The impact of teacher-led discussions on students' subsequent argumentative writing

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THE INTERPLAY OF AUTHORITATIVE AND DIALOGIC INTERACTIONS

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This paper introduces an approach to mathematics teaching and learning which we feel transcends the usual teacher-centred versus student-centred dichotomy by integrating two kinds of mathematics classroom discourse, the authoritative and the dialogic. It is proposed that mathematics teaching and learning should engage students in dialogic communicative approaches to empower them to articulate their ideas and to take more responsibility, but that in order to enable students to build mathematics competences effectively it is also proposed that the teacher should at times involve periods of authoritative discourse on topics prompted by the dialogic discourse.

INTRODUCTION

In recent years, the issue of teaching and learning has often been cast in the context of classroom discourse, which is considered a key element in mathematics education reform (Lee, 2006; Mortimer and Scott, 2003; Nystrand, 1997; Potari and Jaworski, 2002; Wood, 1994). For example, Potari and Jaworski (2002, p. 355) say that classroom discourse is the "essence of teaching/learning processes". Nystrand (1997, p. 29) too makes an explicit link between discourse in teaching and the nature of students' learning:

> Specific modes or genres of discourse engender particular epistemic roles for the conversant, and these roles, in turn, engender, constrain, and empower their thinking. The bottom line for instruction is that the quality of student learning is closely linked to the quality of classroom talk.

Lee (2006, p. 91-99) considers discourse a 'learning tool' that can transform a mathematics classroom into what Sherin (2002) calls a 'discourse community', by changing the roles of both the teacher and students. In Lee's view, the teacher should empower students to take 'ownership' of mathematical ideas through participating in the discourse and articulating their views. Lee insists that this kind of empowerment can advance students' metacognition and improve the quality of their understanding.

In mathematics classroom discourse, and hence in the teaching and learning of mathematics, a central question concerns the location of power and authority, and the extent to which it is necessary or desirable for the teacher to retain control in order to promote the prescribed curriculum, or preferable to pass control to students (Fennema and Nelson, 1997; Smith, 1996). In reform agendas, it is common to suggest a move from one extreme to the opposite extreme, such as the slogans promoting a shift from teacher-centred to student-centred mathematics. However, some studies have sought a reconciliation between them (Sherin, 2002). In this article, we explore this

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particular issue in the context of a teaching intervention in Bahrain, and describe an approach to teaching mathematics that seems to locate power and authority in the unfolding action, rather than in one or other kind of participant, finding the locus of power in how mathematics meaning is made by students, and transcending the teacher-centred versus student-centred dichotomy.

Mortimer and Scott (2003) conceptualize the role of power and authority by characterizing the classroom discourse of the teacher and students in terms of whether student' views are taken into consideration during the progression of the lesson. Accordingly, the authors identify an authoritative–dialogic dimension of discourse. In authoritative discourse, the teacher aims to bring students' complete attention to specific meanings within mathematics as a curriculum subject. This is not to say that students do not or cannot express their own ideas, but that the teacher does not make room for exploring and working on them unless they are compatible with "school math" (Richards, 1991). In contrast, classroom discourse is said to be dialogic when it becomes "open to different points of view" (Scott et al., 2006, p. 610). In a dialogic discourse the teacher always makes room for exploring students' ideas explicitly, even when they do not conform to ideas in the received curriculum.

The distinction can be summarised as follows: in the authoritative communicative approach the teacher focuses on one specific point of view and leads students through a discourse with the aim of establishing and consolidating that point of view; in the dialogic communicative approach the teacher and students consider a range of ideas, exploring and working on different points of view. (Adapted from Mortimer and Scott, 2003).

The desire in teaching to acknowledge and build on students' ideas, and yet to achieve conventional knowledge development, leads to "tension between authoritative and dialogic discourse" (Scott et al., 2006, p. 605).

METHODOLOGY

The extracts reported in this article arise from an attempt to develop inquiry classrooms in schools in Bahrain and to study inductively how power and authority are structured in different episodes of mathematics lessons (Abdul Hussain, 2010). Four teachers in two schools were trained in inquiry methods and supported during an academic year in realising them, with video recordings made of lessons at different stages in the process, and interviews done with them (and with other staff in the school) before, during and after the intervention.

Extracts from the study will be used to illustrate: (1) the nature of and limitations to authoritative discourse; (2) the nature of dialogic discourse; (3) the way teachers in the study developed an interplay of dialogic and authoritative discourse to overcome the limitations of each.

NATURE OF AND LIMITATIONS TO AUTHORITATIVE DISCOURSE

In an authoritative communicative approach, the teacher promotes a particular point of view, and this often acts as a barrier that inhibits any possible attempt to follow a different approach to end up with the solution methods or answers. To illustrate, consider this extract from a lesson by the teacher Moneer. Students are working in groups on the word problem stated in turn 2 of the extract.

Episode 1

1 Moneer: One of your group can read it [again]. Yes, speak.

2 Student1: A group of 732 tourists arrived at Bahrain International Airport. How many buses are required, are required to transfer these tourists if the capacity of each bus 48 passengers?

3 Moneer: What are your ideas?

4 Student2: 732 tourists arrived Bahrain airport and the capacity of the bus is 48 passengers.

5 Moneer: Right, these are the givens of the question (making a gesture by his hands to indicate he wants more details about it). Boys, do you have anything to add more on the ideas of what was given? What did you do here, boys?.

6 Student3: A group of 732 tourists arrived at Bahrain International Airport.

7 Moneer: So at the outset I have to know the number of tourists in the group that arrived at the airport, how many ones. The number of the tourists.

8 Students: 732

9 Moneer: Well, these are 732 who arrived at the airport. Then, what other information is given to us?

10 Students: 48 passengers.

11 Moneer: 48 what

12 Students: Passengers

13 Moneer: The capacity of the bus, other group, because the capacity of the bus for the transfer can take 43, [no] 48 passengers, okay. How many buses were needed?

14 Student4: 15 buses.

15 Moneer: Is that enough to transfer the whole group?

16 Student5: 11 buses.

17 Moneer: Work now…

At this point students were allowed to work out their solutions.

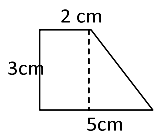
In this lesson, Moneer had a strongly authoritative communicative approach. From the very beginning of the lesson, his point of view was governing the whole process. Students not only had no opportunities to take responsibility to solve their questions but also did not have any real access to what was supposed to be mathematics. Students' actions were constrained strongly by the teacher's successive interventions such as funnelling (Wood, 1994), paraphrasing and/or filtering students' responses (turns 5,7, 9 and 13). In other words, their actions were structured into small steps, and it was only when their answers were approved and/or modified with the help of their teacher that they were allowed to take the next step of completing the answer (turns 15 and 17).

It seems that that there are certain actions embodied in authoritative discourse. Broadly speaking, these involve teachers shaping, selecting/overlooking and paraphrasing (Mortimer and Scott, 2003), which are used to sustain the development of the lesson according to their agendas, and the mathematics instruction is not based directly on students' understandings. As such, this form of teaching and learning constrains students' knowledge construction and meaning making.

NATURE OF DIALOGIC DISCOURSE

In contrast, dialogic discourse (by whatever name) is widely supposed to be helpful for student learning. Yackel et al. (1991) say that small group discussions can create learning opportunities from: (i) verbalising students' thinking; (ii) explaining/justifying answers; (iii) asking for illustrations and (iv) analysing wrong solution methods. Wood and Turner-Vorbeck (2002, p. 194) suggest that through dialogic discourse, individual students can engage in reflection on their ideas in three ways. The first and simplest one occurs when a student is asked to reconsider a solution strategy for the sake of providing additional details to the others. The second happens in a context of "confusion, complexity, or ambiguity" when students are engaged in an inquiry approach of learning. The third form of reflection happens in contexts in which students challenge the thinking of each other and are engaged in agreeing or disagreeing, justifying and critically inspecting their views.

The second episode is an extract from the transcript of a lesson undertaken by the same teacher, Moneer, later in the year, after he and his class had developed an inquiry style. In the lesson, students have been working on finding the area of the trapezium shown, by developing their own method for doing so.



Moneer introduces the plenary by saying:

> "Please before we start [our] discussion, the others should be listening and pay attention. This means any word of the group that comes out must be discussed and [members of the group] are accountable for that. You should not allow passing anything [wrong or non-convincing] for them. Be aware and pay attention. Okay, who would like to start first?"

In this extract, Hussain and Salim came out and presented their solution:

Episode 2

18 Hussain: At first, we constructed a line from the beginning of the angle (pointing to the dotted line). After that we measure it and we got umm 2.5 cm, 2.5 cm.

19 Moneer (to the audience): Oh boys. Pose your questions.

20 Student1: Why, why is it 2.5?

21 Hussain: Because we computed it

22 Student2: Why not 3 so they become equal?

23 Student1: You computed it! Why not 3?

24 Student2: The length of this must be equal to the length of this, why are they different?

[Hussain and his partner Salim discuss it at the board]

25 Student1: I'm saying why is it 3?

26 Moneer: Boys. Do you see what he is doing?

27 Students: Yes.

28 Student2: I asked him, teacher, why didn't you consider them to be 3?

29 Hussain: What we say is: this side is opposite to that (pointing to the sides representing the width in the constructed rectangle) so both of them are equal

30 Student2: Why did you put it 3? 3cm?

31 Hussain: Because here will be 2cm because it equals this one (pointing to the lines of the widths) and it is opposite to that one, so here will be 2 and this one 2. and this side is 5 (the bigger base of the trapezium) and we take away 2 equals 3 (the unknown remaining part from the base)

32 Moneer: What's your opinion, boys, on what he is saying?

33 Students: Not correct.

34 Moneer: We asked him, he said the length of what he has constructed is 3. They asked him why is it 3? First it was 2.5 then he changes his mind and said it is 3 (referring to the constructed line). Well, ask him why it is 3?

34 Student4: Teacher, he said it is 3 which is its opposite

36 Hussain: Because its opposite (pointing to the known side of 3 cm long) it is opposite to, it is opposite (pointing to widths of the rectangle)

The above episode indicates a shift in the nature of classroom discourse in terms of power and authority. Moneer was interested in hearing students' voices and eliciting their multiple solution methods. As it happened, many groups proposed different solution methods and the above episode was just one of them. At the start of the episode, Moneer articulated his expectations about the nature of classroom discussions and subsequently provided more opportunities and freedom for his students to articulate their views. He suspended his evaluative authority and also asked the students to explore, evaluate and reflect on each other's ideas (turns 26, 32 and 34). When Hussain's first idea was challenged, Moneer obliged the other students to understand each other and engage in negotiations, which convinced Hussain and his partner to give up their idea and to adopt the alternative. Moneer subsequently obliged the students to get Hussain to justify his answer further.

ENHANCED DIALOGIC

In the previous episode, classroom discourse had been transformed and became dialogic. Moneer had restructured his power and authority and granted more responsibilities to his students in the evolution of classroom interactions. They gained the floor and their initiations guided the flow of the discourse and the knowledge construction. In addition, Moneer did not restrict his teaching agenda to a pre-established rigid one. Instead, teaching and learning became open to students' input. This dialogic communicative approach was a productive tool to deal with and build on students' understanding and also to engage them in advanced practice such as conjecturing, justifying and convincing, negotiating, evaluating and reflecting on the articulated ideas.

However, the following episode of Moneer's teaching, from a later point in the lesson to find the area of the trapezium, indicates Moneer's felt need to retain some authoritative discourse to extend mathematics meaning making. It occurred when Hussain was recording and communicating mathematically what was agreed upon.

Episode 3

37 Hussain: Now we will multiply, 2x3 [is] 6 cm (writing 2x3=6 on the board).

38 Moneer: What is this 2x3?

39 Hussain: This is its area [rectangular area]

40 Moneer: Boys, I am just posing the questions and the groups don't ask

41 Hussain: This is its area

42 Moneer: Hussain, write the area of the rectangle is such and such, keep a name

[Hussain writes: Area of the rectangle= 2x3.]

43 Student2: Why didn't you multiply 3 times 5?

44 Hussain: Because it is a rectangle, so the length times the width, and this is the length and this is the width

45 Moneer: Excellent, he said the area of the rectangle is the length times the width. Colour the rectangle for them so that they can identify it; put some shading for example so that we know. [Hussain shades in the rectangle]

46 Hussain: The area of a triangle is half of the base times the height. This implies 1.5 here. …

47 Moneer: Hussain, make it clear, clarify your work. Finish your story with the rectangle. Say to them that we calculated its area and it was this much. Now we have the second part.

48 Hussain: We found its area

49 Moneer: The second part of the shape that we made is a triangle. In the triangle we need to know such and such in order to find its area. So explain and illustrate to them. Don't put just numbers.

In this extract, there was an emergent issue for Moneer in how Hussain was recording mathematically and communicating with others. Even though he understood what he had done, Hussain was not recording appropriately. Moneer seemed to feel that without this, Hussain would not be considered as competent and other students would not have had a common understanding about what was going on. In order to address this, Moneer initially used two 'dialogic' strategies. First he drew attention to the ambiguity in Hussain's recording by asking "What is this 2x3?" (turn 38). Second, he tried to engage the students in dialogic interactions by inviting them to reflect on the matter and to pose some questions (turn 40). However, this invitation was ineffective. Moneer felt that he could not ignore this issue as students would not become able to develop their competencies in how to communicate mathematically, so he led an authoritative discourse to illustrate to the students how they should organize their ideas and how to speak and write through using mathematical language.

CONCLUSION AND IMPLICATIONS

In many debates about classroom interaction, the authoritative or teacher-centred and the dialogic or student-centred are seen as conflicting alternatives that must be chosen between. Sherin (2002) proposes establishing a balance between the two faces of mathematical discourse by using a 'filtering approach' in a whole class discussion that follows students' ideas generation. However, filtering implies a prior decision about what is important, which seems little different from an authoritative communicative approach.

Episode 1 represents a 'traditional' kind of authoritative discourse which hindered students' access to meaningful learning. In the approach explored in this article, however, integrating the authoritative with the dialogic is achieved in a different way. The authoritative approach in the teacher dialogue in episode 3 uses student ideas but in a way that is emergent rather than pre-decided, taking the students' ideas but using the teacher's skill and understanding to develop them. The power and authority in the approach did not come from either the teacher or the students, but was centred primarily on meaning making, and as a result could contain both the students' perspectives and the teacher's knowledge. In this, and in other examples in the study, the dialogic and authoritative discourses seeded each other, which productively promoted knowledge construction.

The main guiding principles of the suggested approach require the teacher to engage students in dialogic discourse and also to remain sensitive and responsive to the cognitive side of students' work (Potari and Jaworski, 2002). In addition, he/she has to offer the necessary challenges that extend students' mathematics learning. As such authoritative discourse plays a different function from that in traditional teaching approaches because it is emergent rather than pre-decided, resulting from and supporting dialogic interactions. In this way students gain more epistemic roles (Nystrand, 1997) in classroom discourse, and the discourse becomes a learning tool which allows students to retain ownership (Lee, 2006) of the mathematical ideas.

Our conclusion is that teachers need not be so bound by any felt moral obligation to be student-centred that they do not give their own knowledge a voice. By focusing attention on emergent meaning making, and using their knowledge to develop it, teachers can have it both ways, using their own knowledge to promote student understanding, but in effect subordinating their knowledge to the students' developing ideas.

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PROPORTIONAL REASONING OF PRIMARY TEACHERS

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This paper reports a case study carried out with five in-service primary teachers to whom several kinds of ratio-comparison problems were posed, with different contexts and numerical structures. The obtained results are analysed quantitatively and qualitatively, and a didactical strategy is proposed that can also be applied in teacher training and Professional Development.

Within the frame of an ongoing investigation on the strategies used by subjects of different ages and schoolings when faced to different kinds of ratio comparison tasks, this paper deals with the ability that schoolteachers have for proportional reasoning.

A FRAMEWORK THAT STEMS FROM PREVIOUS WORK

In previous PMEs and other conferences different parts of the research have been put forward. The framework and results obtained by subjects of different ages and schooling have been described in Alatorre (2002) and Alatorre and Figueras (2003, 2004, and 2005). In the following paragraphs we present a succinct summary of these papers; the reader is referred to them for a more complete account.

The problems calling for proportional reasoning can consist of two kinds of task: missing-value and ratio-comparison. We deal with the latter, and focus on two of its important variables: context and numerical structure. According to their context, ratio comparison problems can be classified in three kinds (Freudenthal, 1983; Tourniaire and Pulos, 1985; Lesh, Post and Behr, 1988; Lamon, 1993): Rate problems (which involve two different quantities); Part-part-whole problems (P-P-W, which involve one quantity and which, in turn, can be classified in Mixture or Probability problems); and Geometrical problems. Geometrical problems are not dealt with in this research; examples of Rate, Mixture (from Noelting, 1980) and Probability problems are shown in Figure 1 (each with a different numerical structure, as commented below).

The second variable that we focus on is the numerical structure. The four numbers in a ratio-comparison belong to two "objects" (e.g., in Figure 1, girls, jars, bottles); in each object there is an antecedent (**a**; e.g. blocks, concentrate glasses, black marbles) and a consequent (**c**; e.g. minutes, water glasses, white marbles). If we use the format $\mathbf{(a_1,c_1)(a_2,c_2)}$, the foursomes in Figure 1 are (2,3)(2,2); (2,1)(4,2); (2,1)(3,2). Alatorre (2002) classified all possible such foursomes in 86 different situations and regrouped them in three difficulty levels, L1, L2, and L3, which will be described further on.

Thus, context and numerical structure provide a double classification of the ratio-comparison problems. The possible strategies used by subjects in their answers to the

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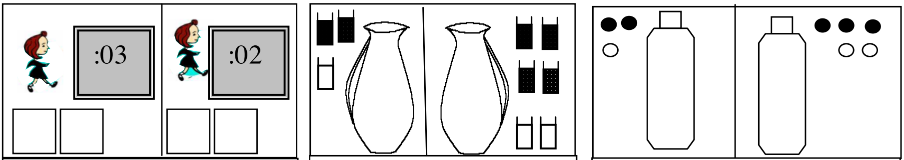

Which girl walks faster? (The squares stand for blocks and the numbers for minutes)

In which jar does the mixture of concentrated juice and water have a stronger taste?

If bottles are shaken with marbles inside, in which one is a black marble more likely to come out at the first try?

Figure 1. Examples of Rate, Mixture and Probability problems

problems have also been classified (Alatorre, 2002; Alatorre et al., 2005). Strategies can be simple or composed; in turn, simple strategies can be centrations or relations. These will be described below; examples refer to Figure 1.

Centrations can be on the totals CT (e.g. "In bottle 2 it is more likely because it has five marbles and bottle 1 only has three"), on the antecedents CA (e.g. "Both girls walk at the same speed because they both walk two blocks"), or on the consequents CC (e.g. "Jar 1 has a stronger taste because it has less water than jar 2").

Relations can be order relations RO (when an order relationship is established among the antecedent and the consequent of each object and the results are compared; e.g. "Girl 2 is faster because she walks for as many minutes as blocks, whereas girl 1 takes more minutes than the blocks she walks", or "In both bottles it is equally likely because both have more black marbles than white ones"), subtractive relations RS (additive strategies, e.g. "Jar 2 has a stronger taste because it has two more concentrate glasses than water glasses, whereas jar 1 only has one more"), or proportionality relations RP (e.g. "Girl 2 walks faster because she takes 1 minute per block, while girl 1 needs 1½ minute per block"; or "Both jars have the same taste because they both have twice as many concentrate glasses than water glasses"; or "In bottle 1 it is more likely to get a black marble because it has two black marbles for each white one, while bottle 2 lacks a black marble to have the same relationship").

Composed strategies are logical juxtapositions of two strategies (e.g. "Girl 2 walks faster, because she walks the same two blocks than girl 1 *and* in less time", or "In bottle 2 it is more likely to get a black marble because it has more black marbles, *although* bottle 1 has fewer white ones").

To the above described system can be added Noelting's (1980) categories *BETWEEN* and *WITHIN*, which describe if the comparisons are between the two objects or within each one: *BETWEEN* strategies compare $\mathbf{a_1}$ vs. $\mathbf{a_2}$ and/or $\mathbf{c_1}$ vs. $\mathbf{c_2}$, while *WITHIN* strategies compare $\mathbf{a_1}$ vs. $\mathbf{c_1}$ and/or $\mathbf{a_2}$ vs. $\mathbf{c_2}$. Centrations are strategies of the category *BETWEEN*, while relations are generally strategies of the category *WITHIN* (such as all the examples of the previous paragraph), but they can also be *BETWEEN* (e.g. "Both jars have the same taste because jar 2 is twice as much as jar 1").

All strategies may be labelled as correct or incorrect, sometimes depending on the numerical situation in which they are used. There are three kinds of correct strategies: **1]** RP in all situations; **2]** RO (*WITHIN*) in situations where **a=c** in one object, as in the Rate example of Figure 1, or where $\mathbf{a_1<c_1}$ and $\mathbf{a_2>c_2}$ (or viceversa); and, **3]** in some situations, some composed strategies (*BETWEEN*, such as the first example of composed strategies). Incorrect strategies are: CT and RS in all situations; CA, CC and RO in most situations; and most composed strategies.

The three difficulty levels for the numerical structure mentioned above refer to which correct strategies may be applied. Level L1 consists of all the numerical situations where, in addition to RP, other correct strategies may be used. In levels L2 and L3 only RP can be used; L2 consists of situations of proportionality (both ratios are the same), and L3 consists of situations of non-proportionality. The Rate, Mixture, and Probability problems of Figure 1 exemplify respectively L1, L2, and L3, although of course the numbers can change and all the levels are possible in all the problems.

METHODOLOGY

A case study was conducted in a suburban lower class area of Mexico City, with five primary in-service teachers (M1, M2, M3, M4, and M5). Each of the teachers was interviewed in two one-hour videotaped sessions. During the first interviews, teachers were posed several questions in each of five sorts of problems, which were two Rate problems, two Mixture problems, and one Probability problem. The Rate problems were the <u>Speed</u> (*S*) problem of Figure 1 and a <u>Notebooks</u> (*N*) problem (two stores sell different amount of notebooks for different amount of coins; in which one are the notebooks cheaper?); the Mixture problems were the <u>Juice</u> (*J*) problem of Figure 1 and an <u>Exams</u> (*E*) problem (in two exams a girl gets different amounts of correct and incorrect answers; in which one did she do better?); the Probability problem was the <u>Marbles</u> (*M*) problem of Figure 1.

Each of the problems was posed in different questions according to numerical structure. Twelve such questions were designed, four in each of the difficulty levels L1, L2, and L3. To each subject all of the problems were posed in *some* of the twelve numerical questions, covering at least a couple of the questions of each level (*some* and not *all* because of time limitations; *which* ones were posed depended on the kind of answers each teacher was giving). Each time, the subjects were asked to make a decision (object 1, object 2, or "it is the same") and to justify it.

For each teacher, between 45 and 51 answers were thus obtained, which were classified using the strategies system described above and also according to correctness. For the second interview, the videotape of the first one was edited according to the answers that each teacher had given, and this custom-tailored tape was shown to the teacher after a general explanation about proportional reasoning; she was then invited to give new answers. Thus, the second interview served the purpose of feedback and a Professional Development experience. Owing to space limitations, here we will focus mainly on the results obtained in the first interview.

The results of the first interview were analysed according to the numerical structure, the context, and the individual teacher. In each case, a two-fold perspective was used: qualitative (strategies used) and quantitative (percentage of correct answers).

ANALYSIS OF RESULTS

The overall success rate of the different context types and difficulty levels is shown in Figure 2. It corroborates previous findings (Alatorre et al., 2004, 2005; Alatorre, Morales & Roldán, 2007): on the one hand the difficulty levels are indeed so; on the other hand Rate problems are easier than P-P-W problems, especially in L2 and L3.

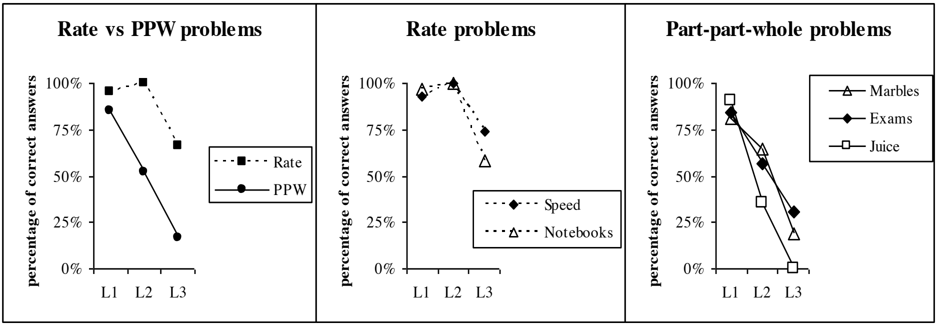

Figure 2. Results obtained in the different contexts and difficulty levels

In level L1 the teachers made very few mistakes; in all five contexts the success rate was above 80%. Often the non-100% was due to incomplete answers, mostly because the questions seemed so trivial that the teacher felt that no justification was needed:

> It is obvious that the notebooks are cheaper in store 2 [M5, *N*(3,3)(2,0)].

Since in level L1 several different strategies can be correctly applied, it is interesting to see which ones were used by the teachers. In P-P-W problems most of the answers (58%) used strategies of the category *WITHIN*, such as

> [*J*(1,4)(3,2)] Jar 2 has a stronger taste because it has more concentrate than water, and jar 1 has more water than concentrate [M1],

while in Rate problems most of the answers (79%) used *BETWEEN* strategies, such as

> [*N*(1,4)(3,2)] When you compare both stores you see that in store 2 you would pay fewer coins for more notebooks [M4].

In level L2 all the answers given to Rate problems were correct: the teachers used the only correct strategy, RP. However, in P-P-W almost half of the answers were incorrect. Most of the errors consisted either on centrations or on additive strategies:

> [*M*(3,3)(1,1)] In bottle 1, because it has more black marbles than bottle 2 [M2].

> [*J*(4,6)(2,3)] The stronger taste is in jar 2, because there's only one extra glass of water, and in jar 1 there are two [M1].

The following excerpt illustrates another erroneous strategy (RO), and also that sometimes a subject can make a correct decision for a wrong reason:

[*M*(2,1)(4,2)] The same, in both sides there are fewer white marbles than black ones [M4]

Level L3, in which the only correct applicable strategy was also RP, was by far the most difficult of the levels. Only 67% of the Rate problems and 17% of the P-P-W ones were successfully solved. Most of the errors are accounted for incorrect attempts at a proportional reasoning, but also there were many additive strategies RS:

[*M*(5,2)(7,3)] In bottle 1 it is two-to-one, two-to-one, one is left. In bottle 2 it is two-to-one, two-to-one, two-to-one, one is left. It is the same in both [M3].

[*J*(5,2)(7,3)] The notebooks are cheaper in store 2, because in store 1 I'll buy five notebooks for two coins, and in store 2 it is five notebooks for two coins. With the extra coin I will buy two notebooks [M3].

[*J*(2,5)(1,3)] Jar 2 has a stronger taste because one glass of concentrate corresponds to one of water and there are two water glasses left, and in jar 1 there are three water glasses left [M2].

In a comparison within contexts types (Figure 2), both Rate problems behave almost identically; there is only a small difference in L3, the Speed problem being slightly easier than the Notebook one. This also corroborates the findings with subjects of different ages and schoolings (Alatorre et al., 2005; Alatorre et al., 2007).

However, the comparison among the three P-P-W contexts yielded bewildering results. In the previous cited studies the Juice problem was generally the easiest and the Marbles problem was generally the most difficult of the three, as is generally the case with Probability problems. The cited studies also gave outcomes congruent with the well-known fact that the familiarity that a subject has with a context is usually determinant in their results; this had been particularly the case with the Exams problem, at which subjects with little or no schooling failed almost unanimously. But in this study with teachers, the Juice problem, with only 35% of successes in level L2 and none at all in L3, was definitely the most difficult of the five; it is also noticeable that more than half of the errors in the Juice problem were due to additive strategies (some examples have already been displayed). In contrast, the Marbles problem was not particularly difficult; the teachers applied different forms of RP; for instance

[*M*(3,6)(1,2)] In both bottles there is two white marbles and one black one; they can have the same proportion for the luck of extracting a black marble [M3].

Finally, the Exams problem, which we expected to be very easy for teachers because of the familiarity, did not exceed the 57% success rate in L2 and 31% in L3. What happened with teacher M3 in this context is interesting. She answered correctly two of the first three questions. Then, on the fourth one, she first used RP correctly:

[*E*(2,1)(4,2)] It is the same: one-to-two in exam 1; in exam 2 it is twice as much [M3].

But when she was asked to grade the girl she assigned the mark 5 out of 10, maybe realizing that in the first exam, with the array (2,1), 1 was the half of 2. Starting from the next question, and probably somewhat based in that same array (2,1), she worked out a general formula: multiply by 2 and divide into 3 (next transcript). After that she applied her formula systematically.

[*E*(2,5)(1,3)] In exam 1 there are 7 and in the second one there are 4. Then 7 times 2, 14; divided by 3, almost 5. Let's see: 7×2÷3=4.6 is what she gets in exam 1. In exam 2 it is 4×2÷3=2.6, which is less [M3].

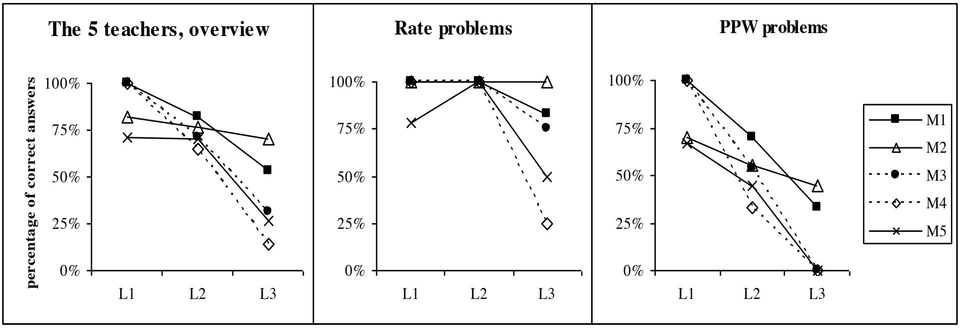

Figure 3. Results obtained by the five teachers

There was of course also a variation among the teachers. Figure 3 shows their results in the different context types and difficulty levels. M1 and M2 had the best results. Both teachers' mistakes in the first interview (mainly additive strategies; in the case of M2 also centrations in the Marbles context) were spontaneously corrected in the second interview after a short explanation. The results of M4 were somewhat poorer. She also used additive strategies, but half of her mistakes were centrations in L2 and L3. She did use RP in all the contexts (especially the Rate ones), but also made several incorrect attempts at RP. In the second interview she managed only partially to correct the errors; oftentimes she substituted a mistake with a different mistake.

M3 and M5 had even poorer results. When confronted in the second interview with her "formula" for marking exams, M3 even changed the mark 5 she had given in the fourth question for 2×2÷3=1.3 in the first exam and 4×2÷3=2.6 in the second. When confronted again ("But you had said that the girl had had the same results in both exams"), she replied "Yes, maybe it is unfair, but if we look at it in proportion those would be her marks". After some help she could apply the rule of three, and she concluded "This could even be of some usefulness in my job, but I prefer to apply exams with ten questions". As for M5, she only managed to use four times RP in L2 (three of which in the very easy structure (3,3)(1,1)), and never in L3. She did not even have incorrect attempts at RP; rather, all of her incorrect answers were due to centrations, with the exception of two RO. She did not follow the explanation of the second interview, and her new answers were of the kind "Let us reason: In which jar is the taste stronger? If I said the second jar, then it must be the first one".

CONCLUSIONS

Before tackling the proportionality issues, we will stress that two methodological features of this research have been extremely useful: 1) The use of the videotape for the analysis of results and of its edited version as a feedback in the second interview, and 2) the interrelation of qualitative and quantitative methodologies in the analysis of the results, where both were mutually enriched by each other's results.

We can only venture some hypotheses to account for the surprising results among the P-P-W problems. The Juice problem was always the first one to be applied (because of the good results it had always had), and that may have triggered an insecurity over what the questions of the interview were about; however, this does not explain why that had never occurred before nor why there were so many additive strategies in this context. The difficulties in the Exams problem could be due, as surprising as this may seem, to unfamiliarity: many teachers apply exams that are easily marked (as in M3's conclusion), or bought from commercial publishers that explain how to mark them. On the other hand, the good results in the Marbles problem could be related to a Probability workshop that these teachers had previously attended in a PD program.

In these interviews we had some glimpses of how school favours routine. The mentioned systematic use by M3 of an incorrect formula was not the only case; even M2 used mechanically an incorrect algorithm to find equivalent fractions.

Almost all the teachers tried to apply (often in vain) the strategy recommended by the Ministry of Education (SEP, 1992) for the teaching of proportional reasoning. It consists of creating tables of doubles, triples, etc of the quantities at stake in a problem. For instance, in the Notebook problem, 2 notebooks for 1 coin is equivalent to 4 notebooks for 2 coins, 6 for 3, etc. (Table 1). In a problem like (2,1)(3,2), which is L3, teachers make a double table such as in Table 2, and then they try to see some equivalences *within the same row*; although it is true that (4,2)(6,4) is equivalent to (2,1)(3,2), it does not solve the problem. However, another equivalent is (4,2)(3,2) (Table 3), which is a L1 problem: in store 1 you get 4 notebooks for 2 coins, and *for the same two coins* you only get 3 notebooks in store 2; the array (6,3)(6,4) signalled in Table 4 is another equivalent and is also L1.

Store	
N	C
2	1
4	2
6	3

Table 1

Store 1		Store 2	
N	C	N	C
2	1	3	2
4	2	6	4
6	3	9	6

Table 2

Store 1		Store 2	
N	C	N	C
2	1	3	2
4	2	6	4
6	3	9	6

Table 3

Store 1		Store 2	
N	C	N	C
2	1	3	2
4	2	6	4
6	3	9	6

Table 4

Thus, the mentioned recommendation is not wrong, but incomplete: Well used, the tables can definitely help in the solution of such problems, because they transform a L3 problem into a L1 problem. With this in mind, we can complete a proposition for the didactical treatment of proportional reasoning in its ratio-comparison modality:

First propose L1 problems to students, then L2, then L3. First propose Rate problems (favouring *BETWEEN* strategies), then P-P-W (and *WITHIN* strategies). When dealing with problems of a certain difficulty, never dump those you have seen before. When dealing with L3 problems, use tables such as Tables 3 and 4, looking in different rows for equivalencies. We sustain that such a proposition can be applied not only to students in grades 5 and beyond, but also to teachers in teacher training and in PD.

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HOW TEACHERS CONFRONT FRACTIONS

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This paper presents the results of a questionnaire about fractions applied, in the context of a Professional Development workshop, to a large sample of in-service primary teachers. The main objective of the questionnaire is an exploration of the teachers' ways of solving difficult situations where they have to put into play their Common Content Knowledge and their Special Content Knowledge of Mathematics.

This paper is centred in the knowledge about fractions held by a particular class of individuals: in-service primary teachers. A great deal of research has been conducted about the difficulties that young children face in the handling and understanding of fractions; however, as is often the case, these and other difficulties in the domain of arithmetic also occur among teachers (see e.g. Southwell & Penglase, 2005).

FRAMEWORK

Since the 80's many authors have highlighted the problems generated in the act of teaching by an erroneous or an incomplete mathematical knowledge of teachers or pre-service teachers. Southwell & Penglase (2005) sustain that "if teachers are not confident in their mathematical knowledge, they may find it difficult to ensure that their students gain confidence and competence." For these authors it is very important to be aware of the actual knowledge of pre-service teachers in order to design mathematical courses for them.

The situation of in-service teachers is somewhat different because they are not in the process of training but working, and even if they are aware that their mathematical knowledge is weak, they have to confront so many labour issues that they become reluctant to mathematical courses. Many teachers think that their difficulties with the maths are not with the contents but with how to teach them. In order to figure out this situation Shulman (1986) proposed a special domain of teacher knowledge that he called pedagogical content knowledge (PCK) as opposed to mathematical content knowledge (MCK); since then much discussion has taken place. Ball, Thames & Phelps (2008) have developed research in order to define more accurately Shulman's model; they have proposed some clarifications to PCK and a division of MCK into Common Content Knowledge (CCK, the word "common" referring to many other professions or people in general) and Special Content Knowledge (SCK, the mathematical knowledge and skill unique to teaching).

Thus, CCK, SCK and their interplay are fundamental in understanding how teachers comprehend and conceptualize the mathematics topics they teach, which is something we take as utterly important. In our research we undertook this in a Professional Development (PD) frame with in-service teachers; we designed and

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applied workshops in topics chosen by them, with activities that made them work with CCK and SCK at the same time. The workshops allowed us to collect information from a large amount of teachers, but unfortunately gave us very little time to further work with them, so, for instance, no interviews were possible. However, references of similar studies (e.g. Zazkis & Siroti, 2004; Southwell & Penglase, 2005) have encouraged us to present or results.

One of the workshops tackled a topic traditionally seen as arduous to teach and/or learn: fractions. It is known that teaching and learning fractions are complex processes, partly because fractions comprise a multifaceted construct. After the seminal identification by Kieren of four subconstructs of fractions: measure, ratio, quotient, and operator, Behr, Lesh, Post and Silver recommended adding as a distinct one the part-whole relationship (Charalambous and Pitta-Pantazi, 2005).

METHODOLOGY

Our Professional Development and Research Study was conducted in a working class zone of Mexico City, where a series of workshops called TAMBA were conducted with 400 to 800 in-service primary teachers of the public schools of the zone (Alatorre, Mendiola, Moreno & Sáiz, 2010). Each workshop was conducted in 20-30 groups of 20-25 teachers in a 2-hour session focused on one of the topics of the Mathematics curriculum of the level. The topics were decided after the teachers were consulted; the one most asked for was Fractions. As all of the workshops, this one consisted of a 20-minute individual task (IT) based on a questionnaire, a team task (TT) as the main activity, and a group discussion (GD); all the tasks were specifically designed for the study and both the TT and the GD were videotaped. In this paper we will report on the IT; it must be stressed that although the main purpose of the IT is to investigate the CCK and SCK related to fractions of the teachers, the videotapes registered the discussions about the processes and solutions of the TT, about issues related to the teaching of fractions, and about the final reconsideration of the IT's questionnaire as a closure.

For this workshop, 429 teachers came to the IT; 203 of them work two shifts in public schools either as regular teachers or in administrative tasks, and 134 of the 203 are in charge of two groups of students (not necessarily of the same grade). Adjacent to the questionnaire, which will be described below, several questions pertaining to the teachers' characteristics were asked: in which school they worked, in what position, etc. Two variables obtained from this information will be used in this paper: the Highest Grade (HG) and the Length of Service (LS). The HG is either the grade taught by teachers who have only one group, or the highest grade when they attend two groups (134 teachers have HG≤3rd and 212 have HG≥4th). The LS was asked directly as the amount of years they have been practicing as teachers (mean=13.4, SD=9.6). There is a high statistical association among LS and HG (F=5.80, df=1, 277, p=0.0166); LS is generally a growing function of HG, with means going from LS(1st)=8.1 to LS(6th)=13.5, with the exception of LS(5th)=14.6 (the 83 teachers only in administrative jobs, with undefined HG, have LS=16.3). Thus, among teachers who work with school groups, the most experienced are those of 5th grade.

The three questions of the questionnaire pretend to explore the teachers' ways of solving difficult situations where they have to use their CCK and SCK. In Q1 and Q2 a mathematical problem is present with simulated Grade 6 students' answers to it, which were designed considering frequent misconceptions and errors; the teachers were asked to mark each one as Right (R✓) or Wrong (W✗). Thus, the teachers not only had to find the solution of the problem but also to analyse and assess the given solutions. In question Q3 a problem is posed and the teacher is asked to say if it can be solved, and to either solve it or explain why not. A brief presentation of the rationale of each question follows (Figure 1 has a reproduction and a translation).

In Q1 the subconstruct is that of the fraction as measure, although this could be debated because of the lack of context. Solutions b(7.5) and e($^{15}/_{2}$) are two correct ways of solving the problem. In solutions a(NP) and c(6) the underlying error is thinking that *the amount of times* must be an integer. In solutions d(0.83) and f($^{5}/_{6}$) it is considered that the result is ⅓ of 2.5; solution d) could also come from considering that $2.5 \div ⅓ = 2.5 \div 3$. Solution g(1.2) responds to the need of operating with the number but not knowing how, and always using the largest number as the dividend.

In Q2 the subconstruct is that of the part-whole relationship. The size of one of the parts is to be found, which generally speaking is a simple task because it is very common in the classroom; the complexity here is that the shared part is in turn a part of the whole and thus of identifying the unit can be a source of difficulties. Solutions b($^{1}/_{8}$) and e($^{1}/_{8}$) are two correct ways of solving the problem. Solutions a($^{1}/_{6}$) and c($^{1}/_{24}$) lose part of the information: in the former the fact that the can only had ¾ of the content, and in the latter the amount of parts destined to the dog and each puppy (although it starts with a correct statement, dividing in 24 parts). Solution d($^{2}/_{9}$) consists of operating with the numbers without knowing what information that gives.

The subconstruct of the problem of Q3 is also that of the part-whole relationship. Here what is to be found is the size of the whole, which is generally a harder task than finding the size of a part, and in this case the difficulty is aggravated by the fact that the whole is a collection of continuous items, of size 11 $^{2}/_{3}$.

Thus, the design of the questionnaire was based on the revised subconstructs of Kieren et al., as presented by Charalambous & Pitta-Pantazi, (2005). The problems used were inspired in many investigations (e.g.Mack, 2001). The form in which Q1 and Q2 were asked allowed not only to obtain information about the CCK of teachers in regard of fractions, but also to emulate the interplay between CCK and SCK: The teachers were confronted with children's solutions to a problem, and these solutions had different levels of correctness and different representations of the numbers and other elements of the problems.

The two-fold analysis of the results was qualitative (also based on Kieren's revised subconstructs) and quantitative.

Questions Q1 and Q2 consist of two problems posed to sixth grade students and the solutions given by some of them. Mark each one as Right (✓) or Wrong (×)

Q1. *How many times does ⅓ fit in 2.5?*

a) No se puede resolver porque sobra una parte

b) 7.5, porque $\underbrace{\quad}_{\text{3 veces}}\underbrace{\quad}_{\text{3 veces}}\underbrace{\quad}_{\text{1.5 veces}}$

c) 6 porque son 3 en cada unIdad

d) 0.83, Porque dividí 2.5 entre 3

e) $\frac{15}{2}$, porque es el resultado de dividir $\frac{5}{2} \div \frac{1}{3}$

f) $\frac{5}{6}$: por esto: 2.5 es Parto cada cacho en 3 y cada pedacito es $\frac{1}{6}$ de . Quedan $\frac{5}{6}$

g) 1.2, Porque hay que dividir 3 entre 2.5

[Translation: **a)** You can't solve it because there's one extra part; **b)** 7.5 because 3 times + 3 times + 1.5 times; **c)** 6 because there are 3 in each unit; **d)** 0.83, because I divided 2.5 by 3; **e)** $^{15}/_{2}$ because that is the result of $^{5}/_{2} \div ⅓$; **f)** $^{5}/_{6}$, because of this: 2.5 is . I cut each part in 3 and each little piece is $^{1}/_{6}$ of . There are $^{5}/_{6}$; **g)** 1.2, because you must divide 3 by 2.5]

Q2. *Juan has a can of dog food. He gave ¼ to his dog and distributed the rest among the 6 puppies. What part of the can did each puppy receive?*

a) $\frac{1}{6}$, porque repartió el alimento entre los seis cachorros y no sobró nada

b) $\frac{1}{8}$, porque repartió $\frac{3}{4}$ de lata entre 6

c) $\frac{1}{24}$ porque si divido la lata en cuartos y después en sextos, la lata queda dividida en 24 partes.

d) $\frac{2}{9}$ porque $\frac{1}{6}$ entre $\frac{3}{4}$ es $\frac{4}{18}$

e) $\frac{1}{8}$ porque si parto la lata en 8 a la perra le toca $2/8$

[Translation: **a)** $^{1}/_{6}$ because he distributed the food among the six puppies and nothing was left; **b)** $^{1}/_{8}$, because he distributed ¾ of the can among 6; **c)** $^{1}/_{24}$ because if I divide the can in fourths and then in sixths, the can ends up divided in 24 parts; **d)** $^{2}/_{9}$ because $^{1}/_{6}$ divided by ¾ is $^{4}/_{18}$; **e)** $^{1}/_{8}$ because if I divide the can in 8 the dog gets $^{2}/_{8}$]

Q3. The following problem was found in a book of mathematical puzzles.

The drawing shows three fifths of the pizzas that a group of young people eat at a party. How many pizzas were eaten?

Does this problem have a solution? If it does, which is it? If not, why not?

Figure 1. Questionnaire used in the Individual Task

ANALYSIS OF RESULTS

Questions 1 and 2

Table 1 reports the frequencies (amount of teachers) that marked each of the children's solutions as Right (R✓), Wrong (W×), or neither (NR=no reply).

solutions	Q1 (⅓ in 2.5)							Q2 (The puppies)				
	a(NP)	b(7.5)	c(6)	d(0.8)	e($^{15}/_{2}$)	f($^{5}/_{6}$)	g(1.2)	a($^{1}/_{6}$)	b($^{1}/_{8}$)	c($^{1}/_{24}$)	d($^{2}/_{9}$)	e($^{1}/_{8}$)
R✓	58	**253**	49	137	**154**	118	41	62	**363**	53	26	**336**
W×	**304**	126	**291**	**213**	182	**219**	**298**	**312**	40	**317**	**334**	58
NR	67	50	89	79	93	92	90	55	26	59	69	35

Table 1. Frequencies of marks for each of the children's solutions of Q1 and Q2.

In Table 1, the frequencies in bold correspond to correct marks. Thus, 253 teachers correctly marked R✓ the solution b) in Q1; this and all other frequencies for correct R✓ marks are screened in dark gray. Also, 304 teachers correctly marked W× the solution a) in Q1; frequencies for correct W× marks are screened in light gray. In Q2 all the frequencies for correct marks (312 to 363) are greater than in Q1 (154 to 304).

But the teachers marking correctly one of the solutions were not necessarily the same who did likewise with the other ones in each question. A separate analysis is conducted for each teacher with the amount of Right solutions s/he recognised as such (R✓), the amount of Wrong solutions recognised as such (W×), and the total amount of correct marks (see Table 2). It must be stressed that among the 119 teachers who correctly recognised both Right solutions in Q1 only 57 also correctly recognised the five Wrong solutions. Thus, the teachers who marked correctly all of the children's solutions were only 57 (13%) in Q1, but as much as 220 (51%) in Q2.

	Amount of correct R✓ marks			Amount of correct W× marks						Amount of correct marks							
	0	1	2	0	1	2	3	4	5	0	1	2	3	4	5	6	7
Q1	137	173	119	62	26	40	91	104	106	23	39	32	51	88	83	56	57
Q2	28	103	298	61	32	77	259			14	32	41	40	82	220		

Table 2. Amount of teachers correctly marking correct/incorrect/total solutions

The teachers showed a large amount of inconsistency in their answers; two indicators of this are shown in the following paragraphs.

1) As many as 194 teachers (45%) in Q1 and 102 (24%) in Q2 marked as R✓ different solutions (here we consider that 7.5 and ${}^{15}/_{2}$ in Q1, 0.83 and ${}^{5}/_{6}$ in Q1, and both ${}^{1}/_{8}$ solutions in Q2 are not "different"). Table 3 shows a summary of this behaviour. Also, for instance, in Q1 120 teachers (28%) marked as R✓at least 7.5 (and/or ${}^{15}/_{2}$) *and* 0.83 (and/or ${}^{5}/_{6}$).

	NR	0	1	2	3	4	5
Q1	13	24	198	143	42	8	1
Q2	8	4	315	80	21	1	

Table 3. Amount of teachers giving R✓ marks to 0, 1, 2,… different solutions

2) In Q1, it could be expected that the 58 teachers marking as R✓ the solution a(NP) would mark as W× all or many of the other solutions; however only 24% had 6 W× marks and 28% had 5, but as many as 48% had 4 or even fewer W× marks.

Question 3 (Q3)

A qualitative analysis was performed with the answers to the third question, and four categories were defined: **AC:** Accepted Correct answer (when the teacher came to an exact or approximate result that seemed satisfying for him or her), **RNI:** Rejected Non-Integer (when the teacher declared that the problem does not have a solution or rejected the solution s/he had found because of a conflict with a non-integer result), **INC:** Incorrect solutions (incorrect solutions, either with incorrect numerical results, erroneous procedures, incorrect attempts at a solution, or only a description of a general procedure), and **NJ:** No Justification (when the teachers did not answer, or answered the first question but failed to explain why, or only gave partial descriptions of the problem's numbers, such as "7= $^3/_5$" or "7 pizzas"). Examples follow:

AC Accepted Correct	$^{65}/_5$=13; $^{100}/_5$=20; $^7/_3$=2.33; 2.33×5=11.65; 11 $^{65}/_{100}$=11$^{13}/_{20}$ 11 $^2/_3$, I will eat the extra ⅓ The fifth part corresponds to 2.33, the whole corresponds to 11.65 Approximately 12 11 pizzas and 2/3 too many
RNI Rejected Non-Integer	It doesn't have a solution as a fraction but it does as a decimal number: 11.65 pizzas The solution is 7 × $^5/_5$= $^{35}/_5$ divided in $^3/_5$ = $^{175}/_{15}$; 175 divided in 15 = 11.66 pizzas, but it does not have a solution because the amount of pizzas is not exact and that causes difficulties to the child's understanding Because you cannot divide in equal parts the 7 pieces in 3
INC Incorrect	7= .60, 1= 0.857142; 7= 0.60; 4285710 7 cannot be divided in 5 and I can't find the way to convert them to $^3/_5$ Because I don't know how many pieces each pizza had (No written response, but a circle divided in 5 pieces, each with the legend 2 ⅓) Using the rule of three

The overall percentages of these categories are: AC (28%), RNI (8%), INC (34%), and NJ (30%), but this varied according to HG, as shown in Figure 2.

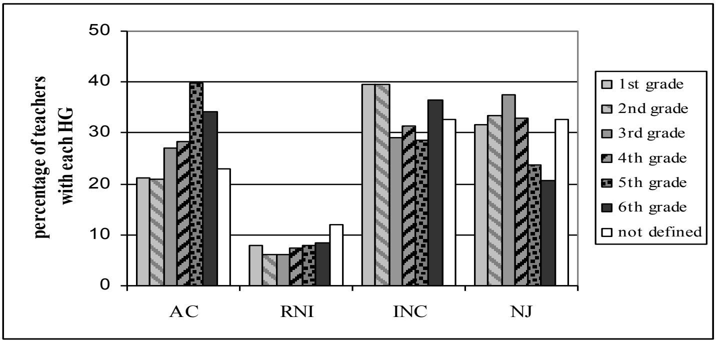

Figure 2. Distribution of Categories for Q3 among teachers with different HG

CONCLUSIONS

Some of our conclusions refer either to CCK or to SCK, but most refer to the interplay between both.

The first two questions were very difficult for the teachers (only 10% marked correctly all twelve children's solutions), and more so Q1 than Q2; this could be explained not only because Q1 is a measure subconstruct task and Q2 is a part-whole one, but also because Q1 lacks a context that Q2 does have. Some teachers seem to be in a very disadvantaged position regarding their CCK; for instance those who marked both b) and e) Right solutions as W× (83 in Q1 and 8 in Q2).

Some of the solutions in Q1, as well as the RNI category in Q3, correspond to the idea that a whole can only be divided an integer amount of times; this misconception could be originated and/or reinforced in the almost exclusive use of the area model, where the whole (a pie, a rectangle) is always divided into an integer amount of parts, and it affects teachers as well as their students. In the case of Q3, the reluctance of teachers to accept non-integer results can be aggravated by a careless reading of the question: many of their explanations mentioned the pizzas *bought* for the party, and thus seemed to ignore that the question referred to the amount that had been *eaten*. Thus, these teachers were correctly trying to make sense of the situation, which is a positive SCK (it is absurd to *buy* a non-integer amount of pizzas), but misunderstood it (it is not absurd for a group of people to *eat* a non-integer amount of pizzas).

One of the most striking results of this analysis is the large amount of what we have called inconsistencies in questions Q1 and Q2. In order to have a good explanation of them we would have needed to interview teachers about their markings, but we can venture some hypotheses. One relates the inconsistencies with a careless reading of the children's solutions; this could be partially or totally due to the questionnaire being answered in the setting of a workshop, not in the teachers' actual practice.

Another possible explanation could lie in the idea held by many teachers that mathematics problems have one and only one correct solution, and as a consequence they mark only the first one they see; this is particularly evident in Q2, where among the teachers who marked as R✓only one solution, 42 marked b($^1/_8$) and only 19 marked e($^1/_8$). Although this effect is also seen in Q1 with the solutions b(7.5) and e($^{15}/_2$), respectively marked by 59 and 7 teachers, this difference could additionally be due to a failure to recognise both solutions as equivalent or as two representations of the same result; the same failure could be present in the case of Q1's solutions d(0.83) and f($^5/_6$), marked in that order by 34 and 16 teachers. These hypotheses are supported by the fact that the difference in the number of teachers marking only one correct answer is much greater in the case of Q1 (59-7=52), where there are two representations, than in Q2 (42-19=23), where there is only one.

Some of the alleged inconsistencies could be seen under the light of an explanation based in SCK. For instance, some teachers could be assessing as correct or non-incorrect some of the children's solutions beside b and e in both cases, particularly when the solutions contain a correct conception or procedure; the teachers doing so would most probably be teachers with a long-time experience. This hypothesis, which refers to SCK, is supported by a highly significant ($p=0.0084$) negative correlation between LS and the amount of W× marks among Q1's solutions a, c, d, f, and g: the longer a teacher has been teaching, the fewer W× marks s/he assigns.

The size of the sample first suggested that many such statistical associations would be found between teachers' characteristics such as their LS and their HG and their performance; however this was not the case. The only other interesting result lies in some significant models of the relationship, shown in Figure 2, between the categories of Q3 and teachers' HG from 1 to 6. There is a linear growing effect of HG in AC (R^2=0.79), which could be the effect of "better" teachers often assigned to 5th grade, because of the widespread idea that it is the most difficult grade in primary school (also teachers with HG=5th have the largest LS). There is also a U-shaped behaviour in INC (R^2=0.69) and an inverse-U shape in NJ (R^2=0.89), which could be due to teachers with higher HG feeling the obligation to answer all questions posed and thus preferring to give an incorrect answer rather than no answer at all.

None of CCK, SCK or PCK is sufficient for a good teaching of mathematics; rather, all are necessary conditions. We have observed serious needs of these in-service teachers with respect to their CCK and their SCK on fractions, which most probably will impact on a poor teaching and learning of the topic.

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RATIO: A NEGLECTED DIVISION AT SCHOOLS

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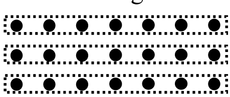
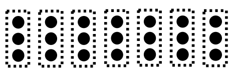
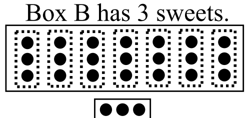
The research results presented in this paper are only a small part of an action research with the main aim of improving student teachers' understanding of mathematics. The re-teaching of mathematics was integrated with the teaching of pedagogy by asking student teachers (STs) to perform children's activities which have the potential to develop understanding of the subject. This report presents some results concerning STs' difficulties in distinguishing and representing different division situations, and some practical solutions proposed to reduce their difficulties within the time available.

SOME RELATED LITERATURE

The theoretical framework adopted this study is based on Goldin (2002), who argues that research on representations in mathematics learning has the potential to generate teaching methods which can make ideas more accessible to a larger majority of students. The literature describes division situations (structures, models, constructs, etc.) using different perspectives and category systems. Comprehensive reviews can be found in Greer (1992) and English and Halford (1995). Figures 1 to 3 exemplify three situations for the division $21 \div 3 = 7$. Division is usually associated with only two situations called: (a) sharing or partition (Figure 1), and (b) measurement or quotition (Figure 2). Yet Haylock (1995) defines a third division situation called comparison or ratio (Figure 3). Comparison situations can also be found in Williams and Shuard (1982), SMP (1987), and Greer (1992). Haylock, (1995) explains:

> The *ratio* division structure refers to situations where we use division to compare two quantities … if A earns £300 a week and B earns £900 a week, … B earns £600 more than A … The '600' is arrived at by the subtraction, '900–300' … B earns three times more than A … The 'three' is obtained by the division, '900÷300'" (p. 57).

(a) **sharing**: 21 sweets are equally shared among 3 boxes.  7 in each box	(b) **measuring**: 21 sweets are packed in boxes with 3 sweets each.  7 boxes are needed	(c) **comparing**: Box A has 21 sweets. Box B has 3 sweets.  7 times more in box A 7 times less in box B The ratio of A to B is 7
Figure 1	**Figure 2**	**Figure 3**

Modelling comparisons with subtraction require the representation of two different quantities (e.g., English and Halford, 1995). This seems to be a strong reason for defining a third division situation in which two quantities are represented and compared (Figure 3). With respect to iconic representations, the literature presents several types of diagrams for division (e.g., Greer, 1992, p. 281-282), but most of them are not related to comparison. Exceptions were found in Williams and Shuard (1982, p. 404) and SMP (1987) in which two quantities or two measurements are represented and compared with division (as in Figure 3). Expressing the quotient of a division as a ratio is indeed a very powerful comparative tool in data handling: "Those people who eat popcorn ingest 2½ times more fibre than those who do not eat it". However, expressing ratio comparisons between two quantities A and B in terms of "A is 2½ times more than B" or "B is 2½ times less than A" is a more meaningful comparison than simply stating that "The ratio of A to B is 2½".

Research tends to show that primary school student teachers (STs) think of division predominantly in terms of sharing situations (forming a certain number of equal groups) and seem unable to access measurement situations (forming groups with a certain number of units) (e.g., Tirosh and Graeber, 1990, Simon, 1993, and Rizvi and Lawson, 2007). No studies were found about STs' knowledge of division in comparison situations (comparing two groups with a certain number of units each). With respect to ratio, Simon and Blume (1994) report that STs have difficulty in recognising ratio relationships. Similarly to school students, they tend to select additive strategies when multiplicative strategies are appropriate. Only 7 out of 26 STs used the ratio width:length to find out the most and the least square of three rectangular shapes. All other STs used the difference between the two sides.

According to Simon and Blume, some of the difficulties faced by the STs seem to be connected to their weak pre-requisite knowledge about comparative structures (i.e., if the ratio height:length is 3:2, the height is 1½ times the length of the base). STs also do not seem to understand the relationship between a symbolic expression for a ratio and the associated real-world situations that it represents. Other more recent reports show similar results (e.g., Ilany et al. 2004). So the particular research question related to the present report is: "In what ways can primary school STs be helped to improve their understanding of division situations?".

METHODOLOGY

An action research was initiated in 1995 with the aim of improving primary school STs' understanding of the mathematics they were expected to teach in the future (Amato, 2004). The action steps of the research are being performed at University of Brasília, Brazil, through a mathematics teaching course component in pre-service primary school teacher education. The component currently consists of only one semester with 60 hours. In the action steps of the research, the re-teaching of mathematics is integrated with the teaching of pedagogical content knowledge by asking the STs to perform children's activities which have the potential to develop mathematical understanding for most of the contents in the primary school curriculum. About 90% of the new teaching program became children's activities. The model of action research adopted is based on McTaggart and Kemmis (1982, p. 5) who emphasize the type of knowledge which must be sought through the action research method: "trying out ideas in practice as a means of improvement and as a means of increasing knowledge about the curriculum, teaching and learning".

Four data collection instruments were used to monitor the effects of the strategic actions in the two main action steps of the research: (a) researcher's daily diary; (b) middle and end of semester interviews; (c) beginning, middle and end of semester questionnaires; and (d) pre- and post-tests. Much information was produced by the data collection instruments, but due to space limitations, only some classroom observations and some STs' responses related to division situations with natural numbers are presented in this report.

A summary of the main (a) research aims, (b) data collection instruments, and (c) teaching activities can be found in Amato (2004). The action steps had the duration of one semester, thus each action step took place with a different cohort of STs. In the third and subsequent action steps, the data collection became more focused on particular mathematics contents and representations which presented more difficulties for the STs during the first two action steps (the first and second semesters). The sequence of activities actually being used for teaching division situations with natural numbers is:

(1) Translating from concrete materials to verbalizations. First I show the class a large transparent plastic bag with 21 "sweets" (21 equal shampoo bottle cups) and ask 3 STs to stand up in front of the class. I take away from the bag a group of 3 sweets and give 1 sweet to each ST. I repeat the same process until the bag is empty and each ST receives 7 sweets. A ST is asked to verbalise the situation. Then I return the 21 sweets to the large bag, take away from the bag a group of 3 sweets, and insert the 3 sweets into a small transparent plastic bag. I repeat the same process until the large bag is empty and 7 small bags are filled. Another ST is asked to verbalise the new situation. Finally I mention that both division situations involve repeated subtraction, are recorded as 21 divided by 3, and write $21\div3$ on the blackboard.

(2) Practical work and discussion about the sharing and measurement situations for division. The STs' bodies are used as units and they are asked to: (a) stand up and organise themselves into 2 equal groups, (b) organise themselves into groups with 2 STs each, and (c) sit down, draw and compare the two division situations. If the number of STs is odd, I include myself as a unit. Finally I mention that the first situation is called "sharing" and the second one is called "measurement".

(3) Practical work and discussion about the two situations of division. The class is divided into pairs. The STs sitting on the left-handed side of the pairs are asked to use 21 units of a plane version of Dienes' blocks (paper squares sized 1cm by 1cm) to model sharing 21 sweets equally among 3 people. The STs sitting on the right-handed side of the pairs are asked to use 21 units to model packing 21 sweets into boxes with 3 sweets

each. Then the pairs are asked to compare the two situations (Figures 1 and 2). Finally I revise that the first situation is called sharing and the second one is called measurement.

(4) Practical work as a home assignment. During the lecture I show the class a big album made with 8 sheets of A3 paper illustrating 4 sharing and 4 measurement situations involving grouping chocolates (grey paper rectangles sized 5.6 cm by 8 cm) for the sums 6÷3, 12÷4, 10÷2, and 8÷2. The STs are asked to verbalise the division situation illustrated in each page. As a home assignment, the STs are asked to manipulate small "chocolates" (grey paper rectangles sized 2.8 cm by 4 cm) in order to construct a similar album using 8 sheets of A4 paper to illustrate 4 sharing and 4 measurement situations for the same sums presented in the big album. On the instructions' page each situation is revised with an example similar to the ones presented in Figures 1 and 2 which include (a) the title "sharing" or "measuring", (b) words describing each situation and (c) iconic representations with lines circling groups of rectangular chocolates.

(5) An exercise to revise the sharing and measurement situations. The instructions include an example for each situation similar to the ones presented in Figures 1 and 2. The STs are asked to represent with pictures and with words the two division situations for five division sums (18÷6, 40÷8, 10÷5, 60÷15, and 60÷6). Each exercise have two parts with the same number of dots (e.g., 18 dots in the case of 18÷6). In part (a) of each exercise, the STs have to circle the dots to illustrate a division in an iconic sharing situation. In part (b), the STs have to circle the same number of dots to illustrate the sum in an iconic measurement situation. At the right side of each set of dots there is also some space for representing each situation with words (i.e., a related word problem).

(6) Another exercise to revise the sharing and measurement situations for the division 14÷2. The instructions ask the STs to: (a) write the name the division situation being illustrated (sharing and measurement), (b) represent each situation with their own pictures, and (c) write a word problem for each situation.

SOME RESULTS

Using children's activities proved to be an appropriate strategy to improve STs' understanding of mathematics since the majority of STs said, and many indicated in the post-tests, that their understanding had improved (e.g., Amato, 2004). In each semester, the majority of the STs mentioned they enjoyed the activities in the teaching programme. As an example with respect to division situations using natural numbers, one ST wrote "I noticed that traditional content can be learned in diverse ways and even with a playing aspect which facilitates the understanding such as using our body to exemplify groupings or divisions" (activity 2). However, the distinction between the sharing and measurement situations for division proved to be one of the most difficult content in the programme. At the beginning of the first semester one ST asked for help in preparing a lesson about division. She was having her first teaching experiences at school. I could notice how difficult it was for her to distinguish between the two situations. For this reason, the initial activities about division were revised once more before teaching the first semester class. Yet some STs presented similar difficulties. In the second semester I decided to perform a systematic research about the frequency of the two situations in some school textbooks. There were far fewer word problems involving measurement situations.

After the first semester it became clear that the number of activities for distinguishing between the sharing and the measurement situations should be increased and spread over a greater time along the semester. One way to provide STs with more activities for those topics without using the short time available in the classroom was to increase the number of home assignments. A new activity asked the STs to construct an album (activity 4) illustrating the two situations for division. In the third and subsequent semesters two other home assignments were included (activities 5 and 6).

After performing an extensive and systematic review of the literature about ratio and proportion in the years 2002-2003, it also became clear that primary school teaching should include activities for comparison divisions and for contrasting the concepts of difference (comparing with subtraction) and ratio (comparing with division). However, new difficulties have emerged with the inclusion of a third situation (comparison) in all the programme activities related to division. For example, at the end of activity (3) after being extended to include a comparison situation, I explain:

> Anne has 21 sweets and Beth has 3 sweets. Anne has 7 times more sweets than Beth. Beth has 7 times less sweets than Anne. When we compare 21 and 3 with subtraction we use the word “difference”. We say that the difference between 21 and 3 is 18. There is also a special word for comparing quantities with division. Try to remember [Time is given. Usually nobody answers the word “ratio”].

Some STs attempt different words and in some classes one ST finally answers “ratio” while some demonstrate surprise in one way or another. In the second semester of 2009 nobody could relate division to ratio and one ST said “that was really challenging! I had no idea about that. Ratio was a mystery for me”. I reassured the class that although I could relate well ratio to division, I could not clearly relate division to ratio (i.e., the opposite relationship) before the literature review in 2003.

In the third and subsequent semesters new tests with more convergent questions were included as another form of diagnosis and assessment. In the year of 2004, a new question was included asking STs to represent $24 \div 8$ with pictures and with words. This question is similar to the ones in activity (5) after the inclusion of comparison situations. Some STs, especially those who had a very good attendance record, could circle groups of dots in different ways in order to distinguish the three division situations (Figures 1 to 3). They could also write three correct comparison questions using the words “times more”, “times less”, and “ratio”. However, other STs continued to present difficulties in distinguishing and in representing the three division situations even in the case of small numbers. When illustrating a comparison with division (as in Figure 3), they did not circle 3 groups with 8 dots in order to show that a group of 24 dots is 3 times more than a group of 8 dots. They simply drew a line separating the two quantities being compared.

Every semester one or more STs usually ask why the results of comparisons (such as the one in Figure 3) were “7 times more” (i.e., “ratio times more”, abbreviated as [rx>]) instead of “6 times more” (i.e., “ratio minus 1 times more”, abbreviated as [r-

1x>]). They believed the word "more" only implied the extra dots in the larger quantity and it did not include the first three dots that were in 1 to 1 correspondence to the smaller quantity. That is, they focused their attention on the number of groups in the difference between the dots as it is done when comparing with subtraction. My usual explanation is:

> I draw two quantities as in Figure 4, circle the sweets (dots) into groups with 3 sweets, and use my hand to separate the first group of sweets in Anne's larger quantity from the other 6 groups. Then I say "Anne has 1 time or the same as Beth". Next I separate the first two groups of sweets from the other 5 groups and say "Anne has 2 times more than Beth or the double". Then I continue to separate the groups and to ask the STs to verbalise the next comparisons until finishing all the groups in Anne's quantity.

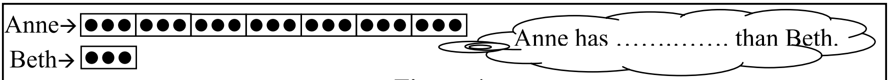


Figure 4

During the classroom activities in the second semester of year 2009, one ST commented that she knew "that the correct answer was 7 times more instead of 6 times more, but her brain still did not agree with it". It was difficult for her and other STs to overcome this misconception as their thinking seemed to be dominated by subtraction comparisons. I decided to perform another systematic research in some primary and secondary school textbooks. Eight collections and a total 32 primary school textbooks (grades 1 to 4, 7-10 year olds) were analysed. There were only two comparison division problems in just one of these textbooks. Even in the chapters about ratio in 27 secondary school textbooks for grade 6 (12 year olds), all the questions asking to compare two quantities A and B were of the type: "What is the ratio of A to B?" or "What is the ratio of B to A?". A few books presented in the initial explanations about ratio the words "A is the double of B", "A is the treble of B", "A is six times B", "A is 1½ B" and "a scale informs us how many times the real object was reduced in the map". However, I could not find any problems asking questions such as: "How many times is A bigger than B?" or "is B smaller than A?".

At the end of the second semester of 2009, I decided to administer a new diagnostic test in order to probe how prevalent this misconception was among STs. The test consisted of 8 similar questions with the following division comparisons: (1) 2 times less, abbreviated as [2x<], (2) 4 times more [4x>], (3) 5 times less [5x<], (4) 3 times more [3x>], (5) 5 times more [5x>], (6) 2 times more [2x>], (7) 3 times less [3x<], and (8) 4 times less [4x<]. The question wording was: "Write how many times more or how many times less Anne has in relation to Beth:" The pictures in the test items were similar to the one presented in Figure 4 for the comparison "7 times more" (abbreviated as [7x>]). In the inverse comparisons such as "7 times less" ("ratio times less", abbreviated as [rx<])", the larger quantity was drawn below the smaller quantity. The categories of responses found are presented in the first and second columns of Table 1.

Table 1: Frequency of responses to questions 1 to 8.

Abbreviations used in table 1: Letters and other symbols are used to abbreviate certain words in table 1: ratio (r), difference (d), times (x), more (>), less (<), and inverse ratio (1/r).
[r x >] means “ratio times more” and [r x <] means “ratio times less”.
[1/r] means the fraction “1/r of”, e.g., a comparison such as “A has 1/5 of B”.
[r-1 x >] means “ratio-1 times more” and [r-1 x <] means “ratio-1 times less”.
[d x >] means “difference times more” and [d x <] means “difference times less”.

		Frequency of responses per question:							
	Question number →	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Cate-gory	**Ratio Anne to Beth →**	6:12	12:3	2:10	15:5	15:3	10:5	4:12	3:12
	Type of response ↓	[2x<]	[4x>]	[5x<]	[3x>]	[5x>]	[2x>]	[3x<]	[4x<]
A	[r x >] or [r x <]	24	26	27	27	27	27	27	27
B	double, treble, … or [1/r]	2	1	1	1	1	2	1	1
C	both categories A and B	1	2	1	2	2	2	1	1
D	[r x a >] or [r x a <]	2	1	1	1	1	1	1	0
E	[r–1 x >] or [r–1 x <]	9	10	11	10	10	7	10	10
F	[r–1 x a >] or [r–1 x a <]	2	4	3	4	4	3	3	3
G	[r a >] or [r a <]	2	1	1	1	1	2	1	1
H	[d x >] or [d x <]	4	3	4	3	3	3	4	4
I	[d x a >] or [d x a <]	1	1	1	1	1	1	1	1
J	Subtraction of fractions	3	2	2	3	2	3	2	2
K	Other responses	3	2	1	0	1	2	2	3
D to I	Sum of categories D to I	20	20	21	20	20	17	20	19

A total of 53 STs answered the test. Categories A, B, and C were correct division comparisons. It should be noted that in Portuguese there are words such as double [2x>], treble [3x>] and quadruple [4x>] up to “10 times more” (see category B in the first column of Table 1). Only a few STs wrote comparisons such as “Anne has half (or ½) of Beth” (in question 1, category B) or “Anne has the quadruple of Beth” (in question 2, category B). Category J was mathematically corrected as in Portuguese it meant a subtraction comparison using fractions such as “6 is 1/2 less than 12” (for ratio 6:12), but it was not what the question was asking for.

Apart from categories E and H, another indication that some STs were mixing subtraction with division comparisons was the existence of responses with an “a” before the words “more” and “less” in categories D, F, G, and I. In Portuguese, subtraction comparisons are written with an “a” before these words: “18 **a** mais” (translation: “18 more”) and “18 **a** menos” (translation: “18 less”). For some STs the wrong comparisons related to subtraction persisted even after the end of the semester.

SOME CONCLUSIONS

Ratio proved to be a neglected division situation in most school textbooks and in the literature about division. STs’ previous experiences with comparison divisions were scarce. The persistent use of comparisons related to subtraction by adults, who will soon become primary school teachers, seems to be more connected to lack of familiarity with multiplicative strategies than to be the result of immaturity. Instructional constraints were, in part, responsible for STs' difficulties in relating division to ratio. Like STs, teacher educators' mathematical and pedagogical knowledge is still under construction and also presents weaknesses that are transferred to their teaching. Revising the literature previously mentioned, which explicitly relates division to comparison situations, was essential in helping me connect these concepts and make the necessary changes in the programme. Teachers need more time to familiarise themselves with measurement and comparison divisions. They also need to acquire the pedagogical knowledge to start teaching comparison divisions to young children. Therefore, it is recommended that primary school teaching and teacher education programmes place a greater time and attention in distinguishing the three division situations mentioned in Figures 1 to 3.

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A MODEL FOR DESCRIBING REASONING IN LOGICAL TASKS

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The paper presents a cognitive model that describes reasoning encompassing formal logic and semiotics, according to two dimensions: formulation and formality. Formulation addresses students' awareness of formal logic rules, referring to both the degree of certainty and Radford's layers of generality. Formality addresses the appropriate use of logic rules. Combining these dimensions, we derive, characterize and discuss five possible behaviours. Evidence is provided by the analysis of written protocols from an experimentation with undergraduate students.

INTRODUCTION AND BACKGROUND

Reasoning plays a crucial role in human activities that are specifically cognitive (such as: learning, development, knowledge processing), but also in those generically creative and social. Amongst reasoning activities, let us consider for example inferences: they allow us to get new information from previous one. Inferences, in fact, help human beings to access knowledge (whether conscious or implicit), and apply it to specific (new) situations. Mathematical formal logic, in any one of its formulations, cannot represent a full formalization of all kinds of reasoning activity. Several researches from the beginning of the 20th century attempt to link reasoning to studies in the field of formal logic (for a review see Casadio, 2006; Toulmin, 1958), but the relationship between reasoning, as an everyday activity, and formal logic in mathematics is still rather complex: according to Dapueto and Ferrari (1988)

> the contexts in which the "daily" reasoning develops and those in which the deductions are built, which (also) mathematical logic deals with, are completely different, with different criteria of acceptability and coherence. (p. 779)

It is possible, however, that some forms of 'daily' reasoning enter also in formal logic tasks. For this reason, we present a cognitive model for describing students' reasoning in typical formal logic tasks, such as syllogisms and if-then statements. Formal logic itself does not only provide a context in which individuals perform tasks, but it can provide tools for analysing such tasks. Indeed, formal logic can be regarded as a tool for studying reasoning processes, when considered in a wider way as the expression of some aspects concerning *language*. Historically, the birth of modern logic and its development are characterized by the change in the role of language in mathematics from being only a communication tool to becoming also a manipulation one. In this paper, formal logic lenses are provided by Reid's model (2002). Since formal logic alone is not enough, we take into account also other Mathematics Education theoretical lenses (Piattelli-Palmarini, 1995; Radford, 2001). All these lenses enter our model as different dimensions and our aim is to study how they are intertwined in framing students' logical reasoning. We do not consider the

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context as a variable in this study, even if we are aware of its importance in mathematics teaching/learning processes.

According to Radford's cultural-semiotic approach that considers cognition as a *reflexive mediated activity*, mathematical concepts are objectified at different layers of generality depending on the semiotic means that mediate activity. Radford (2001) identifies three increasing levels of generality: a *factual generalization,* when the objectification of the general scheme takes the form of a *perceptual/sensorimotor semiosis*; a *contextual generalization* when the general scheme is objectified by more abstract semiotic means that, however, bear the spatial and temporal origin of the situation they come from; a *symbolic generalization* when the general scheme is objectified by symbolic language that does not allow any relation with the spatial-temporal dimension. The learner lives a desubjectification of meaning, namely a rupture with his spatial-temporal and sensorimotor experience.

Reid (2002) describes mathematical reasoning across five dimensions. Among them, "*formulation* refers to the degree of awareness the reasoner has of his own reasoning" (p. 105). Semiotics helps characterizing formulation. According to Radford (2001), a high degree of awareness can be identified with a symbolic generalization, where the cultural logical discourse is objectified by the students at an interpersonal level. On the counterpart, we claim that when there is an unaware inconsistency between the meaning objectified by the individual and the cultural meaning of logical activity, personal opinions and misconceptions play a crucial role in guiding learner's logical reasonings. For instance, several studies show that both children and adults make errors because they infer not only on the basis of the premises, but either introducing other premises or referring to the common sense. The unaware inconsistency can occur both when students use a proper formalism, and when they do not. For this reason, another dimension from Reid's model is taken into account: *formality.* Before talking about it, let us further characterize formulation. Both when there is a good level of awareness and in the opposite case of unaware inconsistency, we claim that the individual tends to feel sure of his resoning. In the first case, such a certainty is provided by the formal logic rules: following them correctly, in fact, leads the reasoner to arrive at a conclusion that is almost always correct and formally grounded. In the second case, according to Piattelli-Palmarini (1995), a student who is leaded by misconceptions and personal opinions tends to be sure of his reasoning, since they seem to be very reliable.

Let us suppose that good awareness and unaware inconsistency represent two poles. We argue that there is also something in between these poles: in this case, students at the same time have a certain degree of awareness that they cannot use only their personal opinions, and they suspect that they do not have enough theoretical tools for reasoning within the formal logic context. It can be considered as a contextual generalization of logical discourse where the individual's experience and opinions are relative to a particular context (Radford, 2001). Indecision and lack of responses are expected in this kind of behaviour. We claim that this situation is didactically the most interesting, since the learner is in Vygotskij's (1978) Zone of Proximal Development (ZPD), where teaching/learning processes are effective. In our case, ZPD is triggered by a "cognitive conflict" caused by the disagreement between an intuitive model and the mathematical model (Fischbein, 1998). Due to the conflict, the individual is induced to reorganize the previous conceptions for integrating new information coming from the new situation (D'Amore, 1999; Perret-Clermont, 1979). The momentary incorrect conceptions, waiting for a more elaborate cognitive arrangement, are a transitory cognitive moment from a naive conception to a more elaborate one and closer to the (logically) correct conception.

We now consider the aforementioned dimension of *formality* (Reid, 2002): it "refers to the degree to which the expression of the reasoning conforms to the requirements of mathematical style" (p. 105). Combining together *formality* and the two poles of *formulation*, there are four possible distinct cases: (1) good degree of awareness and proper use of formal logical tools; (2) good degree of awareness, but incapability in using logical tools; (3) unaware inconsistency and no use of formal logic; (4) unaware inconsistency, but proper use of formal logical tools. Regarding case (4), is it possible, however, to use formal logical rules when misconceptions/personal opinions enter in reasoning? Some researchers, in fact, claim that in everyday thinking people often use logics that are different from the formal one (Ayalon, 2008). In cognitive psychology, Wason (1966) and Johnson-Laird (1983) point out adults' difficulties in doing also simple inferences. According to Johnson-Laird's theory, common reasoning is not based on formal rules, which are independent from the content, but on construction and manipulation of mental models or representations (Girotto & Legrenzi, 1999). Our study highlights cases of students resorting both to formal logical rules and to misconceptions/personal opinions.

THE COGNITIVE MODEL

Is it possible to describe, and to what extent, reasoning according to the degree of awareness and the accordance with formal logic rules? Is it possible to further characterize it though the means provided by the cultural-semiotic approach, and in terms of certainty? These are the research questions that inform our study. The aim of the paper is the construction of a model for describing students' logical reasoning, according to the theoretical background we presented and discussed in the previous section. The interplay of *formulation* and *formality*, along with the role played by semiotics and the reasoner degree of certainty, allows identifying five behaviours.

We address the first one as *R* and it is the case of the reasoner that is aware of his reasoning according to formal logic rules (formulation), and also he properly uses them (formality). Moreover, the reasoner is expected sure of his statements. The second behavior is called *r*: there is the same degree of awareness as in the first one (as well as the same expected certainty), but reasoning has formal imperfections. On the counterpart, when there is an unaware inconsistency between the meaning objectified by the individual and the cultural meaning of logical activity, the reasoner uses misconceptions/personal opinions (formulation) and two behaviours are possible. We refer to *M* when almost none formal logical rule is employed. When we observe some use of formal logical rules, we refer to *m*. According to Piattelli-Palmarini (1995), the reasoner is certain of his statements in both *M* and *m* behaviours. The last behaviour, which corresponds to the intermediate degree of awareness, is called *I*: plausible interpretation. *I* is characterized by containing some dubitative elements (such as conditional verbs). As a consequence, it seems that students perform a reasoning that can be open for being discussed/changed.

We now seek experimental evidence for the relationship between Reid's formulation and formality, their semiotic characterization, and the degree of certainty, according to the five aforementioned behaviours.

METHODOLOGY

A test was administered to 111 undergraduate students (86 in November 2009, 25 in November 2010), with a weak mathematical background, before the beginning of a course in logic. Hence, answers were not influenced by the teaching of logical topics. Students carried out tasks about reasoning. The analysis in this paper refers only to an example of syllogism (task 1), and to one of if-then task (task 4). We choose these tasks because the syllogism is the most classical kind of reasoning scheme and the if-then statement represents one of typical kind of human reasoning. Furthermore, regarding the if-then statements, formal fallacies are very frequent: there is a strong tendency of people to interpret "if-then" statements as "if-and-only-if" statements (Ayalon, 2008; Leron, 2004). In the case of task 4, for example, we predict that a consistent percentage of students would mark the alternative E as the correct one: it would have been true, in fact, if the statement had been an "if-and-only-if" one.

As shown in figure 1, students were requested: (1) to mark one answer, (2) to say how much they feel certain of their response, and (3) to provide a written justification of their answer. On the basis of the written justifications, we make inferences about the reasoning process students activated in their solving activities. According to our theoretical framework, in presenting our results we now provide a detailed analysis of five protocols, respectively classified as *R, r, M, m* or *I.*

Task 1

No ingenuous person is a bad person.
Some bad person is adult.
So??? is not ingenuous.

A) some adult (correct)
B) every adult
C) some ingenuous person
D) some bad person
E) every bad person

How much are you sure of your answer?
Explain your reasoning:

Task 4

Given that *all the dogs that bark do not bite*, one of the following assertions is necessary true:

A) the dogs that bite do not bark (correct)
B) some of the dogs that bark bite
C) all the dogs do not bite
D) all the animals that bite are not dogs
E) if the dogs don't bite, they bark

How much are you sure of your answer?
Explain your reasoning:

Figure 1: tasks 1 (syllogism) and 4 (if-then).

EXPERIMENTAL EVIDENCE

In figure 2 we report two histograms with the relative frequencies of selection of each alternative of tasks 1 and 4. We observe that almost no one omits the answer, and in both cases there is one alternative that has been chosen by a high percentage of respondents (together with the correct answer). In task 4 the choice of alternative E highlights the typical misconception students have, considering the if-then tasks as if-and-only-if ones.

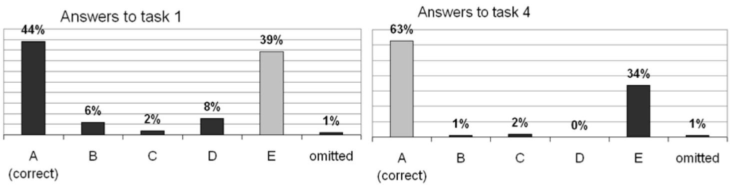


Figure 2: the relative frequencies of selection of each answer.

In figure 3, three answers to task 1 are shown. The English translation is provided in the analysis carried out below.

Let us firstly look at Anna's protocol (figure 3). In providing her justification, she draws a universe set for "human" and inside it she draws three sets for "ingenuous", "adult", and "bad". She matches this graphical representation with the sentence "*Certainly some adult isn't ingenuous because he is a bad person and the bad persons aren't ingenuous*". She answers correctly to task 1. The justification provided by Anna was classified as *R*: her reasoning conforms to the requirements of mathematical style, referring both to *formulation* and *formality*. From a semiotic point of view, she uses a symbolism that is general and desubjectified at an interpersonal level. According to our expectations, Anna is quite sure of her answer, since following a correct logical reasoning provides a good level of certainty.

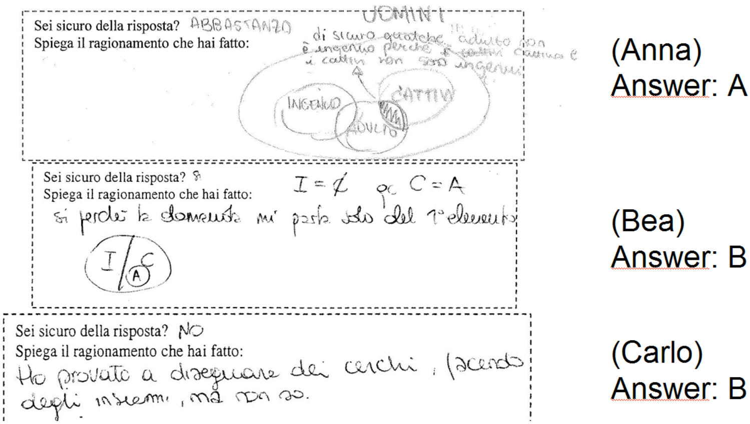
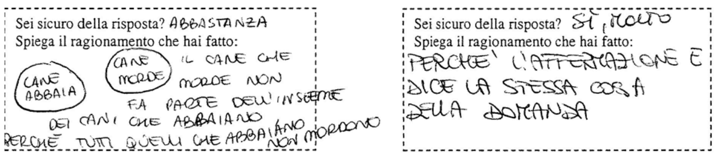


Figure 3: three answers given to task 1.

On the counterpart, Bea answers wrongly to task 1. Looking at figure 3, she makes a partition of the universe in I (ingenuous persons) and C (bad persons). Moreover, she puts the set A (adults) inside the set C, revealing also an incorrect interpretation of the existential quantifier "some" ('qc' in the protocol is an Italian abbreviation for 'some'). She writes also: "*Yes, because the question speaks only about the first element*". The justification is classified as *m*, because Bea fails in interpreting and representing the statements according to *formulation*, but we observe the use of formal logic rules according to *formality*. Bea's mistakes involve both the interpretation of the question ('only the first element' has been interpreted as 'only the first sentence', and the fact the only 'ingenuous' is stated in the conclusion may have lead her to think that only the first sentence is involved), and the representation through Venn diagrams: she represents a partition in C and I, with A included in C. In Bea's case, there is some semiotic manipulation, but misconceptions have a central role in guiding her reasoning. According to our expectations, she asserts to be sure of her (incorrect) answer: when leaded by misconceptions, students feel to be sure.

Let us now look at Carlo's protocol (figure 3). In his justification he uses some dubitative elements. He writes: "*I tried to draw some circles, making some sets, but I don't know*". Mentioned circles and sets are not present in the protocol. We classified it as *I*. We infer that maybe there is a tension between at least two representations in Carlo's mind, hence he is in doubt about his answer. Moreover, Carlo says to be not sure of it, according to our expectations.

In figure 4, two answers to task 4 (an example of if-then task) are shown.


Sei sicuro della risposta? ABBASTANZA
Spiega il ragionamento che hai fatto:
CANE ABBAIA
CANE MORDE
IL CANE CHE MORDE NON FA PARTE DELL'INSIEME DEI CANI CHE ABBAIANO PERCHÉ TUTTI QUELLI CHE ABBAIANO NON MORDONO

Sei sicuro della risposta? SÌ, MOLTO
Spiega il ragionamento che hai fatto:
PERCHE' L'AFFERMAZIONE E DICE LA STESSA COSA DELLA DOMANDA

(Daniela) Answer: A (Elvira) Answer: E

Figure 4: two answers given to task 4.

Daniela answers correctly to task 4 and she represents a set for "dog barks" and a set for "dog bites". There is no intersection between the two sets. This is accompanied by written explanation in a quite formalized daily language: "*the biting dog doesn't belong to the barking dogs' set, because all those which bark don't bite.*" Daniela's justification is classified as *r*: according to *formulation*, in fact, her reasoning is correct, but there is some mistake in her representations. The *barking dogs' set* is not explicitly drawn into the *not biting dogs' one*. She asserts to be quite sure of her (correct) answer.

Elvira provides an incorrect answer: E. In her explanation, Elvira writes: "*alternative E states the same thing of the assertion in the question*". This is a typical misconception about the "if-then" statements: 'if-then' is, in fact, regarded as an if-and-only-if statement. This leads us to classify it as *M*. According to our expectations, Elvira declares to be very sure of the answer.

DISCUSSION AND CONCLUDING REMARKS

In this paper we carried out an analysis of reasoning that pivoted around formal logic. We provided a model for reasoning that, along with formal logic, considers the role played by semiotics and the degree of certainty. The interplay of *formulation* and *formality* allows identifying five reasoning behaviours. It would be interesting to investigate why the use of (proper) logical formalisms does not ensure overcoming the hindrance of misconceptions and personal opinions, as in the *m* behaviour. Data prove the existence of the aforementioned reasoning behaviours and the degree of certainty predicted for each of them. This has been also confirmed by a first quantitative analysis that has not been reported in this paper: both when students are aware of their reasoning, and when misconceptions/personal interpretations play a central role, the majority of them declare to be sure of the answer. Even if we observed few students performing *I*, the majority declares to be unsure of the answer.

In order to corroborate and strengthen our model, it is necessary to both go beyond Radford's layers of generality and analyse students' use and integration of different semiotic registers, and go beyond the mere use of syllogisms and if-then tasks, investigating on the connection between argumentative and proving processes, in order to contribute to the international debate regarding the relationship between argumentation and proof.

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ARGUMENTATION IN EXPLORING MATHEMATICAL MACHINES: A STUDY ON PANTOGRAPHS

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The activities with the Mathematical Machines can promote interesting and important processes of generating conjectures, argumentations, and mathematical proofs. In this paper, we analyse argumentations produced by the students in exploring pantographs for geometrical transformations, showing how the argumentations refer to some specific elements of the pantographs (the structure, the movement, the drawing traced by the machine) and to different components (figural and conceptual) of the geometrical figures representing the linkages.

INTRODUCTION

The Laboratory of Mathematical Machines (MMLab), at the Department of Mathematics of University of Modena and Reggio Emilia (Italy), contains a collection of instruments, called Mathematical Machines, which have been reconstructed with a didactical aim, according to the design described in historical texts. In this paper, we refer to the geometrical machine as a tool that forces a point to follow a trajectory or to be transformed according to a given law (Bartolini Bussi & Maschietto, 2008).

The MMLab organizes activities with the Mathematical Machines for secondary school students, groups of university students, pre-service and practicing school teachers (Maschietto & Martignone, 2008; Bartolini Bussi & Maschietto, 2008). The design and the development of these activities are carried out by the MMLab research group with the aim to provide a suitable learning context in which to activate important processes, such as the construction of meanings and the construction of proof (Bartolini Bussi, 2000).

Our research focuses on machines that establish a correspondence between points of the plan regions, like reflection, central symmetry, translation, rotation, and homothety. These transformations are physically performed through two leads fixed in two plotter points of an articulated system composed by some rigid rods and some pivots (see fig. 1 and 2). These machines, named pantographs or linkages, incorporate some mathematical properties in such a way as to allow the implementation of a geometrical transformation.

