choice of problem situations, the sequence of tasks and the promotion of various representations. Observations of students' work in a spreadsheet environment led the designers to consider additional factors – such as a tendency to employ recursive generalization, a need to deal with large quantities of numbers and a need to understand an additional symbolical representation.

Background

One of the main concerns in algebra is the investigation of dynamic processes of variation, by creating and analyzing symbolical models (Steen, 1990). This view of algebra is particularly amenable to the use of technological tools, such as spreadsheets, symbol manipulators and graph plotters (Sutherland & Balacheff, 1999). Many mathematics educators and researchers (Kieran, 1992; Heid, 1995) analyze the advantages of a dynamic approach to algebra and recommend its teaching in a technological environment. However, we know of very few reports on systematic design of technologically-based curricula, their implementation and specific difficulties that have been encountered.

Generalizing is one of the central activities in algebra and is traditionally interpreted as expressing symbolically patterns, structures or processes (Kaput, 1992; Kieran, 1992). The traditional approach considered the translation of routine word problems into a symbolical equation or expression to be the main manifestation of this thinking skill. Justifying belonged traditionally to the domain of geometry and was seriously neglected in algebra. Justifications were hardly needed, as long as the main concern was translation to a symbolical representation and the manipulation of symbols – both activities being often disconnected of any context or meaning. The focus of a newer approach is on the generalization of variations and patterns, within the context of investigating meaningful situations and use of computers. This approach increased considerably the role and significance of generalizations and justifications in algebra.

In this article, we analyze the structure of an investigative activity from a beginning algebra course (*CompuMath* – Hershkowitz et al., in press). One of the main purposes of the *CompuMath* beginning algebra course was to base student learning on modelling variation processes, and thus to promote the skills of generalization and application of algebraic models. The course was developed in a process of three or four cycles of development, classroom observations and subsequent changes. For a variety of reasons (Hershkowitz et al., in press), spreadsheets were chosen as a technlogical tool to be systematically employed throughout the beginning algebra course. The structure of the activity described here is a result of the designers' attempts to enhance the advantages, and avoid some of the observed difficulties of a spreadsheet-based approach to algebra.

The following sections present an investigative algebraic activity from the *CompuMath* course and describe the considerations that led the designers to choose this particular situation for an algebraic activity and to bring it to its present form. Most of the issues raised in this paper are based on intensive observations made in the framework of evaluating a holistic experimental curriculum.

Choosing the problem situation.

The problem situation of this activity is based on the investigation of a sequence of growing "crossed squares" made of dots, and of the resulting numerical sequence (Figure 1). The activity also contains an introductory section on two other dot sequences dealing separately with squares and crosses.

Sequences of geometrical shapes were chosen as one of the themes suitable for algebraic investigations in general, and generalization skills in particular. This choice can be attributed to several reasons:

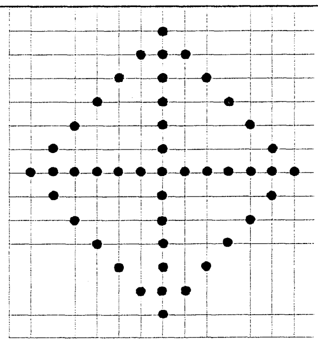
- they require generalization skills (Mason et al., 1985),
- they provide a simple and concrete background for algebraic concepts (e.g., modeling, variation processes, equivalent models),
- they integrate several domains (arithmetic, algebra, geometry and visualization),
- they are amenable to a variety of representations (geometrical, visual, numerical, algebraic, graphical and verbal),
- they are amenable to a variety of solution methods, and therefore are suitable for students of various mathematical abilities,
- they raise student interest.

Sequences of shapes provide a visual representation to various algebraic concepts. In this activity, for example, students counted the number of dots in a shape, by forming "straight branches" of growing length and adding several fixed dots (such as "corners" or "center"). Through this counting process, they became aware of the difference between a variable and a constant in the resulting algebraic expression.

Additionally, tasks 1-3 (Figure 1) led the students through a path of translating a visual pattern of growth into an algebraic model, which reflects the employed counting method. Task 6 shows that the connection between counting methods and the resulting algebraic model works in the opposite direction as well: the counting method can be deduced from the employed algebraic expression.

Thus, generalizing in a geometrical context helps to clarify the meaning of some algebraic concepts, such as variable and constant, equivalence, pattern of variation, generalization, modeling and justification.

This drawing shows the sixth shape in a sequence of crossed squares, that you will investigate next.



1. How many dots are there in the sixth square drawn here?

   Count them in at least two different ways.
2. How many dots are there in the eighth and in the twentieth square?

   Use the two counting methods you found in 1. to check your answers.
3. Write at least two expressions for the number of dots in the n-thcrossed square. Are these expressions equivalent?
4. A crossed square in the sequence has 3,557 dots. What is its place in the sequence?
5. Can you get an even number of dots in this sequence? Explain.
6. Some children in Danny's class found the following expressions for the number of dots in the sequence:

   $4 \cdot n + 4 \cdot n - 3$ $\quad$ $4 \cdot (n - 1) + 4 \cdot n + 1$ $\quad$ $8 \cdot (n - 1) + 5$

   Explain these methods of counting.

   Check whether the expressions are correct.

Figure 1. The *Crossed Square* activity.

Sequencing generalization tasks.

The main issue of the *Crossed Square* activity is pattern generalization. Figure 1 presents a sample of generalization tasks from this activity. We will relate next to some considerations that led to the content, form and sequence of these tasks.

Previous research on beginning algebra students (Friedlander et al., 1989) provided a notion about stages in generalizing a pattern (Figure 2). These stages were implemented as a guiding scheme for the structure of this activity. The sequence is based on a transition from investigation of particular cases to generalizations, then to the justification of the generalized pattern and later on, to its implementation in additional cases. The sequence of tasks illustrates an attempt to follow a path that leads from initial experimentations, to both implicit and explicit generalizations, and then to the use of an explicitly generalized pattern.

Generalizing with spreadsheets.

Students' work in a spreadsheet environment brought to our attention several additional factors that had to be considered in the design process.

Launch The students are presented with specific examples or specific examples are produced by them		
Towards a working generalization		
Producing additional examples.	Producing or solving examples with large numbers.	Solving "reversal" tasks.
Towards an explicit generalization		
Verbal description of the pattern observed.	Symbolical description of the pattern observed.	
Towards a justification		

Figure 2. Stages in generalizing a pattern (Friedlander et al., 1989)

1. *Generalization by recursion versus generalization by position number.* The students obtained a number sequence on a spreadsheet by using one of the following two methods: (a) relating recursively to the previous number in the sequence (usually appearing in the upper cell of the same column) or (b) constructing an explicit formula – that is, using the position numbers (usually appearing in the same row of an adjacent column). Whenever possible, most students used the recursive method. The reason for this preference is a natural tendency to consider the difference between consecutive numbers of a sequence. In a spreadsheet environment, a recursive approach requires a lower degree of generalization. Most students determined the pattern of change at a local level (i.e., the change between two specific cells) and extended the observed pattern by copying (i.e., dragging) the local formula. In some cases, such as an exponential growth, this is the only available way for students of this age. However, the use of recursion does not allow the finding of any data beyond the ones included in the numerical table. We observed students extending their table to thousands of rows in order to answer a question that could be otherwise solved by using a simple position formula. At this point, we should remark however, that a generalization by position number did not necessarily ensure the use of algebra as a solution tool.

   As a result of this observation, the *Crossed Square* problem situation starts with presenting an arbitrary (the sixth) representative, and does not follow the traditional method of showing the first three or four shapes of a sequence. We found that this presentation of a sequence is more likely to elicit generalization by position numbers, whereas the traditional way tends to trigger a recursive generalization.

2. *Abundance of numerical data.* The spreadsheets' ability to produce large quantities of data by simple "dragging" of formulas provides an excellent illustration of the

meaning of variables, algebraic expressions and patterns of variation. Large amounts of data can be a valuable source of cognitive conflicts and insights. We often observed students who revised their work, in view of the numbers produced by their formulas.

On the other side, the same advantage can turn into a disadvantage. Sometimes, we observed students who "abused" the spreadsheet's power. These students chose to construct very large numerical tables, instead of using a more mathematically sound strategy – such as solving an equation. We found that tasks involving large numbers (for example, task 4 in Figure 1) encouraged students to consider this issue. Questions like task 4 caused interesting discussions between students who tended to drag the formula and extend the spreadsheet table and others, who wanted to use the algebraic expression to obtain the desired result.

3. *Spreadsheet formulas versus algebraic expressions.* There is an obvious equivalence between generalizing a pattern as a spreadsheet formula or as a standard "paper and pencil" algebraic expression. However, the difference between the two is sometimes more than syntactic. For example, in order to obtain the multiples of 8, some students used on spreadsheets the (correct) recursive formula $= B3 + 8$, and "translated" it into the algebraic formula of $n + 8$, instead of the expected $8n$. These students disregarded the fact that the variables in the two expressions represent different entities. Initially, we considered the spreadsheet formula as a satisfactory generalization. As a result of this observation, we started to distinguish between the two generalizations and to design tasks for both algebraic and spreadsheet formulas. For example, task 3 (Figure 1) requires explicitly the use an algebraic representation system.

Summary and Conclusions

We presented here several considerations that influenced the design process of an investigative activity in algebra. The variables that influenced the structure of the *Crossed Squares* activity were related to the students' cognitive processes of construction and use of algebraic models. Our initial concerns in the design process were the choice of the problem situation, the sequence of tasks within the problem and the promotion of various representations (particularly visual, numerical and algebraic).

The design of an activity in a spreadsheet environment, involves more than a technical transition from paper and pencil to computers. Computers should not be considered only as "amplifiers of human capabilities", but also as a tool that has the potential to bring structural changes in students' cognitive activity (Pea, 1985). The choice of any tool brings with it advantages, as well as disadvantages and obstacles, and computers are no exception to this fact.

In the process of designing a spreadsheet-based curriculum, we had to consider the impact of several technological factors on students' cognitive activity. As a result, several tasks in the activity related explicitly to the following observed difficulties: (a) students' tendency to employ recursive generalizations, (b) the need to deal with large quantities of numerical data, and (c) the need to work with an additional symbolical representation.

Research findings and classroom observations were crucial in the process of identifying these considerations. In our case, research findings on students' cognitive difficulties in a spreadsheet environment were quite limited. This fact led us to a greater reliance on classroom observations and to the conclusion that farther research in this field is strongly needed.

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Fostering an understanding of algebraic generalisation through numerical expressions: the role of quasi-variables

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In this paper we discuss how algebra is introduced and taught in Japan in elementary and junior high. Many currently used approaches appear to focus exclusively on introducing frame words and later literal symbols as devices for solving simple number sentences. What is often missing is that algebraic thinking necessarily involves students in patterns of generalised thinking. We present several important approaches to introducing algebraic thinking in the elementary and junior high school curriculum using numerical expressions leading to a concept of a quasi-variable.

Introduction

Algebra is not taught as a separate subject in Japanese school mathematics but it is systematically included in various parts of the mathematics curriculum. Foundations for later work in junior high school algebra are treated in the elementary school. In the junior high school, algebra is studied by students in each year of their mathematics program. We describe briefly where it surfaces in the curriculum from elementary to lower secondary level focusing on algebraic expressions and equations.

In elementary school, in third grade, a frame word such as □ is introduced, and in fourth grade, an additional frame word such as ○ is introduced. So that at fourth grade children are able to describe the numerical relationship such that the sum of two quantities is 10 by using the expression: □+○=10. In fifth grade, literal symbols such as x and a are introduced to stand for quantities formerly represented by the frame words. However at fifth graders are not expected to calculate literal symbols, for instance fifth graders are not taught to simplify x+x+x+y+y to 3x+2y.

Beside the frame words, natural language combined with mathematical symbols for four operations and equal sign are used to express quantitative relationship. In Japan, this is called a *kotoba no shiki* or "word sentence". It is useful and handy tool for elementary school children to express quantitative relationship. For example, *A is twice as long as B* could expressed as: Length of A = Length of B ×2 in the word sentence.

In junior high school, in seventh grade, concrete situations expressed in ordinary language are translated into mathematical expressions using literal symbols and vice versa. Calculations of algebraic expressions with one variable are studied in order to solve linear equations using the attributes of equality. In eighth grade, students are required to further develop their ability to find quantitative relationships, and to express such relationships in a formula by using letters. Computation of simple formulas using letters and the four fundamental operations are emphasised. Simultaneous linear equations with two variables and the linear inequations with one variable are studied. In ninth grade, multiplications of linear forms and factorisation of simple quadratic trinomials are studied, as well as solving quadratic equations using factorisation and the formula of solution.

Current research and needed action

In almost all countries mathematics educators note with concern the fact that many students struggle to understand of school algebra, for example, Herscovics (1989,1994), NCTM (1995). There is an abundance of research documents identifying such difficulties in school algebra, for example, Fujii (2000), Dossey (1998), Kaput (1995). However, there are fewer proposals or suggestions so far for improving the teaching of algebra. Some exceptions to this are, for example, Carpenter and Levi (1999), MacGregor and Stacey (1993,1994).

Any improvement in the teaching of algebra must focus on how children are introduced to express quantitative relationships that focus on general mathematical relationships, how they read or interpret algebraic expressions, and how they can calculate algebraic expressions based on the attributes of equality. In the remainder of this paper, our focus is on how children from a quite young age can be introduced to algebraic thinking through generalisable numerical expressions. Our aim is to show that this fundamental aspect of algebraic thinking should be cultivated systematically at all stages of schooling.

There is a reluctance to introduce children to algebraic thinking in the early years of elementary school where the focus for almost all teaching of early number is on developing a strong foundation in counting and numeration. Yet Carpenter and Levi (1999) draw attention to "the artificial separation of arithmetic and algebra" which, they argue, "deprives children of powerful schemes for thinking about mathematics in the early grades and makes it more difficult for them to learn algebra in the later grades" (p. 3). In their study, they introduced first and second-grade students to the concept of true and false number sentences. One of the number sentences that they used was $78 - 49 + 49 = 78$. When asked whether they thought this was a true sentence, all but one child answered that it was. One child said, "I do because you took away the 49 and it's just like getting it back".

The importance of this example for the present paper is twofold. First, it shows that young children, when they are provided with rich material, can engage in quite insightful algebraic thinking. Second, its focus is away from computation. We are deliberately suggesting to teachers that they refrain from treating the numerical expressions we present later in this paper as computation exercises. Computation may be useful in re-assuring children that a particular sentence is true, but the goal is to focus children's attention on the underlying mathematical structure exemplified by that sentence. In the example used by Carpenter and Levi, their focus was not on having children calculate to verify that $78 - 49 + 49 = 78$, although some children might initially need to be reassured that this result is correct. Nor is their example tied to the particular numbers 78 and 49. While they do not use the expression "quasi-variables", we think this term is useful in drawing attention to the fact that the general relationship being exemplified in the above number sentence does not depend on the particular numbers being used.

Numerical expressions and quasi-variables

It was never the intention of Carpenter and Levi to introduce first and second-grade children to the formal algebraic expression, $a - b + b = a$. These children will certainly meet it and other formal algebraic expressions in their later years of school. What Carpenter and Levi wanted children to understand is that the sentence $78 - 49 + 49 = 78$ belongs to a *type* of number sentence which is true whatever number is taken away and then added back. This type of number sentence is also true whatever the first number is, provided the same number is taken away and then added back. We refer to *this* use of numbers as quasi-variables. By this expression, we mean a number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are. Used in this way, our contention is that quasi-variables can assist children to identify and discuss algebraic generalisations long before they learn formal algebraic notation. The idea behind the term "quasi-variable" is not a new one in the teaching of algebra. In his history of mathematics, Nakamura (1971) introduces the expression "quasi-general method" to capture the same meaning. In the same text, Nakamura attributes the use of quasi-general methods to Freudenthal. (We have not been able to locate this original reference.)

We argue that the use of quasi-variables can provide an important bridge between arithmetic and algebraic thinking which children need to cross continually during their elementary and junior high school years. The concept of a quasi-variable provides an essential counterbalance to that treatment of algebra in the elementary and junior high where the concept of an unknown often dominates students' and teachers' thinking. As Radford (1996) points out, "While the unknown is a number which does not vary, the variable designates a quantity whose value can change" (p. 47). The same point is made by Schoenfeld and Arcavi (1988) that a variable *varies* (p. 421). The use of numerical sentences to represent quasi-variables can provide a gateway to the concept of a variable in the early years of school.

The use of frame words in the Japanese curriculum

Frame numbers are often used in Japanese textbooks to signify an unknown number. At grade 3, children are introduced to sentences such as

$$18 + \square = 30 \quad \text{and} \quad \square - 17 = 25.$$

In grade 4, two frame words in the form of a square and a circle can be introduced to denote two numbers that together add to ten. While students readily understand that several number pairs satisfy this relationship, most interpret the problem as having to find missing numbers that satisfy the given relationship. We recognise the usefulness of frame words at this stage of schooling as a means of helping children to consolidate understanding of number facts. However, it is also clear the frame words produce problems for quite a few children. For example in research carried out by Fujii (2000), when fifth and sixth grade students were presented with the sentence using frame words,

$$\square + \square + \square = 12,$$

many responded by giving combinations of numbers such as (1, 1, 10) or (2, 4, 6) to fill the empty frames. It was not clear to these students that the repeated frame might represent the same number. Some would argue that they have quite correctly interpreted the frame as an "empty box" which they have to fill. Teachers frequently use expressions like, "Put a correct number in the box". As a result, it is not surprising that frame words give rise to interpretations that are inconsistent with the later use of literal symbols. Whether one agrees or disagrees with the use of frame words, it is important that teachers ensure that they are not used *exclusively* in contexts where students are asked to find, or decide to guess, a missing or unknown number. Their use must also include contexts that provide children with opportunities to engage in algebraic generalisations such as: $\square + \square = 2 \times \square$.

Using quasi-variables

The problems that occur with the use of frame words do not occur with numerical expressions embodying the use of quasi-variables. In the following three examples we show how in quite different contexts students can be introduced through numerical expressions to the concept of a variable.

Task 1. The following example uses students' knowledge of area relationship. Suppose that the large square has side length 8 units, and the small square a side length of 2 units. The area of the large square is 64 units, and the small one is 4 units.

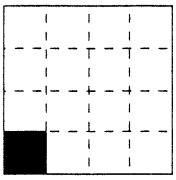

Figure 1

The relationship between the two areas can be expressed as:

$$8 \times 8 = 64$$

$$2 \times 2 \times 16 = 64$$

This can be re-written as

$8 \times 8 = 2 \times 2 \times 16$ which shows that the large square is composed of sixteen smaller squares, a fact that students can deduce from the diagram.

In this sense, we are using 8 as a quasi-variable, since the relationship does not depend on having a square of side length 8 units. If the side length of the larger square is 10 units, the relationship can be re-expressed as:

$10 \times 10 = 2.5 \times 2.5 \times 16$, or even better as $2.5 \times 2.5 \times 4 \times 4$.

Students may verify the correctness of the result if they need to. But the point of the example is not on requiring them to compute the answer. Students can demonstrate using geometrical reasoning that the underlying mathematical relationship is true, provided the small square has sides that are one quarter of the large square. They can show that the relationship holds when the large square has its side length expressed as a fraction or decimal number. They can investigate other cases where the relationship between side length of small and large square is one third, or some other ratio. This kind of treatment is feasible in the elementary school. At a later stage, students will be able to write the relationship using a proper variable, such as a, giving a fully generalised relationship $a \times a = a/4 \times a/4 \times 4 \times 4$; and, at an even later stage, $a \times a = a/n \times a/n \times n \times n$.

At secondary school numerical expressions continue to have an important role of signifying variable quantitative relationships. The next example could be at lower secondary level, to be used to foster an understanding of algebraic generalisation.

Task 2: This task is also taken from a typical textbook problem in the fifth grade of primary school in Japan (Tokyo Shoseki, 1994). Students have learned to calculate the circumference of a circle using 3.14 as an approximation to π. They are asked to compare the length of a semicircle of diameter 60 units with the lengths of three semicircles whose diameters are one third of the large semicircle. A typical calculation would be:

$$60 \times 3.14 \div 2 = 94.2$$

$$20 \times 3.14 \div 2 \times 3 = 94.2$$

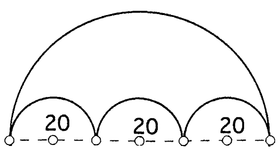

Figure 2

This answer is complete if the focus is on showing both lengths equal to 94.2. Seen as an exercise in applying a formula and calculating the result, that is usually the end of the story. However, the second numerical expression can be seen to be identical to the first by combining 20×3 to be 60.

If there are four semicircles resting on the large diameter, the second line of the calculation becomes

$$15 \times 3.14 \div 2 \times 4 = 94.2$$

If there are ten semicircles, the calculation becomes

$$6 \times 3.14 \div 2 \times 10 = 94.2$$

These uncalculated numerical expressions illustrate the property that, regardless of the number of identical semicircles situated on the diameter of the large circle, the sum of their lengths is always equal to the length of the large semicircle. These quasi-variable relationships allow students to *comprehend* the general relationship of the *type*

$$60/n \times 3.\ 14 \div 2 \times n = 94.2$$

where n is the number of identical semicircles situated on the diameter. Our *immediate* goal is not to have students write this formal expression, although some children may know what this expression means. We want students to be able to articulate the underlying relationship in their normal language. By considering other cases where the diameter is not 60 units, students can be helped to see that the relationship holds for all semicircles. They can also show that the relationship between length of large and small semicircles is true even when non-identical semicircles are constructed on the diameter of the large semicircle.

Task 3: The third task is drawn from a ninth grade textbook where students apply their the knowledge of Pythagoras to recently learned properties of tangents to circles (Kodaira, 1992). The textbook task asks students to find the length of the diameter in the following problem:

In the figure 3, a line has been drawn tangent to a semicircle of diameter AB, intersecting it in at point P. C and D are the points at which this tangent intersects the lines perpendicular to diameter AB at points A and B. If AC = 16 cm and BD = 25 cm, how long is diameter AB?

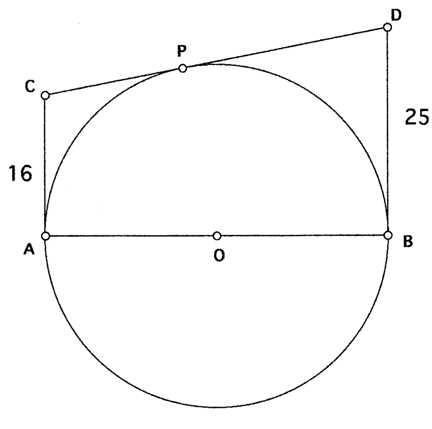

Figure 3

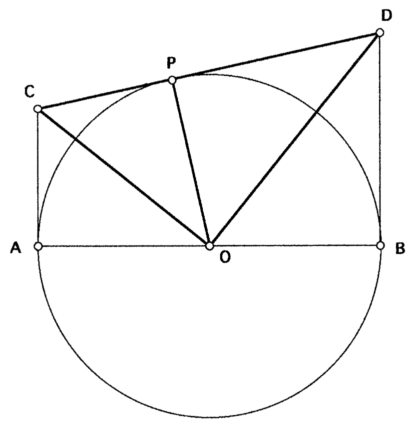

Figure 3a

A typical solution from students could be:

CD is a tangent line to the semicircle, therefore AC=PC and BD=PD.

CD = 16+25 = 41. Now, take a point E on the segment BD that BE =16. DE =25 –16 = 9. On the right-angled triangle CED, CE^2 (= AB^2)=DC^2- DE^2.

Therefore $AB^2 = 41^2 - 9^2 = 1681 - 81 = 1600 = 40^2$, AB =40.

This solution is quite appropriate to a standard textbook task. However we can explore the nature of the task more deeply, in order to express the qualitative and geometrical relationship by using quasi-variables. We need to do this by retaining the given numerals 25 and 16, without calculation, to express the following relationship:

$$AB^2 = (25+16)^2 - (25-16)^2 = 4 \times 25 \times 16 = 40^2$$

where 16 and 25 are lengths of AC and BD respectively.

Suppose the lengths of AC and BD are 10 and 40 units respectively, the same result is found:

$$AB^2 = (40+10)^2 - (40-10)^2 = 4 \times 40 \times 10 = 40^2.$$

Students need to see that the lengths AC and BD are variable depending on the arbitrary point P. On the other hand, AB, the diameter of the semicircle, is constant. Therefore the product AC×BD is constant, in other words AC and BD are in an inverse proportional relationship.

In order to get insight into the underlying geometrical relationship it is important for students not to calculate the numerical expression. Using these uncalculated numerical expressions, as quasi-variables, in fact illustrate the geometrical relationship.

Using dynamic geometry could also help convince students of this underlying relationship. They can also discover that the underlying relationship does not depend on the length of the diameter being 40. They can do this by using other uncalculated numerical expressions for the lengths of the tangents to show other cases of the *type* :

$$AB^2 = 4 \times AC \times BD$$

In this way, they can deduce an underlying relationship between the tangents drawn from a variable point P and the length of the diameter.

When we perceive the geometrical nature of the task, we could go further: The reason why the product of AC×BD is constant is that three right triangles such as COP, DOP and CDO are similar figures, so that CP : PO = OP : PD. From Figure 3a,

$$OP^2 = CP \times DP = AC \times BD.$$

OP is the radius of the semicircle, that is AB/2.

Therefore $AB^2 = 4 \times AC \times BD$.

Concluding remarks

The concept of a *variable* is central to the development of high school algebra and to all subsequent study. Many students appear to experience difficulties in making a transition from working with *unknowns* to working with and understanding the concept of a *variable*. Much of the teaching of early algebra in the elementary and junior high school years appears to focus on working with *unknowns* with only a slight emphasis on introducing children to rich contexts embodying the concept of a *variable*. Many teachers, textbook writers, and syllabus developers have taken the view that introducing *variables* needs to wait until students have been taught formal algebraic notation. In our opinion, this far too late.

Use of numerical expressions as quasi-variables is one way of introducing students to the underlying concept of a variable long before they encounter formal algebraic notation. Many situations in arithmetic and geometry provide rich contexts for such a treatment. The use of quasi-variables in numerical expressions can assist students in the elementary, junior high and even senior high school to deepen their understanding of algebraic thinking. Using numerical expressions in order to illustrate variable relationships requires a shift of thinking away from calculation to investigating patterns of increasing generality and complexity.

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The Method of Analysis as a common thread in the History of Algebra: reflections for teaching

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This paper describes the main outline of a course for teachers on the teaching of algebra. The aim of the course was twofold: to make teachers aware of students' difficulties, and to discuss the nature of these difficulties. To place teachers in a different perspective, from which to look at didactic problems, we decided to use passages taken from the history of mathematics, without claiming that we were offering a course on the history of algebra. We chose the method of analysis as a common thread. This topic proved to be an efficient way of approaching some critical points of algebraic thinking.

Introduction

This paper reports the outline of a course for in-service teachers on the teaching/learning of algebra. It does this from the perspective of using history in the mathematics classroom. Recent literature offers examples of this use within different contexts. In (Furinghetti & Somaglia, 1998) it is explained that the history of mathematics may be a suitable context in which the pupils' "intuitive ideas" may arise and develop. The role played by the historical context is examined by researchers from different points of view. Swetz (1989; 1995) discusses documents of the past suitable for classroom use and emphasizes that providing a feel of contact "with the past" is already a value in itself. Other authors, see (Grugnetti, 1992, Guichard & Sicre, 1988), discuss examples which suggest that the use of sources helps to contextualize mathematics, to focus on the mathematical concepts and to offer opportunities for interdisciplinary work. The basic idea underlying these different approaches is that the history of mathematics may be a strong tool to improve mathematical understanding. In particular, in (Jahnke et al., 2000, p.292) the efficacy of the use of original sources in teaching is advocated since this use promotes *replacement* ("Integrating history of mathematics replaces the usual with something different [...]"), *reorientation* ("Integrating history in mathematics challenges one's perceptions through making the familiar unfamiliar [...]") and *cultural understanding* ("Integrating history of mathematics invites us to place the development of mathematics in the scientific and technological context of a particular time and in the history of ideas and societies [...]").

Many researchers have stressed the importance of the history of mathematics in teacher education and training. We share this position, see (Furinghetti, 2000), for various reasons. The first is the obvious one that without a suitable knowledge the teacher is not able to deal with the use of history in the classroom. The second reason is that we feel that knowing the evolution of some mathematical concepts helps teachers to study students' learning process and, as a consequence, to overcome students' obstacles, where possible. As Sfard (1995) put it, "For those who teach, therefore, familiarity with the history of mathematics is not just optional; rather, it seems indispensable to make them alert to the deeply hidden difficulties concerned with new concepts". (p.34)

Given this theoretical position, we planned and delivered a course for in-service teachers on the theme 'algebra'. We looked to the history of mathematics as an epistemological laboratory to study students' difficulties in learning algebra and to better understand what algebraic thinking is[1].

In organizing our work we have singled out the following critical points in the teaching/learning of algebra:

- symbolism
- the relation between arithmetic and algebra
- the relation between geometry and algebra
- giving meaning to manipulation
  and, as a general theme,
- the obstacle of formalism (generalization, abstraction, ...).

The aim of the course was to make teachers aware of these critical points, exploiting the effects of replacement, reorientation, and cultural understanding provided by history, in particular when using original sources.