The study presented in this paper is part of a wider research about the didactical potentiality of the machines as tools for teaching and learning mathematical proof. Our goal is to investigate the cognitive processes involved in the proof construction in activities with the geometrical machines (some first results are in Martignone &

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Antonini, 2009a, 2009b): in order to do that, we study the processes that can lead to the construction of a proof or that can be an obstacle to this construction. In particular, we deal with the analysis of argumentations proposed by subjects to support their conjectures.

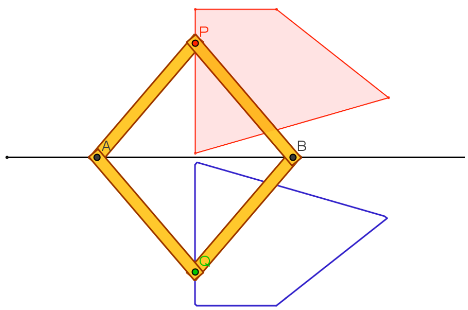

Fig. 1: An image of pantograph for reflection and its products. Two opposite vertices (A and B) of a rhombus, composed of four equal rods pivoted together, can move in a groove (a straight path *r*). The other opposite vertices (P and Q) are corresponding in the reflection of axis *r*.

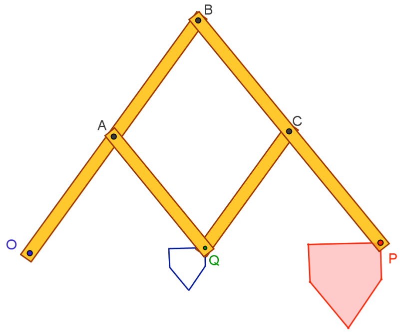

Fig. 2: An image of Scheiner's pantograph and its products. The linkage rods form a parallelogram AQCB. The point O is pivoted on the plane. It is possible to prove that P, Q and O are on the same line and that P and Q are corresponding in the homothety of centre O and ratio BO/AO.

THEORETICAL FRAMEWORK

The study presented here focus on argumentation in geometry context. In the following sections we will expose some theoretical considerations on argumentation and proof and on the geometrical figures.

Argumentation and proof

Many papers have been written on the relationships between argumentation and proof (Mariotti 2006). In general, a mathematical proof of a statement consists of a logical sequence of propositions that states the validity of the statement. Differently, an argumentation consists of a rhetoric means that have the goal to convince somebody of the truth or the falsehood of a statement.

Some authors focus on the differences between argumentation and proof (see, for example, Duval, 1992-93). On the other hand, focussing on the processes of argumentation and proof generation, the theoretical framework of *Cognitive Unity* (Garuti et al., 1996; Mariotti et al., 1997; Garuti et al., 1998; Pedemonte, 2002; Boero, 2007), without forgetting the differences, underlines the analogies between them. In particular, these studies suggest that, in open-ended problems (where students are asked to produce a conjecture, to generate an argumentation and a proof that support the conjecture), a continuity between the argumentation and the subsequent mathematical proof may or may not occur. For these reasons, it is important to identify the factors that can favour continuities and the elements that can lead to a gap between argumentation and proof.

One goal of our research project is to investigate the continuity between argumentation and proof during the machine exploration that led to the identification of the mathematical law made by the machine. In this paper, we focus on the argumentations, in the particular situation in which students are involved with pantographs or linkages for geometrical transformations of the plane.

We underline that a mathematical proof is necessarily linked to a theory, a theoretical framework within which the proof makes sense (see Mariotti et al., 1997), while an argumentation in geometry can also concern some figural aspects (as magnitude, shape, etc.) without reference to a theory. In order to take into account these aspects we refer to the theoretical framework of *figural concepts*.

The theory of figural concepts

The theory of *figural concepts* (Fischbein, 1993) provides us with an efficient theoretical tool, suitable to analyse cognitive processes in geometrical problem solving. According to Fischbein (1993), mental entities involved in geometrical reasoning cannot be considered either pure concept or mere image. Geometrical figures are mental entities that simultaneously possess both conceptual properties (as general propositions deduced in the Euclidean theory) and figural properties (as shape, position, magnitude). Fischbein called them *figural concepts*. A productive reasoning, as an efficient process of proof generation, can be generally explained by the fact that the figural and the conceptual aspect blend in a figural concepts (see, for example Mariotti, 1993; Mariotti & Fischbein, 1997). Our analysis of the argumentation will take into account the distinction and the duality between the components of a figural concept.

METHODOLOGY

The goal of our research is to study the argumentation generated by subjects that are asked to discover what the machine does and to prove it. This is an exploratory study and in this first step of the research we needed to analyse rich argumentation activities linked to the particular situation in which a mathematical machine are the object that has to be explored. For this reason, we interviewed subjects who were familiar with geometry and with problem-solving (three pre-service teachers, two university students and one young researcher in mathematics) but that have not seen these machines before. The task was identifying the geometrical transformation and to prove that the machine performed that transformation. Data were collected through clinical interviews that were videotaped. Subjects were asked to express their thinking process aloud. The analysis of the interviews is based both on the transcripts and on the manipulative activities on the pantographs. In this paper we report some results about the explorations of a linkage for reflection (fig. 1) and of a pantograph for homothety (Scheiner's Pantograph, see fig. 2).

ARGUMENTATION IN EXPLORING THE GEOMETRICAL MACHINES

The argumentations justifying that the machine implements a particular geometrical transformation are closely related to three elements: the drawings traced by the machine, the structure of the machine (as a figural or a conceptual component), and the machine movement. Notice that the same subject could propose more than one type of argumentation. This is common in task requiring the production of a conjecture and a proof, when one arguments with different goals: producing, testing, supporting, and proving a conjecture.

Argumentation that refers to the drawings traced by the machine

These argumentations refer to the shape of the drawings traced by the machine and to their comparison. There are not theoretical references in this type of argumentation. The machine is used to perform effectively the transformation and the argumentation is based on the products of the transformation. We notice that the drawings can be traced by the leads of the machine but there is also the possibility that the subject *sees* the drawings only through the movement of the plotter points.

For example, some of the pre-service teachers, exploring the Scheiner's pantograph, state that the machine performs a homothety because one of the two drawings traced by the machine appears as an enlargement of the other one.

Argumentation that refers to the structure of the machine

These argumentations refer to one or more elements of the static structure of the machine, that is the structure of the machine when it is stopped in some position. We give two examples. In the first, the subject refers to the *figural aspect* of the structure, in the second to the *conceptual aspect* of the geometrical figure represented by the articulated system.

Example 1. Lucia, a pre-service teacher, exploring the Scheiner's pantograph (see fig. 2), conjectures that the transformation is a homothety and she justifies her statement saying that "*it has these two pivots [she points at A and B], and this rod [BP] is longer than this [AQ]*". This, for Lucia, explains the fact that the drawing made by the lead put in the point P is an enlargement of the drawing traced by the lead put in the point Q. According to the theory of figural concepts, the argumentation refers to the *figural aspect* (a qualitative relationship between the length of BP and the length of AQ) of the geometric figure represented by the articulated system, without any reference to a mathematical theory. In a following section we will present a deeper analysis of another example.

Example 2. Anna, a pre-service teacher, exploring the linkage for reflection (fig. 1), justifies that the transformation is a reflection showing that the two plotter points (P and Q) are on the segment whose groove is the perpendicular bisector. Then Anna refers to some conceptual properties of rhombus to support this fact. According to the theory of *figural concepts*, the argumentation refers to the *conceptual aspects* of the geometric figure represented by the articulated system. Another example will be shown in a following section.

Argumentation that refers to the movement of the machine

These argumentations refer to some dynamic properties of the articulated system, i.e. to some characteristics of its movement.

Example. In the transcript we will analyse below, Carlo proposes many argumentations to support his conjecture about the linkage for the reflection (fig. 1). One of these argumentations refers to the movement of the machine, in particular to the fact that if one plotter point approaches the reflection axis, then also the other plotter point approaches that (see below).

ANALYSIS OF A TRANSCRIPT

In this section, we analyse a transcript of an interview in which the subject proposes more than one argumentation to support his conjecture about the transformation made by the machine: one argumentation referring to the *figural aspect* of the structure, one referring to the *movement* and one referring to the *conceptual aspect* of the articulated system. Only after these argumentations he generates a mathematical proof. Carlo is a young researcher in mathematics and this is his first experience with a mathematical machine. He is exploring the pantograph for the reflection.

Carlo: […] I see a line in the centre and I think to a symmetry (*he makes a gesture opening his hands in a symmetric way, like he is opening a book*). I have thought to a symmetry… reflection, because there is this line, it could not be, but… it's all quite… so symmetric that… (*he makes a rhombus with his hands*)

Carlo considers the *figural aspects* of the machine structure. It is the shape, the symmetry, underlined by the groove in the centre, that starts to convince Carlo that the transformation is the reflection (*argumentation referring to the figural aspect of the structure of the machine*).

Carlo: […] well, so, then I remind that this will be the transformation […] symmetry… as it is called... reflection… I had seen it immediately for the shape.

Carlo recalls the previous argumentation and the fact that it refers to the *figural aspect* of the structure of the articulated system.

Carlo: Because these are rigid (*he points at the rods and then he starts to move the articulated system*) then probably there can be some properties linked to the… (*he stops the movement of the machine*)… rhombus.

The observation of the structure of the articulated system ("*these are rigid*") leads Carlo to anticipate that there could be some mathematical properties related to the figure now identified as rhombus. These properties can justify that the machine implements the reflection. There is here an important transition, anticipated and not yet implemented, from an argumentation linked to the *figural aspects* of the machine structure to an argumentation related to the *conceptual aspect* of a geometrical figure (rhombus) that represents the articulated system. For the moment it is only an anticipation that will lead later to the construction of a mathematical proof.

Carlo: Anyway, now we will see…

Carlo postpones the generation of the proof: what he has said about the properties of the rhombus remains only an anticipation. As we can see below, it seems that he feels the need of other argumentations before constructing the proof.

Carlo: And also the movement (*he starts again to move the machine with two hands*), the movement seems to me quite significant for this: if I approach the axis with this point [P], also the other point [Q] approaches it, both perpendicularly ... the fact that there is a movement also in this direction (*he moves the articulated system by sliding the pins, the points A and B in the fig. 1, in the groove*)…

This argumentation is linked to a property of the machine movement, showing a dynamic relationship between the two corresponding points (P and Q) of the transformation; for Carlo it supports again the fact that this is a reflection (*argumentation referring to the movement of the machine*).

Carlo: I do not need the leads.

He does not feel the need to use the leads, he does not consider important to have an argument based on the tracks. The different argumentations produced have convinced Carlo but probably the use of the leads is not taken into consideration also because he is aware that they would not give any new contribution to the knowledge of the machine functioning and then to the construction of a proof.

Carlo: Then, why it works ... so if they all have the same length, we have a figure... this is a geometric figure with four equal sides ... where this (*he follows the groove with his finger*) is a diagonal, therefore it's a rhombus ... for the rhombus properties the two diagonals are perpendicular, this tells me that this diagonal (*he tracks with his finger the diagonal PQ*) is perpendicular and also they (*the two diagonals*) bisect each other.

Here, Carlo proposes an argumentation based on the geometrical properties of the articulated system (*argumentation referring to the conceptual aspect of the structure of the machine*).

We observe that Carlo has no difficulties in the identification of the transformation. Nevertheless he needs to look for additional argumentations before constructing the proof. After a first argumentation based on the *figural components* of the structure of the machines, he proposes an argumentation referring to the relationships between the movement of the two points involved in the transformation. Finally he comes back to the structure of the articulated systems but this time he considers the *conceptual components*, supporting his statement in the geometrical theory.

We think that the different argumentations proposed by Carlo have the goal of strengthening the conviction that the conjecture is true, but also to offer a more complete argumentation that can take into account different perspectives about the machine and its use. It is interesting to notice that, differently from students' processes, the only type of argumentation not generated by Carlo is that linked to the drawings traced by the machine. This type of argumentation is very useful to generate a conjecture and to test it, and for this reason it is very common among students with minor experience. Nevertheless, as Carlo feels, it could not give contributions neither to the explanation of the movement nor to the proof construction.

CONCLUSIONS

The subjects, as we have shown in the Carlo's protocol, produced different types of argumentations, even during the same exploration phase. It is important to underline that mathematical culture, but also familiarity with the machines, seem to promote the emergence and the development of different types of argumentations. In fact, we have noticed that the experts produce several argumentations and they refer to the drawings traced by the machine only when they have difficulties in identifying the mathematical law incorporated in the machine; differently, the argumentations referring to drawings are very common in students' activities.

The proposed analysis of the machines explorations paves the way for the generation of hypotheses on the transition from argumentation to proof. In particular, we hypothesize that, in the case of argumentations referring to the conceptual part of the machine structure, there can be *cognitive unity* between argumentation and proof, in other cases it seems plausible to attend to cases of *cognitive break*. A special case seems to be the argumentations based on movement, because they often lead to further argumentations that explain the motion through the structure of the articulated system, and a *cognitive unity* may or may not occur.

Regarding the transition between argumentation and proof, we mention briefly that in the interviews we have carried on, the interviewer's interventions has been relevant in guiding the students to the proof construction. In fact, it seems that some interventions, aimed to put an emphasis also on argumentations that students do not spontaneously generate, can be educationally effective in the activities with machines oriented to stimulate and to develop argumentative and proving processes.

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THE TEACHER’S ACTIVITY UNDER A PHENOMENOLOGICAL LENS

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According to phenomenological perspectives, students must be educated to see and focus things in ways coherent with the mathematical notions to learn: for example to see the mathematical properties of a function in a graph representing a certain situation. We refer to Rota’s account of mathematical thinking as “disclosure”, and give a phenomenological interpretation of the cognitive processes related to graphical modelling activities, and of the role of the context therein. We contrast the classroom discussions of the same task in two different grades (9 and 11), and show how the teacher uses suitable didactic techniques to promote different “layers” of students’ disclosures of Calculus concepts.

INTRODUCTION

It may be a truism in mathematical education that students must learn to “see the general in the particular and the particular in the general” (Mason, 1996). Recently such an issue has been analyzed from a phenomenological perspective, deepening in particular the relationships between perception and theoretical issues, and focusing on the role of the teacher in promoting connections between them to foster the students’ learning processes. For example, Radford (2010) has pointed out how teachers can

> create the possibility for students to perceive things in certain ways and encounter a cultural mode of generalizing. This new way of perceiving (…) in certain efficient cultural ways entails a transformation of the eye into a sophisticated theoretician organ. (*ibid*., p. 2)

From another perspective, the so-called embodied cognition (Gallese & Lakoff, 2005) claims that the whole of cognition can be understood in terms of perceptuo-motor activity. Nemirovsky (in print) develops a perspective on mathematical embodied cognition consistent with a phenomenological understanding of perception and body motion.

In this paper we follow this last approach and use a phenomenological stance, based on the elaboration given by the outstanding mathematician and philosopher G. C. Rota (1991) to Husserl phenomenology, to analyze how a teacher manages the embodied and theoretical issues while teaching the same lesson in two different classrooms (one at grade 9, the other at grade 11). A major result of the analysis is that many practices of the teacher can be considered examples of what Husserl called

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a 'natural attitude' towards human phenomena (in our case didactical phenomena). Such actions appear to be very common in everyday didactical practices of mathematics teachers, even if they not necessarily know phenomenology.

"SEEING AS": A PHENOMENOLOGICAL STANDPOINT

As underlined by Radford (2010, p. 4), students must be taught to "see and recognize things according to 'efficient' cultural means" and to convert their "eye (and other human senses) into a sophisticated intellectual organ". Namely it is necessary to promote a "lengthy process of domestication" (*ibid.*) of the way they are looking at things while learning mathematics. This process is based on the key phenomenological assumption, pointed out by Rota, that there is "no such thing as true seeing", but "there is only seeing as" (Rota, 1991, p. 239). Hence, learning mathematics requires different modes of *focusing*: "just like seeing is focusing upon some functions which may be present, similarly, remembering, imagining, or visualizing are other modes of focusing" (*ibid.*). Students must be educated to see and focus things in *the right way*, i.e. in ways coherent with the mathematical notions to learn: for example to see the mathematical properties of a function from the graph of a mountain track, as in Fig. 1. This delicate process is far from being natural: on the contrary, to be achieved it requires precise didactical interventions of the teacher.

Consequently, in this paper we refer to two related didactic techniques from the literature: *making present* absent things to students, and *prompting* them in order to direct their *attention*. As pointed out by Ferrara (2006) and following Husserl, we can distinguish two aspects of making present, namely *remembering*, that is "making present the past (the absent being the past)"; and *imagining*, that is "making present the not yet known (the absent being the not yet known)" (*ibid.*). As said above, Rota considers both remembering and imagining as modes of *focusing*.

Focusing *attention* is at the core of the work by Mason (2008), who describes as follows the *prompting* technique of the teacher:

> One of the classic interventions used by relative experts to enculturate novices into particular practices, is often referred to as *scaffolding and fading* (Seeley Brown et al. 1989). A teacher repeatedly uses a particular prompt or question with learners, and then begins to use less and less direct prompts or meta-questions such as "what question am I going to ask you?" or "what did you do last time in this sort of a situation?", until the teacher need only rarely if at all remind learners of the prompt: the prompt has been internalised and become a spontaneous action. (Mason, 2008, pp. 41-42, emphasis in the original)

Prompting the students' attention to the suitable context, possibly enriched with recalled or imagined elements, supports the students towards a progressive *disclosure* of the mathematical objects at stake. Disclosure is a Husserlian concept further elaborated by Rota (1991). It indicates the process by which people make sense of the world and of the situations in context to which they are exposed:

> The world is primarily a world of sense… Our primary concern is with sense itself, how it originates in the world, how it functions in the world. In short how it relevates… The basic relationship to the world is…our senses. (*ibid.*, p. 61).

Disclosure happens when one is able to grasp the *functionality* of the objects in the context, for example those in a didactical situation:

> Sense-making depends ultimately on our own being-in-the-world, on the situation of our interacting, our dealing with the contextual situation in the world […]. If you deconstruct the notion of an object, what you find is pure functionality, the pure 'being good for' of that object or something. So that the world, instead of being a world of objects, will become a world of functions, of tools. (*ibid.*, pp. 156-159, *passim*).

Such functions are related to each other "by a system of references, a network of references among them. […] The world is disclosed to us not just as a system of functions, but as a network of related functions" (*ibid*., p. 159).

Students must be educated by the teacher to make sense of what they perceive/see when exposed to a mathematical situation. Generally a situation may evoke different contexts and so produce a different sense-making, according to the age and the background of the students. For example, the graph in Fig. 1 can evoke a mountain, a graph of a symmetric function, a normal distribution, and so on. Of course, such different contexts are not isolated but are *layered* upon each other; these layers can generate different levels of disclosure in the flow of time:

> Side-by-side with our realization that sense is purely contextual goes the realization that contexts are not units. Contexts themselves are layered upon each other in various ways, and to be in a context is not to be in just one context. … Be-ing in a context does not in any way presume that such be-ing is be-ing in one context at a time. (*ibid.*, p. 126)

Because of the role of contexts, disclosure includes two aspects: grasping a concept requires both an emotional and an intellectual component, which Rota calls *mood* and *grasp*; they can be present in different ways according to the context:

> There are phenomena of disclosure where the actual grasp in the context fits the major role and the mood component fits the minimal role – for example, our approach to solving a mathematical problem. This is not saying that we have to like the problem, but the minimum of mood lets us get involved in it. Unless we get really involved in it, we get nowhere. This is the mood-wise component of the mathematical problem disclosing itself. Without this component of mood, no matter how little, the problem will not be disclosed. (*ibid.,* p. 269).

In our analysis, we will show how the teacher uses the techniques of *prompting* and *making present* to evocate suitable contexts in order to support the students towards the *disclosure* of some basic Calculus concepts.

THE TEACHING EXPERIMENT

In 2009 researchers from New Zealand, Israel and Italy started an ongoing common research project, with the aims of studying the possible benefits of approaching the derivative and the primitive concept in a graphic way.

This paper is based on the teaching experiment carried out in Italy in two classrooms (grades 9 and 11) of a scientifically-oriented high school ('Liceo Scientifico') with the same teacher. In particular, it focuses on the lesson that followed the first task of the teaching sequence. The task was composed of two parts: the first asked to interpret a height-distance graph (see Fig. 1) and to draw the graph that represents its slope. The second part proposed a gradient graph (Fig. 2) and asked to draw a graph whose slope was represented by it (inverse problem).

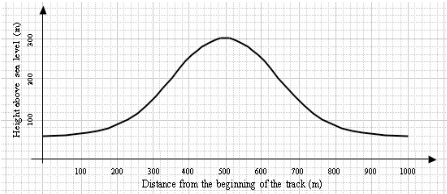

Figure 1: Distance-height graph of track

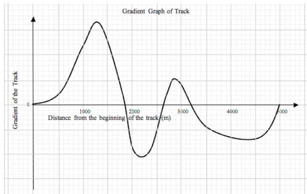

Figure 2: The gradient graph

The students solved the task in groups of 3 or 4, and afterwards were involved in a classroom discussion on the concepts concerned with the task.

The lessons were video-taped by a camera, allowing us to consider the semiotic productions of the teacher and of the students (speech, inscriptions at the blackboard, and gestures). After a first scrutiny, we carried out a semi-structured interview to the teacher to ascertain the reasons of his didactical actions. In the next paragraph we will illustrate and discuss some results of our overall analysis at the light of the phenomenological perspective described above.

ANALYSIS

Sabena (2010) has analysed the students' processes while solving the task, and observed interesting cases of semiotic resources (tipically, words and gestures) that could refer both to the given graph, and to the corresponding imagined track, like:

> If the slope of the tangent line is zero the track is parallel to the x-axis.
>
> The graph goes downhill.

In each sentence there is some element of the track treated as it were part of a Cartesian plane (first example), or vice-versa (second example). Since these signs – intended in the sense of Peirce (1931-1958) – blend the references to two different domains, they have been called "*blending signs*" (Sabena, 2010). Blending can happen since the two objects, though being different, share deep relationships of iconic character (e.g. the highest point of the graph corresponds to the highest point of the track). This feature is specific of the task proposed to the students, and was meant to facilitate their solving activity. On the other hand, it is possible that the students who are using blending signs are not (fully) conscious of their double referencial nature.

From a phenomenological perspective, the blending signs can be interpreted as markers for possible *disclosures* towards the meaning of the graph as a mathematical modelling tool. Disclosure can develop because the students become aware of the double polarity between the objects and their functions, for example the track highness as modelled by a function graph. A blending sign reveals this double polarity between what Rota (1991) calls "the facticity of the context" and its "functionality" (that he calls also "function") that must be disclosed.

The task addresses the students' attention towards the *facticity* of the context (the track in the mountain), but at the same time it is necessary that this facticity "fades before the function" (*ibid.* p. 127), so that the students can achieve the disclosure of the graph as a model of the track. Let us analyse how the teacher helps the students to accomplish this goal.

During the two classroom discussions that followed the task, the teacher refers to the different contexts of the task, to foster two different *layers* of disclosure: a *factual* layer (that of the track, prevailing at the 9th grade) and a *theoretical* layer (that of the functions, prevailing at the 11th grade). For example, in grade 9, the teacher starts the discussion by recalling in an explicit way the context of tracks. In fact, the context can provide meaning to the graph slope, which the students face for the first time. To do that, he uses some blending signs, like saying "the track did something like this", while drawing the graph at the blackboard:

Teacher: You had a function, about… the track, do you remember, the track did something like this, isn't it? (*drawing the graph of Fig. 1 at the blackboard*) Ok? Roughly. So we had the graph of a function.

In grade 11 we observe something different. These students have already some competences about functions, and in particular they have studied the slope features of a graph and know how to compute approximated values of the function slopes. Consequently, during the whole discussion the teacher endeavours to underline that the graph and the track are two objects that belong to different domains:

Teacher: Well, this first task talked about trampers that were following some tracks in the mountains. And we have imagined that in profile the graph…do you remember? This (*drawing the graph of Fig. 1 at the blackboard*), ok? This graph represented, varying the positions along the track, x… x represented the distance from the point?

Students: The starting point […]

Teacher: Then, how is the graph when the slope of the tangent line is negative? You have said well, you have said that it is…

Students: Downhill.

Teacher: Decreasing. Downhill the track; the graph is decreasing. Remember: the properties of the graph are expressed in mathematical language, those of the track on the contrary can be expressed like uphill, downhill.

The teacher stresses the fact that the graph and its parts are signs that *represent* something. Namely the teacher prompts an *un-blending* process in order to reach a further layer of disclosure.

The overall analysis of the two discussions reveals that, starting from the same task, the teacher is working on two different layers of disclosure. For instance, in grade 9 to refer to the point of highest slope the teacher makes present the experience and the *mood* of biking:

Teacher: Look! Graphically (*pointing at the graph in Fig. 1*), can you see here that the slopes are increasing (*starting to surf with his hand along the graph, Fig. 3*)? Can you see that (*moving his hand along the graph*)? And that at a certain point… (*his hand is near the inflection point*)?

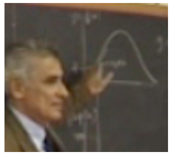

Figure 3

Students: They decrease.

Teacher: They decrease (*taking his hand away from the graph*). This is the sensation that we feel if we bike along this uphill *(the hand again on the graph),* at the beginning it is very hard, isn't it? At the beginning it is very hard because the slope increases *(with a slanted body and the hand as holding a handlebar, he mimes the act of biking uphill, Fig. 4),* then *(the hand along the graph, after the inflection point)* it becomes less and less hard. In fact at the beginning the function increases more and more and then (*the hand has reached the maximum of the graph*)?

Figure 4

Students: It increases less and less.

To grasp the different increasing modalities of the function, and specifically to focus the attention to the point of inflection, the teacher makes present with his words and his body posture the physical sensation of the fatigue ones feels when climbing a steep hill with the bike: it is the *mood* component in Rota's account. Such a sensation may be well known to the students, since they live in a hilly territory: therefore the making present may be accomplished through remembering their lived experiences.

In grade 11, we do not find such perceptual references for the same mathematical concepts. While solving the task, in fact, the students have identified that the point with the steepest uphill corresponds to the point of inflection. In the discussion, the teacher reads their answers and prompts to the fact that they have well done, without making present any experience related to the track. This interpretative hypothesis is confirmed by the interview to the teacher:

> [Asked about the bike episode in grade 9] I often search for the situation that I think it is nearest to their experience: for instance I speak of skiing for those who go skiing, biking, climbing, surfing… so to think to an experience that is very concrete, very perceptive.

> [Referring to grade 11] It is true that the situation was that of tracks, but now the students should understand that they have modelled it with a graph, so we speak of the properties of a graph, with adequate language. […] I think that for the students at grade 11 the concrete situation has not helped them so much. They had the tools to speak in terms of graphs. For the students of grade 9 it is different. I imagine that they have indeed started from the concrete situation and have imagined the person who was climbing with all the problems, then [tried] to eliminate the inessential things and so to think simply to the outline of the track that becomes exactly the outline of the graph.

It is important to notice that the progressive disclosure of the graphs as mathematical objects does not imply a definitive discharge of blending signs. On the contrary, the teacher comes back to blending signs when teaching about new (possibly difficult) properties of the mathematical objects to be disclosed. For instance, in the second part of the task the students have to draw the graph of a function, starting from the graph of its slopes (i.e. to draw a primitive function). Being an inverse problem, this question can raise some difficulty also for the students at grade 11. After drawing a primitive graph, the teacher puts on the table the issue of the y-s of the primitive:

Student 1: [We cannot know the y-s] because the slope graph does not give us that information

Student 2: The differences could be both from 0 to 1 and from 100 to 101

Teacher: By the way, this graph here (*pointing at the primitive graph drawn at the blackboard*) could be an underwater mountain, below the sea-level (*the hand mimes the action of moving the graph below the x-axis*)

It is the teacher himself to introduce a blending sign: in fact, he blends the references to the graph (by means of gestures) and to a concrete imagined context (the underwater mountain). In this way, he has provided a new context by which the students may give sense to a property that regards a relationship between two mathematical objects, i.e. a graph and its primitives.

DISCUSSION

We have sketched an interpretation of the teacher's actions in the classroom, based on the phenomenological notion of *disclosure*, as defined by Rota (1991). Disclosure happens because people are able to grasp the *functionalities* of the context. In this process, both emotional and cognitive aspects are involved. Our analysis was meant to show how the teacher uses didactic techniques like *prompting* and *making present* (i.e. *remembering* or *imagining*) to promote the different layers of students' disclosures. Precisely, we have seen that he fostered the notion of slope of a function in a point through the steepness of a road, on which students are asked to remember/imagine to bike (grade 9); or the fact that all the primitives of a function differ by a constant, imagining a mountain that sinks under the sea (grade 11). In making present these contexts and promoting the disclosure of the related mathematical concepts, we have identified the production of *blending signs* (Sabena, 2010): for the teacher they are tools for fostering students' disclosure process.

Our analysis suggests that in the learning processes, contexts are layered upon each other, and that they are never completely discharged. This causes a complex dynamics in teaching actions. From the one side, when the teacher judges that students have reached a sufficient disclosure of a concept, he pushes towards a more abstract layer, where further disclosure processes can start. From the other side, when some more difficult concept must be faced, the teacher can go back to a previous layer to provoke suitable disclosure processes (e.g. imagining the mountain under the water level to support the disclosure of the existence of infinite "parallel" primitives).

Space does not allow to present things from the side of students, namely to illustrate the extent to which the teacher's actions aimed at provoking learners' disclosures are successful. This problem will be the object of our future research.

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A TOOL FOR ANALYSING MULTIMODAL BEHAVIOURS IN THE MATHEMATICS CLASSROOM

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In this study we take the perspective of multimodality coming from communication design and neuroscience to integrate it with mathematics education. We think that in the mathematics classrooms interaction and communication are multimodal in the same way as text and discourse are in the digital era. Our analysis studies learners' and teachers' cognitive processes by means of a semiotic lens, focussing on their multimodal productions: gestures, words, actions, and inscriptions. We introduce the timeline as an analysis tool that considers the variety of semiotic resources in the classroom over time. The timeline allows us to observe dynamic evolutions of signs, and to better understand the role they play in mathematics teaching and learning.

INTRODUCTION

The shift from purely alphabetic writing to multimodal texts occurred in the digital era requires new abilities by readers and writers. Multimodal texts are made of several modes of communication, like speech, writing, and image, each integrated with the others. Think for example of newspapers today with respect to those published 40 years ago: actual newspapers are also on the web and have videos, texts, images, links to web pages, interactive blogs and platforms, and so on. The capability of reading in an interactive way, integrating information from the various modes is then necessary, and digital natives have this capability more than immigrate ones.

Multimodality is a characteristic of texts, as well as of discourse. A specific case is that of learners interacting with peers or with the teacher at school, to discuss on mathematical tasks. Learners use gestures, gazes, words, sketches and productions on the Interactive White Boards or on a software. In our research, we are interested in this multimodality of signs to study cognitive processes. To this aim, we refer to a theoretical framework on multimodality based on quite recent experimental evidence from neuroscience and psychology. Indeed, the evidence of a multimodal nature of the brain sensory-motor system gives reason of the fact that human cognitive processes are constituted not only by symbolic activity, but also by perceptuo-sensory-motor-imaginary activities (Nemirovsky & Ferrara, 2009).

We use the *semiotic bundle* as an interpretative means for multimodal production. It consists of a model to describe the various *signs* produced by students and teachers over time. The description is given by the *timeline*, a dynamic graphic representation of the signs and their relations, through which we get information about the cognitive processes happening in the classroom.

MULTIMODALITY

In recent years we assist to the introduction in mathematics education of the theory of *embodied cognition* (see Wilson, 2002), which focuses on the role of metaphors and sensory-motor experiences in thinking and understanding. Neuro-scientific studies on *mirror neurons* (Gallese & Lakoff, 2005) pointed out new features of the sensory-motor system of the brain, in relation to its role in conceptual knowledge: "mirror neurons and other classes of pre-motor and parietal neurons are inherently 'multimodal', in that they respond to more than one modality. Thus, the firing of a single neuron may correlate with both *seeing* and *performing* grasping." (*ibid*., p. 457, emphasis in the original). As a result, many modalities like hearing, sight, touch, motor actions, etc., seem to be not strictly separated but integrated with each other when active, each infused with properties of the others. This entails that the brain sensory-motor system has a pre-existing character multimodal rather than modular. Multimodality thus denies the existence of separate modules for perception and action that need to be somehow "associated". Typical human cognitive activities such as visual and motor imagery, far from being of a disembodied, modality-free, and symbolic nature, make really use of the activation of sensory-motor brain regions (Gallese & Lakoff, 2005). Also, correlates of the mirror neurons system have been found in the principles of social cognition. We think these results are very notable in teaching and learning processes and that they may tell much about the resources students and teachers use in communication and in interaction.

Multimodality even appears in the design in communication (Kress, 2004). In this case, the term is used to indicate *multiple modes of representation*, as means to make meanings, whether they are oral or written messages. In the multimodal landscape of communication, the number of ways of expressing and shaping a message implies choices and questions. Text is no longer the main way of communicating, but images and videos are pervasive. As a consequence, the dominant media are no longer books, but videogames, mobile phone, i-Pad, TV, etc. Such aspects should be significant for mathematics education too, in order to study the multimodality of communication in the classroom. For example, they enable us to explain why learners gesticulate a lot or use representational tools during their mathematical activity.

THE SEMIOTIC BUNDLE

The perspective on multimodality, along with other studies, supplied mathematics education researchers with elements to include the body in the act of knowing: "the return of the body is the awareness that, in our acts of knowing, different sensorial modalities—tactile, perceptual, kinaesthetic, etc.—become integral parts of our cognitive processes. This is what is termed […] the multimodal nature of cognition." (Radford *et al*., 2009, p. 92). Within this view: "the understanding of a mathematical concept, rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending on the context." (Nemirovsky 2003, p. 108).

The model we adopt to analyse the phenomena occurring in the classrooms is the *semiotic bundle* (Arzarello, 2006). It is an enlarged notion of semiotic system, which encompasses the classical semiotic registers in mathematics (e.g. the algebraic, and the Cartesian) as particular cases. The semiotic bundle consists in the description of a system of signs produced by one or more interacting subjects according to the comprehensive notion of sign given by Peirce (1931-1958). Elements of the semiotic bundle are the students' and teachers' multimodal productions: words, gestures, and inscriptions (but possibly also gazes, tones of voice, postures, etc.).

Students solving mathematical tasks, which are coordinated by the teacher, produce signs (related to each other) that may be analysed with such a model. In particular, the semiotic bundle does not only take into account the signs at a certain moment (synchronic analysis), but also their evolution over time (diachronic analysis), in a dynamic way. An example of the first case is given by a subject that simultaneously gesticulates and speaks. With the diachronic analysis of the semiotic bundle, we can consider signs produced at different (close or far) times, transformed into other signs. Examples of these transformations are given by gestures shared in the classroom and then performed for solving new problems in various moments (Arzarello & Robutti, 2008), but also by conversions between different registers (Duval, 2006) like from an equation (algebraic register) to the corresponding Cartesian graph (graphic register). Diachronic evolutions can also be found in the semiotic games of the teacher, when she/he "echoes" one or more gestures produced by the students, and at the same time expresses in the verbal register the rigorous mathematical meaning of gestures (Arzarello & Paola, 2007; Robutti, 2009).

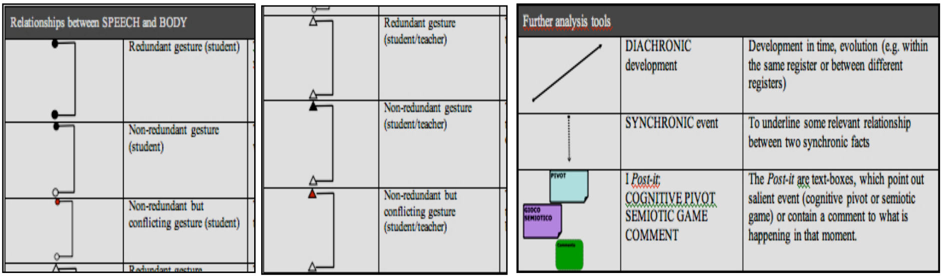
The semiotic bundle thus allows us to analyse the multimodal semiotic activity of the subjects in a holistic manner, showing the dynamic evolution of signs over time.

THE TIMELINE

To represent the evolution of the cognitive processes through the semiotic bundle, we introduce the *timeline* that makes possible to analyse at different grains the complex relations among signs in the semiotic bundle. It consists of a dynamic representation of the multimodal production by students and teacher, made of videos, written or spoken words, gestures, representations and interactions with tools. The timeline provides a *microanalysis* of the evolution on time of the didactic situation in the classroom. In the meanwhile, it offers also a global or a local description of classroom processes. For instance, the timeline allows getting precise snapshots of what occurs at certain instants and for short time intervals (the finest unit of time being 1/24 sec, i.e. the single frame of the video). Or, it may allow having the evolution in time or the morphological similarity of different signs (e.g. gestures) produced by students in their individual story or within the story of the classroom.

The timeline is built up using a spreadsheet. It shows three main areas: speech, body, and inscriptions (see below for details). We observe them for both the students and the teacher. Each word, gesture and sign in the timeline refers to the one that has created or used it. The time grain (appearing at the top of the page) varies according to the evolution of the situation and to the analysis necessities. There could be intervals where it is very detailed and others where it is grosser.

Although the timeline per se is a static product, constituted of a series of static elements (images of gestures, sketches, drawings, and words), scrolling it gives an idea of the complex dynamic and multimodal nature of learning processes. We even introduce codes that permit representing these and other specific aspects of the various components of the semiotic bundle: examples are subjects' postures, gazes, voice tunes, etc. Tables 1 contains the main codes we take into account.



Relationships between SPEECH and BODY	
	Redundant gesture (student)
	Non-redundant gesture (student)
	Non-redundant but conflicting gesture (student)
	Redundant gesture

	Redundant gesture (student/teacher)
	Non-redundant gesture (student/teacher)
	Non-redundant but conflicting gesture (student/teacher)

Further analysis tools		
	DIACHRONIC development	Development in time, evolution (e.g. within the same register or between different registers)
	SYNCHRONIC event	To underline some relevant relationship between two synchronic facts
PIVOT GIOCO SEMIOTICO	I *Post-it*: COGNITIVE PIVOT SEMIOTIC GAME COMMENT	The *Post-it* are text-boxes, which point out salient event (cognitive pivot or semiotic game) or contain a comment to what is happening in that moment.

Table 1: Codes for the speech-body relationships (on the left);
further codes: diachronic and synchronic elements, and Post-it (on the right)

At the top of the timeline, information on time appears: all that is indicated vertically occurs in the same moment. The three major sections said above, investigated at each time interval, are:

- *SPEECH* (spoken productions): transcription of what both the students and the teacher are saying; a progressive number gives the right order of the interactions.
- *BODY* (bodily productions): the main images (even frame by frame) of gestures, gaze, and postures students and teacher use to support their communicative acts.
- *INSCRIPTIONS* (written productions): things written by students and teacher.

There are two complementary sections: one for the *TOOLS* used in the activity (e.g. types of software or materials); the other one for the *LINGUISTIC* and *GESTURAL ANALYSIS* of the speech/body-section.

An example of the appearance of the sheet once the analysis is carried out is given in Figure 1. Table 2 contains other symbols appearing in the timeline, explicitly related to the linguistic and gestural analysis.

LINGUISTIC AND GESTURAL ANALYSIS		
	information (meaning)	Acquiring information (e.g. through specific questions)
	connecting (meaning)	Connecting the acquired information
	seeing a structure	Grasping the structural features of an object, its regularities (e.g. grasping a geometric figure as a whole and the relationships between its parts)
	starting	Starting an argument, sometimes opening so a blocked situation
	conflict	An explicit conflict happens, where there may be a discussion or perplexity
	recalling a structure	The generalisation has already happened, the structure and its relationships have already been acquired: now they are recalled
e	asking a better explanation	Generally it is a teacher intervention, who enters the discussion to support it
v	understanding	The student shows that she has understood
Others		
	internal conflict	E.g. when the student realises to have expressed something, which conflicts with some previous statement
	synthesizing	There is a convergence between different arguments (e.g. expressed in different registers) that produces a shared meaning
L	Locutional	What is explicitly expressed
I	Illocutional	Not explicitly expressed (e.g. through extralinguistic productions like voice tunes, pauses, silences, glances...)
P	Perlocutional	Communication effects that can modify behaviours

Table 3: Codes for gestural and linguistic categories

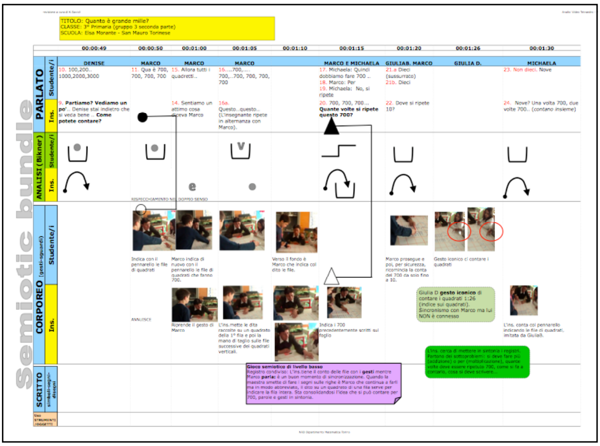


Figure 1: Example of analysis with the timeline

In the timeline we take into account some variables necessary to describe the process in a semiotic way. One variable is the semiotic game introduced by the teacher. It can consist in echoing some signs introduced by the students, in order to reinforce an idea at a certain point of the discussion. Really, according to many studies in a

Vygotskyian approach (Vygotsky, 1978), teaching consists of a process enabling students' *potential* achievements. Within such an approach, the teacher can be seen as a *semiotic mediator*, who promotes the evolution of *signs* in the classroom from the personal senses by the students towards the scientific shared sense. Indeed, having recourse to the semiotic game, the teacher generally judges from the signs produced by the student when they are in Zone of Proximal Development (ZPD), in relation to the concept to be taught. Hence, the teacher mimes one of the signs produced by the students, using words different from theirs. While the students have possibly used an imprecise verbal explanation about the mathematical situation, the teacher introduces precise words to describe it. Namely, the teacher uses one of the shared resources (gestures) to enter in a consonant communicative attitude with his/her students, and another resource (speech) to push them towards the scientific meaning of the concepts they are taking. Such a strategy is suitable when the non-verbal resources students utilise reveal to the teacher that they are in ZPD. Usually, the students explain a new mathematical situation producing at once gestures and speech (or other signs). We can frame them using the semiotic bundle. Students' explanation by their gestures seems promising, but their words are very imprecise, when not wrong. The teacher mimes the former to recall students' behaviour, but pushes the latter towards the right form.

Another specific variable is the kind of gesture. (redundant or non-redundant). Kita (2000) has elaborated on this relevant matter. Every individual gesture is either *redundant* or *non-redundant* with respect to information conveyed in the accompanying verbal clause. Really, for its bigger generality we prefer the use of the terms redundant and non-redundant instead of *match* (if the entire information expressed in gesture is also conveyed in speech) and *mismatch* in the other cases (Goldin-Meadow, 2003). According to Goldin-Meadow (2003), a mismatch is "associated with a propensity to learn" (p. 49), and "appears to be a stepping-stone on the way toward mastery of a task" (p. 51). Gesture-speech mismatch mainly reflects "the simultaneous activation of two ideas" (p. 176). We opt for Kita's categories of redundant and non-redundant, because the presence of a non-redundant gesture does not automatically coincide with a mismatch.

A third variable is the kind of interaction (locutional, illocutional, and perlocutional) (Davis, 1980). In speech, one always speaks in relation to these different levels. At the locutionary level, he/she says something; at the illocutionary level, he/she tells something in a specific manner (e.g. speaking aloud or silently). The perlocutionary level is concerned with effects: "a speaker saying something produces an effect on feelings, thoughts, or actions of the audience, other persons, or himself" (Davis, 1980, p. 38).

DISCUSSION

In this article, we have presented a tool to investigate, through a semiotic approach, students' and teacher's multimodal productions during their mathematical activities in the classroom (bodily, oral and written signs). We have used as a model the *semiotic bundle* that takes into consideration any sign introduced by the subjects. The model may offer a snapshot of signs at a certain moment (synchronic analysis), or a description over time of the semiotic activity (diachronic analysis). Both the analyses give insights into cognitive processes and didactical phenomena. Their illustration is provided by the timeline, through a *microanalysis* and a *macroanalysis* of what occurred in the classroom. The first (micro) furnishes, in a short period, an evolution over time of the signs the students and the teacher introduce. In this way, the timeline gives us a chance to follow processes of meaning making based on: transformations of one sign into another, sharing of signs among people, use of a sign to approach a concept, etc. The second (macro) shows, in the medium-long period, what happens in the classroom or in a group of students, or simply for one student, in their social or individual cultural growth. The dimension of time depicts the processes that support subjects' construction of mathematical meanings. For example, we can see whether a sign is considered by a student, echoed by the teacher in front of all the students, whether the sign becomes a shared sign to recall a concept, to justify a conjecture or to find a thesis (Arzarello & Paola, 2007). As another example, we may scrutinize the complex process through which a student is able to grasp a mathematical concept (e.g. a numerical relationship) starting from a gesture, then transforming it into a key word, and finally converting it in a written inscription, all concurring to explain the concept (see Arzarello *et al*., 2006). In the semiotic bundle, we may also have a sign coming from interaction between a student and a tool (Arzarello & Robutti, 2008). The timeline may show how that sign is re-used in the activity to find an invariant, a pattern or a property of similar situations. Briefly speaking, the timeline provides us with dynamicity, which is an important element for describing processes. It helps understanding how, when, and why social knowledge construction develops. By displaying the ingredients of the semiotic bundle, the timeline orders them in a time dimension, and uncovers the "movie" of a didactical situation, giving the opportunity for a careful analysis.

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TEACHING PRACTICE: A COMPARISON OF TWO TEACHERS' DECISION MAKING IN THE MATHEMATICS CLASSROOM

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In this paper we examine decisions made by two mathematics teachers who presented the same lessons on graphical antiderivative in two different countries, namely Italy and New Zealand. In particular we use a theoretical framework comprising Schoenfeld's Resources, Orientations and Goals (ROG) to describe and analyse the decision each independently made to produce a table of information for their students. A comparison of the orientations and goals behind this decision enables us to uncover how the teachers arrived at their different outcomes and the expectations they had of the students. This shows the value of the framework for making analytical comparisons of pedagogical practice. Understanding the reasons for the different paths taken by the two teachers has possible implications for teacher training and professional development.

INTRODUCTION AND THEORETICAL FRAMEWORK

Since the 1980's, mathematics education research has investigated the competences teachers need in order to promote efficient learning. Shulman (1987) provides an analysis of the kind of teachers' knowledge required. Starting from this, several authors (e.g. see Sullivan & Wood, 2008) have identified a number of important features of effective teaching, one of which is the ability to make appropriate in-the-moment decisions.

A detailed investigation of teachers' in-the-moment decision making led Schoenfeld (2008, 2010) to produce a framework for analysing such decisions. It is based on Resources, Orientations and Goals (ROG) that lead to the production, prioritisation and accomplishment of goals. The framework suggests that teaching (and other activities) begin with the *orientations*, or dispositions, beliefs, values, tastes and preferences, etc, that the teacher brings to bear on pedagogy in the classroom situation. These orientations then "…shape the prioritization of the goals that are established for dealing with those situations and the prioritization of the knowledge that is used in the service of those goals." (Schoenfeld, 2010, p. 29) Thus when appropriate goals (and sub-goals) are established, the teacher brings relevant resources – and this especially means knowledge, but also physical resources such as

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textbooks and technological devices – to bear on the achievement of those goals. This requires decisions to be made, either consciously or unconsciously, about how to reach the goals using the available resources. In familiar situations, such as classroom teaching, automatic scripts, frames, routines or schemata are called upon. For example, the most common routine is the IRE sequence, where a teacher initiates an interaction (I), gets a response (R) from the class or an individual, and then evaluates (E) the response. As a result of the interactions in the goal-oriented behaviour, the teacher's knowledge, goals, and orientations are updated, and the cyclical process continues.

In order to implement the theoretical framework briefly described above, one has to access teacher's ROG. In Schoenfeld's research this is done by inferring the ROG from the video of a lesson. However, Speer (2008, p. 262) concludes that "Analysis at a finer grain size, conducted on interview data from discussions of specific instances of practice may be what is needed to develop rigorous explanations for classroom observational data". Therefore, in this project we have used interviews with the teachers to gather an espoused part of the ROG, as well as inferring part of the ROG from video and transcript evidence.

The framework has been used to link beliefs and goals (Aguirre & Speer, 2000), identifying beliefs about the nature of teaching, learning and mathematics that were influential on teacher practice and became manifest when goals were changed. Also Törner, Rolke, Rösken, & Sririman (2010) show the explanatory power of the theory, and describe the dominance of subject matter goals and beliefs over pedagogical content goals and beliefs. Further research considering a fine-grained analysis of how teacher beliefs play a role in shaping decisions and practices concluded that "…the construct 'collections of beliefs' was proposed as a unit of analysis for beliefs that captures teachers' views about particular issues in ways that make it possible to understand, explain, and even predict teachers' decisions." (Speer, 2008, p. 260).

THE TEACHING EXPERIMENT

Multi-site teaching experiment

The data reported here were collected as part of a larger multi-site teaching experiment involving seven secondary school teachers in Italy, Israel, and New Zealand, who implemented lessons on graphical antiderivatives based on a sequence of four tasks (Yoon, Dreyfus & Thomas, 2008). These tasks were initially written in English, and translated into the local language. The first task in the sequence is a Model Eliciting Activity (Lesh et al., 2000) that asks students to draw the graph of a tramping track whose gradient graph has been given. The second task involves asking students to draw antiderivatives of a number of functions presented graphically, which were intentionally drawn so as to be dissimilar to well known graphs, such as straight lines, parabolas, and so forth. The third task focuses on the graphical interpretation of the constant of integration, concavity and points of inflection, and the fourth task asks students to identify functions, derivatives and antiderivatives that are presented graphically. In this paper, we focus on two teachers – Adam, from New Zealand, and Daniel, from Italy.

Data collection at the New Zealand site

Adam was in his 2nd year of secondary school teaching at the time of the study, and had recent research experience in teaching derivatives using technology that facilitated thinking in graphical representations. Adam's class was a high ability year 12 (age 16-17) mathematics class in a low socioeconomic multicultural school in Auckland, New Zealand, with primarily Maori and Pacific Island students.

The topic of graphical antiderivatives is not in the New Zealand national curriculum, although the related topic of graphical derivatives is part of the curriculum. Therefore, Adam was only able to allocate four 60-minute lessons to each of the four tasks. The four lessons were audiotaped and videotaped, and student work was collected. One videocamera focused on the teacher and the whole class. There were two focus pairs of students, both of which had one videocamera focusing on their faces and another on their written work. After each lesson, the teacher participated in debriefing interviews with the researcher(s), which were audiotaped, in which he described his experience of the lesson, explained certain teaching decisions, and planned for subsequent lessons.

Data collection at the Italy site

Daniel is a teacher with more than 20 years of teaching experience in the secondary school and of research in different subjects of mathematics education (proof, real analysis, technology…). The school where Daniel is working is a scientific oriented high school in Finale Ligure (Genoa). Daniel's class consists of grade 11 (age 16-17) students, who pursue a strong mathematics curriculum (five hours per week). In the traditional teaching praxis the topics of graphical derivative and antiderivative are not studied in the 11th grade. Daniel decided the students would work for 100-minute lessons on the tasks in groups of 3 or 4 students, with very limited support from the teacher, who answered only specific questions. After each task there was a 50-minute lesson to discuss and formalize the concepts in the task (institutionalization lesson). So the whole project took eleven 50-minute lessons.

In the classroom there were two videorecorded focus groups and every institutionalization lesson was videorecorded by a camera focused on the teacher at the blackboard. We analyzed the video, considering all the semiotic productions of the teacher (speech, inscriptions at the blackboard, and gestures), as well as the interventions of the students. At the end of the project the teacher participated in audiotaped debriefing interviews with the researcher: Daniel described his experience of the lesson and explained certain teaching decisions and his main goals.

Data analysis

The data that were collected at each site were analysed by the researchers located at those sites. However, both sites used a similar method for analysing the data. Both sites transcribed the interviews with the teachers (Adam and Daniel), and coded them according to Schoenfeld's ROG model, using the same coding scheme. Next, the two sites transcribed the video footage of lessons in which the teachers created a table of properties (lesson 2 in Adam's case, institutionalization lesson 1 in Daniel's case). These transcripts were annotated with descriptions of the teachers' and students' behaviour, as well as pictures of gestures, boardwork, and student work, and were coded to identify key teaching decisions occurring during the lesson. Finally, the same coding scheme used to analyse the interview data was used to identify the R, O, and Gs that appeared to influence the teacher's decisions

COMPARISON BETWEEN THE ROGS OF THE TWO TEACHERS

Adam's ROG (New Zealand site)

Before the second lesson started, Adam had a planned goal to create a table of properties, and 12 minutes into the lesson, he began copying the outline of the table with the headings and subheadings in the first two rows from a piece of paper he held in his hand. With input from the students, he filled in the rest of the cells in the table over the course of 20 minutes. For each cell, he focused students' attention on a specific point on the graph at $x = a$, $x = b$, or $x = c$ (see Fig. 1a), and asked them about the corresponding points on the antiderivative. When the students responded with the property Adam thought was correct, he wrote it in the relevant cell.

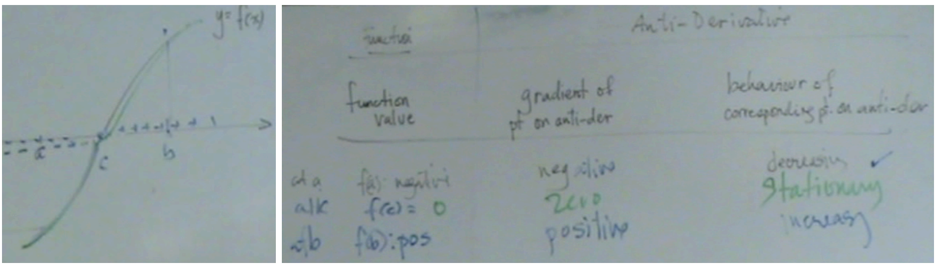

Figure 1: (a) The graph of a function with points $x = a$, $x = b$, and $x = c$ identified.
(b) The table of properties produced by Adam.

	Function	Anti-Derivative	
	function value	gradient of pt on anti-der	behaviour of corresponding pt. on anti-der
At *a*	$f(a)$ negative	negative	decreasing
At *c*	$f(c) = 0$	zero	Stationary
At *b*	$f(b)$:pos	positive	increasing

Table 1: Retyped version of the table of properties produced by Adam.

Adam's decision to create the table appeared to be driven by a goal that influenced many of his teaching decisions – to give students a firm conceptual foundation that would prepare them for the upcoming tasks. This goal is reflected in the following comments from his debriefing interviews: "I'll have to go through this because it would make the foundation more firm", and, "The basic concept of derivative and antiderivative will be firmed up and then that will be a very easy task for them later on.". In the particular case of the table of properties, it appeared that Adam intended the students to use the table as a tool for drawing antiderivative graphs in upcoming tasks. He announced this intention just before he commenced drawing the table, saying, "Just to remind you of the aim for this lesson, we want to become efficient in drawing the anti-derivative. Not in a mechanical way, but in a deeper understanding way." This goal was supported by his knowledge of which students had previously struggled with drawing gradient functions (Resource A1), and his belief that students need to be prepared for upcoming tasks by understanding the concepts that will be used in the tasks (Orientation A1).

Adam makes his goal explicit by instructing the students to use the table on two occasions. After creating the table, he tells the students to draw the antiderivative of the next function themselves, using the table that they had created as a guide, "I created a table here, I mean, I created it with you, so let this be our guide." In another instance, one student explains his solution in front of the class, and is unsure whether the x-axis intercept in the graph of the function corresponds to a maximum or minimum in the graph of the antiderivative. Adam suggests using the table again, saying, "If gradient is zero, we go back to the table to guide us." However, the table only tells him that it corresponds to a stationary point, which the student had previously said, but not whether it is a maximum or minimum. Thus, although Adam made his goal explicit to the students, the students did not appear to use the table in the way he intended.

Daniel's ROG (Italy site)

Daniel prepared for the institutionalization lesson after the first task by reading the students' protocols and considering the problems that arose from the working groups. A driving goal of Daniel's decision to construct the table was to give a summarising tool for the meaning of the concepts involved in the task. This is in line with his main goal: consolidating the students' knowledge. He wished to check whether the students were able to use what they had already done in the years before (they had used TN-spire) and how they reacted to topics already covered in a different way. Daniel began to construct the table after about 20 minutes of the lesson, during which he corrected the first part of the task together with the students. Thus, he created the table after mediating the meaning of new concepts and after a revision of old concepts. His driving goal is reflected also during the lesson in these comments before and during the table construction: "...These things become important…" or " So it's obvious that this thing that we start to put here is important...". This last sentence is matched with his action: to circle the first line of the table on the blackboard. His table consists of two rows (see Figure 2):

First line: “ f increases $\longleftrightarrow$ f’ positive”

Second line : “f with upwards concavity $\longleftrightarrow$ f’ increases”

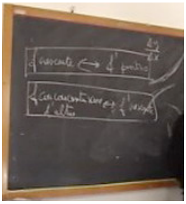

Figure 2

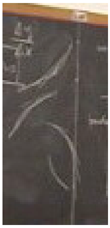

Figure 3

Before writing the first line he drew some pictures to support the students in understanding how the slopes of the tangents in an increasing function first have upward, then downward concavity (Figure 3). Next, he asked some questions of the students in order to make them understand that if f’<0 then the function decreases and vice versa. After that he drew a picture of the slopes of the tangents to a function with downward/upward concavity, asked the students what happened in those cases, and wrote the second row of the table circling it. About 1½ min passed between writing the first and the second line: in fact he wanted to be sure that the students became conscious of the result before writing it down. After about 3 min. Daniel started to solve the inverse problem (second part of the task) together with the students. He did not explicitly invite the students to use the table but suggested they do that by pointing to the table with a gesture and asking: “what information about the slopes are you going to look at to obtain information on the function?”. After this, he and the students started to use the table to get the information necessary for drawing the antiderivative graph.

Comparison of the ROGs that influenced Adam’s and Daniel’s tables

It seemed that Adam had planned in advance the construction of the table and had thus prepared its format, while Daniel planned to systematize students thinking but did not plan in advance the precise time when this would be done. Daniel made the table primarily as a way to summarise the concepts the students had encountered over the course of the lessons on graphical derivatives, so that students could interiorise it as a tool for consolidating their learning. In comparison, the use of the table is quite different in the two teachers: Daniel used it in an implicit way to solve the inverse problem (research of antiderivative graph) and Adam as a tool for preparing the students for success in the upcoming tasks. However, both Adam and Daniel involved students in the construction of the tables.

DISCUSSION AND CONCLUSION

It has been recognised that there is still a pressing need for research to investigate how teacher orientations, and goals arising from them, shape pedagogical practices (Speer, 2008). In this paper we have provided some further evidence that the ROG framework (Schoenfeld, 2010) is a useful explanatory tool for analysing this relationship. This may be especially true of practice decisions that are made in-the-moment. In particular, we have seen how two teachers who were using the same module of work, and who on the surface may have appeared to be doing very similar things, such as making similar decisions, were motivated by quite different beliefs and goals. The ROG analysis enabled us to uncover this distinction and to explain it.

One common outcome was that, while both teachers appear to have achieved their goals in relation to the table construction and use, the value to the students was less clear. It seems that the students in both classrooms did not fully believe in or value the constructed table as a tool for them. This suggests that, as teachers, we need constantly to evaluate our practice and goal setting not only against our own orientations, but also against those of our students. Failure to find out what students value, and show appreciation for it by acting accordingly, could clearly have detrimental implications for their learning.

The apparent surface similarity of the teacher practice disguising underlying beliefs and goals also raises important professional development issues. One is the role of lesson observation by peers, or those training teachers. It is important that superficial observation practice is avoided, possibly, as Speer (2008) agrees, by discussion with the teacher that attempts to address, not only their written, prepared goals as expressed in lesson plans, but also the values and beliefs that underpin such goals. Encouraging teachers to express their orientations and goals will also help with self-awareness of them, which is likely to be a major step towards positive change. Hence, only in this way can professional development avoid a broad-brush approach and provide assistance tailored to individual teacher needs.

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THE IMPACT OF TEACHER-LED DISCUSSIONS ON STUDENTS' SUBSEQUENT ARGUMENTATIVE WRITING

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The present research focuses on patterns of talk in which teachers are involved when they lead discussions in their classrooms in the course of a Grade 8 learning unit on probability and on their impact in subsequent individual argumentative writing. In a previous PME publication, we undertook a qualitative analysis to show that teacher-students interactions were governed by distinctive and relatively stable patterns. In the present study we undertake a quantitative analysis to corroborate those findings. Moreover, we show that the impact of teacher-led argumentative talk on subsequent individual argument elaboration is deep but subtle. Argumentative quality of teacher-led talk was not be detected in student correctness of solutions, on a claims level, but on the quality and frequency of the explanations given to support these claims.

INTRODUCTION

The present paper reports on a study focusing on patterns of talk in teacher-led classroom discussions, and on their impact on subsequent students' written arguments. Since the tasks on which the discussions focused were specially designed to scaffold productive argumentation in probability, we first aimed at observing and analysing teacher-led discussions. We observed four teachers who taught the same sequence of activities. We already reported on how one teacher triggered explanations and how she helped integrating them into coherent arguments (Schwarz, Hershkowitz & Azmon, 2006), and other did not. The major finding of the above qualitative investigation was that teacher-students interactions are governed by distinctive and relatively stable patterns. For example, Teacher A adopted a dialogic-dialectical talk in which she challenged the diverse claims raised by students to encourage construction of knowledge. The talk of Teacher B was governed by an IRE traditional pattern (Cazden, 2001) almost exclusively initiated by the teacher. Because of space limitations, we reported in the 2006 paper and in this paper on Teachers A and B only. In this paper we continue the analyses of three episodes of talk led by the two teachers and investigate quantitatively the impact of the teacher-led talk on individual written arguments of their students in a final exam.

THEORETICAL FRAME

Reasoned discourse is a habit, a way of life. It needs to be socialized, learned daily during years in an environment that expects such behavior, supports it and rewards it in overt and subtle ways. The only venue through which such socialization is done at a widespread scale is the school. Apprenticeship provides the necessary structure to acquire these discourse-based reasoning abilities. The socialization of discourse

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practices requires that members of the school community—student, teachers, administrators—turn these practices to commonplace in every classroom. Opportunities for students to reflect and communicate about their mathematical work have been identified as essential for learning mathematics in a meaningful way, and for effectively implementing high-level tasks. During discussion, students can see how others approach a task and can gain insights into solution strategies and reasoning processes that they may not have considered. By engaging in whole-class teacher-guided reflective discourse, students can explain their reasoning, make mathematical generalizations and connections between concepts, strategies or representations, and benefit from the collective mathematical work of the class for a given lesson or task (Alexander, 2004; Mercer, 2002; Myhill, 2006; Nystrand, 1997). However empirical studies showing the efficiency of talk-based pedagogy in school learning are still missing. Our study attempts to correlate between interactions teacher-students in teacher-led discussions and further learning gains in individual tasks.

THE STUDY

The overall goals of the study were two-fold. First, we aimed at uncovering patterns of teacher-led talk in teachers that taught the same learning unit in their class. Cazden (1988) has shown that asymmetric interactive patterns govern these interactions. Since then, educators have tried to break these patterns (Mercer, 2002) towards interactions in which students and the interactions among them are at the center. Researchers have observed that different patterns may emerge in classes (Nathan, Eilam & Kim, 2007). Educators consider a blended approach, which has more symmetric and interactive patterns and considers the integration between teacher-led and students-led activities, as representing the natural classroom environment. We already reported that some of the teachers who worked with us broke the IRE patterns of interaction, and that, with their students, they adopted other patterns of interaction (Schwarz et al., 2006). In the present paper we first report on findings which support quantitatively the finding from the former PME report. We then report on findings concerning the second aim of the study – investigating the impact of the teacher-led talk on individual written arguments of their students in a final exam. Four teachers, in three different schools, and their Grade 8 students participated in the study. We report on Teacher A and Teacher B and their classes only. The teachers volunteered to teach a designed 10 lessons probability unit (Hadas, Hershkowitz & Ron, 2009). We intentionally did not provide scripts for teachers' structuring of interactions with their students. Rather, they were left free to choose the way to manage their lessons. All 10 lessons were video-taped and transcribed. We chose three episodes from the three parts of the unit for which all teachers taught the same problem situation through a discussion they led. We observed and analysed these parallel episodes qualitatively and quantitatively. As mentioned above, we already conducted qualitative analyses (Schwarz et al., 2006) to show that patterns of talk persisted across all three episodes. In this paper, we first validate this conclusion quantitatively, then study how talk impinged on subsequent individual arguments in a final exam on the same content.

FINDINGS

a. Quantitative analysis of the patterns of talk of teacher-led discussions

In Table 1 we present the numbers and percentages of the categories of teachers' and students' utterances. These data were calculated for each of both classes.

Teachers				Students			
Teacher	Number & % of utterances	Encourage for claim	Encourage for explanation	Number & % of utterances	Claims	Explanations	% of claims turned to arguments
A	88; 54%	10;11%	55; 63%	75; 46%	35;47%	31;41%	89%
B	111; 58%	54; 49%	16; 14%	81; 42%	57;70%	10; 12%	17.5%

Table 1: Numbers (& percentages) of utterances' categories of teachers and their students

Table 1 shows that in both classes, talk was almost equally distributed between the teacher and her students. However, there were big differences in the nature of talk: Prompts for explanations were very frequent for Teacher A. Students were quite responsive as 41% of their utterances were explanations, naturally leading them to complete their claims into arguments in 89% of the cases. In contrast, Teacher B rarely prompted explanations (14%), and unsurprisingly mostly entailed unreasoned claims (82.5%). These findings strongly match the patterns of talk on which we already reported (Schwarz et al., 2006). An important question with theoretical importance is to check whether these distinctive talk characteristics impinge on further students' cognition.

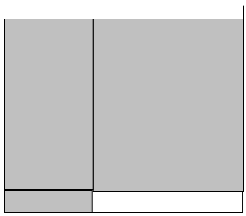
b. The impact of teacher-led talk on individual cognition

The question of the impact of classroom talk on individual cognition is a very hot issue (see for example the recent review of the issue by Resnick, Michaels, & O'Connor, in press). The methodological difficulties are enormous, as it is highly problematic to isolate variables and/or to impose and control teaching methods on teachers. However, in our study we could identify a mathematical problem situation, on which both teachers focused in a teacher led discussion, where both episodes have the same time length. The knowledge which was expected to be constructed in the discussions was checked in a final exam, in which students were asked to write arguments on ideas learned during the learning unit. The final exam included questions which were similar to questions the students dealt with within the learning unit in general and in the three investigated episodes in particular. In this section we examine quantitative data from one item displayed in Figure 1 and which is similar to an activity solved in a teacher-led discussion in part 3 of the learning unit (Schwarz et al, 2006). Three aspects will be separately examined. The first aspect is the correctness of answers (the claims) given by the students (right/wrong). The second aspect is the type of explanations provided, and the third, the richness of students' explanations. Because of space limitations we will deal here with question 9b only (see Figure 1). This question is phrased argumentatively, in order to encourage students to support a particular claim and to provide an explanation for the claim.

The “Nature” school offers a variety of extracurricular programs.
The probability of encountering a child who is in the "turtle nurturing program" is 0.9.
The probability of encountering a child who is in the" Indian music program" is 0.3.
The square below describes the probabilities of meeting children who are in the different programs.
a) Complete the diagram by writing, near the relevant sides of the square, the names of the programs in a way that will represent the correct probabilities.

The ________________ program

The ___________ program



b) The school opened an additional program for "Indian philosophy". The probability of encountering a child who is in this program is 0.2. Gal claims that the probability of encountering a student who is in both the Indian philosophy program and the turtle nurturing program is 0.2+0.9. Yam claims that the probability is 0.2x0.9. Which of them do you think is right? **Explain!!!**

Figure 1: Question 9 in the final exam

The correctness of students’ answers and explanations regarding question 9b:

In this question, two claims were proposed: “Gal claims that the probability of encountering a student who is in both 'the Indian philosophy program' and the 'turtle nurturing program' is 0.2+0.9" and "Yam claims that the probability is 0.2x0.9”. The students were required to decide which of the claims is right. Table 2 shows how students in the two classrooms answered this question.

Teacher	Yam is right	No explanation	Correct explanation	No answer
A	92%	8%	88%	4%
B	94%	11%	89%	3%

Table 2. Correctness of students' claims and explanations in Teachers A & B classes

Table 2 shows that in both classes a very high percentage of students gave a correct answer and explained it correctly. Differences between classes were non-significant.

Differences between both classes concerning the explanation's categories:

Although no differences could be detected at the level of correct answers and proper explanations, we will see that further analysis of the explanations to identify the principles underlying the explanation yielded very interesting results. We could identify three kinds of categories of explanations:

1. *Explanations relying on a multiplication procedure*

This category includes explanations relying on the use of a multiplication procedure. A first possibility is simply procedural. The student evokes explanations such as "*in probability we multiply probabilities*", but, does not show any evidence of understanding why the procedure is justified. A very different way to use a multiplication procedure is to rely on the area model. The students draw an area square diagram, divide it according to the given probabilities, determine the relevant rectangle, and calculate its area. In this case, the multiplication procedure supports the use of a *model that represents the situation* in which the probability is calculated. A variant of this kind of explanation consists of_using "part of the whole" strategy: Students explain that they calculate the required probability according to the portion of the relevant rectangle out of the whole (the square area).

2. *Explanations according to the principle "probability can't be greater than 1"*

Many students chose to support the claim that "Yam is right", by asserting that "Gal is wrong" and backing this claim by a reason such as *"0.2+0.9", the sum of probabilities, will lead to a probability that is greater than 1, and it is impossible that the area square will contain more than 100%.*

3. *Explanations combining both principles*

Many students chose to combine principles. For example, they claimed that Gal is wrong, because the probability can't be larger than one, and they added as further support a reference to a multiplication procedure, by saying for example that the result of the multiplication of 0.9 by 0.2 is 0.18, a probability that is smaller than 1. The distribution of categories of explanations in the two classes for item 9b is presented in Table 3. $\chi 2$ tests were also performed to examine the significance of the differences between classes. It is remarkable that, as shown in Table 2, most students justified their claims in the final exam. But it appears that the distribution is different and that the students whose teacher is Teacher A rely more on the area model, while in the other class they invoke multiplication as a procedure.

Teacher	Multiplication Only	Multiplication For area	Multiplication for part of	Probability isn't more than 1	The 2 principles together
A	12%	38%	4%	13%	21%
B	43%	3%	5%	8%	30%
	P=0.0297	P=0.0023	P=0.5507	P=0.4091	P=0.2957

Table 3. Categories of students' explanations for question 9b in each class

However, while 38% of the students in the class of Teacher A provide a meaningful explanation according to the area model, a far lower percentage of Teacher B's students (3%) do so. In contrast, more than 40% of them provide an explanation relying only on the multiplication principle, while only 12% of Teacher A's students do so. These findings support our hypothesis that the classroom culture as conveyed in its patterns of talk, described by Schwarz and colleagues (Schwarz et al., 2006), impinges on the students' conceptual understanding. In teacher B's classroom, where the patterns of talk are based on teacher-centered encouraging and expressing (short!) claims only, the students find it harder to provide explanations that are beyond the merely procedural. The next subsection comes to illustrate the richness of students' explanations.

Richness of explanations

We adapted the methodology for analyzing the richness of the written justification by *idea units* defined by Mayer (1982). Idea units are the primitive elements that constitute students' justifications. In our case, students' justifications in the exam's items. As an example we report on extreme explanations given by two different students to 9b item: Naomi (from teacher A's class): *Yam is right because Gal action is wrong* (1) *because 0.9+0.2=1.1 and it doesn't make sense that the event will be more than 1*. (2) *and for that reason Yam argues that the probability is getting less and less*(3) *when we multiply 0.2*0.9=0.18, and she is right*. We count in this explanation 3 idea units. Yosi (from Teacher B's class): *Yam, because we should multiply and not to add (1).* Here we count only 1 idea unit. These are two extreme examples which demonstrate this "idea unit methodology".

The counting of the idea units in each explanation was done independently by three researchers. The inter-rater percentage of agreement was 90%. When we calculated the average of the number of idea units per explanation for each class, we found that: the average in Teacher A's class is 1.58 idea units per explanation, and in Teacher B's class is 1.15 idea units per explanation. ANOVA analysis shows that $p < 0.0133$.

CONCLUDING REMARKS

Delegation of responsibilities. The socio-mathematical norms that developed in each class impinged at a macro-level concerning the responsibility of students to their knowledge constructing. For example in Teacher A's class, the challenges of the teacher led students to feel responsibility to provide elaborated explanations. For example: almost all (89%, Table 1) of the claims expressed by the students in the three episodes were reasoned and eventually turned to full-fledged arguments. In Teacher B's class, the elaboration of explanations was not under the responsibility of the students but of the teacher. Only 18% of the students' claims, in the three episodes in class B, were reasoned. Interactions between the teacher and her students appeared as chains of short questions and short claims as answers (70% of the students utterances in the class of Teacher B were claims, and 47% in the class of Teacher A), punctuated by social validation of correctness. Quite naturally, correctness was valorized rather than processes that led to the (correct) result.

The transformatory character of argumentation.

The enactment of dialogic-dialectical talk (like for Teacher A) in a succession of tasks designed to encourage knowledge construction, led students to discuss mathematical principles under the orchestration of the teacher and to co-construct them with the teacher and when left alone in small groups. It turns then that there are strong bonds between teacher-led dialectic argumentative talk and subsequent co-construction of mathematical principles/concepts.

Impact of teacher-led argumentative talk in the classroom, on subsequent individual argument elaboration.

This finding suggests that the impact is deep and subtle. For example: we show the analysis of question 9b in the final exam, where a high percent of students in all classrooms correctly answer to the questions on a claim level, and try to explain the claims. However, the explanations given by the students of Teacher A, a teacher who adopted a dialectical-dialogical style in her interactions (as reflected on her distinctive patterns of interaction), were different from those by the students in Teacher B classroom not because more or less of the answers were correct, but in the fact that those answers integrated more mathematical principles and were richer. This result resonates with recent studies that show the beneficial effects of dialectical-dialogic argumentation on conceptual change (Asterhan & Schwarz, 2009) – the fact that patterns of interaction such as those that took place in the classroom of Teacher A, deepened understanding rather than consolidated acquisition of factual knowledge.

The present research suggests *the importance of the mediation of the teacher*. This mediation seems to be more productive when the teacher acts as an agent that negotiates meanings with students. The argumentative patterns that characterize talk can either give birth to meaningful constructions or to senseless artifacts. The findings suggest the importance of in-service teachers' programs focusing on the animation of classroom discussions for the sake of the promotion of mathematical reasoning and for delegating responsibility to students on their learning.

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REASONING BY CONTRADICTION IN DYNAMIC GEOMETRY

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This paper addresses contributions that dynamic geometry systems (DGSs) may give in reasoning by contradiction in geometry. We present analyses of three excerpts of students' work and use the notion of pseudo object, elaborated from previous research, to show some specificities of DGS in constructing proof by contradiction. In particular, we support the claim that a DGS can offer "guidance" in the solver's development of an indirect argument thanks to the potential it offers of both constructing certain properties robustly, and of helping the solver perceive pseudo objects.

INTRODUCTION

Literature shows that although much research has been conducted on the themes of proof and argumentation in mathematics education, rarely do the studies focus on particular proof structures, such as proof by contradiction. The research centred on proof by contradiction has pointed to various difficulties it presents for students (see for example, Antonini & Mariotti, 2008; 2007; Mariotti & Antonini, 2006; Wu Yu et al., 2003; Leron, 1985) especially the difficulties related to the formulation and interpretation of negation, to the managing of impossible mathematical objects, to the gap between contradiction and the proved statement.

Some literature takes into consideration contributions that dynamic geometry systems (DGSs) may give to students' production of indirect arguments. Within the very little literature in this area, there is a study conducted by Leung and Lopez-Real that describes a proof by contradiction produced by two students working in a DGS. This case study triggered the development of a framework on theorem acquisition and justification in a DGS that the authors used to put together a scheme for "seeing" proof by contradiction in a DGS (Leung & Lopez-Real, 2002). We will illustrate aspects of this framework that we will make use of and develop further in the following section.

With the present paper we intend to contribute to better describing roles that a DGS can have in reasoning by contradiction. We will further elaborate and make use of notions from Leung and Lopez-Real's theoretical framework, in particular that of *pseudo object*, to analyze such roles. Moreover, we will provide analyses of 3 excerpts of students' work to show particular construction choices in a DGS can "guide/promote" significantly solvers' development of indirect arguments/reasoning by contradiction.

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METHODOLOGY

The data presented was collected during two different studies on the role of a DGS in processes of conjecture-generation and proof in the context of open problems in geometry. One study (Leung & Lopez-Real, 2002) was conducted with Form 4 (Grade 10) students in a *Band One secondary school* in Hong Kong. (Hong Kong's secondary schools are streamed according to students' ability. A Band One school is for the most able students). The second study (Baccaglini-Frank, 2010; Baccaglini-Frank & Mariotti, 2010) was conducted with Italian high school students from three different *licei scientifici*, between the ages of 16 and 18. The participants of both studies had been working with dynamic geometry for at least a year prior to when the studies were carried out. Data was collected in the form of: audio and video tapes and transcriptions of the introductory lessons; Cabri-files worked on by the instructor and the students during the classroom activities; audio and video tapes, screenshots of the students' explorations, transcriptions of the task-based interviews, and the students' work on paper that was produced during the interviews.

THE NOTION OF PSEUDO OBJECT

When working with paper and pencil and reasoning by contradiction, slight inaccuracies in the drawing allow the figure to represent properties, which a *proper* construction would not permit. For example, on paper, with no trouble one can assume to have drawn a triangle, of which two bisectors intersect at a right angle. In this case one may easily be unaware of his/her assumption of contradictory properties, and it is completely up to him/her to become aware of a contradiction.

In a DGS a similar situation to that described in paper and pencil occurs when the solver constructs a figure with a robust property (Healy, 2000) while mentally imposing on it a contradictory property without a robust construction. By robust construction in a DGS, we mean construction that can keep the desired properties of a figure invariant under dragging. What happens if, instead, the solver attempts to construct both properties robustly? However, the solver may be uncertain whether such a construction is possible or not, or s/he may realize the impossibility when interpreting the DGS' feedback. Such feedback includes the making explicit, robustly, of all properties that are derived from the properties constructed robustly during the construction steps of the figure. This is the case we find particularly interesting. In this paper we report on ways of reasoning that seem to be induced by the feedback provided by the DGS.

As mentioned above, in a DGS no constructable-figure can be realized by robust contradictory properties. So to represent a geometrical object with contradictory properties (at least) one property must not be constructed robustly, but only conceived (or projected onto the figure) by the solver. Therefore the solver is completely in charge of conceiving any contradiction. In this paper, we will present three DGS cases of (attempts of) reasoning by contradiction by students that involve solvers projected non-constructable properties onto geometrical figures. To analyze these cases, we further elaborated Leung and Lopez-Real's the notion of *pseudo object* (2002) in a dynamic geometry environment as follow:

> A *pseudo object* is a geometrical figure associated to another geometrical figure either by construction or by projected-perception in such a way that it contains properties that are contradictory in the Euclidean theory.

We stress that the notion of pseudo object is solver-centered. Thus, the same dynamic figure can be a pseudo object for one solver, but not for another, depending on whether the solver has projected upon the geometrical objects contradictory properties. In this sense, any dynamic figures defined through a construction have the *potential* of becoming pseudo objects for any given solver. In DGS, this potentiality can be realized through a cognitive process of dragging in which conceiving a pseudo object is critical in reasoning by contradiction. To facilitate the analyses of this process, we introduce a notion of *proto-pseudo object*:

> A *proto-pseudo object* is a geometrical object that has the potential of becoming a pseudo object – such potential is exploited when the solver perceives a property of such object as being contradictory with respect to another of its properties.

Thus a *proto-pseudo object* can become an actual *pseudo object* once (and if) the solver consciously projects a property upon it that s/he is aware of as contradictory. We will use the notions of *pseudo object* and *proto-pseudo object* to show how DGSs seem to provide cognitive support in (1) offering the potential of constructing certain properties robustly, (2) generating feedback in the form of robustness of all properties that are consequences of the constructed ones, (3) and the possibility of dragging parts of the dynamic figure to explore compatibility between the robust properties and those the solver has projected upon it. As we shall see in the following cases, these features seem to guide solvers to conceiving *pseudo objects* in processes of reasoning by contradiction.

THE ROLE OF DYNAMIC GEOMETRY IN THREE SOLUTION PROCESSES

Consider the following task from Lopez-Real and Leung's study (2002):

> Given a quadrilateral in which the sum of the pairs of opposite angles is 180°, prove that it's cyclic.

This task was given to the participants in Leung and Lopez-Real's study. We report on Hilda and Jane's solution.

Excerpt 1: the case of Hilda and Jane

Hilda and Jane construct a quadrilateral ABCD of which the vertices A, B, D lie on the same circle with center E, while C, D, B lie on a distinct circle with center F. Then they mark the measures of the angle in A and in C as *a* and *180°- a*, respectively, and proceed to construct the quadrilateral

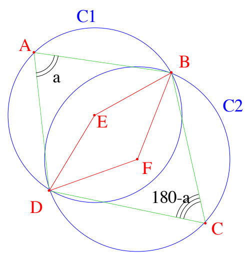

Figure 1

BEDF (Figure 1). If the labeling were Euclidean-correct this construction would not be possible since the circles would coincide, thus ABCD is biased and it leads to the existence of a non-degenerate quadrilateral BEDF, which Hilda and Jane conceive as a *pseudo object*. This can be seen both in Jane and Hilda's proof, and in an excerpt of the transcript of a follow-up interview[1] the researchers had with the girls.

Statement in the proof:

> "From the diagram we see that it has a contradiction as the sum of the opposite angles of the blue quadrilateral (EBFD) is 360° which is impossible."

Excerpt of the interview

7	Int:	So before you did that presumably you first of all drew a circle through 3 of the points and then you did the same for these 3 points.
8	H:	H: Yes.
9	Int:	So then you marked these 2 centers. What did you say after that?
10	H:	Because the angle sum of a quadrilateral is 360 and these two (*referring to* $\angle E$ *and* $\angle F$) already add up to 360 so this is not possible.

Our analysis suggests that the quadrilateral ABCD is initially a *proto-pseudo object*, and it becomes a *pseudo object* for the solvers once EFBD is perceived as "not possible" (10). EFBD possesses two contradictory properties which the solvers perceive simultaneously as (1) a quadrilateral with 4 angles whose sum is 360 degrees (as all convex quadrilaterals), and (2) a quadrilateral in which the sum of only two angles is 360 degrees [statement in proof and 10]. This *pseudo object* EFBD contains the contradiction necessary for a proof by contradiction. The proof is completed by noticing that when EFBD is being dragged to degenerate (disappear) the two circles [C1 and C2 in Figure1] coincide. In other words, the presence of the *pseudo object* implies the negation of the conclusion of the statement to prove. By arriving at the proof, the solvers are aware that their original quadrilateral ABCD also possess contradictory properties; that is, (1) its four vertices are on different circles and (2) the sum of two opposite angles is 180°. Hence the *proto-pseudo object* ABCD becomes a *pseudo object*.

The support offered by the DGS (in this case, Cabri) consisted in guiding the solvers' transition fof ABCD from its status of proto-pseudo object to a genuine pseudo object. In this case the transition occurred through the perception of a *pseudo object* (EFBD) associated with ABCD. Finally, it seems quite remarkable that Hilda and Jane decided to construct two distinct circles [line 7] through two sets of thee points of the original quadrilateral, showing the case in which ABCD is cyclical in terms of coincidence of the two circles via dragging in DGS. In particular, they transform the problem, which does not contain any *impossibility* in its original statement, into a problem of constructing something impossible. This is the type of problem that Stefano and Giulio and Tommaso and Simone were given in the study by Baccaglini-Frank and Mariotti.

Excerpt 2: the case of Stefano and Giulio

Similarly to what we described above, in the following cases awareness of the presence of a *pseudo object* determines the impossibility of a construction, thus validating a statement such as "this construction is impossible." In the following two cases we encounter similar solution processes to those described by Mariotti and Antonini (2009). The solvers conceive a (or various) "new" object(s) that are used to "show" a contradiction. However, having the DGS at their disposal, the solvers make use of it in significant ways that we will describe. In the transcripts below "Int" refers to the interviewer, and the bold letters refer to the solver who is holding the mouse.

The task:

> Answer the following question: Is it possible to construct a triangle with two perpendicular angle bisectors? If so, provide steps for a construction. If not, explain why.

Giulio and Stefano immediately advance the hypothesis that the construction is not possible, but quickly transit to constructing a figure in Cabri to try to explain their intuition.

1 Ste: No, the only way is to have 90 degree angles... [unclear which these may be, as Ste was not constructing the figure nor looking at the screen.]

2 **Giu**: That for a triangle is a bit difficult!! [giggling]…So...they have to be.

3 Ste: If triangles have 4 angles...

4 **Giu**: no, I was about to say something silly...

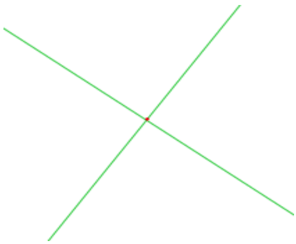

Figure 2

Immediately Giulio starts constructing two perpendicular lines and refers to them as the bisectors of the triangle (Figure 2).

5 Ste: Yes, these are bisectors, right?

6 Int: Yes.

7 **Giu**: So, now we need to get... bisectors... how can we have an angle from the bisector?

[…]

8 **Giu**: the symmetric image? … It's enough to do the symmetric of this one..

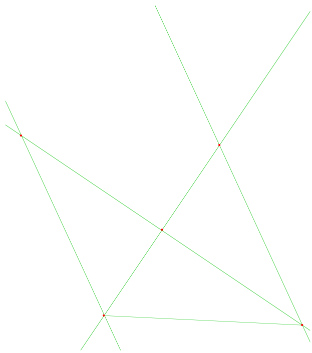

Figure 3

So the solvers have constructed a figure with two robust angle bisectors that intersect perpendicularly (Figure 3).

9 Ste: The only thing is that this [Fig. 3] isn't a triangle!

10 **Giu**: Therefore now we could do like this here [drawing the lines through the symmetric points and the two drawn vertices of the triangle]

11 Int: Yes.

12 Ste: It's that something atrocious comes out!

13 **Giu**: And here...theoretically the point of intersection should be ....the points...very small detail...hmmm

[…]

14 Ste: No, we proved that this is equal to this [pointing to angles], and this is equal to this because they are bisectors... these two are equal so these are parallel.

[…]

15 Ste: These two [referring to the two parallel lines] have a hole so it is not a triangle.

We interpret this episode as follow. The solvers use the DGS to construct two perpendicular lines and the symmetric image to construct the property of them being bisectors. Once the construction is completed they discern properties that are consequences of these two robustly constructed properties, and consequently notice that "the figure must have two adjacent angles with two parallel sides" [9-15]. As soon as they recognize "a hole" in the triangle-to-be [15] the *pseudo object* exists: that is, a figure that has a "base" and two parallel sides, and that has the property "triangle" projected onto it. The appearance of this *pseudo object* reveals to the solvers the impossibility of accomplishing a correct robust construction and thus allows them to solve the problem.

Excerpt 3: the case of Tommaso & Simone

Tommaso and Simone proceed by constructing a proper triangle and two of its bisectors. Then they mark an angle formed by the bisectors and start dragging one vertex of the triangle in the attempt to get the measure to say "90°" (Figure 4).

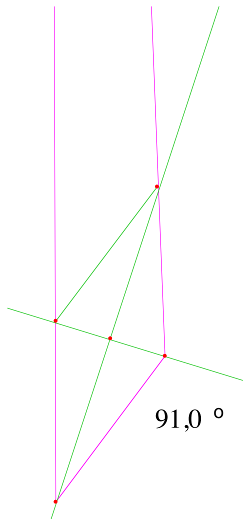

Figure 4

1 **Sim**: It's endless!!

2 **Sim**: 91.2 [reading the measure of the angle between the bisectors.]

3 **Sim**: Well, yes, in any case it will come out!

4 Tom: How do you know? maybe...

5 **Sim**: Well, of course! It's not like it can go on forever! At the end it will make it to be 90!

[…]

6 Tom: I don't think it is possible.

The solvers seem unsure about the possibility of constructing such a triangle, but now seems to think it is not possible. They start reasoning differently.

7 **Sim**: Eh, it is impossible to construct it! Because... I only have these two bisectors.

8	Int:	Hmm.
9	**Sim**:	How can I ....
10	**Sim**:	Since...the perpendicular bisectors...it means here there is a rhombus...or a square
11	**Sim**:	If like here...[he draws a segment]…Here...there were...a rhombus...this would be 90, 90...or a square. And therefore...then…Eh, I mean, if this is like a rhombus, no? here there is 90 and here there is 90, and these are the bisectors.
12	**Sim**:	And then ...and then I bring these up [pointing to the vertical-looking sides of the triangle] and I find their point of ...of intersection.

With respect to Stefano and Giulio, here the solvers choose a different pair of properties to construct robustly: (1) the triangle, (2) the bisectors. They do not construct but (assume) project the property "perpendicular bisectors" onto the figure. Nevertheless they are not able to conceive a contradiction in it or in the new object they conceive: the rhombus. Hence this rhombus is a *proto-pseudo object*, and the solvers do not seem to make the transition to conceiving it as a *pseudo object*. It is significant that the solvers say "it has 4 right angles" [11] pointing to the figure that even has a marked measure of one of the angles, and the measure says " 91° "!! No contradiction among the properties of the "rhombus" is perceived and the solvers are not able to reach a conclusion[1]. They seem to keep on believing that the triangle always has a third vertex "somewhere up high".

When we compare Excerpt 2 and Excerpt 3, a determining difference, from a cognitive point of view, is that in one case the solvers conceive a *pseudo object*, in the other they do not. This can be explained by the solvers' different choice of the properties to construct robustly. The choice determines the type of guidance that the DGS can provide to reasoning by contradiction. In Excerpt 3, starting from the triangle and trying to obtain perpendicularity of the bisectors through dragging allows the solvers to use the DGS (only) as a sort of "amplified paper-and-pencil drawing" in that it allowed the exploration of many cases without having to redraw the figure. On the other hand, in Excerpt 2 the DGS generates two robust parallel lines as a consequence of the constructed properties, thus "guiding" the solvers in perceiving their "a hole" in the triangle-to-be and thus such object as a *pseudo object*.

CONCLUSION

We have introduced the concept of *pseudo object* and illustrated how it can contribute significantly to reasoning by contradiction in Euclidean geometry. In particular, in a DGS environment, construction and dragging strategies leading to degeneration of a *pseudo object* could guide to ascertainment of a geometric theorem or property. The hybrid nature of a *pseudo object* seems to be conducive to formulating an exchange of meaning between dynamic visual reasoning in DGS and theoretical reasoning in the Euclidean axiomatic system (in this case proof by contradiction). We have shown that there can be a strong subjective element in the process of producing a geometrical proof (or a convincing argument) via the solver's conscious choices of construction and dragging in a DGS. We hope this paper will open up a window of discussion to view proof in dynamic geometry environment in ways that can enrich the formal deductive reasoning approach.

Notes

1. We advance the hypothesis that if they had been able to conceive the rhombus as a *pseudo object*, they would have been able to solve the problem geometrically (instead they resort to an algebraic explanation that they cannot coordinate with what they see on the screen).

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PROSPECTIVE PRIMARY SCHOOL TEACHERS' KNOWLEDGE OF THEIR STUDENTS: THE CASE OF MATHEMATICS

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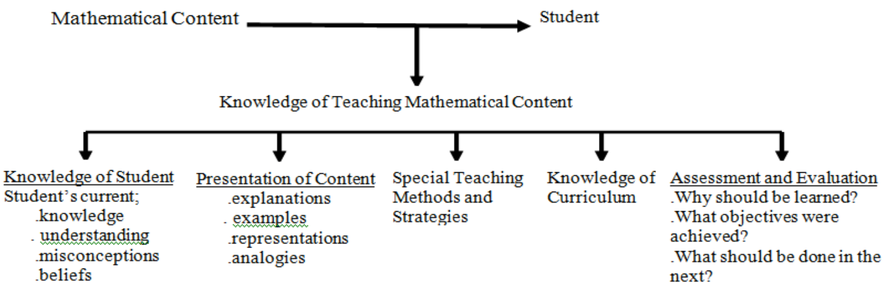
The successful completion of the process of students' achieving the knowledge of mathematics is linked directly to the quality of a teacher's understanding of teaching mathematics. In this process, knowing the student—a component of the teacher's knowledge of teaching mathematics—and its sub-component of utilizing the student's prior knowledge prove to be critical to the overall knowledge base teachers should have. In this study, classroom observations of pre-service primary school teachers were performed to evaluate the teacher candidates' use of their prior knowledge; their lesson plans were examined, and individual interviews were conducted. It was concluded that, although teacher candidates by and large believe that the student's prior knowledge should be taken into consideration during the implementation of a course, they lack the necessary information and experience to know how they might accomplish this in their own classes.

1. INTRODUCTION

Studies on teacher knowledge that Shulman and his associates conducted decades ago still prove to be current (Shulman, 1986; Grossman, 1988; Magnusson et al. 1999; Ann et al., 2004; Ball, 2008). Shulman (1986) divided teacher knowledge into three categories: *content knowledge, pedagogical content knowledge and curricular knowledge.* Shulman defines the knowledge that the teachers are expected to have to make sure the students attain the content information as *pedagogical content knowledge*. The successful completion of the process of students' achieving the content knowledge (knowledge of mathematics) is linked directly to the quality of a teacher's knowledge of teaching the content (mathematics).



As can also be observed in the descriptive chart above, one of the most essential elements within the category of pedagogical content knowledge is familiarity with the knowledge base of one's students. This knowledge entails making subjects that are routinely taught in mathematics easy for a student to comprehend as the teacher gets to know the student (Baki & Baki, 2010). Students carry their previously-acquired ideas, perceptions and past experiences to the classroom. If a teacher is aware of these notions and takes them into consideration when determining the approaches, strategies and practices to be implemented in the classroom, a better organization of learning and teaching environment may be realized. This all hinges on the teacher's knowing his/her students well. Grossman (1998) underlines this fact in his study and argues that the student's prior knowledge of the subject matter has a direct effect on the teacher's actions and execution in the classroom. Thus, *the component of knowing the student*, which is part of the teacher's pedagogical content knowledge, and its sub-component of *utilizing the student's prior knowledge* take an important place within the framework of content with which the teacher should be proficient.

Mopolelo (1999) observed a group of sophomore primary school teacher candidates during their instructional activities in the classroom environment and tried to find out about their common features in terms of their approaches, behaviors and performance in teaching mathematics. The study led to numerous important findings. For instance, it was discovered that most teacher candidates were unqualified in understanding the students' conceptual errors and implementing the type of tasks and activities that are necessary to correct such errors. This demonstrates the fact that knowledge of mathematics is not the only guarantee of effective teaching. Magnusson (1991), in his study with teachers, argued that teachers' content knowledge would not, by itself, be sufficient in estimating the students' prior knowledge of the subject matter and in arranging instructional activities with the students' prior knowledge in mind in order for the desired learning to emerge. Ann (2004) compared the pedagogical content knowledge of primary school teachers in mathematics courses in China and the United States. With this purpose, he examined from many angles the ways in which primary school teachers employed their knowledge of teaching mathematics in order to understand and improve the students' mathematical thinking. At the end of the study, it was concluded that the teachers who were successful in recognizing the students' mathematical analytical thinking, identifying conceptual misunderstandings and facilitating the students' learning from both groups, regardless of their diverse cultures, were those who effectively made use of pedagogical content knowledge along with content knowledge. Hence, as was the case with other studies, this study emphasized that teachers' content knowledge and their pedagogical content knowledge could not be detached from each other in the implementation of effective classroom activities and in designing tasks that would aid students' learning.

Işıksal (2006) investigated, as one of the sub-problems of his study conducted with pre-service mathematics teachers, how informed the teacher candidates were regarding the beliefs and conceptual misconceptions of 6^{th} and 7^{th} grade students about fractions. Işıksal's study revealed that, even though the teacher candidates were proficient in carrying out the procedures related to fractions, they were fairly inexpert in the areas of describing the concepts to the students in depth and explaining the causes of potential conceptual misconceptions of students. All these studies demonstrate to us that teachers and teacher candidates have deficiencies pertaining, in a general sense, to knowing the students, and more specifically, with regard to their perceptions of the students' prior knowledge, beliefs and difficulties. This review of literature also allows us to see another vital point, which is the need to observe how these limitations of teacher candidates are evidenced in actual classroom environments.

In schools of education, pre-service teachers are habitually encouraged, at least in theory, to use student-centered approaches. However, before they initiate their practicum in schools, they are not provided with sufficient opportunities to acquire the training and experience regarding how they should get to know their students and how they can organize their classes around their students' prior knowledge (Baki & Baki, 2010). Because the teacher candidates of today will become the teachers of tomorrow, pedagogical content knowledge of teacher candidates should be explored through in-depth studies; by doing so, it may be possible to implement programs in schools of education that would support teacher candidates in this direction. In this specific study, the answer to the question of "How do pre-service primary school teachers utilize the knowledge of identifying students in mathematics teaching endeavors in their field sites?" will be researched.

2. METHOD OF THE STUDY

This study was conducted with 4 teacher candidate participants taking the "Teaching Practicum" course in the Primary Teaching Program at the Department of Primary Education in the Fatih School of Education during the 2009-2010 school year. The study is a special case analysis within the framework of qualitative research. Observations, field notes, lesson plans and interviews were employed as the data collection techniques in the research. Each of the teacher candidates were observed for two class hours during the entire semester in 5^{th} grade mathematics courses they taught on varied days, and their lesson plans were scrutinized. A voice recording device was used during the observations of the teacher candidates' mathematics courses, and at the same time, field notes were taken by the researcher. Interviews took place right after the activities of the teacher candidates in the school environment. Qualitative data that were obtained from different sources were analyzed using the triangulation method considering the sub-themes of the study.

3. FINDINGS

In this study carried out with primary school teacher candidates, the aim was to evaluate the standing of the teacher candidates on the topic of "being familiar with the students' prior knowledge and linking the students' past and new information" as *knowing the student* component of the pedagogical content knowledge. Data that were collected confirm that teacher candidates are aware of the need to take into consideration the students' prior knowledge in the process of teaching and learning that is aimed at teaching a new skill. Yet, it is clear that teacher candidates experience difficulties in deciding to what extent the presence of students' prior knowledge of a topic should be questioned. Some teacher candidates consider the students' prior knowledge to be similar to that of the previous class. It is believed that this repetition should be adequate to motivate the students of the new class. For instance, indications of this thought is visible in the course Prospective Teacher-1 (PT-1) planned and conducted to teach the skill "finding the value of the whole based on a given fraction." As can be observed in the dialogs below, PT-1 launched the lesson without questioning whether the topics that would set the groundwork of the subject matter he would teach are known by the students:

PT-1: You have studied the notions of improper/compound fractions and equivalent fractions in past lessons. Today, we will find the value of the whole based on a given fraction. Kids, how many calories would be in a whole apple of which one fifth is 25 calories?

PT-1 got an apple in his hand to be able to explain the problem and directed the following questions to the class:

PT-1: Is this apple one whole piece?

Students: Yes.

PT-1: I divided the apple into five. What does one piece represent?

Students: One fifth.

PT-1: If the one fifth is 25, I multiple 25 by 5 to find the five fifth.

To explain this, PT-1 drew the following model onto the board:

25				

1/5

PT-1: Where is 5?

Students: In the denominator.

PT-1: 5x25 = 125. As you can see, I multiplied 25 by the denominator.

It cannot be affirmed that the class fully understood why the given is multiplied by the denominator. If PT-1 had initiated the lesson with the problems answered in the previous years, this would have given the teacher candidate an idea about what the students already knew. Because the students could not relate the newly learnt information to their prior knowledge, there were uncertainties in the classroom. It was clear that the teacher candidate had trouble pulling the class together and organizing the lesson.

Another pre-service teacher, PT-4, prepared a course outline toward teaching the objective of "classifying angles based on their measures." He tried to motivate the students by reminding to them the content of the class taught the previous day.

Because he would classify angles according to their measures, he introduced the protractor.

PT-4: When measuring angles, we use a protractor. Here is our protractor. Take your protractors out and look at it. What do you see on it?

Student: There are numbers up to 180.

PT-4: You see a line right across the 90 degree mark. That is the center of the protractor.

PT-4 drew an angle onto the board and demonstrated to the class, using the protractor in his hand, how an angle should be measured. It was observed that the students did not understand why the line the teacher referred to as the center of the protractor was moved to the vertex of the angle. As is evident here, PT-4, like the other teacher candidates, did not pay enough attention in arranging the course to what prior knowledge the students had. The teacher candidate treated the students as though they knew about angles and elements of angles, and during the measurement process, he gave explanations as if the students already knew the terminology regarding the vertex and degree of an angle. The teacher candidate failed to comprehend the need to check in advance of the class whether the students were familiar with the notions of angles and constituents of angles. This situation brought about confusion during the course of instruction. When the teacher candidate realized that the students did not fully understand what they were being told, he went back and started to tell them about the concept of angles. In the interview conducted after the lesson, PT-4 expressed that he had though the students would have known these concepts.

The majority of teacher candidates report that they have deficiencies in figuring out what students' prior knowledge about a topic should be and that they cannot guess before teaching the lesson what the students knew about the subject. The difficulty that teacher candidates have is how to discern whether the actually students have the knowledge that they are assumed to have prior to the teaching of a topic and how this previous knowledge can be linked to the new knowledge.

When data concerning the knowledge of the teacher candidates about connecting the students' past and present information is scrutinized, it is obvious that teacher candidates have difficulty in referring to previous material in the course of teaching a subject and in relating the new topic to the students' prior knowledge. Correspondingly, it was discovered that some of the teacher candidates acted in an extremely unprofessional manner in such circumstances.

PT-2, who prepared and implemented a lesson plan intended for the acquisition of the objective of *"finding the missing factor in a multiplication operation that has a product with a maximum of four digits"* in the 5th grade mathematics teaching curricula, verified the students' knowledge of multiplication at the start of the lesson. Following the execution of a few simple examples, he asked the students to find the missing factor in the multiplication operation he had written onto the board.

$$\begin{array}{r} 215 \\ \times \qquad ? \\ \hline 645 \\ +\ 430 \\ \hline 4945 \end{array}$$

PT-2 stated that he would demonstrate a different strategy and added that they would find the number at the ones place by diving 645 by 215 and the number at the tens place by dividing 430 by 215. It was observed that the students could not understand why they divided 645 by 215. The teacher candidate appeared to have presumed that the students would be able to find the omitted piece in the multiplication operation with the help of such a strategy as he believed that the students knew the connection between the division and multiplication operations. The pre-service teacher stated that he had chosen the example in the teacher guide book and he had not foreseen that the students would have experienced such a difficulty in understanding this strategy. Because the students did not see a connection between past lessons and this new knowledge, they felt that the problem was too complicated, so they began to feel lost during the lesson.

PT-3 was more successful than the other teacher candidates in taking into consideration the students' prior knowledge during the initial phase of the lesson and in conducting the lesson by relating it to the students' prior knowledge. PT-3 prepared a lesson plan aimed at obtaining the competence of "being able to divide numbers with a maximum of four digits by three-digit numbers." PT-3 reminded the students about the operation of division through concrete modeling during the introduction stage to the lesson. In the beginning of the lesson, he had the students perform divisions with and without remainders with some hazelnuts he had brought to the classroom. He demonstrated the difference between divisions with and without remainders, and then reiterated the terminology of division. However, as the lesson progressed, it was evident that the teacher candidate had difficulty establishing a link between the previous information the students should have and the new topic. As a getaway, he started to explain the operation of division on the board like it was presented step-by-step in the guide book; he assumed that the students knew division in this fashion and thus, went on with his explanations. When PT-3 attempted to explain, together with its associated structure about the notion of digits, the rules of division that the students already knew, things once again started to get mixed up. What PT-3 should have done in this situation was to establish a connection with the students' prior knowledge through breaking the problem down into smaller steps and finding each digit in the answer one at a time, starting with the ones place. These limitations, as was the case in other teacher candidates' classrooms, caused problems for the organization of PT-3's lesson as well.

4. CONCLUSIONS

Although the teacher candidates were generally aware of the obligation to take into consideration the students' prior knowledge in the process of teaching and learning that is prepared, instructed and intended for teaching of a skill, they did not feel the need to question the existence of the students' prior knowledge in the course of the lesson as they considered the students to know the relevant information. This study indicates that the teacher candidates ignored the students' prior knowledge due to the traditional belief that knowledge can transferred to the students directly by the teacher. The teacher candidates, although they shifted to a new topic at the beginning of the lesson, considered starting the lesson by making a reference to the previous lesson to be satisfactory and neglected to prepare groundwork for the new subject matter. Because there is a progression in learning the topics in mathematics, this deficiency led to disorder in the teacher candidates' lessons and therefore caused the lessons to be unproductive. Teacher candidates declare that they experience difficulties during the lessons, but they are unable to identify what leads to this. The teacher candidates justify the difficulties they experience in most lessons with statements such as *these students are not our permanent students; we cannot know what they do or do not know, and this is a challenge for us*. Yet, the problem here is not that the students are not their regular students, but instead by the fact that the teacher candidates are not familiar enough with what the students should already know during the introductory stage to the topic, and that they are poor in determining the related topics before the teaching of a mathematical notion, topic or operation and in terms of the skill of connecting the topics.

The fact that pre-service teachers do not begin the lesson with what the students previously know and they assume that the students already have prior knowledge about the topic leads to problems for teacher candidates as the lesson progresses. Although pre-service teachers might identify the issue and attempt to clarify things to piece the lesson together, this turns out to be not effective since the students are already baffled. As revealed here, the school environment provides teacher candidates with opportunities for further learning. However, because the teacher candidates themselves are still in the process of learning in this practicum period, their practices before the students should be monitored and critiqued more often by their university supervisors and teachers. Although data in this study were collected in a 1-year long teaching practicum, the pre-service teachers' difficulties in terms of getting to know the student persisted and were the same at the end as they were at the beginning of the school year. As a suggestion, lesson plans that are put into practice by the teacher candidates can be carefully looked over by their university and school supervisors, and measures to manage the issues regarding knowing the students can be taken. In this way, the teacher candidates will be aware of their weaknesses and will become better at dealing with and eliminating these difficulties.

Not being able to detect the problems experienced by the students and failing to see that the topic is not clear to the students appear to be critical shortfalls in terms of the pre-service teachers' knowledge of students. It is essential that activities to purge such deficiencies are included in the content of courses in schools of education more often. Because the teacher candidates' micro-teaching applications that are carried out in college classes, such as teaching of mathematics, do not have the student component, they do not feel the need to establish a connection between the topics in accordance with the students' prior knowledge, and therefore, the dimension of knowing the student is missing in these courses. It is essential that, before teacher candidates start their fieldwork, authentic settings that are similar to real-life classroom environments, bringing them face-to-face with students are arranged, and that the lesson plans prepared for micro-teachings with the students' prior knowledge in mind are meticulously evaluated.

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STUDENTS' UNDERSTANDING OF THE NEGATION OF STATEMENTS WITH UNIVERSAL QUANTIFIER

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This paper explores difficulties related to negating a verbal statement with an occurrence of the universal quantifier. The study, carried out with two hundreds and two undergraduate science students, shows that the interference of ordinary speech is a major cause of failure. Some aspects of colloquial registers affecting students' performance were identified. In particular, data shows that lack of awareness of the functions of mathematical language negatively influences the use of logical operators and quantifiers.

INTRODUCTION

Predicate calculus is one of the basic aspects in understanding mathematics and many studies developed in this area, ranging from childhood to undergraduate level and from very different perspectives (see e.g. Deloustal-Jorrand 2002, Hoyles and Küchemann 2003, O'Brien et al. 1971, Wason 1960, Zepp et al. 1987). Mathematical reasoning is affected, on the one hand by logical operations such as implication, negation, conjunction, etc. and on the other hand by the use of quantifiers. For example Dubinsky (1991) wrote "A major difficulty in dealing with the usual formal definition of the limit is the need to cope with the quantifiers". Hoyles and Küchemann (2003) and references therein stressed the importance of the role of logical implication in proving processes. Lin et al. (2003) identified the ability of negating a statement as the first task on processing proof by contradiction.

In this framework this study deals with the logical operation of "negation" and addresses this topic by means of a functional linguistic approach. In particular the study is focused on the ability at negating a statement with an occurrence of the universal quantifier. We have only considered statements given as verbal texts. The aim of the study is to explore the interference of everyday language with the mathematical interpretation of such statements. The following research questions are addressed:

- Which factors of colloquial speech affect students' behaviour?
- Are students aware of the role of mathematical language and in particular of logical rules when facing negation of statements?

Since the study was carried out in Italian language the results refer to the interference of colloquial Italian.

THEORETICAL BACKGROUND

Many studies in educational mathematics pointed out the interference of everyday language with mathematical language as an element of difficulty in mathematics (see e.g. Tall 1977, Cornu 1981, Mason & Pimm 1984, Ferrari 2004, Kim et al. 2005, Bardelle 2010). Such interference is addressed in this paper by means of a functional linguistic approach (Halliday 1985) applied to mathematical language (see e.g. Pimm 1987, Morgan 1998 and Ferrari 2004). The pragmatic view highlights the role of context (including semantic domain, participants and their purposes) in the building of meaning. In this perspective some students' difficulties may be ascribed to the overlapping of *colloquial registers* and *literate registers*. A register (Halliday 1985, Leckie-Tarry 1995) is a linguistic variety based on use, i.e. a conventional pattern or configuration of language that corresponds to a variety of situations or contexts. According to Ferrari (2004) registers customarily adopted in advanced mathematics are extreme form of literate registers and, contrary to everyday-life registers, usually violate cooperation principles (as defined by Grice 1975 and many others). The ambiguities inherent in language may arise from both special vocabulary and the organization of texts. For example in mathematical registers "some" means "at least one" but in everyday-life registers is usually interpreted as "more than one". This holds for the corresponding Italian words [1] too. The "negation" of a sentence in Italian [2] may refer to a particular part of it such as the subject or the verb or an adjective while in mathematics "negation" is a logic operator with its rules. For example possible negations of the sentence "all the children have played football today" are "all the children have played football not today but some other day" (negation refers here to the adverb "today"), "all the children have not played football but they have played volley today" (negation refers here to the object "football"), "only some children have played football today" (negation refers to the subject) and so on.

THE EXPERIMENT

Subjects

The experiment was carried out with 202 Italian science (biology, chemistry, computer science, environmental science, mathematics) freshman students at University of Eastern Piedmont in Italy.

The results come from some questions submitted to students in a written admission test and subsequent interviews. The test was held after two weeks of a precalculus course and predicate calculus was one of the subjects of the course. Both the course and the test were not compulsory but highly recommended. Moreover students could achieve one or two credits according to the results of the test. No debts were given if they failed the test but if so they were highly recommended to attend tutoring sessions.

[1] In Italian: "alcuni" (some), "almeno uno" (at least one) and "”più di uno" (more than one).

[2] The same argument holds for English language.

Tasks

Four questions were developed for the study. Students were asked to recognize equivalent sentences to the negation of statements with a universal quantifier ($\forall x \in X \ \ P(x)$ - universal affirmative sentences "All S is T"). The questions were grouped into two categories. a-questions involve negative sentences i.e. the negation is included in the sentences. b-questions concern affirmative questions to be denied i.e. the negation is not included in the sentences. The questions have a multiple choice format with the possibility of multiple responses. Each question has three options. A): sentence with negation applied to the subject only ($\exists x \in X \ P(x)$ - particular affirmative "Some S is T"). B): the correct one ($\exists x \in X \ \neg P(x)$ - particular negative sentence "Some S is not T"). C): sentence with negation applied to the verb ($\forall x \in X \ \neg P(x)$ - universal negative sentence "All S is not T") given in the form "No S is T"; this last form is more common in Italian language.

The following tables present an English translation of the questions.

Which of the following sentences (multiple choices are allowed) are equivalent to "Not all the animals of the farm are herbivore"
A) Some animals of the farm are herbivore
B) Some animals of the farm are not herbivore
C) No animal of the farm is herbivore

Table 1: Question 1a.

Which of the following sentences (multiple choices are allowed) are equivalent to negating "All the animals of the farm are herbivore"
A) Some animals of the farm are herbivore
B) Some animals of the farm are not herbivore
C) No animal of the farm is herbivore

Table 2: Question 1b.

Which of the following sentences (multiple choices are allowed) are equivalent to "Not all the monomials are polynomials"
A) Some monomials are polynomials
B) Some monomials are not polynomials
C) No monomial is a polynomial

Table 3: Question 2a.

Which of the following sentences (multiple choices are allowed) are equivalent to negating "All the monomials are polynomials"
A) Some monomials are polynomials
B) Some monomials are not polynomials
C) No monomial is a polynomial

Table 4: Question 2b.

Interviews

Ten students were individually interviewed in order to explore their understanding. The interviews were not compulsory and explanations about the experiment were given to students before starting the interview. The interviews were semi-structured. All students were asked: 1. an a-question if they answered a b-question in the entrance test and viceversa; 2. the meaning of "equivalent sentences"; 3. the meaning of "negating a sentence"; 4. whether they perceive a-questions and b-questions as different queries. Such questions were accompanied by flexible ones aimed at explaining students' reasoning. The students were chosen according to the factor analyses of the written responses in order to investigate all the patterns of incorrect answers with more than 10% of frequency.

RESULTS

Table 5 shows students' written responses to the four questions. We recall that the sample was split into four groups. The groups had a similar number of students and each group had to face one of the four questions respectively. In the following tables A, B, C denote the options A), B), C) of the questions respectively, AB denotes that students had chosen both option A) and B) and so on.

Item	Q1a	Q1b	Q2a	Q2b
A	4%	0%	15%	9%
B	24%	32%	28%	14%
C	8%	32%	4%	36%
AB	60%	12%	53%	23%
AC	4%	8%	0%	0%
BC	0%	16%	0%	16%
ABC	0%	2%	0%	0%
Total	100%	100%	100%	100%

Table 5: Responses to written test.

As a first results it is clear that the number of correct responses (less than 32%) is very low as Table 5 shows. Secondly notable differences on percentages are found in relation to a-questions and b-questions. This leads to the conclusion that a-questions and b-questions are perceived as different queries.

Results on a-questions

In a-question the high frequency of AB answers (60% in Q1a and 53% in Q2a) has to be analysed. All the students interviewed who answered AB explained their choice arguments like the one reported below:

I: Why did you choose both A and B [in Q1a]?

S: Not all means some do and some don't.

This is a clear example of *conversational implicature* (Grice 1975) typical of colloquial registers and arising from the adoption of cooperation principles. The expression "Not all x P(x)" such as in Q1a and Q2a if interpreted in a colloquial register suggests the speaker tacitly implies that "Some x P(x)". This fact clearly conflicts with predicate calculus. These data confirm the results on conversational implicature already observed by Ferrari (2004) in a quite similar context. This behaviour is probably aggravated by the fact that the meaning of the queries of the tasks are not well understood or are underestimated. This is confirmed in the interviews. For example

I: What does it mean that equivalent sentences?

S: It means that they say the same thing….they express the same concept.

shows the typical behaviour to the query of explanations about equivalent sentences. Such answers do not contain any reference to the mathematical definition or to any other mathematical aspect related to equivalence.

From the investigation of students' behaviour on a-questions another factor occasionally emerged. It seems that students' chose A because they applied "not" to the subject of the sentences (all the animals, all the monomials). In ordinary Italian language, negation can be applied just to a part of the sentence and not to the its whole meaning. Also this fact conflicts with mathematical language. The following interview describes this phenomenon:

I: Why did you answered A) to the question 1a that is …..?

S: Because "not all" means "some".

Moreover at the end of the interview this student, rethinking to question 1a, said

S: A) is not fine…as a particular case "all the animals are carnivore" can be ok with the statement "not all the animal of the farm are herbivore". Before I have reasoned on the meaning of "not all" only.

Probably the failure of this student stemmed from an imperfect knowledge of the logical rules he used to give answers, combined with behaviours arising from interference with everyday-life registers.

Results on b-questions

The number of students choosing option C (32% in Q1b and 36% in Q2b) is impressive. The same phenomenon was already observed by Lin et al. (2003) in English and Chinese. Also this behaviour could be ascribed to the interference of colloquial registers. The negation of a sentence is not seen as its complementary but as a sentence with the opposite meaning. In particular students seemed to focus on the negation of the quantifier “all” that in everyday-life register can be seen as its contrary e.g. “none” and not as its complementary “not all”. A student explicitly confirmed it in his interview:

I: Why did you answered C) to the question 1b that is …..?

S: Because the negation of “all” is “none”.

In order to investigate what meaning students gave to negating a statement explicit questions were asked them in interviews. Some representative answers are

S1: Negation gives an absolutely negative sense.

S2: “Negation” is stronger than “not”...negation of something means that all the thing cannot be….it is its contrary… “not” may refer to a part only of a sentence.

From these interviews it came out that students talked about negation and the quantifiers according to their colloquial usage.

For what concerns the answer BC, chosen by 16% of the samples both in Q1a and Q1b problems could be ascribed to the colloquial interpretation of the concept of “equivalence” as a student explained in her interview:

S: Negating “All the animals of…..” means ruling out at least in part the truth of “All the animals…”…. If A, B and C are truth then it is not truth that “All the animals ” hold.

Notice that this students chose also option A besides B and C. Some explanations were asked her:

I: Why did you chose A, B and C?

S: C is clear…. B because it says that some animals are carnivore and hence it denies “all the animals are herbivore” and A because “some animals” means “just some animals” and not all the animals and hence it denies “all the animals …..”.

Also in this case we have a use of the quantifier “some” according to colloquial Italian, meaning “just some”, and not to its mathematical sense of “at least one”.

Results on pragmatic awareness

All the interviewed students were asked whether they used logical rules taught during the course to solve the tasks. Some answers are:

S1: I did not think to the rules…I thought to the meaning of the sentences only.

S2: I didn't use rules… these tasks can be solved without logical rules. I do not see what's the point in using them.

S3: Instinctively I don't use the logic rules … I was not accustomed to doing it.

These answers confirm that these students were not aware of the importance of the role that mathematical language plays in solving these tasks.

CONCLUSIONS

The result of our study highlights difficulties in recognizing equivalent statements involving quantifiers and negation given in verbal forms. From the findings of this experiment it might be reasonable to draw the conclusion that the overlapping of everyday-life registers with mathematical ones, combined with a lack of awareness of the functions of mathematical language, is a major cause of failure in students' performance. In particular the meanings assigned by students to "negation" of a sentence and to "equivalence" of sentences often do not correspond to mathematical language but are affected by previous knowledge arising from everyday-life contexts. It turns out then that also the formulation of the tasks can influence students' answers as shows the comparison of the results to a-questions and b-questions. The meaning assigned by students to quantifiers and their negation is another important issue. Also in this case colloquial common sense given to quantifiers and to linguistic expressions with quantifiers are a hindrance in their conceptualization. Such behaviour is sometimes to due cooperation principles such as conversational implicature, typical of ordinary speech. Finally interviews reveal that one of the reasons of failure may be ascribed to the fact that students do not think or do not feel the need to use logical rules for negating sentences with quantifiers.

According to the result of this study one has to take into account that language-related troubles may play a major role in the learning of mathematics and that pragmatic awareness should be regarded as one of the basic aspects in the teaching of mathematics.

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PRE-SERVICE TEACHERS' USE OF VISUAL REPRESENTATIONS OF MULTIPLICATION

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Abstract

We examine pre-service teachers' considerations in using visual representations of multiplication. Based on representations highlighted in the literature, a questionnaire for teachers was designed in order to explore the visual representations that they would use for explanations in different situations. This questionnaire were distributed to 445 pre-service primary/elementary teachers in England (n = 181) and Serbia (n = 264). Rasch analysis was used to analyse and modify the coding of teachers' responses. The analysis of the pre-service teachers' responses highlighted a number of issues, such as the influence by teachers' subject knowledge, limitations in teachers' use of representations, and a difference between teachers in England and Serbia in terms of linking representations of multiplication and division.

INTRODUCTION

Knowing how to use representations of mathematical concepts in the classroom is an important part of a teachers' specialised knowledge of the subject (Shulman, 1986; Ball *et al*., 2008). Leinhardt *et al*. (1991) emphasized the importance of representations in explanations of mathematics, and also highlighted the need to draw on more than one representation of a concept:

> "It is not possible for any one representation to capture all of the salient components of the target material … However, certain representations will take an instructor farther in his or her attempts to explain the to-be-learned material and still remain consistent and useful." (p.108)

Representations are also important for students. They provide a link between the concrete experiences of students and the more abstract world of mathematics (e.g. Bruner & Kenney, 1965; Post & Cramer, 1989; Sfard, 1991; Duval 1999). A representation can constrain interpretation by highlighting a particular aspect of a concept, thereby supporting the understanding of the mathematical idea (Kaput, 1991; Ainsworth, 1999). Furthermore, the linking of multiple representations constitutes a deeper understanding of the mathematical concept (Ainsworth *et al*., 1997; Kaput, 1991):

> "To know a mathematical idea "abstractly" means to have a sufficient rich set of mental structures so as to be able to deal with the idea on the basis of relatively few salient features either in a notation or in a situation to be modeled." (Kaput, 1991, p.62)

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Johnson-Laird (2005) provides a psychological model for thinking, involving the internal manipulation of mental models or representations. Therefore, we can argue that in order to develop mathematical understanding and thinking, teachers need to provide students with a range of representations for a given mathematical concept.

Despite the importance of representations however, Pape & Tchoshanov (2001) have highlighted the controversy regarding the extent to which learners can access mathematical concepts in representations. Researchers such as von Glasersfeld (1987), Cobb *et al*. (1992) and Duval (1999) have emphasized that we cannot make assumptions about the ways that students recognise or interpret representations. Our existing knowledge affects the way that we interpret representations (Lowe, 1993; Cook, 2006) and we need to support students in learning to interpret representations (Flevares & Perry, 2001). One way of doing this is to draw on external representations which are more likely to be part of students' experiences. Paivio (1969) stated that the more concrete the stimulus or external representation, the more likely it is to be associated in learning and memory.

Therefore, in light of the considerations that teachers need to make in using representations with students in the mathematics classroom, the aim of this present study was to examine pre-service teachers' use of representations, specifically visual representations of multiplication. We wished to find out what considerations pre-service teachers made when choosing representations in order to explain particular aspects of multiplication.

METHODOLOGY

The present study specifically examined pre-service teachers' considerations involved in using visual representations of multiplication. This focus developed from our previous research (Barmby *et al*., 2009) that identified the array as being a possibly useful representation that could be used in the mathematics classroom. However, this research also identified difficulties experienced by students when using the array in terms of recognising the array as a representation of multiplication. Other research has highlighted a variety of possible visual representations for multiplicative situations. The situations identified by Greer (1992) included equal groups and Cartesian product situations, which can be represented by diagrams of equal groups of objects and arrays respectively. Greer stated as well that the number line can be used to represent multiplication and quotitive division. In previous work (Barmby *et al*. 2009), we have suggested that an array with spaces at given intervals might support students' reasoning of the properties of multiplication, specifically the distributive properties of the operation. In terms of the simple array, Skemp (1986) also highlights the usefulness of this representation in showing the commutative and distributive laws for multiplication. Anghileri (2000) also highlights the repeated groups or sets representation, with children viewing multiplication as repeated addition in their early understanding of multiplication, and then again the array which is useful for illustrating the commutative law. Outhred & Mitchelmore (2004) stated that the rectangular array model (we adopt the term area representation) is an important model for multiplication, and Battista *et al*. (1998) stated that the area representation is essential for the development of the area concept in students and an important model for multiplicative thinking.

Based on these representations highlighted in the research, a questionnaire for pre-service teachers was designed in order to explore the representations that they would use for explanations in different situations. The representations shown in the questionnaire are given in Figure 1.

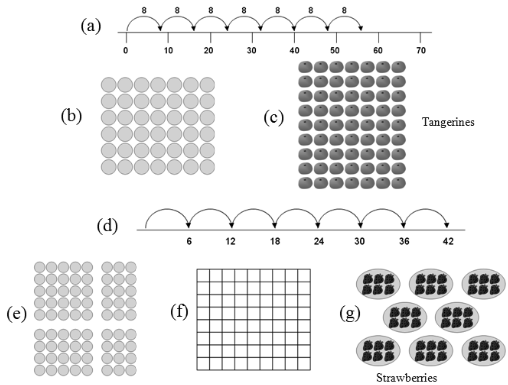

Figure 1: Representations of multiplication

In the questionnaire, the teachers were asked to choose one representation for each of the following purposes: (Q1) To use with pupils to show the commutative law; (Q2) To use with pupils to show the distributive law; (Q3) To use with pupils to show multiplication as repeated addition; (Q4) To use with pupils to show division as the inverse of multiplication; (Q5) To use with younger pupils, for example in Year 2; (Q6) To use with older pupils, for example in Year 6. For each question, the teacher was asked to tick one box corresponding to each of the representations, and a space was provided in each question for teachers to explain their choice. This questionnaire were distributed to 445 pre-service primary/elementary teachers in England (n = 181) and Serbia (n = 264). The English teachers included teachers in the first and final year of a three-year undergraduate course, and also a one-year postgraduate teacher training course. The Serbian teachers included those in the first and final year of a four-year undergraduate course. The teachers completed the questionnaire at the start of one their lectures.

The resulting data from these questionnaires (in terms of the choices (a) to (g) of representations) were then inputted into a spreadsheet for the purposes of analysis. In order to gain insight into the degree of consideration that teachers made when choosing representations, the following framework was used to score the teachers' responses for each question: (Q1) 2D representations (i.e. arrays and area representation) show the commutative law more easily; (Q2) 2D representations show the distributive law more clearly, particularly the array with spaces; (Q3) Multiplication as repeated addition would be shown more by the grouping representations and the number lines, then the 2D representations, and finally the array with spaces; (Q4) Items showing clear groupings would show division more easily, so the grouping representation and the number lines, followed by the 2D representations, then the array with spaces; (Q5) Representations relevant to younger pupils' experiences would be better, therefore the plates of strawberries, followed by the tangerines array, followed by number lines, then the array, then the area representation, then the array with spaces; (Q6) More abstract 2D representations would be more useful, so array with spaces, array, area representation, followed by tangerine array, then number lines, then strawberries. The above framework therefore resulted in the scoring of teachers' responses as shown in Table 1. When respondents had not provided a choice, or provided more than one choice, their score was simply coded as 'missing'.

Table 1: Initial scoring of items

Item	Choice of representation						
	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Q1	1	2	2	1	2	2	1
Q2	1	2	2	1	3	2	1
Q3	3	2	2	3	1	2	3
Q4	3	2	2	3	1	2	3
Q5	4	3	5	4	1	2	6
Q6	2	4	3	2	4	4	1

Following the initial scoring of teachers' responses, Rasch analysis was used to confirm the scoring categories that we had used. Rasch analysis expresses the probability of a person being successful on a given item in terms of a mathematical function of the difficulty of the item and the ability of the person (Bond & Fox, 2007). Rasch analysis is commonly used to analyse dichotomous items, but can also be used to analyse items with a greater number of possible responses, and also allowing for differing numbers of responses on different items. This so called *partial credit analysis* estimates not only the person ability and the overall item difficulty, but also provides estimates for the difficulty thresholds between scoring categories. These thresholds should increase in an ordered manner, in line with the ordering of the scoring categories (Bond & Fox, 2007). Otherwise adjacent categories should be combined and reanalysed. We applied this analysis to the scoring categories given in Table 1, which resulted in the revised scoring of teachers' responses as shown in Table 3.

Table 3: Revised scoring of items

Item	Choice of representation (a)	(b)	(c)	(d)	(e)	(f)	(g)
Q1	1	2	2	1	2	2	1
Q2	1	2	2	1	2	2	1
Q3	2	1	1	2	1	1	2
Q4	2	1	1	2	1	1	2
Q5	2	2	3	2	1	1	3
Q6	2	3	3	2	3	3	1

This simplified scoring differentiated between 2D and non-2D representations for Q1 and Q2, representations that emphasised repeated addition for Q3 and Q4, abstract and less abstract representations for Q5, and again 2D and non-2D representations for Q6 (although differentiating between the number lines and the grouping representation as well).

RESULTS

The revised scoring categories provided the following results for the above items.

Table 4: Percentages of responses and the overall Rasch difficulty (in logits) for each item

Scoring	Q1	Q2	Q3	Q4	Q5	Q6
Missing	2%	7%	1%	4%	1%	7%
Full marks	75%	65%	93%	72%	84%	53%
Partial marks	-	-	-	-	11%	36%
Lowest mark	23%	28%	7%	24%	3%	3%
Overall Item difficulty (logits)	0.53	0.96	-1.31	0.64	-0.74	-0.08

The pre-service teachers found choosing the representation for repeated addition (Q3) to be the easiest item, followed by the choice of representations for younger pupils (Q5), and then for older pupils (Q6). The most difficult items for the teachers involved representing the distributive law, division and the commutative law, where three-quarters or less of the sample of teachers chose the expected representations. We can examine these difficult items further by highlighting the percentages of the different cohorts of pre-service teachers obtaining full marks on the items.

Table 5: Percentage of teachers achieving full marks on particular items

	English			Serbian	
Item	Year 1 UG	Year 3 UG	PG	Year 1 UG	Year 4 UG
Q1	53	77	65	71	91
Q2	54	81	48	57	77
Q4	83	85	81	54	73

With Q1 and Q2, as me might expect, we see that students who have been studying longer on the teacher training courses did better on these items. Turning to the explanations provided by teachers for their choices, for Q2, many of the English teachers who did not get full marks seem to confuse the 'distributive law' with distributing objects evenly into a number of groups. Likewise for Q1, some English teachers did not know what was meant by the commutative law. However, quite a number of these teachers chose the plates of strawberries representation, stating that one would still get the same answer if we had 6 plates with 8 strawberries. Likewise for the Serbian teachers, a significant proportion did not know what was mean by the commutative and distributive law, possibly pointing to limitations in their mathematics knowledge when entering the teacher training course. With Q4, we seemed to find more of a difference between the English and Serbian teachers. Looking at the choices of representations for this question, English teachers were much more likely to choose the number lines or strawberries representations, with the Serbian teachers choosing to a greater extent the two dimensional representations.

DISCUSSION

The analysis of the pre-service teachers' responses to the questions regarding their choice of visual representations in different contexts highlighted a number of issues. Firstly, the choice of representations could be influenced by teachers' lack of subject knowledge. This is shown simply by the case of the distributive law (Q2), and to some degree for the commutative law (Q1), where some teachers did not understand what was meant by these laws. The analysis also highlighted some limitations in teachers' use of representations, for example with the commutative law (Q1) where a simplistic argument was provided for using the groups of strawberries representation, not taking into account that when 6 groups of 8 would look quite different to 8 groups of 6, despite obtaining the same answer. The analysis also identified an interesting difference between the teachers in England and Serbia for the division question (Q4). A possible explanation for this is that students in Serbia are taught more specifically to represent multiplication using the array, and therefore in make the connection of division being the inverse of multiplication, Serbian students are therefore more likely to represent them in the same way, i.e. with two dimensional representations. If this is the case, then our scoring system given above, awarding greater marks to the number lines or the strawberries representation, would need re-evaluating. This highlights a possible further issue that certainly in England, we may not consider these connections to the same extent in terms of representations and connecting the operations of multiplication and division. Although we emphasise the array representation for multiplication, we do not as readily represent division in the same way. We therefore need to explore further how teachers can usefully do this, making the connections between these operations and representations, with pupils in the most effective way.

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PERSONAL CONSTRUCTS OF PLANNING MATHEMATICS LESSONS

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Teachers' personal constructs influence how a classroom is organised and what mathematics will be emphasised and valued. We asked mathematics student teachers in different semesters (N=321) to compare different mathematics lesson plans depending on where they were in their teacher education course. They found quite different characteristics to analyse the lesson plans. In this explorative study we found that personal constructs of planning a mathematics lesson are changing as pre-service teachers move through their course

INTRODUCTION

In different countries TIMSS and PISA showed some deficits in pupils' mathematics knowledge triggering research into teachers' education. For example, Oser and Oelkers (2001) developed a self-assessment for pre-service and in-service teachers to analyse their competencies. Another international study TEDS-M (Blömeke, Kaiser & Lehmann, 2010) focused on content knowledge, the pedagogical knowledge and the pedagogical content knowledge in their tests. Both surveys focus on important parts of the requirement to become a competent mathematics teacher and will affect a teacher's action in class, but each survey alone cannot measure competencies (Weinert, 2002). Thus there is the question:

In which way can competencies of teaching mathematics be measured in a teachers' education?

Which is a very general question, thus we need to focus. For example, the government of Germany and also NSW Institute of Teachers (in Australia) formulated standards for teachers' education. Both require student teachers to demonstrate their knowledge of students' varied approaches to learning, and beginning teachers to apply that knowledge to enhance student outcomes (NSWIT, 2006). It is becoming more important for competence on the part of student teachers in the skills of lesson planning (John, 2006). Thus we have chosen lesson planning as the central research object.

Beliefs of what mathematics teaching is about will determine how student teachers view the whole range of pedagogical issues in the classroom (Barkatsas & Malone, 2005; John, 2006, Nisbet & Warren, 2000), including lesson plans (especially at the beginning of their course). When deciding how a concept or process is to be taught, teachers relay subtle and often unintentional beliefs and attitudes about mathematics (Booker, Bond, Sparrow, & Swan, 2004).Planning to teach includes, at the very least, a knowledge of students and their needs, an overall aim or aims for learning and a set of instructional objectives, and a teacher´s understanding and perceptions of the nature of [mathematics] and learning (Liyanage & Bartlett, 2010). Decisions about materials, activities, and methods are likely to vary greatly depending on the experience of the teacher, whether that be formal teacher education or informal knowledge gained as a student. Writing a lesson plan requires the teacher to effect synergies in their knowledge of content, pedagogy and learning as well as the optimal conditions for learning. To analyse teacher students' competencies in planning a mathematics lesson, we chose the personal construct theory.

THEORETICAL BACKGROUND

Personal construct theory

The theory of personal constructs dates from Georg Kelly (1955). He formulated the following Fundamental Postulate:

> "A person's processes are psychologically channelized by the ways in which he anticipates events" (Kelly, 1995, p.46)

That means our daily action is based on our constructs and interpretation of the world. To characterise the personal construct theory there are eleven corollaries, which explain how constructs emerge, interact, change and influence individual action. To illustrate the construct theory the following example of a mathematics lesson is given:

Example:

The content of the last lessons was integral calculus. The pupils can integrate simple polynomial functions and should now develop by themselves the concept of calculate an integral by substitution. The teacher initiates the aim of the lesson and provides the process by being a moderator. During the lesson most students are overstrained by this task and don't find the solution.

To reconstruct the intention of the teacher in the example we can use the construct theory. The teacher acts as a moderator, because he/she knows that a teacher has to be in the background during an explorative lesson. He/she also realises that the success of self-explorative learning is very high. So the teacher has the construct that students achieve new insights, if mathematical content is developed by the students themselves.

The competence of planning a mathematics lesson is therefore linked with the personal constructs of each teacher. If the development of competencies during a teachers' education are to be analysed, it is necessary to gather individual constructs of teaching mathematics.

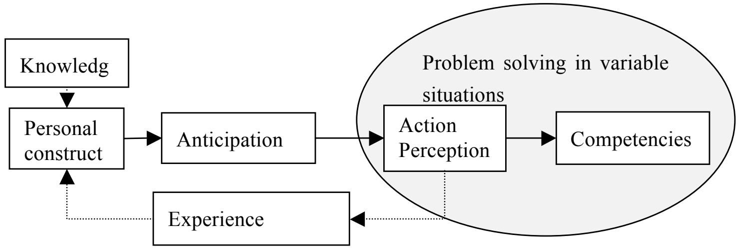


Figure 1: Personal construct theory and competencies

Repertory Grid Method

To gather individual constructs Kelly (1955) developed the repertory grid method. At the centre of this method is the comparison of different objects and definitions of the different constructs. Lengnink and Prediger (2003) report the use of mathematical tasks as objects in an adapted repertory grid survey to study student's constructs about tasks. One grid is represented in figure 2. The advantage of the repertory grid method is that participants give you the information in their own words rather than the words of the researcher. It then allows the researcher to amalgamate the information into categories which can then be compared between individuals and between groups within the participants.

	Task 1	Task 2	Task 3	Task 4
comprehensive task		x	x	
computational task	x			
closed	x	x		x
open			x	

Figure 2: Example of a grid

SAMPLE, DESIGN AND METHOD

Participants

The participants were taken from teacher education courses at the Technische Universität Darmstadt (TUD) and the University of Technology, Sydney (UTS).

The UTS teacher education program is a 18-month (three semesters) Bachelor of Teaching degree which has an accelerated option of 12-months (two semesters). The students already have a degree in mathematics and the Bachelor of Teaching allows them to teach in secondary schools in Australia. The majority of the students from UTS are mature-aged and undertaking a career change i.e. they have been working in a different career for at least 8 years.

The TUD teacher education program is nine semesters long with students completing both the mathematics and the teacher education program concurrently. The majority of the students begin their courses right after they finish school.

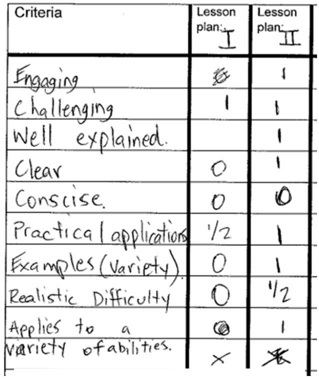
So far we have asked 41 students of the UTS and 280 students of the TUD. The detailed plan in Table 1 shows the structure of the survey. To enable a longitudinal study the survey is embedded in obligatory courses.

Year	UTS Australia		TUD Germany			
	1. semester	2. semester	1. semester	3. semester	5. semester	last semester
Jan. 09	N=10		N=53	N=44		N=10
Oct. 09		N=15				N=16
Jan. 10	N=16		N=67	N=60	N=12	N=8
Oct. 10						N=10

Table 1: Participants

Design

In Germany there are four points of measurement - in the first, in the third, in the fifth and in the last semester. In Australia we have two points of measurement - at the beginning of their first semester and at the end of their second semester. The students were asked to complete the survey taking about 45 minutes. It was conducted during class time but students were free to participate or not.



Criteria	Lesson plan: I	Lesson plan: II
Engaging	0	1
Challenging	1	1
Well explained		1
Clear	0	1
Concise	0	0
Practical application	1/2	1
Examples (variety)	0	1
Realistic Difficulty	0	1/2
Applies to a	0	1
variety of abilities.	x	x

Figure 3: Grid of an Australian student

We adapted the repertory grid method and chose lesson plans as objects which should be compared by the participants. Initially the participants were asked to focus their thoughts on the features of a "good" mathematics lesson, listing them in no particular order. We believed that this initial part of the survey would help them to get started with the analysis of the lesson plans that was important for those students who were in their first teacher education class. They then compared two lesson plans in terms of those characteristics they thought were important. They estimated the occurrence of the characteristics (Figure 2). For example the first set of lesson plans were written on introducing trigonometry and used distinct ways of introducing trigonometry - an historical approach, the right angle triangle and ratios approach, and a problem-solving approach. The other set consists of different ways of introducing newton's formula. Each lesson was considered a valid way to introduce the topic but emphasised different aspects of the topic. To ensure that students did not remember their answers of their first comparison we change the lesson plans in the following surveys.

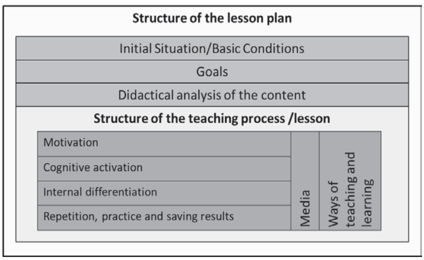


Figure 4: System of categories to analyse grids

To analyse the different grids in a nomothetic way (Scheer, 1996), we compared different descriptions of good mathematic lessons (Leuders, 2005; Helmke, 2009; Bruder,1991) and extracted a system of categories (figure 4) to analyse the constructs of the students. It is obvious that criteria for good math lessons like "good class-climate" or "motivation of the teacher" (Helmke, 2009) are left. These criteria cannot written in a lesson plan, thus we can't use them to analyse students characteristics.

On one hand there are categories which describe the structure and the content of the written lesson plan and on the other hand there are categories which describe the structure and the process of the lesson. An illustration of the categories is written in figure 5.

Category	Example
Structure of the lesson plan	list materials, formular work out, topic
Initial Situation /Basic conditions	list prior knowledge, students background
Goal	clear goal, sub outcomes to monitor progress, objectives of the syllabus
Didactical analysis of the content	correct math, Define terminilogy, explanation of maths principle
Structure of the teaching process	provides time guidance, varied techniques, introduction
Motivation	practical examples, interessting , contextual relevance
Cognitive activation	student involvement, engaging to students, different approaches explanations
Internal differentiation	Applies to a variety of abilities, catering for individual needs
Repetition, practice and results	tasks, methods to remember, exercises homework
Media	transparencies, tools instruments, different technology
Ways of teaching and learning	teacher support, group discussion

Figure 5: Characteristics of the categories

To analyse the classification in the categories we tested the interrater reliability with two trained raters and got a Krippendorfs alpha of 0.81, which is acceptable.

RESULTS

First of all the number of characteristics and categories were analysed. The results are shown in figure 6. In both cases an increase of the number with higher semester was found. The changes with the semester are evidence that the constructs of the students change. To analyse the development of the constructs and to identify typical constructs a principal component analysis (PCA) was used. The aim of the PCA was to find connections between the characteristics. The PCA was practicable because the variables correlate substantially (KMO=0.6) and the Bartlett-Test was significant.

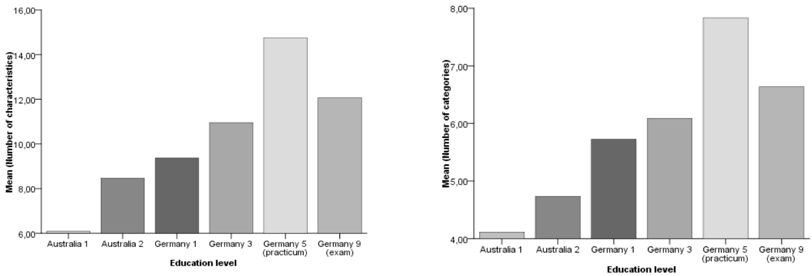


Figure 6: Number of characteristics and categories

With the help of the PCA four components were extracted (figure 7).

1. *Component: Planning and preparation orientated view on mathematics lessons*

   This component is characterised by the characteristics of the categories "Initial Situation", "Didactical analysis of the content", "Structure of the lesson plan" and "Goals". The common point of these categories is that the teacher has to think about the elements of the mathematics lesson before he/she will teach in class. The teacher decides the aims with the background of the initial situation and thereby didactically analyses of the content.

	Components			
	1	2	3	4
Initial Situation /Basic conditions	,732	-,134	,030	,087
Didactical analysis of the content	,720	,029	-,097	-,121
Structure of the lesson plan	,285	-,345	,136	,278
Goals	,704	-,137	,176	,225
Cognitiv Activation	-,121	,748	,053	-,045
Motivation	-,007	,551	,129	,011
Repetition, practice and results	,158	,176	,677	,102
Structure of the teaching process	-,249	-,238	,676	,114
Internal differentiation	,134	,276	,608	-,221
Media	,149	-,076	,013	,823
Ways of teaching and learning	-,139	,550	-,014	,565

Figure 7: Results of the PCA

2. *Component: Activating view on mathematics lessons*

   Characteristics of the categories "Cognitive activation" and "Motivation" build the second component. These categories are pupil-centered and describe the different possibilities to interest and involve pupils in mathematics lessons.

3. *Component: Task orientated view on mathematics lessons*

   This component includes the categories "Repetition, practice and results", "Internal differentiation" and "Structure of the teaching process". A central point of these categories are tasks and their structured use in the mathematics lesson.

4. *Component: Method orientated view on mathematics lessons*

   The categories "Media" and "Ways of teaching and learning" build the fourth component. These characteristics describe methods of teaching which are useful for mathematics lessons. This view on mathematic lesson plans is characterised by general tools to plan and create lessons.

To analyse the connection between the components and the educational level of the students the means of each component are shown in figure 8.

First of all the peak in "Planning" is conspicuous. This peak describes the results of German students who did the survey immediately following a five week practicum in school. It seems that their focus is on the characteristics which describe the process of planning and writing a lesson plan. This could be explained, because the main content of the practicum is to write lesson plans by themselves.

If the view of these students is compared with the German students, who are in their last semester just before their exam, the focus on the component "Planning" does not exist. It seems that the analysis of the lesson plans is more multifarious. The characteristics are nearly distributed equal on each component. The focus of Germans first semester students is on the methods and media, which were used in the lesson plans. By contrast Australian first semester students focus on the activation and motivation of the pupils. This view on lesson plans is intensified after the second semester. Equally the characteristics of the components "Tasks" and "Methods" get higher in the second semester of the Australian students. German third semester students focus the components "Planning" and "Tasks". This view of lesson plans fits with the main content of the course in the third semester.

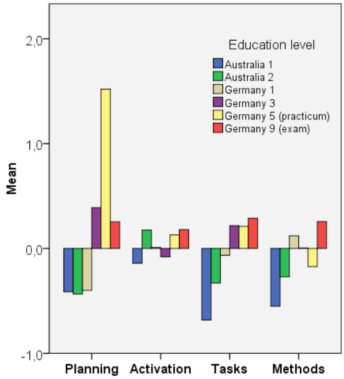


Figure 6: Results of the PCA

CONCLUSION

Through the results of the PCA it can be seen that the constructs of planning mathematics lessons develop with different courses. It seems that the constructs develop through different levels. In the first semesters the students find some different and not linked characteristics to analyse lesson plans. They have a "trial an error orientation" (Galperin, 1974). During their teacher education courses the student teacher learns some examples and models to analyse lessons. Thus their comparison of the lesson plans is conducted by the main content of the last course they had before the survey (cf. the results of German third and fifth semester). They generate a "model orientated level" (Galperin, 1974) to analyse the lesson plans. At the end of university education teacher students summarise theirs special focuses to a multifarious construct system of teaching mathematics, which will affect their action in class.

So far we can describe the development of construct systems about lesson planning. But we have not analysed the quality of the constructs. To do that, it is necessary to analyse the attachment of the characteristics to the compared lesson plans. Students' constructs should also be compared with self-written lesson plans. With this method it is possible to reconstruct students' concepts and find misconceptions and the potential for development of competencies to teach mathematics. To use these insights for supporting the development of competencies during teachers' education courses it is planned to create a feedback for the student teachers. This feedback should help them to reflect their skills in comparison to other students and also in comparison of their individual development.

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BELIEF AND PRACTICE RELATED TO ANALOGY USE IN TEACHING AND LEARNING MATHEMATICS

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Yılmaz Aksoy — Erciyes University/Turkey

This study investigates prospective teachers' beliefs and proficiency in using analogies in teaching and learning mathematics. The research was carried out with 22 prospective teachers. It employed a qualitative inquiry and used written exam and semi-structured interviews as the main source of data. Data were analysed using content and discourse analysis techniques. The participants had strong beliefs in the effectiveness of analogy use in teaching and learning mathematics and, they claimed that analogies would promote meaningful learning and improve students' attitudes towards mathematics. However, they had lack of proficiency in using such kind of instructional tools. Many of the analogues that they used had no content validity to represent the sub-notions of function. They had also difficulty in illustrating the transfer of knowledge from analogue to targeted concept.

LITERATURE REVIEW

The role of instructional analogies in teaching and learning scientific notions has been well recognised. Simply defined, analogy is a process of identifying similarities between two concepts. It entails using a familiar system (*source*) as a foundation for drawing inferences about an unfamiliar system (*targeted concept*) (Spellman & Holyoak, 1996). It is generally believed that analogies are useful instructional tool to facilitate students' acquisition of scientific notions. Analogies constitute one crucial component of the teachers' pedagogical content knowledge that they need most to be effective in teaching practices (Shulman, 1986). Rattermann (1997) states that by means of appropriate analogies teachers could communicate a large amount of information with little explanation and inspire scientific discovery. They allow teachers to organise their subject-matter knowledge in a way that could be grasped by the students of different ability and social background (Shulman, 1986).

Analogies would support learning in three major ways: they facilitate visualisation of abstract notions; they allow comparing new concepts with the ones in students' real world, and they increase students' motivation (Duit, 1991). Despite their advantages, analogies may not always produce intended learning outcomes. This might happen for various reasons including, for instance, constraints associated with the transfer of knowledge from analogue to target and the students' unfamiliarity with the analogues being used. The quality of learning facilitated by the instructional analogies depends upon the transfer of information from a familiar system to an unfamiliar system. Thus, educators place a great emphasis on the structural relations between analogues

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and targeted concepts, suggesting that it is this phenomenon that has significant implication for teaching and learning mathematics (English, 1997).

Instructional analogies have received a good deal of attention from mathematics educators (Fast, 1996; English, 1998; Richland et al, 2004). Fast (1996) indicated that analogies can be used to produce conceptual changes in students' understanding of mathematical notions including probability concept. Kathy et al (1999) conducted an experimental study in which they used seven concrete analogues to investigate the effectiveness of analogies in teaching and learning fractions. Analogies were evaluated with respect to their *ecological validity* – how realistic was the sharing context engendered by the analogues – and their *ease of partitioning* – how easy were the analogues to physically partition into quotients. These features had profound effects on students' ability to draw inferences about the concept of fraction. English (1998) explored the role of analogical reasoning in solving addition and subtraction problems. Most students were capable of recognising surface similarities between the problem situations, yet this did not enable them to resolve the tasks they were given. The author indicated that to be successful in using analogical reasoning in problem solving students should recognise structural relations between the source and target problems and, they should know when and where to apply analogical reasoning.

To sum up, a review of available literature indicates that most of the previous studies dealt with the role of analogies in classroom teaching and in knowledge construction. Very few of them examined the issue from the perspective of pre-service teachers. Thus, the present study takes the interest further and examines prospective teachers' beliefs and proficiency in using analogies in teaching and learning mathematics.

RESEARCH METHOD

This study employed a qualitative inquiry (Yin, 2003) to conduct in-depth examination of the research case at hand. It was carried out with 22 prospective high school teachers. At the time the study was conducted the participants were about the finish their master program (master without a thesis) in mathematics education. They had taken all the pedagogical modules related to mathematics education including *teaching methods*, *school experience* and *teaching practices*. So, it was assumed that the participants had theoretical knowledge and practical experience concerning the analogy use in mathematics education. Data were obtained from written exam and interviews. Students were given a written exam which included two open-ended questions. The first one aimed to investigate teachers' opinions about the instructional gains that the analogies would provide for the students:

> **Q1.** Do you believe in the effectiveness of analogy use in teaching and learning mathematics? If you believe in it, what source of benefits that they offer to the students? Write down your answer with the underlying reasons.

The second question was used to explore their proficiency in using analogies to teach the concept of function and its sub-notions:

Q2. Write down all the analogies that you can you use to teach the concept of function and its sub-notions? Explain, in-detail, how could you use them.

Semi-structured interviews were conducted with three students after the written exam. The line of inquiry developed in accord with the participants' answers. Aspects of clinical interview (Gingsburg, 1981) were considered to reveal the participants' actual thinking processes. Interviews were tape recorded and the annotated field notes were taken for consideration.

Theoretical Framework and Data Analysis

Literature about epistemology of the functions (Dubinsky & Harel, 1992) and the instructional analogies in teaching and learning science and mathematics (Kathy et al, 1999; Podolefsky & Finkelstein, 2006) provided a conceptual base for the data analysis. The literature suggests that the function concept can be interpreted as *a relation that does matching between the elements of two sets* or as *a dynamic process that transforms every input to a unique output*. The concept has two fundamental properties: univalence and arbitrariness conditions. The former states that <u>every element</u> in the domain must have <u>only one image</u> in the co-domain. The latter suggests that a function can do matching in a completely arbitrary manner, not necessarily through an algebraic or arithmetical rule; and the elements of the domain and co-domain could be any kind of entity, not just numbers. On the other hand, analogies can be evaluated with respect to their *purpose of use* and the *content validity* (Bayazit & Ubuz, 2008). The *purpose of use* is concerned whether analogies are offered to illustrate the mathematical notions or to emphasise associated rules and procedures. The issue of content validity can be considered at two levels. The first is concerned with the very nature of an analogue in that the analogue should have intrinsic power to represent the essence and the properties of the targeted concept. The second is concerned with the transfer of knowledge from analogue to target.

Data were analysed in the light of above notions. Content and discourse analysis methods (Miles & Huberman, 1994) were used to discern meaning embedded in the written and spoken expressions. The first phase of analysis included reading thoroughly students' exam papers and writing up a summary of their answers to each question. This process was repeated on different copies of the texts and eventually codes were assigned to the units of meaning inferred from the texts. Some of the codes produced for the students' responses to Q1 included, for instance: **Mean-Lear** (Facilitates meaningful learning); Stu-**Attitu** (Improve students' attitudes towards mathematics); **Visu-Abil** (Promotes students' visual ability).

A sequential approach was employed to analyse students' responses to Q2. First, analogues and targeted concepts were identified. Then, analogies were evaluated with respect to their *purpose of use* – whether they were offered to illustrate function related ideas or to emphasise rules and procedures. If the analogies were offered to illuminate the function concept, then the analysis continued to identify what sort of conception– function *as a relation* or function *as a process* – that they addressed. The issue of *content validity* was considered at two levels. Analogues were examined to see whether or not they were intrinsic power to represent function related ideas. If they were so, then the examination continued to identify the participants' proficiency at illustrating structural relations between the analogues and the targeted concepts. Codes were established to the units of meaning associated with the above notions inferred from the texts. This second process was repeated on different copies of the text and, finally pattern coding was applied to collect units of meaning under more general categories. As it was the manner in the analysis of the students' exam papers, interviews were fully transcribed and considered line by line. Firstly, a summary of students' answers to each question was written up. Then, codes were established to the units of meaning inferred from the texts. Repetition of this second process led to creation of more general categories, which are presented in the coming section.

RESULTS

The results are presented in two ways. First, we consider prospective teachers' beliefs in the effectiveness of analogies in teaching and learning mathematics, and secondly we examine their proficiency in using these tools to illustrate the concept of function and its sub-notions. The research findings indicated that the participants had strong beliefs in the effectiveness of analogy use in mathematics education. They provided several rationales to emphasise the importance of analogies and these were associated with three areas, namely: *cognitive aspect of learning process*, *social and psychological constructs*, and *mathematics-real life relations*. Salient aspects of prospective teachers' beliefs are provided in Table 1.

Instructional Gains	Rationales	Num. of Stud.
Cognitive Gains	• Facilitates visualisation of mathematical notions. • Shortens the learning process. • Eliminates rote learning and facilitates meaningful understanding. • Allows students to connect mathematical concepts to each other. • Facilitates coding, and allows possession of mathematical knowledge in the long-term memory. • Makes it easier to remember mathematical notions. • Facilitates learning by '*doing mathematics*'. • Promotes critical and creative thinking. • Makes it easier to use mathematical knowledge.	21
Social & Psychological Gains	• Increases students' enthusiasm to learn mathematics. • Encourages students' participation in the lesson. • Gets students like mathematics. • Attracts students' attention and interest. • Motivates them. • Eliminates their fear and anxiety.	7
Mathematics & Real Life	• Creates an idea amongst students that mathematics is part of real life.	4

Table 1: Key features of the prospective teachers' beliefs about the effectiveness of analogies in teaching and learning mathematics.

Those who addressed cognitive benefits of analogies provided several rationales each of which highlighted essential features of learning process suggested by the constructivist learning theory. Their comments included, fundamentally, the ideas that analogies would promote conceptual understanding, support development of creative and critical thinking, enhance students' visual ability and, facilitate retention and recalling mathematical notions. The following citation is typical that reflects the participants' opinions:

> ...this is because mathematical concepts are abstract notions and this creates difficult for many students. Analogies make them [mathematical concepts] concrete and, thus, they make it easier for students to learn these notions. …. [Analogies] help learners to develop a meaningful learning…it helps them to recall and use mathematical notions…[11][1].

Bringing social and psychological constructs to the attention seven participants claimed that analogies would improve students' attitudes towards mathematics, eliminate their fears and anxiety and ensure their participation in the lessons. Four participants believed that the use of analogies would enable students to understand the role of mathematics in everyday life.

Interviews with three students complemented the outcomes of the written exam. These three students illustrated cognitive and social-psychological gains that the analogies would provide for students and, this is seen in the following exchange (Episode 1):

> Ilker[2]: Above all, analogies strike students' attention… Suppose that you are giving an example from daily life...; can you imagine how interesting it will be for the students. … They [analogies] eliminate students' fears and anxiety and students start to like mathematics. …
>
> Interviewer: Do you think analogies would facilitate students' learning?
>
> Ilker: … As I said analogies…increase students' enthusiasm to learn… They enhance students' ability to visualise mathematical concept and…[students] develop a meaningful learning. In addition, analogies facilitate retention and recalling mathematical ideas. It is difficult for students to hold mathematical notions, like definition and theorems, in their minds… Illustrating them [mathematical concepts] through analogies makes it easier for students to possess and use this knowledge…

Ilker thinks that psychological advantages precede cognitive ones in that analogies motivate students, increase their enthusiasm and, this eventually stimulates meaningful learning. He stresses that analogies would promote students' visual ability and increases their mental capacity to hold abstract notions in their minds.

The participants were relatively successful in using analogies to illustrate the concept of function. In this respect 7 analogies were offered by 14 participants (see Table 2).

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[1] Refers to particular students from the questionnaire; in this case, **Student 11**.

[2] Students' names are altered.

	Num. of Students	Analogies	Num. of Users	Validity Condition
Function as a ***Relation***	10	1. City-province relation	1	Satisfies the validity condition
		2. Mother-children relation	5	
		3. Mail-address relation	3	
		4. People-destination city relation	1	
Function as a ***Process***	4	5. Factory	2	
		6. Communication system	1	Partly satisfies the validity condition
		7. Washing machine	1	

Table 2: A list of analogies offered to describe the function concept.

Through these analogies 10 students presented the concept as *a relation* doing matching between the elements of two sets and 4 students described it as a process transforming inputs to output. Analogies that entailed the idea of *relation* had the content validity and they addressed the univalence aspect of the function concept. An analogy of *factory* was also considered to be appropriate because students' descriptions clearly addressed the notions of inputs, outputs and the transformation:

> We could think of a function like a factory; a factory takes raw materials, processes them and then gives out products. Function is like that…it transforms inputs to outputs…[3].

The remaining analogies (*washing machine* and *communication system*) did not explicitly state the idea of input, output and the transformation; thus, they were considered to be partially valid.

Nevertheless, they had great difficulty in using analogies to illustrate the sub-notions of the functions. 14 analogies were offered to explain the idea of constant function, yet only two of them (*children-mother relation* and *recycling factory*) were epistemologically appropriate and these were used by 7 students (see Table 3).

	Num. of users	**Num. of Analogies**	**Purpose of use**	**Validity**
Constant function	19	14	concept	2 valid 12 invalid
Piecewise function	11	11	procedures	invalid
1-1 and onto function	1	1	concept	valid
Identity function	1	1	concept	valid
Inverse function	1	1	concept	valid

Table 3: Analogies used to illustrate the sub-notions of the functions.

An example of inappropriate analogies to illustrate the constant function is seen below. One of the interviewees, Ayhan, offered an analogy that "*whoever the president chooses from the national assembly, he/she is still a member of parliament (MP)*". Upon further probing he said (Episode 2):

> Let me define first the constant function..it is $f(x)=c$. ... Here, let MPs be the elements of the domain; here president is the function, *f*...; he is choosing only one MP amongst others and this person is still an MP... I would give this example to illustrate the constant function.

Ayhan does not clarify relations between the example he gave and the idea of constant function. His description incites even an idea that those apart from the elected MP were omitted in the domain, so the analogy cannot be a function.

11 analogies were used to illustrate procedural knowledge about the idea of piecewise function (selection of appropriate rules on the sub-domains) or to emphasise its surface properties (a piecewise function is defined by more than one rule). In the participants' descriptions there was not even an implicit reference to the idea of input, output and the transformation process. For instance, one of the interviewees, Gökçe, gave the following explanation (Episode 3):

> ... Think that we are forming basketball and handball teams. Yet, we have a rule...those who is taller than 180cm shall go to basket team, and shorter than 180cm shall go to handball team. We compare the input with the extreme points of the sub-domain and accordingly we can choose rule we are going to use...

It is quite clear that the premise of this episode is selection of the right formula to operate on each sub-domain.

CONCLUDING REMARKS

The aim of this study was to examine prospective teachers' beliefs and proficiency in using instructional analogies in mathematics education. The research findings indicated that the participants have strong beliefs in the efficiency of analogies in teaching and learning mathematics. Their beliefs prevailed two major fields: social psychology and human cognition. They believed that analogies would motivate students, eliminate their fears and anxiety, and encourage them to participate in the lessons. They did not consider these social-psychological gains in isolation, rather appreciated them to stimulate students' reasoning over the mathematical notions being taught and learned. The participants provided several rationales as to the cognitive gains that the analogies would provide for students – analogies could promote creative and critical thinking, enhance students' visualisation, and promote their mental capacity to possess and preserve knowledge (see Table 1 & Episode 1). These are all consistent with the prescience of major teaching/learning theories, such as constructivism, information-processing theory and socio-cultural theory.

Almost half of the participants offered analogies that presented the concept as a *relation* (see Table 2) and this shows that they were attentive to the consistency between the analogies and the idea of function presented in the Turkish mathematics curriculum. Most participants lacked, however, the ability to use such tools to illustrate the functions, especially the sub-notions of the functions. Many of the analogues were epistemologically inappropriate to represent the targeted concepts. They also had difficulty in explaining structural similarities between the analogues and the targeted concepts (see Episode 2). A number of students offered analogies to emphasise procedures and factual knowledge (see Table 3 & Episode 3). In our view, this could shift students' attention from concept to procedures and, hence, confine their understanding of the concepts to mechanical manipulations.

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USING COMPUTER-BASED INSTRUCTION TO SUPPORT STUDENTS WITH LEARNING DISABILITIES: UNDERSTANDING LINEAR RELATIONSHIPS

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The study of linear relationships is foundational for mathematics teaching and learning. However, students' abilities to make connections among different representations of linear relationships have proven to be challenging. In response, a computer-based instructional sequence was designed to support students' understanding of the connections between representations. In this paper we report on the affordances of this dynamic mode of representation specifically for students with learning disabilities. We outline four specific results identified by teachers as they implemented the online lessons. We consider the educational implications of using online technology in the teaching and learning of mathematics for students with learning disabilities. .

INTRODUCTION

A current focus of mathematics instruction centres on the push for algebra reform and the resulting recommendation from the National Council of Teachers of Mathematics (NCTM 2000) that algebra become an essential strand of the intermediate (Grade 7-8) curriculum, prior to formal algebraic instruction in high school. An understanding of linear relationships is central to the development of algebraic thinking. Expressing an understanding of a linear relationship can be thought of as describing a systematic variation of instances across some domain. The major characteristic of a linear relationship is the covariation between two sets of data represented by two variables, the independent variable, x, and the dependent variable, y. The nature of the relationship is that for every instance of x there is one corresponding instance of y, determined by the underlying linear rule. The relationship that connects the two variables is one of predictable change or growth.

Linear relationships can be represented symbolically/numerically through equations and algebraic symbols using the form $y=\mathrm{m}x+\mathrm{b}$, where m is the coefficient, or multiplicative factor, of x, and b is the additive (sometimes known as the constant) term of the relationship. A linear relationship can also be represented graphically, where m represents the gradient of the slope and b represents the y-intercept. These representations are intertwined, such that a change in one representation leads to a change in the other representation. Mathematics educational researchers stress that it is the ability to make *connections* among different representations, specifically symbolic/numeric and graphic ones, that allow students to develop insights for constructing the concept of a linear relationship (e.g., Bloch, 2003; Evan, 1998).

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There have been numerous studies that have documented the difficulties students have when exploring the connections among representations of linear relationships (e.g., Evan, 1998; Moschkovich, 1996, 1998, 1999). Students have difficulties shifting between different modes of presentation (Brassel & Rowe, 1993; Yerushalmy, 1991). When graphing a linear relationship of the form y=mx+b, researchers have noted that the connections between m and the slope of the line, and b and the y-intercept are not clear (Bardini & Stacey, 2006). Students also have difficulty predicting how changes in one parameter will affect the graphic representation (Moschkovich, 1996; Moschkovich et al., 1993).

Learning Disabilities and Mathematics

Although little research to date has been conducted on the algebraic learning of students with learning disabilities, it would seem predictable that these students would also find the conceptual underpinnings of linear relationships as elusive as typically developing students. Research on learning disabilities in the domain of mathematics is still in its infancy (Gersten, Jordan, & Flojo, 2005). A principal area of consideration is the divide between procedural and conceptual instructional practice and whether explicit and inquiry based instruction can and should be integrated for students with learning disabilities (Pedrotty Bryant, 2005).

Students with learning disabilities often have difficulty retaining facts (Geary, 1993), and so the instructional approach for these students tends toward memorization through repetition rather than the development of conceptual knowledge (Cawley & Parmar, 1992). This rote drill approach may seem a successful strategy as it offers students a means of producing the correct answer, but it is an extremely limited way of understanding complex concepts such as linear relationships.

The National Council of Teachers of Mathematics emphasizes equitable instruction for all students. Given the proportion of students with learning disabilities (5% to 8%, according to Shalev et al., 2000) there is a need to examine mathematics educational for these students, and how these students can be supported to be part of “algebra for all.”

New Instructional Approach for Teaching Linear Relationships

This paper reports on a research study conducted to investigate the implementation of a teaching approach designed to address some of the instructional difficulties outlined above. As part of a larger long-term study, we have been investigating the affordances of an instructional approach that prioritizes visual representations of linear relationships – specifically, the building of linear growing patterns and the construction of graphs (Beatty, 2007; Beatty & Bruce, 2008). Previous research on the lesson sequence has shown that it supports students’ progression from working with linear growing patterns as an anchoring representation to considering graphical representations of linear relationships. Students also make connections among different representations – pattern rules, patterns and graphs (Figure 1).

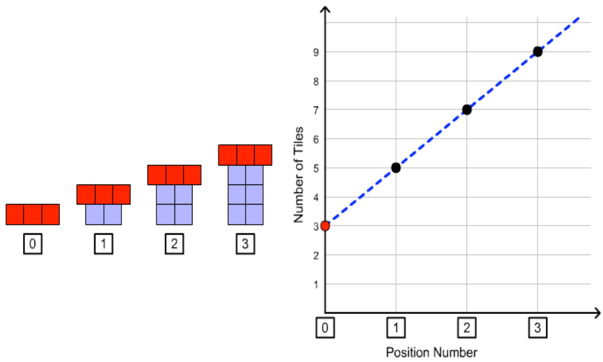

Figure 1: Connecting the linear growing pattern to a graphical representation. Both represent the pattern rule "number of tiles = position number x2+3" or y=2x+3

In our previous study we evaluated the experimental algebra lesson sequence by conducting quantitative analyses of student learning to determine whether there was an increase in student scores from pre to post intervention. We calculated a total pretest and posttest score comprised of 10 sub-items measuring students' ability to find generalized rules/functions for patterns. The results indicate that the mean posttest score (M = 6.72, SD = 2.74) was significantly greater then the mean pre-test score (M=4.21, SD =2.74), t (311)=-14.33, p<.000. The standardized effect size index d., was .99, a high value. The mean difference was 2.51 points between the two tests. Table 1 outlines the results from pre to posttest.

	Number of students who scored between 1 and 5 out of 10	Number of students who scored 6 or above out of 10
Pretest n=295	204	91 (12 achieved scores of 9)
Posttest n=294	66	228 (114 achieved scores of 9

Table 1: Student scores pre to posttest

We then conducted a one-way repeated-measures ANOVA to compare pre and posttest achievement as a function of students' demonstrated achievement level (low n=67, mid n=164, high n=79). Levels were based on teacher rating and report card marks. Students designated as low were on Individual Education Plans (IEP) and most had been identified as having some kind of learning disability. The multivariate test indicated a significant effect, F(1,307) = 159.32, p<.000, but with no interaction of test results by level, F(2, 307) = 1.723, p=.18. These results indicate that students at all three levels increased their test scores from pre to post (see Fig 2). These results suggested that the positive effects for all students, including those identified as having a learning disability, were important enough to pursue further dissemination of the learning sequence by capitalizing on the potential of online learning objects.

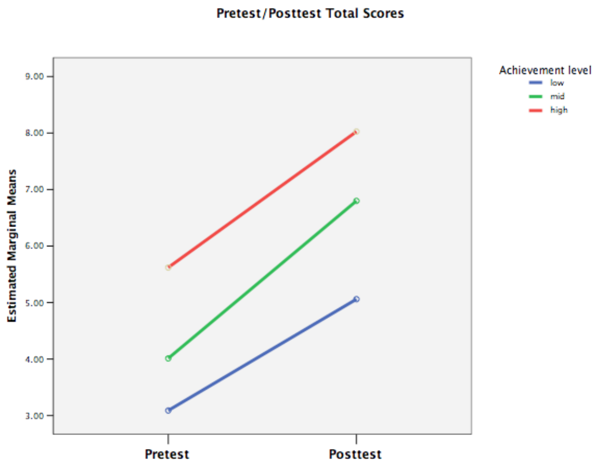

Fig. 2: Estimated marginal means by achievement level pre and posttest.

In the study reported in this paper we enhanced the original instructional sequence by including computer-based dynamic interactive representations of linear relationships. These online learning objects, or CLIPS (Critical Learning Instructional Paths Supports) offered the possibility of combining a proven instructional sequence with unique properties of digital technology.

Unique Support of CLIPS for Students with Learning Disabilities

CLIPS are created using flash animation and incorporate audio narration, offering students the ability to consider mathematical concepts in non-static environments. We hypothesized two specific ways that the affordances that this kind of environment would support students with learning disabilities.

The first affordance is supporting students in focusing their attention. Students naturally focus attention through stressing some features as foreground and ignoring others as background (Mason, 2008). The CLIPS computer animation was designed to direct student's attention, in order that they would discern details and recognize relationships that we, as the educational designers of the activities, believe are important to discern and recognize. In each activity the aspect that we want students to notice – for example the connections between the numeric value of the constant in a pattern rule, the number of tiles in a pattern, and the vertical intercept of a trend line on a graph – becomes the focus of students' attention. As the student works through this activity, the constant in the pattern rule flashes red, the red tiles that "stay the same" in the linear growing pattern flash, and the vertical intercept on the graph has a red flashing ring around it (Figure 3). In addition all activities have audio narration that directs students' attention to particular aspects of the task.

The second affordance of the technology is that mathematical connections can be conveyed to the students interactively. Students move through a series of scenes for each activity, so that the mathematical concepts are introduced in a logical order of increasing complexity. The animation creates opportunities for students to interact with the material by providing activities in which the co-action between user and environment can exist. This co-action takes many forms, from filling in numeric values, dragging words to complete sentences, to more sophisticated and rich interactions such as constructing patterns using virtual tiles or graphs using the graphing tool. Each representation is linked to the other representation so that as students create one, they can see the corresponding changes in the other. Thus the mathematical symbols that students work with are dynamic objects that are constructible, manipulable and interactive. This offers the opportunity for students with learning disabilities to *construct* an understanding of the process of linear co-variation, rather than simply memorizing rote facts.

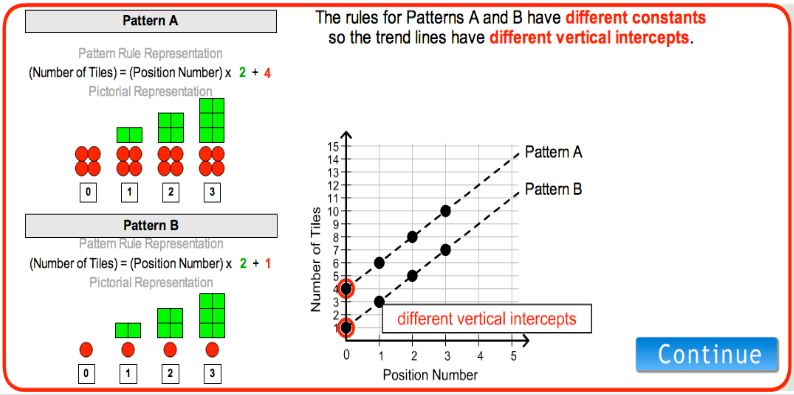

Figure 3: Screen capture of activity to compare pattern rules that have the multiplier but different constants. In this activity, the words "different constants" and "different vertical intercepts" flash, the red circles representing the constant part of the linear growing patterns flash, and the red rings around the vertical intercepts of the graph flash.

METHOD

This study was part of a larger research project in which we investigated the affordance of the CLIPS Algebra sequence for all students. Grade 7 and 8 teachers in two different school boards implemented CLIPS in their classrooms as part of their Algebra unit. For this paper, we focus specifically on data relating to students with learning disabilities. We had hypothesized that the dynamic/interactive nature of the CLIPS learning objects would support students with learning disabilities. In fact, all of the teachers we worked with expressed overwhelming surprise at the levels of learning exhibited by their students who had been identified with learning disabilities.

Participants and Data Sources

Fifteen teachers volunteered to be part of this study. The teachers received three days of professional training and implemented CLIPS in their classrooms. Researchers observed classroom implementation throughout. On two of the three PD sessions we conducted focus group interviews with the teachers. Transcripts from 12 focus group interviews with participating teachers were coded to identify categories and themes. Subsequently, data was transformed to count the frequency of themes and codes in order to identify prevalence of a code or theme.

RESULTS

In this paper we highlight four major themes reported by all 15 teachers.

Teacher's in-class assessments revealed that students with learning disabilities were able to make connections among different representations of linear relationships, and could predict how changes in one representation would affect other representations.

> The concepts were introduced slowly and accessibly and reinforced so that with confidence I can say all my students on an IEP can look at a graph and tell you the rule for that graph, can build a pattern from that graph, and can give you a story related to that graph. I've never had that experience before. On the quizzes and assessments I've been doing, they've all being getting level 4 [out of 4]. (Teacher 1, FG 2.1).

Students in pullout remediation programs were no longer removed for math learning, but remained in the classroom. All of the teachers reported that the sequential nature of the lessons and activities allowed their lowest-achieving students to access the material successfully. This was ascribed to the animated, visual nature of the materials, the voice-overs of any written descriptions or instructions, and the capacity for students to repeat any lesson or activity they did not understand.

No modifications of the material were necessary.

> All my students with learning disabilities were doing what everyone else was doing – all the same lessons. And they're doing fine! That's huge! That these kids can engage in the same activities and communicate their thinking to the class! There was no IEP in place for this. They all did the exact same thing and I did not accommodate any student at any time for this. And everyone did well (Teacher 4, FG 2.1).

Student attendance and contributions to discussion in math class increased.

> The biggest difference for me was seeing IEP kids who are normally petrified of math, and not terribly successful, and believing that they can't do it leading the discussion. One of my self-proclaimed weak math students got the concept and was questioning typically stronger math students in class about their patterns and explaining why it wasn't a linear growing pattern – that the growth wasn't predictable. Our class is a bit class with lots of learning needs and for the first time EVER they ALL get it! (Teacher 3, FG 1.1).

EDUCATIONAL CONTRIBUTIONS

Sequenced dynamic representations of representations of linear relationships had a positive effect on the levels of achievement of students identified as having learning disabilities. The online activities directed students' attention to particular important ideas. The activities allowed students to construct their own understanding rather than memorize procedures. This construction was supported by the ability to "go back and replay if you don't get something." The initial entry point was accessible for all students. The sequence subsequently incrementally built in complexity. Each student trusted that they would be able to continue to successfully work on activities, and that the material would not become too complex too quickly. As a result, these students demonstrated an understanding of the connections among multiple representations of linear relationships that has been shown to be difficult for typically developing students.

This study suggests that students with learning disabilities are capable of learning complex mathematical concepts when given the opportunity to do so. All teachers reported that, subsequent to working with CLIPS, they became more student-focused in their teaching, and that the students who had been in remedial pullout programs were no longer removed from the classroom for math, but remained as contributing members of the classroom mathematics community. This leads us to question whether the learning difficulties for many of these students may have been curriculum or instructional difficulties in addition to learning disabilities.

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USING MATHEMATICAL DISCOURSE TO UNDERSTAND STUDENTS' ACTIVITIES WHEN USING GEOGEBRA

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Sfard's elaboration of mathematical discourse as a well-defined form of communication provides an illuminating framework within which to make sense of students' activities as they engage in mathematical tasks. In this paper I outline key aspects of this framework. I use it to interpret and understand the surprising activities of a pair of in-service South African mathematical teachers' activities as they engage in a mathematical task which allows the use of GeoGebra.

The way in which students use computers as tools for learning of maths has been examined from a variety of perspectives. For example, Kieran and Drijvers (2006) use the instrumental approach to tool use together with Chevellard's anthropological perspective to illuminate the link between theoretical thinking and techniques in a CAS environment. Indeed, Artigue and Cerulli (2008) cite eight major frameworks (including the above) commonly used by researchers when examining teaching and learning mathematics in digital contexts. Although I do not wish to add unnecessarily to this plethora of frameworks, Sfard's (2008) notion of mathematical discourse, which is a key component of her commognitive theory, proved to be a particularly useful framework for exposing what was happening as digital immigrants in an in-service teachers' course at a South African university used GeoGebra to enhance mathematical learning. In this paper I explain this framework and apply it to an episode in which a pair of these teachers engage in a mathematical task which permits the use of GeoGebra.

ANALYTIC FRAMEWORK

Commognition is based on the premise that individual development is individualization of "patterned collective activity" (Sfard, 2008, p. 570). That is, thinking is individualized communication. With respect to mathematics, patterned collective activity takes the form of mathematical discourse and mathematics learning is "tantamount to modifying and extending one's mathematical discourse" (ibid., p. 567). According to Sfard (ibid.) mathematical discourse is far broader than spoken or written words; it is also characterized by visual mediators, routines and narratives. These discourse characteristics have been defined and illustrated in the context of school mathematical activity, for example Ben-Zvi and Sfard (2007). In this paper, I elaborate these characteristics to contexts in which computers are used as a resource in the doing of mathematics.

Sfard (2008) argues that the use of certain **words** or expressions such as equal, function, vertical asymptote in specific ways indicates that we have mathematical

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discourse. Although many of these words also appear in everyday discourse, their use in mathematical discourse is well-defined, albeit often implicitly. For example, in colloquial discourse, we say: "I have eaten half the chocolate bar" to indicate that we have eaten approximately half a chocolate bar. In mathematical discourse, we use the word 'half' to mean 'exactly half', for example, half of eight is four. Word use is very important in that the use of a word constitutes it meaning (Wittgenstein, cited by Sfard, 2008, p. 573). In the task below, the pair of learners initially use the term 'vertical asymptote' in an incorrect way. Indeed they are only able to highlight the distinction between a point where the function has a removable discontinuity (a 'hole') and a point at which there is a vertical asymptote, after they change their use of term 'asymptote'. **Visual mediators** are visible objects such as symbols, graphs and diagrams which participants in a mathematical discourse use to identify the objects of their thinking and/or communication; some of these visual mediators are created especially for mathematical discourse (Sfard, 2008). In GeoGebra context, graphs of various functions may serve as visual mediators. However GeoGebra generated graphs may not reveal key aspects, such as removable discontinuities, of the function. In the task below, we see how the students use hand-drawn graphs in which discontinuities may be represented by a hole as visual mediators, rather than computer-generated graphs in which the discontinuities are hidden. **Narrative** is any text, spoken or written, that is "framed as a description of objects, of relations between objects, or processes with or by objects" (Sfard, 2008, p. 300); it is subject to endorsement and may be labelled "true" or "false". Within formal mathematical discourse, the narratives that are approved by the academic mathematical community according to specific well-regulated rules are called mathematical theories. These theories consist of various discursive objects such as axioms, theorems, definitions (Sfard, 2008). Within the context of computer-based mathematical learning in schools, mathematical narratives are ultimately endorsable only if they conform to the official mathematical narratives. A **routine** is a repetitive and well-defined discursive pattern (ibid.). A routine may be a procedure; it may also be a practice (Ball, 2003) such as generalizing, justifying and endorsing (or rejecting) mathematical narratives. Routines are regulated by certain rules. Within the context of computer mediated learning, computer-generated routines require further scrutiny in that they do not always conform to the rules of mathematical discourse. For example, GeoGebra will generate a continuous graph even if there are removable discontinuities in the relevant function. Consequently the user needs to be aware of how to interpret the computer output so that it is compatible with endorsed mathematical narratives. Also, the user needs to understand when to use computer mediated routines, when not to. The *when* of routines involves the set of metarules that "determine, or just constrain, those situations in which the discursant would deem this performance [routine] as appropriate" (Sfard, 2008, pp. 208, 209). In the task illustrated below, we see how the students flout the metarules (often implicit) that govern the conditions under which it is suitable to use computer-generated routines rather than pen and paper routines.

CONTEXT

For reasons born out of South African history, many high school mathematics teachers in South Africa have a degree in education rather than in mathematics. Unsurprisingly their mathematics content knowledge is generally fairly weak. The events I look at here are part of a larger project in which we examine ways in which in-service teachers are able to enhance and deepen their own mathematical learning. In the particular course in which this research takes place, functions are revisited in the hope of extending in-service teachers' understanding of this fundamental concept. The class meets once a week for a three hour session over eleven weeks. Students, i.e. teachers, are expected to study a specific chapter from the prescribed precalculus textbook (Sullivan, 2008) prior to their weekly session. During the weekly session, the class discusses this chapter for the first hour. Students are then presented with tasks around the topic which they mostly do in pairs. Some of tasks require the use of GeoGebra; others do not. The episode of interest, in which students attempt the task given below, took place during the sixth week of the class. As preparation for the class, students had studied the chapter, 'Polynomial and Rational functions' in the prescribed textbook.