The Method of Analysis

The first problem we had to deal with was to decide which materials to choose for our purpose. To illustrate our problem it is appropriate to quote the metaphor of the famous mathematician Samuel Eilenberg. He has written that

> Among the most conspicuous trends in modern mathematics is the upsurge of "abstract" algebra. Almost every mathematical theory today has an algebraic facet. The structures with which modern algebra is concerned have been compared to the grin of the Cheshire Cat in *Alice in Wonderland*, which remained visible after the cat itself faced away. Algebra's power to generalize often leads to great economics. By emphasizing how seemingly different problems are basically alike, it suggests how the solution of one problem can be adapted to help solve another. (Eilenberg, 1969, p. 153)

This image applies also to the history of algebra. Algebra is always present, even if it is sometimes in shadow or appearing in other forms. To have a guide to identify the algebraic thinking in the history of mathematics we take the method of analysis as a common thread.

The method of analysis is very old and appears in the history of mathematics in different forms. We may see traces of it in the "rule of false position", where one starts from the assumption that the problem has already been solved (taking an arbitrary number as solution) and investigates the consequences of using this starting point. This is considered by Radford (1996) who discusses the problem presented in Babylonian tablet VAT 8389 (*c.* 1900 BC) and the Babylonian tablet BM 13901. Of course, when referring to such ancients documents there are problems of historical interpretation. We also note that the "rule of false position" is an interesting topic from the didactic point of view, see (Arcavi & Ofir, 1992).

A way of reasoning by the method of analysis is found in the famous passage on dialectics in Plato's *Republic* (4th century BC): a complete process of knowledge consists of a double path, from ideas to principles ("ascendant") and from principles to ideas ("descendent") and we can identify analytic thinking with the ascendant path.

In the same period, Leodamas of Thasos presents some scholia with reference to the Book XIII of Euclid's *Elements* in which he develops the demonstration with a double path, analysis and synthesis: in analysis we take as given what is sought, since from there we arrive at the truth which is to be proved; see also (Lelouard et al., 1990). The historian Paul Tannery states that these scholia are the first application of the analytic method.

Hintikka & Remes (Hintikka & Remes, 1974) clarify the role of analysis in relation with synthesis:

> Now analysis is the way from what is sought - as if it were admitted - through its concomitants [the usual translation reads: consequences] in order to something admitted in synthesis. For in analysis we suppose that which is sought to be already done, and we inquire from what it results,

[1] For a discussion of the characterization of algebraic thinking see (Lins, 1990; Lins, 1993).

> and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order. And we call such a method analysis, as being a solution backwards. In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequence the former antecedents and linking them one with another, we in the end arrive at the construction of the things sought. And this we call synthesis. (p.8)

The history of the application of the method of analysis and synthesis transcends the history of mathematics, see (Kleiner & Movshovitz-Hadar, 1997). We can say that analysis is the way of finding demonstrations of propositions or for solving problems and, in so doing, the analytic method contributed to the birth and the development of algebra. As Charbonneau (1996, p.36) put it, analysis is "at the heart of algebra". In fact, to solve a problem with algebra implies translating it into equations. This translation is made by assuming that the unknowns are known and by writing down the conditions which have to be satisfied. Afterwards the equations are solved through algebraic manipulations and so solutions are found. This is a process by analysis. The way of proceeding from the unknown to the known is the mark of the difference between arithmetic and algebra. This fact has been stressed also in studies of mathematics education, as shown in the following passage, taken from (Mason, 1996):

> Arithmetic proceeds directly from the known to the unknown using known computations; algebra proceeds indirectly from the unknown, via the known, to equations and inequalities which can then be solved using established techniques. (p.23)

A sequence based on the Method of Analysis as an approach to Algebraic Thinking

It is important to show students that in algebra, symbols and the method of working are two faces of the same coin and that algebraic manipulations do not exist by themselves, but as a function of the method. In what follows, we give examples of authors of the past who may help us to approach these ideas.

Diophantus

An example of the problems treated by Diophantus is the following (Problem 1 of Book I, *Arithmetica*, Ver Eecke, 1926, p.9, our translation):

> To divide a given number into two numbers with a given difference.
>
> So let the given number be 100 units[2] and let the difference be 40 units. To find the numbers, let the less be taken as 1 arithme. Then the greater will be 1 arithme and 40 units. Then the sum off of the two numbers becomes 2 arithmes and 40 units. But the given 100 units are this sum. Then 100 units are equal to 2 arithmes and 40 units. And taking like things from like: I take 40 units from the 100 and likewise 40 from the 2 numbers and 40 units. The two arithmes are left equal to 60 units. Then each arithme becomes 30 units.
>
> As to the actual numbers required; the less will be 30 units and the greater 70 units, and the proof is clear.

According to Radford (1995*a* and 1995*b*) the process used by Diophantus is different from that used by earlier authors in dealing with analogous problems. Diophantus introduces the new element, the "arithme", on which he performs computations as if it were a number. The arithme is always indicated with the same symbol (ς). Thus in Diophantus there is the unknown, but it is an unknown of the calculation, not the unknown of the problem. Radford uses the expression "operative analyticity" to indicate this way of proceeding by Diophantus.

[2] Diophantus uses symbols (for him "shortened designations") which are letters of the Greek alphabet and some other signs for units, square of units, cube of units, square-square, cube-cube etc.

Al-Khwarizmi

In his treatise of algebra, Al-Khwarizmi deals with second degree equations. For the equation (written in modern terms)

$$x^2 + 10x = 39$$

he finds the solving formula (confined to the positive value) which corresponds to the well known formula used in school at present. His process is described as follows. Firstly he considers the square of side x and the rectangle of sides x and 10, whose area altogether is 39. Afterwards he divides the rectangle into four rectangles of sides x and 10/4 and places them on the four sides of the square of side x. In this way a cross shape is obtained, whose area is again 39. If the cross shape is completed with four squares of side 10/4, a big square is obtained, see Figure 1, whose area is $39 + 4(10/4)^2$. The side of this square is $x + 2(10/4)$ and the formula giving x follows[3].

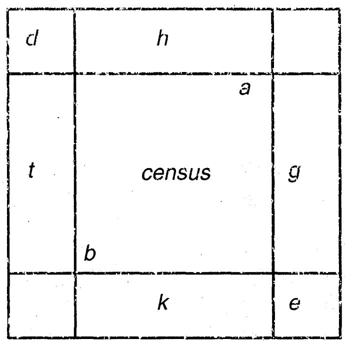

Figure 1: The figure to solve the equation $x^2 + 10x = 39$ as in the Latin version (1838) of Al-Khwarizmi's treatise

Even if second degree equations are studied by Al-Khwarizmi in a geometric environment, the solving process is of an analytical type. This process, indeed, starts with this statement

> The figure to explain this, the sides of which are unknown. It represents the square, the which, or the root of which, you wish to know. This is the figure AB [the square of side x], each side of which may be considered as one of its roots. (Al-Khwarizmi, 1831, p.13).

This passage says that Al-Khwarizmi starts his solving process by taking the unknown to be known. The following passage from (Jahnke, 1994) emphasizes some character of Al-Khwarizmi's process in relation to Euclid's geometric process

> Obviously, there is a certain similarity between this figure [Jahnke is referring to a similar problem] and that in Euclid. Al-Khwarizmi's figure, however, is much simpler, and its meaning is much easier to decipher. It does not aim at a geometrical theorem which would have to be formulated statically, but rather for an arithmetical or geometrical, in any case algorithmic, determination of a certain quantity. Al-Khwarizmi's figure must be read dynamically, and it is incomplete in the sense that it does not contain any equality at all. We must mentally add this equation. Al-Khwarizmi's figure has only an operative meaning, but this meaning is clear and unmistakable. (p.145)

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[3] See (Maher, 1998; Radford & Guérette, 1996) for other uses of Al-Khwarizmi's methods in classroom.

François Viète

Viète's fundamental rules of algebraic manipulation are:
• to transport of a term from one side of the equation to the other
• to suppress a factor common to all terms of the equation
• to divide all members of the equation by an arbitrary term

We may see the application of algebraic manipulation rules in the following problem taken from the first book of *Zeteticorum libri quinque*. The original text in Latin of the problem can be found in [1]. In (Viète, 1983, pp.83-84), which is the translation into English of the van Schooten edition (1646) of Viète's works, the problem is as follows:

> Given the difference between two roots [in Latin: *laterum* = sides] and their sum, to find the roots.

> Let B be the difference between two roots and let D be their sum. The roots are to be found.
>
> Let the smaller root be A. The greater will then be $A + B$. So the sum of the roots is $2A + B$. But this has been given as D. Hence
>
> $$2A + B = D$$
>
> and, by transposition,
>
> $$2A = D - B$$
>
> Having divided through by 2,
>
> $$A = 1/2D - 1/2B.$$
>
> Or [in Latin: *vel*], let the greater root be E. The smaller will then be $E - B$. Therefore the sum of the roots is $2E - B$. But this has been given as D. Hence
>
> $$2E - B = D$$
>
> and, by transposition,
>
> $$2E = D + B$$
>
> Dividing through by 2,
>
> $$E = 1/2D + 1/2B.$$
>
> Given, therefore, the difference between two roots and their sum, the roots can be found, for
>
> *Half the sum of the roots minus half their difference is equal to the smaller root, and [half the sum of the roots] plus [half their difference is equal] to the greater.*
>
> It is this that *zetetics* makes clear.
>
> Let B be 40 and D 100. A is then 30 and E is 70.

In the translation into modern language some notations are slightly different from those used by Viète, e.g. "$2A$" stands for "*A bis*" in Viète, there is not a sign for "equal", etc.

As observed in (Guichard & Sicre, 1988) this exercise may contribute to the reappraisal of algebraic manipulation, which in Viète's work is functional to his method of doing algebra. Moreover, the comparison of Diophantus's and Viète's processes provides hints about the role of parameters.

From (Viète, 1983, p.375) we take the following proposition presented in the part entitled *Effectionum Geometricarum Canonica Recensio* [A canonical survey of geometric constructions]. Note that van Schooten's notations are not those of Viète's original.

> Proposition IX
>
> If there are three proportional straight lines, the square of the smaller extreme plus the rectangle produced by the difference between the extremes and the smaller extreme is equal to the square of the mean. [We would say: "Given BF, FD, FC in proportion then $FC^2 + (BF - FC)FC = FD^2$.]
>
> Set up the standard diagram of three proportional straight lines and call FC the smaller extreme. Assume that BG is equal to it, whence FG is the difference between BF, the greater extreme, and BG (that is, FC) the smaller. I say that
>
> $CF^2 + (CF \times FG) = DF^2$,

for CF^2 is also the product of CF and GB. Thus the two products, $CF \times GB$ and $CF \times FG$, are equal to $CF \times FB$. Consequently DF^2, DF being the mean between the extremes, is equal to this product of the extremes.

Corollary on the Geometric Solution of the Square Affected by the Addition of a Plane Based on the First Power

If it proposed that

$A^2 + B \bullet A = D^2$

D is known to be the mean between [two] extremes and B to be their difference. The extremes are to be found from the mean and the difference between the extremes. The smaller of these will be A, the unknown.

Thus in this case the proportionals BF, FD and FC are constructed from the given terms GF and FD, and FC is the smaller unknown. We were able to show this in the Zetetica and it is now demonstrated by synthesis from a geometric figure.

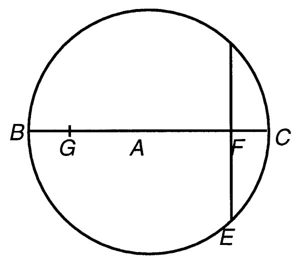

Figure 2: The figure for the geometric proof of Viète's proposition IX as in *Effectionum...*

We see here that algebra and geometry do not communicate directly, but only through the use of the proportion theory as an intermediary.

In the first page of *In Artem* ... Viète clearly places himself within the analytical approach, as evidenced by the following passage (Viète, 1983)

> There is a certain way of searching for the truth in mathematics that Plato is said first to have discovered. Theon called it analysis, which he defined as assuming that which is sought as if it were admitted [and working] though the consequences [of that the assumption] to what is admittedly true, as opposed to synthesis, [...]. (p.11)

Also Descartes adheres clearly to the analytical approach. He writes in *La Géometrie* (1954)

> If, then, we wish to solve any problem, *we first suppose the solution already effected* [our italics], and give names to all the lines that seem needful for its construction - to those that are unknown as well as to those that are known. (p.9)

In Descartes, the algebraic problems are again set in the geometric context, but the introduction of the unit of measure frees them from the need to use proportion theory.

Conclusions

At the beginning we mentioned that the path through history we have proposed in this paper was presented and discussed in a course for in-service teachers on the subject "The teaching of algebra". Our aim was to use history to focus on the critical points stated at the beginning. To choose the method of analysis has allowed us to enlarge the perspective with which the problems of teaching/learning algebra can be studied.

We gave the teachers participating in this course bibliographical information, including the paper (Barrow-Green, 1998) which provides an annotated list of resources where biographical notes on the mathematicians and the works quoted in our course may be found. A general survey of the history of algebra was provided through the web site [2]. But we stress that the focus has not been on giving a complete structured history of algebra, but only on the study of some moments which may establish links with algebra developed in the classroom.

In selecting and presenting the original sources we have followed the advice given in (Arcavi & Bruckheimer, 2000, p.72) that they "have to be used with discretion [...], supported by carefully crafted leading questions". Thus the historical materials we have used were chosen so that teachers could use them themselves directly in the classroom. During the course we suggested some links, which we discussed, between historical passages and critical points in algebraic thinking:

- Al-Khwarizmi and geometric reasoning
- Diophantus and Viète (*Zeteticorum* ...) to analyse the role of symbols and the meaning of parameters
- Viète (*Zeteticorum* ...) to give meaning and value to algebraic manipulation
- Viète (*Effectionum* ...) to see an example of the relationship between algebraic and geometrical frames.

It is also important that, thanks to algebra, we were able to focus on a method (analysis) which transcends different parts of mathematics as a method and which can also be a useful tool for modern students. To conclude, we feel that in our course the main message we addressed to teachers was the idea of looking at algebra as a *method*, and that algebraic manipulation is functional to this method.

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[2] http://www.lib.virginia.edu/science/parshall/algebra.html#BIOS (Here will be found the article Hunger Parshall, K. (1988). The art of algebra from Al-Khwarizmi to Viète: a study in the natural selection of ideas. *History of science*, 26 (72), 129-164.

GEORGE PEACOCK AND A HISTORICAL APPROACH TO SCHOOL ALGEBRA

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ABSTRACT

This article shows the contribution made by the work of George Peacock to school algebra. The author establishes what he calls "Arithmetical Algebra", a science of suggestion that excludes negative quantities restricting the study of equations and the validity of many general results of number theory. Peacock is trying that the limitation of considering only natural numbers and literals conveys the student to the urgent need of introducing the Symbolical Algebra.

INTRODUCTION

A research project was recently carried out whose central problem was the study of negative numbers in their interaction with the languages and methods used to solve equations and problems (Gallardo, 1994). The general methodology of the project dealt with the interaction of these categories on two levels, the historical and the didactic. The historical-critical analysis carried out in this work allows us to conclude that the presence of subtractive terms and the laws of the signs appear in remote times, as do the elements necessary for the operativity of signed numbers. It can be said that a crucial step in the recognition of these numbers is the acceptance of negative solutions. The main difficulty facing medieval mathematicians in the solution of concrete problems was precisely the interpretation of negative solutions. It is the exploration of these facts which has allowed us to locate the historical component of our work in the medieval period in Europe, between the XII and XV centuries. The empirical analysis is based in the preceding historical study of negative numbers in the resolution of algebraic equations. In this research, we identified different stages of conceptualization of negative numbers which appear both in the historical and didactical spheres. The stages are as follows: subtrahend, where the notion of number is subordinated to the magnitude; signed number, when a plus or minus sign is associated with the number; relative number (or directed number), where the idea of opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continuous domain; isolated number where there are two levels, that of the result of an operation or as the solution to a problem or equation. Finally, the formal mathematical concept of negative number is reached, where the same status as that of positive number is acquired (Gallardo, 2000). Once the empirical study is completed, the historical-critical analysis has been continued on the incidence of the negative numbers in the resolution of algebraic equations. The interest has focused in the work "Treatise on Algebra" by George Peacock (1830). This work gave place to controversies on negative quantities in England early in the 19th century. A critical reflection on Peacock's work has provided new analysis elements to explain the difficulties that current students bring up in the solution of equations and problems. In this article, some of the main ideas of the Peacock's work are presented, making emphasis in the construction of the

"Arithmetical Algebra", which is the main axis in the understanding of the transition from Arithmetic to Symbolical Algebra.

A HISTORICAL APPROACH TO SCHOOL ALGEBRA

In the 18th century mathematicians defined mathematics as the science of quantity, and the negatives were considered as "quantities less than nothing" and "quantities obtained by the subtraction of a greater quantity from a lesser". Neither definition was logically satisfying. Furthermore, the latter definition raised another problem – that of the definition of the subtraction operation. (Pycior, 1981, 28)

Early in the next century, Buée, suggested that there were two kinds of algebra: (1) universal arithmetic, in which the signs **+** and **—** stood only for addition and subtraction; and (2) a mathematical language, in which **+** and **—** also represented "qualities". Buée explained that "whenever one has as a result of an operation a quantity preceded by the sign **—**, it is necessary, in order that the result has a meaning, to consider some quality. Then algebra must no longer be regarded simply as a universal arithmetic, but as a mathematical language [...] Likewise, when one has said that a negative quantity is less than zero, one has once more in view his second meaning [**+** and **—** considered as qualities rather than as signs of addition and subtraction]; because it is not the quantity which is less than zero; it is the quality which is inferior to nullity. For example, if my debts exceed my assets I am poorer than if I had neither assets nor debts". (Buée 1806, 25)

In the first half of the 19th century, a number of English mathematicians focused their attention on formulating a clear interpretation of the foundations of algebra. They were faced with explaining the status of entities, like the negatives, which had no place in the quantitative subject matter of mathematics. The work of George Peacock is usually viewed as the first contribution of the foundations of algebra which was developed and revised by a number of his colleagues in the decades following its first publication. While a tutor at Trinity College, Cambridge, Peacock published two works on the foundations of algebra: "A Treatise on Algebra" (1830), followed in 1833 by a "Report on the Recent Progress and Present State of Certain Branches of Analysis".

In his Report of 1833, Peacock explained that the universal arithmetic could not be accepted in place of algebra because

> *"there were a great multitude of algebraical results and propositions, of unquestionable value and of unquestionable consistency with each other, which were irreconcilable with such a system, or, at all events, not deducible from it; and amongst them, the theory of the composition of equations, which Harriot had left in so complete form, and which made it necessary to consider negative and even impossible quantities as having a real existence in algebra, however vain might be the attempt to interpret their meaning. (Peacock 1833, 190-191)".*

On the other hand, Peacock declared the first principles of algebra and the establishment of what he termed 'Arithmetical Algebra' and 'Symbolical Algebra'.

This is Peacock's conception: we pass from the arithmetic of positive numbers to Arithmetical Algebra simply by replacing the numerals by letters. But the results can represent only natural numbers, so that an expression of the form $a - b$, where b is greater than a, has no referent. Here the symbols are general in form but specific in value. In Arithmetical Algebra, he considered that all results including negative quantities which are not strictly deducible as legitimate conclusions from the definitions of arithmetic operations[1], must be rejected as impossible, or as foreign to this science. Then, we pass from Arithmetical Algebra to Symbolical Algebra by allowing the letters to range over any kinds of quantity whatsoever, not just positive numbers or even expressions including $\sqrt{-1}$.

Symbolical Algebra was a considerably controversial subject among British mathematicians of the 1830s and 1840s. This particular aspect of the Cambridge School's approach to algebra, their formalism, was repugnant to Hamilton among others. He desired to assign meaning to the 'impossibles' of Arithmetic, negative and imaginary quantities, via the Kantian intuition of pure time. He was after truth in algebra, and so demanded to know what was signified by its symbols. Hamilton saw algebra as the pure science of number, the number concept being generated from a mental intuition, so that the operands of algebra were not without referent, but rather represented mental constructs.

TREATISE ON ALGEBRA BY GEORGE PEACOCK

We are trying to analyze in this article Peacock's conception regarding the teaching of the current elementary algebra. Therefore, it becomes very relevant to understand the transition from the Arithmetical Algebra to Symbolical Algebra.

In the second version of his Treatise on Algebra (1845), Peacock separated Arithmetical from Symbolical Algebra and devoted the first volume only to the exposition of the former science and the second volume embraces the principles of Symbolical Algebra[2]. He is convinced of this separation *"for it is extremely difficulty when the two sciences are treated simultaneously, to keep their principles and results apart from each other and to obviate the confusion, obscurity and false reasoning which thence arise".* (Peacock, 1845, *iii*)

He considered this second version as entirely a new treatise, a textbook where the student who is familiar with the results of Arithmetical Algebra and with the limitations which it impose, will be in a condition to comprehend and appreciate the whole extent of the legitime conclusions and what is still more important, he will be prepared for the study of Symbolical Algebra. Peacock said that we are perpetually encountering in Arithmetical Algebra examples of operations which cannot be performed or of results which cannot be recognized, consistently with the definitions upon which that science is founded. It is very difficult in innumerable cases, to discover the impossibility of the operation or the inadmissibility of the result, before the operation is performed or the

[1] Whether expressed or understood, for there are never formally enunciated.

[2] The author defined symbolical algebra as "the science which treats of the combinations of arbitrary signs and symbols by means of defined though arbitrary laws" (Peacock, 1830, 71). But in the second version of Peacock's Treatise on Algebra (1845) the author does not include any statement of the arbitrarines of symbolical laws equivalent to those found in the first treatise.

result is obtained. For example, it is required to subtract from **7a + 5b**, the several subtrahends **a + 3b**, **3a - 2b** and **3a + 7b**. We apply the general rule of subtraction, which would give us **7a + 5b – a –3b – 3a +2b – 3a -7b =7a - a – 3a – 3a + 5b – 3b + 2b – 7b =7b – 9b**, a result which indicates that the final operation is impossible in Arithmetical Algebra.

This author wrote that it will be convenient for the sake of greater brevity in describing operations or in the statement of rules, to call all those terms which are preceded by no sign or by the sign **+**, positive terms; and all those terms which are preceded by the sign **—**, negative terms. It is important however that the student should keep in mind that no meaning is attached to the adjectives positive and negative in Arithmetical Algebra. Peacock shifted the emphasis from the meaning of symbols and signs to the law of operations.

As a matter of fact, Peacock did not need negative terms for subtraction[3]. He is forced to define them because they appear in multiplication and division in intermediate processes.

For example, divide $\mathbf{2ab + b^2 + a^2}$ **by a + b**.. The quotient is $\left(\mathbf{2b} - \frac{\mathbf{b^2}}{\mathbf{a}} + \frac{\mathbf{b^3}}{\mathbf{a^2}} - \mathbf{\&c}\right)$ and the process will never finish. The second term $-\frac{\mathbf{b^2}}{\mathbf{a}}$, in the quotient, is the quantity which, when multiplied into a, produces $-\mathbf{b}^2$. For $\mathbf{a} \times \frac{-\mathbf{b^2}}{\mathbf{a}} = -\frac{\mathbf{ab^2}}{\mathbf{a}} = -\mathbf{b^2}$, when we admit the existence of the negative term $-\frac{\mathbf{b^2}}{\mathbf{a}}$. In the course of this example and in those given in the text, Peacock explains

"*we are more or less compelled to consider the sign* **—** *as existing independently, and thus to advance beyond the proper limits of Arithmetical Algebra. The fact is that the mere use of general symbols and signs to connect them, however strictly limited in their primitive meaning and application, conducts us almost insensibly to a science of pure symbols, presenting forms of combination and processes which are both unintelligible and impracticable when considered solely with preference to their simple arithmetical use*". (Peacock, 1845)

On the other hand, Peacock realized that when numbers in arithmetic are added, subtracted, multiplied or divided, we usually obliterate, upon the conclusions of the operation, all traces of the original numbers, and make use of the final result only expressed by the nine digits and zero. But if we employ symbols to denote numbers, we cannot generally, unless in the case of the inverse operations with the same symbols, obliterate in the results the particular symbols which are involved in the operation or operations performed upon them. This fact gives birth to equivalent forms. He showed that looking for all the equivalent forms, we arrive to restrictions in Arithmetical

[3] "If we subtract **b** from **a**, the result is represented by **a – b**, which is the excess of the number **a** above the number **b**". Peacock (1845, 24)

Algebra. For instance equivalent forms in quadratics equations furnish ambiguous and unambiguous solutions. For example:

in **19x – 39 – 2x²** = **6x – 33**, two real and possible values of **x**, are **6** and $\frac{1}{2}$. The solution is consequently ambiguous. For arrived to $\frac{169}{16}-\frac{13x}{2}+x^2=\frac{121}{16}$. Extracting the square root of both members, we gets $\frac{13}{4}-x=\frac{11}{4}$, $x=\frac{1}{2}$. But if we reverse the order of the terms of the equation $\frac{169}{16}-\frac{13x}{2}+x^2=\frac{121}{16}$ we furnish an equivalent form of this equation and obtain the root **x** = **6**.There are two real and possible values of **x**. The solution is consequently ambiguous.

Also equations may be possible and impossible. In the equation $\frac{x+4}{x+1}-\frac{3x-12}{x+8}=6$, we obtain $8x^2 + 33x = 44 - 48$ which involves an impossible operation. The two roots of this equation in Symbolical Algebra would be **–4** and $-\frac{1}{8}$; they are neither of them quantities which can be considered or interpreted in Arithmetical Algebra. The solution of equations generally will require the aid of the principles of Symbolical Algebra.

We also find in number theory restrictions in results. For example, the sum of a decreasing arithmetical series is **36**. The first term is **12** and the common difference **2**. It is required the number of its terms. We replace in formula $S=\{2a-(n-1)b\}\frac{n}{2}$, S by **36**, **a** by **12**, and **b** by **2**. The four terms of the series being: **12**, **10**, **8**, **6** whose sum is **36**. But this series cannot be extended to **9** terms without the introduction of negative terms. The complete series of **9** terms, which Symbolical Algebra would furnish, is **12**, **10**, **8**, **6**, **4**, **2**, **0**, **-2**, **-4**, the algebraic sum of whose terms is equal to **36**, equally with the sum of the four first terms **12**, **10**, **8**, **6**.

Peacock moved from Arithmetical to Symbolical Algebra as follows:

> *The assumption ... of the independent existence of the signs* **+** *and* **—** *... renders the performance of the operation denoted by* **—** *equally possible in all cases: and it is this assumption which effects the separation of Arithmetical and Symbolical Algebra, and which render it necessary to establish the principles of this science upon a basis of their own: for the assumption in question can result from no process of reasoning from the principles or operations of Arithmetic, and ... it must be considered therefore as an independent principle, which is suggested as a means of evading a difficulty which results from the application of arithmetical operations to general symbols.* (*Peacock 1830, viii-ix*)

Alternately, later in the same work, Peacock wrote:

If, however, we generalize the operation denoted by **—**, *so that it may admit of application in all cases, we shall then find the independent existence of this sign which follow as a necessary consequence, and we shall thus introduce a class of quantities, whose existence was never contemplated in Arithmetic or Arithmetical Algebra ... This generalization of the operation denoted by* **—** , *is in reality an assumption, inasmuch as it is not a consequence deducible from the operation of subtraction as defined and used in Arithmetic and Arithmetical Algebra.* (Peacock 1830, 70-71)

Peacock had therefore introduced the symbol **—a** into Symbolical Algebra by assumption and without definition. So in this latter science the symbols are general in form and general in value. In Arithmetical Algebra, the operations are defined beforehand so that these definitions determine its rule of combination. An essential restriction remains, however: that Symbolical Algebra must include Arithmetical Algebra as a "subordinate science of suggestion" or "that science the laws of which suggested those of symbolical Algebra". This restriction was labeled the "Principle of Permanence of Equivalent Forms" by Peacock: *"Whatever equivalent form is discoverable in Arithmetical Algebra considered as the science of suggestion, when the symbols are general in their form, though specific in their value, will continue to be an equivalent form when the symbols are general in their nature as well as in their form*". (Peacock 1833, 199)

Symbolical Algebra adopts the rules of Arithmetical Algebra, but removes altogether their restrictions: thus symbolical subtraction differs from the same operation in Arithmetical Algebra in being possible for all relations of values of symbols or expressions employed. It is in Symbolical Algebra that we form and recognize the result, whatever it may be, without any reference to its consistency with the definitions. We are thus enabled to subtract **a + b** as well as **a – b** from **a**, obtaining by the unrestricted rule

$$\mathbf{a - (a + b) = a - a - b = -b}$$

in one case, and

$$\mathbf{a - (a - b) = a - a + b = +b}$$

All the results of Arithmetical Algebra which are deduced by the application of its rules, and which are general in form, though particular in value, are results likewise of Symbolical Algebra, where they are general in value as well in form. "*This principle,* wrote Peacock, *in my first Treatise on Algebra (1830), I denominated the principle of permanence of equivalent forms and it may be considered as merely expressing the general law of transition from the results of Arithmetical to those of Symbolical Algebra. Upon this principle we shall be enable to give a consistent interpretation to symbolical forms such as* **+a** *and* **–a** *considered with reference to each other. This principles followed in the discovery or determination of equivalent forms*".

The results therefore of Symbolical Algebra which are not common to Arithmetical Algebra are generalizations of form, and not necessary consequences of the definitions. Peacock pointed out that "*it is quite true indeed that writers on algebra have not hitherto remarked the character of transition from one class of results to the other*". (Peacock, 1845).