Task and its pedagogical purpose

> Graph each of the following functions.
>
> $y_1 = \frac{x^2-1}{x-1}$ $\quad$ $y_2 = \frac{x^3-1}{x-1}$ $\quad$ $y_3 = \frac{x^4-1}{x-1}$ $\quad$ $y_4 = \frac{x^5-1}{x-1}$
>
> Is x=1 a vertical asymptote? Why not? What is happening for x=1? What do you conjecture about $y = \frac{x^n-1}{x-1}$, $n \geq 1$ an integer, for x=1? (Sullivan, 2008, p. 209).
>
> Be careful, GeoGebra isn't perfect here.

It was anticipated that students would use GeoGebra to generate y_1. Having observed that the GeoGebra graph of y_1 is the straight line $y = x+1$, we expected students to write: $y_1 = \frac{x^2-1}{x-1} = x+1, \quad x \neq 1$. Similarly for y_2, y_3 and y_4. We hoped that students would notice that each function was not defined at x=1, but that by cancelling factor x-1 in numerator and denominator, the point x=1 could be removed. Graphically this would be depicted by a hole at x=1 (this could be drawn on a print-out of the graph). Alternatively, students could use a theorem (the endorsed narrative) about the location of a vertical asymptote as given in the precalculus textbook to decide that none of the given functions had a vertical asymptote at x=1, or elsewhere. This theorem reads: "A rational function $R(x) = \frac{p(x)}{q(x)}$, in lowest terms, will have a vertical asymptote $x = r$ if r is a real zero of the denominator q" (Sullivan, 2008, p. 188). Since this was a precalculus course we did not expect students to use the language of discontinuities. Rather we expected them to speak of holes and/or functions not existing at a point.

Implementation of task

Eva & Tom, both digital immigrants, were audio-taped and their work was screen-recorded as they worked on the above task. They were told that they should treat this as a normal classroom session and that they should feel free to ask the researcher any question that they would normally ask in a non-recorded classroom session. In an earlier survey Eva claims that she is very confident in her use of the computer; in contrast, Tom claims that he is confident (not very confident) in its use. Eva is a very good student; she obtains 87% in mid-semester test. Tom is one of the weakest students in the class; he obtains 34% in the mid-semester mathematics test.

Description of key events

At the start of the task, Eva and Tom indicate that they will hand-draw the graphs of the four functions and that they will use GeoGebra to check their graphs (line 3, 6). They start by considering y_1. Eva declares that she is aware that the computer generated graph is "not perfect" (line 3). Eva and Tom talk about having a vertical asymptote at x=1 (lines 7 – 11).

3 E: Right? OK. And they say be careful. GeoGebra isn't perfect here, so let's see what happens. We will do it sketching first and then we'll check on GeoGebra. Now if the question is: Graph each of the following functions. So we'll have to take each one and graph it, OK?

4 T: *y* equals to...

5 E & T: *x* squared minus 1 over *x* minus 1.

6 E: Right. Let's draw our system of axes. Drawing our system of axes

7 T: And then x won't be equals to

8 E & T: 1

9 E: OK but

10 T: That is our vertical

11 E: asymptote.

After a little discussion, Eva writes $y=\frac{x^2-1}{x-1}$, $f(x)=\frac{(x-1)(x+1)}{x-1}=x+1$. She does not note that $x\neq 1$. Nonetheless Tom hand draws the line $y = x+1$ with a hole (a circle) at x=1 (lines 36 - 38). He also draws a vertical line at x=1 to indicate a vertical asymptote. See Figure 1.

36 E: So what we're going to do is draw a non-coloured circle through that. So that's the only value *x* is not going to take, right?

37 T: Yes.

38 E: So we're going to draw our circle, right? Right. And then continue that graph. That's it.

Eva & Tom then generate a GeoGebra plot of y_1, presumably to confirm their hand-plot. Eva notes that GeoGebra does not generate a graph with a hole (line 39).

39 E: OK. But you see GeoGebra doesn't do that. Can you see that it's like a continuous graph, né?

For each of y_2 and y_3, Eva factorises the numerator and then cancels factor x–1 in numerator and denominator. In neither case does she write that $x\neq 1$. Eva and Tom then spend much time and energy hand-drawing the resulting functions; they use calculus and point-plotting routines. Tom hand draws each graph correctly with hole at x=1 (but also with vertical line at x=1). See Figure 1. Eva & Tom confirm each hand plot with GeoGebra.

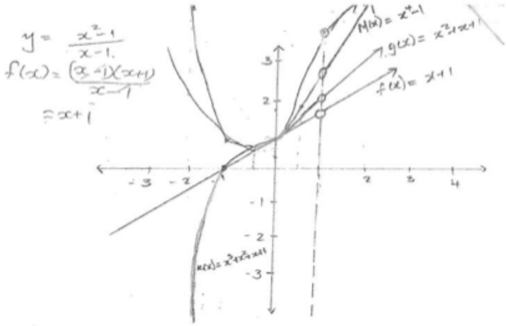

Figure 1: Hand-plots of y_1, y_2 , y_3 and y_4

Eva and Tom persist in talking and writing about x=1 as a vertical asymptote until near the end of the activity at which point the researcher asks the students if they have any questions. Eva states her concern that GeoGebra is not perfect "because the asymptote is not, it's not reflected when x is equal to 1" (line 324). The researcher uses this opportunity to explain that there is no vertical asymptote at x=1: for a vertical asymptote, we must have an expression of form $a/0$, $a \neq 0$, in lowest terms. Although it is unclear whether Tom accepts this explanation (his response is silence), Eva states that she understands. Indeed the students do not speak of or draw an asymptote at x=1 again; rather they draw and speak of a hole at x=1 for y_4. See Figure 1.

Soon after this Eva and Tom discuss their response to the question: "Is x=1 a vertical asymptote? Why not? What is happening for x=1?" (lines 420 – 422, 433 - 436):.

420 E: OK, but now, right. They say is x equal to 1 is a vertical asymptote? We're going to say, no. No, it is not a vertical asymptote.

421. T: No.

422. E: It is not a vertical asymptote.

433. T: For f equals to 1 it's undefined because of division by zero...

434. E: Mmm, because division by zero is… So we can write down it as a hole that x equal to 1. For x equals to 1 then the function is undefined because division by zero...

435. T: zero is not accepted, or however you need to write it.

436. E: is undefined. OK, that's fine 'undefined'. So there is a hole at x equal to 1 for every function.

Eva then writes, while speaking out loud: "x=1 is not a vertical asymptote. The function is not defined for x=1 but is not an asymptote because each function has a factor of (x-1) in its numerator which cancels with (x-1) in the denominator. Therefore it is not an asymptote but function is undefined at x=1".

ANALYSIS

Endorsing narratives in a technological environment

In this task, Eva & Tom do not exploit the affordances of GeoGebra as a tool to support their activities. That is, they do not use the graphs generated by GeoGebra as visual mediators with which to 'do' mathematics; rather they spend a lot of time and effort in hand-drawing each of the graphs and only use GeoGebra for verification. This is despite the fact that they are easily able to use GeoGebra to generate the graphs (evidenced by their quick generation of GeoGebra graphs for verification of hand-drawn graphs). A possible reason for these (unexpected) actions is that the computer-generated graph is not consistent with the officially endorsed narrative wherein a removable discontinuity is represented by a hole. So it is regarded as generally suspect. Another possible reason for Eva and Tom's limited use of GeoGebra is an entrenched cultural attitude (particularly among digital immigrants): mathematics is done by hand. Technology is there as a tool for confirmation of hand-done mathematics and not for doing institutionally-recognized mathematics. Ontologically speaking, mathematical outputs produced by a computer are not part of the official mathematical narrative. Indeed Eva, who takes the leading role (Sfard, 2007) in the discourse, declares right at the beginning of the task, "We will do it sketching first and then we'll check on GeoGebra" (line 3). This attitude was surprising: this was the sixth week of this course and my colleague and I had stressed that technology could be used as a powerful tool in doing mathematics. Its narratives were mostly endorsable although some care had to be taken when interpreting its outputs. An alternate explanation for the privileging of the hand-drawn graphs is the students' reading of the cautionary statement in the task: "Be careful GeoGebra isn't perfect here". This statement was intended to alert the students to the fact that GeoGebra did not reveal removable discontinuities in its graphs of functions. But the statement may have led to an undue mistrust of GeoGebra and may have reinforced the belief (discussed previously) that narratives of computer generated mathematics are not consistent with the official mathematical narrative. Indeed Eva implicitly justifies the hand-sketching of all graphs by invoking this cautionary statement right at the beginning of the task (line 3) and she refers to the warning a further three times while doing the task (for example, line 39).

Visual Mediation

Eva and Tom do not exploit the visual mediation that the graphs of GeoGebra afford. Although GeoGebra generates a visual picture of the graph as if it were a continuous graph rather than a graph with a discontinuity at x=1, it was expected that students would use this graph together with algebraic reasoning, ie $y=\frac{x^2-1}{x-1},=\frac{(x-1)(x+1)}{x-1}$

$= x+1, x \neq 1$ to recognize that x=1 is a point where the function is not defined. So why do Eva and Tom prefer to use hand-drawn graphs rather than computer-generated graphs as visual mediators? Possibly and as previously discussed, Eva and Tom may not accept that GeoGebra graphs can be used in endorsed mathematical narratives. Or they may need to **see** the 'hole' in the graph; a hole which they are able to represent directly on hand-drawn but not on the computer-generated graphs on the screen.

Routines

Eva and Tom execute several routines, each of which they repeat for y_1, y_2, y_3 and y_4. Specifically they use hand graph-sketching techniques such as algebraic simplification, calculus and point-by-point plotting to draw the four functions. As has been discussed, in a context in which the students have access to a tool which they can use to sketch functions (albeit with some imperfections), executing routines to hand-draw the graphs is an inappropriate activity. In this case, I suggest that applicability conditions, that is, the rules that demarcate when a particular routine should be applied (Sfard, 2008, p. 209) are unclear to the students. This may be aggravated by the (unendorsed) narratives which GeoGebra outputs.

Also, when simplifying the expressions for y_1, y_2, y_3 and y_4, the students do not explicitly state where the function is undefined. For this reason, the routines that they execute with regard to simplification, contradict endorsed mathematical narratives. For example, Eva writes: $y = \frac{x^3-1}{x-1}$, $f(x) = \frac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1$. She does not indicate that $x \neq 1$ although she and Tom acknowledge that $x \neq 1$ several times throughout the task, for example, lines 7-8, 36.

A further (non-endorsable) routine involves the students' sketch of the vertical line x=1 to indicate a vertical asymptote at x=1 (see Figure 1). Eva and Tom assume, presumably because the function is not defined at x=1, that x=1 is an asymptote. See for example, lines 7 – 11 and lines 310 – 311.

310 E:	But now again our asymptote is going to be at *x* is equal to 1, so it's not going to pass through *x*. Our graph is not equal to 1. *x* will not equal to 1.
311 T:	This 1, you see.

In the next section, headed 'Words', we see how an intervention by the researcher triggers a change in students' discourse with respect to their use of the term 'asymptote at x=1'.

Words

Initially Eva and Tom use the term 'asymptote' to describe a point (in this case, x=1) where the function is not defined (see, for example, see lines 7 – 11 and 310 – 311). They draw a vertical asymptote at x=1 (although they also talk about and draw a hole at x=1) for y_1, y_2 and y_3. But an intervention by the researcher (described in section headed 'Description of events') triggers a change in Eva and Tom's discourse and after this intervention they no longer talk of a vertical asymptote when referring to an undefined point. For example, when discussing y_4, they specify that x=1 is not an asymptote (lines 420-422). Also, in response to the question: “Is x=1 an asymptote”, Eva (the leader of discourse) presents a written and cogent argument in which she argues that x=1 is not an asymptote since the factor ‘x-1’ in numerator and denominator cancels out. Arguably, it was necessary for there to be a change in discourse (evidenced by the cessation of the use of the term ‘vertical asymptote’ to describe every point at which the function is undefined) before Eva could generate an endorsed narrative which describes the distinctive feature of a point at which the function is not defined but at which there is no vertical asymptote. Here we see how a change in discourse, in this case in word use, constitutes learning.

DISCUSSION

An arguable weakness of the above analysis is that it focuses on the learning of the pair of students rather than on individual students’ learning. In particular it does not illuminate what learning takes place for Tom with respect to the distinction between a vertical asymptote and a hole. However it signals ways in which non-learning may be covered up when students work in pairs in which clear roles are not defined. These interactions are the subject of a further paper.

Nonetheless in this research report we see how Sfard’s framework brings into sharp relief key aspects of the pair’s mathematical discourse (words, visual mediators, narratives and routines). This focus allows an enhanced understanding of both the expected and the unexpected activities of the students, as a pair.

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DEVELOPMENT OF QUALITATIVE AND QUANTIVATIVE INSTRUMENTS TO MEASURE BELIEFS OF PRE-SERVICE TEACHERS ON MATHEMATICS

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As a means of preventing students in pre-service teacher training from developing a static view on mathematics which comes along with a teaching based on rote learning we have implemented a course in reflexive problem-solving including extensive journal writing. The research project 'Mathematics teachers as researchers' is looking into the effects of such a course. This article will give an insight into the development of the complementary set of instruments we were using to measure beliefs about mathematics and their change. First we will present a process of empirical optimisation of Likert-scales and secondly we will discuss the results of an empirical comparison between those classical scales and the so-called "semantic differential".

INTRODUCTION

This study originated from the discussion of differences in Germany's primary, secondary and upper secondary mathematics teacher education – especially the different emphasis on content knowledge and pedagogical content knowledge (Shulman 1986). Teachers who mainly had enrolled in content-related studies in their university phase experience a gap between their studies in mathematics and their every day practice at school (Terhart, Czerwenka, Ehrich et al. 1994). Furthermore "the way most of them studied mathematics leads to a static view on mathematics (…) which is also the reason for the claim that teaching in many cases might be very superficial and concentrating on the rote learning of some procedures and techniques" (Pehkonen & Törner 1999, p.271).

A course model that aims at improving teachers' competences, especially at changing their beliefs about mathematics is a reflexive problem solving course which includes extensive journal writing. This kind of course has already been performed by teacher educators (Lester, Masingila, Mau et al. 1994) though its outcomes have rarely been studied empirically. Therefore the project *Journal writing as an instrument for a self-reflexive development in professionalism in content and pedagogical content knowledge of (pre-service) mathematics teachers*' (University of Education Freiburg, University of Freiburg, Germany; funded by the Federal Ministry of Education and Research Project number: 01JH0913) addresses as a central research question: Which effect does a problem solving course based on journal writing have on the participants' beliefs about mathematics and mathematics teaching?

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This question also comprises the methodological question of how such beliefs can be measured adequately. Hence our further goals were to optimize existing instruments according to the requirements of the study and to find out how different instruments complement each other. This paper will give an insight into the process and the results of this development.

THEORETICAL FRAMEWORK

Some remarks on Beliefs

First of all there is no distinct definition of what is meant by beliefs. Goldin gives a short definition which states that "Beliefs are defined to be multiply-encoded cognitive/affective configurations, to which the holder attributes some kind of truth value (e.g. empirical truth, validity, or applicability)" (Goldin 2002, p.59). Beliefs are seen as a structure of affects (Grigutsch, Raatz & Törner 1998) which is expressed by the term belief systems (Green 1971). These systems are not fixed but "dynamic in nature, undergoing change and restructuring as individuals evaluate their beliefs against their experience" (Thompson 1989, p.130).

Grigutsch et al. (1998) contrast 'mathematics as an action' vs. a 'static view on mathematics'. Studies indicate a congruence between the teacher's beliefs and the teaching practice of a teacher (Thompson 1992). As mentioned above belief systems are dynamic in nature and undergoing change, still the change of deep-rooted conceptions can be considered one of the main problems in mathematics teacher education (ibid.). As already described above, several authors allege that pre-service training can initiate a change of teachers' beliefs (DeBellis & Rosenstein 2004).

Measuring Beliefs

Beliefs have been measured and analyzed by means of a wide variety of techniques. Researchers have been using Likert-scales, semantic differential scales, interviews, observations, content analysis of journal entries, repertory grid techniques amongst others. In German-speaking countries some established Likert-scales are frequently used. Baumert, Blum, Brunner et al. (2009) e.g. use scales in COACTIV which are called 'Mathematics as a system', 'Mathematics as a process', 'Mathematics as a toolbox' and 'Platonist conception of mathematics' (Baumert et al. 2009, Köller, Baumert & Neubrand 2000). This instrument focuses primarily on the cognitive and conscious components of beliefs whereas Semantic differential (SD) scales tend to measure unconscious and affective, i.e. associative-connotative aspects of beliefs (Osgood, Suci & Tannenbaum 1978, Stahl & Bromme 2007). In a SD a series of bipolar adjectives are listed with the same number of divisions (usually 7) between each pair. Stahl & Bromme (2007) have developed a semantic differential called Connotative Aspects of Epistemological Beliefs (CAEB) They have found a two factor solution and labelled it 'Texture' (structure and accuracy of knowledge) and 'Variability' (stability and dynamics of knowledge) (ibid.).

Both types of scales have the advantage that they can be applied to a large sample and that they can be evaluated statistically. But it is disadvantageous in that the scales anticipate certain concepts of beliefs in advance, so that one can have doubts about their validity. By contrast, interviews and journal entries uncover more individual views so that concepts do not have to be given. They can be analyzed through qualitative methods. Its disadvantage is often a restricted comparability of the results due to a lack of standardisation and a restricted sample size by reason of the time and effort that is needed to analyze the data.

Concept of a reflexive problem solving course based on journal writing

The pre-service teacher course we implemented can be compared to the one reported by DeBellis and Rosenstein (2004) who practiced a problem-solving approach in the Leadership Program in Discrete Mathematics to give teachers the possibility to be learners themselves. (DeBellis & Rosenstein 2004). This concept of reflexive problem-solving courses is closely associated with journal writing. In our case we call the journals ‚research journals'. In order to induce reflection, a set of questions is needed to prompt the students' self-reflection (Brouer 2007). In our case the students are asked to reflect on their problem -solving process and on the change of their view on mathematics.

In order to understand the character of the intervention it is important to define the concept of 'problem' we use: „a problem is a situation that differs from an exercise in that the problem solver does not have a procedure or an algorithm which will certainly lead to a solution" (Kantowski 1981 in Heinrich 2004, p.55). The problems were chosen following certain criteria considering the specific objectives of the course. One of the problems used in the pilot study that exemplifies these criteria is 'Stairnumbers' (cf. Mason Burton & Stacey 1991):

> Problem 3: Which are the numbers you can write as the sum of consecutive integers (e.g. 12 = 3+4+5)? Can you tell which numbers can be written in which different ways? When you have worked on the problem to your satisfaction, ask some questions, e.g. "What happens if…?" or vary the problem.

The required prior knowledge for such problems is low, so that the students can easily start with working on the problem. The problem question opens different ways for discovery. The open-endedness is indicated by the last sentence of the instruction.

PILOT STUDY

Methodology

During the seminar 'problem- solving' the students have been working on seven problems during 13 weeks by writing all their ideas and calculations into their journals. These journals reached a volume of about 100 pages. At the end they were asked to reflect on their experiences with the specific problem, with problem- solving in general and on the change in their beliefs about mathematics.

Amongst others, there were two questions in the focus of the study's interest: On the one hand how the students' beliefs changed during the problem solving course, and on the other hand we were particularly interested in the validity of the questionnaire in use. This article will give an insight in the results of the second question.

For the collection of quantitative data we used established Likert-scales in pre- and post-test. They relate to "Beliefs about mathematics" (Grigutsch et al. 1998, Köller et al. 2000, Baumert et al. 2009) and "Beliefs about teaching mathematics" (Köller et al. 2000, Baumert et al. 2009). Additionally, we used the CAEB (Stahl & Bromme 2007) with the scales 'Variability' and 'Texture' for mathematics as a science.

In order to validate and to optimize the scales we analysed the written reflections of the students to the question as to how their view on mathematics had changed. The texts were analysed by 'summarizing qualitative content analysis' according to Mayring (2000) regarding the research question: Which aspects of beliefs on mathematics do the students bring up on their own? In a second step these categories were used to improve the scales.

Results

In the group participating in the course (N=63) there was a significant change away from the view of 'Mathematics as a toolbox' and towards the view of 'Mathematics as a process'. Because of the weak reliability coefficients of some scales they will be modified for the main study in order to detect differences between groups of students (for the results cf. Bernack, Holzäpfel, Leuders & Renkl 2011 in press). This can probably be amended by simplifying the respective questions.

The results of the summarized content analysis of the reflection sections of the research journals are based on N=10 students. Beside five other categories (cf. Bernack et al. 2011), we summarized a large part of the statements under the category 'Mathematics as an activity' with reference to Grigutsch et al. (1998) first, but the frequency was very high and they were not coherent. For that reason, the statements have been summarized under the categories 'Dynamic view on mathematics', 'Activities when doing mathematics' and 'Individuality while doing mathematics'. Going back to the questionnaire, one finds that the categories 'Dynamic view on mathematics', 'Activities when doing mathematics' and 'Individuality in doing mathematics' are all found in the same scale 'Mathematics as a process'. Thus, the number of mentions and their diversity in the qualitative analysis motivate a re-composition of the categories in the scale rather than using the overly comprehensive scale 'Mathematics as a process'. Based on the journal entries we developed some new items for the scale 'Mathematics as a process', changed some of the old ones and divided the scale into the subscales 'Mathematics as a dynamic science' and 'Activities/ individuality when doing mathematics'. The instrument could thus be optimised with regards to measuring the effects of the course during the main study so that we could begin with its implementation.

MAIN STUDY

Methodology

In the main study which took place in summer 2010 we implemented the new scales and in addition to measuring the overall effect on beliefs we added an experimental variation to examine the relative effect of journal writing on one hand and a lecture-oriented component on the other hand. While a control group was given "classical" lecture on 'Mathematical Thinking' with the contents problem- solving, proving and arguing and generating concepts, the treatment group had a reflexive problem-solving course based on journal writing as described above. For ethical reasons the intervention changed after half of the time.

Group	Week 1	Week 2-5	Week 6	Week 7-10	Week 11
1 N=37	Data collection 1 (t_1) Reflection	Journal writing	Data collection 2 (t_2) Reflection	Lecture	Data collection 3 (t_3) Reflection
2 N=33		Lecture		Journal writing	

Table 1: Main study summer 2010

While the main goal of the main study – analysis of the relative effects including identifying moderator effects – is yet to be carried out, the focus of this article is on the instruments in use. Can the reliability coefficient be improved after optimizing the scales? Furthermore we were interested in the relation between the two kinds of scales in use: the scales which we optimized during the pilot study concerning beliefs on mathematics and the scales of the CAEB, also part of the questionnaire. We were asking if they measure the same construct and in which way they differ in order to know more about instruments measuring beliefs and their relationship. Especially the scale 'Variability' and the scale 'Mathematics as a dynamic science' are very similar in their word choice. Because 'Variability' encompasses the degree of dynamics of mathematics as a science, it is very closely linked to the aspect of process. Thus, we hypothesise that there is a strong positive correlation between both of them: The same assumption applies to 'Texture' and 'Mathematics as a system' due to the fact that both include the structure of mathematics. Taking into account that the CAEB is rather measuring underlying concepts (see above) it will be interesting to see how the different types of scales react to the intervention.

Results

Regarding the reliability of the scales, the adjustment of the whole scale 'Mathematics as a process' in the course of the pilot study was successful. The reliability coefficient has increased considerably and the subscales also reached satisfying values. For the sake of completeness, Table 2 shows the results for the other scales measuring beliefs on mathematics.

	Pilot study		Main study		
Scale (Abbreviation, number of items)	t_1 N=63	t_2 N=63	t_1 N=154	t_2 N=109	t_3 N=72
M. as a process, pilot study (4)	.264	.678			
M. as a process optimized (MP, 14)			.830	.838	.889
Subscale M. as a dynamic science (DS, 6)			.679	.724	.817
Subscale 'Activities/ individuality when doing mathematics'(A, 7)			.733	.754	.854
Mathematics as a system (MS, 7)	.659	.664	.754	.768	.743
Mathematics as a toolbox (MT, 4)	.691	.538	.681	.726	.743
Variability (V, 7)	.634	.700	.766	.802	.827
Texture (T, 9 of 11)	.784	.791	.848	.858	.848

Table 2: Reliability coefficients (Cronbachs α)

Table 3 indicates the correlation between the scales measuring beliefs about mathematics. As assumed there is a large positive correlation between the SD scale 'Variability' and the scale 'Mathematics as a process'. When you take the subscales into account, the bigger similarity between scale (DS) and (V) becomes apparent, so that 'Variability' seems to be measuring rather the characteristics of mathematics as a science (which can be dynamic or not) than being in a close relationship to typical process-like and dynamic activities. Nevertheless, (V) and (A) correlate with a medium positive value. The results also corroborate the hypothesis of the correlation between 'Mathematics as a system' and 'Texture'.

	(A)	(MS)	(V)	(T)
Mathematics as a process (MP)		-.258**	.501**	-.267**
Mathematics as a dynamic science (DS)	,633**	-,332**	.581**	-.319**
'Activities/individuality when doing mathematics' (A)		-,165*	.364**	-.188*
Mathematics as a system (MS)			-.445**	.478**
Variability (CAEB) (V)				-.596**
Texture (CAEB) (T)				

Table 3: Pearsons product-moment coefficient r (Main study, T1); N=150; *p <.05; **p < .005

DISCUSSION

The results indicate that the CAEB can be seen as an alternative instrument to measure beliefs about mathematics, especially the aspects ‘Mathematics as a process’ and ‘Mathematics as a system’. Especially the medium correlation coefficient between ‘Texture’ and ‘Mathematics as a system’ is an astonishing result because there are no strong similarities in the word choice of the two scales as in the aforementioned case. Concerning the scale ‘Variability’, however, we have to recognize that it does not take process-like activities into account to the same degree as the subscale ‘Activities/individuality when doing mathematics’ does. It seems to measure rather the descriptive characteristics of mathematics. One advantage of the CAEB is its stable reliability coefficient. Because it tends to measure more connotative attitudes, it will be interesting to see in further analyses which instrument reacts to which part of the intervention in which way. These analyses also can point to further differences between the instruments. These results should make it possible for further studies to choose the adequate instrument to measure beliefs about mathematics knowing its advantages and disadvantages.

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MIDDLE GRADES STUDENTS' EMERGING BELIEFS ABOUT ARGUMENTATION

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Students learn norms of proving mathematically by observing teachers generate proofs, engaging in proving, and generalizing features of proofs deemed convincing by their mathematics instructor or textbook. This study investigated aspects of proofs and non-proofs that were convincing to middle grades students. Fourteen students completed proof evaluation items in number theory and geometry. In each item, both an empirical and a general argument were provided. Students tended to prefer the empirical argument for the number theory statement but valued the explanatory power of an argument when evaluating a proof for a true geometry statement that provided a diagram. Data analysis yielded a descriptive model to illustrate the factors that middle grades students' value when evaluating arguments.

INTRODUCTION

Some mathematics educators define proof as the process one undertakes to remove doubt, or convince oneself and others that a statement is true (Harel & Sowder, 2007). Therefore, learning to do mathematical proof involves adopting the notion of a convincing argument in the discipline of mathematics– namely, an argument constructed with general, established premises in a rigorous and logically deductive fashion (National Council for Teachers of Mathematics (NCTM), 2000). As middle grades students (ages 12-14) have yet to learn the norms of proving in mathematics, this study investigated their emerging conceptions and proof practices to learn what they bring to more formal experiences with proof in settings such as a high school geometry course or in undergraduate advanced mathematics coursework.

Students often learn the norms of proving in mathematics by observing proofs and generalizing features of those proofs deemed to be correct by a mathematical authority, such as their mathematics instructor (Ernest, 1999). Existing work documents that both high school and undergraduate students, even undergraduate mathematics majors, have difficulties distinguishing mathematically correct justifications and proofs from non-proofs (Harel & Sowder, 1998; Healy & Hoyles, 2000; Selden & Selden, 2003; Alcock & Weber, 2005). Weber (2009) acknowledges: "The lack of research on how students *do* read mathematical arguments, as well as how they *should* read them, represents an important void" (p.2). Therefore, understanding what students notice and what they value when evaluating mathematical arguments, especially students who are novices in doing mathematical proof, can support instructional interventions that highlight important distinctions between proofs and non-proofs.

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This study builds upon an emerging research base on students' reading and evaluating of mathematical proof by addressing the following research questions: *What features of mathematical arguments convince students, as evident when they evaluate proofs and non-proofs? To what extent do students utilize these features when modifying arguments to be more convincing?*

LITERATURE REVIEW

Numerous studies indicate that students tend to generate empirical arguments when proving, at the middle school level (Knuth, Choppin, & Bieda, 2009) and beyond (Healy & Hoyles, 2000; Senk, 1985, Weber, 2001). It may be the case that students have difficulty generating mathematically valid proofs because the organization of their mathematical knowledge and the validity of the definitions, theorems, and deductive rules they use has been determined solely by a mathematical authority (a teacher or textbook) and not debated as a community of practice. Balacheff (1987) states that a mathematical proof, "requires a specific status of knowledge which must be organized in a theory and recognized as such by a community. The validity of definitions, theorems, and deductive rules is socially shared" (p. 30). Some existing work has examined the implementation of proof-related tasks from Standards-based curricula at the middle grades level (Bieda, 2010), suggesting that little instructional emphases is placed upon generating and analyzing definitions as a classroom community. Providing example cases to illustrate or verify an explanation or definition is likely a normative practice of mathematics classrooms.

This study explored students' conceptions of proof by eliciting students' evaluations of both proof and non-proof arguments, and also examining the relationship between what students choose as a convincing argument and the modifications they make to an argument that is not convincing. Both kinds of activity – evaluation and production – require students to draw upon their emerging beliefs about mathematical proof and reveal how their conceptions of proving practice aligns with the role of proof in the discipline.

RESEARCH DESIGN

Data collection consisted of obtaining students' responses to proof evaluation tasks using one-on-one, videotaped interviews. These interviews were a part of a larger exploratory study that included administration of written assessments to approximately 200 students, of which the interview participants were a sub-sample, to understand students' proving practices across mathematical domains.

Interviews

Twenty-five 7th grade students (ages 11-12) attending the same junior high school from a small, suburban, Midwestern district in the United States voluntarily participated in interviews. The author conducted the interviews at the students' school over the course of three days. Because of the exploratory nature of this study, different items and interview protocols were used between Day 1 and Day 2 and 3 interviews to determine the best measures. For this paper, analyses focused on students' responses for Days 2 and 3 across the same set of items. Therefore, out of 25 interviewees, only the responses of 14 (those participating in interviews in Day 2 and 3) were considered in analyses for this paper.

The proof evaluation items for both number theory and geometry are shown in Figures 2 and 3. The interview protocols contained, among other items, tasks asking students to choose between an empirical justification and a more general, deductive justification as more convincing in showing a statement to be true. Students reviewed both justifications at the same time, however each argument was printed on a separate page. I posed the following questions to the student about the arguments, in the order given: 1) Which response convinces you that the statement is true? 2) Why is the response you chose more convincing than the other response? 3) If you were to give advice to the student who wrote the response you didn't choose as to how they could improve their response to make it more convincing, what would you tell them?

Data Analysis

Students' responses to the proof production items in the written assessments were transcribed and analyzed using *HyperResearch*. The first step in analyzing students' interview responses involved coding students' responses to Question 1 either as Examples-Based or EB (for choice of Response A) or General or G (for choice of Response B). To analyze Question 2 and 3, the research team used a

The teacher says the following is a mathematical fact: **When you add any two consecutive numbers, the answer is always odd.** Two students offer their explanations to show that this fact is true.

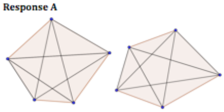
Response A:
5 and 6 are consecutive numbers, and 5 + 6 = 11 and 11 is an odd number. 12 and 13 are consecutive numbers, and 12 + 13 = 25 and 25 is an odd number. 1240 and 1241 are consecutive numbers, and 1240 +1241 = 2481 and 2481 is an odd number. That's how I know that no matter what two consecutive numbers you add, the answer will always be an odd number.

Response B:
Consecutive numbers go odd, even, odd, even and so on. So if you take any two consecutive numbers, you will always get one even and one odd number. And we know that when you add any even number with any odd number the answer is always odd. That's how I know that no matter what two consecutive numbers you add, the answer will always be an odd number.

Which response convinces you that the mathematical fact is true? Explain your reasoning.

Your textbook has the following mathematical fact about pentagons (a pentagon is a shape with five sides): **All pentagons have 5 diagonals.**
Two students offer their explanations to show that this fact is true.

Response A



I drew two pentagons, both different, and they both had 5 diagonals. So, no matter what pentagons I draw, they will have 5 diagonals. The fact is true.

Response B
Pentagons have five sides and five vertex points that connect the sides. To make a diagonal, you just draw a line segment between two vertex points that are not connected, like this:

If you draw a diagonal from each vertex point, each vertex point will be connected to another. Since there are 5 vertex points, and every point is connected to another point by a diagonal, a pentagon has only 5 diagonals.

Figure 2 & 3: Proof Evaluation Tasks for Number Theory and Geometry Statements

constant-comparative method of coding (Strauss & Corbin, 1990) to develop descriptions of aspects of the responses that students noted when describing why they chose a particular argument as convincing.

RESULTS AND DISCUSSION

Features of Convincing Arguments

For middle school students in our sample, examples on their own were often insufficient, and a convincing argument is one that *shows (with examples) and tells (why)* the statement is true to the reader (represented as quadrant I in the model given in Figure 4). This echoes some findings of Chazan (1993), which indicated that high school students prefer mathematical proofs for geometry statements to utilize both explanation and examples.

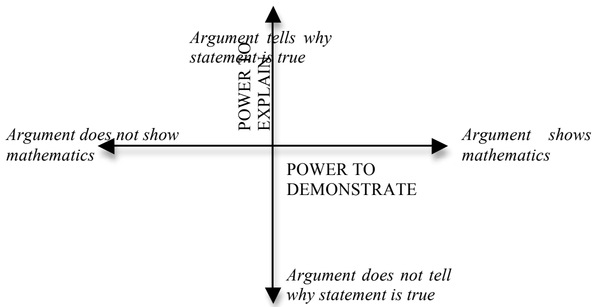


Figure 4: Factors of mathematical arguments that convince students

Arguments Chosen as Convincing

To answer the first question for the number theory and geometry items (see Figure 2 and 3), students chose the response that was more convincing between the general (G) and the examples-based (EB) argument and provided a rationale for their choice. Given the limitations of a small sample size, it appears that

Task	Examples-Based (EB)	General (G)	Both	Neither
Number Theory	11	2	0	1
Geometry	7	5	2	0
TOTAL	18	7	2	1

Table 1: Response chosen across tasks (Day 2 and Day 3 interviews)

students are slightly more likely to choose the G argument as convincing for the geometry task than for the number theory task (5 responses compared to 2 responses). If responses from Day 1 are included, 8 students chose the G argument as more convincing than the EB argument in the geometry task compared to only 4 students for the number theory task.

Mathematical Arguments Must "Show" The Mathematics…And Explain Why

In response to Question 2 of the interview protocol, students explained why they chose one justification as more convincing. In some responses, it was clear that students understood both the limits of just providing examples and the need for a justification to explain, mathematically, why the statement is true:

> Michael: Because it's [Response B, number theory item] explaining it. It's like explaining it other than response A which just picked random numbers showing that it works and B is explaining what it means and how and why it's like that … because it could work for them and not work for others.

Students also attended to the quality of the explanations provided. For the geometry item, Michael chose the EB argument as more convincing, explaining:

> It's [Response A, geometry item] actually showing you like there is five and it's got two different examples to it. And in this one [Response B, geometry item], it's kind of, it's explaining it but it's not doing it as well."

As exemplified by Michael's responses, students evaluated arguments based on two primary categories: power to demonstrate and power to explain. Students' responses indicated that a convincing argument is one that provides a concrete instantiation of the mathematics – a visual containing either a diagram or a numerical example – as stated below by another student:

> I'd say response A [number theory item] because it actually gives you examples. Like response B does but it doesn't show you… well, if you read it, it tells you it but it doesn't show you… Because if you're like a visual person, like me, you have to see it on paper.

Students also attended to the quality of the explanations provided. In comparing Michael's responses across the number theory and geometry items, it seems as if he contradicts himself; for the number theory item, examples are inadequate but for the geometry item, the examples are necessary. He clearly understands that both items present an EB argument as a possible proof. He acknowledges that Response B in the geometry item offers an explanation, but is "not doing it as well" as Response A. Therefore, one possible explanation for Michael's choice of Response A for the geometry item is that he prefers the EB response. On the other hand, he may have chosen Response A simply because Response B gives an unsatisfactory explanation.

Improving Arguments To Be More Convincing

Analyses of Question 3 of the protocol offered an opportunity to determine whether students would be consistent in applying their beliefs about convincing arguments as modeled in Figure 4. We found that students tended to revise arguments using a *show and tell* strategy; that is, if they chose the EB argument, students indicated that the G argument could be improved by including examples (*show*). On the other hand, if a student chose the G argument, they indicated that the EB argument could be improved by providing facts or more explanation in words (*tell*).

A total of 18 students (see Figure 4) chose the EB argument as more convincing across responses on the geometry and number theory items. Not surprisingly, each of the eleven students who chose the EB justification for the number theory item recommended improving the G argument by including examples to create “show and tell”–type arguments. To support their modifications, students claimed that the examples showed the fact, while the general argument explained it. The students indicated a need to “show” or visualize what the statements were referring to in the G argument, so that the reader could actually “see it’s true.”

> If you mixed these two up a little bit, they would pretty much be, like, really good.
>
> The words are like telling you what they’re showing you in the pictures, not just like…like, why.
>
> Give an example. And make sure you’re being specific about this and not that.

Only seven students (see Figure 4) chose the G argument as the most convincing response across both the number theory and geometry items. When asked how the EB response might be improved to make it more convincing, all students stated that there was not enough explanation and suggested adding facts to support the examples. As one student stated: “[Since] any even plus any odd number is an odd number, [they] might want to say that in here somewhere, because that would make more sense to a person who’s trying to understand.” Students also noted the importance of including technical language in the argument, particularly for the geometry item: “They don’t even have the word vertex point, like these sort of points [circles]. They don’t even talk about those when this does. And it doesn’t even say that the pentagon has, like, five sides.”

CONCLUSION

Students’ preferences for “show and tell”-type arguments highlights, as Harel (2006) claimed, the interdependence between how students ascertain the truth of a statement and how they persuade others of the truth of the statement. Although existing work documents students’ tendencies to produce examples-based justifications, the findings of this study suggest that students are not depending solely upon examples to do the work of justifying and proving but are instead employing examples as rhetorical devices to demonstrate the mathematics of the statement being proven.

Further, analyses of students’ responses to proof evaluation items suggest that students may use a different proof scheme to persuade than the one they use to ascertain. The evidence indicating students’ preferences for arguments that provided a concise, yet adequate, explanation of why the statement was true, even when they argued that a proof could be improved by adding examples, implies that an argument must explain to convince them (or others) that a statement is true. While the interview items were not specifically designed to distinguish between the proof schemes students used to ascertain and persuade, the results of this work raise questions as to whether students apply the same scheme when engaging in both processes when proving.

Although this work was designed to be exploratory in nature, these findings expand our notion of what students attend to when evaluating whether or not a mathematical argument is convincing. Although students accept the use of examples in mathematically convincing arguments, they also value an argument's power to explain. The ability of a proof to explain why something is true is one of the fundamental functions of proof (Hanna, 2000) and arguably one of the most important conceptions for students to retain as they continue learning mathematics for understanding.

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TWO TEACHERS AND TWO DIFFERENT WAYS OF HANDLING STUDENTS' DIFFICULTIES DURING MATHEMATICAL TASKS IMPLEMENTATION

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This paper investigates two elementary mathematics teachers' ways of handling students' difficulties during the mathematical task implementation. Two teachers' mathematics lessons are video-recorded, transcribed and then analysed. The data is analysed in terms of teachers' ways of handling difficulties encountered in the classroom. The analysis reveals that one teacher mainly discusses the difficulties while the other totally ignores them. We discuss why this was the case and suggest that the way the task is designed imposes some constraints on how teachers handle students' difficulties and it is hence important to examine teachers' responses to students 'difficulties during the task implementation.

INTRODUCTION

Thanks to the influences of socio-cultural and constructivist ideas that value students' active participation in their learning, considerable attention has been paid to mathematical task design and implementation in the last three decades in mathematics education research (Tzur, Zavlavsky & Sullivan, 2008). Tasks are viewed to offer more than just what to be learnt and that they are considered to structure and shape the way students think as well (Henningsen & Stein, 1997). The way tasks are designed, selected and implemented has hence deep influence on the quality of learning.

Effective task implementation is not an easy endeavour. Unlike traditional ways of conducting the teaching, teaching through tasks requires a great deal of work on the part of the teacher. Such factors as students' prior knowledge, students' learning difficulties, classroom organization, instructional materials, instructional methods and strategies, role assigned to the teacher and roles assigned to the students need to be taken into the consideration if tasks are planned to be used and all these require teachers' readiness and planning beforehand (Swan, 2008; Marx & Walsh, 1988; Henningsen & Stein, 1997). These and similar studies suggest that such factors deeply affect the way a task is implemented and even sometimes they can cause the task not to be implemented the way it is intended at all.

Students' difficulties and misconceptions with the content to be covered is one factor that needs closer scrutiny during the task design and implementation (Swan, 2008). It is important to closely examine how teachers handle students' difficulties and

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misconceptions during the task implementation as their handling types can shape the direction of task implementation. This line of research appears to have received little attention and, in this study; we focus our attention on how two teachers handle students' mathematical difficulties regarding the concept to be learnt during the task implementation and examine the role of the nature of the task on their handling types.

THEORETICAL FRAMEWORK OF THE STUDY

Although not particularly with regard to the task implementation, the issue of handling students' mathematical difficulties has long been the focus of attention. Given that students' difficulties mainly manifest themselves through errors, Ball (1991) points out that errors can be a window into students' ways of thinking and understanding and therefore teachers need to go beyond "right" and "wrong" answer in order to find out the conceptualizations behind the errors. In agreement with Ball, Borasi also (1994) suggests that teachers make use of mistakes as "springboards for inquiry" so that students have the opportunity to get involved in fruitful discussions regarding mathematical concepts. Kazemi (1998) has provided evidence that using errors as "springboards for inquiry" and hence their discussions may cause greater attainment in students' mathematical learning.

A close examination of these and similar studies suggest that discussion of students' errors in the classroom can be fruitful for their mathematical learning. The existing literature, however, reveals that teachers handle difficulties in a number of different ways. In a study on comparing teachers' responses to student mistakes in Chinese and US mathematics classrooms, Schleppenbach, Flevares, Sims and Perry (2007, p. 131) report that US teachers "made more statements about errors than the Chinese teachers, who instead asked more follow-up questions about errors". The Chinese teachers were found to encourage their students to work through their errors more than providing the correct answers immediately after the error occurred. In their analysis, Schleppenbach et al. (2007, p.136) note that teachers respond to students' errors in two distinctive way: i.) making the statements after the error, or ii.) handling the errors with follow up questions. "Telling the student the answer is wrong", "giving the correct answer" and "ignoring the error" are examples of the former type of responses to the errors while "re-asking the question", "clarifying the question", "redirecting the question" are the examples of the latter (ibid.).

In another study, Santagata (2005, p.505) works on Italian and US teachers' ways of handling students' mistakes and reports that "in both Italy and the US, teachers were only randomly observed to organize discussions around students' mistakes". Alongside the similarities, there were also some differences between Italian and US teachers' ways of handling errors. Italian teachers, for instance, were found asking the student who made the mistake to correct his error while US teachers were found asking a different student to correct his/her classmate's error (ibid.).

Similar studies have been conducted comparing Japanese and US teachers' ways of handling students' difficulties (Stevenson & Stigler, 1992; Stigler & Hiebert, 1999).

These researchers report that Japanese teachers used mistakes as sources of discussion and hence take the advantage of encountering them. They report that US teachers, in contrast, had the tendency to avoid the discussion of mistakes and were concerned with students' self-esteem. Unlike US teachers, Japanese teachers were found to integrate students' common mistakes into their lesson plans beforehand.

One can infer form these studies that teachers handle students' errors differently and sometimes teachers' ways of handling can change from one culture to another. In this study, rather than comparing teachers of different countries, we focus on two particular teachers' ways of handling students' difficulties during the mathematical task implementation. The reason that we particularly focus on two teachers' ways of handling difficulties during the task implementation is that every task has 'an agenda' and that may impose some constraints on how teachers handle students' difficulties. In this study, we wonder whether the task agenda has a role in determining how teachers handle students' difficulties encountered during the implementation.

CONTEXT OF THE STUDY AND METHODOLOGY

The theme of this paper emerged from an on-going project concerned with the professional development of elementary in-service teachers in Turkey. A group of 45 elementary teachers (15 classroom, 15 mathematics and 15 science and technology teachers) took part in the in-service teacher professional development programme. These teachers participated in a course on task design principals and implementation that lasted four weeks and it was run by the mathematics education researchers. In first two weeks of the course, teachers got training regarding the task design principles. Such principals as determining the purpose of the task, selection of materials, classroom organizations, students' prior knowledge, learning difficulties etc., were discussed with the teachers. For the third week, teachers were asked to bring a designed task (by themselves) for discussions and two such task plans were thoroughly discussed. After the third week training, all teachers implemented their designed tasks in their classroom and implementation of 9 teachers (3 teachers from each subject area) was video-recorded. Two such videos were used for discussion and recap in the fourth week of course.

In this study, we present the video analysis of two elementary mathematics teachers' task implementation. There were three elementary mathematics teachers and we chose the two as they designed their tasks from the same topic for the same purpose (consolidation). Both teachers implemented a consolidation task related to the solving of equations for 7th grade students (aged 13-14). Even though our aim for the training was to instruct teachers to design the task in light of the task design principals, in this study, we focus our attention on how they handle students' difficulties during the task implementation. We, here, therefore follow an emergent theme.

DATA ANALYSIS AND RESULTS

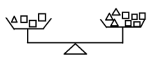
In this section, we present two teachers' (Teacher A and Teacher B) data analysis. We first present how we analysed the data and then provide results related to how teachers handled students' difficulties.

For the data analysis, we first examined both teachers' task plans. Both teachers designed their own tasks and briefly explained what they planned to do in their plans (we provide details for plans below). We then transcribed the video-recording of task implementation of the two teachers. In order to determine the way the teachers handled students' difficulties, we first identified where students had difficulties. For this, we had two criteria. First, when an error was encountered in a student's answer, we regard that as student having difficulty. Second, a situation of lull or student telling 'I don't know' for an asked question was also taken as student having difficulty. This process of identifying difficulties was carried out by the two authors of this paper and there was a consensus for every event regarded as difficulty.

After the identification of the difficulties, we carried on with determining teachers' ways of handling students' difficulties. In line with the related literature, we determined five categories of teachers' ways of handling difficulties: i.) presenting the difficulty to classroom discussion, ii.) asking the questions to the student who had difficulty to overcome his/her difficulty, iii.) ignoring the difficulty, iv.) giving the correct answer and v.) uncategorised. We assume that the first four categories are self-explanatory but uncategorised category stands for handling types that cannot be allocated to the first four. Note that more than one way of handling is used for some encountered difficulties.

Teacher A: Task implementation and ways of handling students' difficulties

Teacher A designed a consolidation task related to the solving equations and its implementation lasted 40 minutes. He started to implement the task by drawing a table on the board first including the following algebraic and verbal expressions: '2x', '3x+9', '$\frac{x}{2}$+4', 'five more than a number', 'three less than two times a number', 'half of a number', 'half of three less than two times a number', 'If five more than three times a number is equal to 15, what is the number?' He later asked the students to state algebraic expressions verbally and state verbal expressions algebraically. Alongside these questions, he drew three balance scales on the board and each scale represented an equation. Students were asked to state the equations and translation of figures on scales is as follows: 'x=3', '2x+2=8', 'x+1=2'. The teacher concluded the task with the following two questions: i) 'If five times three less than a number is equal to 25, what is the number?'; ii.) 'express the figure algebraically':



The task consisted of the above-mentioned questions. The video analysis, however, shows that some difficulties were encountered whilst these questions were solved. The analysis reveals that nine difficulties were encountered during the task implementation. Teacher A shared five of difficulties with the whole classroom and discussed them. In the case of two difficulties, teachers asked the questions to the student who had difficulty to overcome his/her difficulty. Teacher A only once directly gave the correct answer while one of his handling was not categorised.

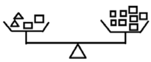
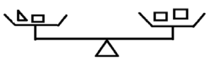
Teacher question	**Student answer**	**Teacher ways of handling the difficulties**
Three less than two times a number	Student-1: x.2- 3 Student-2: 2.x-3 Teacher asks the classroom: ‘Are they the same?’ Some students: ‘No!’	**-** Presenting the difficulty to classroom discussion -Asking the questions to the student who had difficulty to overcome his/her difficulty
Half of three less than two times a number	$2x - \frac{3}{2}$	Presenting the difficulty to classroom discussion
If five more than three times a number is equal to 15, then what is the number?	I do not know how to express this	Presenting the difficulty to classroom discussion
One student asked: “Sir! Can we write this ($\frac{2x-3}{2}$) as $2x - \frac{3}{2}$?”	One student has difficulty in understanding whether ($\frac{2x-3}{2}$) is equal to $2x - \frac{3}{2}$?	Presenting the difficulty to classroom discussion
 How can we express this as an equation?	x.2=6 (although the answer is correct, the student cannot provide justification)	Asking the questions to the student who had difficulty to overcome his/her difficulty
 How can we express this in terms of equation?	One student begins to express the equation as x+1…the teacher reminds that half of the triangle should be thought.	Uncategorised
If five times three less than a number is equal to 25, then what is the number?	One student: 3+5=8-3=5x 5=25	Giving the correct answer.

Table 1. Teacher A’s ways of handling students’ difficulties

Teacher B: Task implementation and ways of handling students’ difficulties Teacher B’s consolidation task is also related to the solving equations and its implementation lasts 40 minutes. He begins with explaining how the task is going to be carried out. Then task implementation unfolds as follows: He writes a question on the board and gives students a period of time to solve it. Students are organised as a group of four and they work together. Once the question is solved, then the group representative tells the answer by putting it on a paper showing to whole classroom. There are 9 groups and each group has a name. When the answer is correct, then the group gets a plus and otherwise a minus. The following questions are posed:

1. If x + 5 = 25 then x = ?, **2.** If 5x = 25 then x = ?, **3**. If $\frac{x}{5} = 5$ then x = ?

4. If 5x-5 = 25 then x = ?, **5**. If 5(x+5)=25 then x = ? **6.** If $\frac{x+5}{5} = 25$ then x = ?,

7. If 5(x+5) = x+45 then x = ?

After taking answers to each question, the teacher asked a student from a group to come to the board and solve it. He then wrote the following questions on the board and explained how they could be solved. But note that he left them unsolved.

1. x+5 = 25, **2**. 2. x – 5 = 25, **3**. 5x = 25, **4.** 5x + 5 = 25,
5. 5(x-5) = 25 **6.** (x+5)/ 5 = 2 **7**. 5(x+5) = x + 45

Following this activity, Teacher B counted the number of pluses and minuses that each group obtained and declared which group came first, the second and the third. The activity ended up with applauding the success of these three groups.

Teacher question	Student answer	Teacher ways of handling the difficulties
If x+5=25, then x=?	One group: 19; One group: 405	Ignoring the difficulty
If 5x=25, then x=?	Two groups: 20; One group: 2 One group: 30	Ignoring the difficulty
If x/5=5, then x=?	Two groups: 5	Ignoring the difficulty
If 5x-5=25, then x=?	One group: 35	Ignoring the difficulty
If 5(x+5)=25, then x=?	One group: 25; One group: 4 One group: 30; One group: 53	Ignoring the difficulty
If (x+5)/5= 25, then x=?	One group: 5; One group: 10 One group: 25	Ignoring the difficulty
If 5(x+5)=x+45, then x=?	One group: 9; One group: 20 One group: 15; One group: 8 One group: 70/6	Ignoring the difficulty

Table 2. Teacher B's ways of handling students' difficulties

As Table 2 suggests, even though 19 different errors were encountered, Teacher B consistently ignored the students' difficulties and did not handle them at all.

DISCUSSION

Both teachers designed a consolidation task related to solving equations and implemented their tasks the way they planned. These are, in fact, what they had in common. What is that they did not have almost anything in common is their ways of handling students' difficulties encountered during the task implementation. In Teacher A's lesson, nine students' difficulties were encountered. He discussed five of them with the whole classroom and did not ignore their existences. Teacher A used 'asking the questions to the student who had difficulty to overcome his/her difficulty' type of intervention as well. Once he also gave the correct answer immediately after the difficulty occurred. Teacher A hence handled all the difficulties encountered and discussed most of them. In the words of Borasi (1994), it can be said that Teacher A used students' difficulties as "springboards for inquiry" in his classroom.

Unlike Teacher A, Teacher B had a totally different way of dealing with the difficulties. In his lesson, difficulties were encountered on seven occasions and in total 19 different students' errors were observed. These errors are rather interesting in two aspects. First, all these errors were the results of collaborative works of a group four student. Second, we do not know what kind of students' thinking processes generated such errors because the teacher did not ask them to explain their answers. What is hence apparent for Teacher B is that he ignored all the difficulties and he was very consistent in terms of ignoring the difficulties encountered in his classroom.

Discussion of students' errors in the classroom has often been regarded as useful for students' learning. This is because errors are considered as sign of misconceptions and lack of understanding and hence overcoming them through discussion is necessary for conceptual learning (Ball, 1991). This perspective is especially valid for common errors. The discussion of errors in the classroom can be risky too. Their discussions, to us, can sometimes cause 'deviation (shift) of task agenda' and this might result in failure in completing the task and following the syllabus as planned.

Looking at the data from this perspective, Teacher A discussed the difficulties and at the same time did not allow 'agenda deviation' to occur. Teacher B did not allow agenda deviation for his task either. But it can easily be argued that Teacher B, in fact, needed an agenda deviation in his lesson because he was conducting a task for consolidation purpose and students still were experiencing serious difficulties. Given that students still had serious difficulties, one would expect that Teacher B would deal with them. Teacher B, instead, completed the task by ignoring 19 serious errors and at the end made the classroom applaud the winners of the task.

But this was so? Why Teacher A always dealt with students errors whilst Teacher B consistently ignored them? This is a complex issue for which we now do not have definitive answer. This, however, we think, can be related to such factors as teachers' pedagogies, beliefs, knowledge for teaching and so forth. Alongside these factors we also think that this is very much related to how the task was designed in the first place. Teacher A designed the task by keeping students' difficulties in mind, as he put notes in his task plan. Teacher B, however, designed his task to have a competitive environment and for that reason he had time limitation for every question in the task. To him, completing the task with deciding which group of students came first was a concern. The nature of the task, to a certain extent, hence imposed some constraints and led Teacher B to ignore students' difficulties. In the words of Stylianides and Stylianides (2008), Teacher B were more concerned with "fidelity of implementation of tasks" in that he exactly implemented the task the way he designed and planned. Although the low fidelity of the task is often criticised in the literature (Henningsen & Stein, 1997), this time it appears that fidelity to the task plan was a serious issue and in fact 'infidelity' was needed as the task was a consolidation one.

As a final point, our findings suggest that teachers' ways of handling difficulties during the task implementation is a complex issue and needs particular attention. The affordances and constraints that the task imposes as well as the way it is designed can deeply influence how teachers handle students' difficulties. It is hence important to examine teachers' ways of handling difficulties with regard to the way the task is designed and planned to be implemented.

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CHILDREN'S, YOUNG PEOPLE'S AND ADULTS' COMBINATORIAL REASONING

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In the present study, 718 participants in regular schooling situation (children and young people in lower or upper Elementary School or in High School), and adults (in initial schooling or in a professionalization course), solved eight combinatorial situations. Schooling had an effect on performance, although similar procedures were used both before and after instruction on Combinatorics. Distinct relations were represented by different procedures but not always in a systematic manner. Cartesian products were more easily understood and arrangements, permutations and combinations were very difficult, especially for adults in initial schooling. This suggests the need of considering in teaching varied meanings, relations and symbolic representations for a broader development of combinatorial reasoning.

COMBINATORIAL REASONING: IMPORTANCE AND PROBLEM TYPES

The present study aimed to investigate combinatorial reasoning in a wide range – throughout distinct schooling experiences and involving different problem types. In general, previous studies (Inhelder & Piaget, 1955; Schliemann, 1988; Bryant, Morgado & Nunes, 1992; Moro & Soares, 2006; amongst others) involved one or more types of problems (Cartesian products, arrangements, permutations or combinations) and were limited to one age range. Thus, the proposal of this study was to analyse students' – children and young people in regular schooling and adults in initial process of schooling and in professional High School – understanding of problems that involve combinatorial reasoning.

Batanero, Navarro-Pelayo and Godino (1997) defend that combinatory capacity is a fundamental component of formal reasoning, as described by Inhelder and Piaget (1955). It is argued that combinatorial reasoning is part of formal thinking because there is a need to deal with hypothetical situations in the raising of possibilities. Combinatorial reasoning is relevant in situations in which combination of elements, analysis and/or categorization is necessary. Thus, it is believed that the study of Combinatorics may be a means to develop logical reasoning and to aid general mathematical development of children and adults.

Combinatorial reasoning is understood in the present study as a kind of thinking that involves counting, but that goes beyond the enumeration of the elements of sets. In Combinatorics, groups of possibilities are counted, based on multiplicative reasoning, by systematic actions that attend the requirements of the different combinatorial

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problems. These strategies involve the constitution of groups of elements, the determination of possibilities and the direct or indirect counting of valid cases, considering choice, order and repetition of elements.

Previous studies and documents (Merayo, 2001; Nunes and Bryant, 1996; Vergnaud, 1983 and the Brazilian National Curricular Parameters – PCN – Brasil, 1997) classify combinatorial problems exclusively in *Cartesian products* (at Primary School) or as *arrangements*, *permutations* and *combinations* (at High School), but in the present study these situations are considered in one classification. This leads to the reflection of the need of considering these four types of problems in the teaching of Combinatorics from Primary to High School – both in children's and young peoples' regular schooling and also in adult education – that in the Brazilian case may involve initial schooling processes, returns to school or professionalization.

The general theoretical approach suggested by Vergnaud (1990) is basis of the analysis performed in the present study. According to the Theory of Conceptual Fields, concepts are present in sets of *situations* that provide *meanings* for the concept, that involve conceptual *invariants*, i.e., logical and operational properties that allow generalization and knowledge transference, and that are *represented symbolically* in varied manners. This theoretical framework was adapted in the present study to analyse performance on combinatorial situations, based on the triplet *meanings*, *invariants* and *symbolic representations*.

Different *meanings* are present in Combinatorics and in these are implied *invariants* (relations and properties that are constant through different situations) that can be *represented by* varied means: drawings, lists, trees of possibilities, tables, formulas and other forms. The four basic combinatorial situations, as presented by Pessoa and Borba (2009) are:

1) Cartesian products: Given two (or more) distinct sets (with n and with p elements), these are combined to form a new set and the nature of the two original sets is distinct from the new set formed. An example: At the square dance, three boys and four girls want to dance. If all the boys dance with all the girls, how many pairs will be formed?
2) Permutations: All the n elements of a set are used and the order of presentation of the elements implies in different possibilities. For example: Calculate the number of anagrams that can be formed with the letters of the word LOVE.
3) Arrangements: With n elements, groups of 1 element, 2 elements, 3 elements.... p elements can be formed, with $0< p < n$ and the order of presentation of the elements implies in different possibilities. An example: The finals of the World Cup will be played by: Argentina, Brazil, France and Germany. In how many distinct ways can the three first places be formed?
4) Combinations: With n elements, groups of 1 element, 2 elements, 3 elements.... p elements can be formed, with $0< p < n$ and the order of presentation of the elements does not imply in different possibilities. For example: A school has nine teachers and five of them will represent the school in a congress. How many groups of five teachers can be formed?

A COMPARATIVE STUDY OF CHILDREN'S, YOUNG PEOPLE'S AND ADULTS' PERFORMANCE

The specific aim of the present study was to verify the performance of students in different schooling situations and on the different problem types whilst solving the same combinatorial reasoning situations. The participants were regular students – children and young people – in three levels of schooling: lower Elementary School (7 to 10 year olds), upper Elementary School (11 to 14 year olds) and High School (15 to 17 year olds), in a total of 568 students. The adult students were also in these school levels and in a professional course at High School level, in a total of 150 adults in these situations. Each student solved eight problems that involved combinatorial reasoning (two problems of each type: Cartesian products, arrangements, permutations and combinations). Comparison of performances and strategies used, by school situation and meanings involved in the combinatorial problems, were performed by analysis of pupils' protocols.

Presentation and data analysis

<u>Performance by school situation</u>

Student performance by school situation is presented on Table 1. Only answers that were totally correct were considered but it is worth mentioning that many students started solving the problems in a correct manner but were unable to reach the final correct answer – mainly in problems in which the total number of possibilities was large. This is evidence that, even before studying Combinatorics at school, students are able to understand what is asked in combinatorial situations but most are not able to systematically list all the possibilities required or to use procedures of indirect counting of all possible cases.

Despite not presenting very good performance, it was observed that schooling had an effect on combinatorial problem solving – both in regular schooling (children and young students) and in adult education. However, it was expected that a stronger effect would be observed, especially amongst High School students that had already studied Combinatorics at school (the 15 to 17 year olds in regular schooling and the adults attending professional High School).

School situation		Mean of correct answers (out of 8)
Regular Schooling	Lower Elementary School (7 to 10 year olds)	0,75
	Upper Elementary School (11 to 14 year olds)	2,68
	High School (15 to 17 year olds)	3,45
Adult Education	Initial Lower Elementary School	0,22
	Initial Upper Elementary School	0,93
	Initial High School	0,77
	Professional High School	2,27

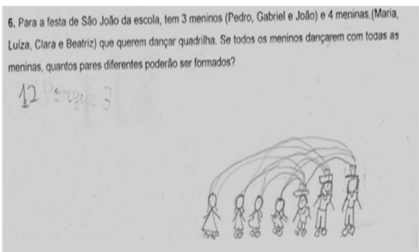
Table 1: Means of correct answers (out of 8) by school situation.

The very low performance of adults entering or reentering school is evidence that combinatorial reasoning is most likely to develop through school situations. Everyday situations – such as work experiences – may have some influence (as observed by Schliemann, 1988), but in the case of adults that left school – mainly because they needed to work – performance in combinatorial situations was only better (but still not ideal) after many years of schooling, that was the case of adults in Professional High School. This finding is in accordance with Fischbein (1975) that pointed out the role of schooling in the development of combinatorial reasoning.

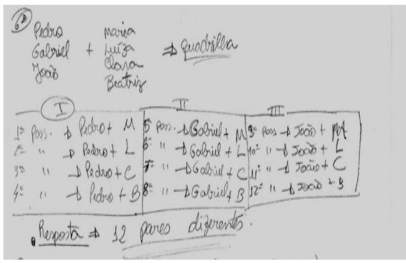
<u>Procedures used by children, young people and adults</u>

Adults in initial schooling were very reluctant in solving the situations proposed. Many times they did not recognize the situations as mathematical problems and questioned how they could solve the problems when numbers were not even mentioned (as in the case of the permutation of the letters of the word LOVE and the arrangement of the three first places in the World Cup).

Children and young people were more used to problems of these types – explicitly and many times implicitly worked at school – and devised diverse procedures to deal with the distinct combinatorial situations, as may be observed in Figure 1. Drawings, lists and simple arithmetic operations were commonly used by these participants before and after specific instruction on Combinatorics.



(a)



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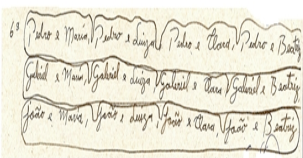

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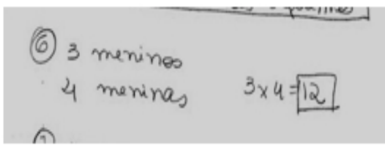

(d)

Figure 1: Examples of students' varied strategies to solve combinatorial situations.

Very few participants used additive procedures (recognizing, thus, that the nature of the problems was not additive) and inadequate and adequate use of multiplication was common, indicating the recognition of the multiplicative nature of the situations.

Listing of possibilities was the most common procedure, both amongst children and young people and also used by adults. Success in situations that resulted in large number of possibilities was not obtained because listing was not the ideal procedure in these cases. Participants were more likely to be successful in these cases when they used procedures in which they recognized regularity in the groups of cases, i.e., they listed some of the possibilities and noticed that they could obtain the total number of cases by simple multiplications.

Formulas were rarely used – only by regular High School students – and the participants that used this procedure were not always sure which was the correct formula for each specific case.

Performance by problem type

The differences in performance in the different school situations, according to meanings involved, are presented on Table 2. These results are evidence that the distinct meanings involved (*Cartesian products*, *arrangements*, *permutations* and *arrangements*) influence student performance. Combinatorics presents different meanings that are not understood simultaneously because different invariants are involved that increase or decrease levels of difficulty. Thus, special attention is required when presenting combinatorial situations in schooling settings, in order that similar aspects may be highlighted but main differences may be pointed out.

School situation	Problem types (2 possible correct answers in each type)			
	Arrangements	Combinations	Permutations	Cartesian Products
Regular Schooling Lower Elementary	0,11	0,17	0,05	0,42
Upper Elementary	0,66	0,25	0,40	1,38
High School	0,85	0,23	0,76	1,62
Adult Education Lower Elementary	0,02	0,02	0,00	0,18
Upper Elementary	0,05	0,05	0,07	0,77
High School	0,13	0,10	0,10	0,43
Professional High School	0,53	0,47	0,47	0,80

Table 2: Mean of correct answers (out of 2) in each school level by in meanings involved combinatorial problems.

Cartesian product was the easiest combinatorial meaning for all students in all school situations and this can be explained either by familiarity or because this is a multiplicative problem in which the implicit one-to-many correspondence is clearer. In terms of familiarity, Cartesian products are the problems explicitly dealt with since lower Elementary School but explanations in terms of cognitive processes implied in combinatorial problem solving must also be considered. Nunes and Bryant (1996) point out that the one-to-many relation is basic in multiplicative reasoning and that this relation marks the difference between additive and multiplicative situations. Thus, a hypothesis defended in the present study is that in *Cartesian products* the one-to-many relation is implicit but can be more easily identified than in other combinatorial problems, such as *arrangements*, *combinations* and *permutations*.

In *arrangement* problems, from a larger set, smaller sets are formed and the orders in which the elements are disposed in the sets indicate distinct possibilities. So, in the case of arrangements, when students list all possibilities, there is no need to set aside some of the sets as has to be done with *combinations* in which it is necessary to observe which cases are equivalent, i.e., despite distinct orders of presentation of elements in combinations these represent equivalent sets. This can explain the better performance – basically of participants in regular schooling – in *arrangement* problems.

Comparing *arrangement* problems to *permutation* problems it is noticeable that permutations require more rigorous systematization in the listing of all the possibilities in which students need to consider the following invariant relations: that all elements must be used, each one only once (in the case of simple permutations – with no repetition of elements), and that the order of presentation of elements is, thus, relevant. In the present study, the main error observed was to list some of the possibilities but not attempting to systematically list all the possible cases.

Combination problems can be very difficult because the pupils need to observe that, similarly to *arrangement* problems, from a larger set of elements, some elements must be chosen to form subsets, but differently, the order of the elements does not imply in new possibilities. In the present study, the students' main difficulty was, thus, to not consider this invariant relation of combinations and to count more than once cases that were the same but that varied in order.

FINAL REMARKS

A contribution aimed at the present research was to examine combinatorial reasoning by means of a study that involved a large group of students from different school levels and situations solving four distinct types of problems. In this sense it was possible to consider the influence that schooling directly has on combinatorial reasoning and indirectly how maturity and out of school experiences may contribute to the development of this kind of thinking.

The main finding considering schooling is that combinatorial reasoning is influenced by school experiences that can lead to a greater systematization and formalization in the understanding of the many meanings involved in Combinatorics. Participants' poor performance indicates the need to aid pupils in the recognition of the multiplicative nature of combinatorial situations, highlighting the implicit one-to-many correspondences present in *Cartesian products*, *arrangements*, *combinations* and *permutations*.

The results presented bring evidence to the need of recognition that students' development of combinatorial reasoning starts at early school years and is not yet concluded at the end of High School. Considering this, at school students' spontaneous strategies for combinatorial situations must be recognised and can be taken as starting points in helping pupils in seeking systematic strategies and the future use of formal procedures – such as the use of distinct formulas that should be used once there is a wider understanding of the variation of different combinatorial situations. Thus, in the teaching of Combinatorics distinct meanings involved, the correspondent invariant relations and varied symbolic representations should be considered in order to provide a wider development of combinatorial reasoning.

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EXPLORING SCHOOL CHILDREN'S OUT OF SCHOOL MATHEMATICS

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The study reports on the preliminary part of an ongoing research study aiming at exploring and characterising the nature and extent of everyday mathematics knowledge amongst middle grade students and their involvement in economic activities. Students' knowledge of numbers related to their currency-denomination knowledge was prominently visible even though they had difficulty in representing positional values of numbers. Students did operations on multi digit numbers on oral mode but not on the school taught methods. Data were collected through interactions with 25 students of grades five and seven in two public schools located in one of the biggest slum dwelling in India that has apparent vibrant house-hold based economy. Categories were created from the obtained data and implications drawn.

INTRODUCTION

The out-of-school mathematical knowledge of children has been studied extensively beginning with the pioneering work of Nunes, Carraher and Schliemann (1985) since it is thought that such knowledge can support the learning of school mathematics. In India, a study of children's knowledge of out-of-school mathematics has been carried out by Farida Khan (2004), in which she explored the mathematical knowledge of child vendors who sold newspapers and betel leaves (*paan*). To our knowledge, there are few studies of children's out-of-school knowledge of mathematics in Mumbai, although it has a large population living in slums (shanties), which are often active centres of house-hold based industry. An exception is the study by Sitabkhan (2009), who interviewed child vendors who sold articles in the local trains in Mumbai.

In this paper we attempt a preliminary characterisation of the knowledge of 'everyday mathematics' prevalent among school going middle grade children (working and non-working) living in a large Mumbai slum that has a vibrant house-hold based economy and offers unique opportunities to its resident children to learn from their environment as well as from their schools. We begin by discussing some of the current literature on 'everyday mathematics' and follow it up with our observations from our interaction with the chosen sample of students. The main part of the paper includes the presentation, discussion and analysis of our observations aimed at exploring the nature, extent and use of oral techniques of solving daily-life problems making use of everyday mathematical knowledge.

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THEORETICAL ORIENTATION

Empirical studies have shown that problem solving in out-of-the-school settings intertwined in everyday activities is quite different from the formal ways of solving them using school taught techniques. The difference also lies in the structure of knowledge and the social conditions of their use. Carraher, Carraher and Schliemann (1987) suggest that situational variables often influence school students' tendencies of using oral calculation procedures based on their everyday knowledge to find solutions and not the strategies learned at school.

School mathematics entails written mathematics while the 'everyday mathematics' involves oral techniques in calculations (Nunes, Carraher & Schliemann, 1985; Resnick, 1987; Saxe, 1988). In 'everyday mathematics', the doer has a continuous engagement with the objects and the situations and she does not burden herself with the extra effort to remember the algorithms, calculation-techniques and the reasoning used – a characteristic that Resnick (1987) pointed out as well. This characteristic of 'everyday mathematics' is in contrast to school mathematics where one does not usually have a freedom of making a choice of using alternate techniques other than those taught in the classrooms. In schools, mathematical activities are based on symbols which get detached from any meaningful context. More stress is usually on symbol manipulation and following rules. School mathematics is aimed at improving individuals' performances and skills, whereas, out-of-school mathematical activities are socially shared. While school mathematics focuses on generalised learning, everyday mathematical ability grows from situation-specific competencies (Resnick, 1987; Resnick & Ford, 1981). The difference also lies in the structure of knowledge and the social conditions of their use.

Most of the studies indicate that participants who were untrained in school mathematics could competently perform the calculations needed in their workplace activities. In contrast, the school students, trained in school mathematics when presented with such problems came up with incorrect solutions or even absurd solutions. School students concentrated more on the numbers given in the problems and paid little attention to the meanings of the problems. (Nunes, Schliemann, and Carraher, 1993, 1985; Lave, 1988). On the other hand, street vendors who with 'impressive ease' solved their routine problems in everyday settings, could not solve the same types of problems which they had earlier solved in their workplace contexts when presented to them as formal word problems without any contexts. Sometimes they gave insensible solutions, for example, getting as an answer a number in a subtraction problem that is bigger than the minuend (Nunes, Schliemann and Carraher, 1993).

SAMPLE & METHODOLOGY

The sample for this study was identified from one grade 7 class of an English medium school and one grade 5 class of a Urdu medium school run by the Municipal

Corporation of Greater Mumbai, located in Dharavi in the north-central part of Mumbai, India. Dharavi is among the largest slums in India. Every third roll number from the attendance register was chosen to form the sample. Most of the students are in the age-group of 10-12 years; different grade years in the English and Urdu medium sections were chosen to achieve parity in age. Discussions were held with 12 students (7 boys + 5 girls) from the Urdu section and 13 students (7 boys + 6 girls) from the English section. Each discussion lasted between 30 minutes and one hour.

The present report discusses preliminary findings from the first phase of the larger ongoing research project aimed at characterising students' knowledge of out-of-school mathematics. The researcher (i.e. the first author) first observed the students in their classrooms and then held informal discussions with them to get a broad picture of the nature of their daily activities that have aspects of mathematics and the nature and extent of their knowledge of everyday mathematics, and to get an initial understanding of the variation among children of out-of-school mathematical knowledge, as well as involvement in economic activity. Hence, an attempt was made to characterise out-of-school mathematical knowledge at the individual level and also form preliminary impressions of the processes by which school-going children acquire them. Attempts were made to identify the opportunities that are available to the children to immerse themselves in elders' pursuits. The discussions were audio recorded after taking the teachers' and each student's consent. The sources of data were students' work-sheets, researcher's field-notes and audio records of the discussions.

LOCATION OF THE STUDY

Dharavi is uniquely different from other slums in the sense that many houses located here run workshops or small-scale factories forming a vibrant economy. Children become engaged in the workshops/factories at an early age. However, there are families which prefer their children to finish studies first before immersing themselves in economic activity. Such parents do not let their children work. However, it is not surprising to find that even such children who are not actively involved in any kind of economic activity have fair knowledge about the activities by virtue of being present in the locality.

Some common house-hold occupations are embroidery, *zari* (needle work), stitching and garment-making, making plastic bags, leather goods (bags, wallets, purses, shoes), dyeing and button-making. Some of these activities are done in the house itself, while some are carried out in “factories” in small-rooms of the shanties. The goods produced are sold not only in Mumbai but sent to many other cities and even exported to other countries, mainly in the Middle East. There are many *bissi* – places where food is prepared in large scale to be delivered to different places. Many children are involved in delivering food (“tiffin”) boxes. Mumbai being the biggest financial hub of India attracts a huge flow of immigrants from different parts of the country, especially from North India. Dharavi is an old, established slum, which continues to receive immigrants and hence has a high population density. The migrant unskilled workers find jobs in the workshops and some of them become apprentices in the small factories. In recent years, there is a move to relocate the population of Dharavi, which is a great source of concern among its residents.

OBSERVATIONS AND ANALYSIS

The interactions with the students indicated that practically all of them have a packed schedule the whole day. The researcher observed the morning-shift school starts at twenty past seven and gets over just after noon at half past twelve. Most students from the English school reported that they go for Arabic classes immediately after school. Many of them go for tuition classes thereafter. Students mostly from the Urdu school are already through with their Arabic lessons and report at their respective workplaces after the school is over. Because of this packed schedule, the students do not get time to play. The lanes and by-lanes of Dharavi are also too narrow for the children to play. However, all students reported that they visit shops in the neighbourhood to buy groceries and other articles that are house-hold daily needs.

Knowledge about currency

All students interviewed had sound knowledge of the various denominations of the currency and could recognise all the currency coins and notes that are currently in use and their conversions. Some students calculated with numbers by thinking of them as money. For example, when asked to divide 981 by 9, one student U1 of grade 5 of the Urdu school looked at the problem as 'equally distributing' Rs 981 among 9 children and arrived at 109 as the answer. His explanation was to divide Rs 900 among 9 children thereby getting Rs 100 for each of them and then divide the remaining Rs 81 among 9 children to get Rs 9 for each. Hence, each child gets Rs 100 plus Rs 9, i.e. Rs 109. Interestingly, when U1 was asked do the calculation on the worksheet he arrived at '19' as the answer, making the common error of omitting the zero (*shown in Fig. 1 above*). When the discrepancy in the answers was brought to his attention, he hesitatingly put a '0' between '1' and '9' probably because he had 'more faith' in the oral procedure than school taught algorithms. The student U1 works in a garment making workshop after the school hours. He had shifted to Dharavi three years ago from Bihar – a North Indian state that is economically backward. His interest in studies brought him

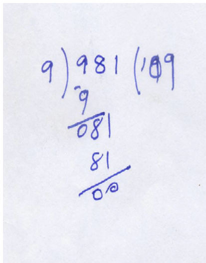

Fig. 1

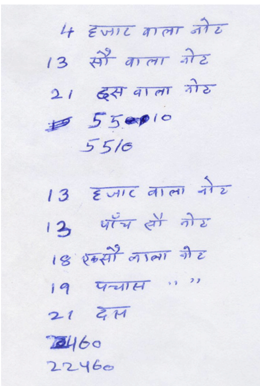

Fig.2

back to studies after a two-year gap when his financial condition of his family forced him to work than attending school. Discussions with U1 had earlier shown that he can add currency-values sometimes involving 5 digit numbers purely mentally. For example, when asked how much money would be, if taken together 4 thousand rupee notes, 13 hundred rupee notes, and 21 ten rupee notes, U1 correctly replied, '*five thousand five hundred ten rupees'* but initially wrote the sum as 550010 and subsequently corrected it to write 5510. When asked to add 13 thousand rupee notes with 13 five-hundred rupee notes, 18 one-hundred rupees notes, 19 fifty rupees notes and 21 ten rupees notes, U1 had the accurate answer as, '*twenty two thousand four hundred sixty*' (*As shown in Fig. 2 in the previous page*).

Number Knowledge

The range of number knowledge varied among the students. This may be related to the extent of engagement in economic activities, but this needs further exploration. Of the 23 students in the sample, 21 had difficulty in writing the numbers dictated to them correctly, making place value errors, especially for numbers bigger than 100. They wrote the numerals reflecting the number-names, i.e. 1001 for 'one hundred one', 10010 for 'one hundred ten', 10051 for 'one hundred fifty one', 20060 for 'two hundred sixty', 10001 for 'one thousand one', etc. Numbers which were multiples of hundred or thousand like 5000 for 'five thousand' were written correctly. However, irrespective of the place-value errors that students made while writing the numbers in figures, they seemed to understand the numbers through their names. This knowledge probably is rooted in their regular use of money.

Interestingly, children expressed the non-integral amount of money (amount that involves 'rupees' and 'paise'; 100 paise = 1 rupee) by juxtaposing the rupee amount and the paise amount by using a 'dot' or 'point' in between to mark the distinction. This is done apparently without the formal knowledge of decimals. Probably, this is based upon the socially accepted meaning drawn from the shared experience while dealing with money in everyday commercial interactions.

Arithmetic Operations on Numbers

Although students in the sample regularly attend school, in several instances they used their out-of-school knowledge of mathematics while solving problems. For example, the student E-10 from grade 7 of the English Medium School belongs to a low socio-economic family of five including her parents. Her father does scavenging work and removes debris from the road sides while her mother works as a domestic help. The student often goes to the shop to buy everyday articles such as kerosene oil for cooking (sold in bottles), milk and other groceries. She informed the researcher that milk is sold for Rs 12 per packet. On asking how much milk a packet contains, she quickly replied *"aadha litre"* ("half a litre"). When asked for the price of 2 packets, she immediately replied, *"24"*. She claimed that she knows this as she often hears the milk-seller telling this to the customers. When she was asked to find the price of 5 packets, she paused and started thinking. She then added 24 and 12 and arrived at 36 and then added 36 and 24 and arrived at 60.

Her strategy was to use to the known values, viz. 12 and 24, adding them to first arrive at the price of 3 packets, and then to add 24 to find the price of 5 packets.

E-10 told the researcher that a bottle of kerosene comes for Rs 28, who asked her to find the price of 5 bottles. She calculated mentally and came up with *"one forty rupees"* as the answer. Her argument was, *"bees ke hisaab se paanch bottle ka hundred aur aath ke hisaab se paanch ka forty"* ("price of five bottles at the rate of twenty is hundred and at rate of eight is forty"). Then for 15 bottles, she added 140 twice and again added 140 to the sum to get 420. To find the price of 7 bottles, she added 28 twice and then added the sum (i.e. 56) to 140 thereby getting 196 as the answer. Similarly for 22 bottles she added 280 twice and got 560 and then added 56 to it to get 616 as the answer.

The strategy to use addition that included 'continuous monitoring' about 'where she is' in the midst of a calculation and gave her confidence in the procedures and meaningfulness in the results obtained.

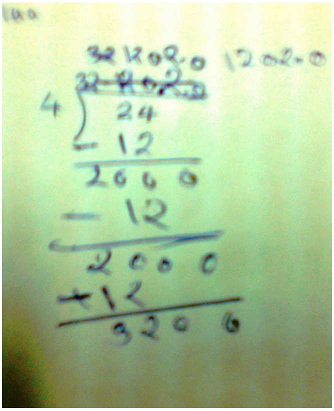

Fig 3

Interestingly, all the students claimed difficulty in the division algorithm though many of them could orally divide two numbers considering them as referents of some familiar contexts. For example, one student (E11) repeatedly obtained absurd results like getting quotients bigger than dividends (for all positive dividend, divisor and quotient). She however, did the seemingly easy division orally in a contextual problem situation instantaneously (As shown in Figs. 3 & 4).

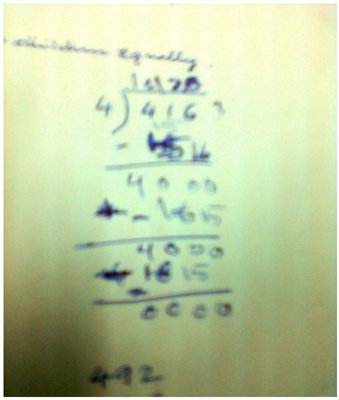

Fig 4

Use of the Units

Discussions with the students showed that children make use of a variety of units mostly based on the convenience and syntactic support from prevalent practices. For example, the student E-12 wrote 'six hundred sixty' as 6005010 and read it as "*chhe sau pachaas aur upar se dus*" ("six hundred fifty and ten more") but for 'one hundred seventy four' she wrote 10074. This probably happens because the student considers the numbers '50' and '10' as 'closed' numbers and took them as units. Several such examples could be seen of different units which bear 'names' in the discourses.

DISCUSSION

Our observations indicate that school going children from Dharavi who have an exposure to currency handling and ensuring its optimal use, can handle operations with multi-digit numbers that represent currency denominations, using the oral mode. Children use different forms of currencies as tools for mental (oral) activities. The resultant cognitive activity of (as in the case of U1, discussed above) involving operations on multi-digit numbers were shaped, dependent and governed by the use of 'currencies' as tools. In lieu of this, when students attempted to write the resultant amount obtained after addition, they expressed the numbers according to the number-names without caring for the multi-digit representations which carry the positional values of the respective digits. This probably happens because of the syntactic as well as semantic differences between the language used in everyday contexts and the language used during classroom-teaching.

CONCLUSION AND IMPLICATION

Multi-digit representation of numbers and algorithms used in the number-operations have remained hard-spots for students in the middle grades. However, in this preliminary work we have found that children having wide exposure of 'everyday mathematics' have sound knowledge about currency handling as well. This includes doing arithmetic operations on the currency denominations including multi-digit numbers. Though the resultant answers were correct when dealt with orally, but their representations in the written form were usually flawed. It remains to be explored how much the teachers are aware about the extent of students' everyday mathematical knowledge and how can such knowledge be brought in the classrooms to facilitate better learning of mathematics.

It also remains to be explored the role of the language in gaining everyday mathematical knowledge in out-of-school contexts and how does language helps in facilitating mathematics learning in the classrooms while drawing upon from familiar contexts. 'Everyday mathematics' (out-of-school mathematics) bears the functional aspect of mathematical knowledge that is available to all and not hidden (Subramaniam, 2010). Bringing together everyday mathematical knowledge and school mathematics possibly can pave way for developing skills and interests in learning mathematics.

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RETHINKING OBJECTIVITY AND SUBJECTIVITY: RE-DISTRIBUTING THE PSYCHOLOGICAL IN MATHEMATICS EDUCATION

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ABSTRACT: *Mathematics in schools exists substantially as pedagogical material crafted for supposed modes of apprehension. But of course such apprehension depends on how we understand mathematical objects and how we understand human subjects. This paper follows the recent work of Badiou whose philosophical model is centred on set theory in defining a new conception of objectivity, and on a Lacanian conception of human subjectivity. In this model objectivity results from counting a set of elements as one, such as in making a mathematical generalisation. His conception of subjectivity comprises a refusal to allow humans to settle on certain self-images that have fuelled psychology and set the ways in which humans are seen as apprehending mathematics.*

Introduction

How do we symbolise mathematical experience? How do we experience symbolisation? Gattegno once spoke of a baby pointing to a fly walking across the ceiling. Each fly position on a continuous path was associated with a particular (discrete) arm position, which Gattegno saw as an algebraic relationship. Meanwhile, Brown and Heywood (2010) depict students carrying out body maths exercises where they walked the paths of geometric loci. In these exercises continuous curves were associated with sets of discrete instructions activating sets of points defined according to certain rules. Gattegno has considered the *algebraization* of geometry, that is, how geometrical experience in school is transformed, perhaps compromised, by an insistence on it being converted to symbolic form (e.g. 1988). A key concern for Gattegno was that in school, such algebraizisation results in a loss. Schubring (2008, p. 140), however, has argued "that the processes of algebraization are among the most marked characteristics of the historical evolution of mathematics". That is, mathematics evolves through successive attempts to algebraizise its objects. The advance of mathematics is defined by the production of its objects. This concurs with certain writers (e.g. Bachelard, Lakatos, Althusser) who see science as a practice marked by the production of *new* objects of knowledge (Feltham, 2008, pp. 20-21). So it might seem that the attempt to create a mathematical object can result in a loss in terms of the experience of mathematics whilst the advance of mathematics depends on this happening. This paper argues that this backstitch approach can be understood as a model for learning as well as a model for the evolution of mathematics. It utilises some apparatus from contemporary philosophy to conceptualise both students and the mathematics that they learn.

Much mathematics education rests on supposed cognitive models (e.g. Piaget/ Vygotsky) in which the human being is understood in a particular way (as an individual, at a particular stage of readiness, in a certain conception of the social world, following certain social codes and expectations, etc). These models are sometimes extended to consider how the body mediates mathematical experience and can occupy similar territory to a variety of such work on gestures and the embodiment of mathematics. Lakoff and Núñez (2000) aspire to a scientific understanding of mathematics grounded in processes common to all human cognition. Nemirovsky and Ferrara, (2008, p. 4) frame their analysis in terms of "perceptuo-motor-imaginary activity" that is "fully embedded in the body". Lappas and Spyrou (2006, p. 12) follow Husserl in proposing a "genetic" conception of embodied mathematics where "man builds his mental representation of the world, through a progressive reorganization of his prior active manipulation of the environment". Meanwhile, Radford (2004, p. 18) in a Vygotskian formulation suggests that we "consider mathematical objects as fixed patterns of activity in the always changing realm of reflective and mediated social practice".

Roth and Thom (2008, p. 2) suggest that Piaget conceives "of mathematics generally and of geometry particularly as paradigmatic examples of knowledge that is independent of sensual experience, though always given in the form of representations that can be related to the things that we come to know through sensory experiences". They contrast his constructivist epistemology with the model of van Hiele: "In the Piagetian model, the human mind necessarily develops to specific endpoints given by classical logic, whereas in the van Hiele model, emphasis is placed on the learning processes that - mediated by language - are specific to the historical period". Roth (2010, p. 8) articulates "a conceptualisation of mathematical knowledge that is grounded in materialist phenomenology". These later comments are more resonant with the recent work by the French philosopher Alain Badiou who also proposes a very different model to Piaget. Badiou's model is centred on Cantorian set theory in defining a new conception of objectivity and a Lacanian conception of subjectivity, where both objectivity and subjectivity are mediated by historically specific depictions. These are both highly complex areas that defy a thorough account here. Yet the basic ideas are within the scope of a short article. The next two sections take objectivity and subjectivity in turn.

Objectivity: That's it

Badiou's (2009) most recent major project is encapsulated in the title of the book *Logics of worlds*. *Worlds* are multiple rather than singular. There are multiple ways of introducing logics into any given *world*. Badiou (2007) commenced with a sheer multiplicity of elements in a pure state of being. In this state the elements are not anywhere. These elements can be combined in subsets of that multiplicity to create or define unities. Badiou's assertion is that any such unity, or object, derives from an operation of "counting as one". "Unity is the effect of structuration – and not a ground, origin, or end" (Clemens & Feltham, introduction to Badiou, 2006, p. 8.).

That is, an *object* is produced by counting a set of elements, within a *supposed* world, as one object. This operation brings the object into existence within a *world*. And in a sense it also brings the world into being. The assertion of an object asserts the world that is the outside of that object, a world that has perhaps been changed a little by the specific noticing of the object. The *world* is itself a result of a wider "counting as one" (of the elements of that *world*).

An assertion of a new object comprises an assertion of a new configuration. This configuration entails "counting as one" a set of elements within the multiplicity. This can be achieved through defining a novel combination of elements. Any element can itself be a set and a potential member of other sets. And within any assertion of a set, yet further possibilities are created, resulting from the construction of subsets producing yet more new entities.

This very proliferation itself defies any final stability in the universe. For this reason there can be no settling or convergence in the meaning of the constituent terms. Badiou contemplates a managed multi-dimensional infinity. Yet forms of knowledge are predicated on a world, comprising specific sets of terms within this world. Such forms of knowledge might be disrupted as they readjust around the ever-expanding set of sets being counted as one. For example, Newton's thought as a supposed universal model was disrupted by quantum physics. Such expansion reveals objects not previously identified within earlier overarching multiplicities.

But how might such an abstract theoretical perspective support the examination of mathematical learning, or more generally the human apprehension of mathematical forms? For a student in school, and probably mathematical learning more widely, mathematics can generally be understood through the pursuit of noticing or asserting generality, a notion resonant with "counting as one". Much mathematics education research has been predicated on enabling students to experience generalisation to emphasise that mathematics has power beyond mere particularities. The noticing of a generality results from an *operation* that apprehends, or perhaps creates, a set of mathematical elements (e.g. points, numbers, shapes) as a unity. This can be geometric as in seeing a circle as a type of ellipse, or algebraic such as assigning a formula to a specific numerical sequence. Whether we are considering students encountering socially known mathematical ideas for the first time, or new innovations by frontier mathematicians, Badiou's notion of "counting as one" provides a technology. A "counting as one" seen as the acquisition of a new generalisation could be understood in either of these two situations in relation to a newly extended situation.