It is in the transition from Arithmetical to Symbolical Algebra, when the symbols or the conditions of their use, cease to be arithmetical, that the meaning of the operations and the quantities must be determined, not by definition, but interpretation. Because the results of symbolical addition and subtraction are obtained from an assumed rule of operations and not from the definition of the operation itself, it will be necessary to resort to an interpretation of their meaning. For example, the addition of a symbol preceded by a negative sign is equivalent to the subtraction of the same symbol preceded by a positive sign and inversely. Thus **a + (-b) = a – b = a – (+b)** ; **a – (-b) = a + b = a + (+b)**. It appears, therefore, that in the case of negative symbols, the operation of addition is no longer associated with the fundamental idea of increase, nor that of subtraction with that of decrease. On the other hand, the signs plus and minus, when prefixed to symbols denoting quantities of the same kind, cannot denote modifications of magnitude[4], but only such affections or qualities of the magnitudes represented, as are convertible by the operations of addition and subtraction: it is on this account that **—a** can admit no interpretation[5], as compared with a or **+**a, when **a** denoted an abstract number, to which no qualities are attributed. However, numerous are the cases in which negative quantities admit of a consistent interpretation. One first example of the existence of qualities of magnitudes will be in expressing the opposite directions on lines in geometry, and which constitutes one of the most extensive applications of Symbolical Algebra. Another example is the symbolization of property possessed and owed. If a merchant possesses **a** pounds and owes **b** pounds, his substance is therefore **a – b**, when **a** is greater than **b**. But since **a** and **b** may possess every relation of value, we may replace **b** by **a – c** or by **a + c**, according as **a** is greater or less than **b**. In the first case we get **a – b = a – (a - c) = c** and in the second, **a – b = a – (a + c) = -c** if **c** therefore express his substance or property when solvent, -**c** will express the amount of his debts when insolvent. And if from the use of **+** and **—** as signs of affections or qualities in this case, we pass to their use as signs of operation, then **a + (-c) = a – c** and **a – (-c) = a + c**. It will follow, that the addition of a debt **(-c)** is equivalent to the subtraction of a property **(c)** of an equal amount. It consequently appears that the subtraction of a debt, in the language of Symbolical Algebra, is not its obliteration or removal, but the change of its affection or character, from money or property owed to money or property possessed. Peacock added, "*the preceding examples of the interpretation of the meaning of negative quantities and the operations to which they are subjected, will be sufficient to show the student that the Symbolical Algebra is not unreal and imaginary*". (Peacock, 1845)

The interpretation of operations must be extended to the equal sign which connects the primitive expression and the result derived from it. This view of its general meaning will include as a consequence, arithmetical equality or algebraic equivalence, according as either one or the other of them may be shown to exist.

It is important to point out that Peacock makes sense of negative quantities and negative terms in Arithmetical Algebra. They become negative symbols in Symbolical Algebra.

[4] For it +**a** and –**b** continue to denote magnitudes of the same kind, they may be replaced by ordinary symbols of Arithmetical Algebra, such as **c** and **d**, when **c + d = a + (-b) = a – b** is always greater than **c – d** or **a – (-b)** or **a + b**, results which are contradictory to each other. (+**a** and –**b** can not denote magnitude of the same kind).

[5] He did not recognized **-a** as a negative number.

He never refer to these negative symbols as negative numbers. These negative symbols are generalizations of form in algebraic expressions and equations.

SUMMARY

In synthesis, the interest of this article has been to show the contribution made by Peacock's work to school algebra[6]. The second version of his Treatise constitutes a text book for students. Once the author has advised that algebra is not Universal Arithmetics, he introduces the study of Arithmetical Algebra separating it from Symbolical Algebra in order to give sense to all expressions that include literals only applying the rules of arithmetic operations. This fact restricts the study of equations and limits the validity of many general results of number theory. Peacock's intention is that students realize the limitations of the Arithmetical Algebra. Thus, the analysis of equivalent forms of algebraic expressions allows the author to pose the transition from the Arithmetical Algebra to Symbolical Algebra through the recognition of the independent use of the signs **+** and **−** that originate negative symbols and a reinterpretation of the meaning of addition and subtraction operations. On the other hand, qualities related to positive and negative symbols arise in geometric contexts and in commercial-type problems. These qualities are once again validated from the supposition of the independent use of the signs plus and minus, which do not have the other arithmetic-algebraic signs.

Once the historical-critical analysis of Peacock's work is concluded, the challenge to be pursued in the didactical setting can be to analyze the convenience to "create one Arithmetical Algebra" looking forward to the construction of a transition bridge between Arithmetic and Symbolical Algebra.

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[6] The Symbolical Algebra established by Peacock does not coincide with the current Symbolic Algebra.

Teaching Algebra in a Technology-Enriched Environment

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This paper presents a review of research reported between 1992 and 2000 in the area of algebra and technology. A total of 19 articles, classified into five categories, are reviewed. Overviews of the role of technology precede research papers involving graphics calculators, computer algebra systems, and the use of the computer as a tool, and a commentary on curriculum issues. It is apparent that further research and development is warranted into the nature of the algebra curriculum and the role that technology might play in the teaching of algebra.

It almost goes without saying that the increasing availability of technological aids is promoting change in the way in which algebra is taught. In this paper, recent research and commentary published in the area of teaching algebra in a technology-enriched environment is reviewed. All articles cited were identified during a search of the ERIC database from 1992 to 2000 using the two descriptors algebra and technology. With the exception of two articles from practitioner-focused journals (*The Mathematics Teacher* and *Australian Senior Mathematics Journal*), all articles were sourced from recognised research journals.

Three of the 19 articles provide overviews of the teaching of algebra and the part that technology might play. Seven have graphics calculators as their focus, five involve computer algebra systems and three are concerned with the role of the computer as a tool to support the development of algebraic concepts. The remaining article examines curriculum issues in the context of the availability of technology. Each of the 19 articles is summarised in this paper and it concludes with a synopsis of their outcomes and the identification of areas for possible future research and development.

Articles Reviewed

Overviews of the Role of Technology

The three articles that fall into this category all emanated from the Algebra Working Group of the Seventh International Congress on Mathematical Education held in Quebec City, Canada in August 1992. Two of the three, those of Harvey, Waits and Demana, (1995) and Arcavi (1995), were keynote addresses presented to members of the Working Group. In the third, Heid (1995) summarised the reactions of its secondary subgroup to keynote addresses presented to the Working Group by Anna Sfard, Alan Bell, and Bert Waits (on behalf of Harvey, Waits and Demena).

Harvey, Waits and Demana (1995) underlined the essential role of technologies in the new algebra curriculum proposed in the NCTM (1989) Standards. They projected that, as a consequence of the availability of technology (in particular the graphics calculator), the study of algebra in schools would focus most frequently on functions and relations and the study of structure, but not completely at the expense of generalised arithmetic and the study of procedures. It was envisaged students would probably solve problems in many different ways and that the role of the teacher would change.

The members of the secondary subgroup of the Algebra Working Group were availed an opportunity in their discussions to react to the keynote addresses of Sfard, Bell and Waits. Heid (1995), in her summary of these discussions, indicated that among the questions the subgroup began to confront were: what is the role of technology in teaching algebra?; and how can and should technology interact with what and how students learn about algebra?

In his keynote paper, Arcavi (1995) attempted to integrate and react to the broad range of issues raised by the Algebra Working Group and particularly in the Sfard, Bell, and Waits keynote addresses. He proposed that technology would assume a protagonist's role in the future development of school algebra. This was because it appeared to have the potential to change curriculum content and to drastically change the ways in which students learn algebra. However, he urged caution in enthusiasm for the graphics calculator, particularly with regard to availability to students, teachers' lack of expertise in its use, and the new learning difficulties its use revealed. Arcavi concluded that mathematics educators were much more knowledgeable about the teaching and learning of algebra than they were ten years previously, but that there was much more work to be done.

Graphics Calculators

Four of the seven articles that have graphics calculators as their focus, namely those of Milou (1999), Merriweather and Tharp (1999), Bethell and Miller (1998), and Slavit (1996), report studies conducted at the secondary school level. The other three, by Hollar and Norwood (1999), Mueller, Pedler, Anderson, and Bloom (1998), and Smith and Shotsberger (1997) were conducted at the college level.

Milou (1999) surveyed 146 secondary mathematics teachers about their use of graphics calculators in the classroom. He found the use of the calculators to still be controversial to many algebra teachers. While there was wide acceptance among high schools and Algebra II teachers, controversy remained among middle school and Algebra I teachers about the appropriateness of the technology. Milou also found algebra teachers were unsure of how to use the calculators in their teaching and that only modest changes were appearing in the algebra curriculum.

In their study with three grade eight general mathematics classes, Merriweather and Tharp (1999) investigated the effects of instruction with a TI-82 graphics calculator. The focus was on changes in students' attitudes toward mathematics and calculator use and changes in the ways they naturalistically solved algebraic problems. The authors reported that, when allowed access to graphics calculators, many students became more excited and involved in mathematics and able to solve problems they were not able to solve previously.

The Bethell and Miller (1998) study was of a single first-year-algebra student (Miller) who improved from an E to A grade with the assistance of a graphics calculator. His success was attributed to the calculator allowing him to explore mathematics without the supervision of an adult and to discover properties of rules, tables and graphs independently.

Slavit's (1996) study focused on the instructional practices of an experienced classroom teacher working with a group of 18 above-average-to-gifted Algebra II students. All students were required to have a graphics calculator. Periodic observations conducted over a year focused on classroom discourse, uses of the graphics calculator, the nature of instructional tasks, and the teacher's questioning patterns. Slavit found that the teacher modified his teaching strategies and the curriculum to accommodate the presence of the graphics calculator, resulting in a blend of traditional and alternative approaches to the teaching of basic elementary functions. The data also indicated the use of the graphics calculator was associated with higher levels of discourse in the classroom, with both higher-level questioning by the teacher and more active learning behaviours by the students evident.

Hollar and Norwood (1999) conducted their study with 90 college students enrolled in intermediate algebra. It was an extension of O'Callaghan's (1998) computer-intensive algebra study (summarised later in this paper) and investigated the effects of a graphics calculator approach curriculum. Students were divided into a treatment (graphing approach) and control group (no graphics calculator access). The authors found students in the treatment group demonstrated significantly better understanding of functions on all four sub-components of O'Callaghan's Function Test than did students in the control group. Further to this, there were no significant differences between the two groups on either a final examination involving traditional algebra skills or an assessment of mathematics attitude.

The implementation of the HP48G graphics calculator as a teaching and learning aid was the focus of Mueller, Pedler, Anderson and Bloom's (1998) study. Students involved were enrolled in an intermediate linear algebra unit at the undergraduate level. The implementation was evaluated in terms of actual student use of the calculator in their final examination and their perception of the usefulness of the calculators as an aid to their learning, as measured by a unit evaluation questionnaire given to the students at the end of the semester. On the ten questions in the final examination, usage of the graphics calculator ranged from 0 to 92% of students. Usage was greatest on questions involving the solution of systems of equations and matrix multiplication, together with the interpretation of results. Questions on which usage was 0% were deemed calculator neutral. Student responses to the evaluation questionnaire indicated the majority found graphics calculators helped them learn the material.

Smith and Shotsberger's (1997) study was conducted with a group of 147 students enrolled in a semester-long college algebra course. Two instructors each taught one section using graphics calculators and one using a traditional approach. Achievement data showed no significant differences between the treatment and control groups, nor did the use of graphics calculators negatively affect student attitude. Students were found to generally favour the use of the technology for an algebra class, although there was a wide range of opinions about the issue. Students using the graphics calculators identified graphing logarithmic, exponential, and higher order functions, and lines, as topics for which the technology was most helpful. The authors concluded that graphics calculator technology could be successfully integrated into college algebra

Computer Algebra Systems

Three of the five articles reviewed in this category (Lindsay, 1999; Zehavi, 1997; Mayes, 1995) concern studies that focused on the use of the *Derive* software package. Tilidetzke (1992) used a computer-assisted instruction (CAI) package developed by Addison-Wesley while, in the Nguyen-Xuan et al (1997) study, the *APLUSIX* learning environment provided the context.

Lindsay's (1999) study was conducted with a group of 70 undergraduate engineering students enrolled in a bridging mathematics subject of one semester of pre-calculus followed by a semester of introductory calculus. Classes consisted of 2 hours of lectures and 2 hours in the computer laboratory, using *Derive*, each week. Analysis of students' performance on a 'technology-neutral' question revealed students using a traditional pencil-and-paper approach were more successful than those using *Derive*. The computer algebra system students were marginally better at answering the part of the question that required them to sketch the graph of a cubic function.

Zehavi (1997) used *Derive* for diagnosis and remediation of basic difficulties in mathematics. Learning activities were developed, each directed at a different area of difficulty, for students in three grades from two junior high schools. Students in the first school, School A, were given a standard sequence of 12 activities in files prepared on *Derive* and based on student knowledge states the author had determined from previous studies. The other students, from School B, were prepared differential files for their teachers to assign 12 appropriate tasks based on their performance on a pre-treatment questionnaire and during the treatment session. There was a marked improvement for all grades, with an advantage to students who were given the differential diagnostic treatment. Zehavi concluded that the linking of conceptual and procedural knowledge within the same framework of tutorial activities using *Derive* appeared to have many advantages for acquiring and effectively using procedural knowledge.

In Mayes' (1995) study, 137 college algebra students were divided into seven groups , including four experimental in which the study of algebra was assisted with the computer algebra system Derive, and three control that were taught in a traditional manner. On a final measure of inductive reasoning, visualisation, and problem solving, students in the experimental group scored significantly higher compared with students in the control group. Although students in the experimental group maintained an equivalent level of manipulation and computation skills, there was a significant decline in their attitude toward the use of the computer in learning mathematics.

Four sections of college algebra students were involved in Tilidetzke's (1992*)* study, with each of two instructors teaching one control and one experimental class. The control classes were taught in the traditional way while, as part of their course, the experimental classes were taught three selected topics using a tutorial software package published by Addison-Wesley. There were no significant differences in mean scores on both a post and delayed post-test between the control and experimental classes of each instructor and for combined control and experimental groups. Tilidetzke concluded that the software package was as effective as classroom instruction on the three selected topics.

Nguyen-Xuan et al (1997)'s study examined the effect of feedback on learning to match algebraic rules to expressions, using the intelligent learning environment, *APLUSIX*. Two groups of grade 10 students were involved on separate occasions in learning to solve algebraic factorisation problems. Some of the system's comments regarding errors and one prompt regarding a particular factorisation rule were modified between the two experiments. When the same error comments were used in both experiments, there were similarities observed in the learning paths of the two groups. Small changes in comments resulted in small changes in learning paths, while significant changes in the prompts and error comments led to significant changes in the learning process. The authors concluded that the feedback provided within the intelligent learning environment had a significant effect on how the students learned to match formal rules to expressions.

Computer as a Tool

The three studies reviewed in this section employed the computer as a tool to support the teaching of algebraic concepts. O'Callaghan's (1998) study incorporated the use of the Computer-Intensive Algebra (CIA) curriculum package, Sutherland and Rojano (1993) employed a spreadsheet approach to solving algebra problems, and Smith (1996) used the computer-algebra system *Derive* to support formal instruction with linear programming.

O'Callaghan's (1998) study involved an innovative curriculum characterised by a problem-solving approach, an emphasis on conceptual knowledge, and the extensive use of technology such as symbol-manipulation programs. Students involved in the study were enrolled in a one-semester college algebra course and were divided into one experimental (CIA) and two traditional classes. Results revealed that the CIA students achieved a better overall understanding of functions. On the four component competencies of a function model developed by O'Callaghan they were better than students in the traditional classes at modelling, interpreting and translating but there were no significant differences for reifying. A higher percentage of CIA students successfully completed the course. They also exhibited significant improvements in their attitudes toward mathematics, had less anxiety about the subject, and rated their class as more interesting.

Sutherland and Rojano (1993) investigated ways in which students used a spreadsheet to represent and solve a range of algebra problems. Their study involved two groups of 10-11-year-old students, one in Britain, the other in Mexico. Both were engaged in two blocks of spreadsheet activities for approximately 12 hours of "hands on" computer time, extended over a period of five months. The authors found that the spreadsheet environment supported students in moving from thinking specifically to thinking generally, both in terms of the unknown and the mathematical relationships expressed in the problem.

Smith's (1996) study involved undergraduate mathematics students and data were obtained from group reports and interviews. As they worked through the different tasks involving linear programming, groups were directed to report their problem-solving methods as well as their numeric results. Smith reported that the use of guided exploration and conjecturing activities, supported by access to *Derive*, allowed students to discover mathematical principles and solve problems in ways they were unable to previously.

Curriculum Issues

Current issues, potential directions, and research questions with respect to technology and algebra curriculum reform provided foci for Dugdale et al's (1995) article. A central tenet was that the advent of computers and calculators in the classroom allows a new approach to algebra. Whereas historically there has been a focus on symbol manipulation, technology facilitates a focus on reasoning with a variety of representations and understanding the relationship between those representations. According to Dugdale et al, function should be established as the central theme of algebra, since "a technologically enhanced curriculum provides an environment in which students can acquire deeper understandings about functions." (p. 334)

In discussing future needs, the authors suggested there may be a greater need for successful implementation models for technological tools in general than for new tools. More attention should be directed toward creating model curricula for use with existing software tools, and toward doing the necessary research to provide guidance to new curriculum designers. They acknowledged that, while the recommendations they provide are not supported by a body of systematic research, they have been influenced by their experiences in developmental projects, isolated empirical studies, and personal use.

Conclusions

At the 1992 Seventh International Congress on Mathematical Education (ICME 7), important issues were raised about the teaching of algebra in a technology-enriched environment. While there was enthusiasm for the role that technology could play in the future (Harvey, Waits and Demana, 1995), a degree of caution was urged by Arcavi (1995). Arcavi was particularly concerned about student access to graphics calculators, to teachers' lack of expertise with them, and to potential learning difficulties inherent in their use.

Some eight years after ICME 7, to what extent have Arcavi's concerns about accessibility been alleviated? Certainly, the graphics calculator is more affordable and in more widespread use. But how accessible does it remain to students, even those living in first-world countries?

On the question of teacher expertise with graphics calculators, Milou's (1999) survey of algebra teachers found they were unsure of how to utilise them in their teaching. This suggests the pace of development of technology may have exceeded that of professional development of teachers.

With the exception of Milou's research (1999), the studies involving graphics calculators reviewed in this paper are, in the main, positive about their use. Calculators expanded the scope of problems that could be solved (Merriweather and Tharp, 1999), facilitated independent learning (Bethell and Miller, 1998), promoted higher levels of discourse (Slavit, 1996), helped college students in their learning of traditional topics (Mueller, et al, 1998), were particularly helpful for graphing (Smith and Shotsberger, 1997), and facilitated better understanding of functions (Hollar and Norwood, 1999). Furthermore, they did not have an adverse effect on the acquisition of traditional skills (Hollar and Norwood, 1999) and they promoted better attitudes toward the study of algebra (Merriweather and Tharp, 1999).

None of the studies identified in this review addressed the issue of learning difficulties associated with graphics calculator use raised by Arcavi (1995). As the use of these calculators becomes more widespread, this appears to be an area fertile for investigation.

The studies involving computer algebra systems produced somewhat mixed results. There were positives in acquiring and effectively using procedural knowledge (Zehavi, 1997); on measures of inductive reasoning, visualisation, and problem solving (Mayes, 1995); in the feedback provided by an intelligent learning environment (Nguyen-Xuan, et al, 1997); and the effectiveness of a tutorial software package (Tilidetzke, 1992). However, where students had access to *Derive*, pencil-and-paper approaches produced more favourable results (Lindsay, 1999) and attitudes toward the use of the computer declined (Mayes, 1995). A disadvantage that computer algebra systems suffer when compared with graphics calculators involves their portability. With computer algebra systems such as *Derive* able to be accessed on a palm-top computer, in the future decisions regarding purchase may come down more to an issue of comparative cost.

The authors of the three studies that incorporated the use of the computer as a tool were encouraged by the outcomes. There were positive effects with modelling, interpreting, and translating functions (O'Callaghan, 1998), students were supported in moving from the specific to the general in a spreadsheet environment (Sutherland and Rojana, 1993), and greater scope in discovering mathematical principles and solving problems was apparent (Smith, 1996). These studies underline that the use of existing computer software: symbol-manipulation programs, spreadsheets, and *Derive*, respectively, can be successfully incorporated into existing practices and curricula in order to facilitate student learning and conceptual development.

Dugdale et al (1995) raised some important issues about the algebra curriculum. Essentially, their three main recommendations were that: technology allows a new look at what algebra should be taught and how; functions should be the central theme; and a pause in the development of technologies in order that curricula and implementation models can be created could be advantageous.

As we look to the future, the broader issues of the role of technology in the teaching of algebra raised by the secondary subgroup of the Algebra Working Group at ICME 7 (Heid, 1995) still appear pertinent. Under the umbrella of the curriculum, the part that technology should play in its teaching, and how it can and should interact with what and how students learn about algebra, maintain their prominence for future research and development.

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Making Algebra meaningful to Pupils

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This paper considers how algebra may be made meaningful to pupils so that they see it is something which is worth learning about. In particular approaches based on games and problem solving are discussed.

Introduction :

Mathematics lessons are very often not about anything. You collect like terms, or learn the laws of indices, with no perception why anyone needs to do such things'.

Cockcroft(1982)

This statement remains for me one of the most powerful comments made in the Cockcroft report. It seems no accident that the two examples given are taken from algebra, an area of mathematics which many pupils find totally meaningless. This paper considers some ways in which pupils may be led gradually to find meaning in algebra. I shall argue that that pupils need to be offered activities which they accept as worthwhile . This is in stark contrast to the lessons of mind numbing meaninglessness that I have often watched in my role as a teacher trainer. My 'favourite' is the lesson in which the pupils first had to solve twenty equations of the type $x + 3 = 12$ and then 20 equations of the type of the type $x - 3 = 12$. Now if you know, structurally, how to solve one of each of these types of equation there is nothing, algebraically, left to learn from the rest. There is even little fluency to acquire!

Some observations from Hungary

I have been lucky enough to watch many mathematics lessons in Hungary and have almost always been impressed by the way in which pupils were able to use algebra and extract meaning from it. For example I saw a class of 8 to 9 year olds solving the problem :

Mother squirrel had a certain number of nuts in her store. She halved the nuts and added 5 nuts and took 19 nuts home. How many nuts were there in her store at the beginning?

A girl came out from the class and wrote : $x : 2 + 5 = 19$ using the standard Hungarian notation for division. She then explained, 'Before adding 5 I had 14. Before halving I had 28.' There was no insistence on notating the way in which the solution was obtained in an algebraic way as would usually be asked for in the UK. The method was *intuitive* (Keiran 1990) The process was clearly meaningful to the pupil. Additionally this was the only problem in the lesson which involved algebra. It was used naturally as a tool when this was appropriate, rather than being a technique which is rigorously practised for its own sake. The regular use of □ in equations and inequalities from early in the primary school and moving gradually to full algebraic notation, and the continual weaving of its use into lessons, seems to foster a natural use of algebra as a language for expressing ideas.

Fluency

Fluency in algebra is clearly important and I would concur entirely with Pimm (1995) :

For me fluency is about ease of production and mastery of generation – it is used also in relation to a complex system. 'Fluent' may be related to efficient or just no wasted effort.

It is often about working with the form. Finally it can be about not having to pay much conscious attention.

In a situation where pupils need to work on fluency and it seems that this can only be achieved by practice exercises, then I believe that if we impose this upon pupils we should take them into our confidence and describe such activities as the building of mental muscles. I was once asked to help a pupil who had just started A level mathematics in the British 16 to 19 system. Unexpectedly she found herself struggling. The previous curriculum she had followed had involved little manipulative algebra and the algebraic demands of the new level were just beyond her. For her the cure proved to be three extremely complex exercises from a very traditional algebra text book, explicitly used as mental muscle toning. She undertook this undoubtedly painful work and her algebraic problems disappeared. In a similar way there seems to be no alternative to building a robust fluency in multiplying matrices, one can get nowhere until the complex process is in some sense automated. However it is clearly to achieve fluency as the by-product of other activities. The two case studies I shall describe in this paper both seemed to produce fluency as a by-product.

This paper ignores the implications of the use of algebraic manipulators, which are not currently widely available in UK schools. I am concerned with the construction of meaning for and a willingness to work with algebraic symbols and this seems to precede the use of technology to carry out the more complex processes. Pupils need to be able to read mathematical symbols and understand the language of the symbolism before they are relieved of routine computation.

Think of a Number

In the context of creating meaning in algebra I see similarities between my first case study and the work of Hewitt (Hewitt 1994), analysed by Pimm (1995). The context is a pair of algebraic *Think of a Number* lessons in which he concentrates on ' *a sense of doing the opposite as a way of undoing or unwrapping linear equations. His focus is on the process of inverting operations rather than finding the 'correct' number.* (Pimm 1995).

The most salient comment in the context of this paper was quoted in Pimm (op cit.) from an interview with Hewitt.

> *They know what it is they want to do and now they are forced to work on the challenge of finding my number. The notation is learnt by sub-ordinating it to the challenge; the notation becomes the vehicle through which they can engage with it.*

In the language of this paper this seems to assert that the acceptance of the challenge is sufficient to give meaning to the task of understanding algebraic notation and thus to the notation itself. The significant difference from an ordinary algebra lesson is that the pupils' attention is not on the learning of the notation but on the higher level task which happens to involve using algebraic notation. It interests me that both in Hewitt's experience and the work I shall describe, it is possible to give meaning to the learning of algebra by using tasks which are themselves purely mathematical but which lift the notation off the printed page and invest it with a dynamic and a spoken aspect as it is used as a vehicle to create a challenge.

Using games to create algebraic meaning

My first case study of the creation of meaning in algebra was part of a piece of research which looked at the response of pupils to curriculum based games in mathematics. Over three years I spent a morning a week in a secondary school working with groups of 6 pupils in the 11 to 14 year old age range. Each group worked with me for 3 weeks. In order to minimise the disruption to the teaching, I undertook to produce games related to the general area being taught in their normal lessons, though often moving beyond what the pupils were actually doing in class. I found myself inventing many algebraic games and their effectiveness led me concentrate in the third year on algebraic games alone.

Since the introduction of the National Numeracy Strategy (1999), there is much emphasis in the UK at present on the teaching of mental arithmetic where computations are done in the head with only informal jottings made on paper if they are needed This is seen as a vital precursor to the establishment of formal methods. I was therefore very interested that when I asked questions about how they felt about the games, one girl told me that we had been doing *mental* algebra. This led me to reflect on the fact that this stage, seen as highly significant in the teaching of number, is usually omitted when teaching algebra. My games involved a lot of talking algebra, very rarely was anything written. It is natural when playing a game to be required to justify one's 'move' to other players and as the games were played it was noticeable that players would read the formula they were dealing with out loud. This I believe is one of the factors which starts to create meaning - because by being verbalised the symbols become a part of familiar every day language in a way which is never achieved when they stay silent on a printed page. It should be noted that Hewitt's work also involved the speaking of algebra. Pimm (1987) supports this view :

> *Articulating aspects of a situation can help the speaker to clarify thoughts and meanings, and hence achieve a greater understanding.*

Another aspect of games which seems to be significant in the construction of meaning for algebraic symbols was that their meaning was seen as part of the rules of the game and thus something which could be queried without loss of face. It fascinated me that the 'rule' most often queried was that 2x means 2 times x. The use of brackets and the use of long and short lines for division fell into place very quickly in comparison with this. I created games which involved equations and expressions such as :

$$\frac{24}{x+2}=3 \qquad 2\left(\frac{N}{2}-8\right) \qquad \text{and} \qquad \frac{30}{N-5}+10$$

In the game context Year 8 (12 year olds) handled these with little difficulty. Even the difficulty of handling quotients with the variable on the bottom line seemed to be minimised. My inference is that the game context enabled them to see the language as worth working on because it enabled them to play the game. In Hewitt's language the use of the symbolism was sub-ordinated to the need to plan a move or acquire a score in the game. Another significant element was that two pupils worked together as one player so that there was always an opportunity for discussion and peer tuition if needed.

Target games

I found that games could also help pupils to create deeper meaning for algebraic symbols. I used several versions of a target game in which the players each held a formula card, a target was announced, for example *the highest number* or *an even number*, and a dice was thrown, this throw to be used as the value of the variable for all players. At the beginning I thought it was a somewhat routine game which involved only substitution, but the pupils suggested a rule in which they were allowed to change their card once only after the announcement of the next target but before the throw of the dice. They perceived that this would allow them to evaluate the scoring potential of their current formula. This changed the nature of the game completely. All players were required to display their formula cards so that each pair could look at the properties of their own function on the domain created by the dice in use, and compare it with those held by others in order to judge what to do. From a research point of view it revealed that at first when the target was 'Not a whole number' the pupils had no sense that they needed, since all the inputs were integers, a formula involving division. Another useful target was 'A negative number' although pupils were much quicker to pick out what was needed here.

A particularly effective version of this game was based on formulae which were all of the form ax + b. Now pupils began to look at 2x+1 and say that it would never be even or at 3x + 5 and say that it would be even half the time. They were able to recognise that 6n + 3 would always give a multiple of 3. Probabilities of scoring with a given card were also calculated. Sometimes the players can work out that it is unnecessary to throw the dice to discover who has scored a point. For example when the target was the smallest number and the cards held where 2n −1, 2n + 3 and 3n + 1 it is clear that 2n−1 is the winning card. On one occasion when the possible dice scores were −6 to + 6, a boy spontaneously announced that the values for his card lay between −30 and 34. I can now see, but have not yet had the opportunity to test with pupils, a possible whole class game in which the pool of targets is declared at the start and each pair of pupils is allowed to select perhaps 4 formulae, here, of course these are effectively functions, one of which must be chosen when each target is announced. It seemed to me that pupils who played this game were well on the way to seeing a function *structurally* rather than *operationally* (Sfard 1991)

Solution of equations

Much has been written about the problems caused for pupils when their view of an equation is limited to the idea that it just states the result of a computation rather than being a statement of equivalence.(Sfard 1994) This, of course, has its roots in a limited interpretation of the equals sign. This causes difficulty once the equations can no longer be solved using inverse operations, but need to be seen as objects which can be manipulated. Sfard (1994) refers to this process as *'reification- our mind's eye ability to envision the results of processes as permanent entities in their own right'.*

I developed an equation card game in which the playing cards each showed an equation, which to begin with could be solved by inverse operations. Watching the initial stages of play, when pupils were using substitution methods before they devised methods of solution involving inverse operations, I had a strong sense that the fact that each equation was a physical object and so had an existence separate from a calculation, helped to begin the process of reification. One heard comments such as 'This one's a 4. I need two 2s and a 5'

In this simple form all pupils seemed able to develop an intuitive method, sometimes with an element of peer tuition to help them. I became very interested in whether pupils would be able to develop intuitive methods for the solution of equations when inverse operations were no longer sufficient. I had been very struck by the research by Whitman (1976) cited in Keiran (1990).This research showed that *'students who learned to solve equations only intuitively performed better than those who learned both ways in close proximity, whereas students who learned to solve equations only formally performed worse than those who learned both techniques. Whitman concluded that formal techniques tend to thwart students intuitive ability to solve equations.'*

Even the oldest pupils I worked with did not seem to have met the standard methods of solution for equations of the form 3x + 10 = 8x. So I decided to try and see what intuitive methods they would invent for such equations. The method they developed for doing this – informally, of course, since we were playing a game, was to see that the 10 had to balance the missing 5x, so they saw the equation as 3x + 5×2 = 8x and deduced that the answer was two. The amount of time I had to spend with them meant that I could not explore further whether they could develop the process so that they could solve e.g. 3x + 19 = 8x by writing this as

3x + **5** ×19/5 = 8x. But it did appear quite a small step to move them to writing in a formal setting that they knew that 5x = 10 and so on. Certainly within this game context they were able to infer meaning, make deductions and achieve a solution.

Algebraic operators

In this context I sought also to create experience of 'add 3x' or 'subtract 7x' as a meaningful operations to carry out on an expression. I hoped that this might then become a weapon in the solution of equations. I created a loop game in which the task was to take an expression like 2x – 3 and '*add 5x and add 7'*. To help both with the computation and the acquisition of meaning, I offered them a number line with a scale marked x, 2x ,3x etc. The hope was that this would connect with jumps on an ordinary number line and so reinforce the idea that x represents a number and that we can therefore jump along a number line in units of x. The feel of an expression changing dynamically was very strong and quite different from the textbook computation 2x – 3 + 5x + 7. By continuing the number line negatively it was possible to work out 2x – 3 'subtract 5x and add 7'. The separation of the x number line from the ordinary number line, used implicitly to calculate -3 + 7, seemed to support the idea that these were two different items which cannot be combined. However I saw no sign of connections being made between this idea and equations and I have no evidence as to whether this would have developed had more time been available to work with the two ideas alongside each other. The pressures of the UK testing and league tables meant that I could not have access to a group of pupils for a longer period.

In this particular piece of research work I was very much concentrating on pupil's acquisition of knowledge of and confidence in algebra as an abstract system. For many pupils the other aspect, that is the use of algebra as a tool for expressing the results of mathematical activity and above all for proof, is perhaps an even more demanding activity.

A problem solving case study

My second case study involved two exceptionally able boys with whom I worked for about an hour per week for the last two years of their time at primary school. They were thus 9 when at the start of the case study and 11 when it finished at the point when they went to secondary school. Tested informally by the primary school at the end of this time using the Key Stage 3 SATS (national tests designed for 14 year olds) they were both performing at level 8, the highest level achieved by 14 year olds who had not followed the extension syllabus. This case study must therefore be seen as a study of possible ways to work with more able pupils on their understanding of this second aspect of algebra.

I decided to work with the boys through problem solving, rather than accelerating them through the school curriculum. Where we touched on standard results, I avoided using conventional approaches. Their performance in the SATS was, I believe, as much to do with their development of thinking skills and their belief in their ability to solve problems, as to do with the actual content they had covered. The test showed them answering questions of types we had never tackled. The problems I used were chosen for variety, for the importance of the mathematical ideas within them, to extend their experience of the number system and to develop algebra and the need for proof. I tried also to ensure that ideas and techniques recurred. The examples I give here are chosen to illustrate the way in which they were able to use algebra fluently as a language and as a tool and to see it as a means of explaining and justifying. Meaning in algebra and a knowledge of algebraic notation arose almost naturally from their acceptance, with pleasure, of problems to solve, and their belief in the need to justify results. In this paper I can only give brief extracts from their development, highlighting their use of algebra.

In the first year of the study they worked on solving addition squares backwards, that is finding the outside numbers for:

+		
	9	15
	17	23

This problem is characterised by the fact any square has either no solutions or an infinite number of solutions. Michael and Martin rapidly discovered this and were able to appreciate the simple proof of the condition for there to be solutions. About a year later I offered them a similar problem in the form of an algebraic multiplication table.

X		
	p^2q	pr
	pq^2r	r^2q

Michael's immediate response was that this was like the addition squares and that he thought therefore that there would be lots of solutions or none at all. His confidence with algebra had led him to hypothesise that the nature of the solution would not change when multiplication was used.

At the end of the first year I decided to use some material from Hoyle and Healy's project on proof to see what level of understanding of the nature of proof they had reached. First I asked them whether they could prove that the sum of two even numbers is always even and they first responded without the use of algebra

Michael *All even numbers are multiples of two so if you add a certain amount of multiples of two then you're going to get another number that's a multiple of two. So that means that will be even.*

Martin *Yeah with two multiples of the same number you are always bound to get another multiple of the same number.*

Then I asked whether they could find an algebraic proof.

GH *If I said write that down in algebra do you think you could?*

Michael *We could try*

GH *What would you have to write?*

Martin 2n add 2k

GH *And what would you have to say about n and k to make it right?*

Michael and Martin (echoing each other) *They're whole numbers.*

After this promising start their algebraic technique broke down. This problem created an opportunity to work on factorisation of the form $2n + 2k = 2(n + k)$. Once this was sorted out they had a correct proof of the result. It was interesting to note that when they took the SATS mentioned above the phrase 'factorise' was meaningful to them for $7y + 14$ but when faced with $6y^3 - 2y^2$, Martin initially wrote $2(3y^3 - y^2)$ and then discarded this in favour of $6\times y\times y\times y - 2\times y\times y$.

We then moved on to some 'proofs (Fig 1) from the project.

Michael *Well what was Bonnie trying to do?*

GH *Trying to prove exactly what you've been doing, trying to prove that when you add up two even numbers you always get an even number.*

Michael *Well it's not really - she's given lots of examples but.......*

Martin *a lot of examples -*

Michael *It might just be coincidence*

Martin *There's a possibility of it being coincidence.*

GH *OK so you don't accept hers.*

Martin *Not quite.*

Michael *It's OK but it's not proving.*

Their response to Yvonne's version was also very interesting with Martin showing an awareness that sometimes the structure of a diagram can make it a general proof:

Michael *It's only one example .*

GH *But I suppose I said to you - what do you think Martin?*

Martin *I think I'd accept it. It's saying - these are going to be - these are even* *because there's like two layers . If it stated that it'd be clearer but you can like just take away the gap and put them all together and it's two even layers, so it's going to be an even number.*

GH *So why did Michael think it wasn't good?*

Michael *I thought it was like Bonnie's, because it's one example but now I see what Martin says.*

Martin *If it's - if it's. If you had the same number twice, like 5 in that case, it's going to be an even number. It's going to be an even number. if they did that I couldn't see anything wrong with it whatsoever.*

They were also quite unworried by the use of different letters in another proof:

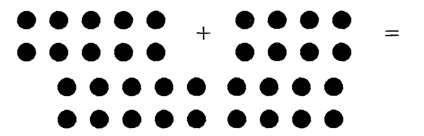
Martin *It's exactly what we've said. a is a whole number - a is any whole number, b is any whole number.*

Bonnie's answer

2 + 2 = 4	4 + 2 = 6
2 + 4 = 6	4 + 4 = 8
2 + 6 = 8	4 + 6 = 10

So Bonnie says it's true

Yvonne's answer



So Yvonne says it's true

Figure 1 Children's proofs

At about this time I gave them a coursework problem taken from the external examination for 16 year olds in the UK) It was about laying a letter T formed from 5 squares over a number grid written in rows of nine. During the time I spent with Martin and Michael we discussed the case with the T written the right way up. They did all the work and very rapidly we arrived at the proof for a grid of width x :

T – (2x + 1)	T – 2x	T – (2x –1)
	T – x	
	T	

T Total for the numbers is 5T – 7x

The problem was a vehicle for working on the difference between T – (2x – 1) and T – (2x + 1). They put in the brackets and this allowed us to check their understanding of them. Back in school, this was their explanation of the upside-down T and a grid of width x.

	32	
	41	
49	50	51

	T	
	T + x	
T + (2x - 1)	T+2x	T + (2x +1)

. *5T + 7x*

To get the marks in an examination I would need to teach them a bit about reporting on an investigation, but their grasp of the algebra is clear.

By the end of the time I worked with them algebra seemed to have become a natural language to them. This was demonstrated when Michael was explaining to me how he knew that 251 ÷ 0.1 was 2510. When I had difficulty understanding what he said, he switched to explaining the general case K ÷ 1/x. Finally I grasped that his argument was 1/x × □ = K, so we multiply by x to get 1 and then by K to get K, so the answer is Kx. I do not know whether he used algebra because he thought I would understand it better or because he thought he might as well do the general case. Either way it showed excellent understanding of algebraic notation .

On another occasion after they had solved a problem using trial and improvement, I asked if they could do it by algebra. I had to restrain them from replacing every number in the problem with a letter immediately and explain that I just wanted them to use algebra to find the number they had just found numerically. Nevertheless their response lead to us going on to solve later an equation for x in terms of three general coefficients, yet another algebraic development.

Martin and Michael are exceptional and therefore the problem solving approach used with them might not work as well in creating meaning for more average pupils. Nevertheless I have some hope that working at a slower rate it might do so. Crucial to its success is the willingness to accept relatively abstract problems as worth solving. In both case studies I used a game which involved using a set of pairs of simultaneous equations such as '*Two needles and three pins cost 12 pence*', and '*Three needles and three pins cost 15 pence*' written on separate cards to play a pairs games. Both groups found this meaningful and were able to develop gradually through versions of the game to solving equations like $4p + 2q = 26$, $7p + 4q = 52$. Both retained a strong sense of what the meaning of the algebra was throughout.

Similarly 11 year olds in the first case study were able to accept the type of puzzle which stated on three cards, three separate simultaneous equations in three unknowns e.g.

$2\square + \Delta = *$ $\Delta + 1 + \Delta = \square$ $\square + * + \Delta = 11$

Indeed this led to vast enthusiasm when two groups were working against each other to be the first to solve them. When I suggested that I did not have time to draw the shapes, they proved able to work with equations such as $2T + S - R = 9$

Conclusion

To summarise, I believe that pupils will make some meaning for mathematics whenever they are asked to work with it in a context which they accept as interesting. It is possible for this to be a highly abstract one provided it is accepted by pupils as worth working with. By using such approaches rather than the modelling of a word problem, like the Hungarian example quoted above we avoid all the translation problems which this produces for many pupils. When pupils use algebra to model and prove a conjecture of their own they are more aware of the structure of the problem and how the variables are related and therefore are more likely to be able to perform this symbolisation required to prove a result in which they have invested their efforts.

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An Examination of Educational Practices and Assumptions Regarding Algebra Instruction in the United States

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Two potential roadblocks to successful implementation of early algebra in the United States are examined. Systematic algebra instruction has traditionally been reserved for students who have achieved arithmetic proficiency and have shown evidence of having crossed into the formal operational stage as defined by Piaget's stage theory. Current theories regarding these assumptions are discussed. Three developmental models are contrasted with Piaget's theory. Several cognitive obstacles to algebra are analysed in relation to the arithmetic-before algebra expectation. Finally, the paper highlights the importance of dialogue regarding the mental models math educators hold about the appropriateness of early algebra.

Consider two children in 8th grade "Algebra 1," both intelligent and motivated. Ana breezes through the course with confidence and insight; Bonnie struggles with concepts, resorts to memorization of procedures and definitions, and fails all year with the task of developing algebraic thinking and understanding. In the past, U.S. educators would have considered Bonnie to have been placed in Algebra 1 too early–she should have spent another year in "pre-algebra" working on her arithmetic skills and maturing into the formal operational developmental stage (Piaget, 1972) that would allow her to succeed in the abstract, symbolic world of traditional Algebra 1. Today, influenced in large part by international studies such as TIMSS (Schmidt, McKnight, & Raizen, 1997), politicians and educational leaders are requiring all students to begin their study of algebra earlier than ever before. These new requirements for early algebra are in direct conflict with the conventional wisdom of our math educators, who have grown up in a system where the rigorous study of algebra is delayed until students are *ready*, usually in 9th or 10th grade, or in many cases, not at all. Curriculum and instruction decisions at the state, district, and classroom levels are being made based on conflicting philosophies, and these philosophic conflicts must be examined and reconciled before changes that are in the best interests of students can succeed.

Decisions about whether and when to place students in a formal algebra course have centered around teachers', counselors', and administrators' beliefs regarding developmental readiness and arithmetic competency. Educators trained in the Piagetian theory of stage changes have been reluctant to engage students in formal algebraic studies without strong evidence that they have made the stage change from concrete operational to formal operational (Case, 1991a). In addition, most math educators operate under the belief that students must have conquered most of arithmetic's complexities in order to be successful in algebra (Lundin & Bruton, 1999, p. 199; Nathan & Koedinger, 2000a; Nathan & Koedinger, 2000b). Substantial effort has been expended over many years to develop assessments that can distinguish students who have passed both the developmental milestone and the arithmetic hurdle.

Asked what it takes for a student to be successful in algebra, high school teachers often say: "Give me a student who knows his multiplication facts, fraction and integer operations, and who is developmentally ready for abstraction, and I can do the rest." Faced with mounting evidence that students from other countries begin their study of algebra before mastering any of these arithmetic topics (with, perhaps, the exception of the multiplication facts), it is time for our math researchers, curriculum designers, and math teachers to examine both assumptions—that students cannot (or should not) begin their study of algebra 1) until they have entered the formal operational stage defined in Piaget's developmental stage theory, and 2) until they have mastered arithmetic. The following two questions can serve as a frame for this examination and dialogue:

1. Is the acquisition of algebraic concepts and algebraic thinking a consequence of repeated experience over time, or is there a developmental shift of some sort that must occur before understanding can transpire?

2. How do understanding of arithmetic concepts and fluency with arithmetic procedures influence success with algebra?

Algebra Readiness: Developmental Considerations

Piagetian Stage Theory

Many children have difficulty learning algebra, even among the college bound who have traditionally been recommended for the year-long high school Algebra 1 course. One common explanation for the poor algebra achievement rates of 9th grade students is that algebraic understanding requires a stage change of the sort described by Piaget. Concrete operational children, according to Piaget and others, tend to be capable of mental operations as long as they relate to real objects, events, and situations (Wood, 1998). As they mature, they are able to work with more abstract concepts without the aid of concrete objects. Although Piaget (1972) postulated that most children transition from the concrete operational stage to the formal operational stage beginning around age 11 and continuing through successive levels until approximately age 15, there is some evidence that many adults never make this transition to generalization, abstraction, and metacognition (Pintrich, 1990, as cited in Sushkin, 2000). If this theory of stage changes is correct, it is easy to see why many children would struggle with algebra, which many U.S. textbooks and teachers present even to beginners in its most pure form, with few connections to context and concrete objects.

Educators trained in Piaget's theory of cognitive development most often structure their selection of cognitive tasks and curricula based on a belief system that children must be *developmentally ready* before they will be capable of understanding new concepts and mastering new skills. Teachers with this philosophical background have a predisposition to expect that arithmetic is the appropriate precursor to algebra, because arithmetic uses specific numbers that can easily be related to concrete objects and contextual situations. These teachers would expect that the systematic study of algebra at earlier ages would be ineffective because students are not cognitively prepared to handle abstraction. Thus, we can predict that teachers who have developed a Piagetian philosophy of education will be resistant to educational reforms that promote early instruction in algebra.

It is likely that a majority of U.S. teachers believe, consciously or unconsciously, in Piaget's stage theory. A survey of 30 5th and 6th grade California teachers conducted in January, 2001, found that 93% of these teachers remembered studying Piaget's stage theory, whereas only 17% indicated any familiarity with alternative developmental theories. Even for teachers who have not been explicitly trained in Piaget's theories, school structures and textbook expectations that implicitly rely on Piaget's theories send a "subliminal" message. Of the 30 surveyed teachers, 73% indicated either high agreement or moderate agreement with the position that "waiting to teach a new topic until children are developmentally ready for that topic will allow them to be more successful.

Observations of teachers in a California school district that implemented a series of algebra inservices for 5th and 6th grade teachers are instructive. Even after more than 25 hours of professional development designed to increase the algebra skills of the surveyed teachers and to provide engaging and age-appropriate lessons for their students, 40% of these teachers agreed that "children need to have entered the stage of abstract thinking before they can successfully master algebra concepts and skills." This belief system was demonstrated several times over the course of the inservices when teachers questioned the appropriateness of spending class time "introducing" algebra when there is clearly not enough time for students to learn all of the state-mandated math standards (which include algebra standards).

Beyond Piaget

Piaget's research and theories have had a profound impact on educational theory and practice in the United States. In addition to his stage theory, Piaget also contributed to the educational philosophy regarding the importance of student construction of knowledge. It is interesting to compare the acceptance (or lack thereof) that each of these two theories has achieved. Although many teachers struggle with the implications of constructivist educational theories, far fewer teachers take issue with his developmental theory. The concept that developmental readiness is determined by maturity seems to fit easily into the historical U.S. structure of universal education, whereby all children were provided a *basic* education in reading, writing and arithmetic, and fewer children were expected to study advanced topics such as algebra.

Unbeknownst to most teachers, however, research findings have demonstrated inconsistencies in Piaget's stage theory. Several competing theories have emerged which either attempt to build on Piaget's basic theory or are based on different assumptions than those of Piaget. Three exceptions to his stage theory demonstrate the reasoning behind current research efforts.

1) Researchers have found a large number of decalages, in which performance tasks predicted by Piaget's theory to be at the same stage of development are passed at very different ages (Case, 1991a). For example, children appear to reach the stage of concrete operations at age 8 or 9 based on a conservation of weight assessment; yet on a conservation of number assessment they appear to reach this stage at age 5 or 6 (Piaget & Inhelder, 1974, as cited in Case, 1991a). This *asynchrony* in cross-task performance has led to a schism in current theories of cognitive development, including the neo-Piagetian, neo-nativist, and information sciences theories discussed below.

2) Research in which children were provided instruction which enabled them to accomplish tasks at a new developmental stage found that this stage change was not universal across all knowledge domains for any particular individual (Case, 1991a). The realization that making a stage change in one knowledge domain does not necessarily transfer to other knowledge domains has caused many theorists to focus on development within specific domains, with little or no reference to "underlying logico-mathematical structures" that Piaget used to explain general stage changes. (Case, 1991b).

3) Cross-cultural studies have also produced data inconsistent with Piaget's stage theory, including evidence that adults in some cultures do not move to the formal operations stage (Dasen, 1972, as cited in Case, 1991a). Moreover, international mathematics comparisons, including the Third International Mathematics and Science Study (TIMSS) (Schmidt, et al., 1997; Silver, 1998) have shown that children from other cultures are successfully learning algebra concepts and skills at earlier ages than Piaget's stage theory would predict.

As a result of these inconsistencies, current developmental theories have been constructed that: 1) continue to build on the notion of general stage changes, or 2) suggest that developmental stages occur within independent knowledge domains rather than across them, or 3) view development as a continuum. These three different approaches to understanding learning and learning difficulties are exemplified in the following paragraphs.

1) Neo-Piagetians, who build on the developmental theories of Piaget, continue to posit some type of central, "domain-general" structure which exerts limitations on the cognitive tasks children can acquire prior to crossing into a new developmental stage (Carey, 1999; Case,

1991a). Each stage is posited to move an individual to a different way of thinking (Wood, 1998). These limitations are often explained in terms of working memory limitations and processing speed limitations which are thought to improve with age (Pascual-Leone, 1969, as cited in Case, 1991a). Most of the current neo-Piagetian theories, although continuing to include three or four proposed stages of development, also include structures that help explain the differences found in different knowledge domains, based primarily on the context and experiences of individual children.

2) Neo-nativists, building on the theories of Chomsky and Fodor, suggest that the human infant is born with a few modular domains of core knowledge such as knowledge of language, physical objects, persons, and space, already in existence. Those cognitive scientists who argue for the existence of various modular domains, both innate and learned, base their arguments on numerous experiments which show related stage-like changes occurring in the same individual at various ages. For example, a child may show a significant shift in understanding in the area of *naïve* physics months and even years before making the shift to a more mature understanding in the domain of *naïve* biology, or vice versa (Carey, 1988).

Within this neo-nativist school of thought, two theories of cognitive growth have emerged. One possibility is that cognitive development occurs through enrichment, achieved through perceptions of experience and reflection. This view of development, referred to as the "continuity hypothesis," relies on the assumption that the core principles and conceptual framework for any particular individual stay constant over time (Carey, 1999; Keil, 1988). The other possibility is that core principles are abandoned as individuals' experiences and reflections cause "incommensurable differences" between earlier concepts and new knowledge (Carey, 1996). When core principles are abandoned, new ways of thinking and reasoning can occur. These conceptual changes, if they occur, are referred to as "discontinuities."

Evidence for significant, qualitative cognitive shifts in the number domain is found throughout childhood, and may possibly even extend into adulthood. Probably the first candidate for a developmental discontinuity in the number domain is the shift from an infant's innate pre-linguistic representation of number (Carey, 1999; Dehaene, 1997) to an understanding of the count sequence that integrates linguistics with a symbol system. Other potential discontinuities in the number domain include the shift from counting strategies to the ability to decompose and recompose numbers in order to efficiently add and subtract (Fuson, 1992), the difficulty U.S students have understanding place value (Fuson, 1992), and the difficulties many students have enlarging their understanding of numbers to include fractions and decimals (Thornton, 1990, as cited in Fuson, 1992; Greer, 1992, p. 245-246).

3) Information science theorists, studying differences between novices and experts in any given knowledge domain, have focused attention on the importance of repeated and prolonged experience, as well as the connections between discrete pieces of information that allow domain-specific knowledge to be accessed in larger and larger *chunks*. Individuals with similar memory storage capacities will thus be more expert when their knowledge structures within a given domain are more extensive and more integrated, allowing them to identify underlying structures and hold and utilize more information in working memory because of the chunks of connected knowledge stored as larger units (Case, 1991a).

Rethinking Early Algebra Success Indicators – Experience, Context, and Culture

Today, theories are being proposed that attempt to integrate the most salient parts of the neo-Piagetian, neo-nativist, and novice/expert theories into one coherent model of developmental growth (Case, 1991b). Future research will help determine this next generation of developmental theories. Even before these theories are fully developed and tested, however, it is clear that experience, context, cultural traditions, and language (Wood, 1998, p. 253-256) are critical factors in cognitive development. Although the neo-Piagetians continue to theorize upper limits on learning and task performance due to maturational stages, even they seem to acknowledge the benefit of appropriate instruction in moving a child to the next stage within a particular knowledge domain (Case, Griffin, McKeough, & Okamoto, 1991).

Novice/expert theorists emphasize the critical role experience plays in developing a new area of expertise, and also of the connections that must be made between discrete pieces of knowledge in order to maximize the working memory limitations that exist at all ages, most particularly children under ages 15 (Jensen, 1998, p. 106). This expectation that learners must interact with new knowledge and skills many times, and must make connections between small knowledge units to create larger chunks of useable information and skills, seems to be widespread in the international educational community. However, U.S. math educators have a well-developed belief system that exposing children to algebra prior to age 13 or 14 will be of little or negative benefit because of cognitive limitations imposed prior to formal operations.

Evidence in support of this position appears abundant—relatively few students achieve solid success in traditional 9th grade Algebra 1 classes. However, considering that these students are typically age 14 or even older, Piaget's stage theory does not seem to be the appropriate culprit. More responsibility for the poor success rate in a typical Algebra 1 class may rest on the fact that students are expected to develop algebraic proficiency in one year, spending extremely short periods of time with each new algebraic concept or skill before moving on to new, and often unconnected, concepts and skills. Interestingly enough, we Americans do not expect our children to achieve expertise in music or sports in one short year—instead, we understand that experience and practice must begin early and must be distributed over many years to achieve mastery. The novice/expert theory provides support for the position that the average student needs much more experience with algebra and functions, beginning several years before that mastery is expected. In order for this to happen, mathematics textbook authors and elementary teachers need to examine their tacit assumptions about cognitive development. Piaget's four-stage developmental theory is not supported by data. Teachers, making decisions every day about which mathematical concepts to emphasize and which ones to skip, need current developmental information in order to consider early algebra instruction as a road to success for their students.

A Second Algebra Readiness Question: Arithmetic before Algebra?

Ninth-grade Algebra 1 students not only lack prior experience with algebra; they may also have developed an over-reliance on arithmetic and *means-end* solution strategies. As mentioned above, there is a widespread belief among U.S. educators that arithmetic competence is a prerequisite to successful study of algebra. Since most students continue to struggle with fractions, decimals and integers into 7th and 8th grades, their opportunity to begin serious and sustained algebra experiences is generally postponed until 9th grade or later. Yet evidence from other countries indicates that students can understand significant algebra and functions concepts prior to mastering the complete range of arithmetic skills and concepts (Schmidt, et al., 1997). An analysis of the major obstacles beginning algebra students face may help identify the relative importance of arithmetic skills to their success or failure. If the resolution of these obstacles requires considerable arithmetic mastery, then the U.S. system of arithmetic before algebra may be justified. If, however, these obstacles have little to do with arithmetical proficiency, this information will be useful as educators continue their dialogue about how to improve students' algebraic skills.

Obstacles to Algebraic Success

Educational researchers have catalogued many of the difficulties algebra students encounter, ranging from concatenation issues to problems with graphing intervals (Chaiklin, 1989; Herscovics, 1989; Kieran, 1989). Some of the difficulties researchers have detailed can be resolved fairly easily with additional practice and teacher awareness of the potential misunderstandings. Other difficulties seem to be long-term barriers that individuals find difficult to overcome, even with appropriate instruction and practice. These barriers may qualify, under the neo-nativist modularity theory of differentiated development within individual knowledge domains, as *cognitive obstacles*. Three areas reported to be pervasive obstacles for algebra students are described below:

Algebra Sign Systems

Many beginning algebra students have difficulty expanding their understanding of arithmetic sign systems to include the new sign systems attached to algebra (Kieran, 1989; Kieran, 1992; Wood, 1998, p. 239). One type of sign system difficulty students often experience is a belief that algebraic expressions such as "3x" represent a numeral "3" next to the substitute numeral for "x". For instance, when x equals 5, beginning students will often write "35". Wenger describes students with "extensive algebra experience ... [who] cannot seem to "see" the right things in ... algebraic expressions and have no clear sense of where they are going" (1987, as cited in Kieran, 1997).

A second significant difficulty related to the symbolic structure of algebra is that of generalizing numerical patterns. Lee and Wheeler (1987, cited in Kieran, 1997) report that less than 10% of 118 second-year algebra students were able to represent a verbal generalization in algebraic notation. When presented with a situation with related unknowns, such as consecutive numbers, students often assign each unknown a different letter (x, y, z), instead of using one letter and expressing the other unknowns in terms of the first (x-1, x, x+1) (Chevallard & Conne, as cited in Kieran, 1997).

A thorough explication of the many difficulties students experience with the algebraic sign system is beyond the scope of this paper. However, difficulties understanding the structure and utility of algebraic symbols do not appear to be dependent on arithmetic computation.

Transitioning from Means-End Strategies to Continuous Translation Strategies

A second major struggle for students is the need to shift from a means-end problem solving approach to a continuous translation and solution system for problem solving. For instance, many U.S. students will break a word problem into parts; the answer to each part will provide the foundation for solving the next part of the problem.

> "Daniel went to visit his grandmother, who gave him $1.50. Then he bought a book costing $3.20. If he has $2.30 left, how much money did he have before visiting his grandmother?"

Kieran (1992) reports that sixth graders commonly solve this problem using the equals sign to "announce" the next result: $2.30 + 3.20 = 5.50 - 1.50 = 4.00$. An appropriate algebraic translation for this problem would "tell" the story from beginning to end, and would place the unknown at the start of the problem: $(x + 1.50) - 3.20 = 2.30$. Neither textbooks nor teachers tend to emphasize this more continuous story translation before Algebra 1.

The shift in thinking that students must make in order to move from solving problems in steps (in a means-end strategy of getting one step closer to the solution with each step), to setting up the entire problem first, is dramatic. It may well qualify as a cognitive discontinuity in terms of modularity theory, as it requires an entirely new way of thinking about operations and equations. However, this discontinuity may not be associated with maturation, but may be more related to the current U.S. emphasis on means-end problem-solving in arithmetic word problems. This means-end approach is typically the problem-solving method of choice for U.S. elementary students and teachers alike (Ma, 1999). In fact, many U.S. teachers find it extremely difficult to write algebraic translations of similar word problems, without solving the problem arithmetically and then creating an inverted equation for the problem. Many elementary teachers would instead translate this problem as $(2.30 + 3.20) - 1.50 = x$. This reliance on *result-unknown* methodologies (Nathan & Koedinger, 2000a) does not appear to be a maturational issue, as researchers have reported that elementary students, with appropriate instruction, are capable of translating similar story problems into appropriate algebraic structures (Cai, 1995, as cited in Cai, 1998; Ma, 1999). Neither does it appear to rely on mastery of higher-level arithmetic skills.