Mathematics can be approached in many ways. For example, to take one object, we all know what a circle is but some people may not know that $x^2 + y^2 = 5$ defines a circle. We could also experience a circle by using a pencil and compasses, by drawing around a coin, by running whilst holding a rope tied to a flag post, by generating a circle on a computer, etc. Or perhaps, it could be experienced in a new way such as by walking in a path defined on a distance ratio 1:3 between two partners. Similarly, all mathematical concepts can be understood from a multitude of perspectives and indeed the concept can often be uniquely a function of that perspective.

In Badiou's framework, the term circle entails an operation to "count as one" the objects of a given set. For example, the set of points on the rim of a bowl may be "counted as one" and given a name, circle. Or the moon and the sun might be seen as displaying a "shape" also occurring in naturally occurring objects, such as, berries, oranges, eyes, etc. The group of objects so classified may be given a name, such as "circular shapes", or "spherical shapes". But thereafter the term can become a member of other sets of objects such as "regular two-dimensional shapes" {pentagons, ellipses, squares, circles etc}) seen as making up a world and utilised in organising our apprehension of the world. Algebraization comprises a similar operation of "counting as one" (e.g. identifying the set of points obeying the relation $x^2+y^2=5$). The objects get to be there, in a world, as a result of the operation. But they need that prior (or simultaneous) construction, of a world (in this instance two-dimensional space, structured according to some rules), to *be there.* In Badiou's account the existence of an object requires a place for it to exist.

In this perspective any mathematical object is a function of its perceived *world*, in contradistinction to so many instances where mathematical objects have been understood in a more ideal sense, as entities in themselves. A circle requires the very human conception of 2D space. Yet in Badiou's formulation a *world* is merely any presented multiplicity, whether that is an assertion of a mathematical object as a generality, or any cultural configuration such as a social structure.

And in this sense learning can be seen as putting things *there*. In Badiou's terminology elements are drawn from an undifferentiated multiplicity of pure being to *produce* objects that *exist* in a *world*. Learning comprises the placing of an object in a *world*. This requires the assertion of an object, and an assertion of a world. Object and world are contingent. They imply each other. With regard to the students moving around according to geometric loci the task is to apprehend continuous movement as a sequence of points. These points are then aggregated to "count as one" object, understood in terms of this mode of aggregation. Retroactively the students can recognise the shape they have walked against a new register and declare "that's it".

That's me: subjectivity

Badiou (2011) draws on Lacan's conception of the human subject. This subject, rather than being seen as a biological cognitive entity, is understood as a reflection of a broader symbolic universe. There are societal demands on the subject that shape who he or she is. The subject is a function of the stories that are told about him or her. In a Lacanian perspective, learning would be understood more as being about an experience through time rather than being about apprehending an object located in a fixed conception of space. The task is to locate education in the formation of objects/events in time/space rather than to see it as an encounter with ready-made objects.

Objects cannot necessarily be apprehended in an instant. Indeed the apprehension may result from a gradual assimilation of the object's components and qualities and how these are combined in forming the object. I may compare new sets with a selection of previously known sets. I may contrast the operation of a newly located function with more familiar functions. The progressive apprehension of the supposed object becomes part of the story of my life, a part of getting to understand who I am and how I fit in to a supposed world or how I might make that world otherwise. That is, this progressive apprehension builds a story around the abstract entities being located, a cultural layer in which any learner is fully implicated since it was integral to their very own constitution.

Lacan's concept of human formation is triggered by a transformation that takes place when a young child assumes a *discrete* image of herself. Lacan's iconic example is that she looks in to the mirror and says "That's me". This allows her to postulate a series of equivalences, samenesses, identities, between herself and the objects of the surrounding world (the equivalence of my movement on the floor, to the drawing on paper, to the image in my mind, seen as continuous movement, or as a configuration of points). The image of self, as characterised by a name, fixes an egocentric image of the world shaped around that image of self. That is, the assumption of a self (a "that's me") results in a supposed relation to the world and a partial fixing of the entities she perceives to be within the world, that the "me" has been gauged against.

In due course these relations become implicated in more overtly mathematical phenomena that underpin the child's formal mathematical education. Unlike Gattegno's baby the older student can become aware of symbolised mathematical relationships, such as how specific bodily positioning responds to a coded spatial environment. And notions of humans and of geometrical objects become relatively fixed in such images with consequential restrictions on how relations between people and geometry can be understood. In Badiou's terminology this assumption of a self in an assertion of saying "that's me" comprises a collation of a set of characteristics, attributes, organs, etc. that make up "me". This set of characteristics is "counted as one" person. Lacan, however, cautions that we should be wary of this image, since it is illusory, a snap shot that never quite works. It never fully captures the real me as it were, rather like the production of a formula not fully capturing the mathematical experience of a curve. In Lacan's model our real self is never fully visible to us.

"That's it" encounters "that's me".

We thus have a situation where an individual human (a set of characteristics that have been counted as one) confronts an object (comprising elements that have been counted as one). A "that's me" encounters a "that's it" and a relation between these two (petrified) entities may be asserted. The image is crafted retroactively (backstitched) within the limits of the apparatus we have available. And this apparatus has a track record of being changed on a frequent basis. The operation of "count as one" can always be performed differently according to new circumstances. The story or image never lasts. It always needs to be renewed.

A new distribution of the psychological

Why is this model significant? School mathematics teaching is often in the business of enabling students to better apprehend and use socially derived mathematical apparatus. And that can drive mathematics into forms more easily managed in the educational contexts concerned, and accountable within the regulative apparatus that doubles to formally assess understanding of the field and student conformity with social norms. That is, in the *world* of teaching situations, mathematical objects are recast as pedagogical and assessment objects that result in the erstwhile mathematical definitions becoming implicated in socially governed processes. The assertion of an object is the assertion of a particular view of the world. Children's mathematical re-productions of such entities are evaluated through filters created from the cultural apparatus. This style of teaching is reproductive of culture, in that it either offers existing culture, or recognises student work only insofar as it is aligned with such culture. At PME Nunes (2010, p. 106) argued: "The frames and analogies used by teachers help them observe students, rendering some things more visible, but others invisible." Within educational contexts the meanings of mathematical objects are necessarily a function of the relationships within such social settings. That has always been the case. The currency in education comprises pedagogically or socially defined objects, not so much ideal mathematical objects. For example, geometry can be converted into particular linguistic forms for accountancy purposes or formal recognition, such as tests or exams. This can compromise aspects of geometrical learning in the way Gattegno highlighted, such as where continuous experience of certain geometric forms is prematurely seen in terms of discrete categorisation, which may obscure or close off potential apprehensions of spatial phenomena.

But teachers and students also find themselves understood in terms of discrete categories with respect to their engagement with mathematical phenomena. Their actions are partitioned according to a discrete mark up of the mathematical terrain. Teachers are not teachers in themselves but teachers subject to particular cultural specifications. They need to be employed in a job with certain social expectations, working practices and responsibilities that restrict how others read their actions and indeed how they assess their own practice. Specifically they work to curriculums that mark out the field of mathematics in particular ways that favour certain priorities or groups of people. And student engagement with mathematics is assessed according to how recognisable it is against this frame. The "that's me" is forced into alignment with the "that's it" within an externally defined register that defines "learners", "teachers", "mathematics" and the relations between them.

Badiou's variation on Lacan's subject is defined by, or comes into being through, an encounter with a new way of being. The individual participates in historical formation rather than apprehending some*thing* fully formed. The subject is only a subject to the extent it participates in renewal rather than reproduction. It is in this sort of model that Badiou's notion of subject leaves all affinity with biological bodies to become more fully a facet of structures guiding our actions.

Understood in this way a learner of mathematics would be seeing and experiencing mathematics as coming into being. The learner would be experiencing mathematics as part of herself, a self that is also evolving in the process. The backstitch movement comprises successive attempts to explain ones understanding that simultaneously petrify the mathematical content whilst alerting the student to a need to move on. The encounter with mathematics is a formative experience for the individual. But her participation in the collective enterprise that is mathematics is also formative of mathematics itself. One might think of such cultural renewal as being consequential to a widespread innovation being introduced in to a given community with more or less unpredictable results. Mathematics as it appears in school, for example, might be seen as resulting from a collective response to curriculum policy, and the attempts made to influence practice across populations of teachers and their students. It is such innovations that activate new modes of mathematical engagement or educative encounters across that community, that define mathematical objects and the worlds that host them, more or less compliant with the policies as envisaged by those who created them.

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PROMOTING TEACHER AND STUDENT MATHEMATICS LEARNING THROUGH LESSON STUDY: A DESIGN RESEARCH METHODOLOGY

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Mathematics design research is garnering the attention of educational researchers (Brown, 1992; Bruce, Flynn & Ross, submitted; Collins, Joseph & Bielaczyc, 2004) because it offers a framework that encourages researchers to work collaboratively with teachers to test and refine theoretical models and practical products. Design research is an ideal fit with mathematics professional learning models such as lesson study because it is situated in complex classroom contexts and generates tested learning trajectories. In this study, 18 teachers engaged in a two-year lesson study process in mathematics. The teachers collaboratively created, tested and refined mathematics lessons, leading to improved student understanding and positive teacher outcomes such as shifts in instructional and reflective practice through collaboration.

BACKGROUND

In this article, we will demonstrate the benefits of applying design research to lesson study as a way of testing and refining lessons and instructional sequences in live classrooms. In our two-year study, the process of lesson study acted as a form of design research for 18 teachers who worked collaboratively in small teams to develop and test lesson sequences.

LITERATURE REVIEW

Lesson study is an intensive professional development model that Stigler and Hiebert (1999) describe as a way for teachers to look at their own practice "with new eyes". It is, essentially, a systematic inquiry into teaching practice and related student learning, carried out by examining lessons. Lesson study is considered a productive professional development model because it "is embedded in the classroom and focused on students, it is collaborative and ongoing, and it is based on teachers' own concerns and questions" (Darling-Hammond & McLaughlin, 1995). In this way, lesson study is a teacher-led activity that has the potential to increase research-based knowledge that is critical to improving instruction (Lewis, Perry, & Murata, 2006). When "teachers engage in lesson study as researchers and scholars of their own classrooms" (Stepanek, 2001), we suggest that this is fits within the methodological framework of classroom-based design research.

Lesson study is a cyclical process that plays out in complex classroom settings, requiring the commitment of teacher participants and an openness to learning more about pedagogy related to the content and concepts in focus. The lesson study working group of researchers of the Psychology of Mathematics Educators of North America identified four critical components of lesson study in 2007. They are:

2011. In Ubuz, B. (Ed.). Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, pp. 193-200. Ankara, Turkey: PME.

1. Goal setting (where a facilitator may assist in setting goals): In the initial goal-setting stage, teacher participants begin by setting a goal for their students that they are aiming to address in their lesson. This is often a topic area that is difficult for the students to learn, or difficult for the teachers to teach. In other words, "the desire to improve is stimulated by seeing what's *not* working" (Lewis *et al*., 2006). Goal setting leads to an exploration for the best instructional strategies that could be used to achieve the goal (Fernandez, 2002).

2. Curriculum planning (collaborative lesson planning): During the curriculum planning stage, the teacher participants benefit from access to outside sources of knowledge – both print (e.g., textbooks, professional resources, outside research articles) and human (e.g., outside educators, content specialists, researchers, knowledgeable others) to support lesson planning.

3. Implementation and observation: Once the lesson is planned, it is taught and observed. Enacting the planned lessons with live observations conducted by participants of the lesson planning, as well as guests, is a fundamental and exciting stage in the lesson study process. The observations are focused on assessing how students respond to the lesson, and whether goals of the lesson were met. Surprises and details of student activity are noted.

4. Debriefing/reflection on the lesson and the lesson study process: The observations of the lesson, students and outcomes are shared in a debriefing session where all participants contribute to reflection and collective knowledge building from the lesson implementation. This debriefing drives the continuation of the cycle as the next set of goals is established.

The interaction between lesson study and design research requires further consideration. Design experiments (now commonly referred to as design research) emerged in 1992 in articles by both Collins and Brown as a way of refining educational designs. Collins, *et al.* (2004) make explicit connections from design research to lesson study, describing the testing and refining processes inherent in both as cycles of progressive refinement. They emphasize that "design research should always have the dual goals of refining both theory and practice" (19). Recent applications of design research by Lamberg & Middleton (2009), illustrate how design research encourages simultaneous generation of instructional tools and theories. In their multi-phased study, a theoretical lesson trajectory in fractions was developed, then tested in a controlled laboratory setting, an then refined in widening iterative cycles that expanded to classroom settings.

RESEARCH METHODS

Participants.

In our lesson study research, eighteen mathematics teachers in four Canadian schools engaged in Japanese lesson study cycles. Three of the sites were secondary schools (students aged 13-18 years) and the fourth site was an urban elementary school (students aged 4-12 years). All four schools were publicly funded and ranked below the standard for student achievement.

Teachers participated in collaborative cycles of lesson study over two years. The first cycle of lesson study was a "familiarization cycle" because teachers were new to lesson study and were gaining familiarity with the process. The second cycle was defined as a "formalization cycle" because the lesson study activity was focused and more nuanced including close attention to student learning. The third cycle added a focus on exploratory lessons with students to experiment with ideas and strategies leading toward the public research lesson. These shifts were in response to teacher learning and researcher learning as the research project progressed, reflecting a refined understanding of the lesson study process both practically and theoretically.

Data collection and analysis.

Sources of data included: transcripts of focus group interviews; field notes and planning materials; video footage and related transcripts of all stages of the lesson study cycle.

The study was a qualitative investigation focusing on two types of teacher and student outcomes: dependent climate variables (teacher engagement, cooperation and risk taking) and dependent learning variables (student mathematics understanding and affect). (See summary table 1).

Dependent variables	**Outcomes based on the intervention**
Teacher variables	Level of teacher collaboration Development of trust amongst team members Risk taking by teachers in the lesson study teams with one another and in the classroom
Learner variables	Student mathematics understanding and achievement Student beliefs and attitudes toward learning mathematics

Table 1. Analysis matrix for outcomes of Lesson Study activity

Initially, researchers read through the text data and viewed the video data. Text and video episodes were then divided into segments and start codes in the coding margins. Codes were reduced through elimination of overlap and redundancy, then collapsed into key themes (Creswell, 2008). Three researchers worked together to identify the primary themes to uncover the essence of the lesson study experience and its effects.

Although the argument of the article is grounded in qualitative data findings, the research team also collected quantitative data in the form of surveys and achievement tests to triangulate student outcomes.

FINDINGS

Teacher outcomes: collaboration, trust, and risk-taking.

Participants indicated that sharing their expertise, their questions and their classrooms with one another was positive. As evidence of the enthusiasm of teachers involved in the process, two teams continued with their core group of teacher participants in a lesson study professional development program beyond the two years of the study, independently seeking alternate funding to support teacher release time. Lesson study had a visible and lasting impact at these schools. (For more on these impacts, see Bruce & Flynn, 2010, who examine issues of sustainability and spread related to this study and describe the ripple effect of lesson study in two school sites.)

Participants learned from one another during their planning days, through observation of classroom teaching and student learning during the formal lesson, as well as during the debriefing sessions.

> In terms of observing a public lesson....I did recognize a lot of the same teaching strategies used by other teachers that I would normally use but there were some I hadn't even considered. So it was really enlightening. (Focus group, June, 2007)

After teaching a public lesson, the lead teacher said to one of her colleagues (who she had observed teaching earlier in the process):

> One thing I learned from your lesson is to wait for the students to respond. I realized I was calling on the same students and now that I wait, the ones who were unsure are gaining more confidence because I am giving them a chance. (Debrief transcript, December 2007)

It was through statements like these that participants affirmed effective teaching strategies with one another, legitimized the purpose of their work together, and built the trust necessary to continue taking risks together. This formal debriefing process provided a venue for teachers to articulate their mutual learning.

Teacher collaboration through co-planning and co-teaching expanded the traditional boundaries of teacher professional activity because the isolation of the individual classroom was broken down:

> As teachers we rarely get the chance to work *with* teachers…like you're a little island but now we have bridges between those islands…it gives us something to talk about ….the best thing about this project is the collegiality. (Focus group, November, 2007)

This high level of collaboration had the effect of expanding the types and refining the quality of instructional strategies the teachers used. Teachers reported they had an increased flexibility in their teaching moves as a result of the lesson study process. The fact that so many of the teacher-participants were struck by the rarity of such collaborative opportunities compared to their previous years of experience suggests that lesson study can offer a pathway for a paradigm shift in the ways in which teachers interact and support one another in developing effective practices.

As a result of their collaborative work, a sense of collective responsibility developed among the teams. This shared responsibility – the motivation to see the difficult work of the project through – emerged directly from the fact that the teams had autonomy in setting the direction for their professional learning:

> There is a lot that is happening in classrooms today that is top down, a lot of things we are told to do. And I respect that. But to have a chance to sit down with my colleagues and say, we know our kids, we know what they need, let's work around that – really validates me, and really makes me feel like I have some control on this process, and makes me more willing to go through it. (Teacher Interview, Feb. 2008)

Getting to the point of shared responsibility, however, was not without its challenges. Teachers identified how difficult it was to be vulnerable to change and to let go of personal responsibility by sharing teaching responsibilities with colleagues:

> It's that piece – for people to let others into their classrooms, into their world … And really, I think as a breed, as teachers, that is a really big hurdle for us to let somebody else in…building up that trust is a really hard thing to do. (Focus group interview, June 2008)

For these forms of collaboration to occur, it was essential that teachers build trust with one another. All participants at one time or another during the focus group interviews brought up the importance of trust and how establishing that sense of mutual trust was a determining factor in the positive outcome of the project.

Even early on in the process, teachers could detect changes in their thinking as a result of participating in lesson study: "I think I was sceptical coming in to this project…how is a formal lesson going to change things? But it does because it changes your thinking" (Focus group interview, December 2007). Later, teachers became even more explicit about the changes in self-perception:

> It all goes back to the fact that the teacher is focused on one thing and re-evaluating their style of teaching, is trying something new, is experimenting with something new – with all the future lessons we are going to be influenced [by this learning]. (Focus group interview, June 2008)

One experienced teacher talked about the enduring impacts on her professionally: "And I feel that I've grown more this year than I have in 17, of taking risks myself and trying things in new ways." (Interview transcript)

Wallace (1999) discusses "true collaborative cultures", which are not driven by specific projects but are "deep, personal and enduring and are absolutely central to teachers' daily work" (67). Little (1990) calls this "joint work". As Puchner and Taylor (2006) explain, joint work means a shared responsibility for teaching, and because it requires shifting from the private isolation of the classroom into the public sphere, it requires teachers to admit that they do not know everything, and that they might need to rely on someone else.

> …how amazing it feels to feel safe taking a risk and trying to become better and not being afraid to make mistakes. And I think the kids are seeing that, and as a result the kids aren't afraid to make as many mistakes. (Teacher Interview, Feb. 2008)

Lesson study may be a way of establishing a culture of collaboration where teachers feel safe taking the kind of risks that allow deep learning, with enduring impacts on teacher perception and practice.

Learner outcomes: student achievement and beliefs.

Although the study was primarily qualitative in design, we collected quantitative data on students to assess the degree of student learning during lesson study cycles. In year one we collected data on the effects of lesson study on student affect. The survey measured student dysfunctional beliefs (the belief that math learning occurs quickly or not at all and that some students are born without math ability Schommer-Aitkins, Duell & Hunter, 2005); and self-reported effort expended in math class (Ross, Xu, and Ford, 2008). Survey items were administered to students in treatment schools and to a matched set of control school students. Schools were matched using provincial math assessment scores and 14 census variables. We conducted a repeated measures analysis of the survey scores in which the within-subjects measures were student affect, repeated at pre and post test occasions. The between-subjects measure was study condition (lesson study or control). All instruments were reliable (alpha=.70+).

Student dysfunctional beliefs declined (a positive outcome) in the lesson study schools, while they increased in the control schools. Further, self-reported effort levels were maintained by students in the lesson study school, while self-reported efforts declined in the control school students. The differences between lesson study and control schools were small but consistent. This population of struggling mathematics students has a long history of becoming more and more disengaged with mathematics through the grades (O'Connell Schmakel, 2008). Maintaining healthy attitudes and beliefs about mathematics learning was an important positive outcome of lesson study activity.

In the second year, teachers in one school administered items measuring grade 7-9 students' conceptual and procedural mastery of volume, which was the focus of the school's lesson study. Students demonstrated modest increments in procedural skills, even though the focus of the lesson study was on conceptual understanding, and had even greater gains in conceptual understanding. The effect sizes (Cohen's d) were small but consistent.

Mandated provincial tests supported the claim that learning improved for students in this same school as scores on the Grade 9 basic level mathematics test improved from 38% reaching the standard before the lesson study project began to 48% after two years of school participation. During the same time period, math scores for the district as a whole declined from 37% to 33%.

SUMMARY

The methodological framework of design research was an excellent fit with lesson study in this research project. Essentially the lesson study activity of teachers acted as a form of design research that generated practical products such as lesson plans as well as theoretical products such as shifting conceptions of mathematics teaching. Simultaneously, the researchers engaged in design research to test and refine the lesson study cycle, developing a deeper theoretical understanding of lesson study as a form of professional learning. From this study we learned that *teacher-directed, classroom-embedded and research-supported* professional learning in the form of lesson study had a positive effect on teacher collaboration and risk-taking, student achievement and student attitudes toward learning mathematics.

Findings from this study suggest that a design research approach to professional learning (using action-oriented models such as lesson study) offers tremendous opportunities for teacher-researcher collaboration with a central focus on ways to improve and refine learning, instruction and research simultaneously.

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SOFTWARE USED IN A MATHEMATICS DEGREE

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This paper examines the software used by staff and students in an undergraduate mathematics degree. The theoretical framework is activity theory. Of software used Excel has a privileged position. We argue that spreadsheet use 'fits' with the objective of the activity and an important rule of the mathematical community and that multiple agents contributed to the privileged use of Excel.

Introduction

In this paper we look at the software used in an undergraduate (UG) mathematics degree programme at Sheffield Hallam University (SHU). SHU is well known in England for deep integration of technology into its UG mathematics programme. The Mathematics Department (MD) at SHU has made extensive use of technology for many years but the extent of use of different types of software has changed over time and the software most used in recent years is spreadsheets (*Excel*). Given that many MDs use specialist mathematical software (e.g., *Mathematica*, *SAS*) we were intrigued as to why *Excel* has a privileged position at SHU. We present an exploratory single case study (Ying, 1994) of software used by staff and students in an UG mathematics programme. We make no attempt to generalise findings but this qualitative study advances knowledge of the complexity of factors influencing the software used in a degree programme.

This paper is a part of a study which examined technology use at SHU MD which, in turn, was part of a funded project which looked at technology integration in UG mathematics instruction in Canada and the UK. Research objectives included: mapping existing international research and literature pertaining to university mathematics teaching with computer algebra systems (CAS); providing an overview of CAS usage in Canadian universities and to compare this with international trends; highlighting, using selected departmental case studies, exemplary practices relating to CAS-based instruction. SHU MD was selected as being useful as a comparative case study. This paper is structured as follows: the next section gives an account of the development of SHU MD; we then consider the use of technology in UG mathematics; there follows an overview of the theoretical framework where we pay particular attention to constructs used in the Discussion section; the next two sections present the methodology used and selected results; the paper closes with a discussion which focuses on issues introduced in the Theoretical Framework section.

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Undergraduate mathematics at SHU

The SHU Mathematics degree has run in its present form since 1996. It was consciously designed to reside at what is now called the practical end of the national benchmark for degrees in mathematics, statistics and operational research (QAA, 2007). This positioning entailed an emphasis on applications and solving practical mathematical problems, and involved considerable and concerted use of a range of technology to support both mathematical learning and the implementation of mathematical techniques, for instance in a modelling context. The concerted use of technology was already ensconced in the MD's practice, for example in encouraging large groups of engineering students to make constructive use of graphic calculators (GC), and using spreadsheets to support statistical learning or implementation of numerical methods with engineers and biological science students. CAS systems were also gradually included over time, with practice in all technologies transferred back and forth between 'service' courses and the Mathematics degree.

The practical emphasis fitted well with the university context. SHU was one of the largest providers of 'sandwich education' in Europe, and the Mathematics degree includes a popular optional one year job placement. An impetus for including the use of spreadsheets came from a perception that they provide a tool for elucidating mathematical concepts in linear algebra, numerical methods and statistics. However feedback from employers soon suggested that a facility with technology in general, including spreadsheets, was a valuable 'employability skill' in its own right.

THE USE OF TECHNOLOGY IN UG MATHEMATICS

Perceived weaknesses in students' mathematical preparedness for university study and the availability of new technology has prompted numerous mathematicians to experiment with innovative teaching and a number of them have turned their attention to pedagogy (Buteau et al. 2009). In many cases the use of technology into undergraduate teaching is seen as way to revitalise teaching and to assist students in raising their level of mathematical understanding (Devlin 1997). Although university-level mathematics teaching is undergoing considerable changes, little attention has been paid to teaching issues at this level by the educational research community. In particular, little has been known about the current extent of technology use and mathematicians' practices in university teaching. An exception is Lavicza (2008) who surveyed 4500 mathematicians from Hungary, the UK and the USA. Lavicza claims that: (i) many mathematicians, applied and pure, use technology regularly in university instruction; (ii) the use of CAS in one's own research is the strongest predictor for the use of technology in one's teaching; (iii) university teachers' international mobility and awareness of trends in research and teaching strengthen their ability to implement significant innovations in technology integration into educational settings; (iv) mathematicians are less bound than school teachers by centralised curricula and tests and have greater freedom to develop their own course materials, teaching approaches and assessment.

Theoretical framework

Our theoretical framework is activity theoretical but activity theory (AT) takes several forms and we use constructs from two forms in the Discussion section, so we explain these forms and constructs. Engeström (2001) describes three generations of AT: (i) Vygotsky and mediation where the unit of analysis was focused on the individual; (ii) Leont'ev's expansion to account for individual action in collective activity; (iii) multiple interacting activity systems. Engeström's (2001) version of third generation AT has five principles: the collective activity system (AS) is taken as the unit of analysis; the multi-voicedness of ASs; ASs take shape over time; the role of contradictions as sources of development of ASs; the possibility of expansive transformations (re-organisation of object and motive) in ASs.

A debate in AT concerns the 'unit of analysis'. Daniels (2001) contrasts Wertsch's focus on mediated action with Engeström's focus on activity systems. Cole (1996, p. 334) claims "Mediated action and its activity contexts are two moments of a single process". Like Cole we recognise there are times (within research) to focus on mediated action and times to focus on ASs. Wertsch's (1998) work on mediated action considers Vygotsky's construct *internalisation* via two aspects, mastery and appropriation of mediational means (tools). Mastery relates to the skilful use of a tool and appropriation relates to ones disposition towards a tools. "In most cases … [they are] … intertwined, but … [they are] empirically distinct" (p.53). In the Discussion section we claim that mastery and appropriation of *Excel* co-evolved in the AS SHU MD. In the Discussion section we also consider agency. Agency, as a construct, began as personal agency (free will/choice) but has grown to include collectives (what others want us to do), agency of tools (the constraints and affordances of tools) and disciplinary agency, e.g., mathematics (Pickering, 1995).

METHODOLOGY

Two visits to SHU MD were conducted in 2009 by pairs of authors DJ, ZL and JM: to interview 'core' MD staff; to observe teaching activities with particular emphasis on final year students; to interview students during or after these activities and to collect documentary data (department handbooks, assignments, student projects, etc.). For this paper we report on interviews: eight with MD staff, six with students and two with SHU leaders. All staff interviews were semi-structured and used the hierarchical focussing technique (Tomlinson, 1989) whereby a structured interview is planned but executed through open questions. The MD staff interviews had one open question, 'tell me about your department', with follow up questions on: the interviewee, the staff (teaching, research, networks), mission statement, place of MD in SHU, degree programmes, students, use of technology and room allocation. The two interviews with SHU leaders were not initially planned but MD staff interviews suggested such interviews would be important to position the MD within the university structure (an AS). The students were only asked questions about their courses and technology.

Neil Challis is an author of this paper and is the head of the MD and this is a possible source of bias. But he was not involved in data analysis and the study was an exploration of technology use in his department, not an evaluation of this use. Further to this he was able to provide useful comments on our interpretation of the data.

Two methods of analysing interview transcripts were used to provide the account of technology use which follows. The first was open coding (à la Strauss and Corbin, 1998) but stopping short of a full grounded theory approach as we already had a theoretical framework. Coding was done through *Atlas.ti* and 12 codes and documents were produced, of which 'technology used' was one. The second method, by a different researcher, was done as a reliability check on the 'technology used' document, and was completed by highlighting all words related to technology in the transcripts (e.g., technology, computer, *Derive*, *Excel*, etc.). The account of technology use in the two documents was deemed consistent.

RESULTS

We present four issues arising from data (analysis) to illustrate the embeddedness of technology and then explore the place of *Excel* within the technology used. These issues inform discussion in the next section. The four issues are: the presence of technology in every code; examining specific software/hardware (SW/HW) other than general technology comments; an investigation of whether *Excel* is privileged because of the importance attached to graduate employment; interview extracts that present an 'anti-black box rule'.

The presence of technology in every code

Technology was present in the interview extracts assigned to each of the 12 codes. At one level this is not surprising as staff interviewees knew that our interests centred on the use of technology in the degree programme, but interviews were sustained and wide- ranging and there was no reason why technology had to enter the excerpts assigned to every code, but they did. In the following we present each code (other than 'technology used') followed by an interview except.

<u>History</u> We weren't allowed to run a maths degree until about 10 years ago … and people were realising computers were quite important, that we wrote our course.

<u>Structure</u> In the final year we have a large 30-credit project module and many of them do a lot of programming in that, particularly with *Excel*.

<u>Rationale</u> [See inset excerpt in the final sub-section below]

<u>Instructor background</u> [After my degree] I got out into industry and the first thing I had to do was to learn the numerical method and … Fortran programming.

<u>Obstacles and challenges</u> We ran a seminar for our engineering colleagues … we got into all those arguments about the role that technology can play.

<u>Recruitment</u> Our programme is very much based around the use of technology. We make this clear to students from the outset … We've got an open day.

Learning community The log book, it's a log centre on our website … there are a lot of tools we use for tracking what they're doing and communicating.

Mathematical concepts I think most of us are genuinely excited about what you can do with a mathematical approach using technology.

Assessment We … let them use calculators without the algebraic facility in exams.

Sustained departmental shift A vast bulk of the group that are absolutely committed to that technology and to that philosophical approach.

Student project [We] use *Excel* for image analysis … an add-in for image analysis which means we can just manipulate images, blur them, structure them.

Examining specific SW/HW other than general 'technology used' comments

A lot of SW/HW is used: "the balance is 50% maths, 50% technology" (final year student). "There is a computer programme modules where you will learn *Excel* but it also does XTML, PHP, Java Script. There is also a computer programme where you use C+, Visual BASIC, the actual Visual BASIC not in *Excel*." (final year student) Students mentioned *MathLab*, *Excel*, *SAS*, *GeoGebra*, *Derive* and *Front_Page*. *Excel* (19) and *SAS* (19) account for the vast amount of student references to SW. Students also mentioned HW: their own laptops, SHU computers and GCs.

In this paper we are interested in what mathematical SW was used and why specific SW was used. However, except when following up on an interviewee's comment on specific SW, interview questions were framed in terms of technology, not software. The following summarises descriptive statistics from all interview transcripts regarding general key words (technology, computers, ICT, calculators) and specific SW keywords (*Excel*, *Derive*, CAS, *SAS*, etc.).

Over all interviews general key words were used much more often than specific SW keywords. There were 'trends' in the interviews: the two leaders did not refer to any specific SW/HW; students talked about specific SW more than any other group. Of the lecturers, two only talked in terms of generalities. Of the other six there was still a trend to talk in of generalities, but of specific SW keywords used, 34 were to *Excel* or spreadsheets compared with 16 to *SAS*, *Derive*, CAS, TI GCs and the web.

Is *Excel* is privileged due to the importance attached to graduate employment?

Interviews with some staff suggested that the answer to the question above is 'yes'. For example, a lecturer paraphrasing a businessman happy with a student placement, noted, "He revolutionised our store's record keeping with the spreadsheet that he designed" and another talking about students claimed, "They'll always get the job because they can talk and communicate … and they can play with spreadsheets very effectively". To explore this further we constructed a document in which we included all references to a word containing 'employ' in it (e.g. employment, employability). We retained meaningful text preceding and succeeding these words to put the 'employ words' in context. There were 71 'employ words' in this document, 19 uttered by interviewers. Of the 52 uttered by participants, 13 were from one 'leader', seven from students and 32 by mathematics lecturers. There were nine references to either *Excel* (five, all by students, four by a single student) or spreadsheets (four, all by lecturers). The only other reference to mathematical SW (MSW) in this document is to *SAS* (two instances, each by a student). So, spreadsheets are the dominant MSW reference but if spreadsheets were used just for the employability objective, then one would expect more reference to spreadsheets by staff. However, whilst there is not evidence that spreadsheets are used simply because of the employability objective, there is clearly some relationship here. We explore this further in the next section where we use the word ‘fit’, i.e. spreadsheet use fits with the employability objective.

Rules

The term ‘rules’ can mean different things in different theoretical frameworks. To Ostrom (2005), ‘or else’ rules are particularly important in studying institutions. Rules in AS analyses consist of templates for action by the community for realising the objective of the AS. In SHU MD, as a part of SHU, there are many ‘or else’ rules regarding, for example, attendance at lectures and the conduct of examinations. Below we consider an unwritten rule concerning MSW and the transparency of mathematical operations. This rule (though expressed as an aim) was described clearly by the leader of the MD:

> Our aim, my aim, has always been—not that the technology would de-skill people, so you don't use it as a black box, what you do is you use the technology which forces you to really understand what you're doing before you can use it. And that's why I like the spreadsheet so much, because you've got to understand what the inter-relationship is, and you really need to think about, you know … So they're learning a tool which is widely used in the industry, but they're also using it … enhance what they're learning mathematically as well. … modelling is a terrific area for doing this kind of stuff.

This excerpt illustrates a positive attitude to spreadsheets from a negative – they are not black boxes, i.e., one must program them (mathematically) rather than the programming being obscured in code that the user does not provide. This ‘anti-black box rule’ was expressed in similar terms by another member of staff, “The danger is that you’ve got that black box that gives you all the answers and that’s certainly not a way to train mathematicians—maybe some engineers—the understanding of the process”. Other members of staff expressed it in slightly different terms, e.g., with regard to coding logic in *Excel* for mathematical modelling, “We are actually using the logic functions in the cells rather than macros, and seeing how you can get those to inter-relate to form a model”. Students were also aware of this rule, e.g., a final year student in talking about learning, “If you can put the maths into a computer program they’ll understand it more”. We return to this rule in the next section.

Discussion

We consider constructs raised in the theoretical framework, activity systems (AS) and mastery/appropriation/agency, with respect to SW used (as a subsets of tools used in the AS), in the light of the results we have presented.

In terms of Engeström's (2001) version of third generation AT, both SHU and the MD can be viewed as an AS. Our primary unit of analysis is the MD, its people (with motives), mediating artefacts (tools), rules, and division of labour. The object, clearly stated by leaders and MD staff, is employable mathematics graduates. Interviews evidenced the voices of the staff and students, of SHU leaders and of employers. We view the current AS SHU MD to be the product of its history, with its tensions, and periods of expansive learning. We now focus down to the tools in this AS. The tools are diverse and interrelated: lecture/workshop formats; assessments; student log books; as well as mathematical tools including software. In as much as technology permeates all of the codes/categories, technology permeates many if not most of the tools in the AS. We now focus further down to MSW and begin by noting two things: (i) MSW is a proper subset of the technological tools used, e.g. the MD web site includes a number of tools that are not MSW; (ii) apart from a few modules which introduced SW, MSW was used as considered appropriate by students in mathematics modules, e.g. students could use *Excel* or *SAS* for statistical purposes. Within the set of MSW, *Excel* use is privileged but not exclusively so. We consider *Excel* with regard to the 'anti-black box rule' as presented above in the AS and the object of the AS. Interviews revealed that MD staff regarded both *SAS* and *Derive* but not *Excel* as 'black box' MSW. There is, of course, other MSW which is not black box SW but *Excel* 'fits' with this rule. *Excel*-use also 'fits' with the object of the AS, graduate employment. We have argued above that while there may not exist a direct relationship between *Excel*-use and graduate employability, 'fit' seems an appropriate word to describe this relationship. The use of *Excel* is thus consistent with a rule and the object of the AS. We regard this as an important point with regard to the use of MSW in UG mathematics programmes and why we refrain from any attempt to generalise about such MSW use, for had the rules and object been different, then *Excel* may not 'fit'. For example, *Excel*-use is unlikely to fit with an UG programme with an objective of 'maintaining traditional standards with regard the content' and concomitants rules such as 'must pass exams in real analysis and linear algebra before proceeding to their final year'.

We now consider mastery, appropriation and agency with regard to *Excel*-use. We agree with Wertsch (1998) that mastery and appropriation of tools may develop unevenly, e.g. one may master a tool that one does not appropriate, or that the development may go hand-in-hand. In our opinion SHU MD staff mastery and appropriation of *Excel*-use not only went hand-in-hand but 'evolved' over years: MD staff mastered *Excel* and did not have a disposition against it; initial use of *Excel* fitted with the anti-black box rule and the employability object; use of *Excel* in modules increased and again this use fitted with the rule and the object. This appears to have stabilised to the current system of *Excel*-use being privileged.

We now turn to 'agency' and our argument is that MD staff never decided (on their own) to privilege *Excel*-use in the degree programme, the voices of others had a say in this. The voices of others were: employers who were delighted to have placement students with such strong *Excel* skills; and the voices of students who were pleased to have their *Excel* skills valued by others. Further to this, we argue that the anti-black box rule and the employability object exerted agency with regard to choice of tool. This statements may need clarifying with regard to the anti-black box rule. Pickering (1995) was the first to suggest disciplinary agency. The argument can be put in terms of 'are you free to put any answer down when given $a + a$?' Of course you can write down something other than $2a$ but the discipline of mathematics leads you to write $2a$. We believe the discipline of mathematics in SHU MD includes "understand why the output is the output" and this leads to the use of a tool which enables this to be realised. The upshot of this is that multiple agents impacted on use of *Excel* at SHU.

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PROSPECTIVE TEACHERS' WAYS OF MAKING SENSE OF MATHEMATICAL PROBLEM POSING

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This study investigated prospective teachers' ways of making sense of mathematical problem posing [PP] and the impact of posing various types of problems on their learning. Focus was on the generation of new problems and reformulation of given problems. Participants were 40 prospective elementary teachers. They were required to pose problems for diverse specified situations. Data included their problems, reflective journals, and interviews. The findings provide insights into possible ways prospective elementary teachers could make sense of problem posing of contextual problems and the learning afforded by posing diverse problems. Highlighted are five perspectives of problem posing and nine categories of PP tasks important to support their development of proficiency in problem-posing knowledge for teaching.

INTRODUCTION

This paper is based on a larger, ongoing project that investigates mathematics teachers' sense-making of contextual problems, problem solving, and problem posing and their development of problem-solving proficiency and knowledge for teaching. The project involves prospective and in-service elementary and secondary school teachers. The focus here is on prospective elementary school teachers and their mathematical problem-posing knowledge for teaching.

Problem posing [PP], like problem solving, is promoted as an important way of learning and teaching mathematics (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics [NCTM], 2000). But whether or how this view gets implemented in the classroom will depend on the teacher and how he or she understands it. Thus, it is important to understand teachers' sense-making of PP and ways to help them to develop meaningful PP skills. This study contributes to this through the investigation of prospective elementary teachers' sense-making in posing word/contextual problems and the impact of posing various types of problems on their learning.

RELATED LITERATURE

Since the 1980s, there has been increased attention in promoting PP as an important aspect of school mathematics. The NCTM (1989, 2000) has proposed increased emphasis on PP activities in teaching mathematics. Kilpatrick (1987) and Silver (1993) have suggested that the incorporation of PP situations into mathematics classrooms could have a positive impact on students' mathematical thinking. Brown and Walter (1983) have also identified important aspects of PP in mathematics. Many benefits are gained from PP, such as enhancing problem-solving ability and grasp of mathematical concepts, generating diverse and flexible thinking, alerting both teachers and students to misunderstandings, and improving students' attitudes and confidence in mathematics (English, 1997b; Silver, 1994). PP activities reveal much about the understandings, skills and attitudes the problem poser brings to a given situation and thus is also a powerful assessment tool (English, 1997a; Lowrie, 1999).

Studies on prospective elementary mathematics teachers have raised issues about their knowledge of problem solving. While such studies imply related issues with their PP knowledge, this is an area that is under-explored. Studies on PP tend to focus on students at school levels. Such studies have increased attention to the effect of PP on students' mathematical ability and the effect of task formats on PP (Leung & Silver, 1997). Some studies have investigated the extent to which children generate problems (Lowrie, 1999; Lowrie & Whitland, 2000; Silver et al., 1996). One finding is that unless children are encouraged to talk about problem solving (Lowrie, 1999) and share ideas during mathematical activities (English, 1997a) they tend to pose traditional word problems that are variations of those found in textbooks. Lack of exposure to meaningful contexts for problems was also found to restrict students' ability to pose problems (Stoyanova, 1998). Since students grow up to become teachers, it is likely that prospective teachers maintain some of these issues that will then continue the cycle unless they are helped in appropriate ways.

PERSPECTIVE OF PROBLEM POSING

Dunker (1945), and more recently Silver (1994), described PP as referring to both the generation of new problems and the reformulation of given problems. Stoyanova (1998) defined it as the process by which, on the basis of concrete situations, meaningful mathematical problems are formulated. For English (1997a), generating new questions from given mathematical tasks is considered to be the main activity of posing problems. However, as Silver et al. (1996, p. 294) explained, "The goal is not the solution of a given problem but the creation of a new problem from a situation or experience." Importantly, the problem poser does not need to be able to solve the problem in order for positive educational outcomes (Silver, 1995).

In this study, the focus is on the generation of new problems and reformulation of given problems. The relevance of this is associated with the teacher's role in selecting, creating, or posing appropriate problems to engage students in meaningful problem-solving experiences (NCTM, 1989; 1991). To promote diverse and flexible thinking for students, it is critical for teachers to be able to generate diverse problems. They need to be able to generate a broad range of problems to widely combine situations with mathematical concepts or solution methods. For example, for mathematics teachers to develop quality-structured PP situations, they should be able to pose problems based on textbook problems by modifying and reshaping task characteristics; formulate problems from every-day and mathematical situations and different subjects' applications; restart ill-formulated or partially formulated problems and pose complex and open problems as well as simple problems.

Problem posers have to appropriately combine problem contexts with key concepts and structures in solutions along with constraints and requirements in the task. Thus, both contextual settings and structural features of problems are recognized as crucial. Comparison between problems is also important. As Gick and Holyoak (1983) demonstrated, similarity judgement between problems facilitated the induction of schemata, that is, general information about key elements and their relationships in the problems. In PP, it is important to identify key elements and their relationships embedded in problems (English, 1997b; Leung & Silver, 1997).

The preceding theoretical background about PP provided the basis for selecting PP tasks used in the study and for framing the research method.

RESEARCH METHOD

Participants were 40 prospective elementary teachers in the second semester of their two-year post-degree Bachelor of Education program. They had no instruction or exposure to formal theory on problem solving or PP prior to this PP experience. This timing of the study was intended to capture their initial ways of making sense of PP.

The PP experience included comparing problems of similar and different structure and responding to PP tasks involving posing a problem: (i) of their own choice, (ii) similar to a given problem, (iii) that is open-ended, (iv) with similar solution, (v) related to a specific mathematics concept, (vi) by modifying a problem, (vii) using the given conditions to reformulate the given problem, (viii) based on an ill-formed problem, and (ix) derived from a given picture. Table 1 offers examples of the tasks.

1. Create a "word problem" of your choice for students in a grade of your choice.
2. Create a "word problem" that you think is open-ended.
3. Create a "word problem" that you think is similar to the following problem:
Tennis balls come in packs of 4. A carton holds 25 packs. Marie, the owner of a sports-goods store, ordered 1600 tennis balls. How many cartons did she order?
4. Create three "word problems"; each related to a different meaning of multiplication of whole numbers.
6. Create a "word problem" for the following situation:
Some students held a bake sale to raise money for a local charity. They sold fudge, brownies, and cookies. Each type of treat was put into paper bags and the students were allowed to keep the leftovers. They started out with 110 cookies, 130 pieces of fudge and 116 brownies.

Table 1: Examples of the PP tasks

These PP tasks were presented one at a time in an intentional sequence to minimize the influence of one task on participants' thinking of another. Participants were also required to focus on their thinking as they created the problems in order to notice and document it. They were told to interpret "word problem" in flexible ways that made sense to them. It was not intended to mean only traditional-style problems.

Data sources were the participants' written work for the PP experience and reflective journals of their thinking. Upon completing all tasks, they wrote journals describing what they learned in general and about mathematics, PP, problem solving, and teaching and learning mathematics. Six of the participants whose thinking seemed to be representative of different ways of making sense of PP were interviewed to further explore and clarify their thinking. Interviews were audio taped and transcribed.

Data analysis began with a process of open coding (Strauss & Corbin, 1990). In addition to the researcher's coding, two research assistants conducted this open coding independently of the researcher, and independently of each other. Only after initial categories had been identified were the results discussed and compared and revisions made where needed based on disconfirming evidence. Themes emerging from the initial coded information were used to further scrutinize the data and then to draw conclusions. There was triangulation among participants' problems, interviews, and journals. Coding included identifying (i) the types and nature of the problems the participants posed based on guidelines developed from the literature and (ii) participants' sense-making and learning based on significant statements in their thinking and the knowledge implied in the context and structure of problems. The coded information was summarized and categorized for each participant and compared for similarities and differences in their thinking, knowledge, and learning.

FINDINGS

The findings represent the participants' ways of making sense of PP prior to taking any mathematics education courses. The focus here is on their sense-making of PP in general and a sample of the PP tasks and their learning from the PP experience.

Sense-making of PP in general

Collectively, the participants' thinking displayed the following five perspectives of posing "word problems" that related to their sense-making of PP. While these were partly influenced by the PP task, they all emerged in tasks where there was no problem to influence their choice or thinking (e.g., Table 1, #1 & #2) and prior to seeing the other tasks.

(1) A paradigmatic perspective that emphasizes PP as creating a problem with a universal interpretation, a particular solution and an independent existence from the problem solver. This was evident in some of the participants' problems of their choice and reflected their experience with traditional word problems.

(2) An objectivist perspective that is similar to (1) but is specific in considering PP as creating an object involving a mathematical fact. Thus the goal is primarily to work backwards from the fact that needs to be computed or determined in the problem. For example, start with a number sentence (2 x 3 = 6) and clothe it with a context.

(3) A phenomenological perspective that emphasizes PP as creating a problem that is meaningful from the learner/student's perspective and provides a lived experience, i.e., allows students to interact with problem contexts in a personal way and produce personalized interpretations and solutions. This was common for the open-ended problem. For example, "John is going to the grocery store and needs 6 fruits total. How many apples, oranges, and pears did he buy?" This participant explained that students could choose any amount for each fruit as long as the total was 6. Another participant explained that students could decide who get how many in the following:

> Gary received a package of jelly beans for his 9th birthday. He decides to share them with his 3 friends, Brad, Gilles and Monica. If there were 26 jellybeans in total, how many would each person receive?

(4) A humanistic perspective that is similar to (3) but is specific in considering PP as creating situations directly related to personal aspects of the students' experience; for example, their interests, meanings, creativity, and choices.

(5) A utilitarian perspective that emphasizes PP as creating problems in terms of their worth based on their contributions to students' learning.

Sense-making of each PP task

Only three of the tasks, which the participants considered to be the most challenging, are discussed here to highlight the uniqueness of their thinking. For posing an open-ended problem, their common thinking was that open-ended meant more than one answer but there was uncertainty about what this meant mathematically. One explained, "Open-ended means more than one answer, but when I think of math I can only think of one answer, so I couldn't provide an example." Some of the problems they posed were ill-formed, not mathematical, or lacking sufficient information, but not done intentionally or with awareness of these features. Other problems involved multiple operations (but not open) and potentially yes/no/don't-know answers. For some problems, open-endedness involved any interpretation/solution whether or not appropriate for the given conditions. Examples of their open-ended problems:

> If the population of the earth increases every year by 500000, does the mass of the earth increase?
>
> A teacher creates a lesson on study of fish. Students are to observe the fish over the year. Will there be fish babies at the end of the year?
>
> How many times does Ben have to bounce his basket ball before he refills it with air?

For the multiplication task (Table 1, #4), 40% of participants created one problem, 40% created two problems and 20% created three problems. Collectively, they produced one meaning for multiplication – combining equal groups. The problems involved multiplication only, division only, or various combinations of two or three of the four arithmetic operations. For many problems, participants did not attend to relationships among numbers, operation, and context and whether the problem made sense structurally. Their thinking and problems indicated that they were unaware of their focus/interpretation/use of "times" (e.g., 3 times more; how many times; 3 times 6; 3 times older), which resulted in the various combinations of operations and not necessarily attending to the meanings of multiplication. Their problems included:

> How many books are on the shelf if each book is an inch thick and the shelf is 15 inches wide?
>
> A mouse has 3 babies in January. If a mouse is pregnant for 3 months at a time and has a litter of 3 every time, how many children will she have in September?

The picture task, which represented a comparison meaning of subtraction (i.e., one column of eight objects compared to a parallel column of five of the same objects showing how much more) was not interpreted this way by any one. They focused on pairing, "left over" and other interesting possibilities as in these examples.

> How much energy/force would be required to move the marbles in the left column to where they are in the right column?
>
> Mrs. C found 5 pairs of gloves and 3 toques in the lost and found. How many items did she find altogether?
>
> I have 13 players in a tennis tournament. I need 3 score keepers. How many games of tennis would be playing with the remaining players?

Learning from the PP experience

The participants focused on self-awareness in describing their learning. They became aware of what they could or could not make sense of, were uncertain of, and wanted to learn more about regarding PP and the mathematical concepts they encountered in the process. They developed awareness of the importance of context in PP. They realized that PP can be challenging and developed a different understanding of it and appreciation of its importance in learning mathematics. As one participant explained:

> I learned how difficult it is to write math questions that are open-ended and require thinking rather than memorization. … I learned the differences between thoughtful questions and questions that I experienced that can make math stressful and boring for students. … I learned that math is not just memorizing multiplication tables and adding at the elementary level. It can be creative and have problem solving at a very young age. … I learned that by writing questions properly, students can be given the opportunity to share their own good ideas on how to deal with problems. … I learned how problem solving can be presented as more about memorization of skills, like the way I learned it, than about creating problem-posing abilities.

Participants also gained self-understanding of limitations of important aspects of their mathematics knowledge for teaching. The tasks required understanding of different mathematics concepts and provoked different ways of thinking about and reflecting on PP which allowed them to engage in mathematical thinking in a variety of ways.

CONCLUSIONS AND IMPLICATIONS

The participants' sense-making of PP was dependent on their mathematical knowledge, imagination or creativity, and past experience with problem solving.

They were challenged most by the tasks to pose questions that were open-ended, related to a specific mathematics concept (meanings of multiplication), and derived from a given picture of a mathematics concept (comparison subtraction). These tasks conflicted with their prior experience that exposed them mainly to closed problems and one meaning of each operation. Many of them were able to imagine and create interesting problem situations but, generally, their sense-making of posing "word problems" often excluded intentional or conscious consideration of mathematical structure or context of the problems or the relationship to the problem situation. The five perspectives of PP identified in the study (i.e., paradigmatic, objectivist, phenomenological, humanistic, and utilitarian) indicate ways of thinking about "word problems" and posing problem situations they can make sense of and thus provide a meaningful basis to build on to enhance their PP skills for teaching. In spite of this range of perspectives, individually, their initial ways of making sense of PP on entering the education program was limited by their lack of experience with PP and exposure to mainly traditional ways of experiencing problems and problem solving.

The study suggests the need to attend to the PP knowledge prospective elementary teachers bring to teacher education in addition to addressing PP as an explicit topic in order to help them to build on, reconstruct, and extend their sense-making of it. The five perspectives of PP provide a basis to compare and unpack their ways of PP. All five need to be explored in order to allow the teachers to understand how each could support or inhibit students' mathematical understanding and mathematical thinking. The nine categories of PP tasks provide a meaningful basis of prospective teachers' self-understanding and self-study of PP. The examples provided of the participants' thinking for three categories of tasks (i.e., open-ended, meaning of a concept, and picture of a concept) draw attention to potential areas of concerns that are important to address explicitly in teacher education. These examples, linked to mathematics concepts, also imply that it is necessary for PP to be integrated as part of prospective teachers' learning of the mathematics concepts they are expected to understand for their teaching. Their relational understanding of such concepts is needed to support their PP knowledge and vice versa. This blending of the two could allow them to develop the flexibility to engage students in PP not only in terms of being able to create and select worthwhile tasks, but also on an impromptu basis during mathematical discourse and teaching problem solving.

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PROBABILITY ZERO EVENTS: IMPROBABLE OR IMPOSSIBLE?

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In this study we examined prospective secondary mathematics teachers' ideas related to probability zero events. We asked prospective teachers to provide examples of such events and analysed their example spaces according to the sample spaces of the events. The results reveal strong dependence on the classical perception of probability and a rather limited set of attributes used in responding to example-generation tasks. Several issues related to the vagueness in the definitions of probability concepts are discussed.

BACKGROUND

Extended attention to probability and statistics in school curriculum resulted in renewed interest in these topics in mathematics education research. Despite the growing number of studies that explore understanding of probability concepts among students of different ages, the research on teachers' knowledge and beliefs related to probability is still rather limited (Jones, Langrall, & Mooney, 2007). Furthermore, among specific probability topics under investigation, no explicit attention has been paid to events with probability zero. Our study aims at addressing this deficiency, while also contributing to research on teachers' knowledge of probability.

THEORETICAL FRAMEWORK

Within a variety of educational uses of examples in mathematics, we view examples as "illustrations of concepts and principles" (Watson & Mason, 2005, p. 3). This study is conducted within the framework of *learner-generated examples* (LGEs). Watson & Mason (2005) considered LGEs – an approach in which learners are asked to provide examples of mathematical objects under given constrains – as a powerful pedagogical tool, through which learners enhance their understanding of the concepts involved.

Watson and Mason also introduced the construct of *example space* as collections of examples that fulfil a specific function, and distinguished among several kinds of example spaces. Of our interest in this study are *personal example spaces*, triggered by a task as well as by recent or past experience, and *collective example spaces,* local to a classroom or other group at a particular time. When invited to construct their own examples, learners both extend and enrich their personal example spaces, but also reveal something of the sophistication of their awareness of the concept or technique (Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006). In accord with this observation, Zazkis and Leikin (2008) suggested that LGEs provide a

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valuable research tool as they expose learner's ideas related to the objects under construction. Goldenberg and Mason (2008) illustrated how the construct of example space can inform research and practice in the teaching and learning of mathematical concepts.

METHODOLOGY

The participants of this study were pre-service secondary school teachers (n=30), holding majors or minors in mathematics or majors in science. There were asked to respond in writing to the following task:

- Give an example of an event with probability zero.
- Give an example of a more complicated event with probability zero.

The time for completing the task was not limited. The task was followed by a classroom discussion that addressed the examples provided by the participants and the general notion of probability zero events.

The research questions we address in this study are:

- How do pre-service teachers interpret and exemplify probability zero events in variety of situations?
- What are characteristic elements of their personal example spaces, as well as of their collective example space, regarding events with probability zero?

In what follows we analyse participants' responses to the task in order to identify and describe (a) their perception of probability zero, and (b) their understanding of "more complicated". Furthermore, we attend to several issues of interest that emerged in the data analysis and in the discussion that followed the task. These include the use of numbers in the examples, the chosen context of examples and vagueness of probability definitions.

RESULTS AND DATA ANALYSIS

The data were first analysed in terms of the respondents' perception of probability of an event, as derived from their examples. Their ideas of probability appeared to be in accord with the classical interpretation, that is, the ratio of favourable outcomes to all possible outcomes. However, the set of "all possible outcomes", which refers to the sample space of an event, can be either finite or infinite. Further, the infinite sample space can be either countable or uncountable, the issue which considerably complicates the consideration of a probability as a ratio.

What is considered a probability zero event?

Logically impossible events – exact zero

We first focus on examples of events with finitely many outcomes in their sample space, such as flipping a coin, rolling a fair die, or picking a random integer from the set of ten positive integers. Most of the examples provided by the participants (50 out of 60) fell in this category, which we entitled "logically impossible events". Several examples are provided below:

- *Rolling a 7 with a standard die*
- Rolling a sum of 13 with two standard fair dice
- Having head and tails at the same time when flipping a fair coin

In these cases the denominator of the fraction that represents probability of an event is a natural number, so the probability ratio (fraction) can be equal zero if and only if the numerator is zero. Within the provided examples the numerator was considered to be zero if the described event did not belong to the set of possible outcomes. While this kind of examples dominated the collective example space of this group, we show below that during classroom discussion this view was strongly criticised by some participants, with the reference to the definition of the "event".

Very small probability events – estimated zero

While "exact zero" value of a fraction can be obtained with numerator zero, a "close to zero" value can be obtained with a "very large" denominator. Three participants provided examples of events with "very small" probability, which was estimated to be zero.

- Getting all 6 in 6 roll of a fair die.
- Getting a pattern (123123) in 6 roll of a fair die.
- Getting 200 tails in 200 tosses of a fair coin.

Theoretically, the probability of these events is very small (for example, $1/2^{10}$ for 10 heads in 10 flips, which is about 0.00097) and can be considered zero for "practical purposes".

Very small probability events – converging to zero

The previous two categories attended to finite (even though occasionally very large) sample spaces. We turn now to infinite sample spaces in the participants' examples, where seven examples fit into this category. In these cases consideration of probability as a ratio is more complicated. While a numerical value cannot be assigned to the expression $\frac{1}{\infty}$, the limit of 1/x, where x is tending to infinity, is zero ($\lim_{x\to\infty}\frac{1}{x}=0$). This is exemplified by the following:

- Tossing a fair coin infinitely many times, all of them resulting in Heads.
- Rolling sixes an infinite number of times with two fair dice.
- In a roulette game betting on same number every time and always winning when playing infinitely many times.

In fact, these examples do not describe a single event (such as tossing a coin, or tossing a coin 3 times, where the probability can be calculated), but a sequence of events where the probability is converging to zero as the size of sample space is tending to infinity.

Measure-theoretical probability zero

Considering infinite sample spaces, there is also a possibility of infinite and uncountable ones. From a theoretical account we introduce a fourth type of examples as “measure-theoretically explainable probability zero” (Example: picking a certain number from a given interval of real numbers). Two examples provided by the participants could fit this category as they relied on an uncountable sample space; however, no evidence of a reference to measures in the sense that distinguishes a set of countable points versus a set of uncountable points was given.

- *Picking the number 1.0000097 from [1,2]*
- *Picking 4.7123 when picking a random number between 1 and 10.*

However, the classroom discussion suggested that this type of probability zero events could be understood from the point of view of each of the three aforementioned categories. While from a measure theoretic perspective the probability is 0/1 (a single point has a measure of 0 and the interval of real numbers has a measure of 1) it was considered as 1/infinity, that is, picking one possible number from infinitely many numbers. From this perspective two different interpretations of infinity was distinguished: one that treats infinity as an unknown arbitrarily large but fixed number (students: we don’t know how many numbers exist between 1 and 2, but whatever the number is, it is really huge!). For this group of respondents it is one case out of a fixed large number and therefore they perceive the probability as estimated to be zero. Another take of infinity considers it as a sequence of growing numbers, the common approach to infinity when dealing with it in calculus (students: it is one over infinity, it means the limit will tend to zero). For this group however they don’t explain how, but one over infinity provokes the concept of converging to zero.

The results are summarized in the diagram presented in Figure 1.

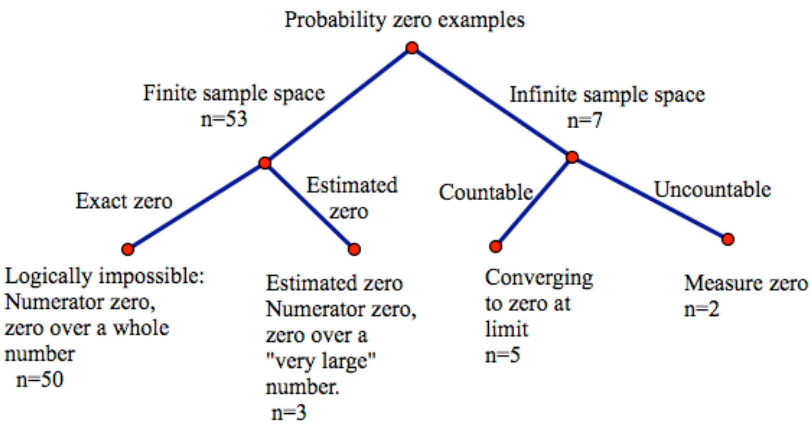

Figure 1: Categorization of examples for probability zero events

What is considered 'more complicated'?

Watson and Mason (2005) discuss the "give another example" strategy as a powerful instructional tool that may direct learners' attention to unifying features of different examples. We modified this strategy by a specific request of making the second example "more complicated" with an expectation for more variety within a larger pool of examples. However, the examination of second examples in this study revealed that in 24 out of 30 cases the first and second examples fell in the same category with respect to the sample space. We further examined how the participants have made their second example "more complicated". It turned out that combining events was a popular technique to describe more complicated events. In 20 examples out of 30 the participants combined two events in order to provide an example of a more complicated event.

Three different types of combination have been identified in the data:

The impossible-possible combination:

In this type of examples the 'impossible' event described in the first example was frequently used as the impossible component in the combination, as in the following pair:

> First example: Rolling a 7 with a fair die. Second example: Rolling a 5 and then rolling a 7 with a fair die.

The impossible-impossible combination:

Some participants have conceived "more complicated" as an event even less likely to happen than their first impossible event. In the following pair the second example is a combination of two probability zero events.

> First example: Getting infinitely many 1's when rolling a fair die infinitely many times. Second example: Getting all faces when flipping a coin infinitely many times while getting infinitely many 1's when rolling a fair die at the same time.

The possible-possible combination with empty intersection:

Another way to get a "complicated" event was to combine the possible events in the sample space such that their intersection is empty, which at the same time makes the combined event logically impossible. The frequent example of this type was getting both 3 and 4 at the same time when rolling a fair die once.

As a second technique for adding complexity, some participants have used generalization, that is, presenting their second example as a generalized form of the first. As such, it satisfied both the conditions: probability zero event and a more complicated one, as in the following pair:

> First example: Rolling two dice and getting (6,7). Second example: Rolling two dice and getting (i,j) such that i+j=13.

As Watson and Mason (2005) suggest, leading the learners toward generalization is one of the merits of asking for another or for a more complicated example.

On context and numbers

The provided examples were examined in terms of the probability generators used to describe events. From the 60 examples, 32 used dice, 14 used coins, and 8 used marbles in a bag (or equivalent variations of it). The remaining 6 examples included a spinner, a deck of cards, and picking a random number from some interval. Two examples involved "real life" situations, such as a vending machine and street crossway. The impact of conventional textbook objects for teaching probability on the example spaces of teachers is conspicuous.

Moreover, any task designed for research that deals with numbers can reveal by-product facts about people's perceptions of numbers and part of their number sense. The task described in this study is no exception. One of such interesting by-products is the different treatment of numbers found in two of examples: in both examples the participants described an experiment of picking a random number from a real number interval and the probability zero event was to pick a certain pre-determined number, 4.3275 and 1.0000097 respectively. It is evident that the examples are of the same nature: they provide "safe" examples of numbers that are not likely to be picked. However both respondents – as was clarified in classroom discussion – were aware of the fact that picking any number has the same probability zero, but they may have felt that numbers like 0,1,2 or $\frac{1}{3}$ were not "safe enough" to mention.

Our unsupported conjecture is that this preference is based on the fact that in the past these students were asked to locate integers and simple fractions like $\frac{1}{3}$ on the number line, but they have never been asked to locate on the number line 1.0000097. As such, some numbers have been "exposed" as bold dots or thick dashes on the number line, whereas others remained "hidden". In short, the participants' examples were influenced by intuition that a number with several decimal places is less likely to be picked at random than an integer, even when their formal knowledge suggested otherwise.

WHAT IS PROBABILITY ZERO? ISSUES FROM CLASSROOM DISCUSSION

As mentioned above, the data collection was followed by a classroom discussion of the examples as well as of the notion of probability zero in general. Several issues of interest related to the definition of probability and of basic probability related concepts surfaced in this discussion. The probability function assigns a value between zero and one (inclusive) to any event. As such, it is reasonable to consider – as did participants in our study – that an event with probability 1 is a "sure" event that will always happen, whereas an event with probability zero is an "impossible event", that can never happen. As shown, most of the examples referred to logically impossible events with a finite sample space. A small group of students carefully avoided this type of examples, and argued that rolling a 7 on a standard dice is not a good example of a zero probability event, since getting a 7 is not an event. They relied on the definition of sample space to be the set of all possible outcomes, provided that they are equally likely. The issue appeared to be with the definition of an event, which is, conventionally, any subset of outcomes of the sample space. While rolling 7 is not among possible outcomes – they argued – it is not in the sample space and this makes it not eligible for being considered as "an event". Accordingly – considering probability as a ratio of favourable outcomes to all possible outcomes – a probability zero event could be found only if the "favourable outcomes" are represented by the empty set, or in other words, the event that neither of the possible outcomes happens.

Moreover, following the definition of the sample space as a set of all possible outcomes, another awkward situations was identified in case of an infinite sample space. When flipping a fair coin twice the sample space is {HH, HT, TH, TT}, a set of four (2^2) equally likely outcomes, each having a probability of $\frac{1}{2^2}$. Following the same reasoning, tossing a coin infinitely many times is an experiment with an infinite sample space, where each event is an infinite sequence of heads and tails. Since the probability assigned to each of these events is zero ($\frac{1}{2^\infty}$), they are all "impossible events" by definition. Therefore – a student wondered – the sample space then should be empty, since it should include only all possible outcomes.

Another issue that was attended to in class discussion related to the perceived relationship of "impossible" – supported by the "logically impossible examples" – hand probability zero. On one hand the participants agreed that the event of picking a "2", or any given number, from the interval of real numbers that includes this number has a probability zero. On the other hand, it appeared as "still possible".

The above-mentioned arguments give evidence of vagueness in definitions of sample space and event and the confusing effect that the word "impossible" (that seems prudent to be replaced with "improbable" in textbooks) has on the understanding of these concepts. While the "advanced" theory of probability that relies on measure theory provides theoretical solutions to these issues, it is important for teachers to be aware of potential pitfalls in the conventionally used definitions. The discussion of probability-zero provided an avenue for the teachers to raise their awareness.

SUMMARY AND CONCLUSION

In this study we examined ideas of prospective secondary mathematics teachers related to probability zero based on the examples of probability zero events that they generated and the subsequent classroom discussion. The results reveal that the collective sample space of this group includes mostly events that are "logically impossible" and that are situated in a conventional context, such as flipping a coin or rolling a dice. Additional examples included events with probability that was "almost zero" or zero at a limit. The request to exemplify a more complicated event resulted mostly in events in the same category that either combined two events or generalised an event exemplified previously.

The subsequent classroom discussion revealed complexities in the conventional interpretation of probability related definitions in general, and potential problematics in identifying "impossible" with "improbable" with respect to probability zero in particular. We suggest that explicit awareness to such subtleties is an important aspect of teachers' knowledge. Further research will attend in detail to the distinction between infinite sample space and "very large" sample space, and in such between probability zero and probability that is "very small", estimated to be zero or zero at limit.

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AN INVESTIGATION OF RELATIVE LIKELIHOOD COMPARISONS: THE COMPOSITION FALLACY

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The objective of this article is to contribute to research on prospective teachers' understanding of probability. To meet this objective, we presented prospective mathematics teachers with a novel task, which asked them to identify which result, from five flips of a fair coin, was least likely. However, unlike previous research, the participants were presented with events (i.e., sets of outcomes) as opposed to sequences, which have dominated previous literature on relative likelihood comparisons. Given that previous changes to the task have resulted in new areas of research, we utilize a new lens – the composition fallacy – when accounting for participants' responses. Use of the new lens also allows us to contend that logical fallacies are a potential avenue for future investigations in comparisons of relative likelihood and research in probability.

In a recent, comprehensive synthesis of research in probability, Jones, Langrall, and Mooney (2007) declared, "research on teachers' content knowledge in probability is sobering at best" (p. 934). However, they also noted, "research on teachers' mathematical content knowledge, pedagogical content knowledge, and knowledge of student learning[, which, collectively, they referred to as teacher's probabilistic knowledge] is limited" (p. 933). Recognizing the former point and given the dearth of research documented above and elsewhere (e.g., Stohl, 2005), the objective of this article, in general, is to contribute to research on teachers' probabilistic knowledge. More specifically, the objective of this article is to (1) contribute to an emerging thread of investigations into prospective teachers' probabilistic knowledge (e.g., Chernoff, 2009; Zazkis & Chernoff, in press) and (2) contribute to an established thread of investigations into comparisons of relative likelihood (e.g., Borovcnik & Bentz, 1991; Chernoff, 2009; Cox & Mouw, 1992; Hirsch & O'Donnell, 2001; Kahneman & Tversky, 1972; Konold, Pollatsek, Well, & Lohmeier, & Lipson, 1993; Rubel, 2006; Shaughnessy, 1977; Tversky & Kahneman, 1974; Watson, Collis, & Moritz, 1997).

To realize the general and specific objectives we, first rationalize and subsequently present a novel task – the relative likelihood of events task – for (present and future) research investigating comparisons of relative likelihood. Second, we demonstrate that particular responses to the relative likelihood of events task fall prey to the *fallacy of composition* (i.e., because parts of a whole have a certain property, it is argued that the whole has that property). Our accounting for responses via logical fallacies also, potentially, paves the way for a new thread of investigations – as

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responses, in the past, have traditionally been accounted for with normative reasoning, heuristics (e.g., Tversky & Kahneman, 1974, LeCoutre, 1992), and informal reasoning (e.g., Konold, 1989).

A REVIEW OF THE LITERATURE

Nearly forty years ago, (psychologists) Kahneman and Tversky (1972) asked a group of individuals whether there would be more families with a birth order sequence (using B for boys and G for girls) of BGBBBB or GBGBBG. In a second, related question, the same individuals were asked whether there would be more families with a birth order sequence of BBBGGG or GBGBBG. Kahneman and Tversky argued that individuals who declared one sequence as less likely were reasoning according to the *representativeness heuristic*, where one "evaluates the probability of an uncertain event, or a sample, by the degree to which it is: (i) similar in essential properties to its parent population; and (ii) reflects the salient features of the process by which it is generated" (p. 431). Despite the subsequent permeation of the representativeness heuristic, there were concerns associated with the inferential nature of responses to the task.

In order to address the inferential concerns associated with the task, Shaughnessy (1977) introduced three developments: provision of an equally likely option; request for reasoning; reworking from a *least* likely version to a *most* likely version of the task. Despite these developments, the framework of the task remained, essentially, the same. For example, Shaughnessy had individuals compare the birth order sequence BGGBGB first to the sequence BBBBGB and, second, to the sequence BBBGGG. However, with the "provide a reason" development to the task, Shaughnessy was able to reinforce inferred results from Kahneman and Tversky's (1972) research and establish new areas for investigation. For example, certain individuals determined, correctly, that the sequences BGGBGB and BBBGGG were equally likely, but according to their incorrect justification, because both sequences had the same ratio of boys to girls (3:3).

Presenting an entirely different version of the relative likelihood task than had been seen in the past, Konold et al. (1993) provided individuals with four sequences and the equally likely option. For example, Konold et al. (1993) asked individuals "which of the following is the most likely result of five flips of a fair coin?" and provided them with the following options, "a) HHHTT b) THHTH c) THTTT d) HTHTH e) all four sequences are equally likely" (p. 395). Further, the researchers gave students a most likely version of the task followed by a least likely version. They found, for the most likely version, certain participants answered using *the outcome approach* – "a model of informal reasoning under conditions of uncertainty" (Konold, 1989, p. 59) – and for the least likely version subjects answered using the representativeness heuristic.

The framework presented in Konold et al.'s (1993) iteration of the relative likelihood task has, for the most part, been adopted by all subsequent research on comparisons of relative likelihood (e.g., Cox & Mouw, 1992; Chernoff, 2009; Hirsch & O'Donnell, 2001; Rubel, 2006). Alternatively stated, the relative likelihood task has not undergone any major alterations in nearly 20 years. To address the issue raised, we, in the next section, present and rationalize a major alteration to relative likelihood tasks.

TASK DESIGN

As seen in Figure 1 below, our task development for comparative likelihood research is heavily influenced by previous versions of the relative likelihood task.

Which of the following is the least likely result of five flips of a fair coin?

a) three heads and two tails.
b) four heads and one tail.
c) both results are equally likely to occur.

Justify your response...

Figure 1: The relative likelihood of events task

Our version of the task, denoted the relative likelihood of events task, is a unique blend of particular components found in the original task and subsequent developments to the task. First, in a throw back to the original version of the relative likelihood task, the present task asks individuals to compare two events, as opposed to a larger number of events or sequences. Second, two of the three task developments, introduced by Shaughnessy and used by all subsequent research (e.g., Cox & Mouw, 1992; Chernoff, 2009; Hirsch & O'Donnell, 2001; Konold et al., 1993; Rubel, 2006), that is, the equally likely option and the opportunity for response justification are present in the current iteration. Third, the wording and framework of the task are similar to Konold et al.'s (1993) iteration of the relative likelihood task. Fourth, given that Rubel (2006) found "very few instances of such inconsistencies" (p. 55) between the least likely and most likely versions of the task and, further, given the lack of subsequent research confirming or denying the inconsistencies experienced, the present iteration of the task asks individuals which event is least likely. In essence, the relative likelihood of events tasks is similar to all previous iterations of the relative likelihood task, except for one major difference: instead of presenting individuals with sequences of binomial outcomes, they are presented with events (i.e., sets of outcomes), which are subsets of the sample space.

THEORETICAL FRAMEWORK

As demonstrated in the review of the literature, changes to the relative likelihood task have established new domains of research. For example, Konold et al.'s (1993) research, which asked participants to determine which of the sequences was *most* likely, led to the now ubiquitous outcome approach. In a similar vein, given that we are introducing a new iteration of the relative likelihood task, we have also decided to depart from past theoretical frameworks for our analysis of results. Instead of using "traditional" theoretical frameworks (e.g., the representativeness heuristic (Tversky

& Kahneman, 1972), the outcome approach (Konold et al., 1993), and others) to account for participants' responses, we contend, and will subsequently demonstrate in our analysis of results, certain logical fallacies (e.g., equivocation, begging the question, the fallacy of composition, the fallacy of division, and others) can be used to account for participants' responses to the relative likelihood of events task. Alternatively stated, while, in the past, it has been argued that participants' responses to comparisons of relative likelihood are a result of heuristic or informal reasoning, we are contending that certain responses, to our task, are a result of falling prey to particular informal fallacies.

Given the boundaries associated with the present venue (i.e., the 8 page limitation), we had decided to limit our scope and, as such, our theoretical framework (i.e., the multitude of informal fallacies which can account for relative likelihood responses) will consist of one particular fallacy: the fallacy of composition. Put simply, *the fallacy of composition* occurs when an individual infers something to be true about the *whole* based upon truths associated with *parts* of the *whole*. For example: Bricks (i.e., the parts) are sturdy. Buildings (i.e., the whole) are made of bricks. Therefore, buildings are sturdy (which is not necessarily true). As we will now demonstrate in the analysis of results, certain participants in our research inferred certain truths associated with individual coin flips to be true for events, that is, sets of outcomes.

PARTICIPANTS

Participants in our research were (n =) 63 prospective mathematics teachers enrolled in a methods course designed for teaching middle-years (i.e., ages 10 to 15) mathematics. More specifically, the 63 participants were comprised of two classes, containing 26 and 37 students, taught by the same instructor. Participants were presented with the relative likelihood of events task and were allowed to work on the task until completion. Of note, the participants had not answered any of the other versions of the relative likelihood task prior. Further, the topic of probability had yet to be discussed in class at the time of the research.

RESULTS AND ANALYSIS

Responses from the 63 participants fell into three categories. First, five individuals (or 8%) responded incorrectly that three heads and two tails is least likely to result after five flips of a fair coin. Second, 12 participants correctly responded that four heads and one tail is the least likely result. Third, the majority of participants, 46 (or 73%), responded incorrectly that both results were equally likely to occur.

Inconsistencies between responses and justifications were witnessed in both the normatively correct and incorrect responses to the task and helped further classify responses within each of the categories into subcategories. For example, of the 46 participants who responded that both results were equally like to occur, 20 of the 46 (or 43%) response justifications evidenced Lecoutre's (1992) equiprobability bias where the notion of equiprobability is misconstrued as anything can happen. Further, a consistency between justifications was evidenced for individuals who (1) declared four heads and one tail as least likely and (2) a sub-group of individuals who responded that both results are equally likely to occur – each of which are now commented on in turn.

Four heads and one tail is least likely to occur

All 12 of the participants who declared, correctly, that the event four heads and one tail is least likely to occur after five flips of a fair coin, were unable to provide proper normative justifications for their responses. In what follows, we analyse three, exemplary responses, which evidence the fallacy of composition.

> Rupert: four heads and one tail are least likely to result because the coin is two-sided. Because the coin is two sided and has two different sides there is an equal chance that either side will result. The chance that the outcome will be tails is equal to the chance it will be heads.

> Robert: Answer b) is least likely to occur. It is unlikely that by flipping a coin five times your answer would result in four heads and one tail. Since the coin has a head side and a tails side there is a fifty percent chance you will get either heads or tails. It is just very unlikely that when flipping a coin it would result in four heads and one tail.

> Amber: The least likely to occur is b because it would be more in favour of an equal end result.

As seen in both the responses of Rupert and Robert, they pay particular attention to the characteristics of the fair coin. More specifically, they reference that the coin has two sides and that either side has equal chance of occurring or, as Robert states, “there is a fifty percent chance you will get either heads or tails.” Further, the fairness of the coin, for Rupert and Robert, influences the ratio of heads to tails they are expecting in the events presented. In other words, given that the coin is 50-50 or has a heads to tails ratio of 1:1, they are expecting the ratio of heads to tails in the event to be close to 1:1 (exemplified in Amber’s response). Given that the ratio of 4:1 isn’t as close to the expected 1:1 as 3:2, they declare that the event with four heads and one tail is less likely than the event with three heads and two tails.

Presented within the fallacy of composition framework, Rupert and Robert’s responses, declare that the ratio of heads to tails for fair coins is 1:1 (i.e., the brick). Further, they note that the event (i.e., the building) is comprised of five flips of a fair coin. Therefore, the event should also have a ratio of 1:1. For Rupert, Robert, and Amber, the expectation of a 1:1 ratio of heads to tails for five flips of a fair coin leads them to declare that the event with a head to tails ratio of 4:1 is least likely. The fallacy of composition was also present in certain responses from individuals who incorrectly declared that both results were equally likely to occur.

Both results are equally likely to occur

As mentioned, 46 of the 63 respondents declared incorrectly that both results are equally likely to occur and, while 20 of the 46 responses are accounted for with

Lecoutre's (1992) notion of the equiprobability bias, the majority of the other responses are accounted for with the fallacy of composition. In what follows, we analyse and elaborate upon 4, exemplary responses, which, again, evidence the fallacy of composition.

Evidenced from Randy and Kelly below, their responses are similar to those of Rupert, Robert, and Amber; however, the "bricks" of the "building" are slightly different.

> Randy: Because there is equal chance, one head and one tail

> Kelly: both results are equally likely to occur because you have flipped the coin five times, there is a chance each time that you can get either heads or tails, so there is an equal chance of coming out with the outcome of a) or b)

Randy and Kelly, in their responses, declare that both results are equally likely to occur because of the equal chance of one head and one tail for the flip of a fair coin. More specifically, Randy and Kelly's response note that there is an equal chance (i.e., the brick) of heads and tails for the fair coin. Further, the event (i.e., the building) is comprised of five flips of a fair coin. Therefore, the event should also have an equal chance of occurring or, in other words, both results are equally likely to occur. While (somewhat) implicitly presented in Randy's response, Rudy's response, like Kelly's, further evidences particulars associated with the notion of composition.

> Rudy: Because each time you flip the coin there is a 50/50 chance of the coin being heads or tails. Therefore each pattern that is created by flipping the coin (answer a and b) is equal in happening.

Rudy's response is presented, nearly verbatim, within the framework of the fallacy of composition. For example, (1) "each time you flip the coin there is a 50/50 chance," (2) "each pattern that is created by flipping the coin," (3) "Therefore each pattern...is equal in happening." Further, Rudy's response, as well as the response from Richard presented below, sheds light on how the participants take into consideration each individual toss of the coin (i.e., the brick) as part of the event (i.e., the building).

> Richard: When flipping a coin there are only 2 outcomes: heads or tails. Therefore, there is a 50% chance of getting heads for one flip and 50% chance of getting tails for the same toss. It is just as likely to get 4 heads and one tail as it is to get 3 heads and 2 tails because looking at each individual toss the coin has equal chance of going heads or tails.

As presented in the four responses analysed above, the fallacy of composition is able to account for particular incorrect responses to the relative likelihood of events task. More specifically, the equal likelihood of the fair coin, which is flipped five times, is, in essence, transferred to the event.

CONCLUDING REMARKS

Demonstrated in the analysis of results, the fallacy of composition accounts for particular responses to the relative likelihood task (which we have introduced in this article). In particular, the fallacy is present in (certain) response justifications, which, correctly, declare that for five flips of a fair coin four heads and one tail is less likely than three heads and two tails; and, also, the fallacy is present in the justifications for incorrect responses, which declare that both results are equally likely to occur. Further, correct responses with incorrect justifications note that the ratio of heads to tails for a fair coin is 1:1, that the event is the result of five flips of a fair coin, and, as such, the ratio of heads to tails in the event should be close to 1:1. (Of the options participants were presented in this task, the event three heads and two tails is closer, as a ratio, than the event four heads and one tail.) Alternatively, the justifications associated with (certain) incorrect responses, which declare that both results are equally likely to occur, note that a fair coin has equal likelihood of heads and tail, that the event is the result of five flips of that fair coin, and, as such, each of the two events presented are both equally likely to occur. In other words, the fallacy of composition is present in both the former and latter justifications.

DISCUSSION

Research involving comparisons of relative likelihood has, historically, been focused on accounting for individuals' responses – both correct and incorrect. Developments associated with the task have allowed for parsing between the answer an individual gives and the justification for their answer. As a result, related research has developed a variety of theoretical models (e.g., representativeness, the outcome approach, and the equiprobability bias) to account for incorrect, sometimes incomprehensible, responses. However, in more recent years, there has been a lack of developments to tasks that investigate comparisons of relative likelihood. In line with this point of view, we have presented the relative likelihood of events task for use in future investigations. Our analysis of the results has also opened, we contend, a new area of investigation for future research on comparisons of relative likelihood: the use of logical fallacies, as opposed to traditional methods. More research and, we would contend, more variations to relative likelihood comparison tasks, will determine to what extent logical fallacies are a part of teachers' knowledge of probability.

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A SHIFT IN ONTOLOGY: MATERIAL AGENCY AS AN INFLUENCE IN IDENTITY FORMATION

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This research report presents a new framework for identifying and analyzing material agency (Pickering, 1995) and its influence on student's experiences in mathematical practice. This framework is then implemented in attending to affective attributes of students, particularly positioning attitudes and identity formation. Students engage in a problem solving activity with a technological tool. Analysis of the students' engagement with the tool, in terms of the different types of agencies, is based on their spoken words.

INTRODUCTION

The idea of allocating agency to materials has a sparse academic history but current conceptions about human activity are challenging traditional perspectives (Coole, 2010; Malfouris, 2004; Pickering, 1995). Material agency has been given credence in practical areas such as in the relationship between technology and organizations in information systems environments (Rose & Jones, 2005), in ecological approaches to environment issues (Oliver, 2009), as well as in feminist studies (Grosz, 2010). Coole & Frost (2010) presents a more philosophical perspective when they refer to the ontology of materials as an essential perspective in these post-modern times. They describes a new materialism, arguing that "…reconfiguring our very understanding of matter are prerequisites for any plausible account of coexistence and its condition in the 21st century" (p. 2).

Although the notion of material objects is often at odds with the idea of mathematics as a more "mental" discipline, the role of manipulatives and, more recently, digital tools, has paved the way for a reconceptualization of the status of such artefacts in mathematical thinking and learning. One of the purposes of considering the mathematics object is that it can be associated with an emotional dimension of human experience (Turkle, 2007). In that having something to hold or touch or engage with can have an affective appeal. Papert (1993) states that "…working with an electronic sketchpad…" students acquire "…a new image of themselves as mathematicians" (p. 13). Boaler (2002) makes a clear connection between practices and identity formation arguing for a shift from focusing on knowledge to one that attends to inter relationships of knowledge, practices and identities (p. 47). This study recognizes that the tool itself does more than mediate mathematical learning; it has a much greater role in its influence of a student's identity. Indeed, Tim Lenoir, speaking of our technological world, states: "The materiality of media rather than their content is

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what matters. Communicational media are machines operating at the heart of subject formation" (p. xii, in Rotman).

While there are other frameworks to analyze student's engagement with objects, such as the instrumental approach of Rabardel and Verillion (1993), or the mediational approach of Vygotsky (1978), they have tended to almost exclusively focus on conceptual cognitive development. This study looks at student's engagement with a mathematical object in an attempt to explore and account for the idea of material agency as well as to pursue the material influence of the tool and how it shapes the identity of the student. The research question has two parts: one is to determine if this is a viable perspective, that is, whether there is evidence of an influence of material agency; and, the other is to examine what happens to student identity while interacting with a mathematical object.

THEORETICAL FOUNDATION AND FRAMEWORK

Pickering has classified three types of agency: individual, disciplinary and material. He describes disciplinary agency as the negotiated rules and algorithms of a conceptual system, such as mathematics, and he describes material agency as the resistant capacities manifested in the engagement with a tangible object. He argues that all advancements and discoveries are a synthesis of individual agency interacting with either disciplinary or material agency. Pickering describes the synthesis from the individual perspective as a "dialectic of resistance and accommodation" (p. 52). Pickering has referred to this interplay of resistance and accommodation as a "dance of agency".

Boaler (2002) uses Pickering's framework to describe different practices in mathematics classrooms. She argues that disciplinary agency often determines the practices in a traditional classroom, leaving no space for student agency. If students are not given the chance to act independently, the math is given the status to direct and determine the practices of math classroom activity. Boaler describes that practices, that leave no room for student agency, have a positioning effect. This positioning involves students viewing themselves as receivers of knowledge. She argues further that good classroom teaching would engage a balance of disciplinary agency and student agency in which case students would position themselves as participants who have a voice (p. 46). Both Boaler and Pickering, however, do not refer to material agency in mathematics. Wagner (2007) also uses Pickering's framework by acknowledging disciplinary agency and its role in his research. He questions the role of materials in mathematics practices when he asks, "What is the nature of material agency in mathematics?" (p. 43).

Malafouris (2004) describes agency as not being "properties of things or humans but are properties of engagement" (p. 22). Agency is a result of activity; it is an emerging product resulting from an interaction. This perspective makes agency a challenging word to define as it depends on the actors and context in which it is present. For this study Pickering's basic definition will be sufficient: who is the cause of the doing?

With this definition, materials can be understood as having agency when their structures, make-up and design restrict the subject within a context of activity. The subject, consequently, has to adapt to the form of the object in activity.

This study operationalizes Boaler's connection of practice and identity formation. While she has focused more on the disciplinary agency of mathematics and its relation to knowledge, this study focuses on the material agency of a mathematical tool and its relation to identity formation. Identity, in this context, takes into account both the narrative and positional dimension of identity; how the student develops their relationship to mathematics and how the student is assigned a position in that environment (Horn, 2008). It is within activity that I employ a discursive analysis to indicate the positioning and orientation of the student with respect to the practices they will engage. Practice, in this context, refers to the social interaction in the moment.

METHODOLOGY

Discourse is not just a unit of language but a social process (Herbel-Eisenmann et al.). Discourse analysis is an expressive and social perspective. It commits to language as being representative of identity and positioning (Herbel-Eisenmann). It is these particular aspects of discourse analysis that will guide an understanding and an account for material agency and its influences on student agency. It is a useful methodology as it allows one to trace how students are thinking. Language choice is crucial to tracing how speakers shift positions, identities and alignments towards the words they speak (Morgan, 1998).

Herbel-Eisenmann and Wagner (2010) discuss how positioning occurs in a fluid activity in moments of action, and in relations to the figures in a scenario. Positioning indicates agents of change who have the authority. Personal pronouns identify markers of positioning (Fairclough (2001), in Herbel-Eisenmann et al.) so when students use "I" it is an expression of themselves and refers to their identity. Keane, as well, describes that voice represents the one who is speaking and points out that voice can direct attention to positioning and identities (in Herbel-Eisenmann).

The materiality in this study is manifested in a technological tool. Papert (1993) describes this as an expressive technology and it is a way to see how technological and social processes interact. The agency of computers is particularly interesting, given its range of expressive possibilities and feedback. It presents an environment where students can make choices giving them the freedom for expression and exercise their agency.

RESEARCH CONTEXT AND PARTICIPANTS

The data collection took place in a Vancouver high school with some students who had been working in an environment using The Geometer's Sketchpad (GSP). A pair of students worked after class, although it was not in the natural environment it still allowed an attending to the idea of material agency and to the student's experiences.

Data was collected by means of a software capturing software, Jing. Jing recorded all the activity on the screen as well as recorded the verbal utterances of the two students.

The task given to the students was in the form of a black box sketch. Initially two points were visible and when the student dragged either point the other point would move in a deterministic path. The points were related by a mathematical relationship. The problem posed was to identify the relationship, either in words, or as an equation, between the two points. This activity was chosen for it was a challenging but accessible problem that related to previous curricular work on the topic of transformations. Initially these points, A and B, were visible on the screen (Figure 1). In this particular case, the points were related by circle inversion defined by $(AO)(BO)=r^2$ (O is the centre of the circle of radius r). It might be of interest to note that the points A, B and O are collinear. Later on during the activity, after experiencing some difficulty with the problem, the previously hidden circle was revealed (Figure 2). The girls worked for 15 minutes.

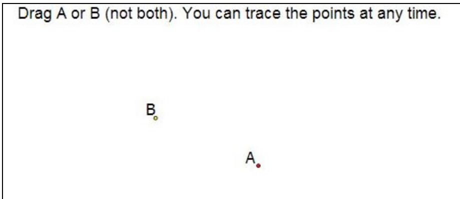

Figure 1

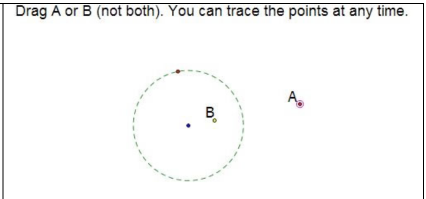

Figure 2

Jessica and Mark working with the black box sketch.

This data represents some short excerpts of the students' work. Jessica had control of the mouse.

Jessica initially moves point A around the screen and both students observe what B is doing. As soon as she moves B far from the unseen centre of the circle B is not moving very much since the inverse relation keeps it still near the centre.

1 Jessica: How come I can't see anything?

2 Jessica: How come it's not moving when I'm over here. Then it moves when I get closer. See…oooo

3 Jessica: If you move a lot far from B it doesn't really look like it's moving but if you go closer to it. It moves really far away.