Transitioning from Process to Object Conceptions

A third potential cognitive discontinuity is related to the second—students have great difficulty thinking of algebraic expressions as objects that can be manipulated as if they were individual numbers. Kieran (1992) argues that, "for most people, ... the transition from a 'process' conception to an 'object' conception is accomplished neither quickly nor without great difficulty. Sfard (1991, as cited in Kieran, 1992) suggests that this discontinuity, caused by a "deep ontological gap between operational [procedural] and structural concepts...," is related to maturation. Not only do many adults fail to cross this developmental hurdle, but the historical evidence also supports a centuries-long transition from procedural to structural understanding of algebra (Kieran, 1992). So although this obstacle may not be easily conquered, the problem does not seem to be caused by inadequate arithmetic skills.

Rethinking Algebra Prerequisites

The argument that children must first master arithmetic before they begin a serious study of algebra is not supported by the above analysis. The significant obstacles discussed above relate to sign systems, arithmetic means-end habituation, and the related need to gain procedural confidence prior to transitioning to structural understanding. These obstacles, at least, do not appear to occur because of arithmetic insufficiencies. They seem, instead, to rely on algebraic ways of thinking.

Although high school algebra teachers usually identify rational number operations (primarily integers and fractions) as a critical obstacle to algebraic success, it seems unlikely from the research documentation cited above that these arithmetic skills are the primary impediments to algebraic understanding. In general, teachers working with traditional textbooks do not identify algebraic conceptual issues as causes of student difficulty. For these teachers, algebra is the study of procedures, which students memorize and then imitate, with computation providing the main obstacle to success. When educational researchers, however, examine students' broad understanding of algebra in situations where students do not know which chapter is being assessed, their difficulties with sign systems, algebraic translation, and structural understanding are apparent.

More experience with arithmetic will not help students shift their thinking in these three areas. In fact, as evidenced in international studies (Schmidt, et al., 1997), students who experience algebraic thinking earlier are more likely to make these cognitive leaps. The question arises, then, as to whether so many years spent focusing only on arithmetic may not, in fact, cause an over-reliance on arithmetical understanding of mathematics. In either case, a strong argument can be made for systematic, ongoing algebraic experiences much earlier in a students' educational career.

Conclusion

The call in the United States for more students studying algebra, and more students studying algebra earlier, provides an opportunity and an incentive. We have an opportunity to examine the actions that are currently being taken in response to the mandates for higher levels of algebra achievement. These implementation strategies, and their short-term and long-term results, will provide a wealth of data regarding belief systems, leadership characteristics, and student achievement requirements. We have an incentive to educate teachers, math educational leaders, and math textbook authors and publishers, in the areas of cognitive development. Political and educational leaders have made it clear that they believe intensive teacher training in algebra skills and concepts is the critical element in successfully implementing higher algebra standards. However, to the extent that we disregard the philosophical beliefs of teachers who have grown up in an era of *algebra readiness* expectations, we travel down the road of many other well-intentioned educational reforms. Teachers are the ultimate decision-makers regarding which reform efforts are well implemented. Many of their decisions are based on deep-seated philosophical beliefs, of which even the individual teachers may be unaware.

Researchers and K-12 math leaders continue to propose "algebraic thinking" and "algebraic reasoning" as the appropriate topics for elementary students. Carpenter, Levi, & Farnsworth (2000) express what seems to be the prevailing sentiment:

> This study showed that young students can learn arithmetic in ways that provide a foundation for learning algebra. Recognizing young students' ability to reason algebraically does not suggest that elementary students should learn high school algebra. Rather, this study showed that a broader conception of algebra can be a part of elementary instruction that builds on students' implicit mathematical knowledge and increases their ability to understand, reason, and engage in challenging problem solving. (p.3)

This philosophy speaks to the ubiquitous belief that math educators should provide positive and successful arithmetic and problem-solving experiences for young children until they are *ready* for symbolic algebra. The mandate to provide our students with mathematicall opportunities equal to students in other countries is in conflict with our philosophical and historical traditions. When curriculum authors and elementary teachers read the new *Principles and Standards for School Mathematics* (NCTM, 2000), an awareness of their underlying beliefs and current research will assist them in interpreting passages such as this with open minds:

> All students should learn algebra. By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school. For example, systematic experience with patterns can build up to an understanding of the idea of function, and experience with numbers and their properties lays a foundation for later work with symbols and algebraic expressions (p. 37).

Important and far-reaching educational decisions are being made about algebra education in the United States without full awareness of the mental models educators hold about the teaching and learning of algebra. Senge (1990) includes awareness of mental models as one of the five disciplines required of any successful *learning organization*. "Many insights into ... outmoded organizational practices fail to get put into practice because they conflict with powerful, tacit mental models" (Senge, 1990, p. 8). If we want children in the United States to have the same mathematical opportunities as children in other parts of the world, math educators of all types (teachers, researchers, curriculum developers, staff developers, and educational policy-makers) need to engage in a dialogue about our mental models surrounding developmental theories and the arithmetic-before-algebra system. With or without this dialogue, changes in the expectations, curricula and teaching of algebra are going to occur. The goal of the dialogue, however, will be to ensure that those changes are thoughtful, informed, and well-implemented in classrooms across the country.

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On learning to adopt formal algebraic notation

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This paper considers some of the difficulties students face with algebra and differentiate between algebraic activity and learning algebraic notation. Through recognising that students already have experience of both algebraic activity and learning notation within their early pre-school learning, I offer approaches to the teaching of algebraic notation which help students become comfortable with using and interpreting relatively complex algebraic expressions.

Introduction

There are many difficulties which students face when learning algebra. The Cockroft report in the UK (DES, 1982) highlighted the fact that *algebra is a source of considerable confusion and negative attitudes among pupils* (p60), and Herscovics (1989) said that *algebra is a major stumbling block for many students in secondary schools* (p60). Difficulties with algebra have not been restricted to secondary schooling with the London Mathematical Society, et al (1995) raising issues of serious problems perceived in higher education including *the ability to undertake numerical and algebraic calculation with fluency and accuracy* (p2). Some particular difficulties that students experience have been reported such as students interpretation of algebraic expressions as a process to be carried out (Ginsburg, 1977), in particular the interpretation of the equals sign as signalling the requirement to 'do something' (Behr, Erlwanger and Nichols, 1976; Freudenthal, 1973; and Kieran, 1981). Sáenz-Ludlow and Walgamuth (1998) also discussed several different meanings which can be brought to the equals sign. The issue of students wanting to carry out a process also relates to Collis' (1974, 1975) identification of some students having difficulty with an acceptance of lack of closure. Sfard and Linchevski (1994) have discussed further the need for students to see algebraic expressions as both process and object, and the vital role that notation plays in the reification process. Such difficulties have led Filloy and Rojano (1984) to talk of a didactic cut between arithmetic and algebra.

Students can end up feeling as if algebra concerns a meaningless game played with various symbols where the roles are never really understood. In a lesson with 12-13 year olds (third set out of five) a student teacher was working on the way in which the multiplication sign is not usually written. He had 'a x b =' written on the board when one student suggested that the answer was c. Rules for what to do with the algebraic symbols can be invented by students, and often are when they are unsure of the meaning being placed on these symbols by a teacher. A little later in the same lesson the student teacher tried to explain why a x b = ab by saying that x is often used as a variable and so x x x x x (written on the board) would be confusing. I overheard one student look at this expression and say *times times times times times*. As Freudenthal (1973) commented:

> With the upmost patience teachers have tried to engrave in their pupils' minds that letters in algebra mean something, that no formula is meaningful unless the meaning of its components is told, and that algebra is not a meaningless game with 26 letters. It was to no avail. (p290)

I want to differentiate between difficulties with notation and cognitive difficulties in working algebraically. Some problems with algebra are actually problems about notation rather than algebraic activity. Goodson-Espy (1998) said that *In our attempts to explain student learning of algebra, we often find ourselves dealing with student interpretations of algebraic symbolism that are based in arithmetic notions* (p219). Thom (1973) commented that *the 'meaning' of an algebraic symbol is established with difficulty or is non-existent* (p207). For example, in a different class of students of a similar age, one girl, on seeing x written on the board, said that x is *a bloke with no head*. This statement reveals a meaning she gave to that particular symbol, rather than revealing anything about her ability to deal with the notion of an unknown or a variable. Likewise, another girl in the same class was looking at the following:

<u>Example 1:</u> $x + 3 = 9$ $x = 6$ <u>Example 2:</u> $x + 7 = 10$ $x = 3$

I asked her whether she understood what was written on the board and she replied *I don't understand, I just copy it down.* I then asked *What is the 'ex' on the board?* To which she replied *What 'ex'? That's a times.* Although I didn't ask at the time, I feel confident that she would have successfully told me my number if I had worded the first equation in Example 1 in terms of: *I am thinking of a number and when I add three I get nine, what is my number?*

Issues relating to notation have been highlighted by Tall and Thomas (1991), for example, who described the conflict between the left-right ordering of symbols in natural language and the not so straight forward ordering within some algebraic expressions, as the *parsing obstacle*. Whilst acknowledging this significant difference for someone who can already read and write, it also can be noted that young children learn arguably more complex rules concerning the ordering of words within sentences in their natural language. So why do issues of ordering appear to be such an obstacle for those same children now that they are teenagers in mathematics classrooms? Might this be accounted for by the difference in which the students are asked to work on ordering within mathematics classrooms as opposed to the way they worked out ordering of language as a young child? Common pedagogic approaches to teaching algebraic notation are significantly different to a young child sorting out sentence construction through little, if any, explicit teaching. Tall (1989) discussed empirical research on relative levels of difficulty of certain algebraic tasks (Küchemann, 1981), and how a different teaching approach (Thomas, 1988) can reverse these traditionally accepted levels of difficulty. Teaching approaches can significantly affect the difficulty students experience with their learning of specific topics. Before discussing further the issue of pedagogy, I will develop a way of viewing algebra which differentiates between algebraic activity and using algebraic notation.

What is algebra?

Freudenthal (1980) commented that:

> Many times I have pointed out that expressions like 'language', 'music', 'mathematics' mean not only a stock, the result of some activity, but also the activity itself. And though everybody would admit this is a triviality as far as language and music are concerned, it is not the same with mathematics. In fact, mathematics as a human activity is little known… (p3).

Rather than stress the results of mathematical activity, I wish to stress the activity itself. This stressing has implications for ways of viewing the task of teaching algebra to students as the educational task becomes developing the algebraist within each student, rather than helping a student come to know some external 'stock' which is the algebra curriculum. I will not ignore the existence of a curriculum which is expressed in terms of algebraic techniques or facts. Instead I will explore arriving at students knowing such a curriculum through the results of their own algebraic activity. Pimm (1995) commented: *the algebra takes place <u>between</u> the successive* *written statements and is not the statements themselves* (p89, his emphasis). This not only includes the algebraic awareness required to perform a manipulation of one algebraic statement into another, but also includes the work carried out leading up to the writing of an algebraic statement in the first place. For example, the expression $2n(n+1)$ may be the result of the task of expressing a formula for the number of 'matches' required to produce an n by n square of which the 5 by 5 version is indicated in Figure 1.

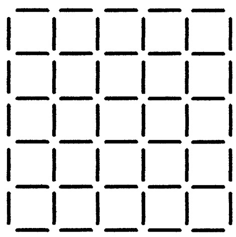

Figure 1. How many matches?

The final expression of $2n(n+1)$ is an algebraic expression, but the work carried out by students to get themselves into a position to write such an expression is algebraic activity. It may come from, for example, the result of looking at patterns of numbers from results of drawing different sized squares; or from a particular way of viewing the drawing above such as seeing it as a series of $n+1$ vertical lines, each one of which is comprised of n matches, and this being doubled since the same happens with horizontal lines. Mason (1987) describes this as seeing the general through the particular.

One way to help students to see the general through the particular, such as the 5 by 5 square in Figure 1, is to ask students how many matches there are in the figure. When they have worked out their answer, ask them not to give the answer but to say how they counted them. Given the number of matches, it is unlikely that the students will literally have counted them one by one. They are likely to have developed a particular way of counting. Describing this way of counting, and expressing the calculation involved without actually carrying out any calculations, is one step towards an algebraic statement. An expression of the arithmetic to be carried out, such as 2 x 4 x 5, reveals a structured way of seeing the matches. A general expression comes from the awareness that a particular structuring of the counting can be carried out in the same way no matter what the size of the square. Algebraic activity is concerned with structuring ways of viewing and ways of carrying out processes. Sfard and Linchevski (1994) commented that an *equation requires suspension of actual calculations for the sake of static description of relationships between quantities* (p211). The carrying out of a calculation is one step away from revealing algebraic structure.

Different ways of 'seeing' result in different algebraic expressions. For example, someone who sees the matches as a square array of L's - with each L comprising of two matches - added to one outside row and column of matches, might write $2n^2+2n$. Likewise a student, who has found a pattern in the numbers entered within a table, will write an expression which represents the arithmetic process they have noticed connecting the size of the square to the number of matches.

It is the work leading up to the writing of an expression which is algebraic activity. This same work is carried out by a student who ends up writing n x n + 1 x 2; or another student who verbally says *you multiply the size of the square by one more and then double it.* The difference between these students, and a student who writes $2n(n+1)$, only concerns the form of the expression and not the quality or quantity of algebraic activity. It is a matter of notation - the adoption of a conventional form of language - which is the issue here. Thus there are two, sometimes separate, tasks for students: one is to carry out algebraic activity; and the other is to adopt conventional notation. I suggest that the teaching of notation offers a separate pedagogic challenge to that of developing students' algebraic skills.

Algebra and young children

There are many academic papers which talk about the transition from arithmetic to algebra (for example, Slavit, 1999; Filloy and Rojano, 1984) and there are assumptions that arithmetic comes first and algebra is often built upon structures which are found within arithmetical work (for example, Linchevski and Livneh, 1999). However, I wish to challenge this notion. I have already argued that algebraic activity is concerned with the structures and processes involved in carrying out, for example, arithmetic. In Hewitt (1998) I argue that the successful carrying out of an arithmetic calculation requires a person already to have created a structure on the numbers involved in order to decide a process by which to carry out the calculation on those numbers. I argue further that algebraic activity is common amongst all children well before they enter formal schooling. The learning of their first language, for example, involves such activity as abstracting rules from examples and applying those rules in novel situations. Ginsburg (1989) said that:

> The same sort of thing happens when children learn to talk. They are immersed in the world of language when they hear people speak. This provides them with abundant evidence from which they must abstract the underlying rules of the language. From hearing "walked" and "talked" and the like, they must perceive a rule for constructing the past tense. The perception of such rules is required for efficient speaking. It is easier to remember a rule for the past tense than to remember the past tense of each verb separately. (p186).

Further evidence that children carry out this type of activity comes from them mis-applying these abstracted rules to irregular verbs such as *we goed to the park yesterday.* Children have not heard adults regularise these irregular verbs, and so this is something they have applied themselves. The ordering of words in sentences and transformation of sentences from statements to questions, all involve activity which would be described as algebraic if it were carried out by older students in a different context, such as finding number patterns and applying rules to new numbers. In fact I suggest that there is a lot which can be gained from recognising similarities between the formal algebraic activity of mathematics classrooms and the informal algebraic activity of young children going about their daily life. Such a shift in thinking leads to a recognition that all students have carried out algebraic activity since an early age and indeed continue to do so within their current daily life. I am not claiming that students have a meta-level awareness of such activity but that the natural algebraic activity of being a human being can be utilised and formalised in mathematics classrooms. Sáenz-Ludlow and Walgamuth (1998) pointed out that:

> Peirce (in Hoopes, 1991), Dewey (1938), and Hawkins (1966) noted that the learner's present experiences take up something from those which have gone in the past and modify in some way the quality of those which come after. Peirce calls it synechism, Dewey calls it the principle of continuity of experience, and Hawkins the cognitive history of the learner (p171).

The algebraic experiences students bring with them from their everyday learning as part of being human can be utilised when working on current experiences within a mathematics classroom. This contrasts with an image that algebraic activity or learning notation is something which is new and unknown to students, and as such ignores Dewey's notion of continuity of experience by presenting algebra as if it bore no relation to previous experiences in their lives. Young children are used to adopting apparent arbitrary notations, and a recognition of this in a teaching approach can help students deal with algebraic notation in a similar way to how they dealt with the written notation of their first language.

Pedagogic approaches to the teaching of notation

Notation is sometimes stressed when a student has found a rule within an investigation and is expressing that rule on paper. There is nothing else a student is doing at the time except trying to express that rule in a way told to them by their teacher. That algebraic expression does not form an integral part of some activity whereby it is going to be used within mathematical activity. It is a statement of an end point in mathematical activity. As such there is little need to conform to the quirks of a particular notation other than wishing to do as a teacher asks. The notation is not being used and practised to any great extent. At other times, students are given an equation and spend their time trying to work out what to do with it. For example, a boy in a class of 12-13 year olds was solving a linear equation and had written:

$$7x + 2 = 5 - 3x$$
$$7x + 3x + 2 = 5$$
$$10x = 5 + 2 = 7$$

He was then stuck. I wrote down:

$$10x + 2 = 5$$

I said that I agreed with his 10x but I noted that there was a '+2' before it equalled 5. He asked *What should I do next? Should I add 2 to 5?* I covered up the 10x with my finger and asked what was under my finger. I said that I know it says '10x' but what number is really under my finger? He had no problem in saying 3. So I said that the 10x must equal 3 and wrote:

$$10x = 3$$

He was then not sure what to do. He asked *Is it 3 divided by 10 or 10 divided by 3?* He ended up feeling that he would do 10 divided by 3. This is an example of someone who was working with algebraic notation but not sure what he was trying to do at certain stages.

An alternative approach is only to provide the written notation after students already know what they want to do with it. An example of this approach (Hewitt, 1996, for a fuller description) involves students being asked a carefully constructed sequence of questions along the lines of *I am thinking of a number, add 3, times by 2 and get 14. How can you get my number from the numbers I have given you?* This question is designed to stress operating with the numbers involved in the verbal equation and so helps students to focus on inverse operations. The detail of this part of the lesson I will not give here, but it is possible for students to feel quite confident that if I give a list of things I do with my number, they would be able to articulate what they would have to do to the answer to get back to my number. A significant aspect is that the lesson up to this stage is carried out purely verbally. Nothing has been written down. At the stage when students feel confident of saying how to find my number, the next sequence of operations carried out on my (potentially new) number is deliberately too long for them to remember. They then want me to write down what I have done. I write on the board the written equation equivalent to what I had said, in 'correct' algebraic notation (I note here that the equation is written in order of operations carried out on x, with each part of the equation being written at the same time as its spoken equivalent. This helps students to begin 'reading' the equation). For example:

$$3\left(\frac{2(x+3)-4}{5}+17\right)-7=50$$

Whether the students like it or not, this is all that is available to help them remember the sequence of operations. This may be the first time that the students have met such notation, and the first time they have met a letter in an algebraic context. So although the notation may be new, the algebraic activity of solving the equation is not. They are already familiar with the task of saying how to find my number (*solving the equation for x* in more formal language) and already know how they are going to do this. Thus the equation is immediately being read with a purpose. They are already aware of the significance of order and so begin reading the equation in terms of order of operations. As they say what they have to do to find my number, I write in formal notation what I am told:

$$\frac{5\left(\frac{50+7}{3}-17\right)+4}{2}-3=x$$

As I write at the same time as they say, and as they say in the order that the operations are carried out, I will write the expression in order of operations as well. Again, this helps students with their task of reading order into the notation. Rather than being confronted with algebraic notation and then working on what it means, the students already know what the notation is meant to represent before the notation is even written. So when it does appear, students have the different task of placing the meaning they already have into the arbitrary notation. In this way the notation is immediately being 'read' rather than just 'looked at'.

Quite young students can become familiar with relatively complex notation using such teaching approaches. A Masters student, Ann Boyle, wrote in an assignment about work she was doing with a class of 9-10 year olds following an approach similar to that outlined above: *In most cases children are using correct standard notation because they are now familiar with the conventions.* She gave an example of an equation produced by an average ability girl who passed on to her friend who the correctly wrote down the solution for x in one step:

$$\frac{(8x^2+10)^2-1}{2}=834 \qquad \sqrt{\frac{\sqrt{2(834)+1}-10}{8}}=x$$

Another approach to algebra where students can be familiar and confident with algebraic activity, and so know how to 'read' the associated algebraic notation, is in the software *Grid Algebra*[1]. This is based upon movement around a grid based on rows of multiplication tables (first row the one times table, second row the two times tables, third row the three times table, etc), although it is a matter of choice which, if any, numbers can be seen at any one time. A journey from one number in the grid to another represents a process to be carried out. So, for example, going from a number (say 6) in the 3rd row (three times table) to the number (12) two more cells to the right along that row, might be seen as 6+6 (going straight there) or even 6+3+3 (by going one cell to the right and then another cell to the right). An alternative journey with the same start and finish points could be 6+12-9+3 (see Figure 2).

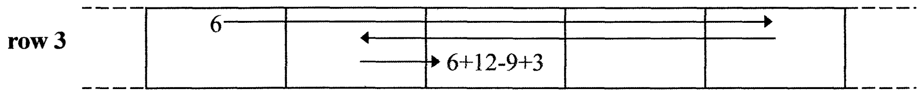

Figure 2. A three stage journey from one cell to another.

Students can pick up and drag a number from one cell to another, and what appears is an (algebraic) expression which represents the operation carried out on that original number. A series of such movements results in a series of operations on the original number, as in Figure 2. With the additional dynamic of moving up and down (division and multiplication respectively), more complex expressions can be built up through a series of movements (see Figure 3).

Because students make the movements themselves and can soon relate movements to operations, the notation which is provided by the software can be read and interpreted in the light of what they already know about the algebraic activity (the movements/operations carried out).

Since attention is with the operations and not calculations (hence algebra rather than arithmetic), it makes relatively little difference whether a student started with '5' in the original cell or something like 'x'. The software allows a letter to be placed in a cell and movements carried out so that expressions involving a letter can also be created. This is an environment where students confidence with the algebraic activity (explorations and challenges relating to making journeys on the grid) helps them to read the accompanying algebraic notation and hence students have little difficulty in creating and interpreting relatively complex algebraic expressions.

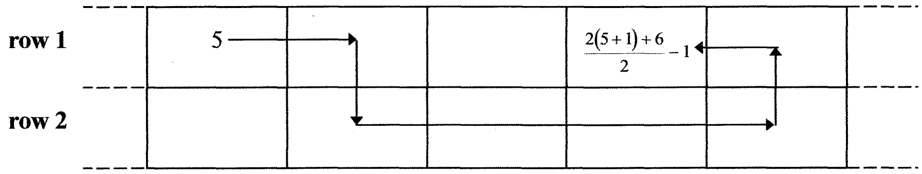

Figure 3. A journey involving five movements/operations.

Summary

I have indicated some of the common difficulties found with students learning algebra and raised the issue of whether some of these problems are concerned with notation rather than algebraic activity. By differentiating between these two, the different pedagogic issues facing teachers are stressed. Algebraic activity is concerned with awareness and thus a teacher is concerned with educating that awareness. Notation is a social convention and as such students need to be informed and helped with adopting the conventions. These two, quite different, tasks for a teacher can be supported by recognising that all students from an early age have successfully carried out algebraic activity, and have also successfully adopted a far more complex notational system in learning to speak, read and write their first language (amongst other things). I have offered two examples of pedagogic approaches to teaching notation which contrast with the way in which notation is often met for the first time by students. Notation is only provided once students have already involved themselves in algebraic activity and have confidence in making statements about that activity. Notation is then introduced as a medium through which statements and further work is carried out. The fact that students are already confident with the algebraic activity means that when the associated notation is introduced they already have meaning with which to interpret and 'read' the notation. In this way, notation does not have to be the confusing collection of symbols which is the experience of many students. Instead, students of a relatively young age and attainment can become confident with interpreting and using quite complex algebraic notation.

[1]The software *Grid Algebra* was developed originally at the Open University, UK for Archimedes and Nimbus computers. An updated version is currently being prepared. For details, contact the author.

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A scaffolding for linear equations.

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A framework of understanding in algebra is necessary to provide a scaffolding for teachers and draw attention away from the procedural approaches the outcomes based curricula of the nineties has encouraged. This paper suggests such a framework using linear equations as an example of one domain.

Algebra occupies a large part of the mathematics syllabi in schools for 11-16 year olds around the world. There have been changes to the curriculum over the years. Rather than having just a content list, curriculum in most countries has moved to include general principles such as application to real world problems and interpretation. For example the NCTM standards in Algebra (NCTM, 2000) requires the instructional program to enable all students to "Represent and analyze mathematical situations and structures using algebraic symbols" which is then exemplified by expectations such as "solve linear equations". In Australia changes to more general principles came with *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991), and following this most States developed new curriculum documents (e.g., Board of Studies, 2000). The National Statement provided a framework that divides algebra into the sub-strands expressing generality, functions and equations. The framework is an outcomes based structure that manages to give general outcomes showing broad development. Sweden has a similar set of general guidelines including modelling and problem solving. The curriculum, *LpO 94*, (Skolverket, 2000) revised in year 2000, points out the importance to work with algebraic concepts, expressions, formulas, equations and inequalities. As the new curriculum was presented in Sweden 1994 the National Board of Education supported a group of researchers and teacher educators to sum up the research so far. This was presented in a book, Algebra för alla (Bergsten, Häggström & Lindberg, 1997) to the curriculum users.

While these curriculum documents provides a useful scaffolding for curriculum development and reflects ideas of increasing understanding, they do not effectively provide a means of monitoring children's understanding of algebra in grades 7-10, based on the considerable work of researchers in the area of cognition in algebra. All three countries mentioned above have testing programs which translate these general statements into specific outcomes measured by paper and pen tests. This means that teachers often focus on the procedures rather than on the general relational understanding that the general principles espouse. What is needed is a framework built on the development of students' thinking and strategies rather than outcomes.

A framework which would be useful to teachers and researchers should reflect the findings of relevant research, should emphasise the big ideas, the structure of algebra and movement towards abstraction, and form the basis for planning and teaching enabling teachers to identify both improvements and difficulties for individual students with whom they work. In the Early Numeracy Research Project (Clarke, Sullivan, Cheeseman & Clarke, 2000) a framework of growth points was developed for numeracy. These growth points have been described as " key 'stepping stones' along paths to mathematical understanding. This does not mean that all children necessarily pass all growth points along the way but rather that the order is close to the "order in which strategies are likely to emerge and be used by children. . . .

intuitive and incidental learning can influence these strategies in unexpected ways" (Owens and Gould, 1999, p. 4). The growth points are moving towards a higher level of abstraction and generality within a number of different domains. As mentioned they are big ideas or concepts and they are not necessarily discrete so a student may operate at a number of levels depending on the context. There are also many possible "interim" growth points between them. As Clarke et al note "a child may have learned several important ideas or skills *necessary* for moving to the next growth point, but perhaps not of themselves *sufficient* to move there" (P 185). One major purpose of the framework was to provide a structure for teachers which focussed on understanding, moved towards abstraction, provided key growth points and above all was easy to remember and use in the classroom.

There are many parallels between the teaching of arithmetic and of algebra apart from the idea of the framework of growth points which have informed this paper. The framework is part of a larger project looking at the development of algebra sense in years 7-10. Algebra sense has been defined as an understanding of the objects of algebra and the different representations, and the ability to sense the form of the result of a particular process. It is not so much the ability to work with the objects to produce the required solutions as the ability to visualise the nature and form of the solution and to move readily between the representations or mathematical sign systems (Horne & Maurer, 1998). Just as number sense, estimation and mental strategies are part of arithmetic alongside the use of technolgy so algebra sense becomes critical with the technological advances in Computer/Calculator Algebra Systems (CAS).

The learning of algebra is a large field with many different groups of researchers who have focussed on particular aspects of algebra such as the modelling used when solving word problems (e.g. Lemut & Greco, 1998; MacGregor & Stacey, 1993, 1998), the understanding of the equals sign (e.g. Kieran, 1981; Pillay, Wilss & Boulton-Lewis, 1998), the move from arithmetic to algebra (e.g. Bednarz, Radford, Janvier & Lepage, 1992, Linchevski & Livneh, 1999), the translation from tabular form to symbolic form (e.g. Ryan & Williams, 1998; Warren, 1998), the solution of linear equations (e.g. Linchevski & Herscovics, 1994, 1996; MacGregor & Stacey, 1995) and functions and graphs (e.g. Herscovics, 1989; Garcia-Cruz & Martinon, 1998).

There are many different aspects associated with algebraic cognition. In algebra we operate with a range of mathematical sign systems such as graphs, algebra code and tables on a variety of mathematical objects (Filloy & Sutherland, 1996). Meaning is given as we translate within a mathematical sign system, between systems and between mathematical and non-mathematical sign systems and as we move towards abstraction (Kaput, 1989). For example a student learning about a linear function may see that initially as a table of values. Within this sign system the student can translate information and make generalisations. The student also develops meaning by translating between systems such as from the table of values to another sigh system such as a graph or a symbolic expression of the function. The translations between these systems and natural language is a third way of adding meaning. All the time the student is moving towards abstraction.

As the mathematical sign systems develop and the abstractions occur there is an interaction between many of the aspects of development. For example the understanding of the equals sign develops alongside understanding of aspects of linear equations in an interactive way rather than linearly. Within each area a hierarchical development can be seen. What is critical is that the development is moving always towards abstraction and recognition of the underlying structure. In order to monitor students' progress through algebra in the early years in particular, a framework is needed which highlights some of the growth points in development and shows the hierarchical structure while at the same time acknowledges the separate developments in different areas.

A number of possible domains with a related structure of growth points were proposed by Horne (1999). These included generalised arithmetic; concept of "="; identity; equations;

function-graphs; function-tables; inequalities; and modelling. In this paper one domain from that proposal will be presented – that of the solution of linear equations. Linear equations have been used to show a development from arithmetic through pre-algebra to algebra. Pillay, Wilss and Boulton-Lewis (1998) have developed a schema for this based partly on the cognitive load faced by the students. During the transition from arithmetic to algebra they suggest that the equals sign changes from meaning each side has the same value to equivalence. They also at the same time see equations moving from purely numeric to simple linear one variable equations to linear equations where there is more than one unknown or variable. Their structure recognises the increasing complexities of the tasks, but also increasing sophistication of the strategies students use. The structure presented here is based partly on their ideas. A few researchers have developed hierarchies for linear equations that indicate the development and the cognitive complexities of the situations. The understanding of the equals sign and the understanding of operations and the laws of arithmetic (and algebra) are two aspects critical to further development in algebra. Often success in algebra is hampered by poor arithmetical skills and understanding (Horne, 1994; Kieran, 1989; Linchevski & Herscovics, 1994). The development of operations and rules associated with arithmetic, and the development of understanding of the equals sign from a sign that an action is needed to an understanding of equivalence with symmetric and transitive properties, occur concurrently with interaction between these separate but related areas and other areas such as the solution of equations hence they have been included in another domain (Horne, 1999). Some support for the structure of growth points in the table can be found in Linchevski and Herscovics, 1994; MacGregor and Stacey, 1995; and Pillay et al, 1998.

Figure 1 below shows the proposed growth points for Equations. The descriptions indicate both the nature of the mathematics and the types of thinking the student may be using to do such a task. The levels are moving towards greater abstraction and generalisation.

Equations

1. Numerical understanding 1 step or two steps using box (Completion of fill in gaps in arithmetic (or use of box) in one step equation format.)
2. Solution of one step and 2 step equations using algebraic symbols with whole numbers by trial/knowledge of number facts
3. Use of backtracking to solve 2 or more step equations with whole numbers and inverse ideas
4. Solution of equations with non whole number solutions using the approaches in levels 2 and 3
5. Use of (inverse operations/) balance to solve 2 step equations
6. Simple equations with more than one occurrence of the variable on one side (solved by methods beyond substitution such as balance and inverse operations).
7. Simple equations with the variable appearing on both sides (solved by methods beyond substitution such as balance and inverse operations).

Figure 1. Sequence of growth points for linear equations

One difficulty is that within a particular area growth points are chosen to show an order. There will be other aspects of learning as well but the idea of these growth points is to choose key aspects of development. The growth in any area is from limited arithmetic and concrete understanding towards abstract and structural understanding. It is not clear for students when each of these areas begins and ends, nor is it always clear which growth points in one area should or do precede a growth point in another area. These need further research to accept or reject their position in such a heirarchy.

The critical difference between this approach and that used in curriculum and standards frameworks is that the concern here is not just whether the student can answer the question correctly but rather how the student approaches the task, what strategies they use and what understandings they show. Asking questions in interview illustrates this point.

In the interview a number of students in year 7 (aged approximately 12) were asked (among other tasks) to solve an algebraic equation of the form:

Q1. If $c = 5b + 2$ and c is 27, what is b?

The students were asked to explain both what they were asked to do and how they obtained their answers. This equation is transparent in that any-one understanding the nature of the question and with a good arithmetic knowledge looking at 27 = 5b + 2 can see the answer straight away without actually doing any calculations. Those who gave the correct answer to this transparent two step equation were then asked one in a similar form but with a non-integer solution.

Q2. If $g = 4f - 3$ and g is 6, what is f?

These questions were given to the students in written form, paper was provided if they wished to use it and a calculator was available if desired by the student. The interviewer was instructed to allow the student some thinking/working time then to ask them to explain their answers. It is important to realise that only those students who answered Q1 correctly were asked Q2.

The following approach by student L illustrates a student operating at level 2 on the proposed heirarchy.

L (In answer to 27 = 5b+2) b equals 5
I. (Interviewer) How did you get that?
L I times. They went up by 5's and then stopped at 25 and added 2 equals 27

Notice L did not have difficulty with the form of the question. Nor the reversal of the order but the approach was one based on knowledge of number facts, in particular the five times table. The more complex question requiring a non-integer solution led to L changing the operation of the 4f to addition so that the task could be completed showing clearly that while L could operate at level 2, on this day L was not operating beyond level 2..

L (In answer to 6 = 4f-3) I think 2, but it can't be, because if you multiply 4 times 2 equals 8 take 3 equals 5
I What else can you do now?
L 4 plus 5 equals 9 take 3 equals 6 and then the f equals 5
I That's if you make it 4 plus something, are you happy with that?
L Can't work it out with times.

It is not necessary for all students to use all approaches during their learning. While the idea of backtracking is mentioned in some syllabus documents, not all students need to use it. It seems to be more of a syllabus artefact than a necessary step in learning. Student V illustrates the class taught process of backtracking though V was the only student interviewed to fully use this approach and it was only used on the simpler problem.

V (In answer to 27 = 5b+2) See we've been doing equations or something so I tried to figure out there but I couldn't because like we go backwards so I did with a box number and I found out the number in here; it's a box story sort of thing and you set out like that but you don't have a real number and it's like that and then you put times 5 plus 2 and when you go back you go backwards and you put take away 2 because it's the opposite of minus you put divide in opposite of plus you put take away and then you get 27 take 2 equals 25 then divide that by 5 and you get 5. And then I worked it out here 5 times 5 equals 25 plus 2 equals 27.

Student A also approached the first task partially by backtracking but again with the second task could not manage the non-integer aspect so just removed the difficulty by removing the 4. Students were quite accepting of different possible answers to the same questions and changing the question to a form that suited them when they had difficulty solving the question as asked.

A (In answer to 27 = 5b+2) I didn't do this very well. This is how our teacher told us how to do it; c equals 27 so b equals 27 take 2 equals 25 or you could have b times 5 or 5 times 5

...

I. How did you get that?

A I did a rule that my teacher told me how to do; I worked out if g equals 6 I'd do 6 plus 3 so you do the inverse which is minus the 3 would work out f equals 9 so 9 take 3 equals 6

I What have you done with this 4 here?

A I scrubbed it out

Some of the students interviewed were able to cope with the non-integer aspect of the harder problem. Student B jotted 9 on the paper and then gave an explanation illustrating both a willingness to accept non integer answers and some aspects of operating beyond backtracking and substitution, though not clearly operating fully at level 5 on this problem.

B (In answer to 6 = 4f-3) 2 and 1/4

I How did you find that?

B g equals 6 but you have to 4 times something minus 3 equals and it can't be a whole number because if 4 times 2 equals 8 minus 3 doesn't equal 6 and I thought it must be a mixed number and then 4 goes - doesn't go into 9 ; 9 take 3 equals 6 so I thought that there's one quarter of 9 and then 2 and a quarter

These student responses illustrate how one or two questions can show different levels of strategy and understanding. Some students could not interpret the question so were below level 2. Some of their explanations of the question show misunderstandings which provide information for other domains. For example the lack of understanding of the meaning of the variable and the notation 5b shows clearly with these responses from students T and S. It was not uncommon for this misunderstanding of 5b as being fifty something even though this had been discussed in all classes and is also used commonly in their textbooks. These students were from different schools.

I (In answer to 27 = 5b+2) Why couldn't we do it?

T Because the b was after the 5

I ... if the b was before the 5 what would b be?

T 20, no 2

S (In answer to 27 = 5b+2) ... I don't get how if c=27 what would be the value of b because if the b is after the 5 that would be 50+2 ... Because if the b's after the 5 like it would be 50 something

The interview approach gives a clear picture of students' thinking which is not available from pen and paper instruments. It is not usually possible for time and cost reasons to use interviews for national testing. Pen and paper tests can provide some information about the students' thinking but interviews provide teachers with more information and can focus their attention on the strategies students are using and the nature of misunderstandings rather than just pen and paper procedures. This is particularly useful if teachers have a framework which enables them easily to interpret and record such information. Teachers report that the interviews are revealing of student mathematical understanding and development, in a way that would not be possible without that special opportunity for one-to-one interaction.

Concluding remarks

The proposed structure is intended to focus on student's understanding rather than on their skill at reproducing taught methods. It should provide a means of monitoring students' development in moving towards the abstraction of algebra in the area of linear equations. Such a framework needs a considerable amount of research but could provide a useful structure for other researchers and for teachers in schools who are trying to assist students to learn algebra. We are currently trialling interview questions to provide data to enable us to refine the structure of growth points in this domain and are planning to extend it into other domains of algebra.

A framework such as the one suggested above, with appropriate corresponding interview questions, does reflect the findings of relevant research, does emphasise the big ideas, the structure of algebra and movement towards abstraction, and form the basis for planning and teaching enabling teachers to identify both improvements and difficulties for individual students with whom they work.

However, in this paper only one domain was presented. Other domains such as function are being developed. Once a framework with a range of appropriate domains is developed and substantiated, along with corresponding assessment, then links between the domains can be investigated and a network of interactions built which will in turn provide information that will assist teachers to structure appropriate classroom experiences for each of their children.

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TRACING DEVELOPMENT OF STUDENTS' ALGEBRAIC REASONING OVER TIME

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We report on the performance of high attaining Year 8 students on two open-response written algebra questions that formed part of a nationwide survey. Preliminary findings suggest that performance is not always consistent between classes, between questions and compared to an overall measure of mathematical attainment. These findings will be further investigated through multilevel analysis and case study

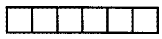
The findings reported in this paper arise from a study that aims to understand how students develop their competences in mathematical reasoning over time and how schools and teachers promote this development. Here we focus on the algebra part of the study1. A 50 minute written Proof Survey was administered in June 2000 to high attaining Year 8 students from 63 randomly selected schools within nine geographical areas that spanned England. The same students will be tested again in the summer term of 2001 (and again in 2002) using new and modified questions designed so we can trace development in algebraic thinking. Responses from the Year 8 survey have been coded and analysed using descriptive statistics2. We report here on two open-response algebra questions from the survey. Frequencies for the sample as a whole are given (2797 students) and for four groups of students (P1, P2, Q and R). Groups P1 and P2 are parallel top sets from a non selective suburban school, Q is a group of 25 high attaining mathematics students selected from four mixed ability classes in a highly selective grammar school, and R is a top set from an urban comprehensive school.

Responses to a question about generalising a structure

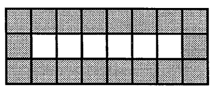
Question A1 (Figure 1, below) is concerned with generalisation in a familiar setting (tile patterns). The question shows a single drawing of a tile pattern and a description of how it is formed, and students were asked to make what Stacey (1989) calls a 'far generalisation'.

A1 Lisa has some white square tiles and some grey square tiles.
They are all the same size.

She makes a row of white tiles.



She surrounds the white tiles by a single layer of grey tiles.



How many grey tiles does she need to surround a row of 60 white tiles? ....

Show how you obtained your answer.

Figure 1: Question A1: Generalising structure

As well as providing an answer, students were asked to show how it was obtained. Responses were coded into 5 broad categories[3]. These are shown in Table 1 (below), together with their frequencies for the total sample. Correct answers, with clear evidence that a correct structure had been used, were coded 3, 4 or 5, depending on the degree of generality with which the structure was expressed (42 % of the total sample, $N = 2797$).

Code	Code description	Frequency
Code 1	Incorrect answer (180); use of an incorrect number pattern	37 %
Code 2	Incorrect answer (eg 120); partial use of correct structure (eg doubles but does not add 6)	8 %
Code 3	Correct answer (126); use of correct structure in the specific case of the question with no extra indication of generality	32 %
Code 4	Correct answer (126); use of correct structure indicating its generality	4 %
Code 5	Correct answer (126); use of correct structure (expressed in variables)	7 %
Code 9	Miscellaneous incorrect answers (including no response)	13 %

Table 1: Response codes for question A1

Recognising Structure

Figure 2 shows a typical, if rather minimal, code 3 response. Though there is little in the way of explanation, it is clear that the pattern for 60 white tiles has been seen as two rows of 60 grey tiles (one above the row of white tiles, the other below), with a total of 6 tiles at the ends of the rows. A less common, but equally valid way of reading the structure, is shown in Figure 3.

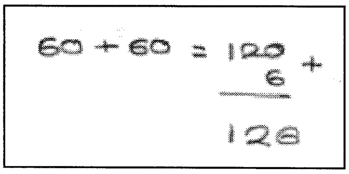


Figure 2: A typical code 3 response

A characteristic of code 3 responses is that they are couched in terms of a specific number (60) of white tiles. Some students expressed the pattern in more general terms, though it is worth pointing out that this is not necessary to answer the question correctly. Responses like "double and add 6" were coded 4, while responses involving a named variable, like "double the number of white tiles and add 6" and "$2 \times w + 6$" were coded 5.

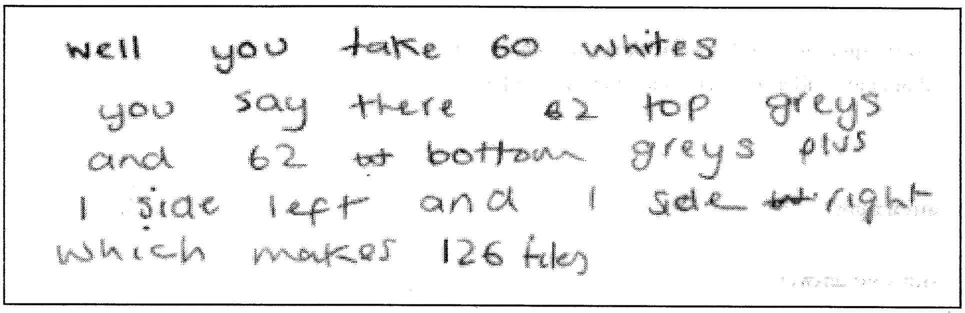


Figure 3: An alternative code 3 response to question A1

Spotting an incorrect pattern

The given diagram in question A1 shows 6 white tiles surrounded by 18 grey tiles, and students were asked for the number of grey tiles needed to surround a row of 60 white tiles. A substantial number of students gave the answer 180, on the basis that since 60 is 10 times 6, the number of grey tiles will be 10 times 18, or (less commonly) on the basis that since 18 is 3 times 6, the number of grey tiles will be 3 times 60. The first pattern spotting response can be classed as scalar, as it maps grey tiles onto grey tiles, while the second is functional, as it maps one variable (the number of white tiles) onto the other (the number of grey tiles). Both were given a code of 1. Pattern spotting responses were not unexpected, though we were surprised by their overall frequency (37 % of the total sample, N = 2797).

Pattern spotting responses have been found in a number of studies (for example, Stacey, 1989; MacGregor and Stacey, 1992; Orton and Orton, 1994). However usually in such studies, students have been asked to generate numerical data from a pattern sequence, before making a far generalisation. It is not surprising, perhaps, that in these circumstances many students focused just on the numbers and attempted to make an empirical rather than a structural generalisation (Bills and Rowland, 1999). As Orton and Orton (ibid) rather forlornly comment with regard to patterns similar to Stacey's:

It had been hoped that the experience of actually handling the matches and building the next shape would help pupils to focus on the matches and make use of the structure of the shapes but, once the numbers had been made explicit, it often appeared that the matches were set aside.

Noss, Healy And Hoyles (1997) have argued that pattern spotting is a result of disconnections: between actions (or virtual actions) that build a structure, the visual, narrative and/or algebraic descriptions of the structure and the output of the actions (eg, the number of matchsticks).

Focussing on numbers rather than structure will not necessarily lead to errors, but it would seem that errors are more likely, at least in situations where the structure can be construed easily. Thus Stacey (ibid) found that 50 % of a sample of 70 Year 8 students, having been shown a drawing of a matchstick ladder with 3 rungs and with 4 rungs, could successfully determine the number of matches needed for a ladder with 20 rungs, dropping to 24 % for 1000 rungs. The students were selected randomly from a suburban (Australian) high school, so their average mathematical attainment would probably have been lower than for our Year 8 sample.

Stacey found four common methods, which she called Counting, Difference, Whole-object and Linear. In the counting method, students would draw a ladder with the required number of rungs and count all the matches, or they would consider a shorter ladder and add on the appropriate number of matches (3 per extra rung). The counting method was widely used for 20 rungs (near generalisation) but not for 1000 rungs (far generalisation). In A1, students were asked to consider the pattern containing 60 white tiles. As with Stacey's 1000 rungs, we classed this as requiring a far generalisation as we felt that 60 was large enough to deter students from counting. This was largely borne out by our results, with just 9 of our 2797 students finding or getting close to the correct number of grey tiles by drawing and counting.

In the ladders question, 3 matches are needed for each extra rung. Stacey found that some students overgeneralised this relationship to argue that 3 matches are needed for each rung. She called this the difference method, which was commonly applied to 20 rungs and 1000 rungs. In A1, the number of grey tiles increases by 2 for each extra white tile, so the difference method would give 120 grey tiles when applied to 60 white tiles. Interestingly, this answer was rare (less than 5 % of the total sample), perhaps because the question did not encourage students to consider differences (see below).

In the whole-object method, students attempt to scale-up from one term to another, involving a false assumption of direct proportionality; for example, given that a 5-rung ladder requires of 17 matches, then a 20-rung ladder is deemed to require 4×17 matches. This method was common for both the 20- and the 1000-rung ladder and corresponds to the scalar number pattern response in A1 (10×18) which was given by 26 % of our total sample. Stacey makes no mention of a method corresponding to the functional number pattern response (3×60) in A1, which was given by 9 % of our sample, perhaps because the numbers of rungs and matchsticks in the ladders question did not lend themselves to this approach.

The relationship between the numbers of white tiles and grey tiles in A1 is linear and can be expressed in terms of multiplication and addition (for example, double then add 6; add 3 then double; add 2 then double then add 2). Such responses were coded 3, 4 or 5, depending on the degree of generality with which they were expressed and correspond to Stacey's linear method.

In A1, we deliberately avoided presenting a pattern sequence (although of course, students were free to generate their own sequence if they wished, and indeed some did - see below). Also, we opted for a tile pattern rather than, say, a matchstick or dot pattern, as it seems likely that a tile pattern will tend to focus attention on the required relationship (namely, the relationship between the independent and dependent variable - the number of white and grey tiles) rather than on the way the variables change from one term in the sequence to the next (ie a tile pattern will tend to encourage what Healy and Hoyles, 1999, call intra-term as opposed to inter-term chunking). In the light of this, the frequency of pattern spotting responses to A1 for our high-attaining Year 8 sample seems to be surprisingly high, even bearing in mind these prior studies and the fact that we chose numbers that leant themselves to both scalar and functional pattern spotting strategies.

The next phase of the current research will involve interviews with teachers and students in some of our sample schools, to throw light on the occurrence of these strategies. A likely influence is the teaching materials used in schools. MacGregor and Stacey (ibid), writing about a pattern-based approach current in Australia in the early 1990s, state that:

> One common textbook approach takes a geometric design, immediately derives from it a table of values and then seeks an algebraic formula which will produce the numbers in the table. Students may work out the formula in a rote fashion by using the constant difference as the coefficient of x and then adjusting the values by adding or subtracting a constant.

They go on to contrast this with a second pattern-based approach which:

> does not derive the algebraic rule from the table of values. Instead, the geometric features of a pattern or design are directly translated into a statement about the relationship, first in an English sentence and then in algebra.

The first of these approaches has long been familiar in the UK. It is present even in such stimulating materials as the DIME project (Giles, 1984) and can be found in many of the materials that are used today (including national tests and examinations). This approach might well encourage erroneous pattern spotting, something we hope to investigate.

The textbook is unlikely to be the only influence, however. Figure 4 shows the frequency of the erroneous pattern spotting (ie code 1) responses for question A1. The solid black column is for the total sample, the next columns for groups P1, P2, Q and R respectively. Of particular interest is the difference in frequency for groups P1 and P2, which, it will be recalled, are parallel top sets in the same school. This strongly points to the operation of teacher influences, over and above any textbook influence, though at present we do not know the nature of these teacher influences . It is also interesting to note the relatively high frequency of code 1 responses for group Q (48 %), bearing in mind that the students in this group were selected for their high mathematical attainment in an already selective school. Many of those who gave a code 1 response to question A1 gave high level

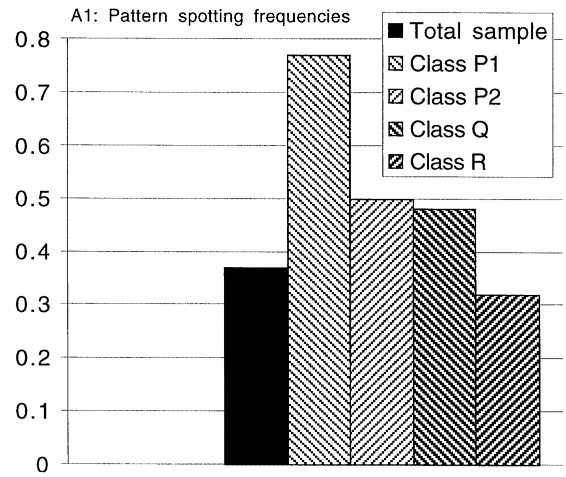


Figure 4: A1 pattern spotting frequencies for total sample and for four groups

responses to other questions on the proof survey; many also scored very highly on a Baseline Mathematics Test[4] that we had given them a few weeks before the Proof Survey to assess their overall mathematical attainment (see Table 2). Again, the reasons for the high frequency of code 1 responses for group Q are not yet known, though the Year 8 textbook used in the school might provide a clue. The book devotes several pages to number sequences, and though these are presented in a fairly open way; the setting is nearly always purely numerical, rather than involving spatial patterns as in A1.

Group Q	A1 codes					
Baseline Mathematics Test scores	c1	c2	c3	c4	c5	c9
15			1			
16						
17						
18	1	1				
19	5	1	1			
20	2	1	1			
21	3		1		3	
22	1		1	1	1	

Table 2: A1 codes against Baseline for group Q (N = 25)

As part of the coding scheme for question A1, where students answered the question correctly (ie gave a code 3, 4 or 5 response), note was taken of whether they had written down any extra values for the numbers of white and grey tiles, other than those in the given example of the tile pattern (6 white tiles, 18 grey tiles) and in the one that was asked for (60 white tiles, 126 grey tiles). A response of this kind is shown in Figure 5 (below). The response is typical in that the student has produced an ordered table of values, with the number of white tiles increasing by 1 each time. It is typical also in that the constant difference (2) between successive numbers of grey tiles has been identified.

In the sample as a whole, just 43 students gave responses of this kind, but interestingly 6 of these were in group R (with another student in Q but none in P1 or P2). The textbook used by group R, though recent, adopts precisely the first of the two pattern-based approaches

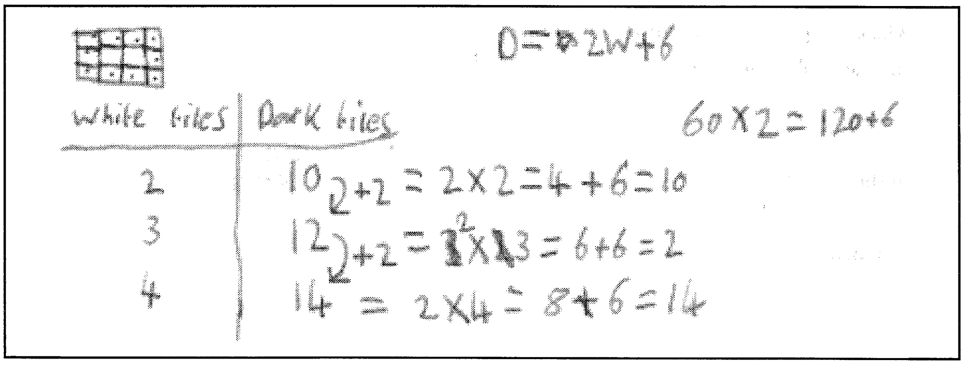


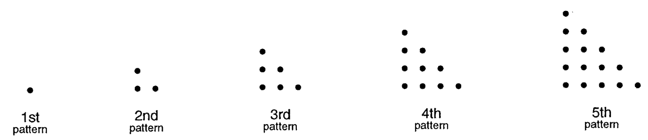
Figure 5: Use of extra data generated by the student in question A1

described by Stacey, above, including the rote use of the constant difference for determining the multiplier in linear relationships. This gives some indication of the potential strength of the influence of a textbook (and teacher). It should also be said, of course, that for these 6 students at least, the textbook's approach proved to be successful; however, this still does not mean that the approach is necessarily efficient or beneficial, especially in the long term (something we can explore in our longitudinal study).

Responses to a question about generalising a method

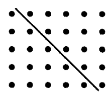
The second algebra question in the survey, A2, is shown in Figure 6. Unlike A1, successive terms of a pattern are shown and the structure is quadratic rather than linear. Students are presented with a method ('Rachel's method') for calculating the 5th term, with the aim of seeing whether they can generalise this method by applying it to the 20th term. The 20th term was large enough to discourage students from finding the number of dots just by drawing, although 4 % of the total sample made an attempt at drawing the 20th dot or rectangle pattern before giving up without finding the total. Another 6 % attempted to find the number of dots by 'long addition' (for example, 15 + 6 + 7 + ...+ 19 + 20). Both these methods would seem to fit Stacey's counting method and were coded 2. Interestingly, many students, despite being able to apply Rachel's method, made a 'skeletal' drawing of the 20th term first.

A2 These are the first five patterns in a sequence of dotted triangle patterns:



a) Rachel wants to *calculate* the number of dots in the 5th triangle pattern.

She draws the 5th pattern twice. She explains why:
"I can calculate the number of dots in this rectangle pattern.
I can then calculate the number of dots in the triangle pattern".



i. Show how Rachel calculates the number of dots in the rectangle pattern.

ii. Show how Rachel calculates the number of dots in the triangle pattern.

b) Rachel wants to find the number of dots in the 20th triangle pattern.
She imagines drawing it twice to produce a rectangle pattern.

Use the number of dots in the imagined rectangle pattern
to find the number of dots in the 20th triangle pattern. ....
Show how you obtained your answer.

Figure 6: Question A2: Generalising a method

Student's responses were classified into five broad categories (Table 3)[5] similar to those in A1. Answers showing correct (or near correct) use of Rachel's method were coded 3, 4 or 5. A similar percentage of the total sample gave answers with these codes (52 %) as in A1 (42 %).

Code	Code description	Frequency
Code 1	Incorrect answer (eg, 60); use of an incorrect number pattern	9 %
Code 2	Correct or incorrect answer; recognition of structure but use of drawing or long addition rather than the given method	11 %
Code 3	Correct (or near correct) answer (190, 200 or 210); use of the given method in the specific case of the question with no extra indication of generality [eg, writes $20 \times 21 \div 2$]	45 %
Code 4	Correct answer (210); use of the given method indicating its generality [eg, writes $20 \times (20 + 1) \div 2$]	4 %
Code 5	Correct answer (210); use of the given method (expressed in variables)	2 %
Code 9	Miscellaneous incorrect answers (including no response)	28 %

Table 3: Response codes for question A2

Code 1 responses were those where students scaled-up from one pattern (usually the 5th) to the 20th (for example, the 5th pattern has 15 dots, so the 20th has 4 × 15 dots). This matches the scalar number pattern approach found in A1 (Stacey's whole-pattern method), though there were far fewer answers of this sort than in A1. This is perhaps not surprising as A2 directs students to follow a particular procedure, whilst the choice of method in A1 is left completely open. The numbers in A2, particular those for the 5th pattern, would also seem to lend themselves to the functional number pattern approach that we found in A1 (the 5th pattern has 3 × 5 dots, so the 20th has 3 × 20 dots). However, we found no evidence of this. We did however, find occasional examples of other number pattern approaches, for example a 'two dimensional' scaling-up (the 5th pattern has 5 × 6 ÷ 2 dots so the 20th has 20 × 24 ÷ 2 dots). These were not coded separately but subsumed under code 9.

Figure 7 shows the frequencies of the codes for A2, for the total sample (the solid black columns) and groups P1, P2, Q and R. The differences between P1 and P2 are less marked than in question A1 and the performance of group Q students on A2 compared to A1 is much closer to what one might expect from their high overall mathematical attainment.

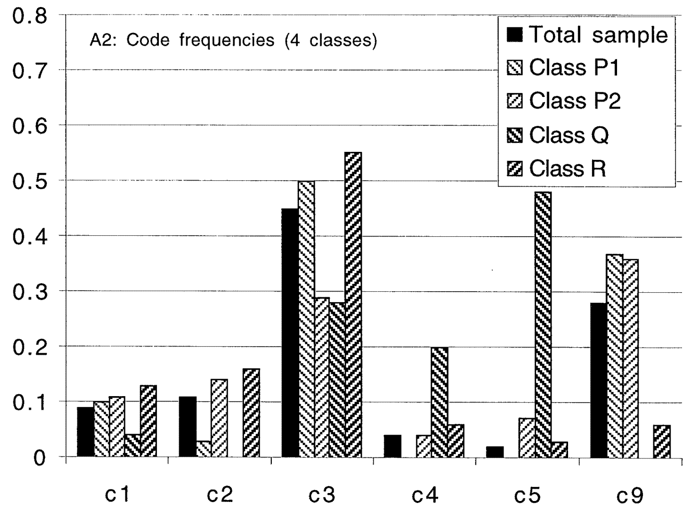


Figure 7: A2 code frequencies for total sample and for four groups

The performance of the group R students is of interest, especially with regard to code 3, 4 and 5 responses. Twenty of the 31 students in group R gave such responses for A2, compared to only 11 for A1. However, only 1 of these students used variables in their response in A2 (that is gave a code 5 response). In contrast, 7 students gave such a response in A1 and 6 of these used a table of values. Of course, it is not necessary to use variables in A1 or A2. Nonetheless, these findings, and the fact that using a table is not likely to be helpful for most students in A2 (as the relationship is not linear), suggest that for many group R students, the use of variables is strongly tied to using a table of values, and has not transferred to other methods.