Jessica seems to be experiencing resistance to how she expected the points to move. Although she is just moving points on the screen, she is coming to recognize what the software will allow her to do and what it will not. Between line 1 and 2 she is moving the point A. In a situation of resistance, where B is not moving, she accommodates by moving the mouse as well as attempting to explain in line 3.

At this point the show all function is turned on and the circle appears as well as its centre and a point on its circumference which changes the size of the circle. See Figure 2.

4 Mark: Move A to that dot (referring to the centre but Jessica thought it was the point on the circle)

5 Jessica: This dot?

6 Mark: No that one.

Jessica accidentally highlights the point and drags it outward. The circle gets bigger.

7 Mark : Ooooh!

8 Jessica: Woah, oh!

9 Mark: I think you're moving the wrong dot. (she laughs)

As Jessica makes the circle bigger B moves off the screen.

10 Jessica: I'm moving B away, Bye bye.

Mark puts himself in a position of being able to act in this interaction. It leads to a misunderstanding and both girls are surprised by this new change.

At Mark's request Jessica drags the point A to the centre of the circle. Since the circle is bigger from the last episode, B moves off the screen much quicker when A is dragged toward the centre of the circle.

11 Jessica: Where did it go?

12 Mark: It went that way. Move here (pointing at the centre, B moves off the screen)

13 Jessica: Huh, didn't it just go that way? (B appears at the top of the screen but it had left the screen on the bottom.

14 Jessica: How come it wouldn't ..ok… let me try that again? (She drags A through the centre)

15 Jessica: You saw that right? B went down but then it came from above right? Okay, let's do that again. 1, 2, go, go, go, right, B gone.

16 Mark: B went out that way.

17 Jessica: But then over here. Ahh! So it's like drawing a circle right but it went like this. Don't' you think? It went like this. (she gestures with her hand, drawing a big circle in the air)

This particular episode indicates that Jessica and Mark are both engaged as well as personally connected to the points.

Jessica gets an idea near the end of their activity. She wants to translate the circle from one side of the screen to the other.

18 Jessica: Oh what if you moved the circle over here? I want to see what happens to this.

Though very short, these episodes provide sufficient evidence of material and personal agency. The utterances "I'm moving B away" (line 10) or "Let me try that again" (line 14) both employ the "I" voice, indicating an agency, an expression of oneself. In questions like "How come I can't see anything?" (line 1) or " Where did it go?" (line 11) are an example of resistance. They are also a form of positioning because it is clear that they are not, currently, exercising their agency. They are allowing "it", the computer, to perform its act. Their voice has clearly changed from an "I" voice to the third person pronoun, "it".

DISCUSSION

In terms of Pickering's distinction of agency and his methodological approach of resistance and accommodation the data reveals this kind of activity. There was evidence of student's agency mixed with resistances from the computer. These different agencies indicate more than a back and forth mechanical activity. There is a motivation evident in this exploration. As Papert claims the students creates a new self-image, one that allows them to negotiate their expression within the environment.

It is worthy to note the attitude of wonder and connection such as in line 7, 8, and 10. Ooooh! Woah, oh! Bye bye. These utterances reveal a level of commitment and motivation to the activity. It is interesting to note that these students were not engaging with circle inversion but with moving points and changing circles. The fluid dynamic activity with the technological tool allowing for instant feedback offered an environment of exploration and an opportunity for expressing agency.

Both narrative and positional identity categories were enacted. Students positioned themselves in a much different way than in what Boaler termed a traditional classroom. Their roles allow for expression and this negotiation with the program offers a new perspective of practicing mathematics. These students were not aware of the resistances a priori, their continued engagement showed a motivating force inherent in the activity.

CONCLUSION

This study is part of an ongoing research plan. This study has presented a new framework for operationalizing a dimensionality of materialism that is considered relevant in identifying the constructions of student's identities and positions. With an attentiveness to students that have a tool to play with, students have access to a range of opportunities of expressiveness, authority as well as resistance and challenge. This paper has shown that material is important and that it does play a role in developing student identity and that there is evidence of agency from the technological tool. Reminded of what Boaler described as good mathematics teaching these student's experiences were balanced with another form of agency. The results indicate that there is evidence of resistance and accommodation when engaging with a technological tool and that the framework also raises attentiveness to student identity expressed in the practice they participated.

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"I'M LIKE THE SHERPA GUIDE": ON LEARNING TO TEACH PROOF IN SCHOOL MATHEMATICS

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This article describes the experiences of a beginning mathematics teacher, Matt, across his first three years of teaching proof in a high school geometry course. Matt's past experiences with mathematics influenced his beliefs about what he could and could not do to help his students learn how to prove. During his first year of teaching proof, Matt claimed that you cannot teach someone to write a proof. Over time, however, Matt eventually developed some strategies for teaching proof to his students. Within this work is an interest in learning more about how a teacher learns to teach proof to students who are just learning how to construct a formal proof. This case highlights the importance of pedagogical content knowledge.

Learning to think and reason both formally and informally is an important goal in the mathematics classroom. On the formal end of reasoning, students must learn to understand and write a proof (NCTM, 2009). Over the past few decades, proof has been given increased attention in many countries around the world (see, e.g., Knipping, 2004). This is primarily because "proof is the basis of mathematical understanding and is essential for developing, establishing, and communicating mathematical knowledge" (Stylianides, 2007, p. 191). In the *Reasoning and Sense-Making* document (NCTM, 2009), formal reasoning (i.e., proof) was situated as the final of three stages in the reasoning progression required for increasing levels of understanding in the high school mathematics classroom. The authors pointed out that the effort to help students progress from less formal to more formal reasoning requires that "teachers play an essential role in encouraging students to explore more sophisticated levels of reasoning and sense making" (p. 11). One might wonder, however, how and how well are teachers being prepared to play this essential role? A more relevant question to this study might be: Is prior experience with mathematical proof as a student sufficient preparation for teaching it?

In this paper, I use data from a longitudinal case study designed to learn more about how a beginning teacher learns to teach proof in Euclidean geometry to address this question. At the onset of the study, Matt (a pseudonym) was teaching proof in geometry for the first time. Here I address the following research questions: (1) How did Matt introduce proof to his students? (2) What limitations did Matt believe that he had with regard to teaching proof? (3) What strategies did Matt develop to overcome these limitations?

Before I explore these questions, I review some literature on learning to teach mathematics and on proof as problem solving. After, discussing the methodology of the study, I present and discuss some findings.

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LEARNING TO TEACH MATHEMATICAL PROOF

Shulman (1986) described three types of knowledge that are necessary for effective teaching: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. According to Shulman (1986), to present specific content to particular students, teachers need a special blend of content and pedagogy that he referred to as “pedagogical content knowledge.” This includes the ways of representing and reformulating the subject that make it comprehensible to students (Shulman, 1986). Influenced, in part, by Shulman’s conceptualization of pedagogical content knowledge, researchers in the 1980s and 90s sought to identify what teachers know (or should know) to teach mathematics (Hill, Sleep, Lewis, & Ball, 2007). This area is an important body of work that has provided frameworks to investigate the various kinds of knowledge that teachers must acquire to maximize student learning. In the interest of brevity, however, in this paper, discussions of mathematics knowledge for teaching will include only references to Shulman’s subject matter content knowledge (CT) and pedagogical content knowledge (PCK). Of particular interest in this paper is the knowledge needed to teach mathematical proof.

A number of studies have already reported that proof is a difficult topic, both for students to learn (e.g., Senk, 1985) and for teachers to teach (e.g., Knuth, 2002). Some research has suggested that perhaps the reason that teachers have not moved their students beyond the traditional two-column approach to proof is related to teachers’ beliefs about the purpose of proof and their students’ abilities to complete a proof (Knuth, 2002). Additionally, teachers may not have had opportunities to consider alternative ways of teaching proof that fall outside of the “apprenticeship of observation” (Lortie, 1975) experienced in their own mathematics backgrounds. Finally, when we think of proof as problem solving, it is easy to understand why it is a challenging area in mathematics education.

PROOF AS PROBLEM SOLVING

A review of the current proof literature illustrates that some researchers are beginning to take the stance that proving is a form of problem solving. By its very definition, a task is only a “problem” when there is no immediate, clear solution or a known path or strategy that sheds light on the appropriate mathematical action required to complete the task (Weber, 2005). Weber (2005) argued that “focusing on the problem-solving aspects of proving allows insight into some important themes that other perspectives on proving do not address” (p. 352). One example of such a theme is the exploration of reasons that students reach impasses in proof where they do not know how to proceed (Schoenfeld, 1985).

In order to solve a proving task that is truly a “problem” as described above, successful students eventually have a breakthrough where they progress from not seeing a path or strategy to developing one that will assist them in writing a correct proof. These kinds of breakthroughs have been described in the literature. For example, Barnes (2002) wrote about a student named Naidra who described his lack of insight on one particular day as not having "anything magical" happen. When pressed further, Naidra said that "flashes of understanding can happen" and "lots of different things can spark that off" (p. 83). This sudden flash of understanding that Naidra described as magical is often referred to as an 'Aha!' or 'Eureka!' experience (Barnes, 2002). Mathematicians writing about the creative process have also described these kinds of moments. For example, Polya (1965) wrote about "a sudden clarification that brings light, order, connection and purpose to details which before appeared obscure, confused, scattered, and elusive" (p. 54). In the context of this study, these descriptions beg the question: What can teachers do to support their students in having these kinds "magical," "aha" discoveries when they are first learning to prove?

"Discovering" a Proof

The idea that there are different phases or activities in proving has been tacitly acknowledged by various sources. For example, in textbooks, the problem solving aspect of proving has been called developing a "Plan for Proof" (Larson, Boswell, & Stiff, 2001), "analyzing a proof" (CME project), "scratch work" (Velleman, 2006), and so forth. The idea that *doing* a proof and *writing* a proof are two different activities was explicitly noted by Farrell (1987) who portrayed both of these activities as important. The *doing* requires good problem solving skills, and the *writing* requires rigor and precision. Farrell claimed, however, that prospective teachers needed to learn that the *writing* takes a back seat to the generation of ideas. Because I call on Herbst and Brach's (2006) work related to "doing proofs," which they describe as the range of practices carried out by students and their teacher, I do not reference the problem solving part of proving as "doing" a proof as Farrell did. Rather, I refer to the problem solving, finding a proof phase of proving as *developing* a proof. As Farrell noted, this activity is the more difficult phase of proving. The development precedes writing up the proof, an activity that is important in terms of mathematical communication, however, it is more about expressing yourself clearly, rather than a problem solving endeavour. This construct is useful for describing a practice of this study's participating teacher.

METHODOLOGY

This longitudinal interpretive case study (Stake, 1995) focuses on the classroom experiences of a high school geometry teacher, Matt, over a three-year period (2005-2007) in which Matt taught Euclidean proof to students (ages 15-16) in the regular track of a geometry course. At the beginning of the study, he had just taken a new position at a public high school in a suburb of a large U.S. city. Matt was chosen to participate in this study because of his new teacher status, his willingness to share his experiences, and his interest in studying his own practice. He provides an interesting case because he had a strong mathematics background as well as a Masters degree in teaching. According to the teacher preparation literature, Matt's background represents the "best-case scenario" (Gay, 1994) in terms of beginning high school teacher preparation. Therefore, the case of Matt presents a "well-prepared" teacher who is learning to teach proof in school mathematics.

Data Collection and Analysis

Data was collected across three years and analysed using qualitative methods. The primary data sources were classroom observations, in-situ field notes, and interviews with Matt. All interviews were semi-structured and audio recorded. For three years, I visited Matt's classroom during lessons when he introduced proof to his students. Each lesson was audio and video recorded. In the interest of illustrating change over time, I only report on the classroom observations from Years 1 and 3 here.

FINDINGS AND DISCUSSION

In the interest of space, in this section, I present data intertwined with some brief discussion. I first describe Matt's early experiences with mathematical proof in high school and at the university in order to shed some light on Matt's preparation for teaching proof. I then describe the ways that Matt introduced proof to his students during Year 1 (Y1) and Year 3 (Y3) of this study. Finally, I provide some interview data to shed light on the changes observed between Y1 and Y3. Following the presentation of findings and discussion, I close with some concluding thoughts.

Matt's Early Experiences with Proof

Matt did not follow a traditional path through mathematics in high school. He completed geometry in the 8th grade (age 13-14) as an independent study which was two years earlier than most students in the United States. Matt said that he was never asked to develop a proof during his school mathematics experience, and he did not recall even being shown a proof in high school. As a mathematics major in college, however, Matt said:

> I was immediately asked to do all sorts of proofs, which now, looking back at it, I can see as not being so bad, but at the time I'm like, this is a joke. I'm like, this is impossible. You know, you can't do this? (Interview, 6/21/06)

The difficult transition that Matt experienced from school to undergraduate mathematics is not uncommon. The paucity of proof in school mathematics coupled with the fact that even in the lower-level university courses, few, if any, proofs are required of students (Moore, 1994) helps us understand why Matt felt that developing proofs was "impossible." During Y1, Matt compared the challenge of doing his first proof (as a student) to walking through a wall. This, he said, caused him to rethink his major in mathematics. These comments may seem surprising given that Matt was clearly above-average in school mathematics, evidenced by (among other things) his being two years ahead in his studies prior to graduating from high school. As Moore (1994) explained, however, "This abrupt transition to proof is a source of difficulty for many students, even for those who have done superior work with ease in their lower-level mathematics courses" (p. 249). Matt said that even though he did not take any sort of an introductory proof course, eventually he was "able to understand or *believe* that [proof] was something that [he] could do" (Interview, 6/21/06). The experiences described here caused Matt to begin to think about mathematics in ways that were different from his conception of mathematics prior to his university coursework. Next, I describe the ways Matt introduced proof to his own students.

Introducing Proof to His Students

Year 1. When presenting the first proof to his students in Y1, Matt told a story about Pokémon (an anime series, film, and video game from the U.S. and Japan):

> If [the line segments] have the same length, then they have to be congruent. So, the definition of congruence, I choose you...Nobody in here watches Pokémon? Ever? Are you kidding me? Are you serious, nobody watches Pokémon?....We're gonna have to rent it. Alright? Piccachu, I choose you. Right? That's how you wanna think about this. I remember this. In college, my roommate one time, he was a good friend of mine....And we're sitting there and one of his internet browsers wasn't working, so he totally decides to switch his internet browser, and of a sudden we're sitting there working and he goes "Minsky, I choose you."...That was really funny. But I remembered that last night. That's the way you want to think about this, right? Definition of congruence. Go, right? Symmetric property. Go. Definition of congruence. Go. Now I'm done, right? That's how we proved this. Okay. (Y1, 9/30/05)

In this example, in the absence of tools to introduce proof, Matt attempted to connect with the students by referencing Pokémon. Even after realizing that the students did not understand the reference, Matt continued to connect to Pokémon, saying "Symmetric property. Go. Definition of congruence. Go." Also, in Y1, the students were not given very many opportunities to participate in the development of proofs.

Year 3. Three specific changes were observed in the way that Matt introduced proofs in Y3: (a) what Matt wanted students to do before they started writing their proofs; and (b) the flexibility Matt stressed related to the *form* of the proof (c) the confidence shown in the way that he spoke about proof which did not involve seemingly random analogies. Rather than using the example proofs from his textbook (as he did in Y1), in Y3, Matt wanted to start with a proof that was "more interesting." Matt began the lesson by talking about what the students should do before they write a proof:

> Before you ever write a proof, you want to make sure that you can convince yourself that it's true, okay? No one learns anything by writing a proof. They just write down what they already know has to be true. So look at number 12, here. Look at that problem for 25 seconds. See if you can convince yourself that it has to be true. (Y3, 9/21/07)

Rather than Matt immediately demonstrating proof as he did in Y1, Matt gave the students time to think about the proof. Matt attempted to involve students by giving them this time and then *asking* them if they were convinced of the truth of the proposition that they were supposed to prove. He then asked the students *why* the statement was true, and then he called on a student to provide an explanation. Matt the proceeded to tap into the students' thinking as he simultaneously led them through a two-column and a flow proof of the theorem.

Exploring Observed Changes through Interviews

During an interview at the end of Y1, Matt discussed how students either see or do not see how a particular proposition can be proved.

> To do a proof in a real mathematical way is very, it's very isolating. You can't teach somebody how to do a proof….I mean if a student's really gonna do a mathematical proof, you look at the problem and you either see how you do it or you don't. After that, the writing it down, although an important exercise in communication really is sort of pointless. I mean it's not pointless, but it's trivial. You know. If you can see how to prove something, then you can see how to explain it to somebody else and the seeing or not seeing it is nothing that I can teach you. (Interview, 6/21/06)

After hearing Matt say that "seeing it is nothing that I can teach you," I asked him if there was anything that he could do, as a teacher, to provide students access so that they could progress at the pace that was dictated by the demands of the school context. To this question, Matt replied:

> I mean you don't want to go so far as to say it doesn't matter what I do, but the reality is that I can't prove it for them. You know, simply showing somebody how to do a proof will help, but only up to a certain point. Only until they understand…the way in which a proof becomes a proof. (Interview, 6/21/06)

Here, Matt expressed what he saw as a limitation for him as the teacher. After teaching proof for the second time, I, again, asked Matt about the comment, "seeing it is nothing that I can teach you." I was curious as to whether Matt still believed that there was nothing or even very little that he could do to help students learn to prove. I was interested in his answer to this question because, at that point, I had observed Matt teach proof for the second time, and he had made changes that I thought might be designed to help his students "see it," whereas the previous year he said that there was very little that he could do. I was trying to understand if there was a shift in his thinking. After discussing the analogy of teacher as coach, which did not seem to resonate with him, Matt initiated a new analogy:

> I'm like a Sherpa. Okay? That's the word I'm looking for. So…you know, I've been up and down the mountain 50 times. And if you didn't have me, you could make it to the top of the mountain. 'Cause I'm not a requirement, right? But it'll probably be a lot uglier and take a lot longer. And, there's a good possibility that you would freeze to death and never get to the top. Right? So. Yeah, I'm like the Sherpa guide who like, you know, just walks with you up the mountain, but then at base camp I just, I go off and meditate somewhere else and I really don't pay attention to what you're doing. Right?....And I don't just have one person, right? I'm trying to herd like 30 people to the top of the mountain before next Friday. (Interview, 4/19/07)

So, although, Matt could not climb the mountain for his students in the same way that he could not "see it" or "prove it for them." He seemed to view his role as one of being there and knowing (or believing) from experience that it was possible to get to the top of the mountain. He also noted the reality of the classroom when he said that he had to herd 30 people to the top of the mountain "before next Friday."

Significance

Although there is widespread agreement that novice teachers lack a number of important skills, only a few researchers have sought to understand how beginning teachers develop their knowledge of and for teaching (Brown, 1993). Researchers in the area of science education are beginning to explore the challenges that new science teachers face as they begin their teaching careers (Luft, 2007). Similar to Luft's (2007) work with new science teachers, studies such as this one are important because they reveal the complexity of being a beginning mathematics teacher in the context and setting in which the new teacher works. In this study, data were presented to illustrate the ways in which CK is not necessarily sufficient preparation to teach proof. Even with a strong mathematics background, Matt still struggled to develop tools to support his students through the discovering phase of doing proofs. This study illustrates the need for additional studies that seek to observe teachers introducing and cultivating proof. It could be helpful to understand what successful, experienced teachers do to scaffold proof-development practices in their classrooms. In practice, more support should be provided to beginning teachers in their preparation to help their students develop proofs.

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UNIT ELICITING TASK STRUCTURES:
VERBAL PROMPTS FOR COMPARATIVE MEASURES

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The purpose of this report is to compare three different unit eliciting task structures for measurement comparison tasks. Twelve students ranging from grades 2-4 were presented length, area, and volume tasks. Student responses were coded for correctness and comparison type. The results indicated that students were most successful with task structure 1, "How much longer/bigger?" and least successful with task structure 2, "How many times longer/bigger?" Students tended to produce additive comparisons in response to task structure 1 and tended to produce multiplicative comparisons in response to task structure 3, "If this has a(n) length/area/volume of 1, then what would you call the length/area/volume of this?" Suggestions for use and modification of the three task structures are discussed.

INTRODUCTION

Researchers, educators, and policymakers have identified measurement as an essential topic in school mathematics because of its practical applications, connections to other areas of mathematics, and links to other disciplines (e.g. Clements & Sarama, 2007; Lehrer, Jenkins, & Osana, 1998; National Council of Teachers of Mathematics, 2000; National Governors Association & Council of Chief State School Officers, 2010). This connectivity of measurement can be attributed to its nature. According to Battista (2007), "Measurement plays a central role in reasoning about all aspects of our spatial environment" (p. 891). Lehrer, Jenkins, and Osana (1998) concurred, claiming, "Children's reasoning provides the foundation for instruction about the mathematics of space" (p. 137). The National Research Council (2009) also emphasized the pervasive nature of measurement, identifying measurement as a system for "describing, representing, and understanding the world" (p. 35). Unfortunately, despite the importance of measurement knowledge, instruction of measurement concepts is often inadequate or overlooked entirely (Clements & Sarama, 2007; Kordaki & Potari, 2002).

For the purpose of this study, we use a definition of measurement based on one proposed by Sarama and Clements (2009): measurement is the process of quantifying an attribute of an object in reference to a chosen unit. In other words, we recognize that the motivation or end product of measurement is usually a comparison. Piaget, Inhelder, and Szeminska (1960) noted the importance of comparison in measurement. Their claim was that students compare in three distinct ways, "bringing the objects

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themselves together, using another object as a common measure, and finally the construction of units to measure any distance by stepwise movement, i.e. unit iteration" (p. 30). According to Sarama and Clements' definition these first two comparisons are not considered measurement because no unit is chosen. In these instances, only a qualitative comparison, for example this object is longer, can be produced. In the case of the third, with the introduction of a unit and the change in position (unit iteration), the comparison can be a more sophisticated quantitative comparison. Research suggests students can be motivated to engage in the more sophisticated, quantitative comparisons through the prompt, "How much longer/bigger…" is object 1 than object 2? (Cullen et al., 2010). We see this as a unit-eliciting task because a correct response requires the selection and use of a unit of measure. However, it is not clear what type of comparison students are likely to produce in response to this prompt, i.e. additive or multiplicative.

The purpose of this study was to compare three different unit-eliciting task structures based on the type of comparison created. We have noted three distinct comparisons students tend to generate, additive, multiplicative, and excess. If a student were presented with a 3-inch segment and a 9-inch segment, the additive comparison would be 6 inches longer, and the multiplicative comparison would be 3 times as long. In the case of an excess comparison, the student would produce a comparison similar to a percent increase. To clarify, s/he would report that the 9-inch segment was two times bigger, essentially noting that it would take two more 3-inch segments to be as long as the 9-inch segment. Although we do not consider this comparison to be incorrect, we find the production of multiplicative comparisons to be more useful because they lay the foundation for measurement as a ratio between object and unit.

In each of the three task structures, students were first asked to compare two objects by some attribute, length, area, or volume. Secondly, we prompted students for a quantitative comparison with one of the following task structures (TS):

TS 1: How much longer/bigger is object 1 than object 2? (Cullen et al., 2010)

TS 2: How many times longer/bigger is object 1 than object 2?

TS 3: If this has a(n) length/area/volume of 1, then what would you call the length/area/volume of this?

From our preliminary work with students, we anticipated that TS 1 was likely to evoke additive comparisons, TS 2 was likely to evoke comparisons of the excess, and TS 3 was likely to evoke multiplicative comparisons. For example, assume a student was presented with a task of comparing two line segments (6 inches and 18 inches). For TS 1, we anticipated a response of "This is 12 inches longer." For TS 2, we anticipated a response of "This is 2 times longer." For TS 3, we anticipated a response of "This would be called 3," or "This would have a length of 3."

RESEARCH QUESTION

How do students' quantitative comparisons of lengths/areas/volumes differ under three conditions of verbal task structure?

METHODOLOGY

The data for this study was collected through clinical interviews with 12 students, four each from Grades 2-4, during the fall semester of 2010. Each student was presented with nine measurement tasks: three length, three area, and three volume. These interviews lasted approximately 10 minutes per student. Each of the nine items is presented below.

For each length item, the student was presented with the two indicated objects and asked to compare them by their lengths. This was followed by one of the three task structures described above.

Length Item	Object 1	Object 2
1	2 inches	6 inches
2	3 inches	12 inches
3	1 inch	5 inches

Table 1: Length items

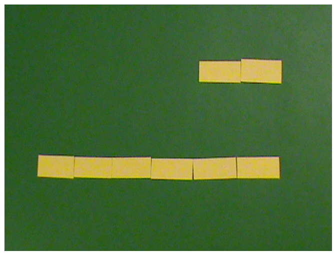

Figure 1: Length item 1

For each area item, the student was presented with the two indicated objects and asked to compare them by their areas. This was followed by one of the three task structures described above.

Area Item	Object 1	Object 2
1	1 in x1 in	2 in x 2 in
2	2 in x 1 in	4 in x 4 in
3	1 in x 3 in	4 in x 3 in

Table 2: Area items

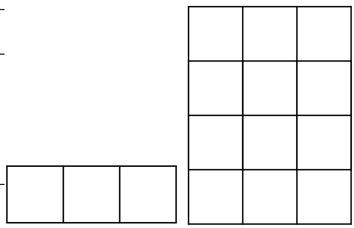

Figure 2: Area item 3

For each volume item, the student was presented with the two indicated objects and asked to compare them by their volumes. This was followed by one of the three task structures described above.

Volume Item	Object 1	Object 2
1	1 in x 3 in x 1 in	1 in x 3 in x 2 in
2	1 in x 2 in x 2in	2 in x 3 in x 2 in
3	1 in x 2 in x 1 in	1 in x 10 in x 1 in

Table 3: Volume items

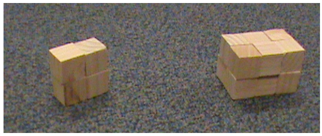

Figure 3: Volume item 2

Student Selection

The participants for this study were selected from a public suburban school in the Midwestern portion of the U.S. The students selected for this study were a subset of students participating in a larger study characterizing cognitive developmental stages while extending, amending, and improving existing hypothetical learning trajectories for length, area, and volume (Sarama & Clements, 2009). We randomly selected two females and two males from each of the three grade levels represented in three classrooms.

Data Collection

These twelve students were posed the set of nine measurement comparisons tasks in individual interviews, each lasting approximately 10 minutes. The interviews occurred during the normal school day and were videotaped. Each of the twelve interviews was conducted by the first author and was observed and videotaped by the second. For each interview, the length items were presented first, then area, and finally volume. Because the focus of this study was the structure of the task rather than the tasks themselves, we varied the task sequencing for each interview. Thus the first student interviewed was presented length, area, and volume item 1 with TS 1, length, area, and volume item 2 with TS 2, etc. The second student then was presented with length, area, volume item 2 with TS 1, length area, volume item 3 with TS 2, etc. This allowed us to isolate the effects of the three task structures from the tasks themselves.

Data Analysis

One researcher watched the videotaped interviews and transcribed each student's response to each item. All responses were then coded by a pair of researchers for correctness as well as for the type of comparison produced, additive, multiplicative, or excess. Any disputes were resolved through discussion between the two coding researchers. This data was then organized and analysed by task, task structure, grade level, and dimension (length, area, or volume).

RESULTS

An analysis of the interview transcripts revealed that the participants utilized four comparison types across dimensions and task structures. We categorized them as additive, multiplicative, excess, and other. We also classified responses as correct or incorrect.

The tables below present some of the results. Table 4 displays correctness of responses per grade level and TS. This table revealed which of the three task structures students were most successful with as well as if student age affected success with any one of the three task structures. Table 5 presents the comparison type by TS. We used this table to determine if any of the three task structures was more or less likely to elicit a specific type of comparison.

	TS1	TS2	TS3	Total
Grade 2	83%	8%	33%	42%
Grade 3	67%	8%	50%	42%
Grade 4	83%	42%	100%	75%
Total	78%	19%	61%	53%

Table 4: Correctness by task structure

As demonstrated in Table 4, students were most successful with TS 1, "How much longer/bigger…?" and least successful with TS 2, "How many times longer/bigger…?" This pattern also held for each individual grade level. Additionally, we note that there was substantial growth in correct responses to TS 2 and 3 as the students' age increases, while TS 1 remained more consistent across all three grade levels. Finally, we note that every grade 4 student produced a correct comparison when presented with TS 3 regardless of which attribute was identified for comparison, length, area, or volume.

Comparison	TS 1	TS 2	TS 3
Additive	75%	36%	6%
Multiplicative	3%	19%	61%
Excess	6%	25%	0%
Other	17%	19%	33%

Table 5: Comparison by task structure

From Table 5, we note three important results. First, TS 1 elicited an additive comparison 75% of the time, which was more than 12 times as much as the next most popular identifiable comparison, excess. Second, 61% of the time TS 3 elicited a multiplicative comparison, which was more than 10 times as much as the next most popular identifiable comparison, additive. Third, TS 2 did not clearly elicit a consistent comparison strategy; the three identified comparison strategies were used at least 19% of the time and none more than 36% of the time.

The analysis of the interview transcripts also revealed several interesting themes related to individual trends. For example, John, a male third grader, gave an answer of "2" every time TS 3 was posed and reported the length, area, or volume of the larger object every time TS 1, “How much longer/bigger…?” was posed. Rebecca, a female second grader, was the only participant unable to give a single multiplicative comparison regardless of attribute or TS.

Another theme revealed through analysis of the transcriptions related to how students understood the attributes. Several students required instruction as to the correct definition of volume before proceeding to the volume tasks. Rebecca initially defined volume as the amount of sound the blocks made when dropped, and five students (three second graders, one third grader, and one fourth grader) defined volume as the height of the collection of cubes. Only one student was unable to understand the attribute of area despite interviewer interventions. Samuel, a male second grader, tried to compare areas of regions by attending to only one dimension. Thus, for Area Item 1 with task structure 2, Samuel claimed the 2 in by 2 in region was “one bigger” than the 1 in by 1 in unit.

DISCUSSION

The purpose of this study was to compare student responses to three different, unit-eliciting task structures. Our results allowed us to compare these three task structures in two different ways, first, by correctness and second, by the comparison type typically produced. In general, we are pleased with each of the three task structures in that they do prompt students to shift from comparing by “bringing the objects themselves together [or] using another object as a common measure” (Piaget et al., 1960, p. 30), which can only be qualitative, to a comparison based on the selection and iteration of a unit, which is quantitative.

Students were most successful with TS 1 and produced correct responses 78% of the time. This task structure also proved to be extremely efficient for prompting students to see the need for a unit and producing an additive comparison. Students provided an additive comparison 75% of the time when presented with TS 1. Again, we do not feel that an additive comparison is inherently less desirable than a multiplicative comparison, however, we do feel that students should have experience producing both comparison types.

Students were least successful with TS 2, producing a correct response only 19% of the time. Many student comparisons in response to this TS 2 were based on quantifying the excess. In fact, more than four times as many excess comparisons were given in response to TS 2 than TS 1. We initially intended task structure 2 to be a prompt that would motivate the production of a multiplicative comparison, however, we found that it lead to more additive and excess comparisons than multiplicative. While reflecting on TS 2 we propose a change in the wording to “How many times as long/big…?” rather than “How many times longer/bigger…?” It seems reasonable that in response to a prompt of “How many times longer…?” a student is likely to attend only to the “longer” part of the longer object and to quantify the excess, as they did 25% of the time.

Students did not perform as well on TS 3 as they did on TS 1, however, they performed more than three times as well on TS 3 than they did on TS 2. More important to us is the fact that students provided more than three times as many multiplicative comparisons to TS 3 than to TS 2. We see both of these task structures as prompts for a multiplicative comparison, but we note that TS 3 is far better at eliciting this type of comparison. We also note that no student ever produced an excess comparison in response to TS 3.

CONCLUSION

Comparison and unit are both essential to the teaching and learning of measurement. According to Piaget et al. (1960) students can compare in three distinct ways, "bringing the objects themselves together, using another object as a common measure, and finally the construction of units to measure any distance by stepwise movement, i.e. unit iteration" (p. 30). Sarama and Clements (2009) describe measurement as the process of quantifying an attribute of an object in reference to a chosen unit. With this definition of measurement, we see that it is not until a student produces Piaget et al.'s third type of comparison that they are engaged in measuring. As we strive to engage students in meaningful measurement tasks, we have followed the two-step process described by Cullen et al. (2010), which we have found to be effective at motivating students to select and iterate a unit but not effective at motivating students to produce multiplicative comparisons.

As we continue to strive to engage students in meaningful measurement tasks, we recommend two task structures to motivate students to produce quantitative comparisons. We recommend task structure 1, "How much longer/bigger…?" as an effective prompt for the production of an additive comparison and task structure 3 and "If this has a(n) length/area/volume of 1, then what would you call the length/area/volume of this?" as an effective prompt for the production of a multiplicative comparison.

Questions for further research

We recommend further research to explore modifications of the wording of TS 2 from "How many times longer/bigger…?" to "How many times as long/big…?" We feel that the structure as we have presented here may have been drawing students' attention to the excess because of the use of the words longer and bigger. We also recognize that our sampling was small, so further work is needed to check the generality of the findings.

Finally, we are interested in exploring variations of TS 3 to focus on proportional reasoning and on the production of ratios between zero and one. For example, when a student was presented with area task three (Figure 2) with TS 3, they were told that the 1 in x 3 in shape had an area of one and asked to find the area of the 4 in x 4 in shape. In this case, the student could be guided to see that 3:12 as 1:4. The focus on proportional reasoning could be stressed more by telling the student the 1 in x 3 in shape has an area of 7 and asking them what the area of the 4 in x 4 in shape would be. This could help the student to notice that 1:4 as 7:28 or 3:12 as 7:28. As an extension, we are interested in modifying TS 3 to explore students' work with rational numbers between zero and one. This can be achieved by presenting students with two objects to compare, telling them that the larger object has a length, area, or volume of 1 and asking them to find the length, area, or volume of the smaller. For example, if a student is presented with a 4 in by 4 in square and a 1 in by 1 in square, told the larger square has an area of 1 then the area of the smaller would be ¼.

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A COMPOSITE MODEL OF STUDENTS' GEOMETRICAL FIGURE UNDERSTANDING [1]

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The main aim of this research study was to confirm a composite theoretical model concerning middle and high school students' geometrical figure understanding. Data were collected from 888 middle (grade 9) and high (grade 10, grade 11) school students. Structural equation modelling affirmed the existence of nine first-order factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification, construction of a figure and proof. The four second-order factors which represented the perceptual, operative, sequential and discursive apprehension were regressed to a third-order factor that corresponded to the geometrical figure understanding. Data analysis provided support for the invariance of this structure across the three age groups.

INTRODUCTION AND THEORETICAL FRAMEWORK

In geometry three registers are used: the register of natural language, the register of symbolic language and the figurative register. In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws (Mesquita, 1998). As a representation, it becomes more economically perceptible compared to the corresponding verbal one because in a figure all the relations of an object with other objects are depicted. However, the simultaneous mobilization of multiple relationships makes the distinction between what is given and what is required difficult. At the same time, the visual reinforcement of intuition can be so strong that it may narrow the concept image (Mesquita, 1998). Geometrical figures are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). Therefore the double status of external representation in geometry often causes difficulties to students when dealing with geometrical problems due to the interactions between concepts and images in geometrical reasoning (Mesquita, 1998).

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[1] This paper is a part of the research project "Ability to use multiple representations in functions and geometry: the transition from middle to high school" (0308(BE)/03) founded by the Research Promotion Foundation of Cyprus.

Duval (1995, 1999) distinguishes four apprehensions for a "geometrical figure": perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, *perceptual apprehension* refers to the recognition of a shape in a plane or in depth. In fact, one's perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. *Sequential apprehension* is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. *Discursive apprehension* is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). The epistemological function of the discursive apprehension is the proof. However, it is through *operative apprehension* that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refers to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one makes the figure larger or narrower, or slant, while the place way refers to its position or orientation variation.

Recently, some researchers (Deliyianni, Elia, Gagatsis, Monoyiou, & Panaoura, 2009; Elia, Gagatsis, Deliyianni, Monoyiou, & Michael, 2009) made an effort to verify empirically some of the cognitive processes underline the geometrical figure understanding proposed by Duval (1995, 1999). Elia et al. (2009) gave emphasis on the cognitive processes involved in operative apprehension. Furthermore, Deliyianni et al. (2009) affirmed the existence of a third-order model that involved six first-order factors indicating the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concept, three second-order factors representing perceptual, operative and discursive apprehension and a third-order factor that corresponded to the geometrical figure understanding. The study also suggested the invariance of this structure across elementary and secondary school students. The model emerged in this research study took into account Deliyianni's et al. (2009) findings and moved a step forward by involving sequential apprehension dimension, the three ways of figure modification in operative apprehension dimension and the deductive reasoning dimension. Specifically, keeping in mind the underlying cognitive complexity of geometrical activity (Duval, 1995) and the transition problem from one educational level to another universally (Mullins & Irvin, 2000) the main aim of this research study was to confirm a composite theoretical model concerning middle and high school students' geometrical figure understanding which involves the whole spectrum of geometrical figure apprehension types, i.e. perceptual, discursive, sequential and operative apprehension.

HYPOTHESES AND METHODOLOGY

In the present paper the following hypotheses were examined: (a) Perceptual, sequential, operative and discursive apprehension influence middle (grade 9) and high (grade 10, grade 11) school students' geometrical figure understanding, (b) Perceptual and recognition abilities have a differential effect on perceptual apprehension, (c) The three ways of figure modification (i.e. merelogic, optic and place way) have a differential effect on operative apprehension, (d) The abilities to construct and describe a figure's construction differentially affect sequential apprehension, (e) Inferences based on definition and inferences based on procedures for proof differentially affect discursive apprehension and (f) There are similarities between grade 9 to 11 school students in regard with the structure of their geometrical figure understanding.

The study was conducted among 888 students, aged 15 to 17, of middle (grade 9) and high (grade 10, grade 11) schools in Cyprus (319 in grade 9, 304 in grade 10, 265 in grade 11). Taking into account, Duval's (1995, 1999) apprehensions for a geometrical figure the a priori analysis of the test (Appendix) that was constructed in order to examine the hypotheses of this study is the following:

1. The first group of tasks includes task 1 (PE1a, PE1b, PE1c, PE1d, PE1e, PE1f, PE1g), 2 (PE2a, PE2b, PE2c, PE2d, PE2e, PE2f) and 3 (PE3a, PE3b). These tasks examine students' perceptual apprehension of a geometrical figure. The task 1 examines students' ability to identify and name the squares in a complex figure. The tasks 2 and 3 examine their ability to discriminate and recognize in the perceived figures several subfigures.
2. The second group of tasks includes task 4 (OP4), 5 (OP5), 6 (OP6), 7 (OP7), 8 (OP8), 9 (OP9), 10 (OP10) and 11 (OP11). These tasks examine students' operative apprehension of a geometrical figure. The tasks 4, 5 and 6 require a reconfiguration of a given figure, the tasks 7 and 8 an optic way of modification, while the tasks 9, 10 and 11 demand the place way of modifying figures
3. The third group of tasks consists of the tasks 12 (SE12), 13 (SE13), 14 (SE14), 15 (SE15) and 16 (SE16) that examine students' sequential figure apprehension. The tasks 12, 13 and 14 require students to construct a figure, while the tasks 15 and 16 investigate students' ability to describe the construction of a figure.
4. The fourth group of tasks includes the verbal problems 17 (DI17a, DI17b, DI17c), 18 (DI18), 19 (DI19), 20 (DI20) and 21 (DI21) that look into consideration students' discursive apprehension. Concerning discursive apprehension Harada, Gallou-Dumiel and Nohda's (2000) conceptualization is used, who indicated that the hypothetical-deductive proof is produced by this kind

of apprehension. In fact, the discursive apprehension is produced by inferences based on definitions and valid procedures of proof. Therefore, on the one hand, the problem 17 demands inferences based on definitions in order to be solved. On the other hand, the problems 18, 19, 20 and 21 require inferences based on procedures for proof for their solution.

Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning students' answers to the tasks were codified with PE, OP, SE and DI corresponding to perceptual, operative, sequential and discursive apprehension, respectively, followed by the number indicating the exercise number.

Confirmatory factor analysis (CFA), by using the EQS program, was used to explore the hypotheses about the structural organization of the various dimensions investigated here (Bentler, 1995). The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI and RMSEA. The following values of the three indices are needed to hold true for supporting an adequate fit of the model: $x^2/\mathrm{df} < 2$, CFI > 0.9, RMSEA < 0.06.

RESULTS

Confirmatory factor analysis model. Figure 1 presents the results of the elaborated model, which fitted the data reasonably well [$x^2(532) = 1021.58$, CFI = 0.96, RMSEA =0.03]. The first, second, third and forth coefficients of each factor stand for the application of the model in the whole sample (grade 9 to 11), grade 9, grade 10 and grade 11 school students, respectively. The errors of variables are omitted.

Particularly, the third-order model which is considered appropriate for interpreting geometrical figure understanding, involves nine first-order factors, four second-order factors and one third-order factor. The four second-order factors correspond to the geometrical figure perceptual (PEA), operative (OPA), sequential (SEA) and discursive (DIA) apprehension, respectively. Perceptual, operative, sequential and discursive apprehensions are regressed on a third-order factor that stands for the geometrical figure understanding (GFU). Therefore, it is suggested that the type of geometric figure apprehension does have an effect on geometrical figure understanding, verifying our first hypothesis.

On the second-order factor that stands for perceptual apprehension the first-order factors F1 and F2 are regressed. The first-order factor F1 refers to the perceptual tasks, while the first-order factor F2 to the recognition tasks. Thus, the findings reveal that perceptual and recognition abilities have a differential effect on geometrical figure perceptual apprehension (hypothesis b). On the second-order factor that corresponds to operative apprehension the first-order factors F3, F4 and F5 are regressed. The first-order factor F3 consists of the tasks which require a modification of a given figure in a mereologic way. The tasks which demand an optic way of modifying a given figure compose the first-order factor F4 and the tasks demanding the place way of modifying constitute the first-order factor F5. Therefore the results indicate that the ways of figure modification have an effect on operative figure understanding (hypothesis c). The first-order factors F6 and F7 are regressed on the second-order factor that stands for sequential apprehension. The first-order factor F6 refers to the tasks which demand the construction of a figure, while the first-order factor F7 consists of the tasks in which the description of a figure's construction is needed. Thus, the results indicate that these two abilities differentially affect sequential apprehension (hypothesis d). On the second-order factor that stands for discursive apprehension the first-order factors F8 and F9 are regressed. The first-order factor F8 refers to the tasks which require inferences based on definition, while the first-order factor F9 to the tasks which inferences based on procedures of proof are needed. Thus, the findings reveal that the kind of inferences has a differential effect on this kind of apprehension (hypothesis e). Loadings indicate that operative and discursive apprehension is more strongly related with geometrical figure understanding than perceptual and sequential apprehension.

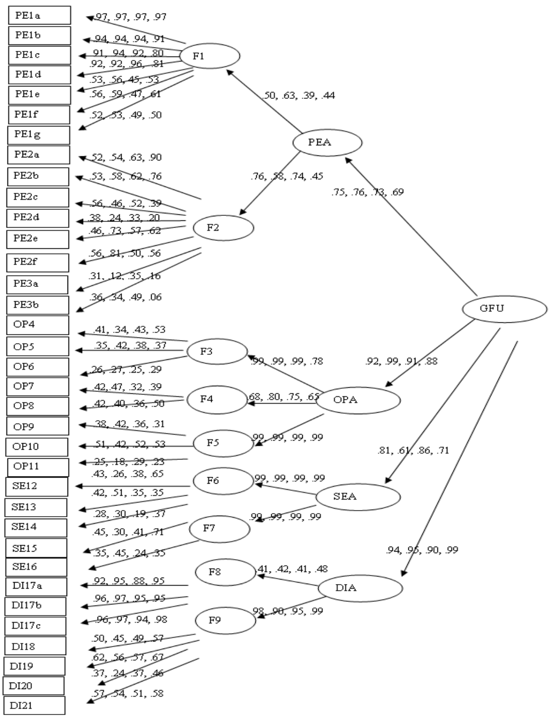

Figure 1. The CFA model of the geometrical figure understanding

To test for possible similarities between the three grades concerning students' geometrical figure understanding the proposed three-order factor model is validated for grade 9, 10 and 11 school students separately. The fit indices of the model tested for grade 9 [x^2 (538) = 747.96, CFI= 0.96, RMSEA= 0.04], grade 10 [x^2 (529) = 694.65, CFI= 0.96, RMSEA= 0.03] and grade 11 school students are acceptable [x^2 (539) = 773.33, CFI= 0.94, RMSEA= 0.04]. Thus, the results are in line with our hypothesis that the same geometrical figure understanding structure holds for both the middle (grade 9) and the high (grade 10, grade 11) school students. It is noteworthy that some factor loadings are higher in the group of the high school students suggesting that the specific structural organization potency increases across the ages. Besides, the factor loadings in grade 10 and 11 regarding perceptual and operative apprehensions are lower than in grade 9, while the factor loadings for sequential and discursive apprehensions are higher than the corresponding loadings in grade 9. These findings indicate that as students grow up are based more on mathematical properties and less on perceptual laws and cues.

CONCLUSIONS

In this research study a comprehensive model for geometrical figure understanding was constructed and verified using structural equation modelling. Moving a step forward in relation with previous studies (e.g. Elia et al., 2009; Deliyianni et al., 2009) which verified Duval's (1995, 1999) taxonomy, the proposed model involves the whole spectrum of geometrical figure apprehension types, i.e. perceptual, discursive, sequential and operative apprehension. Specifically, structural equation modelling affirmed the existence of a model with nine first-order factors, four second-order factors and one third-order factor. The four second-order factors correspond to the perceptual, operative, sequential and discursive apprehension of the geometrical figure, respectively. Perceptual, operative, sequential and discursive apprehensions are regressed on a third-order factor that stands for the geometrical figure understanding. Besides, findings affirmed the existence of nine first-order factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification, the construction of a figure and inferences based on definition or on procedures of proof. The model also suggests the invariance of this structure across middle and high school students. Thus, emphasis should be given in all the aspects of geometrical figure apprehension in both educational levels concerning teaching and learning. Findings reveal also that operative apprehension is the one which contributes the most to geometrical figure understanding. Taking into account that visualization consists only of operative apprehension (Duval, 1999) the important role of this kind of apprehension confirms empirically Duval's (1999) opinion that there is not understanding in geometry without visualization. The specific result indicates also that teaching and learning should give emphasis in this kind of apprehension since visualization is not primitive. In fact, the use of visualization requires specific training, specific to visualize each register (Duval, 1999). However, the model points out the important role of the other types of geometrical figure apprehension, as well, taking into account that even though coordination between them is needed each one is distinct from the other (Duval, 1999).

In addition to extent our knowledge about students' geometrical figure understanding, this study may give valuable information to curriculum designers and teachers of both middle and high school education. The elaborated model offers teachers a framework of students' thinking while solving a wide range of geometrical tasks in a systematic manner within and between the two educational levels. Therefore, the proposed framework may be used as a tool in mathematics instruction and designing tasks on geometry in both middle and high school. The framework of this study appears to be useful from an assessment perspective, as well. It may provide teachers with valuable and specific information on students' thinking in geometry based on prior knowledge and enable them to enhance this thinking by giving appropriate support through the tasks focused on the competences and cognitive processes for the geometrical figure understanding.

Concerning age, it is important to stress that the structure of the processes underlying the geometrical figure understanding was invariant across the three age groups tested here. These findings enhance the validity of the proposed framework and support its potential to coherently describe and predict students' understanding in geometry irrespectively of their grade, even during the transitional phase from middle to high school. However, findings reveal that some factor loadings are higher in the group of the high school students, indicating that overall cognitive development and learning take place. Furthermore, the results provide evidence for the existence of three forms of elementary geometry, proposed by Houdement and Kuzniak (2003). We may assume that in this research study, middle school teaching is mainly focused on Geometry I (Natural Geometry) that is closely linked to the perception. On the other hand, high school teaching gives emphasis to Geometry II (Natural Axiomatic Geometry) that it is closely linked to the figures and privileges the knowledge of properties and demonstration. Further research is needed to evaluate the feasibility of using this framework for developing effective instructional programs for the teaching of geometry in regular classroom situations in middle and high education.

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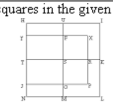
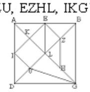
APPENDIX

1. Name the squares in the given figure:



(PE1a, PE1b, PE1c, PE1d, PE1e, PE1f, PE1g)

2. Recognize the figures in the parenthesis (KEZL, IEZU, EZHL, IKGU, LGU, BEL)



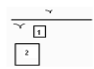
(PE2a, PE2b, PE2c, PE2d, PE2e, PE2f)

3. Underline the right sentence

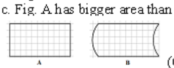
(OP4)

a. Fig. A has equal perimeter with Fig. B
b. Fig. A has smaller perimeter than Fig. B
c. Fig. A has bigger perimeter than Fig. B

4. Paris is looking the box 1 and 2 in the horizon. He says that the box 1 has exactly the same size with box 2. Is his opinion right? Explain your answer. (OP7)

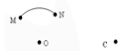


5. Underline the right sentence
a. Fig. A has equal area with Fig. B
b. Fig. A has smaller area than Fig. B
c. Fig. A has bigger area than Fig. B

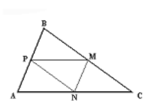


(OP9)

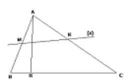
6. a)Draw an arc AB with center C, equal to the arc MN with center O. (SE12)
b) Describe the construction of the figure. (SE15)



7. The points M, N and P are the midpoints of the sides of triangle ABC. Show that the quadrilaterals APMN, BMNP, and GNPM are parallelograms. (VE17a, VE17b, VE17c)



8. In the figure below:
– AB and AC are equal
– Line (e) is parallel to BC
– AH is perpendicular to BC



Underline the right sentence
The length of NH
a) is smaller than MH
b) is equal to MH
c) is bigger than of MH
d) it cannot be determined (VE20)

METONYMY AND OBJECT FORMATION VECTOR SPACE THEORY

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Paper discusses the use of existing metonymies in reasoning with advanced mathematics tasks, specifically linear algebra topics, and the role of metonymies in the formation of a new object (metonymy).

INTRODUCTION

Various theories such as APOS (Parraguez and Oktac, 2010), Balacheff's theory of conceptions and Fischbein's theory of tacit models (Maracci, 2003) are used in interpreting and understanding the cognition of linear algebra concepts. We use the framework of metonymy as cognitive construct (Presmeg, 1998; 1997) in our interpretation of a linear algebra student's interview responses. Studies on metonymy have mainly been at the pre-secondary level documenting the role of metonymy in reasoning and object construction. There have however not been very many studies documenting its function at the college level especially at the advanced level topics such as linear algebra and analysis. Linear algebra is one of the advanced mathematics courses with high degree of abstraction and symbolism, which require learners to be able to comprehend abstract representations. Our work identified the frequent use of metonymic reasoning while working with abstract language. In this paper, we discuss a linear algebra student's reasoning with metonymies and his attempt to use these metonymies to construct another metonymy, thus a new mathematical object.

Framework

Work on metonymies mainly focuses on them as literary devices, rather than cognitive constructs that are used to encode information. Presmeg (1998; 1997) and Lakoff &Johnson (2000) on the other hand view metaphor and metonymy as cognitive structures. The act of using one object to stand for another is considered as functioning with metaphors or/and metonymies. Presmeg (1998; 1997) considers two types of metonymies. One of which, namely metonymy proper, is defined as "*a figure by which one word is put for another on the account of some actual relation between the things signified*" (Webster). An example of this kind is "*We studied Gauss*." Here, the word "*Gauss*" is used to indicate Gauss' work (*Gauss* $\rightarrow$ *Gauss' work*). Moreover, mathematical symbols can be put for various mathematical entities such as number families. The symbol "x" for example can represent real numbers (*x* $\rightarrow$ *real number*) even though the symbol x and the numbers are two unrelated objects. Another example of a metonymy proper may be a geometric image of a plane which may represent the mathematical attributes of vector spaces. The attributes of the mathematical object recognized from the image however are subject to the individual's interpretation of it. The geometric image may in fact be both a metaphor and a metonymy (Dogan-Dunlap, 2007; 2010). An individual may first consider the image as having similarities with various aspects of vector spaces, and after the initial consideration of the image as a metaphor, the same individual may begin considering the image as an object that is solely put for the concept itself. Second type of metonymy is considered as figure of speech. In this type, a part is used to represent the whole or vice versa (Presmeg, 1998). An example of this kind may come from the sentence, "*I've got a roof over my head*." Here, the part "*roof*" stands for the whole "*house*" (*roof* $\rightarrow$ *house*). An illustration of a circle taken to represent the class of all circles can also be considered as the metonymy of this kind. Presmeg (1997) however argues that this example may go beyond the figure of speech type to metonymy proper for the signifier may not be an element of the class represented. In other words, because the elements of classes are mental constructs, and an act of interpretation by an individual is involved in setting up the metonymy, individual may use the illustration to consider a class of circles that are not closely related to the figure. Hence, the illustration may become an example of a metonymy proper.

Method

The data discussed in this paper came from our work with two groups of students enrolled in three sections of a matrix algebra course at a Southwest University in USA—one traditional and the other two implementing an interactive web-module that provided the geometric representations of abstract linear algebra concepts. Students volunteered for a set of interviews conducted during spring 2009. The student whose interview responses discussed in this paper is from a module section. We use an alphanumeric name "SA21," to refer to him throughout the paper. He is an Hispanic-American majoring in mathematics with a secondary education minor. He was interviewed toward the end of April, 2009. Interview began with a set of pre-determined questions on basic vector space concepts such as linear independence, span and spanning set, and continued with follow-up questions. Pre-set questions were structured based on the learning difficulties reported in the literature (Dogan-Dunlap, 2010; Maracci, 2003; Sierpinska, 2000). A qualitative approach, namely the constant comparison method (Glaser, 1992), is used to analyze the responses.

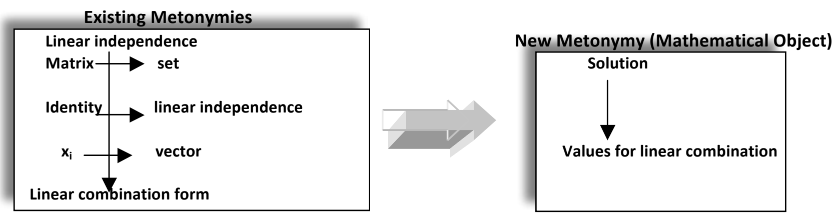


Figure 1. Metonymies displayed in SA21's reasoning.

Results and Discussion

Data provided in this section came from SA21's interview responses to a question "*Define the linear independence of a set of vectors*." Interview began student sharing his definition of linear independence, and continued with follow-up questions. SA21 shared two main notions with his metonymies embedded in for the linear independence/dependence of a set of vectors. One was his notion of linear combination. With this idea, SA21 was able to accurately identify linearly dependent sets provided that he could obtain a linear combination among the vectors of a set resulting in another vector of the set. The second idea he held throughout the interview focused mainly on the identity form of a matrix. Whenever a set with vectors given, where a linear combination is not easily accessible, SA21 proceeded directly (skipping vector equations) to representing vectors with a matrix and searching for an identity form via Gauss-Jordan elimination process, which is a part of one of the approaches included in the textbook (Johnson, Reiss and Arnold, 2001) and covered in class. Using the two notions, SA21 was able to accurately identify linearly independent/dependent sets. The two ideas however appeared to have been unrelated entities for the student. Throughout the interview SA21 was prompted by the interviewer to discuss his understanding of the two concepts by comparing his parametric representation of solutions (see figure 2, IV) and its connection to the linear combinations he provided. During his attempts, toward the end of the interview, he began to apply his existing metonymies, and came up with a new notion of how the two ideas may be related. We believe that SA21 was, at the start of the interview, unaware of any connections between the two objects, but toward the end he began to consider the potentiality. Before proceeding with SA21's responses, in order to provide a context for the responses included in this section, let's present one of the examples student gave. After sharing his notion of a linear independence, SA21 was asked to give an example. He gave the set $\{u=\begin{bmatrix}1\\1\end{bmatrix}, v=\begin{bmatrix}2\\3\end{bmatrix}, w=\begin{bmatrix}4\\5\end{bmatrix}\}$ and continued to explain how he identified this set as a linearly dependent one: first, he considered a matrix whose columns were formed by the vectors of the set. After applying Gauss-Jordan elimination process, he obtained the row reduced echelon form (rref) of the matrix, which is $\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\end{bmatrix}$, circling the identity form as seen in figure 2, I. Furthermore, using the rref form he identified the set as linearly dependent reasoning that the last column of the matrix is depended on the first two columns since only the first two columns form an identity. He proceeded to write both parametric and vector representation of the solutions (see figure 2, IV, III respectively). Next, when asked to explain how his solutions may imply the linear dependence of the set, he wrote the linear combination 2u+v=w directly using the numerical entries of the vectors of the set ignoring his solution representations (see figure 2, II). In fact, many of the excerpts provided in the results section are revealing SA21's attempts to explain how one may obtain a linear combination among vectors of a set using parametric or vector solution forms.

At this point, let's share our perspective of how a linear combination can come from a solution. Consider the parametric representation of the solutions seen in figure 2, IV. One solution would be (-2, -1, 1) with variable x_3 assigned value 1. That is, this particular solution satisfies the vector equation $x_1u+x_2v+x_3w=0$ for the vectors u, v and w, thus -2u-v+w=0 (This connection appeared to have been missing in SA21's knowledge during the interview). Solving the equation for w, one would obtain the vector equation w=2u+v offering a linear combination of the vectors u and v for w. Considering x_3=2, as another example, one would obtain the equation -4u-2v+2w=0 leading to the vector u written as a linear combination of the other two vectors. That is u=-1/2v+1/2w. For the remainder of the paper, we will discuss SA21's use of his existing metonymies while responding to interview questions and portray a picture of his effort to form a new metonymy (a mathematical object)

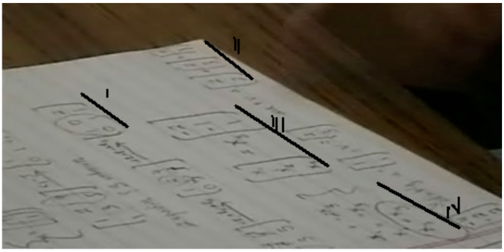


Figure 2. View from SA21's work from his

Existing Metonymies

Linear Independence $\longrightarrow$ ***Linear combination***

SA21's interview displays a frequent use of metonymies in his reasoning. Moreover these metonymies appear to constitute his knowledge of linear independence. As depicted in figure 1 above, linear combination form appears to be the overarching metonymy SA21 functions with. When student SA21 was asked to share his definition and his understanding of linear independence, SA21's initial response indicated that he was considering the term "*linear independence*" to stand for "*linear combination*" ideas, which can be seen in the following excerpt (some of the phrases are made bold by the authors in order to put emphasis on):

SA21: *Okay ….I think of linear independence so… I think we have a set of vectors, so I'll just write… like you have **u1, u2, so we can go** all the way to however many we want. Then I… so, I know that they are independent if, suppose we have, **so we have a1** which is like some **real number**… times **an** and we'll just keep on going…So I think that's kind of close to what you wanted. Since this is **the key component*** [pointing to $\mathbf{a_1u_1+a_2u_2+\ldots+a_nu_n}$; $a_1, a_2, \ldots, a_n$ are real numbers].

Even though later in the interview SA21 stated that his initial description was for linear combination and not for linear independence, his responses throughout the interview however appeared to sustain the view that the metonymic use of "*linear independence*" for "*linear combination*" has been a more dominant factor in his reasoning.

Matrix $\longrightarrow$ ***Set***

SA21 was regularly prompted to elaborate his responses. While elaborating, he integrated other metonymies into his reasoning. His metonymic use of "*matrix*" standing for "*set*" for example fits well with his overall notion of linear combination ideas. He consistently regarded matrices as representing vectors of sets, and looked for linear combinations among the columns. This can be observed in his response "*So, then here, so I would to determine independence or dependence I know so I **just build my coefficient matrix**…"* Here, SA21 goes straight to a matrix whose columns are the vectors of his set. Next, he points to the rref of this matrix and states that "*it is linearly dependent*." When asked what he means by "*it,*" he says " *Uh… for the set? I would I don't know if I would say for **the set** or for **the matrix**…"* It is obvious that SA21 does not distinguish sets from matrices and that for him matrices are sets.

Identity Form $\longrightarrow$ ***Linear independence***

SA21 continued to consider matrices as sets throughout the interview. While searching for linear combinations among the columns of a matrix, he introduced another metonymy, use of "*identity form*" for "*linear independence*." In the conversation below, for example, when SA21 is asked to explain how he identified the linear dependence of a set without considering solution forms, he reasons with his metonymy of "*identity form.*" Here, he focuses on the identity form among the columns of matrices to identify "*linear independence*." Furthermore, he uses these columns to come up with linear combinations.

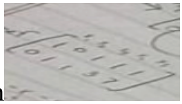
I: …you stopped you did not write it [meaning a solution set]. *You directly said this* [pointing to a set of vectors] *is linearly dependent, and reasoning for that was?*

SA21. …I cant express these other vectors [pointing to the last three columns of a 2x5 matrix] *as **identity** …What I would want is I want **identit**y that is **the key**, for a 3 by 3…we want something like this* [meaning an identity form] *to me that* [meaning **identity form**] *says that that* [pointing to the vectors of an identity form in a matrix] *is **linear independent**…*

He next gave, after prompted to provide a solution set, the following response still functioning with his metonymy of identity form.

*SA21 ... we **have identity** here* [pointing to the first 2 columns of a 2x3 matrix], *but this is not* [points at the last column with values (2, 1), see figure 2, I] *and this **means that this is dependent on this*** [meaning that the last column of the matrix is dependent on the first two columns] *so I like to write what we have, so I'll write **x1, I like to use xs,** equals minus **x sub 2, x sub 3,** **x sub 2** equals minus **x sub 3** and then **x3 is our independent vector*** [see the parametric representation in figure 2, IV] *So then, from here* [pointing to the parametric form seen in figure 2, IV] *I can just see that we have a dependent … linearly dependent set…*

x_i $\longrightarrow$ ***vector***

The excerpt above also reveals another metonymy, "x_i*s*" set forth for "*vectors*." This appeared to be the most influential metonymy in SA21's reasoning. He, in fact, seemed to attribute symbols with fixed meanings and reason with these meanings throughout the interview. Initially in the interview, SA21 considered "a_i " as symbols representing known values in a linear combination but later reserved them for unknowns and chose the symbol "x_i" to stand for "*vectors*." His preference to use x_is in representing vectors is apparent in the phrases "*I like to use xs*" and "*x_3 is our independent vector*." Furthermore, his persistence to attribute a fixed meaning to x_is can also be observed in the excerpts below. In this response, SA21 forms a matrix using a set of vectors and labels each column as x_1, x_2, x_3, x_4 and x_5 respectively as seen in . It is clear from these responses that student is using the symbols, x_1, x_2, x_3, x_4 and x_5 to represent the five vectors of a set.

SA21: *Now, if I was to write,* ***like how I did that last one*** [pointing to a set **with three vectors]** *so you have I have* ***five vectors.*** *So I have* ***x1, x2*** [marking each column on a coefficient matrix with x_1, x_2, x_3, x_4 and x_5]. *I'm thinking… so I'm already saying that I think I'm saying that* ***my x1 and x2 are independent vectors… and that x sub…*** *Well, the way I can think about. It is. I know we would rewrite this as* ***x1, x2****, this is gonna equal some* ***x3*** *and this is gonna be -2, -1, and 1 so then I just see* ***that x3*** *or* ***our third vector*** *will be dependent that's kinda like how I think about it.*

Formation of New Metonymy

After prompted for further explanation on the potential connection between a vector equation and a parametric representation of solutions, toward the end of the interview, SA21 began comparing the roles of the symbols a_i and x_i. One can clearly observe, on his responses below, his metonymic use of the symbols and how each symbol continues to hold a distinct meaning.

SA21: *…we know, and* ***these are unknowns*** [pointing to $\mathbf{a_1}$ in $a_1u_1+a_2u_2+\ldots+a_nu_n=0$]. *So I want to say that… I think* ***that a1 and this*** [pointing to the parametric solution form in figure 2, IV] ***should be the same****, is that what you are trying to say?*

It is evident with the phrase, "*is that what you are trying to say?"* that up to this point in the interview, SA21 was not considering the symbols a_i and x_i holding the same meaning. After this point however he began to consider the potentiality of them relating. For example, in the excerpt below, while attempting to connect the two symbols, he uses his existing metonymies. He begins with reiterating his metonymy of x_is representing vectors. Next, he uses the metonymies of "*columns*" for "*vectors,*" and "*identity*" for "*linear independence*." That is, SA21 considers the first two vectors of a set (which forms the first two columns of a matrix) as linearly independent vectors reasoning with his metonymy of "*identity form*," and concludes the linear dependence of the last three vectors of the set. He next, for the first time in the interview, begins considering x_is as unknowns at the same time reserving them as signifiers for vectors. For SA21, x_is now embrace two meanings. Furthermore, the two meanings appear to imply that x_is may also be representing coefficient values for linear combinations.

SA21: *So I am really saying that I think of saying that my* ***x sub 1 and x sub 2 are independent*** *vectors, and that ….or I would say these two* [pointing to u1 and u2 in the set $\{u_1 = \begin{bmatrix}1\\1\end{bmatrix}, u_2 = \begin{bmatrix}2\\3\end{bmatrix}, u_3 = \begin{bmatrix}4\\5\end{bmatrix}, u_4 = \begin{bmatrix}5\\6\end{bmatrix}, u_5 = \begin{bmatrix}9\\10\end{bmatrix}\}$] *because I put them in* ***this order*** [implying they would lead to **identity form**]. *So I would say that* ***these two*** *would* ***be independent*** *and these* ***independent vectors these*** [pointing to the last three vectors in the set]

dependent on these *[pointing to the first two vectors in the set] so and I think this is telling me that* ***I wonder*** *I was to put of x sub 3 like think some value?...I know this* [**pointing to** x_1 and -2 in the vector equation $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$] *is my* ***unknown for my very first vector this is the second one, this is the third and fourth*** [pointing to x_2, x_3 and x_4 in the same vector equation]. *So I wonder if I was to put. I think may be telling me that if I was to* ***look at it this way****. so I am thinking if I want to* ***express the third vector*** [**circling** x_3 in the same vector equation] *I wanna say this* [pointing to u_3 in the set] *is my third vector* ***my x sub 3*** *because I gave it this so I wanna say that suppose I wanna write this as* ***a combination of this*** *it is telling me that if I was to have that. If I pick any value for* ***x sub three****, suppose I want two so I want* ***x sub 3*** *equal just some two. It is telling me that I can write* ***a linear combination of this third vector [****pointing to* x_3 *in the vector equation****]*** *as* ***a combination of all of these*** [pointing to the vectors in the set above] *then I can substitute this two into here* [pointing to the vector equation $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$] *so it is going to give me…since this* [pointing to -2 in (-2,-1,1,0,0)] *belongs to my first* ***unknown.*** *I wanna say that I can* ***express u sub one as a minus two u one*** [writes $-2u_1$] *…since we are adding them* [meaning $-2u_1$ and $-u_2$] *it is telling me that* [$-2u_1-u_2$] *will equal my* ***u sub 3*** *my third vector...*

Our inference, in fact, is validated by the response below. In this excerpt, SA21 is using the values in (-2,-1,1,0,0) as coefficients to form a linear combination for the third vector signified by x_3. Furthermore he is considering each value in (-2,-1,1,0,0) associated with one of the symbols x_1,x_2, x_3, and x_4 respectively. That is, he now identifies the first vector with x_1 and considers the first component value, -2, as the coefficient value for the first vector and so on.

SA21: *I am focusing on since I want to express these I know somehow this has to. This* [pointing to the vector equation, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$] *telling me that I can* ***express this third vector*** [**pointing to x3** in the same vector equation] *somehow with* ***a*** **combination** with ***these numbers*** [pointing to (-2,-1,1,0,0)]…

From this point on in the interview, SA21 consistently considered "*solutions*" (represented by parametric representations or vector forms) as values for "*coefficients of vectors forming linear combinations*." This new notion appeared to have developed into a new metonymy and a new mathematical object for SA21.

Conclusion

There are many studies documenting metonymic reasoning at the pre-college level, but very little at the college level, especially in advanced mathematics topics. This paper outlined one case to discuss the metonymic use at the college level, specifically with linear algebra topics. SA21's interview responses provided a portrait of how a mathematics student may be functioning with metonymies. We observed among the particular student's interview responses that metonymic reasoning may have led to the formation of a new metonymic knowledge.

The case we discussed in this paper further supports the earlier studies in that metonymies appear to be cognitive constructs with meanings associated to (Presmeg, 1998; 1997), not just literally devices to aid with recalling. Moreover, they appear to be instrumental in forming new knowledge thus need to be taken with utmost importance and paid close attention to their role in one's knowledge of advanced mathematics concepts.

Finally, this paper reported findings of one linear algebra student's metonymic reasoning. They by no means can be taken as generalization to all linear algebra learners. Future research, utilizing the work reported here, is in need with a larger sample group.

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THE EFFECTS OF PHYSICAL MANIPULATIVES ON ACHIEVEMENT IN MATHEMATICS IN GRADES K-6: A META-ANALYSIS

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The use of manipulatives in the teaching of mathematics is supported by theory, by professional organization' recommendations, and by some research, but there is no conclusive evidence that their use results in higher achievement. The goal of this meta-analysis was to synthesize existing research comparing the effects of instruction using manipulatives to the effects of traditional mathematics instruction in the USA. A systematic search of databases and journals located 31 studies conducted between 1989 and 2010 that met specified criteria for inclusion in the meta-analysis, The mean effect size for these studies was 0.50 with a CI of (0.34, 0.65), indicating that students using manipulatives scored statistically significantly higher than students who did not. The effects of nine moderator variables were also investigated.

INTRODUCTION

For many years, the use of concrete physical manipulative materials in the teaching of elementary school mathematics has been encouraged by mathematics teacher educators, mathematics education researchers, and professional organizations of mathematics teachers such as the National Council of Teachers of Mathematics (2000). Often these recommendations have been supported by psychological theories put forward by Piaget (1965), Bruner (1977), or Dienes (1973). Proponents of manipulatives have argued that manipulatives help students in the transition from concrete to abstract, that manipulatives help strengthen multiple representations, that manipulatives help students understand mathematics, and that manipulatives help increase students' achievement in mathematics. Others have noted, on the other hand, that many US teachers avoid using manipulatives and that students sometimes have difficulty using manipulatives effectively. For example, Kaput (1989) pointed out that one problem sometimes encountered when using manipulatives is that connections between actions on the manipulatives and actions on symbolic notations are unclear to students, and he hypothesized that the cognitive load imposed during activities with manipulatives may be too great for some students. Other authors have made similar observations about the difficulties teachers may have in using manipulatives effectively, and some researchers and commentators have described these problems as cases of improper use of the physical materials in teaching.

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PREVIOUS REVIEWS OF RESEARCH ON MANIPULATIVES

Since the 1970s many articles have been written about the effects of using physical manipulatives in teaching mathematics. In this section two narrative or "vote counting" reviews (Fennema, 1972; Suydam & Higgins, 1977) and three meta-analyses (Parham, 1983; LeNoir, 1989; Sowell,1989) are discussed.

Fennema (1972)

Fennema (1972) examined 16 studies carried out from 1950 to 1966 on the effectiveness of learning mathematics with the use of concrete manipulatives for students in Grades 1-8. Results from this review supported the use of physical manipulatives in the teaching of mathematics in the early elementary grades, especially first grade. However, Fennema found that the use of physical models appears to neither improve nor hamper the learning of mathematical ideas for older children. Fennema concluded that as children move through elementary school to higher grades, physical models should be replaced by symbolic ones since older learners' background in mathematics is richer than that of younger learners.

Suydam and Higgins (1977)

Suydam and Higgins (1977) did a comprehensive review of studies on activity-based programs in mathematics conducted between 1933 and 1976 in Grades K-8. They provided an annotated bibliography of 235 studies and reviewed research studies on manipulative materials and on activity programs and modes of instruction..

Suydam and Higgins' main research question was, "Does the use of manipulative materials help student achievement in mathematics?" To answer this question, they identified 23 studies conducted from 1957 to 1976 in Grades 1-8 comparing lessons in which manipulatives were used with lessons in which manipulatives were not used. Approximately half (11 of 23) of the studies favored the use of manipulative materials; ten studies (43%) reported no statistically significant difference in achievement between lessons using manipulatives and those not using manipulatives; and two studies (9%) favored lessons in which manipulative materials were not used. After taking a closer look at these studies, Suydam and Higgins concluded, "Lessons using manipulative materials have a higher probability of producing greater mathematical achievement than do non-manipulative lessons" (p. 83).

Parham (1983)

A meta-analysis of of research on the use of manipulative materials in Grades 1-6 was conducted by Parham in 1983. Parham examined 64 research studies carried out between 1960 and 1982 that compared achievement in mathematics using manipulatives to not using manipulatives. Parham's study yielded 171 effect sizes for six selected study characteristics: assignment of students to teachers, control for author bias in evaluating achievement, number of treatment groups, overall study quality, source of study, and publication year. Parham obtained an overall mean effect size of 1.03 for the achievement scores of students who used manipulatives as compared to students who did not use manipulatives. Parham interpreted this effect size to mean that, the average student who used manipulatives in learning of mathematics scored at approximately the 85th percentile whereas the average student who did not use manipulatives scored at the 50th percentile. Parham also conducted a step-wise multiple regression analysis to account for the variation in effects among the different studies. The regression analysis indicated that grade level and type of study had a significant impact on the outcome. Parham concluded that based on the average effect sizes obtained from her two types of analyses, the use of manipulative materials does have a positive effect on student achievement.

LeNoir (1989)

LeNoir (1989) conducted a meta-analysis on the effects of manipulatives on the acquisition, retention, and transfer of mathematical concepts from kindergarten through college. LeNoir selected 45 studies (41 dissertations and four journal articles) carried out between 1958 and 1985 that met his selection criteria. The independent variables in LeNoir's meta-analysis were: form of publication, date of publication, grade level, length of treatment, length of retention period, experimenter bias, type of manipulative, subject matter studied, and type of use (teacher or student or both). One or more effect sizes were calculated from each study, bias was removed, and correction was made for reliability of the instruments.

LeNoir (1989) performed three sets of analyses on the data, one each for acquisition, retention, and transfer of a mathematical concept. He also tested each set of effect sizes for homogeneity. If a set was not homogeneous after outliers were removed, LeNoir subcategorized the set according to grade level, content area, and length of treatment/retention period. He then repeated the testing for homogeneity. As a result of these three sets of analyses, LeNoir had three specific findings: (1) Grades 10 to college students who used manipulatives in learning mathematical concepts achieved and retained more than students who did not use manipulatives, (2) Grades 6-9 students who used manipulatives in learning measurement achieved more than students who did not use manipulatives; and (3) Grades 6-9 students who used manipulatives in learning various mathematical concepts retained more after 34-112 days of instruction than students who did not use manipulatives.

Sowell (1989)

Sowell (1989) conducted a meta-analysis of studies conducted between 1954 and 1987 in order to examine the effect of using concrete manipulatives on student achievement and attitudes toward mathematics in kindergarten through college. Sixty studies (38 journal reports, three unpublished reports, and 19 dissertations) fit the inclusion criteria which were: being a comparative study of manipulative use versus non-use, using manipulatives in learning mathematics, involving a treatment that lasted at least a week, and providing data from outcome measures. Seventeen studies (28%) were conducted in Grades K-2, 17 (28%) in Grades 3-4, nine (15%) in

Grades 5-6, 11 (18%) in Grades 7-9, and six (10%) at postsecondary level. Treatment length varied from one to 72 weeks, with a median of six weeks.

Sowell (1989) conducted several analyses to determine the effects of treatment length and grade level on the acquisition of specific and broadly stated objectives for students who used manipulative, pictorial, or abstract methods. The only result found to be statistically significant was that students in Grades 1-6 who used manipulatives for a whole school year or longer improved in their achievement of mathematics.

Conclusions from previous reviews and meta-analyses

The studies discussed above provide some evidence supporting the use of manipulatives in mathematics instruction, but the evidence is not consistently strong. Efforts to examine results by grade level, different mathematical topics, or in other categorizies have not yielded many positive and statistically significant findings. Although teacher educators and professional organizations have become increasingly vocal in advocating the use of concrete manipulative materials, especially since the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) were published, we could find no meta-analyses of research on this topic published since 1989.

RESEARCH QUESTIONS

The main research question addressed in this study is ”What is the effect of the use of manipulatives on achievement in mathematics for students at the elementary school level (grades K-6)?” In addition to the main research question, the effects of nine moderator variables are also explored.

METHODOLOGY

This study uses a meta-analytic approach as defined by Glass, McGaw, and Smith (1981) and elaborated by Lipsey and Wilson (2001) and Cooper (2010). According to Glass et al., “The essential characteristic of meta-analysis is that it is the statistical analysis of the summary findings of many empirical studies” (p.21). Meta-analysis involves the following steps:

- formulating the research questions,
- developing a coding form
- gathering research studies by searching the literature,
- carefully coding appropriate information in each research study,
- calculating effect sizes,
- analyzing the effect sizes using conventional statistical techniques, and
- interpreting and reporting the findings

In addition to formulating and delimiting the research question the meta-analysis is designed to answer, the researcher must also develop a set of criteria for deciding which research studies to include in the meta-analysis. The inclusion criteria for this study were the following:

- Eligible studies must involve the comparison of manipulative use to manipulative non-use in mathematics classes. Manipulatives must be used by the students, and not just by the teacher.
- Eligible studies must include students in Grades K-6 only.
- Each treatment group must contain at least ten students.
- To be eligible, studies must report scores of achievement in mathematics. In addition, sufficient statistical data must be reported to allow the calculation of an effect size (or sizes).
- Eligible studies in this meta-analysis must use a treatment group / control group design. The treatment condition could be any mathematics topic and teaching method in which the use of manipulatives was incorporated for at least one week. The control condition should be the same mathematics topic and teaching method for the same amount of time, but not involving the use of manipulatives.
- Only studies conducted in the United States were eligible.
- Only studies published in 1989 or later were included.
- Both published and unpublished studies were eligible for this meta-analysis, including articles published in refereed and non-refereed journals, unpublished dissertations, and unpublished works such as conference papers.

A coding sheet was developed to record the key features of the studies considered for inclusion. Examples of the characteristics coded include the following: form of publication, date of publication, grade level, student ability level, socioeconomic status of the students, gender, ethnicity, language, community type, measurement instrument(s), reliability of instrument(s), mathematical topic, method of instruction, type(s) of manipulatives used, length of treatment, amount of teacher training in the use of the manipulatives, teaching experience, sampling procedures, experimental design (pretest-posttest or posttest only), etc.

FINDINGS

Thousands of studies about manipulativs were located as a result of searching electronic databases, tables of contents of journals, and reference lists of articles. A total of 1035 abstracts were examined closely, but 885 were rejected after it was determined that they did not meet the criteria listed above. After carefully reading each of the remaining 143 studies, 111 studies were rejected, leaving 32, one of which was later determined to be an article based on a dissertation already included. A list of the final 31 primary studies is available from the authors. Four of the studies included two dependent variables each, resulting in 35 different effect sizes.

Overall Effects of Manipulative Use

The figure below shows the 35 sample sizes and effect sizes and their associated 95% confidence intervals. The dashed vertical line indicates an effect size of zero. The effect sizes ranged from -0.22 to 1.52, effect sizes greater than zero favoring the groups of students who used manipulatives, and effect sizes less than zero favoring the groups that did not use manipulatives. The median effect size was 0.52 and the unweighted mean effect size was also 0.52. The weighted mean effect size was 0.50. An effect size this large is considered to be a medium effect size. Five effect sizes (14%) were negative, and 30 (86%) were positive. The distribution of effect sizes is skewed positively.

In the forest plot the wider confidence intervals are from studies that have smaller sample sizes and low precision, and the narrower confidence intervals are from studies that have larger sample sizes and high precision. If the confidence interval includes zero, then the effect size is not statistically significant, but if the confidence interval does not contain zero, then the effect size is statistically significant at $p<.05$;. In this meta-analysis, about half of the confidence intervals (18 out of 35) do not include zero, which means that those effect sizes are statistically significant. The very short confidence interval at the bottom of the forest plot represents the overall weighted mean effect size and its confidence interval. It is short because it is based on the total sample size from all 35 dependent variables in the 31 studies.

Analysis of Moderator Variables

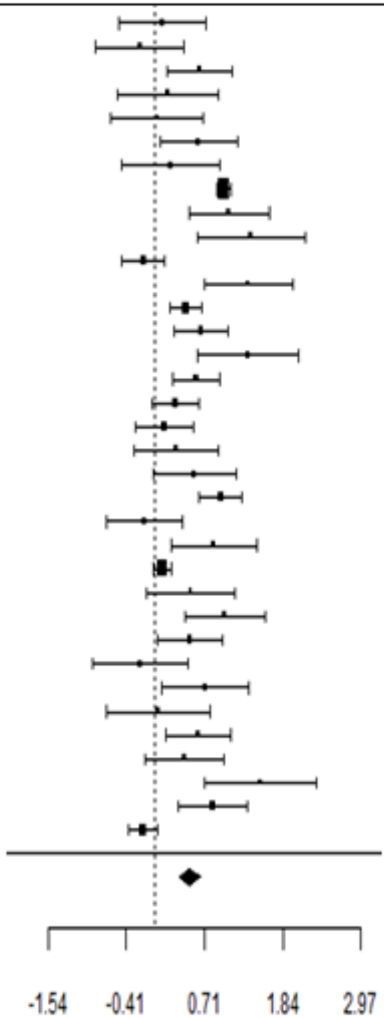
Moderator variables reflect characteristics of the primary studies that may affect the observed effect size in the study. Although 25 different potential moderator variables were coded, only nine were analyzed because very few researchers reported sufficient information about the other potential moderators. Moderators were tested using homogeneity analysis (Cooper, 2010) and the Q_b statistic. A significant Q_b indicates that the mean effect sizes for the groups of studies vary more that would be expected by sampling error alone. All of the nine potential moderator variables had statistically significant Q_b statistics, indicating that they accounted for significant amounts of variance in the mean effect sizes.

The quality of the studies in this meta-analysis is reflected by the moderator variable, type of design (pretest-postest or posttest only). For the pretest-posttest studies the mean effect size was about half of that of the studies that employed a posttest only design (0.39 *vs.* 0.73). In both cases the mean effect sizes were significantly different from zero. Of the 35 effect sizes in this study, six were from published journal articles and 29 were from unpublished dissertations, theses, or ERIC documents. The mean effect size for published studies (0.64) was significantly larger than for unpublished works (0.47). The correlation between year of publication and effect size was 0.051 (ns) indicating no linear relationship between these variables, even though the Q_b statistic based on groups of studies in four intervals was significant.

Figure: Total Sample Sizes, Effect Sizes, 95% Confidence Intervals, and Forest Plot for 35 Effect Sizes from 31 Studies.

First Author	Total N	Effect Size	95% Confidence Interval Lower Limit	95% Confidence Interval Upper Limit
Ananth 1	42	0.1104	-0.51	0.73
Ananth 2	38	-0.2174	-0.86	0.42
Busta	102	0.6514	0.19	1.12
Casey 1	30	0.1633	-0.56	0.89
Casey 2	35	0.0240	-0.64	0.69
Chester	52	0.6288	0.07	1.19
Cotter	31	0.2252	-0.48	0.93
Cramer	1666	0.9953	0.89	1.10
Daniel 1	53	1.0713	0.49	1.65
Daniel 2	32	1.3899	0.62	2.16
Drickey	165	-0.1654	-0.47	0.14
Fanelli	46	1.3458	0.71	1.99
Getgood	325	0.4451	0.22	0.67
Gilmore	104	0.6680	0.27	1.06
Harger	36	1.3394	0.62	2.06
Hawkins	139	0.6023	0.26	0.94
Hiebert	151	0.2914	-0.05	0.64
Hougas	86	0.1326	-0.29	0.56
Jabagchourian	42	0.3070	-0.30	0.92
Johnson	45	0.5670	-0.03	1.16
Jordan	183	0.9492	0.64	1.26
Kanemoto	51	-0.1493	-0.70	0.40
Kipfinger	44	0.8435	0.23	1.46
Mayo	917	0.1094	-0.03	0.24
McLaughlin	40	0.5157	-0.11	1.15
Moore	52	0.9856	0.41	1.56
Nute	72	0.5102	0.04	0.98
Omniewski	33	-0.2054	-0.89	0.48
Rust	42	0.7224	0.10	1.35
Taylor	29	0.0468	-0.69	0.79
Thomas 1	72	0.6289	0.16	1.10
Thomas 2	55	0.4256	-0.14	0.99
Tournaki	30	1.5174	0.71	2.33
Witzel	68	0.8330	0.34	1.33
Woodworth	380	-0.1724	-0.37	0.03

Forest Plot



The effect of students' ability levels was statistically significant, with learning disabled students and high ability students having the largest mean effect sizes (1.10 and 1.07 respectively. Students with low and average abilities showed lower effect sizes (0.39 and 0.44), but the mean effect sizes for all four groups of studies were significantly different from zero. The analysis by type of achievement instrument showed that the mean effect size was highest for researcher- or teacher-made tests (0.66) as compared with standardized tests (0.34) and textbook tests (0.40).

CONCLUSIONS

These results indicate moderate variability in the results of the primary studies, but an overall moderately strong positive effect of using manipulatives in elementary school mathematics instruction. Teacher educators and mathematics supervisors can confidently recommend their use.

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RBC EPISTEMIC ACTIONS AND THE ROLE OF VAGUE LANGUAGE

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In this paper, the RBC framework developed by Hershkowitz, Schwartz, & Dreyfus (2001) is used to analyse and describe construction of mathematical knowledge by primary pupils in a whole-class setting. The lesson concerned the development of an explicit formula that could be used to solve what is commonly termed the Handshakes problem. On the basis of mathematical principles developed for the lesson, transcripts of whole-class discussion were coded using the RBC framework. Some of these epistemic actions were inferred by the language used by pupils – for example, they tended to use linguistic hedges when conjecturing (building-with) but used language of greater certitude when constructing. It also emerged that vague language was central to the collaborative construction of mathematical ideas.

INTRODUCTION

In research related to mathematical abstraction, a theoretical framework that has received considerable attention is 'Abstraction in Context' (AiC) (Schwartz, Dreyfus, & Hershkowitz, 2009). The three epistemic or observable actions identified by Hershkowitz and her colleagues (Hershkowitz, *et al*., 2001; Schwartz, *et al.*, 2009) as giving a strong indication that mathematical abstraction is taking place are 'recognizing', 'building-with' and 'constructing'. This RBC framework has been applied to the construction of knowledge by individuals (e.g., Hershkowitz*, et al.*, 2001) and by small groups of students (e.g., Hershkowitz, Hadas, & Dreyfus, 2006). In this paper I extend and illustrate its application to the construction of mathematical ideas in the context of whole-class conversation. Attention to the role played by vague language further enhances the analysis and description of the constructing process.

THEORETICAL FRAMEWORK

As mentioned above, the epistemic actions related to AiC are 'recognizing', 'building-with' and 'constructing'. Recognition of a familiar structure occurs when a student realizes that the structure is a component of a given mathematical situation. This is not the first time that the student has met the structure. When 'building-with', the student is not enriched with new, more complex structural knowledge but is using available structural knowledge to deal with the problem at hand. This stage is evident when he or she is involved in an application task or making a hypothesis or justifying a statement. Constructing, the most significant of the epistemic actions that are

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constituent of abstraction, is a process of building more complex structures from simpler structures. It involves the reorganization of mathematical elements so that a more refined structure emerges. In order to distinguish between 'building-with' and 'constructing', it helps if the goals of the particular activities are considered. In constructing, students use a new mathematical structure to attain their goal. In 'building-with', a goal is attained by combining existing structures. These three epistemic actions are not linear but nested. In other words, 'recognizing' (R) and 'building with' (B) do not precede the process of 'constructing' (C) but are rather nested within it. Furthermore a construction (or C-action) might subsume not only a large number of R- and B- actions but also other C-actions.

While the RBC model of abstraction described by Hershkowitz *et al*. (2001) was based on data derived from a teaching interview with one student who had a computerized tool at her disposal, the effect of peer interaction on abstraction was also investigated by the team. Hershkowitz *et al*. (2006) found that the interactive flow of knowledge among groups of three students afforded the co-construction of knowledge. Hershkowitz (2009) has lent the term 'collective abstraction process' to the situation where different students contribute different building blocks to the constructing activity.

RESEARCH PROCESS

In order to investigate the construction of new mathematical ideas by pupils I conducted a 'classroom design experiment' (Cobb, Gresalfi, & Hodge, 2009) in three different primary schools in Ireland. I taught 32 lessons in all to pupils aged 9 – 11 years[1]. Data collected included field notes, audiotapes of whole-class and group interactions, pupils' written artefacts, digital photographs of blackboard recordings, interviews with teachers and, in two of the schools, pupil diaries and post-lesson interviews with small groups of pupils[2]. Data collection and data analysis were interwoven. Retrospective analysis was conducted on micro- (between lessons) and macro-levels (between and after cycles of research in the three classrooms). For each lesson, I identified mathematical principles, that is, the constructs that pupils might be expected to develop over the course of a lesson, and these informed a hypothetical learning trajectory. Other principles arose *a posteriori* and were included in the analytic framework. Using the computer aided qualitative data analysis software package, Nvivo, I first coded all pupils' turns as 'R', 'B' or 'C'. While use of the principles provided a guiding framework, a difficulty I encountered was that I had to infer epistemic actions from pupils' verbal protocol. Pupils' use of 'hedges' and pronouns facilitated the coding process.

Hedges and Pronouns

A 'hedge' according to Lakoff (1973: 471), is a word "whose meaning implicitly involves fuzziness - … whose job is to make things fuzzier or less fuzzy". A modifier such as 'sort of' is an example of a hedge that a speaker might use to indicate a degree of uncertainty around class membership, e.g., "A whale is sort of a fish".

Rowland (2000) developed a taxonomy of hedges with reference to the discourse of mathematical conjecture. The first major type of hedge, a 'shield' indicates some uncertainty in the mind of the speaker in relation to a proposition. In the statement, "I think that a square might be a rectangle", the speaker injects a level of vagueness into his/her mathematical assertion and thus implicitly invites feedback on his/her conjecture about a relationship between the two shapes. There are two types of shield: (a) a 'plausibility shield' (e.g. 'I think', 'probably', 'maybe') which can suggest some doubt on the part of the contributor that the statement will withstand scrutiny and (b) an 'attribution shield' (e.g. 'According to') in which some degree or quality of knowledge is implicated to a third party. The second major category of hedges are termed 'approximators'. The effect of the approximator is to modify the proposition rather than to invite comment on it. One subcategory of the approximator is the 'rounder' which comprises adverbs of estimation such as 'about', 'around' and 'approximately'. The second type of approximator is the 'adaptor' – it indicates vagueness concerning class membership such as 'somewhat', 'sort of', e.g., "A square is sort of a rectangle".