Discussion

A full scale statistical analysis of these data through multilevel analysis (Goldstein, 1995) will enable us to identify significant predictors of the different types of response described in this paper. In particular it will provide further evidence for or against the findings from the descriptive statistics; namely how far performance in algebraic reasoning is related to overall mathematical attainment and how far it is shaped by by textbook and teachers. The analysis will not only take account of student variables (such as gender), but school (such as text book used) and teacher variables to explain responses. Through multilevel analysis, we are also identifying a small number of schools that are exceptional in their promotion of mathematical reasoning. Case studies of these outlier schools will be conducted in spring and summer 2001. These will consist of interviews with selected students to probe the answers and explanations given in the survey and to seek further insight into patterns of response, teacher interviews and documentary descriptions of policies and practices to illuminate the combination of factors that might come together to contribute to a schools' success in this area. We will in particular seek some explanation for the widespread use of pattern spotting in data regardless of mathematical structure. In the Year 9 survey (that is at present under design) we will include questions about generalising structure and generalising method that are similar to questions A1 and A2 described here so we can trace changes in students' response profiles both quantitatively and qualitatively.

Acknowledgement

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Notes

1. This study follows on from a survey conducted in 1998 of Year 10 students' conceptions of proof (for details of the algebra results, see Healy and Hoyles, 2000).
2. This analysis forms the basis for this paper, although a presentation in December would include further analysis and preliminary results from the Year 9 survey.
3. Most categories were divided further, giving 22 subcategories in all.
4. This test consisted of 22 multiple-choice mathematics item selected from TIMSS (IEA, 1996).
5. As with A1, categories were divided further, though not to the same extent (there were 13 subcategories in all, compared to 22 for A1).

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Synchronization of Algebraic Notations and Real World Situations from the Viewpoint of Levels of Language for Functional Representation

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Introduction

In the Discussion Document for this ICMI Conference, the section 'Algebra with Real Data' states that "Modeling the behaviour of real things with algebraic functions is fundamental to applications of mathematics". Further, a number of questions are posed including: "What new opportunities for using real data have proved to be successful, and how do they relate to research on students' learning of functions and other algebraic concepts?", and "What are the strengths and weaknesses of using real data and how are these best managed in the classroom and in the curriculum?".

In the 1990s, new classroom technologies were developed to enable students to deduce the equation of function from data through regression or graph fitting (eg. Zbiek, R., 1998). This purports to be new teaching content but in which contexts is it meaningful? Is this a better way to relate mathematics to the real world? There are obstacles to finding the equation of a function from real data and then interpreting the mathematical result in the real world. For example, good real world problem solvers with arithmetic are not always good real world problem solvers with algebra. A good equation solver may be unable to formulate an equation or interpret its solution. New technologies may appear helpful in overcoming such obstacles but do they establish enough of a relationship between algebraic notation and real world situations?

This paper explores obstacles to relating mathematics to the real world from the viewpoint of language hierarchy and illustrates the need for activity that synchronizes different structures. Using a number of illustrations, the paper asks whether the new movement towards real world problem solving lacks this synchronization and raises the importance of constructing synchronization.

Levels of Language for Functional Representation through to Calculus

Levels of language for functional representation have been demonstrated (Isoda, M. 1996)[1] to explain different ways of reasoning and the difficulties that inevitably exist in the teaching-leaning process (figure 1). This has been shown in two ways: first by investigations of the development of students' language for describing functions in the Japanese curriculum and of the history of functional reasoning from the age of the ancients; second by comparison with generalized forms of the van Hiele levels in geometry[2]. Because *function* is identified as a concept whereas *geometry* is seen as an area or a subject in mathematics, the idea of language levels for functional representation is unclear. However, the curriculum reform movement has been more clear about the sequence of teaching function in relation to arithmetic, algebra and calculus. The Japanese national curriculum has had the area (or conceptual field) of *function and relation* at elementary and junior secondary school since 1958[3]; similarly, NCTM Standard 2000 has the Standard for the area of ALGEBRA[4] which emphasizes the understanding of pattern, relation and function.

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[1] Related research reports were written in Japanese since 1985 at Journal of Japan Society of Mathematical Education.

[2] In van Hiele theory (1958), levels are described using generalized students' activities. These generalized descriptions mention the features of language rather than each student's thinking itself. Indeed, depending on the context/educational situation, students could do much higher level activity. In this meaning, levels function as channels for reasoning which are fixed depending on the context or aims of reasoning.

[3] The area was clearest in curriculum reform in 1968. From 1978, further reforms reduced teaching time for mathematics.

[4] ALGEBRA Standard includes some traditional areas of pre-calculus and calculus. It is much broader than algebra.

Major Language of the Level	Major **Object** and *Method* for Reasoning on the Level	Major Activity on the Level
Level 1 Everyday Language	Students explore **phenomena** (objects) using immature *relations* or *variation* (method).	Students describe relations in real world phenomena using everyday language immaturely. They can discuss changes in numbers using calculations, but usually their descriptions are done with or focused on one physically evident variable, the dependent variable. Even if they are aware of covariation, it is difficult for them to explain it appropriately using two variables because their descriptions of relations are done immaturely using everyday language. So it is difficult for them to compare different phenomena.
Example of conflicts between Level 1 and Level 2	In Japanese, we use "2 BAI, 3 BAI" to mean "two times, three times" on level 2. But in everyday Japanese (level 1), Japanese use "BAI" to mean any of "double", "plus" or "more". Eg. "Hito (person) Ich Bai (one time)" means "more than other person" or "two times the other person" in Japanese usage. A child on level 1 says "BAI, BAI" ("plus, plus") to mean three times the original amount. But some students think "BAI, BAI" ("double, double") means four times. On the other hand, at level 2, students have to use "2 BAI, 3 BAI" to explain proportion as a covariance and they say three times as "3 BAI", not as "BAI, BAI".	
Level 2. Arithmetic Language	Students explore **relations** using *rules.* The **object** on Level 2 was the *method* on Level 1.	Students describe the rules for relations using tables. They make and explore tables with arithmetic. Their descriptions of relations in phenomena are more precise with tables than just with the everyday language of level 1. Students have general concepts about some relations, for instance, proportion. Students can compare different phenomena using such rules. They describe rules for relations as covariation and when reading tables, their interpretation of the covariation of variables is at least as strong as their interpretation of correspondence. Students begin to use algebraic formulas and graphs to represent rules and relations in phenomena but it is difficult for them to translate between notations without any phenomena.
Example of conflicts	The constant function, y = constant for any x, is a function on level 3 but 'constant' means no relation in the phenomena on level 2 because students on level 2 try to find a relation of covariational change in the phenomena.	
Level 3 Algebraic or Geometric Language	Students explore **rules** using *function notation.*	Students describe functions using equations and graphs. To explore functions, even where there is no reference to real world phenomena, they translate among *the notations of tables, equations and graphs and use algebra and geometry.* At this level, their notion of function, which they already understand well, involves the representation of different notations already integrated as a mental image. For example, they can easily find the equation corresponding to a graph, and a graph from an equation.
Example of conflicts	On level 3, algebraically, a tangent line of a quadratic function can be deduced using the property that there is only one common point (ie. a multiple root). On level 4, using calculus, the tangent line does not always have this property.	
Level 4 Calculus with algebraic or geometric notation	Students explore **functions** using the *derived* or *primitive function.*	Students describe functions using calculus. In calculus, functions are described in terms of derived or primitive functions. For example, to describe the features of a function we use its derived function (which has already been learned). The theory of calculus is a generalized theory of such descriptions.

Figure 1. Levels of Language for Functional Representation through to Calculus

Using the idea of levels in figure 1, we can characterize mathematical models in ALGEBRA. For example, historical models about motion are characterized as follows.

Level 1. Zeno, of the Eleatic school, raised the paradox of how Achilles could not catch a tortoise. The relative motion of Achilles and the tortoise were discussed but their movements were not described in terms of time as an independent variable.

Level 2. Archimedes defined *Archimedean* using the idea of proportion between angle and length.

Level 3. Galileo showed that the locus of a cannon-ball is the parabola of Apollonius and that the cannon's elevation is the tangent to the parabola. In his time, geometry was the main language used to describe motion. Today, we describe it using algebraic notation.

Level 4. Newton described motion using the idea of fluxion[5]. We describe it using algebraic notation.

If we postulate level 5 along the lines of van Hiele levels in geometry, then general theories of calculus such as functional analysis may be considered. For example, J. Bernoulli posed the problems of the brachistochrone and the geodesic line. These variational problems were the origin of functional analysis and differential geometry.

In figure 1, each example of conflicts between levels illustrates that some of the knowledge at a lower level may act as an obstacle to learning the next level's language. Below, this feature helps to explain obstacles to mathematical modeling at each level, and it is used to illustrate the need for cognitive synchronization in mathematical modeling at each level.

Different Ways for Getting the Equation of Proportion Depending on the Levels

In my mathematics education methods class for third graders of an undergraduate mathematics major, I pose formulation problems relating to differential equations;

"Problem 1. If we define the growth of a microorganism as proportional to the amount X of a fungus at time t, find the differential equation"

"Problem 2. If we define the growth of a microorganism as proportional to the amount X of a fungus at time t and *at the same time as proportionally decreasing depending on how 'close' to a maximum quantity Xmax it reaches,* find the differential equation"

For university entrance examinations for mathematics majors, students had learned a lot of calculus in high school, so we believe that they can use language at level 4; further. they took units in differential equations and functional equations at undergraduate level. The result was that 77% students got a correct equation in Problem 1 but only 15 % students got a correct equation[6] in Problem 2 – most of them could not understand the meaning of the italic part in Problem 2. This suggests that even though students achieved at a higher level and learned much higher level mathematics, they cannot formulate real world equations that involve proportion. This fact can be

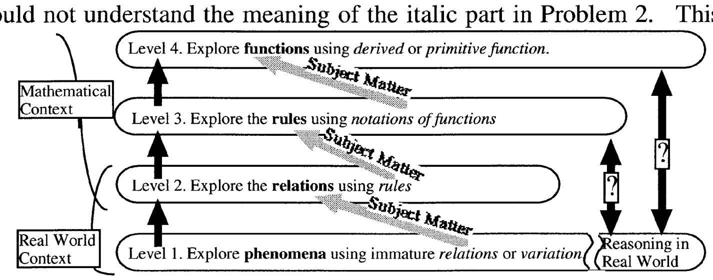

Figure 2. The development of levels does not guarantee applicability.

[5] Newton used to use algebraic notation but in the last half of his life he returned to geometric representations and wrote Principia. He got $(1+x)^{-4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 \ldots$ from algebraic notation. In this expansion, he tried to represent an unknown function by known functions (the algebraic sum of x^n). This method can be generalized using calculus.

[6] $dx/dt = kx(1-x/x_{max})$

explained with levels as follows.
Van Hiele (1958) describes a teaching sequence based on levels such that the method at one level is explored as the subject matter at the next level. Indeed[7], in the Japanese curriculum and textbooks up to 2001, the sequence in the area of *relation and function* has retained a structure as in figure 2. For example, to help students move from level 2 to level 3, the teacher asks them to explore a rule such as "the result of addition is constant" using notations involving function such as linear equation, table and graph. It is used as a method to explore hidden relations in a table of real world phenomena on level 2. In the teaching sequence in figure 2, students have the opportunity to relate their knowledge at a lower level to the next level. On the other hand, in this sequence, Japanese students[8] were not guaranteed to relate their knowledge to reasoning in the real world ('?' mark in figure 2) because reasoning in the real world, which is necessary for applying upper level knowledge, changes depending on the levels. This is illustrated below. Thus, if students are well versed in proportion at level 1, we still must teach them how to use proportion with the language of other levels[9]. The difficulty with problem 2 emphasizes this point. Indeed, the next example illustrates evidence to demonstrate the difference in proportional reasoning between levels 2 and 3.
Problem 3 and 4 (next page) were posed to grade 4 to grade 9 students[10]. The results of problem 3[11], table 1, show that students' proportional reasoning looks the same[12] after they learned the formal concept of proportion via situations in grade 6 and after they re-learned the concept via the functional rule $y = ax$ in grade 7. But the results of problem 4 show the change in their reasoning from grade 6 to grade 7. Q3 in problem 4, table 1, shows that grade 6 students' proportion of correct answers was higher[13] than in grade 7 and 8, but is the same as grade 9. Graph 4 indicates that, to get a correct answer, grade 6 students' solution methods for problem 3 and for Q3 of problem 4 were more different than grade 7 students' methods. Q2 of problem 4, Graph 3, shows that many grade 7 students still recognized this situation as dealing with proportions. Graph 5 shows that half of them could not write a correct answer to Q3. The difference between problem 3 and problem 4 is that problem 4 was posed via a real situation. This result suggests that many grade 7 students, in the process of reconstructing their conception of proportion as a function, lost their ability to apply proportion to the real world. Indeed, Graphs 1 and 2 for Q1 show that after learning proportion, grade 6 students could describe and analyze the situation itself exactly, while grade 7 students, having re-learned proportion in a functional sense, could not.
This result was due to teaching. The Japanese curriculum is structured to achieve level 2 as far as grade 6 and then, from grade 7, to develop students to level 3. Up to grade 6, elementary school teachers teach relations and rules in real situations, and from grade 7, junior high school teachers begin to teach functions as special rules of relations. Indeed, in figure 3, many grade 6 students get the equation of proportion from the table with the situation via the idea of unit, but most grade 7

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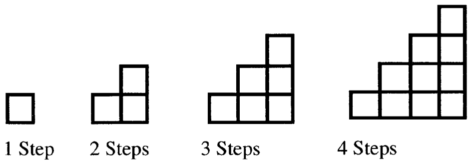
[7] The following discussion assumes the Japanese case. In footnotes, what is assumed is noted.
[8] Japanese school textbooks are oriented towards pure, higher or abstract mathematics and do not treat mathematical modeling and real world problems as special teaching content and thus are very far removed from US textbooks such as Mathematics in Context and Core-plus Mathematics Project which describe mathematics in the real world. On the contrary, from the comparison of Japanese and US textbooks, such US textbooks are treating the part of '?' mark in figure 2 more than the step by step development in the levels as do Japanese textbooks.
[9] In the US, because teaching content is not fixed to grades, this point may seem obscure. Japan has a national curriculum and textbooks do not repeat the same content in different grades.
[10] This data was collected in large city downtown areas and each grade's population was greater than 150. Students had already learned each grade's function or functional thinking content area in the national curriculum.
[11] This problem is the same as a problem in the Second International Mathematics Study.
[12] The probability of no difference is 0.6. There is no significant difference.
[13] The probability of no difference is 0.00015. There is a significant difference.

students get it via a constant quotient because junior high school teachers approached the equation of proportion as a function like this (Isoda, M. 1991). Even in the case of applying proportion to real world situations at each level, different ways are used depending on the level. Thus, even if students are well versed in proportion at level 1, we have to teach students how to use proportion with the language of other levels. We conclude that the many ways of finding the equation from real world situations are different depending on the levels. However, 'different ways' are not only

Problem 3. In the table at the right, if y is proportion to x, find the values for P and Q in the table.

x	3	6	P
y	7	Q	35

Problem 4. Let's make stairs using squares with sides 1 cm as follows.

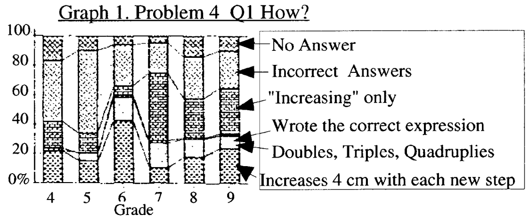


Q1. How does the perimeter change as the number of steps increases? Why do you think so?
Q2. How can we relate the number of steps and the perimeter?
Q3. What is the perimeter if there are ten steps?

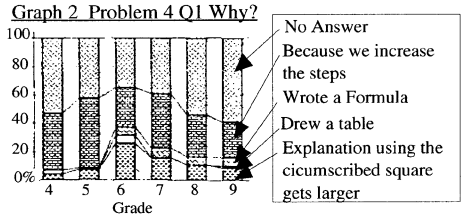
Table 1. Different results between real world problem and table

	Grade 4	Grade 5	Grade 6	Grade 7	Grade 8	Grade 9
Problem 3, Right Ans.	0.7%	3.1%	61.0%	58.5%	58.1%	60.8%
Problem 4, Q3, Right Ans.	35.9%	24.2%	64.4%	42.6%	43.5%	59.5%

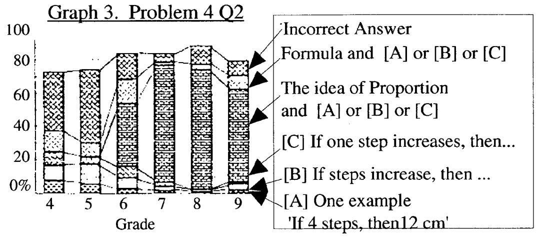
Graph 1. Problem 4 Q1 How?



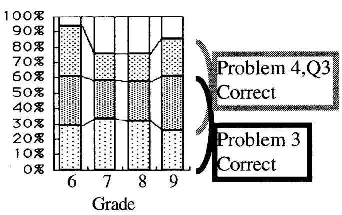
Graph 2 Problem 4 Q1 Why?



Graph 3. Problem 4 Q2



Graph 4. Problem 3 and Problem 4, Q3



Graph 5. Problem 4 Q2&Q3

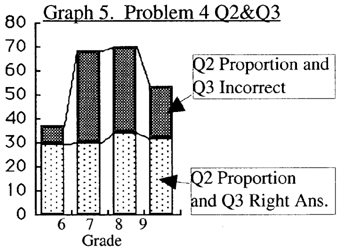
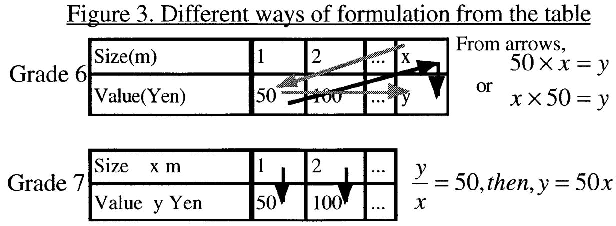


Figure 3. Different ways of formulation from the table



focused on equation formulation processes such as those in figure 3. The development of reasoning in real world situations depends on the level. In the following, we clarify this point.

Synchronization of the Algebraic Representation and the Situation

To get a mathematical model, the interpretation process is also necessary for the formulation of the model. Through the cycle of formulation and interpretation, we teach students to synchronize the algebraic notation and the real world situation. The following example in figure 4 (Isoda, M. and Matuzaki, A. 2000) demonstrates the difficulty of getting synchronization of algebraic representation on level 3 and real world situations (see '?' mark in figure 2).

At the start of the task, grade 11 high school students were asked to construct the mechanism of a wooden-horse on a merry-go-round with LEGO. They made an actual horse because their reasoning with visual images was not based on the crank mechanism. At the next stage, students were given the crank mechanism made with LEGO. They observed the motion of the mechanism but in the beginning, they suggested that the motions of horse were similar circles – they needed some discussion before they understood that the motions were 'egg-shaped curves' with the same amplitude. Even after they operated the mechanical structure, they still used inadequate visual images until they could reason with the actual structure. At the third stage, students were asked to represent the piston motion of the crank with the equation of a function. Students drew the graph of the function with a graphics calculator and recognized that the wave shape graph given by the equation of the function represented the piston motion very well, so they believed their equation to be true. But when students got non-continuous graphs in special cases of the equation's parameters, they thought that their equations were not appropriate for the mathematical representation of the mechanical structure because they were only reasoning with the mathematical representation, the equation of the function, and without the non-mathematical structure. At the fourth stage, students were asked to make the crank mechanism using LEGO so that they could interpret the meaning of non-continuous graphs. They could recognize that the equation of the function represented the real structure well and deeply. Thus we can say that they could reason with mathematical structure corresponding to the mechanical structure.

In this example, the crank mechanism is one physical structure in the real world and the equation of the function that represents piston motion is the mathematical structure. Until synchronizing algebraic structure and real world structure, students have to abandon their several familiar notions (or

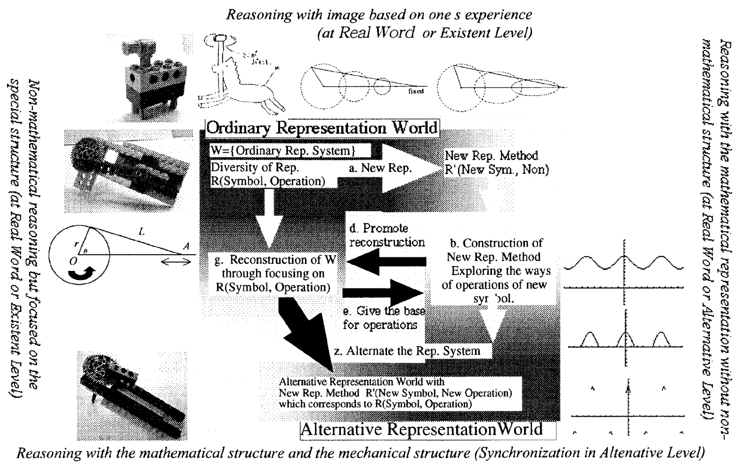


Figure.4 The Process of Mathematization from the view point of a representational system (Isoda 1991) using the example of Crank Mechanism

structures) and focus on a special structure in the real world; this is the left side of figure 4. In the example, the algebraic structure that synchronized with real world structure is constructed through changing the equation's parameter and its interpretation into the real world structure. The achievement of synchronization of structures is confirmed through changing the parameters of an equation and observing it in real world; this is right side of figure 4. This synchronization illustrates that teaching methods for finding algebraic equations in real world situations is not sufficient – we have to reconstruct or develop reasoning in real world situations that synchronizes with the reasoning on that level.

Synchronization of Structures at Different Levels

One of the corresponding structures for differentiation and integration in real world situations is the notion of increment, for example acceleration, and its sum in motion (see Clement, J. 1989), or rate and total (see Thompson, A.G., Thompson, P.W. 1996, Kaput to appear). In recent research by the Algebra, Geometry and Calculus for All project (Isoda, M., Takeuch, N. et al) it is shown that grade 5 and 6 elementary school students at level 2 do not know with any accuracy the meaning of velocity and acceleration. However, they have an immature view of a body's motion via their senses such as their sensation of speed and acceleration as expressed in everyday language. Many secondary school students at level 3 lose their everyday language to represent their feelings about motion and cannot connect their sense of motion with an algebraic representation because familiar functions such as the linear function was learned through ignoring non linear or variable sensations.

Motion digitizers and graphing tools (such as CBL, CBR, TI; EA-100, CASIO) are expected to enable students to become aware of their sensation of motion as intermediaries for translation among distance, velocity and acceleration via their own motion (eg. walking) and to develop their notion of increment and its sum in real situations. To test this, in one session of 100 minutes, we set a student activity to explore their motion (walking) by graphing distance-time and velocity-time (see figure 5) and by translating between graphs with the intermediation of their actual motion. Problems 5 and 6 showed the poorest performance in translation problems. Through 100 minutes of the digitizing-graphing experience of their walking, students are better able to tackle other problems such as explaining their walking pattern from a distance or velocity graph. On the other hand, the results of grade 5, 9 and 12 students in table 2 and 3 shows that translations between a distance graph and velocity graph were not achieved in the one session. Grade 12 and 12* students had already learned the calculus of algebraic functions and most could perform differentiation and integration, yet their achievement is no different from grade 9 students. The results of grade 12 and 12* illustrate that the teaching of calculus at level

Problem 5. When Tom walks in a straight as in the following graph, draw his velocity-time graph.

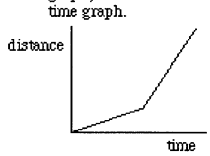


Table 2. Problem 5; Before and After Treatment

Grade & their leaning Level	5 Level 2	9 Level 3	12 Level 4	12* Level 4	10** Level 3
Number of participants	22	35	17	21	4
Right Ans. Before	0	5	7	4	0
Right Ans. After	0	6	8	--	4
	Only CBR 100min.			CBR+SimCalc	

Problem 6. When Tom walks in a straight as in following graph, draw his distance-time graph.

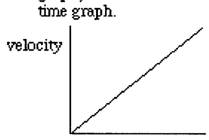


Table 3. Problem 6; Before and After Treatment

Grade & their leaning Level	5 Level 2	9 Level 3	12 Level 4	12* Level 4	10** Level 3
Number of participants	22	35	17	21	4
Right Ans. Before	0	6	10	7	0
Right Ans. After	1	8	11	--	4
	Only CBR 100min.			CBR+SimCalc	

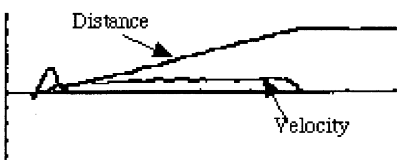


Figure 5. Graph of Walking

4 in figure 2 is not a guarantee of success in the development of reasoning with the notion of increment and its sum in a real world activity.

In using digitizing-graphing tools, it is difficult to explain graphs of motion and translation between graphs numerically because the digitized graph itself is unclear (figure 5). Students could only use their acceleration or no-change feelings for translations of their real world activity (a level 1-like activity) but to solve problems 5 and 6, a numerical and arithmetical approach (a level 2-like activity) is also helpful. For grade 10** students of tables 2 and 3, we ran 6 sessions, 50 minutes each, of activities: the first session was a digitizing activity, the next three sessions were simulations with SimCalc (see Kaput), and the last two sessions were introducing the definition of differentiation and integration by referring to previous experiences. Using SimCalc, students could explore translation problems via a numerical and arithmetical approach with ratio and total. The results in grade 10** of table 2 and 3 illustrate that performance developed perfectly, and in addition, the interpretation of relations between the distance graph and velocity graph, and between the velocity graph and acceleration graph worked as a metaphor to understanding the meaning of differentiation and integration.

The results of both sessions illustrated that there are two reference structures which are expected to synchronize with differentiation and integration; the first is our notion of increment and its sum such as sensation of acceleration in the real motion (a level 1-like activity) and the second is numerical-arithmetical reasoning with rate and total used to describe data (a level 2-like actibity). Both structures do not develop via ordinary teaching until level 4 but develop via special sessions for structural synchronization.

Conclusion

In the first part of this paper, the existence of levels in the area of 'functions and relations' was illustrated. Using these levels, the need to teach methods of getting equations of functions in real world situations depending on the levels was indicated. It was asserted that the development of reasoning in real world situations depends on these levels. Further in the paper, synchronization of the algebraic notation and the situation was discussed and illustrated with examples. In the teaching of mathematical modeling, formation and interpretation processes are treated. The example of the crank mechanism illustrated that only focusing on finding an equation from the real world situation is not enough. New technologies enable students to fit functions to data but if we only rely on them we lose the possibility of synchronization, necessary to using algebraic equations in real world situations. Synchronization of different structures reflects the operational nature of algebraic notation at each level and at the same time, implies the need to develop methods of reasoning in real world situations depending on algebraic notation at each level.

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Learning to Teach Algebra in the UK : Trainee Teachers' Experiences

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This paper uses an investigation into trainee teachers' experiences of learning to teach algebra as a basis for raising questions about how best to design courses for maximum impact. Results indicate that assumptions of expert guidance and debate within existing training models are not borne out in practice. Trainees more frequently operate in isolation using textbooks and schemes or their own 'common sense' for guidance. This raises questions about how best to prepare new teachers to teach algebra effectively.

Introduction and context

It is generally acknowledged that algebra is difficult to teach effectively. Teachers' early experiences set the foundations from which they learn to improve their expertise. The empirical study reported briefly here arose from concerns about the quality of trainee teacher's early experience of learning to teach algebra and whether, as a training providers we have 'got it right'. It sought to find out about trainees actual experience across different secondary school contexts. Knowing more about what constitutes their reality may help us to formulate proposals about what might be a more appropriate beginning experience for the majority of new teachers. For course designers the common experience must be taken as a starting point. Current approaches to initial teacher training are grounded in models that emphasise reflective practice.

Learning to teach algebra

The growing body of research into the ways in which trainee teachers learn to teach does not in general use the teaching of algebra as a domain of study. There are some exceptions to this, for example the work by Even et al (1993). In that particular study, they explored the differences in ability to make lesson and content connections between experienced and novice teachers of algebra and suggested that trainee teachers needed to engage in consideration of lesson and content connections.

At an individual psychological level, Schulman (1986), Turner-Bissett (1999), and others have distinguished various domains of knowledge acquired and used by the teacher. Expert teachers have more extensive knowledge bases and ones that are better structured and organised.

More widely across professions, Eraut (1994) discusses a range of theories of professional expertise. Those focussing upon skill or competence acquisition emphasise the increasing capacity to act in complex decision making contexts. In such models, problem solving and analytic behaviour moves to the state of intuitive action. Expertise is often not articulated easily by experts who in gaining expertise, 'automate' and render inaccessible much of their in situ decision making.

By contrast, Schon (1987) and Van Manen (1977), among others, emphasise the importance of reflection both in action and on action for the acquisition of teaching skills. In their models, professional practice is grounded in experience and inseparable from it. The expert supports the novice through 'coaching' – *helping the trainee teacher to see on his own behalf and in his own way the relations between means and methods employed and results achieved* (Schon, 1987, p.17).

Within a wider theoretical framework of situated cognition, Collins, Brown and Newman (1989) propose a cognitive apprenticeship model. Here the learning of novice teachers is inextricably linked to the context and activity of expert practice through making explicit the thinking underlying expert practice. The emphasis lies in the importance of the articulation of expert practice and its modelling by the novice.

All of these models suggest that the trainee teacher needs to engage in forms of in-depth discussion and reflective practice of both pedagogy and mathematics. They also assume that the new teacher's experience grows in breadth and depth as s/he becomes more experienced – implying at one level that such pedagogical and mathematical debate continues beyond the initial experiences. They recognise the significance of articulation of expert practice and presume the existence of a common discourse.

Focussing upon the learning of algebra, there is a significant body of research that identifies the significant issues facing pupils when learning algebra or learning to think and operate algebraically. Again, depending upon the wider theoretical framework, authors may focus upon meaning making, language and notation, misconceptions, conceptual models, the nature of learning activity and so on.

Investigating the experience of one year trainee teachers

In the UK, government reforms have promoted schools as the best place for initial teacher education. Typically, trainee teachers spend 24 weeks out of a 36 week training period in schools. A simplified version of Schulman's knowledge bases is embedded within the UK national curriculum guidance for training of new mathematics teachers (1998) where a distinction between 'subject', 'pedagogic', and 'pedagogic subject' domains is made.

Training programmes include consideration of issues about the teaching and learning of algebra including research findings. Some require trainee teachers to complete assessed written tasks in this area. All consider some approaches to teaching some algebraic concepts and topics.

Programmes place trainee teachers in schools under the care of a mentor – the expert teacher- whose role is both to guide, coach and support the trainee teacher but also (in most programmes) to assess their teaching competence. The practice of internship inherent in many training programmes, can be viewed as a cognitive apprenticeship model.

In this approach then, the teachers in school provide the main expert role models for trainee teachers, and people with whom debate, discussion and reflection can be shared. What models of teaching algebra do they offer and are these discussed and analysed with trainee teachers as they begin to learn how to teach algebra effectively?

Data

The data reported here is provided by 145 trainee teachers from 12 training programmes following a one year course in the UK to become secondary mathematics teachers. Participants in such training programmes are normally graduates with a mathematics or mathematics related first degree. The data was collected over a three year period. In 1997/8 from trainees at one institution (36 trainees). In 1998/9 (49 trainees) from 7 Institutions and 1999/2000 (60 trainees) from 12 Institutions.

Trainees were given a questionnaire at the end of their 1 year courses that contained a mixture of fixed and open response questions organised in 4 sections:

- Experience of learning to teach algebra during the University based part of the training,
- Experience of Teaching algebra in school/college yourself,
- Experience of Observing others teaching algebra in school,
- Your current views on algebra teaching/learning.

In addition a small number of interviews was conducted to help illuminate some of the general responses.

A summary of responses

University based work

Despite the best efforts of mathematics teacher educators to focus upon subject pedagogical issues relating to algebra, a proportion of trainee teachers (25%!) did not acknowledge that this focus had ever existed. We might speculate why this was the case but it remains a salutary reminder that learners do not necessarily have the same agendas as their tutors!

Across the various institutions, pupils' difficulties and common mistakes, were identified by about 80% of trainees as issues they had focussed upon. Other elements that more than one third of trainees identified were; research literature on teaching and learning algebra, the teaching of specific topics in algebra, language issues and approaches to generalisation through investigational work. Most felt that the elements they recalled best were the difficulties pupils had with algebra.

Observation of experts teaching algebra.

Only just over half the trainees had observed anyone teach an algebra lesson during their training year. Those that had reported that what they observed tended to fall into one of the following 'traditional' categories:

teacher explaining a new idea in algebra (55%)

teacher demonstrating how to perform a technique in algebra (81%)

pupils working on exercises individually (77%)

though a smaller proportion (35%) had observed expert teachers supporting pupils working on algebraic generalisation through investigative work.

Only a tiny proportion (5%) reported having observed pupils tackling a discussion task or pupils doing group tasks in algebra and only 17% had observed pupils using IT (eg spreadsheets) or graphic calculators to do algebra. 20% reported seeing pupils being asked to explain publicly how they had tackled a problem.

This data would suggest that the models of teaching of algebra by expert teachers are in general more likely to be traditional exposition and practice approaches. The practice of sharing meaning making appears to be limited as does the significant use of ICT tools in ways suggested by Sutherland (1991). This of course limits trainee teachers experience alternative teaching approaches. It also suggests that large numbers of trainees never see anyone else teach algebra. Opportunities can sometimes be provided through alternative means, video for example, by mathematics education tutors, although the value of such activity is diminished if it is not supported by similar experience in schools – which are increasingly viewed as 'the real situation'. Moreover, access to good video material is still difficult. Perhaps in future more effective use could be made of videoconferencing, though I would suggest we still need to work out effective ways of using such tools.

Trainee teachers' perception of the quality of teaching they observed tended to be negative. At most a fifth of the trainees felt that lessons had been challenging, motivating and interesting, or whilst routine, had been made interesting by the teaching approach. Half expressed the view that lessons observed had been routine, hard or rather boringly, if efficiently taught, with a few (4%) even feeling lessons observed had been confusing and pointless. This creates further difficulties for trainees whose confidence in the expertise of existing teachers may be threatened. Do trainees focus sharply enough upon the nature of the learning taking place or do they focus more superficially upon the global learning environment? More positively, how can we best help them to articulate their critique of existing practice in ways which help them to teach effectively?

Discussion of own teaching with experts

90% of trainee teachers in all 3 years had taught algebra themselves during their training year. The most common topics to be taught within algebra were Linear equations (49%), Simplification of expressions (inc collecting terms, brackets) (51%), Substitution into expressions(35%), Generalising from patterns (23%), Simultaneous equations(20%), Graphs of functions(20%).

Trainees were asked to identify how they had known what to teach and in what order, how to approach the teaching and what they had discussed with experienced teachers or mentors beforehand, during the teaching or after they had finished that section of work.

Less than 40% the trainee teachers claimed they had made decisions about content or order of content after or in consultation with teachers in the school. In 99/00 this proportion was as low as 25%. Most (60%) cited **only** schemes or textbooks or the National Curriculum as their source of knowing what to cover within a topic that had been given to them.

It would appear that trainee teachers were largely given a topic, a scheme of work and/or a text book and asked to cover a section. What that constituted, what a desirable order or sequence might be and what connections there might be with other topics or within algebra itself may or may not have been embedded in the particular material supplied to trainee teachers. We should not assume however, that this would be discussed with them prior to them starting to teach. What would we want to recommend is embedded in such written materials anyway?

There was more consultation about teaching methods, with 55% stating discussion with teachers as one of the ways in which they knew what teaching methods to use. Nonetheless, over a quarter of trainees claimed **only** to have worked it out by themselves.

Nearly 30% claimed to have had no discussion with teachers during or after their teaching of algebra, and of those that did, a third categorised this discussion as feedback and evaluation of teaching performance generally, rather than discussion about pupil responses, teaching approaches etc.

Despite the fact that the majority of trainee teachers were asked to teach algebra during their training year, the fact that so many are apparently doing so without discussion with more expert teachers is of concern. Some trainee teachers admitted that they didn't know what teaching methods to use yet had still not had or taken the opportunity to discuss this with more experienced staff. Perhaps staff appear too busy, perhaps trainee teachers fear that asking will signify weakness on their part.

If trainee teachers do not gain support and insight from practising experts at this early stage in their careers, we might conjecture that the pupil learning experience is likely to suffer. Whilst we still have limited knowledge about the most effective ways to teach algebra we do have some. University mathematics education tutors may engage trainee teachers in debate and discussion about existing research findings. However, if that is not consolidated at early stages of practice through in-depth coaching and analysis of practice, the research literature on the acquisition of expertise suggests that this will not occur. How then are new teachers to gain in expertise – through their own experience only? If so, how do we help them?

Trainee teachers views about algebra teaching

All but 2 trainees thought that it was important for pupils to learn algebra. The majority (85%) thought that **all** pupils should learn algebra, with only a few (8%) suggesting that the weakest should not or, more rarely (3%), that algebra was really only an appropriate topic for the most able pupils. Their reasons were varied, but largely focussed upon (a) its importance in mathematics and problem solving (b) its value in real life or use in other subjects and (c) its importance as a gatekeeper for higher levels of study in mathematics and related issues of equal opportunity. This mirrors wider views of the education community although the issue of the value of algebra in real life to many pupils is problematic. This view creates dilemmas and tensions for trainee teachers as they struggle to find meaningful contexts.

Trainees' perceptions of the current focus of school algebra in the UK were that it focussed upon equation solving (38%) and manipulative techniques (23%) with generalisation, use of functions (18%) and key ideas in algebra, processes, structures (5%) being seen by the fewest as having a major focus. They reported that they felt that algebra was currently taught mainly abstractly (68%), with a lower percentage gaining the view that it was taught mainly through investigative approaches (10%) or through problem solving and application (16%).

These perceptions may be related to the topics they themselves were asked to teach during their training year and/or the ways in which they were advised to approach them, and for some, the practice of experienced teachers that they observed.

Their comments on teachers' views on how best to teach algebra reflected a pragmatic approach. A higher level of non response to this question may simply suggest that trainee teachers didn't know as they had had minimal discussion with teachers about this:

- Teaching algebra is difficult
- You need to do theory/examples on the board, let pupils copy them down and then do plenty of practice (60%)
- Relate algebra to real life eg '*Use balancing and let letters represent the price of things that the students would like to buy'. (25 %)*

  This was complicated by trainees' view that *' the majority (of teachers) still use a= apple, b= banana.... '*
- Do algebra through investigations eg *' through investigative work, given a set problem, turning it into algebra, looking for patterns'*

Trainees own views on how algebra is best taught could be categorised in four ways :

- an emphasis on requiring pupils to think for themselves and break problems down appropriately
- constant connections with real-life contexts and concrete situations
- through small incremental steps
- by involving pupils actively, being varied in one's approach and making the subject interesting

The first and last category may reflect an awareness by trainees of the limitations of current teaching approaches. The remaining two categories reflect what they commonly quote as advice from teachers and in some senses reflect trainee teachers practical dilemmas of attempting to keep trainees on task, helping them succeed in the moment, keeping them motivated and a desire to give purpose to the learning of algebra. This dilemma led some trainees to make ridiculous connections particularly when working with lower attaining students;

> *'Supposing that you're looking at a combination of different ingredients in a cake and I said you could replace flour by f and you could replace the eggs with e and I said if you want to stick in the whole cake with a big letter C and you could start talking about one f as given in this recipe and one e and whatever equals C (writes 1f +1e = C), so if you double everything and have two of all these, what are you going to get in the end; and we did manage to get through a class discussion and get the answer 2C....*
>
> *....for children who are having difficulty with the concepts you really have to introduce real problems so they can get their heads round, so they can have some conception of what's going on.*

It was disappointing, given their awareness of the need for variety, motivation and interest that few trainees (less than 10%) had used IT (Computers or Graphic Calculators) to help them teach algebra, and few (only 5%) thought that this was a good idea – claiming largely that pupils still needed to learn the skills and techniques.

Discussion

The prevailing culture of initial teacher education programmes in the UK emphasises the importance of reflective practice, the significance of the mentor and the articulation of expert practice. By contrast, it would appear that, where the teaching of algebra is concerned, the experience of a high proportion of trainee teachers is one of isolation, being left to work it out on their own and limited opportunity to share, let alone debate, expert practice. The data suggests that algebra is a topic where trainee teachers gain insufficient experience of observing experienced practitioners, are left to develop their skills with insufficient discussion and rely on text books and schemes and their own common sense to select material, structure that material and choose a teaching approach. In-depth discussion of pupil conceptions, or the impact of particular teaching approaches on such conceptions is rare. The use of IT tools has yet to make a significant impact in the learning and teaching of algebra.

One might be optimistic and hope that debate about expert practice in the teaching of algebra will occur once trainee teachers are in post and gaining experience. Anecdotal evidence however, suggests the opposite – pressures upon teachers would appear to have forced a reduction in staffroom debate about practice in very many secondary schools.

Given the amount of time available on 1 year training programmes, perhaps it would be best to focus initial experiences upon a more in depth consideration of those topics which trainees most frequently get asked to teach – solution of linear equations, algebraic manipulation, generalisation. Tirosh et al (1998) suggest further that it could be fruitful in pre-service courses to choose to focus upon particular pupil conceptions, acceptance of lack of closure for instance. They add that the research literature does not often help novice teachers to identify approaches to teaching that might enhance pupil conceptions in algebra, nor help them develop a repertoire of approaches from which to select with particular conceptions in mind. Graebner (1999) identifies 5 'big ideas' about forms of knowing for preservice teachers but recognises the limitations of time. She asks similar questions – are these the most important ideas?, what are effective ways of helping preservice teachers to know how and when to utilise such knowledge?

Trainees would appear to enter teaching with some knowledge of pupil difficulties and misconceptions and to have considered a selection of research findings and approaches to teaching algebraic topics. For some there has been a focus on language and meaning making. The training model presumes the opportunity for observation of and discussion with experts that can provide the practical contexts in which to explore those issues. However, the reality appears to be a predominance of traditional exposition and practice approaches and limited opportunity for debate and discussion.

Trainees' reality does not match the assumptions in the training model. We need to reconsider the assumptions we make or we need to change reality to better match the assumptions. Perhaps we need to reconsider the training model. Perhaps we should start with the assumption that trainees have to build expertise without expert help. Grenfell (1996) outlines an alternative structuralist model for teacher training, following Bordieu, that suggests that it is important that trainees occupy an autonomous space: *'By having both schools and higher education institutions involved in training, there is a space called 'nowhere' where students* *have to decide for themselves'* (Grenfell, 1996, p300). If course designers started from a presumtion of isolation, what then would be the priorities for training programmes in preparing trainees for the teaching of algebra? What will help trainee teachers to make 'good' decisions?

Trainee teachers often find it difficult to articulate their own understandings of algebra. Few can talk about the nature of the subject, many lack an ability to distinguish between the objects and processes of algebra or any appreciation of, say, algebraic structure, in a global sense. Many fail to display the symbol sense discussed by Arcavi (1994). Perhaps course designers would use time more effectively by focussing upon trainees own conceptions of algebra, or perhaps the focus should be engagement in the discourse. 'Lack of closure', 'reification', even 'algebraic structure' is the discourse of research literature – it is not the discourse of mathematics teachers. Trainees struggle to acquire the language, teachers don't recognise it. Learning to talk about algebra and the pedagogy of algebra presents its own difficulties both for trainee teachers, but also for teachers themselves.

If research is to impact upon practice then we must start with the realities of that practice. How best to equip new teachers to teach algebra effectively within that reality then becomes our question.

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Algebrafying the Elementary Mathematics Experience
Part I: Transforming Task Structures

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We describe 3 dimensions of algebrafying the elementary school mathematics experience for students and teachers: (1) the process of building task-opportunities for generalization and progressive formalization of mathematical patterns & structures; (2) building teachers "algebra eyes and ears" so that they can recognize opportunities for such work in daily practice, & (3) creating classroom practice and culture to support such work. We set the context goals for a systemic approach to algebrafying elementary mathematics and illustrate with a few examples. A companion paper (Part II) deals with the development of elementary school teachers' capacity for algebrafying their practice.

Introduction: The Algebra Problem and the Algebrafying Strategy

Difficulties in learning algebra occur worldwide, but are especially acute in countries such as the United States which delay he learning of algebra till secondary school and isolate the learning of algebra from other mathematics. Our approach to this "Algebra Problem" centers on articulating a coherent and integrated approach to algebra that begins with the development of algebraic reasoning in the early grades in ways (a) that dramatically deepen students' understanding of elementary mathematics by fostering fundamental skill in generalizing, expressing, and systematically justifying mathematical generalizations; and (b) that are scalable and sustainable within the constraints of typical elementary schools, teachers, administrators, and instructional materials. As we have argued elsewhere (Kaput, 1998; 1999; Kaput & Blanton, April, 1999), elementary teachers are in the critical path to longitudinal algebra reform, yet they typically have had little experience with the rich and connected activities of generalizing and formalizing that need to become the norms in schools of the 21st century. Hence we focus on the growth of typical elementary school teachers within the constraints of their daily practices, resources, and capacities to grow mathematically and pedagogically. More particularly, we have developed an explicit "algebrafication" strategy, which involves classroom-grounded teacher development in three dimensions:

1. The process of building algebraic reasoning opportunities, especially generalization and progressive formalization opportunities, from available instructional materials, including the

"algebrafication" of existing arithmetic problems by transforming them from one-numerical-answer arithmetic problems to opportunities for pattern-building, conjecturing, generalizing, and justifying mathematical facts and relationships;

2. The building of teachers' "algebra eyes and ears" so that they can spot opportunities for generalization and systematic expression of that generality (including written expression) and then exploit these as they occur across mathematical topics;

3. The process of creating classroom practice and culture to encourage and support active student generalization and formalization within the context of purposeful conjecture and argument, so that algebraic reasoning opportunities occur frequently and are viable when they do occur.

In particular, (1) allows us to work with teachers to enhance their own instructional resource base while simultaneously building their subject matter knowledge. Just as we focus on *student* generalization within mathematical activity, we frequently ask *teachers* to generalize *tasks*. (2) is a shorthand for understanding the particular forms of student thinking, representation and, argumentation that we take as central to building algebraic reasoning skill. (3) integrates the development of classrooms that promote understanding with improvement of subject matter learning for both teachers and students (Kaput & Blanton 1999a).

Why Early Algebra?

"Solving the Algebra Problem" ensures simultaneous progress on four major goals:

- Opening curricular space for 21st century mathematics desperately needed at the secondary level, space in many countries such as the United States locked up by the 19th century curriculum now in place;

- Adding a new level of coherence, depth, and power to elementary mathematics, which tends to be centered on procedurally oriented arithmetic;

- Integrating more deeply two separate domains of mathematical experience, arithmetic and algebra, that have been inadequately related to each other for centuries;

- Democratizing access to powerful ideas by transforming algebra from an engine of inequity to an engine of mathematical power, thereby making opportunities for achievement more equitable.

What Do We Mean by "Algebra"—A Content Analysis

A narrow view of algebra as primarily syntactically-guided, symbolic manipulations not only grossly understates the multiple sides of algebra historically as mathematics, it is also an inadequate foundation for reform of algebra in school. We need a broader and deeper view of algebra that can provide *school* mathematics with the same depth and power that the many facets of algebra have historically provided mathematics and that can support the integration of algebraic reasoning across all grades and all topics. We have argued (Kaput 1995; 1998; 1999) that mature algebraic reasoning is a complex composite organized around five interrelated forms, or strands, of reasoning listed below. However, we see generalization (which includes deliberate argumentation) and the progressively systematic expression of that generality, especially in the symbolic systems of mathematics as underlying all the work we do, and underlying all by #2 below, although even there, it is essential in order to build referential meanings for the symbolic objects that are manipulated. We take this kind of thinking to be essential to deepening the school mathematical experience of the elementary grades beyond arithmetic proficiency and to building knowledge and habits of mind that can support learning the more complex and abstract mathematics that will grow in importance in the coming century: The "algebrafication" strategy outlined above has the following mathematical content at its base.

1. Algebra as Generalizing and Formalizing Patterns & Constraints, especially, but not exclusively, Algebra as Generalized Arithmetic and Quantitative Reasoning. This includes using arithmetic as a domain for making, expressing and arguing generalizations. This kind of activity seems to fall into two closely related categories. (a) Generalizing arithmetic operations, their properties and in some cases, reasoning about the more general relationships and their forms (e.g., properties of zero, commutativity, inverse relationships, etc.) This is algebra as generalized arithmetic. (b) Building generalizations about particular number properties or relationships (e.g., a sum of odds is even; finding regularities in the 100 table; determining general properties based on the placeholder system, such as what happens when you multiply by a power of ten, etc.) This is the development of algebraic ways of establishing generalizations about particular properties of numbers. It is different from (a), where the focus is on properties of the number *system* – thinking of algebra as generalized arithmetic. Both (a) and (b) have been a focus of the Madison, WI, work and that of the Madison *Project*, on which the former draws (see the accompanying paper by Carpenter & Franke). Both also involve generalizing the notions of equivalence associated with the "=" sign.

2. Algebra as Syntactically-Guided Manipulation of Formalisms.

3. Algebra as the Study of Structures and Systems Abstracted from Computations and Relations. This includes thinking with and generalizing with (relatively) more abstract objects and systems, such as the mod-n numbers of clock arithmetic and their properties, operations on different classes of objects, including strings of letters, as well as objects such as matrices. This is one of the roots of "abstract algebra" and algebra as a mathematical discipline. In a sense, the objects and operations in this kind of thinking are being thought of in an abstract and relational way.

4. Algebra as the Study of Functions, Relations, and Joint Variation. One aspect with which we are particularly concerned as a starting point is generalizing from numerical patterns (sometimes generated geometrically), typically to provide function descriptions (e.g., work with

triangular or figurate numbers, patterns in areas of figures, and more generally, "patterning" activities). Descriptions might also include iterative approaches: how the "next" state or number can be described in terms of the current state or number. Here, the domain in which assertions are made is primarily mathematics itself. Both kinds of description are important to building concepts of function, an important ingredient of school algebraic thinking.

5. Algebra as a Cluster of Modeling and Phenomena-Controlling Languages. We focus on using modeling as a domain for making, expressing and arguing generalizations. This activity seems to be of two broad types. (a) Generalizing patterns and regularities built from mathematized situations or phenomena, where the generalization is putatively about the *situation or phenomenon*, and the numerical patterns and relationships are in a support role to the larger modeling task. Here, knowledge of the situation or phenomenon interacts intimately with knowledge of the mathematics involved, and typically, the mathematical and extra-mathematical knowledge co-evolve during the process. In particular, the generality of the mathematical assertions interacts with the generality of assertions about the modeled situation or phenomenon. (b) Generalizing from solutions to single-answer modeling problems by relaxing the constraints of the given problem to explore its more general form, scope and deeper relationships—including comparisons with other models and other situations. Here, the domain of generalization is the situation being modeled.

This list emphasizes algebra's deep, but varied, connections with all of mathematics. This depth is exactly why algebra can play the key role across K-12 mathematics that we and others suggest. This content analysis is consistent with that provided by the National Council of Teachers of Mathematics Algebra Working Group and appearing in various reform documents e.g., (NCTM, 2000). A collection of downloadable papers [http://www.simcalc.umassd.edu/EABook.html], discuss and provide concrete illustrations of these forms of reasoning at the elementary school level. Our work focuses on #1, #4 and #5, but is intended to provide foundations for #2 and #3 as well..

Using Numbers and Arithmetic Operations in Algebraic Ways

Generalizing in the context of elementary mathematics often puts the teacher, and the students, in the position of using numbers and operations in an algebraic way. For example, one of our 3rd grade teachers (reported on in the accompanying paper by Blanton & Kaput) reported on an episode that involved building students' notions of even and odd numbers—a kind of activity that would fall into category 1(b) above.

A student determined that 5+7 was even by first doing the computation and seeing that 12 belonged to the (visible) list of even numbers. She then gave him the task of adding 2 really large (3 digit) odd numbers, so that the child had to think in terms of even/odd properties and not depend on the computation. While she used actual numbers, they were really being used as placeholders for *any* odd numbers, i.e., as variables. In this way she was able to avoid the semiotic complications of using literals, which require writing an arbitrary odd number as something like 2n+1, and a second one would be 2m+1, leading to the familiar sum-of-odds-is-even sequence of maneuvers. This example illustrates both how the abstractness of numbers gets built during use and how a teacher can set the stage for the next move, the formal expression of the generalization (which has not yet occurred in this case). Blanton & Kaput report how this teacher approached this challenge over an extended period in her classroom.

Just as numbers can be treated algebraically, an operation on numbers can be treated algebraically by deliberately leaving it in "indicated form," unexecuted. These algebraic ways of approaching arithmetic occur frequently in #1 above, and indeed occur in our examples below concerning a patterning type of activity. And, of course, these have been widely studied by others, including in ways that differ from ours in their deliberate use of literals to represent variables at an early age, e.g., Bodanskii & Davydov (Bodanskii, 1991).

Some of the generalization activities are natural sites for learning more specific things at the heart of algebraic reasoning, such as the expanded roles and properties of the "=" sign, inverse operations, identities, and so on. Most of them are also places where opportunities abound for enhancing number sense, problem solving and communication. We now provide a bit more detail on the kinds of teacher experience that we regard as essential foundations on which our algebrafying strategy can be based.

Concrete Illustrations of the Algebrafication Strategy with Teachers

A central feature of our work with teachers is to begin with carefully selected generalization and formalization problems that are mathematically challenging to them and that they can modify for use in their own classrooms at each level K-5. Such problems embody important mathematical ideas, are approachable at different levels, can generate mathematically rich conversations, typically involve substantial quantitative reasoning and computation, and hence can be seen as building number sense and even computational skill. Such flexible, multipurpose problems are not rare, indeed some are well-used favorites and frequently occur in the professional literature (e.g., NCTM journals for teachers and enrichment materials). See, for instance, the "Handshake Problem" revisited in Yarema, Adams & Cagle, 2000:

> *Twenty people are at a party. If each person is to shake everybody else's hand once, how many handshakes will be needed? How many handshakes will be needed for 21 people? How does the number of handshakes grow every time someone new is added to the group?*

Despite the fact that such problems are quite common, they are rarely systematically used in extended, classroom-oriented professional development on a large scale.

We provide teachers with a first-hand, authentic mathematical experience that reflects the kinds of thinking and behavior that we wish to stimulate and support. The teachers are not only engaged in learning and reflecting upon powerful mathematical thinking and representation strategies themselves, but they are actively constructing materials whose use in their classrooms is designed to promote the kinds of thinking that they are thus predisposed to anticipate, recognize, and promote. They then report on their students' approaches to the problem, and cross-grade comparisons are made to help identify growth trajectories as well as more general strategies for scaling problems upward or downward in difficulty.

A second related feature of our work with teachers is to support them in finding or extending opportunities for the same kinds of generalization and formalization activities with their students. Our approach is to use their available instructional materials, especially their basal textbook series, so that activities of generalizing and formalizing become part of their mainstream instruction, *not a form of enrichment.* In particular, we engage explicitly in transforming naturally occurring arithmetic problems into algebraic reasoning problems, usually asking "How can this one-numerical-answer problem be 'algebrafied' into a problem that involves building and expressing a pattern or generalization of some kind?" This can take the form of allowing one of the numerical "givens" of the problem to vary and examining the patterns in the resulting series of number sentences and calculations; or it can take the form of changing the conditions of the problem in some systematic way. While a problem such as the Handshake Problem is transparently a pattern-building activity, most arithmetic problems, such as the following missing addend or subtraction problem, are not normally treated in this way.

> *I want to buy a tee shirt that costs $14 and have $8 saved already. How much more money do I need to earn to buy the shirt?* We could allow the cost of the item to be purchased to vary: *Suppose it cost $15, or $16, or $17 or $26. Using P (or BOX) for the price of the item I want to buy, write a number sentence that describes how much more money I need to buy the item.* Or, a more interesting problem might ask, *Assuming I make $2 per hour for babysitting, how many hours do I need to work to have enough money to buy the $14 shirt? If it cost $20? If it cost $P? What if I earned $3 per hour? Then how many hours do I need to work to buy the $14 shirt?*

Clearly, most problems can generalize in many ways, and how an algebra-capable teacher algebrafies the problem depends on the background and interests of the students as well as her goals—which might simply be to provide a context for skip-counting, or to introduce division, or practice subtraction, etc.

Such generalization and formalization activities build the notions of variable, function, and "=" as equivalence rather than as a symbol separating procedure from answer. The latter is especially true when teachers turn the situation around and ask students to write open number sentences that help describe which input-value is needed to produce a given result (e.g., How many people would require 171 handshakes?). Many of these problems also involve treating arithmetic statements in an algebraic way. For example, the approach to the generalized Handshake Problem (How many handshakes for any given number of people?) based on creating a sequence of numbers (1, 3, 6, 10, 15, 21, ...) and writing either a closed form or recursive formula for the sequence is less revealing and powerful than an approach that uses "uncomputed" sequences of sums. Here, where the number in parentheses is the number of people in the group, we write (1) 0+0 = 0 (2) 0+1 = 1, (3) 1+2 = 3, (4) 3+3 = 6, (5) 6+4 = 10, (6) 10+5 = 15 reflecting the fact that each time a person is added to the group, the new person must shake the hands of all the people who were already in the group, thereby adding a number of handshakes equal to the number of people who were already in the group.

But even more revealing is to write these sums in completely uncollapsed form, say (5): 0+1+2+3+4+5. This not only reveals the pattern more explicitly, it sets the stage for connecting this problem to others, such as sums of integers, triangular numbers, Pascal's Triangle, and so on. Importantly, the key here is to *treat the sum as an algebraic object*—that is, to use *the form of the symbols* as a resource for reasoning rather than simply to compute the numerical sum to fill out a sequence (Kaput & Blanton, October, 1999). This illustrates treating arithmetic objects in an algebraic manner, a key aspect of relating arithmetic to algebra in a fruitful way.

Similarly, we sensitize teachers to the "algebra opportunities" that abound in familiar materials such as Hundred Charts and multiplication tables. An important measure of whether a given teacher's change is generative and self-sustaining is in the extent to which she spontaneously generates or exploits such "algebra opportunities" (Blanton & Kaput, October, 1999). For example, one of our teachers built a captivating activity around the "Twelve Days of Christmas" song and related it to ... yes, the Handshake Problem!

The explicit linkage into and enrichment of existing instructional materials by the teachers has two critical effects: (1) it provides them with a broadly applicable, generative skill rather than a closed set of materials, and (2) it integrates our approach with their daily practice, their instructional materials base, and district curricular responsibilities. A secondary effect is to help gain the acceptance and assistance of administrators, who see our work as directly in line with and building upon their student achievement objectives.

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