In the analysis of lesson transcripts, it emerged that pupils tended to use this kind of vague language (e.g., 'probably' 'might' 'I think') when conjecturing, an action coded as 'building-with'. In turn, the language of a constructing action was marked by greater certitude – in particular, pupils often used pronouns such as 'it' or 'you' to signify generalisation (Rowland, 2000). There follows an account of a lesson on a Chess problem (a variation of 'Handshakes') with a 4th class, in which the analytic framework is exemplified.[3]

THE CHESS LESSON

The Chess problem read as follows:

> In a chess league each participant plays a game of chess with all other participants. How many games will there be if there are 3 participants? 10 participants? 20? Is there a way to find the number of games for any number of participants?

One way to solve this problem is to consider the number of games played by each person. In the case of 8 participants, the first person plays a game of chess with seven others, the second with six more, the third with five more and so on. The solution for eight people then is 7 + 6 + 5 + 4 + 3 + 2 + 1 giving a total of 28. Of particular relevance to this Chess problem were three lessons entitled 'Friendship Notes' that I had taught one month previously to this class and which I described at PME 33 (Dooley, 2009). These lessons concerned the number of notes applicable in the event of each individual in a group of size n giving a note to each other individual in the group. In the second of these lessons, most pupils generated an explicit rule (that is, $n(n -1)$). Both Chess and Friendship Notes are characterised by non-reflexivity (that is, no element of a set relates to itself). The main difference between the activities lies in the property of symmetry. 'Chess' is symmetrical because if A relates to ('plays a game with') B, then it follows that B relates to A. However, in 'Friendship Notes', if

A relates to ('sends a note to') B, the reciprocal relationship is not implied. Therefore, in a group of size *n*, the number of friendship notes is double the number of handshakes. The function mapping *n* (the number of people) to *y* (the number of games) in the Chess Problem is $y = {}^{n(n-1)}\!/_{2}$. This might emerge from inspection of the relationship between the *n* and *y* values as listed in a table of values or if consideration is given to the symmetric nature of the activity.

The lesson extended over two periods (Chess 1 and Chess 2), each having a three-part structure – introductory whole-class session, group work and final plenary (when results were discussed). In Chess 1, pupils were introduced to the problem and in the concluding plenary session, they explored the number of games in the case of 20 competitors. One pupil, David, conjectured that the number of games for 20 competitors might be found by multiplying 20 by 19 and halving the product. However, he was unable to verify this formula structurally.[4]

In Chess 2, some revision of the previous day's work took place. In the 'group work' phase of the lesson, pupils considered the number of games required for 11 to 20 competitors – as extension, they were asked to calculate the number of games for 40 and 100 competitors. The phase of the lesson that is examined hereunder occurred during the final plenary session and concerns the development of new ideas by one pupil, Enda. However, his construction had embedded within it the contributions of others in the class and thus their input is also described and analyzed.

ANALYSIS OF ENDA'S CONSTRUCTION OF INSIGHT

Enda's construction of mathematical insight in these lessons could be traced in the whole class discussion. In Chess 1 he made some faulty conjectures - for example, he thought that the solution for 20 competitors might be found by adding 45 (the number of games for ten competitors) and 19.

Towards the end of Chess 2, when an explicit rule for *any* number of competitors was being discussed at a plenary session, David developed an explicit formula for Chess, that is, "Multiply it by the number less … and then half it". Shortly after, Enda made a connection between Chess and the Friendship Notes activity and this led to justification of David's formula at a structural level. Since I coded his contribution as 'Construction' in relation to the mathematical principles for the lesson, a transcript of this phase of the lesson is now presented, and analysed in more detail.

Turn	Transcription [5]	Pupil Action	Epistemic Action (RBC)
639	Enda: It looks like … it's pretty much the very same as the friendship cards, it seems kind of like that.	Enda made a connection with 'Friendship Notes'.	Building-with
640	TD: Right, Enda, do you remember the friendship notes, that's a good thing. Do you remember the friendship notes? Do you remember what you did for the friendship notes? What did you do for the friendship notes? Do you remember the rule? Barry?		

Turn	Transcription [5]	Pupil Action	Epistemic Action (RBC)
641	Barry: It's kind of the same thing as, eh, you wouldn't have to do themselves so there's going to be one less.	Barry referred to the non-reflexive nature of both activities.	Building-with
642	TD: Ok, so but the rule … according to David, when we were doing the friendship notes, [ ] For example in friendship notes if there were three children than how many notes would there be for three children? … Do you remember? … Right, Barry? Does anyone remember how many notes there were for three children in the friendship notes? … Yeah?		
643	Colin: Em, six.	Colin recalled number of friendship notes for six children.	Recognising
644	TD: Six but see in the chess game it's only three. So why is it a bit different? Does anyone know why it's a bit different? […] Myles?		
645	Myles: Em because in chess you will just have to play them, if they played you one time then you have kind of played them once.	Myles referred to symmetric nature of 'Chess'.	Building-with
646	Ch: Ah!		
647	Myles: In friendship notes you have to play them kind of again so like you give them your note and they will have to … they will still give back you a note.	Myles referred to the asymmetric nature of 'Friendship Notes'.	
648	TD: Ok, so once you play the game you don't play it back, isn't that what you are saying?		
649	Myles: Yeah.		
650	TD: That's what you are saying. Yes?		
651	Colin: Em, well cos in the friendship notes you have to give two because if there were three you would have to give one to each person …	Colin referred to the asymmetric nature of 'Friendship Notes'.	Building-with
652	TD: Hm, hm.		
653	Colin: … and everyone has to give one to each person, so it's the same as three by two.		
654	TD: Hm, hm.		
655	Colin: Eh, and in chess you only have to play them once even if they challenge you.	Colin referred to symmetric nature of 'Chess'.	
656	TD: Hm, hm.		
657	Colin: So eh …		
658	TD: And what does that mean for the chess game then? What does it mean for the chess … rule?		
659	Colin: Eh, you don't … you don't play them twice.		
660	Ch: Ah!		
661	TD: Ok, so what happens then, what's the rule for the chess? Enda?		

Turn	Transcription [5]	Pupil Action	Epistemic Action (RBC)
662	Enda: Eh well, I actually definitely agree with David's way by doing the friendship notes, the same way as the friendship notes and halving it …	Enda made a connection between David's formula and 'Friendship Notes'.	Building-with
663	TD: Hm, hm.		
664	Enda: … because all of the things we get in that are half what we get in the chess thing.		
665	TD: Hm, hm.		Building-with
666	Enda: So I definitely agree with David's way by multiplying by one number less and halving it. I definitely agree with that now.		Constructing

Epistemic Actions

Initially Enda's conjecture about a possible relationship between Chess and Friendship Notes was marked by uncertainty:

> 639 Enda: It looks like … it's **pretty much** the very same as the friendship cards, it seems **kind of** like that.

He used two adaptors, 'pretty much' and 'kind of' which, according to Rowland (2000), indicate vagueness of class membership – in this instance vagueness about the relationship between the two activities. Enda's 'it seems' is an example of plausibility shield indicating an awareness on his part that his conjecture might be false. In the next pupil turn, Barry also made use of an adaptor:

> 641 Barry: It's **kind of** the same thing as eh you wouldn't have to do themselves so there's going to be one less.

His uncertainty also centres on the extent of the similarity between the two activities as he spoke confidently about the unreflexive nature of both (that is, the need to multiply n by $n-1$). Myles' input (turns 645 and 647) also indicates some level of uncertainty but this seems to be more around the verbs he has chosen to use:

> 645 Myles: Em because in chess you will just have to play them, if they played you one time then you have **kind of** played them once.

His 'kind of' (both here and in his next turn) appears to relate to his concern about the appropriateness of the verb 'played' and to the idea that both competitors 'play' simultaneously. However, he has discerned the essential difference between the two activities (that is, the symmetric property) and has thus built-with Enda's proposal. Colin further elaborated on this input by ratifying it with an example (that is, three players). His input in turns 651, 653, 655 and 657 contains no adaptors or plausibility hedges, suggesting greater conviction on his part. Interestingly, a hallmark of the assertions made by Barry, Myles and Colin is the presence of the pronoun 'you'. This 'you' was not used to address me or others in the class but rather as a pointer to generalities – what happens 'every time' (Rowland, 2000). Enda, who has demonstrated in previous contributions that he is not easily convinced by superficial arguments, has now been persuaded that David's rule is viable. He has justified this on the basis that all the numbers in the Chess are half those in Friendship Notes (although his description of this in turn 664 is inaccurate). While his initial conjecture about the connection between Chess and Friendship Notes was tentative, his assertions in turns 662, 664 and 666 are marked by certitude ("I definitely agree ..") and a lack of vague language. Although it is not completely clear that he has taken on board the structural justification offered by other pupils in the preceding turns, his contribution in turn 666 is coded as 'construction' because it is an articulation of the relationship between the formula provided by David and 'Friendship Notes'. Nested within this construction are the 'building-with' actions of Enda himself and of Barry, Myles and Colin. It thus exemplifies the distributed and nested nature of RBC. Although the above analysis concerns only a few pupils, the construction can be traced back to David's earlier development of a formula and to a constructing activity (involving these and other pupils) that took place in Friendship Notes. The analysis therefore supports the applicability of the RBC framework to a whole-class context.

CONCLUDING REMARKS

In previous research involving the RBC framework, Williams (2002) identified the use of common language rather than precise mathematical language as indicating the presence of an amorphous mathematical idea as opposed to one that is well structured. In this paper I have developed this idea further by utilizing the categories of vague language generated by Rowland (2000). What is shown is that vague language fosters construction of new mathematical ideas. It is not that such language always implies 'building-with' but, in analysis based on mathematical principles, it facilitates RBC coding. Furthermore, because it allowed pupils to suggest ideas without fully committing to them, other pupils appeared to feel free to build-with them further. The development of this 'conjecturing atmosphere' (Mason, 2008) where ideas are tested and later modified makes particular demands of a teacher, particularly in the context of whole-class conversation, and such demands are in need of further analysis.

Notes:

[1] Some lessons extended over more than one class period.

[2] Video data were not collected due to ethical constraints.

[3] Pupils were aged 9 – 10 years.

[4] One form of generalization (empirical) is achieved by considering the form of the results, whilst the other (structural) is made by looking at the underlying meanings, structures or procedures (Rowland, 1999).

[5] Transcript conventions are: TD: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings; … : a short pause; […]: a pause longer than three seconds; [ ]: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

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UNJUSTIFIED ASSUMPTIONS IN GEOMETRY

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We investigated unjustified assumptions (UAs) made by students when proving geometric statements. UAs can originate in partial content or logical knowledge. UAs can be used in a forward or a backward way. We thus asked how UAs arise and in what ways they are used in the proof. Data were collected by means of written questionnaires and interviews. The main findings are that UAs arose when students misused theorems or assigned extraneous properties to geometric objects, and that UAs were made with the purpose of reaching a critical step in the proof.

THEORETICAL BACKGRUND

Geometry and Proof

Research has shown that high school students as well as university students have great difficulty with the task of proof construction (Chazan, 1993; Moore, 1994; Weber, 2001): Students don't have an appropriate conception of what constitutes a proof; many students believe that verifying a theorem by specific instances is a sufficient proof; others believe that a proof of a theorem is valid if and only if it follows the format of 'two column proof' taught in geometry (Healy & Hoyles, 1998). Moore (1994) reports on students who do not understand the theorems and the concepts involved and misapply them. They lack the language needed to express mathematical ideas and they do not provide the justification for each step in a proof. When learning theorems, students often incorporate information contained in a specific diagram as part of a theorem which later constrains the application of the theorem. In such cases, concepts are introduced by prototypical examples. According to Hershkowitz (1989), and Yerushalmy and Chazan (1990), these prototypes may induce inflexible thinking, thus preventing the recognition of a concept in a non-standard diagram. Students' definitions may include irrelevant characteristics of the diagram. Reliance on prototypes may also lead to the expansion of the definition to include non-critical attributes or to its narrowing by omitting critical attributes. Apart from these content aspects, students may lack more general skills and strategies that influence the proving process. Among these are the availability of working backwards and forwards, using symmetry and other patterns, and eventually adapting the plan for the proving process. Weber (2001) names this knowledge 'strategic knowledge', i.e. the ability to distinguish helpful proof steps from irrelevant ones.

These difficulties often lead students to assume properties that are not given or not essential when proving geometric statements. Dvora and Dreyfus (2004) called such assumptions *Unjustified Assumptions* (UAs); we focused on UAs that were based on diagrams and asked whether the way in which a statement was presented (with or without diagram) had an effect on UAs. The findings were that in almost each task, the diagram affected students' way of thinking and making UAs.

Since then, we have expanded the investigation by looking at UAs that have other causes such as partial content and logical knowledge. The aim of the study presented here was to investigate how UAs arise and how they are used in the proof.

METHOD

Data were collected by means of a written questionnaire and by means of individual and pair interviews. Questionnaires have been administered to 93 students in Israel who were enrolled in a full-year 10^{th} grade geometry course. Eight individual interviews and four pair interviews have been conducted. More questionnaires will be administered and more interviews will be conducted during the coming months.

Questionnaire

The goal of the questionnaire was to investigate to what extent students made UAs when proving geometric statements. The questionnaire consisted of six geometric statements and proof tasks: two on triangles, two on quadrilaterals and two on circles. The tasks were chosen on the basis of the following criteria:

- The tasks were within the field of experience of the students and of a level they could be expected to prove in class or in an examination. The tasks were chosen to be identical or similar to tasks from the students' textbook.
- The tasks had the potential for inviting UAs; for every task, an a priori analysis was carried out on the expected UAs.
- The tasks were varied so that different types of UAs could be expected.

Three versions of the questionnaire were used. Each version included two proof tasks from different topics. This was intended to make the questionnaire appropriate for a 45 minute lesson while eliminating the influence of any particular task or topic. The first task of the questionnaire (see Figure 1) will be analysed in detail below.

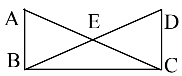
Description of the task	UAs expected on the basis of a priori analysis
Given: BD=AC, AB⊥BC, DC⊥BC Prove: DE=AE 	- ABCD is a rectangle - ∠ABE=∠ DCE - BE=ED, AE=EC

Figure 1: Task 1

Interviews

Two kinds of interviews were conducted: Individual interviews following the administration of the questionnaire and pair interviews while answering the questionnaire. The individual interviews were audio-recorded and the pair interviews were video-recorded. All interviews were transcribed.

The goal of the individual interviews was to investigate students' explanations of how they dealt with the questionnaire tasks. Therefore, these interviews were conducted right after the students had finished answering the questionnaire.

The pair interviews provided a live opportunity to observe students while they were dealing with a task in order to get information about the process they were going through: how they used the data, how they drew conclusions, why they made the UA, whether they had disagreements and how these were resolved.

We analysed both types of interviews with the purpose of investigating how UAs arise and how they are used in the proof. The analysis related to the reasons for making the UA, the missing content constructs that might have led to the UA and the ways the UA was used in the proof: in a forward or in a backward way. We interpret as forward students' actions based on to the given data, and we interpret as backward students' setting a goal and planning their proof to reach this goal. In addition, we were looking for different types of UAs such as: UAs that originate in adding extraneous data or in misapplying theorems or UAs that indicate a jump in the proof. For each interview, we created a profile based on this analysis. These profiles included information on the nature of the UA, how it arose and how it was used in the proof.

FINDINGS AND DISCUSSION

In this paper, we present findings about task 1 only. Questionnaires including task 1 were administered to 30 students. Among them, 17 produced incorrect proofs, 14 of them making UAs (see Table 1).

Unjustified assumptions	Number of students
i) AD∥BC	7
ii) ABCD is a rectangle	4
iii) AD⊥AB, AD⊥DC	2
iv) E is the midpoint of BD	1

Table 1: UAs in task 1

The following discussion of an individual interview and a pair interview with students about task 1 is intended to shed light on students' UAs when dealing with task 1.

Individual interview with Ro

Ro was interviewed individually after having answered the questionnaire. She referred to her UA right away at the beginning of the interview:

2	Ro:	I am given that AC=BD and that AB⊥BC and DC⊥BC, so ∠B=∠C=90^0 and I have to prove that AE=DE. I know already that ∠B=∠C=90^0 so I drew an auxiliary line AD, that AD is perpendicular to AB and to DC.
10	Ro:	and then ∠A=D=90^0 so I get that ABCD is a rectangle since all the angles are 90^0, and in a rectangle the diagonals are congruent and bisect each other so AE=DE.

The UA that Ro made was that line AD was perpendicular to AB and to DC (line iii of Table 1). Ro added extraneous data to those that were given.

How did the UA arise?

12	Ro:	I saw in my mind a rectangle with one side missing so I drew AD and I realized that I can prove that half of the diagonals are congruent.
13	In:	When you read the task, you thought of reaching a rectangle because…
14	Ro:	because its diagonals bisect each other
15	In:	and then you added AD so that AD is perpendicular to AB and to DC?
16	Ro:	that is right
17	In:	so you can get two right angles?
18	Ro:	Yes

Lines 10, 12 and 14 demonstrate that Ro made the UA with the purpose of reaching a critical step in the proof; she was directed by the goal of proving that ABCD is a rectangle. For this goal she needed three right angles, she was already given two (∠B, ∠C) and the UA provided two more (∠A, ∠D). Then she concluded that AE=DE due to the property of the diagonals in a rectangle.

While making the UA, Ro neglected the given data that AC=BD. She needed to add extraneous data to compensate for this. During the interview, it seemed that Ro was unaware of making the UA; she was very sure, her answers were quick and she had no doubts about her proof [20]:

19	In:	How would you evaluate your proof? Do you think it is correct?
20	Ro:	I believe it is very good.

How was the UA used in the proof?

21	In:	You saw at the beginning that it is worthwhile to prove a rectangle?
22	Ro:	Yes, I saw it right away from the diagram

Ro used the UA in order to reach a specific stage in the proof; she claimed to prove that ABCD was a rectangle and then concluded that AE=DE. The UA was made while thinking backward from the goal.

Pair interview with Mi and Ne

Mi and Ne were interviewed while attempting to prove the statement in task 1. They made two UAs. These two UAs do not appear in Table 1 since they have so far been observed only in this interview. The first UA was assuming that BE=CE. This UA arose after they tried but did not manage to prove that ABCD is a rectangle [27-81]:

27	Ne:	We need to say that this [ABCD] is a rectangle and then the diagonals bisect each other or a parallelogram, I don't know
35	Ne:	Ah, no, wait. In what quadrilateral the diagonals bisect each other?
36	Mi:	in a parallelogram and in a rectangle
61	Ne:	so, I think let's prove first that this is a parallelogram and then it is a rectangle
63	Ne:	so how we prove it is a parallelogram?
74	Ne:	we do not remember well, we want to prove a parallelogram and then a rectangle
81	Ne:	we do not remember the theorem; I mean the properties that prove a rectangle

They then abandoned this idea and looked for another one.

93	Mi:	Maybe we can prove congruence?
94	Ne:	No
95	Mi:	congruence, congruence
96	Ne:	no, we can subtract congruent segments from congruent segments and then look, we can write, look, AC minus, AC minus CE equals BD minus BE and then we get that AE equals DE, right? What do you say?
97	Mi:	Ah, right because it says this [AC=BD] is equal
98	Ne:	subtracting congruent segments from congruent segments
99	Mi:	but how do you know that this [CE] and this [BE] are congruent and this [AE] and this [DE] are congruent?
100	Ne:	subtracting congruent segments from congruent segments
101	Mi:	but how do you know that this [CE] and this [BE] are congruent, I mean all of them…
102	Ne:	because it says that BD=AC
103	Mi:	o.k., but how does it tell you that this [CE] and this [BE] are congruent
106	Ne:	you know that AC=BD and then you can subtract congruent segments from congruent segments, got it?
107	Mi:	but you do not know that CE=BE=DE

Ne came up with the idea of subtracting congruent segments from congruent segments. The goal was to prove that AE=DE; AC=BD was given, so Ne proposed to argue that AC minus CE equals BD minus BE. This statement relied on the assumption that BE=CE. This is considered an UA since it constitutes an addition of extraneous data of congruence to given lines that neither follows from previous statements nor relies on given properties. However, the students' final proof was not based on this UA because Mi objected to it and confronted Ne [99, 101, 103, 107]. The excerpt 93-107 illustrates that Mi seemed to understand the conditions under which the rule of subtracting segments should be used, i.e. not only should the larger segments be congruent but the smaller ones should be as well. That is why Mi was not convinced by Ne's argument.

How did the UA arise?

Mi and Ne could not prove that ABCD is a rectangle. They explicitly said that they did not remember what properties were needed to prove a rectangle [74, 81]. One may claim that this lack of content knowledge had no direct influence on making the UA, but it seems that it did at least have indirect influence: Due to this lack, Ne had no tools to deal with the task and therefore assumed that BE=CE.

This nature of the UA was using the rule of subtracting congruent segments from congruent segments without regard for the conditions under which this rule is valid; Ne neglected the condition that CE and BE had to be congruent and was satisfied with the fact that only AC and BD were congruent [96].

As mentioned above, the students' final proof was not based on this UA since Mi objected to it. We therefore have no definite answer to the question how the UA was used in the proof. From the conversation between the girls, we interpret that the UA was made in a backward way: Ne wanted to prove that AE=DE, and since it was given that AC=BD the idea of subtracting segments seemed tempting: It immediately provided the statement to be proved.

The second UA that Mi and Ne made was that the following pairs of angles are complementary angles (i) ∠BAC and DAE, and (ii) ∠CDB and ∠ADE (see Figure 1). This UA arose during the following exchange:

119	Ne:	Do you have something better to offer?
120	Mi:	Ahh, I think I know
121	Ne:	What?
122	Mi:	I know, I know
123	Ne:	What?
124	Mi:	listen, we can prove congruence, triangle ABC and triangle BDC

Mi and Ne then tried and succeeded to prove that these triangles are congruent, although Ne seemed not to see to what use this could be put:

139	Ne:	But how is this going to help us?
140	Mi:	because then you can do, this [∠BAC] equals to this [∠CDB], right?
141	Ne:	well
142	Mi:	you can, then you can subtract, no, then you do, look, look, you write AD it is an auxiliary line and then you say that ∠BAC=∠CDB, o.k.?
143	Ne:	well

144	Mi:	O.k. and then you s.., then you say that, then it has to be that ∠DAE=∠ADE
145	Ne:	Why? Why it has to be? You do not know that these angles are congruent
146	Mi:	it has to be
147	Ne:	why?
148	Ne:	maybe, ah, there is this theorem, if two angles are congruent, then their exterior angles are congruent too
149	Mi:	What? What?
150	Ne:	if the interior are congruent, then the exterior are congruent too
151	Mi:	what is exterior?
152	Ne:	here, this is exterior [∠DAE is exterior to ∠BAC]
153	Mi:	well, yes, because they are complementary angles
154	Ne:	Can we write it down?

After proving that triangles ABC and DCB were congruent, Mi and Ne concluded correctly that ∠BAC=∠CDB. From this, Mi and later Ne concluded ∠DAE=∠ADE. This conclusion is unjustified, and hence an assumption. (We already note that it was also used, shortly afterwards, as an assumption – see below.) The reason they gave for making this assumption was that "complementary angles to congruent angles are congruent". The students treated those pairs of angles as complementary angles. The UA arose and was made with the purpose of reaching a critical step in the proof: Mi was directed by the goal to prove that ∠DAE=∠ADE [144, 146]. She insisted that is “has to be”, and we interpret this as expressing that she realized it was a critical step since it would lead immediately to the conclusion that triangle DAE is isosceles, i.e. AE=DE which was the statement to be proved. Mi and Ne were looking for a justification for this assumption until Ne came up with the idea of the exterior angles [148, 150, 152]. Mi appeared to be relieved that a justification was available and offered only a few doubts [149, 151] before accepting it [153].

Proving congruence between triangles ABC and DCB was a good decision of Mi. However, both Mi and Ne were lacking strategic knowledge (Weber, 2001), i.e. the ability to distinguish helpful proof steps from irrelevant ones; they did not conclude that ∠ACB=∠DBC which would have led to BE=CE and then to AE=DE. Instead they concluded that ∠BAC=∠CDB, which was not only not helpful but was a step towards making the UA.

This UA was used in order to reach the stage that ∠DAE=∠ADE which immediately provided the statement to be proved. This UA, like the previous ones, seems to have been made in a backward way, and then used accordingly, though the situation is more complex in this case. Mi correctly claimed that ∠DAE and ∠ADE had to be congruent – she probably realized that this would let her complete the proof. Hence, she set a goal to prove this statement; the decision to prove triangle congruence seems to have been taken as a step to reach this statement: she probably saw right away that she would get that ∠BAC=∠CDB and then expected she could somehow prove that ∠DAE=∠ADE.

SUMMARY

In this paper, we showed that students make different types of UAs when proving geometric statements: UAs that are based on adding extraneous data to given geometric objects, and UAs that are based on applying theorems under wrong conditions.

Furthermore, we showed how UAs arise: UAs arise when students want to reach a critical step in the proof, when students lack the necessary content knowledge, when students use theorems and rules without considering the conditions under which they are valid, when students neglect some given data, or when students lack strategic knowledge. Of course, combinations of these circumstances also occur when making UAs.

Finally, we showed how UAs are used in the proof: UAs are used in reaching the statement to be proved either in a forward or in a backward way. In a forward way, students move on according to what they get from the data, while in the backward way, students set a goal and plan their proof to reach this goal.

While this study provides insight into several aspects of UAs, further research is needed to investigate which missing knowledge elements entice students to make UAs, to distinguish and confirm different categories of UAs, and to identify characteristics of tasks that invite UAs.

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EMBODIED COGNITIVE SCIENCE AND MATHEMATICS

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The purpose this paper is to describe two theories drawn from second-generation cognitive science: the theory of embodiment and the theory of conceptual integration. The utility of these theories in understanding mathematical thinking will be illustrated by applying them to the analysis of selected mathematical ideas and processes, including proof. The argument is made that mathematical ideas are grounded in embodied physical experiences, either directly or indirectly, through mechanisms involving conceptual mappings among mental spaces.

INTRODUCTION

The goal of this paper is to clarify the central concepts and potential utility of two existing theories from outside of mathematics education for understanding mathematical thinking. The "outside" theories derive from what is known as "second generation cognitive science," which includes neurophysiology, emotion, perception and the body in its models of cognition, in contrast to first generation cognitive science, which built a model of thinking based on the metaphor of mind as computer, employing rules, physical symbols and productions systems (Lakoff & Johnson, 1999). In this paper, we will look at the theory of embodiment in cognitive science (Johnson, 2007; Lakoff & Johnson, 1999), and the theory of conceptual integration (Fauconnier & Turner, 2002). The first theory takes a broad view in connecting cognition and meaning to bodily origins and existence, and the second offers a specific set of analytical tools and proposed mechanisms for explicating how new meanings are generated from existing mental structures. The paper presents a description of each theory, and also relates them to other influential theories in mathematics education.

EMBODIMENT THEORY

Embodiment theory offers an answer to the question of how meaning arises, and of how thought is related to action, emotion and perception. Embodiment theory proposes that meaning and cognition are deeply rooted in physical, embodied existence, on at least three levels:

1. Phylogenetic: At the level of biological evolution, our particular capacities for perception, emotion, cognition and the construction of meaning are both enabled and constrained by the current evolutionary state of our bodies, including our modes of movement, organs of perception, and nervous systems. Furthermore, this state is the result of millennia of interaction with various environments, in which the

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capabilities that permitted survival (including pattern-noticing, inference and problem-solving) were selected for and refined over evolutionary time.

2. Ontogenetic: At the level of the individual organism, a child is born with (or soon develops) a set of basic perceptual and cognitive capabilities as a result of the evolutionary processes described above. Recent research in infant cognition has demonstrated what could be called "proto-arithmetic", in that they can detect changes in small numbers of objects as well as "impossible" changes in the number of objects displayed to them (Deheane, 1997). The development of the individual, however, is also based to some degree on his or her specific physical experiences within a particular environment. Thus, embodiment theory would predict that children who learn mathematical concepts and procedures with hands-on mathematical manipulatives would have a different conceptualization of them than those who are taught only with symbols and two-dimensional representations.

Given the fact that humans are bipedal, symmetric with respect to left and right, with a distinct front and back, and live in a world with gravity, a relatively small collection of common perceptual/cognitive constructions that seem to be common across cultures has been delineated. These constructions are called *image schema*, and are defined as "recurrent, stable patterns of sensorimotor experience...[that] preserve the topological structure of the perceptual whole [and have] internal structures that give rise to constrained inferences" (Johnson, 2007, p. 144). A simple example of an image schema derives from the fact that we stand in a vertical relationship to the ground, and that if we stack objects one on top of another, the pile becomes taller. As a result, we develop the image schema UP IS MORE, which is used, unconsciously, in numerous situations where we want to express an increase, both linguistically and in conventions of representation. We say, for example, "The numbers go up" when we mean, "The cardinality of the numbers increases." In the Cartesian coordinate system, the y-axis displays numbers that increase "upwardly," although, in theory, this convention could have been reversed.

A second example of an ubiquitous image schema is the SOURCE-PATH-GOAL schema, which is based on our basic experience of goal-directed movement, which originates at a particular physical location, proceeds along a given path (possibly encountering obstacles or detours) and arrives at the goal location. Thus, image schema are common (but generally unconscious) building blocks of cognition available to all thinking humans living on earth. This construct from embodiment theory has already been used in the analysis of mathematical ideas (e.g., Lakoff & Núñez, 2000).

3. Microgenetic: At the level of individuals in interaction with each other, or the immediate environment, embodiment is also omnipresent. Humans must use the bodies we have to do things in the world, and also to engage in social interaction and symbolic production. Although in studying human interaction and symbolic production, many have reduced these phenomena to the words exchanged or written inscriptions produced, these things do not happen without the engagement of concomitant modalities, including physical motion, bodily stance, gesture, facial expressions, prosody and rhythm. These modalities, in particular gesture, are now being analyzed as part of a move toward a more complete understanding of human cognition and communication (Edwards, 2008; McNeill, 1992, 2005).

Embodiment proposes a theory of meaning that contrasts with both traditional linguistics, in terms of the definition of meaning, and with a representationalist, information-processing view of the mind. Rather than an objectivist view of meaning as a connection between concepts in the mind and objects in the world, mediated by symbols and words that somehow "carry" meanings, the theory of embodiment sees meaning and thought as emerging from interactions between the knower and the environment. Similarly, cognition is not seen as the manipulation of internal representations of the outside world by an internal "processor" or viewer, but as the dynamic interaction of patterns of neural activation, responding to perceptions (or equivalent re-imaginings) and preparing for action. There is support from recent neuroscience for an embodied theory of cognition and mathematics, whether from the discovery of mirror neurons that are activated when one simply thinks of an action as well as when one enacts it (Gallese & Lakoff, 2005), or the fact that the area of the human brain responsible for counting is the same as that which controls the fingers (Dehaene, 1997).

Johnson (2007) summarizes what he calls an "embodied, experientialist view" of meaning, based both on the work of pragmatists philosophers like Dewey and William James, as well as empirical work in contemporary cognitive science:

> Meaning ... arises through embodied organism-environment interactions in which significant patterns are marked within the flow of experience. Meaning emerges as we engage the pervasive qualities of situations and note distinctions that make sense of our experience and carry it forward. The meaning of something is its connections to past, present, and future experiences, actual or possible (p. 273).

One of the central principles of embodiment theory, as well as of pragmatism, is that of continuity. Under this principle, thinking is an activity that is fundamentally connected to other life activities, like moving, perceiving and feeling. In addition, human cognition may differ in complexity from that of other living things, but it arises under the same circumstances described above by Johnson, and shares many common features (see, for example, research on counting abilities among primates and certain birds (Dehaene, 1997)). The principle of continuity breaks down the long-held distinction between body and mind, where image schemas and other concepts are products of the interaction between the thinker and the world, not disembodied abstractions. The mind is the on-going cumulative trace, in the form of neural patterns, of experienced and imagined action. Under the principle of continuity, "concrete" thought is not ontologically different from "abstract" thought, and mathematics is not ontologically different from other realms of thought. Instead, one of the tasks of cognitive science is to delineate how it is that the mechanisms that have allowed people to survive and thrive have also supported the creation of art, language, monuments, music and mathematics.

The principle of continuity has relevance to theories of mathematical thinking. Under some theories, there are different kinds of mathematical thought, some embodied and some not (e.g., Tall, 2007). However, the theory of embodiment, as originally conceived in contemporary cognitive science, does not recognize kinds of thinking that are ontologically distinct in this way. Although there are certainly more and less complex kinds of thinking, all cognition is "built" using the same set of mechanisms and working from the same "raw materials;" all thinking is ultimately embodied. The next section considers cognitive mechanisms utilized in the construction of ideas and inferences, whether in mathematics or in other domains of thought.

THE THEORY OF CONCEPTUAL INTEGRATION

The theory of conceptual integration was developed in order to explain how ideas emerge from other ideas, and how the inferential structure of one domain can be imported or mapped to another, permitting logical reasoning and the construction of more complex networks of thought out of simpler ones. The theory is based on the construct of "mental spaces" (Fauconnier & Turner, 2002). Mental spaces (which can be compared to the notion of "schema" in cognitive psychology) are partial conceptual structures, made up of elements and relations among them, derived from and elicited by our experiences and interactions. Fauconnier and Turner (2002) call them "small conceptual packets constructed as we think and talk, for the purposes of local understanding and action" (p. 40). Examples of mental spaces from mathematics are legion: we are presumed to construct mental spaces corresponding to everything from whole numbers to polygons to proofs (Lakoff & Núñez, 2000). The interesting question is how these mental spaces relate to each other, and how they are constructed. It is assumed here that the construction of mental spaces is constrained and facilitated by multiple influences, including the physical body, social interactions and cultural contexts. Taking these influences as a given, we focus on a specific mechanism for creating new mental spaces, conceptual integration.

As described by Fauconnier and Turner (2002), conceptual integration “connects input spaces, projects selectively to a blended space, and develops emergent structure” (p. 89). In other words, conceptual integration (also referred to as conceptual mapping or conceptual blending) begins with one or more mental spaces, designated as "input spaces." Selected elements, inferences and relationships within the input space(s) are mapped to a newly created mental space, referred to as a conceptual blend or blended space. An example of a conceptual blend in mathematics is the number line (Lakoff & Núñez, 2000). A number line is neither strictly an arithmetic entity nor a geometric one - it has elements drawn from both domains. It conceptually "maps" numbers to points on a line, blending properties of numbers (for example, that 2 is greater than 1) with properties of points on a line (for example, that point B is to the right of point A). The resulting conceptual blend (that is, the mental space for "number line") has a useful emergent inferential structure that is not found in either of the input spaces.

A conceptual blend that maps a single input space ("source domain") to a single output space ("target domain") is called a single-scope blend or a conceptual metaphor (Fauconnier & Turner, 2002). An example of a conceptual metaphor for mathematical proof, based on the image schema SOURCE-PATH-GOAL, is shown in Figure 1.

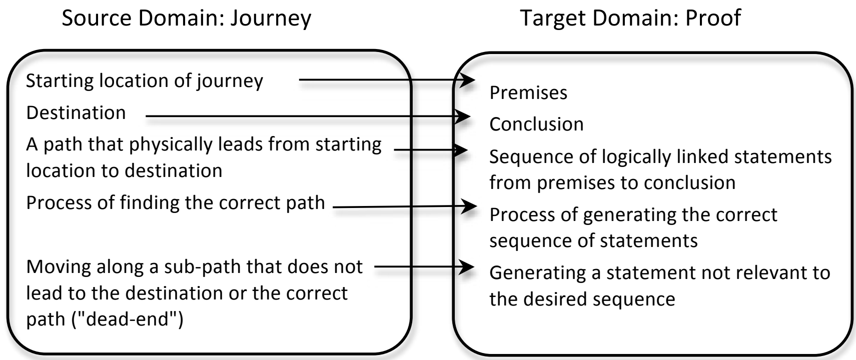

Figure 1. The "A Proof is a Journey" Metaphor

RELATIONSHIPS AMONG THEORIES

Embodied cognitive science, including the theory of conceptual integration, came onto the mathematics education scene at a time in which well-established theories were already doing useful work. These theories included radical constructivism, socio-cultural theory, various specific theories of reification (process-concept transformations), semiotics, and information processing theory. The theory of conceptual integration is based on the principles of embodiment, but offers specific mechanisms to account for how our embodied experiences become reflected in our thought and language (where language is taken broadly to include such things as gesture, written inscriptions and external imagery).

But how do these theories related to the major theories in mathematics education? At a foundational level, embodied cognition is incompatible with any theory that views meaning as an objective coupling between the external world and internal representations, or cognition as a set of rules that could be instantiated in silicon chips just as well as in the brain/body. In embodied cognitive science, cognition requires an active organism (a brain within a body) engaged in ongoing interaction and adaptation within an environment, and thinking, even logical reasoning, is ultimately rooted in physical experience. Thus, certain perspectives from information processing psychology (for example, the notion that reasoning can be modelled solely by the manipulation of propositions) would contradict embodiment theory.

However, embodiment is compatible with many other theories used in mathematics education. In my view, the situation is like that of the blind men grasping the elephant, with each theory giving only part of the whole picture. Embodiment is consistent with the tenets of radical constructivism that hold that there is no "gods-eye," objective view of reality, but only the individual's constructions based on his or her experience. However, it proposes a grounding for these constructions in physical experience. It is likewise consistent with models of intellectual development in which more complex thinking and capabilities emerge from simpler ones. The theory and constructs of conceptual integration offer a mechanism for the construction of new ideas (mental spaces) that is compatible with schema theory. However, Piaget's discontinuous stage theory, in which strict demarcations between levels of conceptual development are proposed, would be rejected, on the principle of continuity.

The theories of embodiment and conceptual integration are also fully compatible with socio-cultural theory, situated cognition, and theories that emphasize discourse. Embodiment and conceptual integration acknowledge that the environment in which cognition develops in humans includes other people, as well as the cultures and institutions they have created. It also stipulates that language and discourse are a vital part of the medium within which thought develops. Mental spaces and conceptual blends do not emerge in isolation from the surrounding culture; in fact, they fully reflect (and contribute to) that culture. Returning to proof as an example, for many secondary school students in the United States, a proof must be presented in a two-column format, with statements on the left and justifications (in the form of already-proved theorems) for the statements on the right. This convention, which is culturally specific, would form part of their mental space for proof.

What the theory of embodiment insists is that although intellectual constructions, including mathematical ideas, are socially constructed, they are not unconstrained or arbitrary. Instead, they are made possibly by, grounded in, and constrained by physical realities (Nuñéz, Edwards & Matos, 1999). As noted in the introduction, these realities include the way our bodies and brains have evolved, how they develop throughout our life spans, and how we learn through multiple modes of engagement with our environment. From the perspective of embodied cognitive science, the human intellectual product, mathematics, is grounded in embodied physical experiences, either directly or indirectly, and grows through the mechanism of conceptual integration as well as other transformations of mental spaces (Fauconnier & Turner, 2002).

The recent work utilizing the theory and tools of semiotics in the analysis of mathematics shares the goals of embodiment theory and conceptual integration, to understand the construction of mathematical meaning, including attention to the important roles of shared signs and symbols. However, embodiment looks for meaning beyond relations among signs or within semiotic systems, and is careful to avoid the objectification of these human constructions. That is, signs and symbols are not characterized or investigated as formal systems, or as the "carriers" of meaning (indeed, the idea that any kind of linguistic expression can "carry" meaning is a pervasive objectivist metaphor). Instead, according to embodiment theory, the physical (and possibly even the social) world is first experienced at a non-linguistic level, and such experiences are needed in order to attach meaning to culturally created semiotic systems.

Anna Sfard (1994) and others have highlighted an important construct in mathematics, the idea that mathematical activities and processes are often reconceptualized and treated as "objects" by mathematics learners and thinkers. Sfard has called this conceptual process "reification" or "objectification", and has proposed that this process is the source of a basic metaphor in mathematics, that of the mathematical "object" (loc. cit.). From the point of view of embodiment theory and conceptual integration, it is more likely that the situation is reversed. That is, our knowledge of physical objects and actions provides a foundation for thinking about and "manipulating" (metaphorically) the results of abstract mathematical processes as if they were objects. In fact, Sfard herself, in a later paper, stated that, "the existence of some special beings (that we call mathematical objects) implicit in all these questions is essentially metaphorical." (2000, p. 322). Font, Godino, Planas, & Acevedo (2009) have elaborated what they call the "objectual metaphor" in mathematics: "The objectual metaphor is a conceptual metaphor that has its origins in our experiences with physical objects and permits the interpretation of events, activities, emotions, ideas... as if they were real entities with properties" (p. 985). This metaphor allows someone carrying out mathematical work to treat symbols as well as abstract ideas as objects, thus radically reducing the cognitive load that would be required if every mathematical sign had to be grounded in its logical or prior mathematical definition.

This paper has only sketched the general outline and basic constructs of the theories of embodiment and conceptual integration, and has attempted to bring the ideas of these theories into (metaphorical) contact with those of other influential theories in mathematics education. One of the goals of carrying out research and building theory in mathematics education is, presumably, to reach a more complete understanding of mathematical thinking, learning and teaching. Although we may be a long way from seeing the whole elephant, I would argue that to reach, eventually, an integrated and comprehensive theory of mathematical thinking we shall need to incorporate the kind of knowledge gained from contemporary work in embodied cognitive science.

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INCORPORATING LESSON STUDY IN PRE-SERVICE MATHEMATICS TEACHER EDUCATION

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This study elucidates the potentialities of incorporating the essential elements of Lesson Study in the pre-service mathematics teacher education in the desire to ameliorate student teachers' facility in realizing their theoretical knowledge in the actual practice of teaching. A single case of student teaching program in a fuzoku school in Japan was highlighted in the hope to elevate certain ruminations on how Lesson Study might be accessed and nurtured in pre-service teacher education.

INTRODUCTION

Field experiences, most popular of which is the student-teaching practicum, subsist as a very important component of virtually every teacher education program in the world. It is deemed to expose pre-service teachers face-to-face with the complexity of the classroom; and in the process, allow them to meld theory into practice. However, there always seems to have a certain schism between the theoretical preparation of teachers and the practice of teaching as especially tangible during the first times the (student) teachers stand in front of the classrooms. Student teachers usually struggle to reach an acceptable level of harmony between their conceptions and their teaching practice, not to mention the limitations and conditions imposed by the school context (Georgiadou-Kabouridis & Potari, 2002). In this light, interventions that would facilitate the transition from the theoretical phase of learning to be a mathematics teacher to the beginning phase of the actual practice of teaching must be instituted in mathematics teacher education environments.

The perceivable distinctiveness between the positioning of prospective teachers as learners in the university and as student teachers in placement schools allows a stance for teacher education stakeholders to probe the relationships and transitions that dwell in both contexts. Consequently, the theoretical preparation in university courses and the prospective teacher's practicum in schools must be able to engender a compatible and reciprocating relationship. Drawing from the sociological research traditions, Ensor (2001) utilized the notion of *recontextualizing* in order to explicate the movement of a teacher from methods courses into actual classrooms. The notion of recontextualizing highlights the "transformation of discourses as they are disembedded from one social context and inserted into others" (Ensor, 2001, p. 297). On the other hand, deliberating on the upbringing of every prospective mathematics teacher as organic individual entities has implications for the implementation of teacher education curriculum. Cobb (1994) articulates that attention must be given to teacher cognition and the conditions and opportunities that facilitate their learning in

2011. In Ubuz, B. (Ed.). Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, pp. 305-312. Ankara, Turkey: PME.

order to understand how teachers learn to teach. This intimates that in teacher education, it is also necessary to consider the (student) teachers' learning processes. This includes considering their previous knowledge, beliefs, and conceptions, and valuing the role of their activity and reflection on the activity; promoting the construction of meaning through classroom interactions, between instructor and student teachers and among student teachers; and appreciating the heuristic value of an investigative dimension in learning as well in the teachers' work (Ponte, 2001).

This study explores how some elements of '*Lesson Study'* (LS) are being utilized as a powerful intervention in order to facilitate the transition of prospective mathematics teachers from being students in methods courses into undergoing actual teaching practice in schools during the student teaching program.

LS IN THE STUDENT TEACHING PROGRAM AT A *FUZOKU* SCHOOL

LS is a teacher development model that originates from Japan. The history of LS dates back from the Meiji period (1868-1912), when it was developed as an educational practice which function was to enable the teachers to develop and study their own teaching practices (Baba, 2007, p. 2). LS is a process that follows a cycle by which teachers of mathematics in the same community of practice work together by identifying a goal or a problem for the lesson, collaboratively developing a lesson plan, implementing the lesson with observation by colleagues and other experts (also called the *research lesson*), analytically reflecting on the teaching and learning that occurred, and revising the lesson (Stigler & Hiebert, 1999). It must be noted that each of the components or steps in the LS cycle signifies indispensable features for the anticipated fruition of the activity. It is not surprising, therefore, that in the course of replicating the practice in other cultures, the inclination towards superficial adaptation of the structural or procedural features is readily perceivable. Thus, it would be imperative to clarify the deeply rooted principles behind LS by explicating on its essential elements.

The simple premise behind conducting LS is that in order to improve teaching, the most effective place to do so would be in the context of a classroom lesson (Stigler & Hiebert, 1999). Moreover, Isoda, Stephens, Ohara and Miyakawa (2007, p. xvi) identify several underlying ideas about the practice of LS: (1) teachers can best learn from and improve their practice by seeing other teachers teach; (2) there is an expectation that teachers who have developed deep understanding of and skill in subject matter pedagogy should be encouraged to share their knowledge and experience with colleagues; and (3) while the focus appears on the teacher, the final focus is on the cultivation of students' interest and on the quality of their learning. Furthermore, Watanabe (2002) expresses that through the LS experience, teachers realize possibilities wherein they could examine all aspects of their teaching, such as the curriculum, lesson plans, instructional materials, content, teaching strategies, etc.

The essence of LS lies in the amount of intellectual and affective engagement of its participants who engender a spirit of collaboration – working on a shared goal that they themselves generated. A great deal of contextual generation and edification of knowledge, together with critical reflection, are elicited and nurtured in the process. The main objective of doing LS is not being able to come up with the best "lesson"; instead, the lesson just serves as a vehicle towards achieving intersecting goals on the improvement of teaching and student learning, and/or evincing contextual mathematical knowledge for teaching. Moreover, LS is driven by making sense of a range of resources, starting from student abilities, curricular pre-requisites, school context, reform-oriented research recommendations, etc.

In the Japanese educational system, national universities that offer courses in Education have attached institutions that serve as laboratory schools for student teachers, among some of their functions. These schools are called *Fuzoku* Schools, which exemplify a well-defined function in the pre-service (and in-service) teacher education in Japan. Whereas LS is widely performed in in-service mathematics teacher education in Japan, the student teaching program in *Fuzoku* schools integrate some vital elements of LS as a part of the practicum.

It is believed that undertaking an investigation on how LS is being incorporated and nurtured in pre-service mathematics teacher education in Japan will contribute to the ongoing reflections on how pre-service mathematics teacher education could be made relevant towards a smooth transition of perspective teachers into becoming in-service mathematics teachers. In this regard, this particular investigation tackles a focal research question: *What skills, competencies, or habits of mind are needed to be cultivated in pre-service mathematics teacher education in order for perspective teachers to successfully participate in LS?*

METHODOLOGY

This inquiry is a part of a larger phenomenological study that seeks to understand the underlying principles behind the accession of LS in pre-service mathematics teacher education. Prior to undertaking this particular investigation, the researcher has already been amply acquainted to the social context of the practice by observing a number of LS in several schools in Japan for about four years. In this report, a pre-service teacher was observed daily in his activities as a student teacher (ST) over the span of the practicum, which lasted for four weeks. As the objective of the researcher is to make an inquiry regarding pre-service teacher education in mathematics as a specialized area, the subject for this investigation was purposively selected to be a prospective middle school mathematics teacher. The other criterion for the selection of the subject was the willingness to be a part of this investigation. The placement school was a *Fuzoku* Middle School of a national university named Saitama University, which is located in Saitama (a prefecture that borders the north of the Tokyo Metropolitan Area). It is composed of three year levels, each with four sections and an average class size of 40.

The observation allowed the researcher to become acquainted with the contextual environment of the practicum, and to be able to generate conjectures regarding the underlying principles that are engendered in the program. Moreover, the interviews, informal conversations, and the analysis of the activities that the ST underwent and several artifacts (the ST's observation notes, daily journal, etc.) were utilized as rich sources of data that would facilitate in the crystallization of findings. The interviews with the subject and the cooperating teacher (CT) were audio-taped, along with the reflection meetings and the *Research Lesson* (RL, an important component of LS) undertaken by the ST. Though it was not possible for the researcher to videotape all the activities and the RL, some photographs were taken when they were allowed. Van Mannen's (1990) phenomenological method was employed in analysing data from interview transcripts, observations, and pertinent artifacts. Significant statements, utterances, and actions were highlighted to provide an understanding of how the participants experienced LS in the *Fuzoku* school. From these, clusters of meanings were formulated into the emergent themes that pertain to skills, competencies, or habits of mind that were nurtured and cultivated in the ST during the practicum.

SUMMARY AND FINDINGS

The subject for this inquiry was assigned to the CT who handles all the four classes of first year; and he was to handle only one section. On the other hand, another ST was also assigned to the same CT; she was supposed to teach another class. The CT and the two STs together formed a group, wherein, with the guidance of the CT, they observed, commented, discussed, and reflected on each other's lessons. This small community of practice simulated a small LS group. The ST engaged in a series of classroom observations, lesson preparations, actual classroom instructions, and *hanseikai* (reflection meetings held after every lesson done by the STs). A RL was also undertaken towards the end of the practicum. All the 7 STs in mathematics observed each other's lessons, together with all the 3 CTs, and 2 mathematics teacher educators from the university. They all participated in the *hanseikai* after all the lessons were delivered. Indeed, as the elements of LS are embedded in the programme, it appeared that it's already in the tacit core of beliefs and practices of the ST. He said, "I didn't treat the RL as something special. But I really believe that I always have to think of the students' learning, be it a RL, or just a simple day."

It can be said that the strong linkage between the mathematics teacher educators of the university and the teachers at the *Fuzoku* Middle School has been greatly beneficial in nurturing a shared common teaching culture between the institutions. Thus, the process of recontextualization and enculturation were well coordinated, fortifying the socialization of STs from the context of the university into the teaching culture of the school. For one, the requirements in university courses were arranged so as not to interfere with the rigorous obligations of the ST in the whole duration of the practicum. Nevertheless, as it was apparent that the ST was able to develop most of the practices that the CT was usually doing in the classroom, the propensity of the apprenticeship model of learning to be a teacher in the student teaching programmes surfaced. In any case, it must be noted that the teachers of *Fuzoku* schools themselves are actively and continuously participating in research activities (e.g., LS) that espouse reform-oriented views on subject matter teaching and learning, which, in one way or another, addresses the danger of socializing STs into continually adhering to traditional ways of teaching.

Four interconnected themes that pertain to skills, competencies, and habits of mind were elevated from the investigation: (1) making sense of powerful resources for classroom instruction; (2) utilizing the school and classroom contexts as venues of inquiry; (3) engaging in critical reflections; and (4) forging the spirit of collaboration.

Making sense of powerful resources for classroom instruction

The practicum prompted the ST with opportunities to analyze and make sense of student abilities, classroom context, and mathematical or didactical/pedagogical stances as rich resources for the development of lessons. This is a substantial habit of mind integral for doing LS. Though these resources might have posed potential constraints in the recontextualization of the ST's acquired knowledge from the university, he was able to fabricate a rich personal repertoire of mathematical knowledge for teaching. Say, when he made use of the example given in the textbook as the main task for his first actual lesson, the CT questioned the meaningfulness of the task considering the capabilities of the students. Hence, emphasis was given to inquiry and making use of all legitimate resources for lesson development. The level of aptitude demonstrated by the ST in being able to channel his observations, lesson planning, lesson delivery, and reflection to the more important aspects of mathematics classroom instruction might have been influenced by his educational biography, school context, or the university courses. As it is not definitive as to which of these influences shape the process of recontextualization, Ensor (2007) suggests that teacher education is rationalized through access to *recognition* and *realization rules* (italics from the original, p. 314), and through development of mathematical discourses. In selecting tasks for lessons, the ST said:

> In the desire to come up with a lesson that promotes conceptual understanding, I was impelled towards understanding the mathematical content of the topic by myself, and at the same time, think of how it could relate to the students' understanding. To do so, I review what the students learned in the past, think of ways to allow the students to summarize the day's lesson using their own assimilations, and make use of aids that let the students visualize concepts that are hard to understand.

Reinvesting principled rationalizations and critical reflections in amalgamating intertwined resources for lesson preparation and actual classroom instruction evoked practical considerations on his pre-active and interactive decision-making process, which rendered reinforcement in forming his own identity as a teacher.

Moreover, the opportunity for his actual classroom practices to be available for evaluation by the mathematics teacher educators during the RL might have fortified reflections on the reproduction of tasks based on the principles engendered in the university. Through the comments rendered by the mathematics educators, and also by all the CTs, the ST was introduced to relating the lesson to institutional/

contextual/national stances in mathematics education. However, the level of sensitization of these bigger mathematical/pedagogical issues on his sensibilities still calls for further investigations, especially regarding his beliefs and conceptions of these issues and his appraisal of doing RL in the practicum.

Utilizing the school and classroom contexts as venues of inquiry

Performing the actual lessons has proven to be a legitimate avenue for the ST to have access in the *realization* of the theoretical knowledge he acquired from the university, validate his assimilations from the prior observations he made, and assess the soundness of his decisions based on the enacted tasks he planned for instruction. This provided an implication on the development of capacities in generating one's own pedagogical content knowledge based on the authentic contexts. Utilizing the practicum and the classroom as a legitimate venue for inquiry led the ST into active engagement in his induction to mathematics classroom practices. Evidences on the students' engagement and mathematical learning in connection to the classroom milieu and mathematical, didactical, pedagogical and institutional decisions embodied in the tasks were continuously gathered and deeply internalized.

The actual instances that happened in the classroom elicited in the ST a necessity to further look into his engendered beliefs and practices. For example, it was visible from the observations of the ST's first actual lessons that his concern was more on the completion of the planned tasks for the day. This somehow banished him from paying much needed attention to the students' responses to the tasks. After his first lesson, his written reflection on his journal read, "Even though I was feeling nervous, I think that the first parts of the lesson went well because I was carefully thinking of the flow of the lesson." However, the inability to anticipate students' possible responses on the tasks he prepared revealed the inadequacy and/or the inappropriateness of his personal analysis of the students' abilities during his prior observations. He said, "As the students were able to generate more sophisticated responses than what I expected, I wasn't able to respond well and it seemed that the lesson was not sustained." Though his inadequacies surfaced from the actual classroom setting, an opportunity to address the problem presented itself. It was through the actual classroom experience that his conceptions were affirmed or challenged, which provided trajectories for future growth. This exemplifies that classroom experiences do not only subsist for students' learning, but for teachers' as well.

Engaging in critical reflections, personally or within a group

With the realization of the complexity of the school and classroom environment, critical reflections have been nurtured and made attendant to the ST's engagement in the practicum. It prevented the ST from the trappings of conceptualizing teaching mathematics as a technicized endeavour wherein the STs merely imitate the techniques of the CT. Furthermore, it has ushered him into further inquiry regarding how he could improve his class, with implications on continuous and recursive learning. It also indulged him to emotional engagements and brought him to a certain commitments. He intimated,

> During the *hanseikai*, the realization of so many things I was not able to consider in the actual class even made me feel like crying... Consequently, I am resolved in making the next lesson a better one.

Critical reflections served as a way to address the emancipatory development of the ST. Ascertaining others' perspectives assisted the ST into being able to consider things that he wasn't able to discern in his personal observations and in the enactment of his decisions. This draws implications on the role that social dimension plays in an individual's journey into becoming a mathematics teacher. For example, engaging in *hanseikai* posed an opportunity for the ST's articulated observations and thoughts to be challenged or reinforced, wherein opportunities on making connections between actions and theoretical underpinnings were fabricated with the guidance of the CT, and also with the support of the other ST.

Forging the spirit of collaboration

Working in a community of practice presented certain transformative values in the ST's development towards becoming mathematics teachers, as they learn from the diversity of each other's perspectives. In order to be able to do so, the ST's were impelled into expositions of their thoughts and making their learning public.

The issue on clarity, in written or verbal form, had been valuable in cultivating in the ST the ability to precisely convey his ideas, intentions, beliefs, and thoughts, which would allow the creation of a platform for collaboration and critical reflections. This was addressed while the ST was in the process of writing the lesson plans; and was even reinforced during the RL as the lesson plan must be able to convey a clearer and more comprehensive picture of the learning context for the benefit of the 'outsiders' who were supposed to observe their lessons. Also, verbal explicitations engendered during the *hanseikai,* and even during informal discussions with other STs, provided opportunities to reflect and learn from each other's feedbacks.

Moreover, systematically structuring the practicum to be conducive for collaborative explorations of mathematical classroom norms afforded certain benefits on being able to nurture a shared knowledge-base for teaching practice. In the ST's words:

> In observing the classes of other ST, I can objectively consider some good aspects of the instruction which might be missing in my own lesson; and by being able to relay feedback regarding some questionable aspects of the lesson which might not be noticed by the other ST, I think that the quality of both of our lessons can be improved.

DISCUSSION

The findings generated from this particular investigation raise issues on being able to balance the purposes and implementation of the student teaching practicum with the beliefs and perceived needs of STs when they appear to be located in a polarized continuum. Several factors, such as time (length of the practicum) and opportunities for reconstruction of one's beliefs and classroom practice, might have exiled the ST from being able to make sense of the reform-oriented stances that could be envisaged from the process of undergoing the RL. The ST said, "I wanted to deliver the same lesson again. In this way, I would have the opportunity to reconstruct and rectify my lesson based on the discussions during the *hanseikai*."

Indeed, certain skills or habits of minds that might lead to positive transformations in mathematics classroom practices could be cultivated in the student-teaching program using the elements of LS as a powerful intervention. More than sensitizing the STs to the pedagogical stances of mathematics teaching, deeper understanding and continuously recursive learning of the mathematics content and socio-mathematical norms were also reinforced through the facilitation of the creation of local theories in the collaborative process of preparing, enacting, and reflecting on each lesson.

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UNPACKING *MATHEMATISATION*: AN EXPERIMENTAL FRAMEWORK FOR ARITHMETIC INSTRUCTION

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Research is reviewed that emphasises mathematisation, that is, students bringing increased mathematical sophistication to their activity. A framework is described of ten dimensions of mathematisation that are important for learning arithmetic: complexifying arithmetic, distancing the setting, extending the range, formalising arithmetic, organising and generalising, notating, refining computational strategies, structuring numbers, decimalising numbers and unitizing numbers. The paper draws on a corpus of videotape of interactive teaching from an on-going design research project. Uses of the framework are illustrated in analyses of two lesson episodes, and in a map of ten ways a teacher could develop the instructional task—8+5. Finally, four questions are posed on the potential of the framework to inform instruction.

The Numeracy Intervention Research Project (NIRP) is a design research project developing pedagogical tools for intervention in learning arithmetic (Wright, Ellemor-Collins, & Lewis, 2007). Central to our instructional design is the aim of cultivating mathematisation. The purpose of this paper is to describe an experimental framework of dimensions of mathematisation to support instruction in arithmetic.

BACKGROUND

Mathematisation in learning arithmetic

Students *mathematise* by bringing more mathematical sophistication to their activity. *Progressive mathematisation* means the development of mathematical sophistication over time: for example, developing from adding with counters through to adding bare numbers. Freudenthal and others have argued that the central task of mathematics instruction is to support progressive mathematisation (Beishuizen & Anghileri, 1998; Freudenthal, 1991; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). The emergent modelling heuristic (Gravemeijer & Stephan, 2002), for example, seeks to design instruction that supports progressive mathematisation from context-bound activity to more formal and more sophisticated reasoning.

Research has established, for learning number and arithmetic, the importance of several particular forms of mathematising, such as: symbolising and formalising (e.g. Gravemeijer, et al., 2000; Gravemeijer & Stephan, 2002); generalising (e.g. Carraher, Schliemann, Brizuela, & Earnest, 2006); flexiblising of computation strategies (e.g. Beishuizen & Anghileri, 1998); structuring numbers (e.g. Ellemor-Collins & Wright, 2009; Gravemeijer, et al., 2000); decimalising to develop base-ten thinking (e.g. Cobb & Wheatley, 1988; Freudenthal, 1991); and unitising numbers (e.g. Cobb &

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Wheatley, 1988; Mulligan, Mitchelmore, & Prescott, 2006). Thus, designs for arithmetic instruction need to address several forms of mathematisation.

Dimensions of mathematisation in interactive instruction

In moment-to-moment instruction, teachers can observe students' responses to a task, and pitch a subsequent task or comment just beyond the cutting edge of the students' current knowledge, to elicit mathematisation. Such interactive instruction could be described in terms of scaffolding or micro-adjusting (Wright, Martland, Stafford, & Stanger, 2006). Such instruction requires, in part, that teachers be aware of the different ways a task could be developed or extended, and the different forms of potential mathematisation involved, similar to what Chick has investigated as the *affordances* of tasks (2007). Hence, designs for instructional tasks and procedures could involve articulating the potential *dimensions* for developing tasks to elicit mathematisation.

A framework of dimensions of mathematisation

Within the NIRP, an experimental framework of five key domains of arithmetic knowledge has been developed (Wright, et al., 2007). Instructional tasks and procedures have been developed for each of these domains (Wright, Ellemor-Collins, & Tabor, in press). Design of instructional procedures has sought to promote students' progressive mathematisation, and has included the explicit development of tasks along particular dimensions of mathematisation. Examples of dimensions include: in instruction for addition and subtraction in the range 1 to 20, the progressive distancing of the setting (Ellemor-Collins & Wright, 2008); and in instruction for conceptual place value, the distancing of the setting, the extending of the range of numbers, and the complexification of increments (Ellemor-Collins & Wright, in press). However, we were aware that significant dimensions of mathematisation have remained less explicitly articulated in the instructional design within each domain. Also, dimensions of mathematisation appear to be common across the domains. We became interested in unpacking the significant dimensions of mathematisation for the learning of arithmetic across all domains, and becoming more explicit and systematic about addressing each of these dimensions in instruction.

To this end, we have developed an experimental framework of ten dimensions of mathematisation for arithmetic instruction. Research-based frameworks can be effective for guiding instruction (Bobis, et al., 2005). The experimental framework of dimensions described in this paper is intended to indicate productive dimensions for developing tasks to elicit mathematisation in interactive teaching, within all domains of arithmetic. More broadly, we intend the framework to characterise the key dimensions of progressive mathematisation involved in learning whole number arithmetic. The purpose of this paper is to describe the dimensions in the framework, and to illustrate the potential of the framework in the development of interactive instruction.

METHOD

The NIRP adopted a method based on design research (Lesh, 2002), incorporating extensive teaching experiments (Steffe & Thompson, 2000) over three one-year design cycles. Each year involved an experimental intervention program, including professional development of teachers, student assessments, and a term of intensive teaching in classes of one or three students. In total, the project has involved 25 teachers and 200 students in intensive intervention teaching (Wright, et al., 2007). All individual classes and assessments were videotaped, providing an extensive corpus of video data for analysis.

The development of the experimental framework of dimensions of mathematisation is a form of instructional design (Gravemeijer & Stephan, 2002). The corpus of videotape of intensive interactive teaching is a rich context for unpacking mathematisation. Developing the framework has involved an iterative process of analysing teaching episodes and teaching procedures for common dimensions of mathematisation, devising a conjectural framework, returning to check the framework against further teaching, and revising the framework. Thus the design process is similar to the method of Cobb and Whitenack (1996) for analysing longitudinal teaching data. Nevertheless, as instructional design, our final criterion for the success of the framework is not the fit with current data. Rather, the criteria are the significance of the dimensions of mathematisation for student learning of arithmetic, and the pragmatic usability of the framework by teachers. Thus the framework should describe what teachers see students doing and how they think of their practice. The framework is experimental in the sense that the design process is on-going: we will trial it, and teachers will trial it, in professional development projects.

FRAMEWORK OF DIMENSIONS OF MATHEMATISATION

Table 1 lists the ten dimensions of the framework. Each dimension is given a one-letter code for ease of reference. Below we briefly describe each dimension, and give examples of the development of tasks to elicit mathematisation along the dimension.

(C) Complexifying arithmetic. By complexify we mean: develop more parts or more directions. Common ways to make a more arithmetically complex task include changing: from counting forwards to counting backwards; from adding to finding a missing addend; from adding a single ten to adding multiple tens; from tasks that do not involve regrouping to tasks that do; and from division without a remainder to division with a remainder.

(D) Distancing the setting. In the instructional design, initial tasks often involve an instructional setting such as ten-frames or base-ten materials. The student can be progressively distanced from the setting through steps such as: (1) manipulating the materials; (2) seeing the materials but not manipulating them; (3) seeing them only momentarily; and (4) solving tasks posed in verbal or written form without materials.

C Complexifying arithmetic	N Notating
D Distancing the setting	R Refining computation strategies
E Extending the range	S Structuring numbers
F Formalising arithmetic	T Decimalising numbers (T for Tens)
G Organising and generalising	U Unitizing numbers

Table 1: Framework of dimensions of mathematisation for arithmetic instruction.

(E) Extending the range. Tasks can be posed using higher numbers. The range of numbers can progress through: 1-5, 0-10, 0-20, 0-100, 0-200, 0-1000, beyond 1000.

(F) Formalising arithmetic. Formalising means investing more significance in form, especially in notations and language. Formalising arithmetic can involve: developing more formal notation, such as shifting from idiosyncratic notation, to informal arrow notations, to formal number sentences; developing more precise terminology, such as shifting from 'take 'away' to 'subtract'; and developing more standardised arrangements of materials, such as a practice of grouping counters in rows of five.

(G) Organising and generalising number relations. Organising can involve making categories: for example, from a set of ten-frame cards, separate the five-wise and pair-wise configurations. As well, organising can involve making an ordered list: for example, list the partitions of 6 in order: 0+6, 1+5, 2+4, 3+3. Organising is closely aligned with generalising about number relations. For example, considering the ten-frames, how can we characterise five-wise configurations? Considering the ordered list of partitions of 6, how many partitions of 7 or of 8 would there be?

(N) Notating. A teacher can notate, or can ask the student to notate. Arithmetic tasks can be notated: for example, the task "16 and how many more to make 20?" presented on the arithmetic rack can be notated as $16 + \square = 20$. Also, computation strategies can be notated: a jump strategy for solving 34+19 can be notated as jumps on an empty number line, or with number sentences: 34+10=44, 44+6=50, 50+3=53.

(R) Refining computation strategies. Students can reflect on and discuss their computation strategies. In discussions, teachers can draw attention to curtailed procedures or the use of number relations, and encourage efficiency, flexibility, and insight in computation. Teachers can also pose tasks selected to elicit particular strategies, for example, posing the set of tasks 34 + 19, 64 − 18, 49 + 27, 57 + 28, to elicit strategies capitalising on numbers near a decuple.

(S) Structuring numbers. By structuring numbers we mean noticing and using number relations, and developing an increasingly dense network of number relations. A common task to elicit structuring is, in a setting such as a ten-frame, to describe a number as a combination or partition of other numbers: for example, describing an 8-dot ten-frame as 5-and-3, or as 10-less-2. Also, students can be asked to use a number relation to solve a task: for example, can you use that 8 is close to 10 to solve 8+7?

(T) Decimalising numbers (T for Tens). By decimalising we mean developing the practice of organising numbers into ones, tens, hundreds, thousands and so on; developing *base-ten thinking*. Tasks can involve incrementing and decrementing by 1s, 10s, and 100s. Tasks can involve arranging materials in groups of ten. Tasks can emphasise decimalised language—"how many tens, how many ones?"—or the decimalised numeration system—"why are there zeros in the numeral 1007?"

(U) Unitizing numbers. By unitizing we mean students coming to regard numbers as units, that is, as single whole objects that can be counted. For example, when a student counts how many 3s in 12 as one 3, two 3s, three 3s, four 3s; the 3s are regarded as units. Unitizing can involve, for example, reasoning that if there are four 3s in 12, then there are eight 3s in 24, which involves counting units of units. Tasks to elicit unitizing include counting rows in arrays, and drawing attention to the unitary aspect alongside the composite aspect of numbers.

LESSON EPISODES ILLUSTRATING THE DIMENSIONS

Below we describe two lesson episodes. For each episode, we analyse how the teachers develop tasks to elicit particular dimensions of mathematisation. These accounts serve as illustrations of how the dimensions arise in interactive teaching. Further illustrations of the dimensions in teaching are available in earlier papers (Ellemor-Collins & Wright, 2008, in press).

Episode 1: Subtracting nine

Mr Benz used an arithmetic rack, a frame with two rods each of ten sliding beads. Mr Benz posed 16–9, which Alan did not solve. With 16 on the rack (10 upper & 6 lower beads), Mr Benz asked Alan to take away 9 on the rack. Alan shuttled all 6 lower beads, and 4 upper beads, and after questioning, changed to 3 upper beads. Mr Benz said "You've taken away nine. And how many's left?" Alan looked at the rack and said "Seven!" Mr Benz built 16 again and indicated 9 upper beads. "You take away those nine." Alan shuttled 9 upper beads. Mr Benz asked, "What's left?" Alan answered "Seven!" Mr Benz made 16 again, saying "You can say, well if I took away ten (shuttling 10 beads), but I leave one (returning 1 bead) and that's like taking away nine." Next, Mr Benz asked Alan to write 14−10, and momentarily displayed 14 on the rack. Alan answered "Four," checked with the rack unscreened, and wrote "= 4". Mr Benz asked Alan to write 14−9, and momentarily displayed 14 on the rack. Alan answered "Five." Mr Benz unscreened the rack, asking "Where's the big nine you can grab?" Alan shuttled 9 upper beads, then confirmed and wrote "= 5".

Analysis of dimensions. Mr Benz suggested that taking nine beads can be related to taking ten, eliciting structuring of the numbers (S). He posed 14−10 followed by the more complex 14−9 (C). He micro-adjusted the distance of the arithmetic rack setting (D), between Alan manipulating the rack, looking at the rack, and only briefly seeing the rack. He also had Alan write each task and each answer (N), and prompted him to write formal number sentences (F).

Episode 2: 75 – 39

Connor solved two 3-digit addition tasks mentally. In previous lessons Connor was unsuccessful when attempting 2-digit subtraction tasks. Mrs James posed "75 take away 39" (with 75–39 written), asking Connor to start with 75. After ten seconds Connor answered "36", and Mrs James wrote 36. She proceeded to jointly work through the solution:

Mrs James:	I want to keep the 75 as a whole number. 75 take away 30 equals …? (writing 75–30=).
Connor:	45.
Mrs James:	(Writes 45 to complete the number sentence 75–30=45.) 45 take away...? (writing 45–).
Connor:	Five (pointing at the 9 in 39).
Mrs James:	Okay. (Writes 5 & 4 under 9.)…five is 40 (completes writing 45–5=40). 40 take away…? (writing 40–).
Connor:	Four (pointing to the 4 under the 9).
Mrs James:	…4 gave you the 36 (completes writing 40–4=36). Well done.

Analysis of dimensions. Mrs James shifted from posing addition tasks to posing a subtraction task (C). At the same time, she retreated from 3-digit numbers to 2-digit numbers (E). She asked Connor to start with the 75, and after he solved the task, she led him through a jump strategy, to encourage a preferred strategy for subtraction (R), which organises the subtrahend in tens and ones (T). Mrs James notated the strategy (N), using standard number sentences (F), with an extra notation to record the partition of 9 into 5 and 4 (S).

APPLYING THE FRAMEWORK IN INSTRUCTION

Our instructional aim is to encourage students to mathematise their arithmetic activity. In interactive teaching, the framework dimensions serve a dual purpose: to make explicit the dimensions along which students might mathematise their activity, and to indicate ways a teacher can develop tasks to elicit such mathematisation. Teachers might elicit mathematisation by commenting on a student's response to a task, or by posing a similar task with one dimension ratcheted up a level. The framework of dimensions can serve as a map of these possibilities at any given moment in interactive teaching.

To illustrate, imagine the task 8+5 has been posed with visible ten-frames, and a student has responded. At this point, possible teacher's comments and task developments to elicit particular dimensions of students' mathematisation include: (C) posing 8+?=12 with ten-frames; (D) posing 8+4 with screened ten-frames; (E) posing 28+5 with ten-frames; (F) posing 8+4 in standard written form; (G) posing "Make all the pairs of numbers that sum to 13"; (N) asking the student to notate her strategy on an empty number line; (R) comparing a counting-on strategy (8: 9, 10, 11, 12, 13!) with an add-through-ten strategy (8+2→10+3→13); (S) asking what number relations could be used to help solve 8+5; (T) drawing attention to regrouping the numbers using 10; (U) asking how many fives there are (one in the eight, and one in the five). Thus, drawing on the framework, we can devise up to ten ways to develop an instructional task, with each development directed toward significant mathematisation.

SUMMARY

As described earlier, particular forms of mathematising are well established in the research literature. The potential contribution of the framework is to synthesise into a coherent and instructionally useful form an account of the significant forms of mathematisation for learning arithmetic. We expect the framework of dimensions to enhance our broader instructional framework, and to serve as a map of instructional possibilities in interactive teaching. As well, the framework will be of interest to others developing arithmetic instruction based on mathematisation.

Having developed this experimental framework, we will continue to trial and revise it in professional development settings. Questions arising include: How can we refine our current instructional procedures using the framework? How can teachers use the framework to inform moment-to-moment teaching? How does the framework clarify the link between moment-to-moment teaching decisions and medium-term goals for student learning? What is the potential of a framework of several dimensions of mathematisation, as an alternative to one- or two-dimensional learning frameworks?

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SUPERVISORY KNOWLEDGE AND PRACTICES OF A MATHEMATICS COOPERATING TEACHER IN A SUPERVISION PROGRAM

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This paper reports on how a program based on educative supervision supported the supervisory knowledge and practices of a mathematics cooperating teacher. The role of cooperating teachers in the professional development of student teachers is important; however, research indicates that cooperating teachers need support in transitioning from being classroom teachers to being teacher educators. We adopted an emergent perspective to understand how one cooperating teacher developed his supervisory knowledge and practices as he engaged in the program activities.

INTRODUCTION

As a result of a shift in many countries towards school-based teacher education, there has been a need to support cooperating teachers [CTs] as teacher educators (Koster, Korthagen, & Wubbels, 1998). Despite their significant impact in the professional career of student teachers [STs], research has found that CTs need support in enacting their role as teacher educators, such as in delivering feedback to STs (Clarke, 2006). Several researchers recommended that university supervisors [USs] might provide guidance and modelling for effective supervision to CTs as the teachers transition from being classroom teachers to being teacher educators (Borko & Mayfield, 1995; Koster, Korthagen, & Wubbels, 1998; Fernandez & Erbilgin, 2009). This might also help develop partnerships between schools and university programs. Aligned with this suggestion, we designed a program where the university supervisor [US] assisted the development of the supervisory practices and knowledge of mathematics CTs. This paper reports on the development of supervisory knowledge and practices of one mathematics cooperating teacher as he engaged in the program activities with other CTs and a US.

THEORETICAL FRAMEWORK

In the field of mathematics teacher education, Blanton, Berenson, and Norwood (2001) advocated "educative supervision" as an approach to supervise STs. An educative supervisor supports the ST's growth by asking open-ended questions, engaging the ST in discussions of critical incidents from the ST's teaching, and being sensitive to the ST's developmental level. Educative supervision is in sharp contrast to 'evaluative' supervision through which the supervisor is focused on conducting authoritative evaluations of STs (Blanton et al., 2001). We used the construct of educative supervision and developed the program discussed in this paper to help

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mathematics CTs implement more educative supervision approaches in their supervisory practices. The theoretical basis of educative supervision is the notion of “zone of proximal development” [ZPD] defined by Vygotsky (1934/1978). ZPD can be drawn on as a basis for the learning of the CTs engaged in our program and supported by a US knowledgeable in educative supervision. In designing the program activities, we adopted an emergent perspective and sought to integrate Vygotsky’s social learning theory with an individualistic view of learning. According to the emergent perspective, learning is a process that involves both social and individual aspects (Stephan, 2003). From this perspective, a group comes to a shared understanding through discussion and participation by its members while at the same time individual participants reorganize their own understanding, and contribute to the group’s evolving practice. Individuals’ construction of knowledge and the group’s construction of shared knowledge is a reflexive process.

THE SUPERVISION PROGRAM

The program was designed for a period of 15 weeks to fit with the student teaching experience. It was designed as a kind of practicum for CTs: a program providing CTs with opportunities to improve their supervisory knowledge and practice while working with a student teacher as recommended by Fernandez and Erbilgin (2009).

The CTs participated in three online and four face-to-face meetings. The content of each program meeting was as follows in the order of occurrence within the program. The first face-to-face meeting included an introduction to and negotiation about the program. The first online discussion was about current reforms in teaching mathematics. The second online discussion involved readings and discussions about educative supervision. The second face-to-face meeting focused on learner-centred supervision practices in support of STs’ growth. The third online discussion was about fostering STs’ reflection on their own teaching. The third face-to-face meeting included activities that engaged participants in reflecting on, analysing and discussing their present supervisory practices. The fourth face-to-face meeting focused on supervision approaches from the perspectives of CTs and STs.

Over the course of the program, the CTs were asked to conduct a formally planned post-lesson conference with the STs every week. Additionally, the CTs participated with the US in triad post-lesson conferences with the STs that provided opportunities for the US to model aspects and facilitate the use of educative supervision.

METHODOLOGY

The participants of the supervision program were three mathematics CTs. The CTs had over 30 years of teaching experience and had each supervised between 3 to 8 mathematics student teachers prior to this study. Pseudonyms are used throughout this paper for the participating CTs: Denise, Lauren, and Andrew. One of the researchers assumed the role of the US. She had 3 years of mathematics teaching experience and had supervised three mathematics STs prior to this study.

In this paper we report on an investigation of Andrew's learning about and implementing educative supervision. We chose to report on Andrew's case because although each teacher changed their supervisory practice as the program progressed, the changes were more marked in Andrew's case. We used the emergent perspective to interpret the development of Andrew's supervisory knowledge and practices as he participated in and contributed to program activities. We took into account both Andrew's individual construction of knowledge and the social context of the program meetings in which he was a participant.

The primary data for this paper comes from the post-lesson conferences between Andrew and his ST, Alison (pseudonym). The post-lesson conference communications were analysed from three perspectives based on literature (Shulman, 1986; Blanton et al., 2001; Fernandez & Erbilgin, 2009) and open coding. First, we determined the amount of communications contributed to the conferences by the CTs and STs through the use of the "word count" function of a word processor. Second, we determined the content (i.e. mathematics, pedagogy, mathematics pedagogy, teacher-student relationship, classroom management, and general teacher growth) of the post-lesson communications. Third, we determined the types of communications (i.e. questioning, assessing, suggesting, describing, explaining, and emotional talking) used by the CTs and STs. The description of each category can be found in Erbilgin (2008). A combination of the three types of analyses helped us understand how the supervision style of the CTs changed, if any, throughout the semester. We cross-checked the consistency of the findings by using triangulation of sources.

RESULTS

Andrew's Supervision Style before the Program was Implemented

Andrew supervised about seven STs prior to this study. He attended a course on supervising STs less than 10 years prior. He learned how to supervise STs mainly by attending that course and through repeated experiences as a CT. In his initial interview, when asked about how he communicated feedback to his STs, Andrew said that he communicated his thoughts explicitly to the student teachers in a positive manner and encouraged them to ask for his feedback. In her initial interview, when asked to describe a typical post-lesson conference with her CT, Alison said "He always points out all the positives first, he says this is going very well…maybe you might want to look at this or maybe put it in a different way." Providing feedback in the forms of assessment and suggestion in a positive manner seemed to be the norm in Andrew's initial supervision style.

Another piece of information about Andrew's initial supervision style came from a role play experience. In both the initial and final interviews, the CTs were asked to watch a mathematics lesson clip, imagine that the teacher in the video was their student teacher, and explain what and how they would communicate with this ST during a conference. In his initial role play, Andrew focused on pedagogy (e.g. getting every kid involved) and somewhat on mathematics pedagogy (e.g.

significance of the volume formula). He posed two questions in the beginning, and then communicated his suggestions (three of them) and explanations (five) with positive assessments (four) between his comments. Offering suggestions, explanations, and positive assessments was a common trend that we found in his initial post-lesson conferences with Alison. In summary, Andrew did not seem to be implementing an educative supervision approach before he participated in the program.

Changes in Andrew's Supervision Style as he Participated in the Program

Andrew and Alison had five audio-taped post-lesson conferences; the first two conferences were conducted before educative supervision was discussed in the program. We sought to understand if there were any changes or not in Andrew's supervision style by analysing his post-lesson conferences with Alison from three perspectives: amount of communications contributed to the conferences, content of the post-lesson communications, and types of communications. Andrew had the following talking percentages from the first conference to the fifth conference respectively: 98%, 98%, 81%, 50%, and 46%. Alison's corresponding percentages in these conferences were 2%, 2%, 19%, 50%, and 54%. The calculation of talking percentages by Andrew and Alison revealed that Alison's voice in the post-lesson conferences drastically increased from the first conference to the fifth conference.

Having student teachers voice their ideas, an attribute of educative supervision, was a focus of the program. For instance, in the second face-to-face meeting during the discussions about the meaning of educative supervision, the US elaborated that educative supervision values that the student teachers express their ideas to analyse their own teaching. Denise contributed that "Like you let them speak instead of giving them the answers." Lauren added that "I usually, when we first meet then, I have her reflect on her lessons." As part of the meeting, the teachers were provided a lesson plan of a student teacher, read a transcript of the post-lesson conference between the student teacher and her cooperating teacher, and critiqued the post-lesson communications. One focus of the CTs' responses in this activity was on the amount that the ST spoke. For instance, Lauren said that "I think that there was too much talking by the cooperating teacher and not enough by the student…" Andrew spoke as follows:

> Yeah, and I'm not saying the lesson was not a good lesson. It's just that she pointed everything out to the student teacher and those were definitely directed questions with a short response and you never really got to know what she thought, good or bad.

As evident in the above excerpt, Andrew observed that the ST was not given opportunities to express her thinking. The program's continuous focus on helping STs voice their opinions in the post-lesson conferences is aligned with Andrew's change toward encouraging Alison do more talking in their post-lesson conferences.

Figure 1 shows the percent of communications in each content category across five post-lesson conferences between Andrew and Alison. A big change from the first conference to the fifth conference was that the talk about mathematics pedagogy increased considerably.

	General Pedagogy	Mathematics Pedagogy	Mathematics	Classroom Management	General Teacher Growth	Teacher-Student relationship
Conf-1	52%	8%	0%	3%	23%	5%
Conf-2	21%	17%	0%	11%	33%	8%
Conf-3	0%	94%	3%	0%	0%	0%
Conf-4	10%	54%	0%	18%	5%	2%
Conf-5	39%	44%	0%	8%	8%	0%

Figure 1: Types of Content in Post-lesson Communications

Not only did the quantity of communications in the mathematics pedagogy category increase, their quality also improved. During the first two post-lesson conferences, Andrew and Alison discussed general ideas rather than talking about the specifics of the lessons. For example, Andrew talked about explaining the concepts in more detail in the first conference, but he did not provide examples and observations from Alison's lessons. In contrast to the general ideas discussed in the first and second meetings, Andrew and Alison had discussions related to specific details of the lessons that Alison taught, during the third, fourth, and fifth post-lesson conferences. For instance, in their fourth conference, after Alison made an evaluation of her lesson, Andrew described a mathematics problem from the lesson, and asked Alison to compare how the students in the two classes handled it. This gave Alison an opportunity to think about what modifications she did from the first period to the second period and how it affected the students' understanding. It was valuable from the program's perspective that Andrew and Alison started talking about specifics of Alison's lessons because talking about classroom incidents might help student teachers to reflect deeply on the day and is a key element of educative supervision (Blanton et al., 2001). The quantitative and qualitative improvements of the mathematics pedagogy category in Andrew and Alisons' post-lesson conferences might be a result of the program because in several face-to-face meetings, mathematics pedagogy was put forward as an important teaching domain to be discussed in the post-lesson conferences. Other program activities supported this idea as well, such as the articles that the teachers read and the triad conferences that the US led during her visits to the STs' classrooms.

Figure 2 represents the percentages of communications in each category used by Andrew in the post-lesson conferences. Figure 2 shows that Andrew used communications in the questioning category in the third, fourth, and fifth conferences while he did not pose any questions in the first two conferences. He focused on transmitting his opinions to Alison in the first two conferences.

	Questioning	Assessing	Explaining	Describing	Suggesting	Emotional Talking
Conf-1	0%	14%	27%	18%	32%	9%
Conf-2	0%	32%	24%	35%	8%	0%
Conf-3	32%	0%	32%	0%	36%	0%
Conf-4	52%	18%	18%	9%	3%	0%
Conf-5	44%	19%	13%	13%	6%	6%

Figure 2: Types of Communications used by Andrew

The analysis of how Andrew used questioning communications and what type of questions he used revealed a gradual change. In the third conference, out of the nine questions Andrew posed, only two of them requested Alison's reflection on her teaching. The two questions were asked in the beginning of the conference. The rest of the conference included mostly suggestions and explanations by Andrew along with confirming or requesting information type questions. Alison's voice was mainly heard in the form of "uh, huh." This was also evident in the talking percentages (19% for Alison). The third conference represents a transition for Andrew from traditional supervision to educative supervision. Changing beliefs and practices is a slow process (Gregoire, 2003). Thus, if there was a change in the supervision style of Andrew, it should have occurred slowly. In the fourth and fifth conferences, he used questioning communications frequently to help the student teacher reflect on that day's lesson. One finding about his questioning style in the fourth and fifth conferences was that Andrew did not use follow-up questioning regularly in the conferences. In other words, he posed a question, received Alison's opinion, and then offered his opinion rather than posing further questions. We believe that using follow-up questions might give CTs opportunities to dig at the STs' thinking. This might be considered in designing future supervision programs.

Asking open-ended questions to help the STs think deeply about their teaching is a key component of educative supervision and was a main focus in the program activities. In the second face-to-face meeting, the CTs wrote down questions that a CT could have asked the ST in a transcribed conference. Here is a part of their discussion:

> Denise: You know, if we want to have post-lesson conferences with a student teacher, there would be exactly those questions asked each time. What went good with this lesson? What were you happy about? What did you see as problems? What did the kids seem to learn from the lesson?
>
> Andrew: Tell me why you'd do it again? Why you wouldn't do it again?
>
> Denise: Yeah, what would you do differently?

The CTs supported each others' thinking, brainstorming about possible open-ended questions to ask the STs in the post-lesson conferences. In the third face-to-face meeting, the CTs shared a video segment from their own recorded post-lesson conference, a segment where they felt they implemented educative supervision. The shared segments included many open-ended questions posed by the CTs. We feel that the discussions in the program activities provided the CTs with opportunities to gain new knowledge, share ideas on supervision, and implement educative supervisory practices. In summary, data analysis revealed that as Andrew engaged in the program activities, he started asking more questions to Alison, they talked more about mathematics pedagogy in detail, and Alison spoke more in their post-lesson conferences. Hence, Andrew started implementing a more educative supervisory approach.

Andrew's Supervision Style after the Program was Implemented

In the final interview, Andrew watched the same lesson video that he watched in the initial interview and explained what and how he would communicate during a conference with the ST. Compared to his focus on general pedagogy and somewhat mathematics pedagogy in the initial interview, Andrew showed a deeper focus on mathematics pedagogy (e.g. connecting the volume formula to real life, the sequence of mathematics topics taught, and relating mathematical topics with each other) in the final interview. Regarding the types of communications, Andrew used 7 questioning, 5 assessing, 5 suggesting, 3 describing, and 2 explaining communications. His use of questioning communications increased in the final interview (32%) compared to initial interview (14%). These findings are aligned with our observation that Andrew started implementing more educative supervisory practices.

Interviews with both Andrew and Alison confirmed that Andrew's supervision style changed in the direction of the program's goals. In her final interview, Alison elaborated that their conversations were based on more structured questions about the lessons later in the semester. Alison's perception of change in their post-lesson conferences was shared with Andrew during his final interview. He explained that the program's goals made sense to him and it became part of his supervisory practices. He wrote the following in his reflection survey. "The adjustments I have made are a direct result of what I learned from this program. They are very sound and effective modifications."

DISCUSSION AND CONCLUSIONS

In this study, we examined how a program that focused on educative supervision supported the supervisory knowledge and practices of Andrew. The analysis of the five post-lesson conferences from the three perspectives revealed that the supervision style of Andrew changed throughout the semester towards educative supervision. We observed similar changes in the supervisory practices of the other two participating teachers as well (Erbilgin, 2008). We believe that this study contributes to efforts to create learner-centred student teaching experiences through program activities that help CTs progress toward supervising from an educative approach and supports ways that CTs and USs can work together as proposed by previous studies (Borko & Mayfield, 1995; Koster, Korthagen, & Wubbels, 1998; Fernandez & Erbilgin, 2009). Supervising a ST at the time of the study helped the CTs implement what they learned in the program and reflect on their supervision.

The results of this study suggest further questions to be examined by possible future research. For instance, what might be perceptions of STs about their growth during student teaching when working with CTs implementing and those that are not implementing educative supervision? Another question might be, how does a program that focuses on educative supervision work with prospective CTs? Also different designs of similar programs (e.g. more than one US working with CTs) might be examined.

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DEVELOPMENT OF PROSPECTIVE MATHEMATICS TEACHERS' PROFESSIONAL NOTICING IN A SPECIFIC DOMAIN: PROPORTIONAL REASONING

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The aim of this research is to identify aspects that support the development of prospective mathematics teachers' professional noticing in a b-learning context. The study presented here investigates the extent to which prospective secondary mathematics teachers attend and interpret secondary school students' proportional reasoning and decide how to respond. Results show that interactions in an on-line discussion improve prospective mathematics teachers' ability to identify and interpret important aspects of secondary school students' mathematical thinking.

THEORETICAL BACKGROUND

A relevant skill of mathematics teachers is the professional noticing (Jacobs, Lamb, & Philipp, 2010). Although this skill has been conceptualized from different perspectives in the last years, the connexion is making sense of how individuals process complex situations (Mason, 2002; van Es & Sherin, 2002). Mason (2002) considered noticing as a fundamental element of expertise in teaching characterized by: (a) keeping and using a record, (b) developing sensitivities, (c) recognizing choices, (d) preparing to notice at the right moment and, (e) validating with others. On the other hand, van Es and Sherin (2002) considered that noticing in teaching involves (a) identifying significant events of a classroom situation, (b) using knowledge about a context to reason about these events, and (c) making connections between the specific classroom events and broader principles of teaching and learning. These approaches are concerned with the development of perceptual frameworks that enable teachers to view mathematics teaching situations in a particular way.

A particular focus for mathematics teacher's professional noticing is children's mathematical thinking. In this context, Jacobs et al. (2010) conceptualize professional noticing of children's mathematical thinking as a set of three interrelated skills (i) attending to children's strategies: the extent to which teachers attend to the mathematical details in children's strategies; (ii) interpreting children's mathematical understandings: the extent to which the teachers' reasoning is consistent with both the details of the specific child's strategies and the research on children's mathematical development; and (iii) deciding how to respond on the basis of children's understandings: the extent to which teachers use what they have learned about the

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children's understandings from the specific situation and whether their reasoning is consistent with the research on children's mathematical development.

Researchers have focused on how professional noticing is developed (Llinares & Valls, 2010; Star & Strickland, 2008). A relevant issue in this context is the characterization of the development of professional noticing of children's mathematical thinking in specific mathematics domains (Levin, Hammer, & Coffey, 2009). So, in this study, we are going to focus on prospective mathematics teachers' professional noticing of children's mathematical thinking in the context of the transition from students' additive to multiplicative thinking, since researchers have shown the difficulty of students in discriminating additive from multiplicative situations (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2010; Modestou & Gagatsis, 2007).

A context for the development of prospective mathematics teachers' professional noticing

Recently, the development of on-line approaches in mathematics teacher education and specific contexts of blended learning approaches (b-learning) have generated particular issues about how the professional noticing skill is developed in these new learning contexts. From a social learning perspective (Wells, 2002), participations in social interaction spaces are considered as a way of learning. Furthermore, the characteristics of on-line environments seem to influence the way in which prospective teachers interact with knowledge and the professional vision of classroom events and children's mathematical thinking. In a specific way, Mason (2002) underlined that the validation of the records and interpretations of mathematics teaching situations with others is an important aspect of the structure of teachers' attention.

RESEARCH QUESTIONS

In this study, we analysed how prospective mathematics teachers' professional noticing of children's mathematical thinking is developed in the context of the transition from students' additive to multiplicative thinking. We are also interested in how the participation in on-line discussions, the analysis of secondary school students' answers to proportional and non-proportional problems and the resolution of tasks collaboratively could support the development of prospective teachers' professional noticing o secondary school students' proportional reasoning.

METHOD

Participants and context

Participants were 7 prospective secondary school mathematics teachers that were enrolled in a post-graduate program. This program qualifies them to teach mathematics in Secondary Education. This study was carried out in one of the subjects of this program called "Learning of mathematics in Secondary Education". One of the aims of this subject is that prospective teachers learn to identify and interpret characteristics of secondary school students' mathematic thinking. A specific subject-matter refers to the relation between the additive and multiplicative thinking on secondary school students (12-16 years old) in the context of proportional reasoning.

A b-learning (blended learning) environment was designed to this part of the subject integrating face-to-face and on-line activities in a web platform. In the face-to-face activities, prospective teachers worked collaboratively in the classroom in order to solve and discuss the proposed tasks. In the on-line activities, they shared and synthesized their ideas into a final joint report. The learning environment consisted of five face-to-face sessions during five weeks in which prospective teachers had to read theoretical papers about the transition from additive to multiplicative thinking, analyse video-clips where secondary school students solved problems with additive and multiplicative structures and analyse written student work. The prospective mathematics teachers began these tasks in a face-to-face context but they continued them in an on-line context that included the discussion in on-line debates. In this paper, we are going to focus on the resolution of the initial task and its discussion in the on-line debate.

The initial task

Prospective teachers solved an initial task that consisted of the analysis of four secondary school students' answers to two proportional problems (modelled by the function $f(x) = ax$, $a \neq 0$) and two non-proportional problems with an additive structure (modelled by the function $f(x) = x+b$, $b \neq 0$) (Figure 1). Prospective teachers had to analyse a total of 16 secondary school students' answers (four problems × four secondary school students). Secondary school students' answers were selected taking into account the different profiles of primary and secondary school students when they solve proportional and non-proportional problems (Van Dooren, De Bock, Gillard, & Verschaffel, 2009). These students' profiles are: students who solve proportional and additive problems proportionality, students who solve proportional and additive problems additively, students who solve both type of problems correctly and finally, students who solve problems with integer ratios using proportionality (regardless the type of problem) and problems with non-integer ratios using additive strategies.

Prospective teachers were asked to answer three questions related to the three component skills of professional noticing of students' mathematical thinking in each student answer: (i) "Please, describe in detail what you think each secondary school student did in response to each problem" (prospective teachers' expertise in attending to students' strategies); "Please, indicate what you learn about secondary school students' understandings related to the comprehension of the different mathematic concepts implicated" (prospective teachers' expertise in interpreting secondary school students' understanding), and (iii) "If you were a teacher of these students, what would you do next?" (prospective teachers' expertise in deciding how to respond on the basis of students' understandings).

Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded 40 boxes, Tom has loaded 100 boxes. If Peter has loaded 60 boxes, how many boxes has Tom loaded? $100 - 40 = 60$ $60 + 60 = 120$	Jean and Paul are swimming. They started together but Jean swims slower. When Jean has swum 25 m, Paul has swum 75 m. If Jean has swum 125 m, how many meters has Paul swum? $75 = 3 \cdot 25$ $125 \times 3 = 375$
Paul and Tom are climbing the wall of a skyscraper. They climb equally fast but Paul started later. When Paul has climbed 3 m, Tom has climbed 9 m. If Paul has climbed 6 m, how many meters has Tom climbed? $9 = 3 \cdot 3$ $6 \cdot 3 = 18$	Laura and Peter are pasting stamps on postcards. They started together but Laura pastes slower. When Laura has pasted 80 stamps, Peter has pasted 280 stamps. If Laura has pasted 120 stamps, how many stamps has Peter pasted? $280 - 80 = 200$ $120 + 200 = 320$

Figure 1. Part of the initial task: Example of a student who solve problems with integer ratios using proportionality (regardless the type of problem) and problems with non-integer ratios using additive strategies.

Analysis

Firstly, we identified the mathematical significant elements that prospective teachers should identify in each problem and strategy (for instance if the ratio or the difference between quantities remains constant, or if the function passes through (0,0) or not). Secondly, we determined whether prospective teachers' answers indicated attention to these mathematical details. This led us to identify each participant attention to secondary school students' strategies. We also considered the extent in which prospective teachers identified the different profiles mentioned above. This provides information about if prospective teachers were able to discriminate proportional from additive problems using the relevant elements identified. For example, how prospective teachers identified if the additive strategy was used correctly in the additive problems but also incorrectly in the proportional problems. In that way, we analysed how prospective teachers interpret students' understandings. Finally, we analysed if prospective teachers were able to include considerations of students' understandings in their decisions of how to respond.

RESULTS

In the first part we describe how prospective teachers attended and interpreted secondary school students' strategies and, in the second part, we show how the discussion in an on-line debate helped prospective teachers to develop the professional noticing of students' mathematical thinking.

Attending and interpreting secondary school students' strategies and deciding how to respond

Initially, most of the prospective teachers were able to recognise and describe some of the secondary school students' strategies but had difficulties in discriminating proportional and additive problems and in relating the types of problems and the characteristics of strategies in order to interpret students' mathematical thinking. Only one of the prospective teachers was able to identify a student profile (student who solved proportional and additive problems additively). For example, the prospective teacher P6 only described the operations that the secondary school student made to solve the first problem (Figure 2) and was not able to recognize the additive structure of the second problem.

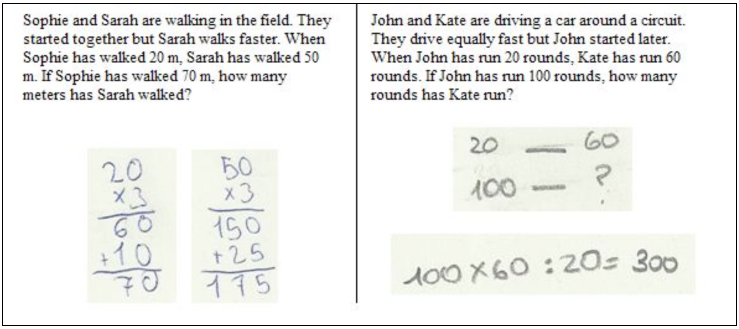

Figure 2. Part of the initial task.

Problem 1. P6 The student tries to solve the problem using proportions. He/she tries to go from 20 to 70 using multiplications and additions. The student knows that has to go from 20 to 70 so he/she multiplies by 3 and then adds 10. So we have to do the same operations with 50. We obtain 175. Therefore, I think that this student does not know proportions but he/she solved the problem correctly.

Problem 2. P6 This student used the method of proportions. Although he/she did not write 20:100 = 60: x, he/she wrote 100×60/20.

On the other hand, prospective teachers' interpretations of students' mathematical thinking influenced their teaching decisions. So, when prospective teachers did not identify additive and proportional problems then they indicated general teaching actions such as asking to the students more explanations about their answers or explaining the use of procedural approaches to solve proportional problems.

The development of prospective teachers' professional noticing of students' mathematical thinking

Through the on-line debate, prospective teachers were able to focus on the characteristics of the problems and to identify some secondary school students' profiles. Therefore, the interaction motivated by the interpretation of secondary school students' answers made prospective teachers to start to attend and interpret jointly secondary school students' answers. For example, prospective teacher P1 discriminated proportional from non-proportional situations in the initial task underlying the importance of the sentences "they load equally fast but Peter started later" and "they started together but Jean swims slower" but prospective teacher P4 did not discriminate them. Interactions in the on-line discussion between prospective teacher P1 and prospective teacher P4 led prospective teacher P4 to start to discriminate both type of problems identifying relevant aspects of the situations.

P1 Students use elemental operations (such us addition, subtraction…) correctly. However, they do not usually read well the problem and interpret, in the same way, the fact to start later and the fact to be slower.

P4 I agree with you. Students do not differentiate between "doing equally fast an action but starting at different times" and "starting at the same time but doing an action faster". We have to find out if students did not read well the problem or they had difficulties in understanding the concept of proportionality (the difference between proportional and non-proportional problems).

An example of how interaction led students to identify secondary school students' profiles is the interaction between prospective teachers P7, P3 and P1. This interaction started with the participation of P7. This prospective teacher identified that the secondary school student solve one of the two proportional problems correctly but the other incorrectly and the same happened with the additive problems (Figure 1). The participation of the prospective teacher P3 was not relevant. However, P1 focused on the multiplicative relationships between quantities. In that way, P1 indicated that the secondary school student solved the two problems with an integer relationship between quantities (triple) proportionality but when the relationship was non-integer the student solved the problems additively.

P7 This is a strange case because there are two proportional problems but one is solved correctly and the other incorrectly. And there are two problems where *they do not start at the same time* and again, one is solved correctly and the other incorrectly. How can we explain it? The student could not understand the problem or he/she could have some difficulties. We have to ask students for explaining their answers.

P3 It is true that it is a strange case. As you said, if we ask for more explanations, students could understand when he/she can use the strategy. For example, when he/she wrote 100-40 = 60, he/she should have written "60 boxes loaded by Tom when Peter start to load"

P1 Respect to this student, we could say that he/she did not discriminate proportional from additive problems. However, two problems were solved by the same strategy because the multiplicative relationship between quantities is integer ("the triple", the multiplicative relationship between 25 and 75 and the multiplicative relationship between 3 and 9). The other two problems have a non-integer multiplicative relationship between quantities and they are solved looking for a difference and using it. So when students had difficulties in looking for the relationships between quantities they used a constant difference instead of a multiplicative relationship.

Moreover, teaching decisions changed after the participation in the on-line discussion. All prospective teachers stressed the necessity of differentiate proportional and non-proportional problems and when prospective teachers were able to identify the secondary school students' profiles they proposed to focus on the type of ratio and on the use of qualitative problems instead of missing-value problems.

These data indicated that the participation in the online discussion and the fact that prospective teachers had to write a joint report with the conclusions of the on-line discussion allow them to begin to develop the professional noticing of students' mathematical thinking focused on the proportional reasoning.

CONCLUSIONS AND DISCUSSION

This study contributes to the research base on how prospective teachers see and make sense of classrooms, particularly in how they begin to develop making sense to students' mathematical thinking. New in this line of research is that we characterise the professional noticing of children's mathematical thinking in a specific mathematic domain: the transition from students' additive to multiplicative thinking and we also provide a specific context for the development of the professional noticing integrating on-line discussions.

Results show that initially prospective teachers had difficulties attending and interpreting students' mathematical thinking. They described students' answers without including mathematical significant aspects about the structure of the problem or about students' strategies, and therefore they were not able to identify secondary school students' profiles. However, the participation in the on-line discussion led prospective teachers to begin to develop the professional noticing of students' mathematical thinking.

A characteristic of the on-line discussion is the progressive discourse that it was built facilitated by the interaction and the integration of ideas related to proportional and non-proportional situations and to the characteristics of secondary school students' proportional reasoning. Therefore, the on-line debate and the characteristics of the task played a relevant role in the construction of knowledge.

Finally, our results also indicate that professional noticing can be learned (Jacobs et al., 2010) and that the b-learning environments could help to develop this skill (Llinares & Valls, 2010). However, it is necessary more studies about how some characteristics of the learning environment (such as the specific use of on-line discussions and the characteristics of the task) could support this development.

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YOUNG STUDENTS THINKING ABOUT MOTION GRAPHS

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This article considers an individual activity given to grade 2 students at the end of a teaching experiment lasted about four months. The students have extensively used a motion sensor to work with graphical representations of position versus time. The activity asks the students to compare two graphs, choosing two cartoon characters, animals, and vehicles, as subjects of possible corresponding motions. Analysing their written arguments, we look at the thinking strategies the students adopt to solve the task. A common factor characterizes these strategies: the meaning of the horizontal straight line in terms of motion modelling is used as a key to the understanding.

INTRODUCTION AND THEORETICAL FRAMEWORK

"In the mathematics laboratory, the construction of meanings is strictly tied to the tools used in the activities, on the one side, and to the interactions among people that develop during those activities, on the other" (Anichini *et al*., 2004, p. 28). The idea of the *mathematics laboratory* is not that of the traditional mathematical lesson that entails an imparted teaching. On the contrary, learners are like apprentices in a Renaissance workshop: they learn by doing, seeing, and communicating with each other and with the experts. An example is given by the series of experiences about which we speak in this paper. A group of fifteen grade 2 (7-year old) students took part in activities on modelling motion through graphing. Graphing is intended as "drawing graphs, reading graphs, selecting and customising graphs for particular purposes, and interpreting and using graphs as tools." (Ainley, 2000, p. 365). Previous research has analysed the cognitive processes of students studying motion, mostly focussing on individual interviews, and on older students (e.g. Nemirovsky *et al*., 1998; Ferrara, 2006). In general, this research has shown that representing motion situations through the use of bodily actions enriches the ways students encounter and make sense of ideas relative to kinematics, like distance, speed, acceleration, and time. On the other hand, motion embodies the cognitive roots of the basic concepts of Calculus (Tall, 2000).

The students involved in our activities used a motion sensor called CBR (Calculator Based Ranger) to display in real time graphical representations of their movements in front of the sensor. Through this kind of experiences, they started to grasp meanings related to the function concept (Ferrara & Savioli, 2009). Particularly, the students developed a covariance view of the functional relationships (in the sense of Slavit, 1997). For instance, they were able to understand the meaning of a horizontal straight line. This is a delicate matter: one sees a horizontal straight line originating point by point on the screen, although there is absence of motion. No one is moving in front of

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the CBR, but something moves: the point marking the origin of the curve of position versus time. Time is the key for comprehension. Not an easy task at so early age.

In this paper, we centre on a specific activity of the series that required the children to compare two graphs related to similar but different movements. If we analyse the children's written productions, we can shed light on the meaning they were able to give to the horizontal straight line in terms of motion, and on how they used this meaning in a functional way to support their arguments.

METHODOLOGY AND ACTIVITY

The activity we consider here comes from a teaching experiment lasted about four months, from February to May 2006. The experiment is part of a 4-year longitudinal research study that finished in 2009. The study was conducted in a primary school in Chieri, a small town in the surroundings of Torino, in the Northern Italy. At the time of the experiment, the fifteen students were attending the 2nd grade, and they had no experience with motion phenomena and graphs describing them. All the activities were carried out during regular mathematics lessons, and designed by the authors (respectively, university researcher and teacher). In these lessons, the students spent much time working in groups of 2 or 3, and in individual tasks, being always required to explain their reasoning. The researcher also directed some general discussions that permitted the students to communicate and compare their different solutions.

In the course of the experiment, the students worked with graphical representations of position vs. time related to movements they performed in front of the motion sensor. They could watch the real time origin of the graphs thanks to the use of a view-screen that allowed projecting on a wall the screen of a calculator connected to the sensor. The calculator processed the position-time data coming from the sensor, displaying the corresponding position-time graph. Sometime the students were asked to interpret graphs associated with specific movements; at other times, they had to anticipate the movements connected to given graphical shapes. In this way, they were involved in the double passage, from motion to model, and from model to motion. Skills on both the passages are fundamental ingredients to construct a sense for the graph, in terms of functional relationships. In order to collect data, we used two video cameras: one filming the movements of the students and the subsequent graphs; the other one filming the groups' dynamics, both in the case of small groups and in the case of the whole classroom. We further used all the written productions coming from both pair and individual written tasks solved by the students (only seldom from small group tasks). This article focuses on one of the individual written activities we proposed to the students toward the end of the experiment, when they had extensively used the motion sensor for tackling problems of different type.

Those problems largely included as graphical shapes horizontal and slanted straight lines, which describe respectively absence of motion and uniform motion (walking away from, or approaching the CBR). In some of the activities, the students had to choose at random a card that hid a cartoon character, an animal or a vehicle, and to move in front of the sensor (according to the trajectory they prefer) assuming to be that character/animal/vehicle, while the classroom mates had to draw the expected corresponding graphs. The data we present here comes from the written arguments produced in particular by two students (Elisa and Gaia) to solve the following task: to associate two animals, two cartoon characters, two vehicles with two given graphs, and to explain the choices. In this case, both graphs represent piecewise functions that consist of a first increasing slanted piece and a second horizontal piece. They are similar in shape, but differ from each other since one has a steeper slanted piece, and a longer horizontal piece than the other one. Figure 1 shows the (hand made) graphs. The labels on the axes are letters for the variables, as they appear on the screen of the calculator: position (distance to the sensor) in meters (*D*[m]), time in seconds (*t*[s]).

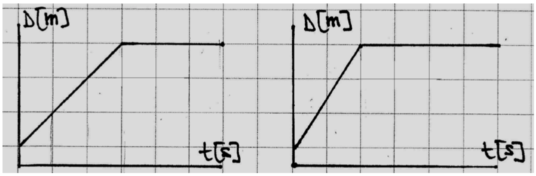

Figure 1. The two graphs of the activity

The complete text of the activity is as follows ("below" refers to the inserted Fig. 1):

> *Choose two animals to be associated with the drawings below: Explain your reasoning and the motive of your choices.*
>
> *Repeat the reason using, instead of animals, two cartoon characters, and two vehicles.*

Investigating the written explanations that support the choice of the moving subjects, we look at the thinking strategies the students adopt to solve the task. Particularly, these strategies are expressed by, and contained in the ways the students compare the two graphs. A common factor features the explanations. Indeed, most students tend to collate the horizontal pieces of the graphs, and to give reason of their choices through reference to such pieces. Few other arguments compare the slanted pieces instead. In general, the reports well highlight the link between the shape of the graphs and the speed at which the chosen subjects move. Focussing on the language of the selected arguments and on inscriptions, if possible, the analysis will mark how the meaning of the horizontal straight lines as models of motion is clearly interiorised.

ANALYSIS AND DISCUSSION

We consider the strategies produced by two young girls (Elisa and Gaia) to solve the first section of the activity: they account the horizontal pieces of the two graphs as functional to the thinking processes. In each of the two extracts below, punctuation in the text and references in parentheses are added for the sake of clarity; underlined words are key words for the investigation. We start analysing Gaia's explanation.

Gaia

As regards the choice of the two animals, Gaia gives a verbal description but does not introduce any sketch. Gaia writes:

> *For the first drawing* [Figure 1, left] *I chose the pig 'cause it is slow and the line that came, it* [the pig] *started near the CBR, it walked and then it stopped at the end of the red band, and a horizontal line came. In the first drawing I chose the pig 'cause it goes slowly and the drawing shows that it* [the pig] *takes longer to arrive at the end of the red band, and a shorter horizontal line came 'cause it* [the pig] *took longer than the horse. The horse is faster* [than the pig] *and it went faster, a longer horizontal line came* [Figure 1, right]*, and the horizontal lines are equal in height, 'cause they* [the pig and the horse] *both stopped at the end of the red band, and they stopped at the end of the red band and they waited for the 15 seconds to finish.*

Gaia chose a pig and a horse and distinguished them assigning each a colour: pink to the pig, orange to the horse. She coloured the lines and the animals' names written under the horizontal pieces, which are what discriminate between the graphs in the argument. Gaia does not look at the graphs globally but locally: she draws attention just to their horizontal part ("*the line*", "*a horizontal line*"), and this is enough to justify her choice of the two animals. The slanted pieces are not of interest to her. Since the beginning, Gaia uses the third person singular "*it*" to refer to the pig in an implicit way. Indeed, at a first reading of the expression "*the line that came, it started near the CBR*" in the original Italian language (where punctuation is not used), it was not so evident that "*it*" alluded to the pig. We initially thought of a fusion of "*the line that came*" with the moving subject that "*started near the CBR*" (Nemirovsky *et al*., 1998). But we solved the doubt as soon as we read the first segment about the choice of the cartoon characters Micky Mouse and Beep Beep, the name for Road Runner from Wile E. Coyote ("*as before*" refers to the situation above): "*I wanted to put Mickey Mouse in the first drawing 'cause it is slower than Beep Beep, and as before they both started near the CBR, they walked away and they stopped at the end of the red band*". The experiences with the motion sensor play a pivotal role in the account. Naturally, Gaia makes present in the graph the image of a real pig that moves so to obtain a shape like the one "*that came*". In so doing, Gaia imagines the actual movement as embodied in the graphical representation: "*it started near the CBR, it walked and then it stopped at the end of the red band*". Even the red band is made present in this imagination process (a red band was put on the floor to mark the spatial range where the children were free to move – a range that cannot be over 6 metres for the functioning of the sensor). Imagining the movement of the pig is the first step to start the real explanation of the choice for the graph on the left, advanced since the very beginning ("*I chose the pig 'cause it is slow*"). After this imaginative step, Gaia is able to give an effective justification that links a real quality of the pig's motion ("*it goes slowly*") with a quality of the horizontal piece on the left (being a "*shorter line*") with respect to the horizontal piece on the right. Again, the image of motion is made present in the graph to express that link: "*the drawing shows that it takes longer to arrive at the end of the red band*", "*it took longer than the horse*" (associated with the graph on the right). From this moment on, the horse enters the argument to complete it. The fact that one line is "*longer*" than the other one depends on the speed Gaia imagines the animals moving at: "*the horse is faster*", "*it went faster*". Reference to the animals' motion for interpreting and explaining the shape of the graphs in a global manner rather than local, is present up to the end ("*'cause*" repeated many times). So, the same "*height*" for the graphs is linked with the stop of both the pig and the horse "*at the end of the red band*"; having horizontal pieces means in terms of motion that "*they waited for the 15 seconds to finish*" (in fact, the sensor collects data in real time for a 15-second time interval). In this conclusion, Gaia leaves implicit that being faster for the horse/slower for the pig corresponds to waiting longer/shorter for the 15 seconds to finish.

Elisa

Elisa produced an interesting argument, in which again the horizontal pieces are used in a functional way, and the slanted pieces are not taken into account. A new strategy is adopted with respect to Gaia to differentiate the lines, though. Elisa chose a frog and a snail, respectively for the graph on the right and the graph on the left (Fig. 2).

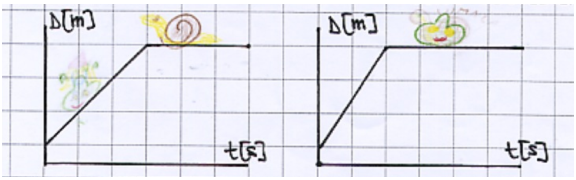

Figure 2. Elisa's sketches of a snail (on the left) and of a frog (on the right)

Even if the argument is mainly expressed in verbal form, Elisa also drew sketches of the chosen animals just over the horizontal parts of the two lines (Figg. 2 and 3). She writes:

> *The drawings are equal in height but there's a thing that is not equal, I chose a frog and a snail* [Figure 2]*, then I looked at the drawings and the first one on the left, where there's the horizontal line I counted the big squares and they were 3 and a half, and it means that the snail went slowly. The second one on the right, the horizontal line and I counted the big squares again and they were four and a half, and it means that the one went more slowly and the other fast; the frog fast, the snail very slowly 'cause the frog goes fast and if it goes fast it stops earlier than the snail. And the snail goes slowly and arrives later, if the frog goes fast it arrives earlier and the 15 seconds are not finished.*

Elisa's strategy is developed (like Gaia's) by focussing on the length of the horizontal parts of the graphs ("*where there's the horizontal line*"). Nonetheless, it differs from Gaia's strategy for the fact that the two lengths are distinguished through the count of the big squares filling the Cartesian space just under the horizontal pieces ("*I counted the big squares*"; see the circled regions in Fig. 3).

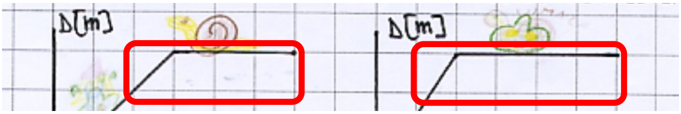

Figure 3. The regions where Elisa counted the big squares

We may suppose that the count of the big squares was the very first step of Elisa's thinking process, and that Elisa introduced on the graphs the sketches of the snail and of the frog as soon as she ended the count and understood which one of the horizontal pieces is the longest. Like for Gaia, even for Elisa the previous experience with the motion sensor plays a pivotal role in the argument. As a matter of fact, Elisa thinks of each number of big squares as associated with an animal moving at a certain speed ("*slowly*" or "*fast*"). So, the "*3 and a half*" squares counted under the piece on the left correspond to the choice of the snail ("*it means that the snail went slowly*"). This is because three and a half are less than the "*four and a half*" squares Elisa has counted on the right (exactly, that region contains four squares and a portion of a fifth square, but the reasoning is still valid). In terms of motion, counting less squares on the one side with respect to the other "*means that the one went more slowly and the other fast*". As a consequence, the frog that goes "*fast*" is associated with the "*four and a half*" squares, while the snail that goes "*very slowly*" is connected to the "*three and a half*" squares. Up to here, the report has a fuzzy nature: we may state that Elisa was likely to have clear in mind the motive of her choice, but she was not yet able to express it fine to her readers. She repeats more times the subjects ("*the one*"/"*the snail*", "*the other*"/"*the frog*"), referring to the different qualities of their movements in terms of speed ("*went more slowly*"/"*very slowly*", "*fast*"). At this point, the causal use of the conjunction "*'cause*" transforms the fuzziness of the former description into the effectiveness of the justification. The change is also marked by the presence of the "*if*"–form. This form may be seen as an implicit 'if… then' that provides the reader with the real explanation of Elisa's choice: "*if it goes fast it stops earlier*", "*if the frog goes fast it arrives earlier*". Quite the contrary, "*the snail goes slowly and arrives later*". As a result, since "*the 15 seconds are not finished*" the graphs contain two horizontal parts, one longer than the other due to the speed the frog and the snail move at.

CONCLUDING REMARKS

Gaia and Elisa show to have understood the graphs' shape in terms of motion, even if they do not make any reference to the slanted parts. Gaia and Elisa have interiorized the meaning of the horizontal straight line as a model of absence of motion, so much to use it in a functional manner in order to compare the situations described by the two graphs, and to justify their choices. For what concerns Gaia's language, the sense of the comparatives "*shorter*" and "*longer*" to speak of the horizontal parts is soon connected to the pig's and the horse's speed. Concerning Elisa's argument, the count of the "*big squares*" filling in the region under the horizontal pieces allows her first to discriminate between the lengths of the two lines, and then to associate each with the right animal – the rightness being determined by the speed at which the frog and the snail move. The previous bodily experience with the motion detector is recalled by imagining the animals' real movements on the graphs. The latter thus become objects of consciousness (condensing qualities of motion and mathematical qualities) through language but also through: the two colours for the different animals, and for the different shapes, in the case of Gaia; the sketches (over the horizontal pieces just as if they were moving along), in the case of Elisa. On the other hand, imaginary activity constitutes mathematical thinking beside perceptuo-motor-sensory activities (Nemirovsky & Ferrara, 2009). In both cases, the young girls provide a real dress to the abstract shape of the graphical representations, a dress certainly influenced by cultural factors (like knowledge of animals, and beliefs on the speed at which animals move). Gaia's and Elisa's behaviour is not an isolated example. Looking at all the written explanations, we find other instances of analogous strategies concentrating on the horizontal pieces. For example, Marco chose a mole for the graph on the left and a horse for that on the right. He stresses: "*I chose the mole 'cause it goes less fast than the horse but it goes enough fast. I chose the horse because it goes very fast, and in fact in the second drawing it is motionless longer*". Marco makes present the image of a horse that "*goes very fast*", and does not move for a "*longer*" time than the mole. Similarly, Manuele illustrates his choice of a cat and a turtle to be linked with the lines: "*The cat goes slowly 'cause it is in the first path, instead the turtle went faster 'cause the straight line is longer; for the cat the seconds are more instead for the turtle they are less but the turtle waits longer since the line is longer 'cause the turtle has been motionless*". The explanation is untidy (particularly when the "*more*" and "*less*" seconds for the slanted pieces are introduced), but at the end the length of the line on the right (its being "*longer*") is related to the fact that "*the turtle waits longer*", meaning that it waits longer for those 15 seconds to finish also considered by Gaia. These examples all evidence that 7-year old students can grasp meanings related to the covariance of variables, and make sense of graphical representations of motion. Non-standard laboratory activities on graphing motion enabled our grade 2 students to make sense of horizontal straight lines as models of absence of motion, coming to use them in a functional way for supporting their thinking processes.

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ALGEBRAIC THINKING OF PRIMARY STUDENTS

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On the one hand, algebra plays an important role in mathematics teaching at school. On the other hand, it often still proves to be particularly difficult to many students. This is why efforts have been made especially over recent years to connect arithmetic and algebra as early as primary school. Less frequent are studies which systematically explore the natural capacities of primary school pupils regarding their algebraic thinking patterns.

This contribution attempts to describe algebraic thinking at primary school age. In addition, we report of a study which captures the relevant abilities of primary school pupils by looking at the way in which they handle potentially algebraic problems.

INTRODUCTION

Of course, there are varied and partly profession-specific perceptions of algebra, for instance by mathematicians, teachers, pupils, educators ...

A more functional perspective in particular stresses the significance of algebra for mathematics as a whole. So, Ball (2003) describes algebra as a tool for representing and analysing quantitative relationships, as a technical language which makes it possible to model situations as well as formulate and prove general statements. Furthermore, algebra provides fundamental methodology and concepts for several branches of mathematics, thus contributing profoundly to the coherence of mathematical subdomains.

A psychological and didactic perspective places a stronger focus on the processes involved. Kaput (2007), for instance, mentions two core aspects of algebraic work: firstly, an explicit generalisation of patterns and connections by means of a system of symbols which is increasingly becoming more differentiated and conventionalised. Secondly, he points to the syntactically guided handling of symbolic generalisations. It is important to keep in mind that making generalisations explicit is necessary to be able to speak of algebraic activity. We believe, however, that this is also sufficient, especially with regard to primary school. The use of symbolic systems and a syntactically driven handling of symbols usually develop only at a later stage and only gradually so.

These core aspects manifest in various spheres of activity and general fields of competence within mathematics, e.g. in the exploration of patterns, functions and relations, in modelling, arguing and problem solving (Kaput, 2007; Kieran, 2004).

On this basis, algebra seems to be an important access path to "higher mathematics", but it has also proven its role as a so-called gatekeeper: Coping with algebraic demands is crucial to the overall success in mathematics lessons at secondary level of

2011. In Ubuz, B. (Ed.). Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, pp. 345-352. Ankara, Turkey: PME.

education (Cai & Knuth, 2005). It is thus alarming that students experience numerous difficulties with algebraic content, a fact repeatedly shown in case studies and large-scale comparative studies (e. g. Carraher & Schliemann, 2007; Kieran, 2007).

The main reason for these difficulties is seen in the discontinuities between arithmetic and algebra, and also seemingly seamless continuations often involve shifts in and expansions of meaning (Carpenter, Levi, Franke, & Zeringue, 2005). This is supported by the findings that even those students who are good at arithmetic in class (during primary school), frequently perform poorly in school algebra (Kieran, 2004).

An initial assumption was that overcoming these discontinuities was exclusively impeded or hindered by the students' limitations in terms of developmental and cognitive psychology. Newer studies now suggest, however, that mathematics lessons themselves play a significant role in the manifestation of the frequently observable difficulties. While recognizing that preconditions exist regarding developmental psychology, these studies find that an important reason for these difficulties is that the semantic and conceptual differences between arithmetic and algebra are addressed at too late a point in time (Schliemann, Carraher, & Brizuela, 2007).

This is why several Early-Algebra-programmes have been developed over the past years (e. g. Blanton, 2008). In this context, relevant works in the field of didactics of mathematics have largely been more in the shape of intervention studies. Descriptive studies depicting algebraic abilities of young students without any special prior intervention programmes, are scarce up to now.

ALGEBRAIC THINKING

In 1998, Hewitt wrote: Working algebraically means "awareness of awareness" (cited in Mason, Graham, & Johnston-Wilder, 2005, p. 309). This clearly shows what can be taken for granted at primary school: Algebra is more than just knowing how to deal with terms, equations or functions. Algebraic abilities at primary school age become apparent in *algebraic thinking*, for which we postulate the following components:

Handling operations (as objects) and their inverses (Mason, Graham, & Johnston-Wilder, 2005; Kieran, 2004): This predominantly includes knowledge regarding arithmetic operations and their application, e. g. in calculation rules (inversion tasks, commutative, associative, distributive law, identity elements, …), but also solving simple linear equations by "backward calculations" with the inverse operation. Reifying arithmetic operations makes it possible, for example, to compare them regarding their characteristics and effects.

Establishing relationships between numbers, sets and relations (relational thinking) (Warren, 2003): At primary school age, this can show especially in the use of relationships between numbers (e.g. $12+13=12+12+1$ or $9+6=10+5$) or operations (e. g. $2+9+6-9=2+6$) for advantageous calculations, i. e. in the rearrangement of arithmetic expressions for an easier subsequent calculation.

On the basis of these first two components of algebraic thinking, students can derive general transformation rules for algebraic expressions in the long run.

Generalising (Blanton, 2008): This component, detaching the thought from the concrete object, is of great, or even constitutional, significance to algebra. In interplay with the first two components, it enables a generalised arithmetic by the transition from an arithmetic-empirical perspective of concrete objects and processes to an algebraic understanding of relationships and structures. At primary school age, the "generalising" component can show especially in the identification or construction, and use of relationships, for instance in exemplary arithmetic-geometric sequences, or in operationally structured exercises (Wittmann, 1992). We can here differentiate between induction based purely on empirics and a structure-based generalisation in Radford's sense (2006).

Dealing with changes (Zevenbergen, Dole, & Wright, 2004): This includes identification and use of functional dependencies, e. g. in simple (linear) correlations, especially in dynamic situations or those thought to be dynamic. The concept of variables as changeable numbers or quantities, which is highly demanding for primary school pupils, is relevant to this component too.

Dealing with unknowns: Here, we can further differentiate, for instance between constellations (1) where an unknown can be determined at the end of the calculation process, (2) where a relationship between two unknowns has to be established, and (3) where unknowns must indeed be treated as known mathematical objects. For the latter, one could consider equations such as $ax+b=cx+d$, for example, for which equivalence transformations of the equation are necessary in order to solve it. Experience has shown that this is extremely challenging for primary school children, and some scholars speak of a "cognitive gap" (Herscovics & Linchevski, 1994) or "didactical cut" (Filloy & Rojano, 1989) in this context.

Using (symbolic) representations (Kieran, 2004): This component generally includes algebraic expressions or terms, but also the (related) perception of the equal sign as a relational sign, and the dual character of terms as processes and products.
The symbolic language of algebra is not only demanding, it also has a supportive character, and the syntactically driven handling of symbols is extremely important for algebra as a whole (Kaput, 2007). However, algebraic thinking is possible even without letters as variables (Linchevski, 1995). At primary school age, this component is shown especially in using not necessarily symbolic representations of characteristics and relations (Radford, 2006).

If we compare the components of algebraic thinking with characterisations of mathematical giftedness at school age (e. g. Krutetskii, 1976), implicit similarities with characteristics of mathematically gifted children become apparent. On the other hand, algebraic thinking is not explicitly referred to in existing attempts to describe mathematical giftedness. It thus still remains unclear which relationships exist between a student's apparent mathematical talent and his capabilities of algebraic thinking.

RESEARCH QUESTIONS

Within the framework of a qualitative case study, we have two main goals:

By analysing the way in which Year 4 primary school students of varying proficiency levels deal with specific mathematical problems, we want to identify and describe natural elements of algebraic thinking in detail. The primary goal is not to assess students according to their respective abilities. Rather, as the main result of the study we strive for an age-group-specific ascertainment of natural components of algebraic thinking. In addition, we want to find out whether – on the basis of observed individual cases with regard to the components of algebraic thinking – certain ability profiles can be determined.

In doing so, our study wants to give empirically substantiated suggestions regarding the components of algebraic thinking and the content areas of primary math classes in which linking arithmetic and algebra could be particularly fruitful. It can thus create a foundation for possible intervention programmes on the basis of current curriculums and practices in mathematics lessons.

We look at the way in which students handle specific mathematical problems both among mathematically gifted students as well as among students who approximately represent the respective form's spectrum of proficiency. Through a comparative analysis of algebraic thinking among these two groups of students, we want to examine, from a differentiated perspective, whether and to which extent according abilities can also be considered as indicators for the presence of mathematical giftedness.

METHODOLOGY

To answer the research questions, we conduct (semi-standardised) diagnostic interviews, in which students work on selected mathematical problems with potential for algebraic approaches. The sessions are videotaped and analysed by content.

Some specialised German grammar schools with focus on mathematics and natural sciences require applicants to pass highly demanding entrance exams in mathematics. In our current research phase, we collaborate with such a school in Halle and identified some of their mathematically gifted Year Four applicants. Of these students, we picked ten to participate in our study, and additionally chose three other students from each of these students' forms, respectively, who roughly represented the form's spectrum of proficiency. In total, we thus conduct about 40 diagnostic interviews, each with duration of 30 to 45 minutes (max).

The interviews are analysed on the basis of the "algebraic thinking"-construct as depicted above, with its various degrees of manifestation at primary school age, while the components of algebraic thinking are differentiated and specified for each respective problem used during the interview. The system of analytical categories and the basic theoretical assumptions, on which it is based, however, is kept generally open. Particularly during this current first phase of the study, aspects of analysis are also derived from the material itself and the categories used for evaluation are inductively complemented or refined.

PRELIMINARY EXPERIENCES

During the diagnostic interviews, we used, among others, the following problems. They had also already been tested in a preliminary study.

> Marie and Alec collect football cards which come in big and small packs. Marie has one small pack, two big packs and 5 individual cards. Alec has one small pack, one big pack and 9 individual cards.
> How many cards are in each big pack if both children have an equal amount of cards, respectively?

Figure1: Problem 1 – Trading cards

> Erika and Paul are saving up their pocket money for the summer holidays. There is already five times as much money in Paul's piggy bank than in Erika's.
> Erika receives 10 EUR for helping out with garden work, and puts the complete amount into her piggy bank.
> Who has more money now?

Figure 2: Problem 2 – Piggy banks

Fill in the blank in the second expression so that the result is the same.

a) $362 + 157$ $\qquad$ $359 + \boxed{}$

b) $639 - 215$ $\qquad$ $\boxed{} - 219$

c) $14 \cdot 8$ $\qquad$ $5 \cdot 8 + \boxed{} \cdot 8$

Figure 3: Problem 3 – Advantageous Calculations

Put in different numbers in the double computation chain and compare the results. What do you notice?

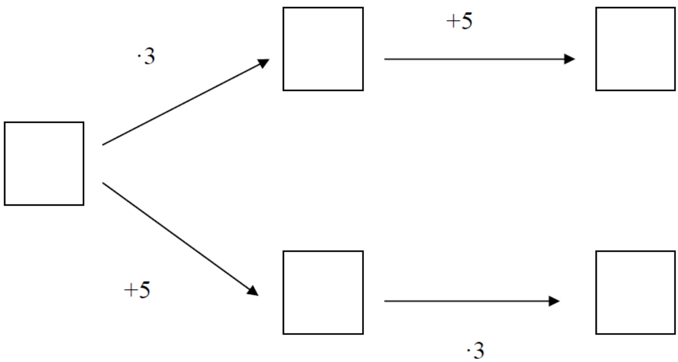

Figure 4: Problem 4 – Double computation chain

In the interviews conducted so far, we found that these problems carry potential for an algebraic approach by primary school students, and that they address the following components of algebraic thinking in particular: “relational thinking” (problem 1, problem 3), “generalising”, “dealing with operations” (problem 4), “dealing with change”, “working with unknowns” (problem 2).
On the other hand, it has already become clear that the problems can, in fact, distinguish different degrees of manifestation of algebraic thinking among the observed age group.

Due to the limited scope of this paper, we can only discuss selected results we obtained for Problem 3 during our preliminary study. It addresses the component "relational thinking", i. e. determining relationships between numbers, sets or relations. This is the case because by putting numbers into relation with each other ($359 = 362 - 3$, $219 = 215 + 4$, $14 = 5 + 9$), it is possible to avoid a time-consuming calculation of solutions, and on this basis, a calculation of the missing numbers in the respective terms on the right side of the equation.

The following categories may be useful to evaluate the students' work regarding this component:

1: Exercises a) and b) are partly solved incorrectly by means of written arithmetic.
2: Exercises a) and b) are solved correctly by means of written arithmetic.
3: Student makes opposite changes to numbers in exercise b) after giving advice on exercise a).
4: Student makes uniform changes to numbers in exercise b) after giving advice on exercise a).
5: Advantageous calculations applied in exercises a) and b).
A: Exercise c) is not understood.
B: Order of operations not followed in exercise c).
C: Exercise c) correctly solved by computation.
D: Number bonds (to 14) used in exercise c).

For the Year 4 students that participated in the preliminary study, who demonstrated very high (mathematically gifted), good and average mathematical performance, respectively, we obtained the following results for their work on Problem 3:

	Math. gifted students	Good students	Average students
Boys	5C, 5B, 5D, 5D	3D, 5C	2B, 2B
Girls	2C	3-	2B, 1C, 1B, 3B, 2B

Table 1: Categories for work on Problem 3

Table 1 clearly shows the differences between the individual students. It also becomes clear that this problem distinguishes between groups of students with varying levels of performance regarding the observed component of algebraic thinking.

Overall, the existing results of the current research phase also indicate that abilities concerning the several different components of algebraic thinking are surprisingly weak among students who were perceived to have not a special mathematical talent. The opposite is the case for mathematically gifted students in the same group.

Further problems used in this study as well as important results of our current research phase will be presented and discussed in the presentation.

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SUPPORTING STUDENTS TO OVERCOME CIRCULAR ARGUMENTS IN SECONDARY SCHOOL MATHEMATICS: THE USE OF THE FLOWCHART PROOF LEARNING PLATFORM

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The extent to which students are competent in identifying circular arguments in mathematical proofs remains an open question, as does how it might be possible to enhance their competency. In this paper we report on a study of learners encountering logical circularity while tackling geometry proof problems using a web-based proof learning support environment. The selected episodes presented in the paper illustrate how learners who have just started learning to construct mathematical proofs make various mistakes, including using circular arguments. Using the feedback supplied by the web-based proof learning support environment, and with suitable guidance from the teacher on the structural aspects of a proof, learners can start bridging the gap in their logic and thereby begin to overcome circular arguments in mathematical proofs.

INTRODUCTION

Bardelle (2010) provides an example of some undergraduate mathematics students in Italy being presented with the diagram in Figure 1 as a "visual proof" of Pythagoras' theorem. The students were asked to use the figure to help them develop a more formal written proof of the theorem.

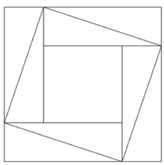


Figure 1: a "visual proof" of Pythagoras' theorem

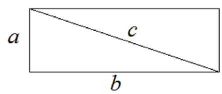


Figure 2: a rectangle from Figure 1

Bardelle relates how one student focused on the rectangles that surround the central square. By defining *a* as the short side and *b* the longer one (as in Figure 2), the student used Pythagoras' theorem to get $c=\sqrt{a^2+b^2}$ and thence, by squaring both sides, the student obtained Pythagoras theorem $c^2=a^2+b^2$. This is an example of a student using a circular argument or *circulus probandi* (arguing in a circle). It entails *assuming just what it is that one is trying to prove* (Weston, 2000, p75). In logic, circular reasoning is considered a fallacy as the proposition to be proved is assumed (either implicitly or explicitly) in one of the premises.

In a comprehensive consideration of the key questions for mathematics education research on the teaching and learning of proof and proving, Hanna and de Villiers (2008, p333) raise the issue of the extent to which students are competent in identifying circular arguments in proofs. They also ask how it might be possible to enhance such competency in students. In this paper we report on a study of learners working with logical circularity while tackling proof problems. Our research questions encompass how it is that they create a proof which has a logical circularity, and how they modify their thinking through to constructing a correct proof. To answer these questions we analysed selected episodes collected as students work on geometry proof problems using a web-based proof learning support environment (for more details of this web-based system, see Miyazaki et al, 2011).

CIRCULAR ARGUMENTS IN DEDUTIVE REASONING

Rips (2002) has argued that the psychological study of reasoning should have a natural interest in patterns of thought like circular reasoning, since such reasoning may indicate fundamental difficulties that people may have in constructing and in interpreting even everyday discourse. However, Rips claims that up until his study in 2002 there appeared to be no prior empirical research on circular reasoning. While Rips reports on a study of young adults, Baum, Danovitch and Keil (2008) report findings with younger students - indicating that by 5 or 6 years of age, children show a preference for non-circular explanations and that this appears to have become robust by the time youngsters are about 10 years of age.

While learners' preference for non-circular explanations may be robust by the time they are ten years old, within mathematics education Kunimune, Fujita and Jones (2010) report on data on Grade 8 and 9 pupils showing that as many as a half of Grade 9 students and two-thirds of Grade 8 pupils are not able to determine why a particular geometric proof presented to them was invalid; that is they could not see the logical circularity in the proof. Likewise in Germany, Heinze and Reiss (2004) report that from Grade 8 to 13 an unchanging two-thirds of pupils fail to recognise circular arguments in mathematical proofs. Such evidence illustrates that pupils are in need of considerable support in order to identify and overcome circular arguments in mathematical proofs. As Freudenthal (1971, p427) observed "you have to educate your mathematical sensitivity to feel, on any level, what is a circular argument".

THEORETICAL FRAMEWORK

We take as our starting point that a mathematical proof generally consists of deductive reasoning starting from assumptions and leading to conclusions. Within this reasoning process, at least two types of deductive reasoning are employed: universal instantiation (which deduces a singular proposition from a universal proposition) and syllogism (where the conclusion necessarily results from the premises).

In order to understand the structure of proof, students need to pay attention to elements of proof such as its premises and conclusions and their inter-relationships. Both Heinze and Reiss (2004) and McCrone and Martin (2009) identify appreciation of proof structure as an important component of learner competence with proof. In this paper we use the following levels of learner understanding of proof structure elaborated by Miyazaki and Fujita (2010):

- *Pre-structural*: this is the most basic status in terms of understanding of the proof structure, where learners regard proof as a kind of 'cluster' of possibly meaningless symbolic objects and they cannot see that within the structure of proof 'singular propositions' are those which are universally instantiated from 'universal propositions', that 'syllogism' is necessary to connect 'singular propositions', and so on.
- *Partial-structural*: given that a proof consists of elements of proof such as singular and universal propositions, deductive reasoning, and their relational network, if learners have started paying attention to each element, then we consider they are at the Partial-structural elemental sub-level. To reach the next level, learners need to recognise some relationships between these elements (such as universal instantiations and syllogism). If learners have started paying attention to each relationship, then we consider them to be at the Partial-structural relational sub-level.
- *Holistic-structural*: at this level, learners understand the relationships between singular and universal propositions, and see a proof as 'whole' in which assumptions and conclusions are logically connected through universal instantiations and syllogism (much like the 'warp' and the 'weft' when weaving textiles). Once learners have 'Holistic-structural' understanding, they should be able to start refining proofs, become aware of the hierarchical relationships between theorems, be able to construct their own proofs, and so on.

The Pre-, Partial-, and Holistic-structural levels of understanding of proof structure is summarised in Figure 3.

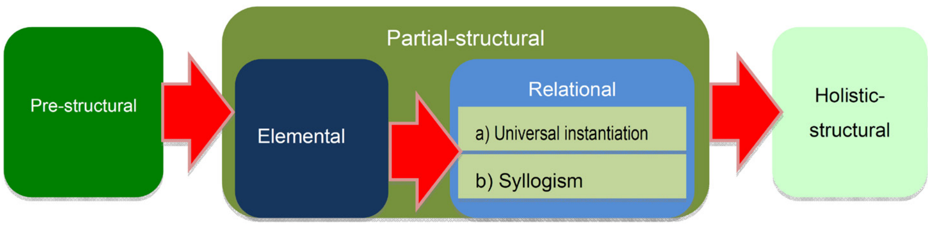


Figure 3: Pre-, Partial-, and Holistic-structural levels of understanding of proof

According to this framework of Pre-, Partial-, and Holistic-structural levels of understanding of proof, most learners who are just starting to learn proofs would be at either the Pre or Partial-structural level. In particular, if learners do not fully understand the role of syllogism, then they would be likely to accept or construct a proof which includes logical circularity.

METHODOLOGY

To investigate students' understanding of logical circularity in mathematic proofs, a web-based learning platform (hereinafter the system) was utilised (for details of this, see Miyazaki et al, 2011). The current version is online at:
http:// www.schoolmath.jp/flowchart_en/home.html

For this learning platform, flow-chart proofs are adopted (see Ness, 1962) and both open and closed problems in geometry are available to learners, including ones that involve the properties of parallel lines and congruent triangles. Learners tackle proof problems by dragging sides, angles and triangles to cells of the flow-chart proof and the system automatically transfers figural to symbolic elements so that learners can concentrate on logical and structural aspects of proofs. The geometry problems that student tackle when using the learning platform include both ordinary proof problems such as 'prove the base angles of an isosceles triangles are equal' (we call these 'closed' problems) and problems by which students construct different proofs by changing premises under certain given limitations (we call these 'open' problems). Each time the learners selects a next step in their flow-chart proof, the web-based system checks for any error via a database of possible next steps. If there is an error, the learners receive orderly feedback in accordance with the type of error (such as error in the deductive chain, error in selecting the appropriate theorem, error in the antecedent and the consequent of a singular proposition, and so on).

For data collection, a range of individual or grouped learners (up to 4) tackled one or more mathematical activities with the web-based system and their conversations were recorded by video camera and then transcribed. In the next section we report selected cases involving five learners: two high-attaining secondary school students aged 14 years old (WS1 and WS2) and three undergraduate primary trainee teachers (an individual, R, and a pair, J1 and J2). None of these learners had prior experience of mathematical proof in geometry.

DATA ANALYSIS AND DISCUSSION

In the problem in Figure 4 (lesson 2-b00), the learners are asked to prove 'AB=CD', with reasoning in both universal instantiation and syllogism being required to deduce a proper conclusion. This is an example of an open problem in that while learners have to use 'AO=CO' for their proof, they can decide for themselves which other properties to use. In this problem, they could either consider AO=CO, BO=DO and $\angle AOB=\angle COD$ (the SAS condition) or use AO=CO, $\angle AOB=\angle COD$ and $\angle OAB=\angle OCD$ (the ASA condition).

Case 1: after practicing with an introductory problem, and understanding that there are three conditions that can be used to say that two triangles are congruent, two 14-year-old students, WS1 and WS2, undertook the problem in Figure 4.

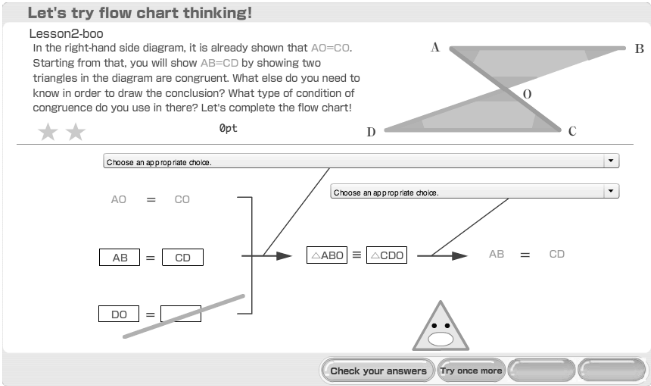
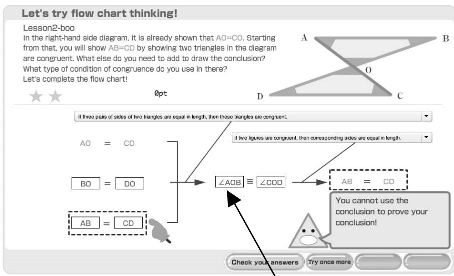


Figure 4: System interface and Lesson 2-b00

Without any hesitation, their first attempt involved using the SSS condition as follows (I: Interviewer)

50	WS2	That one and that one (BO and DO)? That one looks bigger than that one. Is it that one (∠AOB=∠COD)? [student chooses SSS condition, and checks answer] No.



51	WS1	I don't think that angle is … [indicating]
52	I	What does it say?
53	WS2	[Reading the hint] You cannot use the conclusion to prove your conclusion.
54	I	What do you want to prove?
55	WS1	We want to prove that the three pairs of sides …..I don't know, I am really confused.

They made a mistake (line 50) as they put ∠AOB=∠COD are congruent, rather than ΔOAB and ΔOCD. More importantly, they failed to notice that they should not use 'AB=CD' in their proof. This is evidence that they did not have good understanding of universal instantiation (line 55) or of logical circularity (line 50). The system highlighted the use of logical circularity by showing a box saying "you cannot use the conclusion to prove your conclusion". After receiving this hint from the system, and with additional support from the interviewer, the students started considering that ‘AB=CD’ should not be used in their proof. With this they began to understand, as shown below in the dialogue immediately below, why AB=CD should not be used.

86	WS1	It is the same as that. [reviewing WS2’s answer] You have done AB=CD again!
87	WS2	Why can’t we do that?
88	WS1	Because it is the same conclusion.

After realising that AB=CD should not be used, they finally constructed a correct proof. Nevertheless, the above example illustrates that understanding the meanings and roles of premises and conclusions are difficult for learners who have just started learning mathematical proof. Moreover, from the structure of proof point of view, our evidence shows that learners who cannot see the whole structural relationships between premises and conclusion (namely that they are not at the Holistic structural level) cannot identify the logical circularity. In order to identify logical circularity as a serious error, learners need to understand at least the role of syllogism which connects premises with conclusions. It means learners need to understand the aspect of syllogism included in the relational Partial-structural relational sub-level (see Fig 3).

Case 2: in the episode below, student R, a first year student on a primary teacher training course, first considered that it would be possible to use SSS condition as a way to tackle the open problem to prove ‘AB=CD’. This indicates that R is lacking understanding of logical circularity. After making several mistakes, including logical circularity, student R finally reasoned why it was not possible to use SSS (see lines 34 - 40 below). This shows that student R was in the upper level of the Partial-structural relational level involving the understanding the aspect of syllogism at least.

34	R	I don’t think anymore answers.
35	I	Are you confident to say so?
36	R	Yes.
37	I	If you choose ∠AOB&∠COD, and ∠ABO&∠CDO, then…
38	R	We need to use BO and DO but …
39	I	No, we can’t use them as AO=CO is already assumed. Also we can’t use AB=CD, because this is…
40	R	What you are trying to find! [laughs]

Case 3: J1 and J2, two first year students on a primary teacher training course, are towards the end of their work on the proof problem. In the extract below, they are not

only considering why they cannot use the SSS condition for the problem (lines 149-151 below), but also eliminating other possibilities for answers (lines 152-157). This illustrates their capacity to identify logical circularity in proofs, and that their understanding of structure of proof is almost at the Holistic structural level as there is evidence that they have started grasping the relationship between premises and conclusion.

147	J1	Um, try again?
148	J2	You could do all the …
149	J1	All the sides?
150	J2	Yes… actually no, because..
151	J1&J2	You are trying to prove [AB=CD] …
152	J1	And if you can't use this line [AB] then we can't use the other angle… because it is not included…
153	J2	You mean those [∠ABO&∠CDO]?
154	J1	Yes, it is not included [as AB cannot be used]… and we've already got others…
155	J1	How about AO-∠OAB-AB?
156	J2	You cannot use these, because…
157	J1	Because these ones [AB&CD] which we are trying to prove…

This example shows that students J1 and J2 could overcome the logical circularity gradually by considering possible combinations of premises and conclusion and checking whether their proof fell into logical circularity or not. This might mean that the kinds of activity available with the web-based flow-chart proof system are useful to understand the whole structural relationship between premises and conclusions more deeply, to encourage learners to shift the level of the understanding of proof structure, and that this may lead to them, in the end, overcoming the error of logical circularity.

CONCLUSIONS

The selected episodes presented in this paper illustrate how learners who have just started learning to construct mathematical proofs make various mistakes, including using a conclusion to prove the same conclusion. Our conjecture is that the cause of this is their incomplete understanding of whole structure of proof, especially their lack of understanding of the role of syllogism. The web-based learning environment with its open problem situations using flow-chart-type proof, as we show in this paper, can reveal learners' naive status of understanding, in particular their lack of understanding of syllogism (for example, cases WS1 and WS2, and R).

While it is appears difficult for learners to consider why logical circularity cannot be used in a proof (see the example of WS1 and WS2), to overcome such difficulties it is important for teachers to encourage learners to attend to the structural relationships between premises and conclusion and how they could be bridged (via syllogism). As support, the feedback supplied by the web-based proof system provides guidance on what help might be given learners to help develop their understanding. By focusing on the structural aspects of a proof, the learners start bridging the gap in their logic in syllogism (see example R, lines 34 - 40). For some learners (for example, J1 and J2), by using the open problem situation, logical circularity is eliminated by considering possible combinations of premises and conclusion (see case J1 and J2, lines 141-157). This suggests that both considering possible combinations of premises and conclusion, and checking whether the proof falls into logical circularity or not, are useful for overcoming errors of logical circularity.

ACKNOWLEDGEMENT

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STUDENTS' MEANING MAKING IN A COLLABORATIVE CLASSROOM PRACTICE AS INITIATED BY TWO TEACHERS

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This paper reports the nature of classroom practice that afforded students' meaning making at an upper secondary mathematics classroom in Norway. The participation of both teachers and students in the collaborative classroom practice they jointly establish is outlined. A longitudinal and person-in-practice view sheds light not only on the meaning producing foreground that was initiated, but also the nature of its growth. An artefact of instructional practice of two teachers Olaf and Knut is thus evidenced. In this there is opportunity to appreciate mathematical content, pedagogy and students' thinking in an integrated manner – making such knowledge useful and usable by practising as well as prospective teachers of mathematics.

INTRODUCTION

Drawing upon a year-long doctoral study at an upper secondary mathematics classroom, I report on one of four themes found grounded in my data (Gade, 2006). These themes emerged in response to my research question – Within a collaborative teaching-learning practice in the mathematics classroom, how do artefacts and activity mediate: meaning making in participation, consolidation of meaning made, development of problem solving know-how, and cooperation in problem solving. In reporting on the first of these themes, I describe the nature of classroom practice that was initiated by Olaf and Knut who shared teaching in the class I conducted my study; and whose prior objective it was to have their students cooperate in small groups within instruction. I label the classroom practice they established jointly with their students as a collaborative one, since it provided students opportunity to cooperate not only with each other in their groups, but also with students from other groups. At times groups of students presented solutions to tasks they had cooperated upon to other groups in the classroom. I explore how such practice was conducive to students meaning making in relation to the mathematics being demanded of them.

LITERATURE REVIEW

The need to attend to practices in mathematics classrooms has been recognised in PME and wider literature (e.g. Seeger *et al*., 1998; Boaler, 2003; Forman, 2003; Morgon, 2009). Socio-cultural-historical and activity or CHAT perspectives have also enabled classroom research to move beyond claims about individual cognition alone. A larger analytical zoom and holistic understanding has enabled research to inform a person-in-practice view of students and their teachers (Lerman, 2001). A relational view of mathematics, students, teachers and material aspects of practice has

2011. In Ubuz, B. (Ed.). Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, pp. 361-368. Ankara, Turkey: PME.

in turn led to insight about the nature of participation and negotiation of meaning that is capable of leading to greater knowing in classrooms. Greeno (2003) has argued that such a situative view has consequences, not only for what students learn but the kind of learners that students become. Attending to changes in patterns of discourse, it has also become possible to address critical aspects in mathematics education, especially those inherent in the complexities of everyday classrooms. The study of any sphere of practice which provides experience and enables students to bring forth their intention and foreground their personal meaning making has thus been argued for as desirable (Skovsmose, 2005). The demand to understand mathematical content, pedagogy and students' thinking in an integrated manner has also been recognised (Silver, 2009). Such practice-based knowledge has potential, Silver points out, to be useful and usable by practising and prospective teachers, as well as in professional development. Though the theoretical benefits of such knowledge have been recognised, it is time Silver argues, to reap these benefits in empirical terms. I respond to this demand by reporting on the nature and growth of an instructional practice that Olaf and Knut initiated and steered in my study.

THEORETICAL FRAMEWORK

CHAT perspectives offer constructs that enable appreciation of the participation of individuals in practices. Built upon Vygotskian assumptions, which are fairly broad and under current scrutiny, these perspectives consider the social environment as the provider of cultural tools and resources that mediate psychological processes and determine development. Resting upon innate biological functions, such psychological processes identified as higher mental functions, are mediated by cultural tools and resources. CHAT perspectives thereby analyse how higher mental functions emerge and became functional in individuals. These perspectives premise human engagement in practical activity as the means by which individuals transform themselves, at the same time as they are transforming external reality (Steksenko 2004). Analysis of various activities in which individuals objectify their psychological processes as they participate, thus forms methodological basis. Participation as unit of analysis in my study is informed also by Rogoff (2003) who distinguishes this unit with individuals' actions such as remembering or planning, where transformations in these are dynamic and indicative of participation in cultural practices. Laying emphasis on relationships rather than transformations Lave (1993) identifies participation, in addition, with relations entered into in local social practices. Her emphasis on relationships is not on those entered into between participants and their contexts, but on those relations that contextualise the way people act both within and across those very contexts.

Participation with meaning of individuals in either a cultural or social conception is guided further by Bruner (1996) who argues that the role of culture in general and education in particular, is the idealisation and consolidation of personal meaning into academic forms. In treating education as embodiment of culture and not a preparation for participating in one, Bruner says it is thus demanded of pedagogy to select ways of negotiating the academic meaning that is to be made. Teachers and students are thereby treated as if they had intentional states and considered as acting with purpose. Extending Bruner's arguments, Olson (2003) has singled out the need for teachers to make timely decisions within instruction so as to bring together students' minds as well as various cultural tools and available resources. In particular Olson draws attention to the formation of joint intentions between teachers and students in practice - characterised by their sharing a common vocabulary and the eventual taking over of responsibility by the students of their own learning. In adopting a person-in-practice (Lerman, 2001) and situative (Greeno, 2003) view of the mathematics classroom, it is my intention to evidence the nature of one such practice steered by Olaf and Knut. In this I outline how their students transformed themselves within instructional activity (Stetsenko, 2004) and formed joint intentions with their peers and teachers through participation (Olson, 2003). I turn to methods adopted for studying such participation which resulted in students taking over responsibility of their own learning.

METHODS

The principal means with which I investigated Olaf and Knut's classroom practice was ethnographic. Such a stance not only drew upon my experience as a teacher but also became a way to embed methods that were necessary for studying various other units of analysis found necessary in my study. Enabling me to bring ground to figure such an approach was question driven, wherein I could match evolving models about teaching-learning with events that transpired in subsequent practice (Weisner, 1996). Of the three specific methods I utilised – field notes, survey response by students to group-tasks, and transcriptions of problem solving conducted with student groups; I now explain my collection of field notes that informed participation as unit.

I made field notes as a participant observer during the length of data collection. This enabled me to appreciate Olaf and Knut's bilingual instruction in Norwegian and English of 32 students seated in 8 small groups. Making field notes enabled me to record not only that discourse which transpired in English, but also make additional notes about material aspects that accompanied teaching-learning. Inclusive of who was speaking, from which group and whether one was stationed at their desk or blackboard, I obtained a thick description of actions, events and cultural artefacts that constituted instructional practice (Geertz, 1973). The corpus of data collected was thus naturally occurring and included my interpretation of the experience of teachers, students and myself (Silverman, 2001). While I draw extensively on field notes in this report, I acknowledge having arrived at current interpretations with multiple levels of triangulation by deploying units of analysis other than participation as well. When Olaf taught alone at the beginning of the academic year, I made notes by seating myself to one side of his classroom. Upon Knut joining teaching duties, which coincided with commencement of the second chapter of the textbook, I sat beside one particular student group. This enabled me to view classroom teaching-learning as much as was possible from that group's point of view. I sat with a new group with every subsequent chapter of instruction and report in this paper from events that transpired in the first three chapters.

RESULTS AND DISCUSSION

I sketch the nature of classroom practice established within my larger study in three sections. First, when Olaf taught alone, next when he and Knut initiated group work, ending with students' discussion and formalisation of rules for group cooperation.

A single teacher

Olaf began instruction with the chapter titled *Number Understanding* while stationed near the blackboard. With his students seated in groups around tables they had pulled together, Olaf's proximity to his students and their workings was restricted. Beyond greeting students on the first day of instruction Olaf began his instruction as below:

Olaf: Turn to page 14 ... there are some rules in the box

Olaf: [A while later] If you have a problem, box first, partners next, then me.

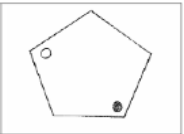
The very brief exchange above is indicative of the nature of relationships that Olaf was forging with his students in his classroom. Aware of talk that could arise when students worked in groups, Olaf was guiding the manner in which his students were to speak with each other and seek guidance when in doubt. The rules in the box that Olaf drew attention to, demonstrated how one could obtain equivalent fractions and how one could reduce a fraction to its simplest form. Olaf's drawing the attention of students to these rules had a two-fold purpose within classroom practice. First, these rules reminded students of procedures they would have been familiar with even prior to his classroom. Second, Olaf signalled his intention of having students utilise rules even before seeking assistance from peers in their group or even him. It was with such advice that Olaf embedded classroom practice with his intentions (Olson, 2003). Being new to his classroom, Olaf's students were now participating in an instructional practice that he was laying out. Their making of meaning in mathematics was made between the cultural resource of the textbook, their peers and him (Greeno, 2003). The participation of Olaf and his students was therefore not independent, but anchored in a specific kind of classroom practice that Olaf had initiated.

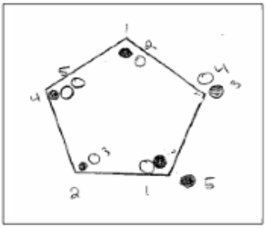
A team of teachers

There were several changes in classroom practice when Knut joined teaching at the commencement of chapter *Equations and Proportionality*. In line with their stated objective of having students cooperate in groups, Olaf and Knut conducted two tasks *When Together* and *How Heavy* in consecutive sessions of teaching-learning. The tasks and sample solutions evidenced in Table 1 are indicative of two aspects. First, that the use of diagrams in the two tasks was different. Where in the first, the given diagram was used to cooperate, by the second, students had to provide a diagram or equation in order to cooperate. Second, group cooperation was also different. Where in the first, cooperation was initiated by Olaf and Knut, by the second, students took for granted and consolidated group cooperation. It was in the conduct of these tasks that Olaf and Knut realised their objective of having students cooperate in small groups, which was to become the norm of instructional practice in the classroom.

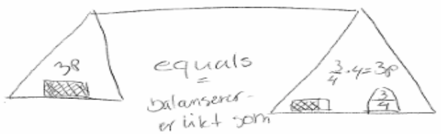
When together

- In the pentagon are two dots, black and white, on the move.
- The black moves two corners counter clockwise. The white moves three corners clockwise.
- After how many moves are the two dots together?





After the fifth move the dots are back in their starting positions! Answer: They can never meet.



How heavy?
If a brick balances with three-quarters of a brick and three quarters of a pound, then how much does the brick weigh?

Table 1: *When together* and *How Heavy* – Tasks and sample solutions

The changes in classroom practice just outlined changed the participation of Olaf, Knut and their students in four distinct ways. First, corresponding to changes in the manner students worked with say diagrams, I record Olaf and Knut noticeably work as a team and complement each other while say teaching at the blackboard. These transformations corresponded to their continued participation in the changing practice (Rogoff, 2003). Second, Olaf and Kunt's students also had opportunity to bring forth personal meaning and knowledge they had prior to participation in this sphere of practice (Skovsmose, 2005). Third, in privileging the use of a simple equation by one student group, Olaf and Knut led their students to utilise forms of societal knowledge that were acceptable beyond the context of the classroom (Lave, 1993). By this Olaf and Knut guided various versions of meaning students had about balancing to greater academic forms, as was the case with a simple equation (Bruner, 1996).

Group cooperation is formalised

From students in my study being asked to turn to a particular page in the beginning of the year, their manner of participation gained far greater freedom by the third chapter *Scale factor in similar figures*. Illustrating one such instance, I relate how Olaf both accepted and acknowledged the accuracy of personal meaning made by Levi - which was independent of the one Olaf was discussing with Levi's other classmates.

Olaf:	What is the scale factor of the side?
Jan:	Three
Tove:	Three
Levi:	Or one by three [Belonging to the group I was sitting beside]
Olaf:	What is the scale factor of area?
Researcher:	[Records Olaf to extend this discussion with Levi's other classmates in relation to the scale factor of area, as well as that of volume]

Olaf:	So you were right Ulrik ... So if you have the volume of one of them we can calculate the value of the other
Levi:	What if we do it the other way? [Persisting to question Olaf]
Olaf:	If we know volume of the larger we find the volume of smaller
Olaf:	[Demonstrating correctness of Levi's scale factor on the blackboard]
Olaf:	Good question [Addressing and accepting Levi's version of scale factor]

My sitting beside Levi's group allowed me to observe and record how Levi's version of scale factor was independent of the one being discussed by Olaf with the whole class. While Olaf and Levi's classmates were working with a numerical value of three as scale factor, Levi was working with its reciprocal. In presenting the above extract I evidence the manner in which Levi pursued Olaf, seeking to ascertain the correctness of his version of scale factor - one which Olaf accepted and demonstrated as accurate on the blackboard. From offering explicit instructions with regards to how students were to make meaning in the first chapter, by the third chapter Olaf acknowledged the personal meaning that Levi had independently made. Illustrative of the kind of timely decisions that Olson (2003) said a teacher needed to make within one's own instructional practice, my final extract shows how Olaf was working with one version of scale factor as cultural tool (with Jan, Tove and Ulrik) and its reciprocal as another acceptable cultural tool (with Levi). In guiding the utilisation of different forms of academic meaning (Bruner, 1996) and the formation of corresponding higher mental functions (Stetsenko, 2004) Olaf's role by this time in practice had now shifted from being custodian, to arbitrator of alternate kinds of mathematical meaning being made by his students. By the end of this chapter, Olaf and Knut also had all students groups discuss arguments and counter-arguments in relation to working in small groups, so as to formulate ways in which such manner of working was best possible. I present guidelines that the eight student groups together agreed upon in Table 2:

Cooperative learning in mathematics	
1	Everyone must be treated with respect
2	Everyone must contribute
3	All ideas must be considered by the group
4	Everyone must be aware of what transpires before the group moves ahead
5	Everyone must be able to present the work of the group
6	Everyone must ask the others in the group before seeking help from the teachers

Table 2: Students guidelines in relation to group cooperation

Put up in large letters on their pin-up board these rules became part of the new norm in instructional practice. In Olaf and Knut thereafter encouraging students from across groups to present their group work either at the blackboard or to each other, I witnessed classroom practice to progressively became a collaborative one.

CONCLUSION

With participation as unit of analysis it has thus been possible to appreciate a person-in-practice view of the classroom (Lerman, 2001). Such analysis viewed the efforts Olaf and Knut made to guide the meaning being made by students within their sphere of practice (Greeno, 2003; Skovsmose, 2005). There was opportunity for either, to participate in the intentions of others (Olson, 2003). Olaf was first seen establishing his own intentions. With Knut he then guided the sharing in groups of the meaning students were making with their peers, affording opportunity for them to participate in their own as well as others intentions. Finally, Olaf's student Levi had occasion to externalise the meaning he had personally made. This was representative, more generally, of independent meaning making by students and coincided with a shift in responsibility to them for their own learning. I summarise the growth of the instructional practice that Olaf and Knut so initiated as follows:

Chapter number and topic	The collaborative practice
1: Number Understanding	Establishment by the teacher of his intentions
2: Equations and proportionality	Participation by students in their and other's intentions
3: Scale factor in similar figures	Participation by students with independent intention

Table 3: Nature of growth of Olaf and Knut's classroom practice

A person-in-practice study has had two implications for my ongoing research. First, based on opportunities that students had for imitation, I have since shown how a *zone of proximal development* or *zpd* was formed when students cooperated as well as collaborated within such an instructional practice (Gade, 2010). The corresponding development of higher mental functions resulted in students becoming independent. Second, analysing day-to-day material practices in relation to meaning making has provided me with researcher strategy that is conducive to the conduct and sustenance of action research (Gade, 2011). While I respond in this report to Silver's (2009) call of the need for empirical examples of practice-based studies, I also illustrate how the mathematical content taught; pedagogy and students' thinking were interrelated in one such practice. I thereby underscore the need to recognise classroom practices in general and the role that these may have in the meaning being made by teachers and students in particular classrooms. Towards this, I have argued for the benefits of one such practice that two teachers Olaf and Knut had instituted in my study.

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TOWARDS A COMPREHENSIVE THEORETICAL MODEL OF PRE-SERVICE TEACHERS' CONCEPTUAL UNDERSTANDING OF FUNCTIONS [1]

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The aim of this study is twofold, to confirm a model for the structure of the conceptual understanding of functions related to multiple representational flexibility and problem solving ability and to investigate its stability across pre-service teachers from two countries. Confirmatory factor analysis (CFA) affirmed the existence of five first-order factors representing the concept definition and examples, the recognition of the concept, the conversions, the vertical transformations and problem solving, two second-order factors representing multiple representational flexibility and problem solving ability and a third-order factor that refers to the conceptual understanding of functions. Results provided evidence for the invariance of this structure across the two countries.

INTRODUCTION AND THEORETICAL FRAMEWORK

There is a basic difference between mathematics and other domains of scientific knowledge as the only way to access mathematical objects and deal with them is by using signs and semiotic representations. Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2006).

Nowadays the centrality of different types of external representations in teaching and learning mathematics seems to become widely acknowledged by the mathematics education community. Furthermore, the NCTM's Principles and Standards for School Mathematics (2000) document includes a new process standard that addresses representations and stresses the importance of the use of multiple representations in mathematical learning. In fact, recognizing the same concept in multiple systems of representations, the ability to manipulate the concept within these representations as well as the ability to convert flexibly the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh, Post, & Behr, 1987) and allow students to see rich relationships (Even, 1998). Moving a step forward, Hitt (1998) identified different levels in the construction of a concept, which are strongly linked with its semiotic representations. The particular levels are as follow: 1) incoherent mixture of different representations of the concept, 2) identification of

[1] This report is related with the research project "Ability to use multiple representations in functions and geometry: the transition from middle to high school" (0308(BE)/03) founded by the Research Promotion Foundation of Cyprus.

different representations of a concept, 3) conversion with preservation of meaning from one system of representation to another, 4) coherent articulation between two systems of representations, 5) coherent articulation between two systems of representations in the solution of a problem. In this research study the term multiple representational flexibility is used extensively and it is defined as the ability to switch mental sets in response to within- and between-representation alterations (recognition, treatment, conversion) of the same mathematical object. In other words, it is assumed that multiple representational flexibility refers to switching between different systems of representations of a concept, as well as to recognizing and manipulating the concept within multiple representations (Gagatsis, Deliyianni, Elia & Panaoura, 2010).

In this research study we incorporated a synthesis of the ideas articulated in previous studies on learning with multiple representations to capture pre-service teachers' processes in multiple representations tasks. This may enable us, firstly to gain a more comprehensive picture of function understanding related to multiple representational flexibility and problem solving ability; secondly, to understand pre-service teachers' multiple representational flexibility in a more coherent way; and thirdly, to find out more meaningful similarities in Cypriot and Italian pre-service teachers' representational thinking and problem solving ability. In particular, two hypotheses were tested: a) multiple representational flexibility and problem solving ability influence the conceptual understanding of functions and b) there are similarities between Cypriot and Italian pre-service teachers in regard with the structure of their conceptual understanding of functions.

METHODOLOGY

The participants of this research study were 279 Cypriot and 206 Italian pre-service teachers. The subjects were admitted to the University of Cyprus and to the Universities of Bologna and Palermo on the basis of competitive examination scores. The above investigation is conducted in two countries in order to explore if there are differences between the Cypriot and Italian pre-service teachers concerning their cognitive structure of the various dimensions of the conceptual understanding of function. The participation of pre-service teachers from two countries will give further validation to the model. The fact that it is a comparative study is quite important, since these two countries have cultural similarities. It is noteworthy the fact that the impact of cultural tradition is highly relevant to mathematics learning. However, despite cultural similarities, differences are observed in the educational systems of the two countries. Two tests, consisted of nine and fourteen task, were administered to the teachers by the researchers in two 90 minutes sessions. The tests that were constructed in order to examine the hypotheses of this study included:

1. Five tasks demanding a definition or examples of the concept of function (D1, D2, D3, D4, Ex).Examples of these tasks are:

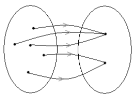
Task D2: Does there exist a function all of whose values are equal to each other? Explain your answer.

Task D3: Can f be a function, if $f(-2)=3$ and $f(-2)=0$? Yes or No

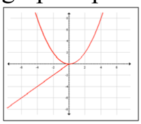
Task Ex: Give two simple examples from the applications of functions in everyday life.

2. Four tasks involving recognition of functions given in different modes of representation. There were given five Venn diagrams (Red1-Red5), six graphs (Reg1-Reg6), six symbolic expressions (Res1-Res6) and four verbal expressions (Rev1-Rev4). Examples of these tasks are:

Task Red1: Examine if the correspondence presented in the form of a Venn diagram is a function. Give an explanation.



Task Reg1: Examine if the following graph represents a function. Explain.



Task Res1: Examine whether the following symbolic expression may define a function and justify your answer.

$$5x+3=0$$

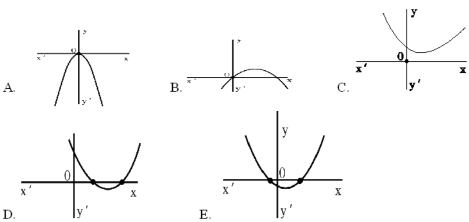
Task Rev1: Explain whether we define a function when:

a) In the set of the girls of a class, we correspond a girl with different classmates of hers (George, Homer, Jason, Thanasis, etc.) with whom she will probably dance at a party.

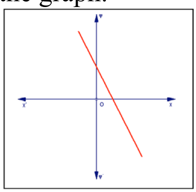
3. Four tasks involving six conversions, three of them from an algebraic expression to a graphical representation and the other three from a graphical representation to an algebraic expression (Coag1, Coag2, Coag3, Coga1, Coga2, Coga3). Examples of these tasks are:

Task Coag1:

The function $ax^2 + bx + c$ is given. For this function $a \neq 0$ and $a.c < 0$.
Which of the following graphs represents the above function?



Task Coga1: Choose the function that corresponds to the graph.

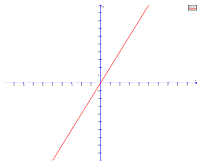


a) $y+5x=0$ b) $y=-5x-2$

c) $y+3=2x$ d) $y+3x=1$

4. Four tasks involving vertical transformations of functions. In each task, there were two linear or quadratic functions. Both functions were in algebraic form and one of them was also in graphical representation. There was always a relation between the two functions. Teachers were asked to interpret graphically the second function. (Trans1, Trans2, Trans3, Trans4). An example is:

Trans1: In the following diagram $y = 2x$ is given. Draw the function $y = 2x + 1$.



5. Six complex problems with functions. Examples of these tasks are:

Task Pr3: The function $f(x) = ax^2 + bx + c$ is given. Numbers a, b and c are real numbers and the $f(x)$ is equal to 4 when $x = 2$ and $f(x)$ is equal to -6 when $x = 7$. Find how many real solutions the equation $ax^2 + bx + c$ has and explain your answer.

Task Pr4: A parachutist jumps from an airplane which is in 3000 m height (above the earth). The parachutist falls with stable speed 30 m/s. (a) Express the parachutist's height as function of time, (b) Draw the graph of the above function, (c) Find the parachutist's height (from earth) 1 minute after his/her fall and (d) In what height the parachutist will be 20 minutes after his/her fall? (Give an explanation).

Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning pre-service teachers' answers to the tasks were codified with D (Definition), Ex (Example), Re (Recognition), Co (Conversions), Trans (Transformations) and Pr (Problems), followed by the number indicating the exercise number.

RESULTS

In order to explore the structure of the various dimensions of the conceptual understanding of function a third-order CFA model for the total sample was designed and verified. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI (Comparative Fit Index) and RMSEA (Root Mean Square Error of Approximation). The following values of the three indices are needed to hold true for supporting an adequate fit of the model: x^2/df < 2, CFI > .9, RMSEA < .06.

A series of models were tested and compared. Specifically, the first model involved only one first-order factor associated with all of the tasks. The fit of this model was poor [x^2(276) =2133.13; CFI=.74; RMSEA=.12, 90% confidence interval for

RMSEA=0.116-0.126], indicating that a single common factor is not sufficient to describe the solution of all the tasks in the test.The second model involved five first-order factors and one second-order factor on which all of the first-order factors were regressed. The first-order factors stand for the concept definition and examples, the recognition of functions given in a diagrammatic, a graphical, a symbolic and a verbal expression, the conversions from a graphical to an algebraic representation and vice versa, the vertical transformations and problem solving. The second-order factor stands for the conceptual understanding of functions. The fit of this model was also poor [$x^2(270)$ =1203.01; CFI=.87; RMSEA=0.087, 90% confidence interval for RMSEA=0.082-0.092].

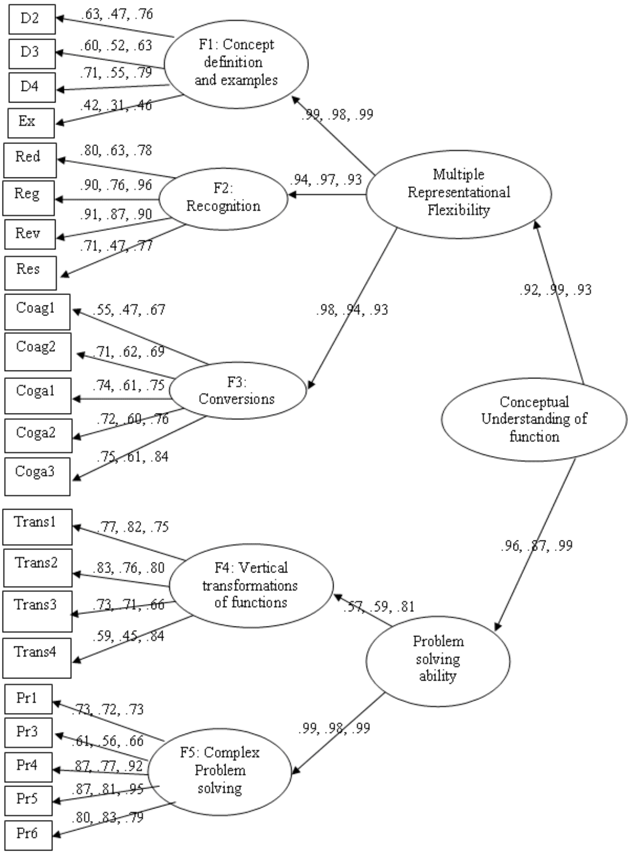


Figure 1. The confirmatory factor analysis model accounting for performance on the tasks of both tests by the whole sample, the Cypriot and Italian pre-service teachers separately

Note: Three tasks were omitted due to low loadings (D1, Coag2, Pr2)

Figure 1 presents the results of the elaborated model, which fits the data reasonably well [x^2(178) =360.92; CFI=0.97; RMSEA=0.047, 90% confidence interval for RMSEA=0.040-0.054]. The third-order model which is considered appropriate for interpreting the conceptual understanding of function, involved five first-order factors, two second-order factors and one third-order factor. The two second-order factors that correspond to the multiple representational flexibility and problem solving ability, respectively, regressed on a third-order factor that stands for the conceptual understanding of function. On the second-order factor that stands for the multiple representational flexibility three first-order factors (F1-F3) are regressed. The first first-order factor (F1) referred to the tasks involving the definition end examples of the concept of function, the second first-order factor (F2) to the recognition of functions given in various representations and the third first-order factor (F3) to conversion tasks from an algebraic to a graphical representation of function and vice versa. The first-order factor F1 to F3 loadings strength revealed that multiple representational flexibility constituted a multifaceted construct. Thus, the findings revealed that concept definition and examples, the recognition of the concept given in various representations and the conversions from a graphical to an algebraic representation of the concept and vice versa have a differential effect on the multiple representational flexibility concerning the concept of function.

On the second-order factor that corresponds to problem solving ability two first-order factors (F4, F5) were regressed. The first first-order factor (F4) involved the vertical transformations of functions and the second first-order factor (F5) consisted of the complex problems. Therefore the results indicated that vertical transformations of functions and the complex problems have an effect on problem solving ability. It is noteworthy that complex problems loadings are higher than the respective vertical transformations loadings, indicating that in order to be solved extra mental processes are required since more complicated processes are demanded. The two second-order factors that correspond to the multiple representational flexibility and to the problem solving ability regressed on a third-order factor that stands for the conceptual understanding of function. Their loadings values are almost the same revealing that pre-service teachers' function understanding is predicted from both multiple representational flexibility and problem solving ability.

To test for possible differences between the two countries in the structure described above, multiple-group analysis was applied, where the higher order model was fitted separately on each group. The model was first tested under the assumption that the relations of the observed variables to the five first-order factors would be equal across the two groups. The fit of this model was quite good [x^2(369) = 548.29; CFI=.97; RMSEA=.046, 90% confidence interval for RMSEA=0.038-0.054]. In order to achieve an improvement of the model some of the equality constraints were released. Releasing some of the constraints resulted in a considerable improvement of the model fit [x^2(366) = 508.17; CFI =0.98; RMSEA = 0.041, 90% confidence interval for RMSEA=0.032-0.050]. Although the same structure holds for the two groups, in the multiple group model (see Figure 1), many of the factor loadings are stronger in the group of the Italian pre-service teachers. This finding indicated that the dependence of the conceptual understanding of function varies across the two groups.

CONCLUSIONS

The main purpose of this study was twofold, to test whether multiple representational flexibility and problem solving ability have an effect on function understanding and to investigate its factorial structure within the framework of a CFA, across pre-service teachers from two countries. The results provided a strong case for the important role of the multiple representational flexibility and problem solving ability in Cypriot and Italian pre-service teachers' understanding of the concept of function. Specifically, CFA showed that two second-order factors are needed to account for the flexibility in multiple representations and the problem solving ability. Both of these second-order factors are highly associated with a third-order factor representing the conceptual understanding of function. CFA also showed that three first-order factors are required to account for the second-order factor that stands for the multiple representational flexibility and two first-order factors are needed to explain the second-order factor that represents the problem solving ability. This finding is in line with the results of previous studies that underline the important role of multiple representations (Even, 1998; Lesh et al., 1987) and problem solving (Schoenfeld, 1992) in the understanding of mathematical concepts. Furthermore, the important relation between the representational flexibility and problem solving is highlighted, verifying the results of previous studies (Gagatsis et al., 2010). Particularly, Gagatsis and Shiakalli (2004) and Hitt (1998) claimed that the ability to translate from one mode of representation to another is closely related with function problem solving.

On the second-order factor that stands for the multiple representational flexibility the first-order factors referring to the concept definition and examples, the recognition of the concept given in various representations and the conversions from an algebraic to a graphical representation of the concept and vice versa are regressed. The important role of recognition, treatment and conversion in representational flexibility was also highlighted in other studies (Gagatsis et al., 2010; Duval, 2006). In this study a new dimension of the multiple representational flexibility emerged, that is the concept definition and examples of functions and this was expected since in order to give a definition and examples of the concept pre-service teachers had to use and flexibly manipulate various representations. On the second-order factor that stands for problem solving the first-order factors referring to the solution of complex problems and the vertical transformations of functions were regressed. This is in line with the results of Lage and Gaisman-Trigueros (2006) that showed that transformations of functions, that are considered to be problem solving tasks and are strongly related with the problem solving ability since they can be used in the solution of many problems, are also strongly related to the understanding of the concept of function.

Furthermore the results of their study showed that flexibility with the use of different representations is highly related with the transformations of functions.

It is noteworthy that the structure of the processes underlying the function understanding is the same across the two countries. Even though some factors loadings are higher in the group of the Italian pre-service teachers, probably due to the differences exist in the educational systems of the two countries, the results provided evidence for the stability of this structure. This fact gives further validation to the model emerged in this study. The results of this study have direct implications for teaching and assessment. One must remember that in order to teach functions, it is important to include the different dimensions emerged in this model. It seems that there is a need for further investigation into the subject. In the future, it is interesting to conduct the same research with students attending middle and high school and examine whether the model for the conceptual understanding of function proposed here applies and remains invariant for these students.

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GENERAL VS. MATHEMATICAL GIFTEDNESS AS PREDICTORS OF THE PROBLEM SOLVING COMPETENCE OF FIFTH-GRADERS

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Project MALU (Mathe-AG an der Leibniz Universität) is a mathematical enrichment program for fifth-graders. Our research interest lies in investigating differences between the cooperation of exceptionally vs. normally gifted students, see the research report of Lange in these PME proceedings. To that end, it is appropriate to select first a sample of differently gifted students from a representative sample. Certainly a test can be only one method of assessment for the multi-faceted construct of mathematical giftedness (Sriraman 2008), but it will suit the need to differentiate rather quickly between a large number of individuals. Children with varying degrees of giftedness are invited to take part in project MALU. In retrospect, the degree of success of the students will shed some light on the predictive validity of the test.

THEORETICAL BACKGROUND

The literature on mathematical giftedness is exceedingly vast. However, there seems to be no consensus on some main points concerning the definition of this construct. Following Heilmann (1999), we view mathematical talent as the potential for future high achievement. Tests are then means to predict future achievements by present ones. One point of interest here is whether one should use tests for general intelligence or rather tests that were developed specifically for mathematics. This depends of course on the theoretical conception of mathematical giftedness. In principle, three different models to explain mathematical achievements are conceivable: Namely, interindividual differences for achievement in mathematics could be explained by

- different degrees of mathematical giftedness,
- different degrees of general giftedness or
- a combination of both.

The first opinion is hold for instance by Krutetski (1976), the second one by Rost (2009). Empirical research (e.g. Lubinski & Humpreys 1990) indicates the concomitance of high general and high mathematical giftedness, thus supporting 3. General giftedness is here as often understood as intelligence: „A very general mental capability that, among other things, involves the ability to reason, plan, solve problems, think abstractly, comprehend complex ideas, learn quickly and learn from experience. It is not merely book learning, a narrow academic skill, or test-taking smarts. Rather, it reflects a broader and deeper capability for comprehending our

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surroundings – "catching on," "making sense" of things, or "figuring out what to do“ (Gottfredson 1997). The enumeration suggests, that intelligence has various aspects, which in turn are operationalized by different subtests of intelligence tests. For a long time it used to be an issue whether the manifest variables given by such tests scores are better conceived of as manifestations of one latent variable „general intelligence“ (namely Spearman’s general factor g) or rather by several latent variables representing different abilities (e.g. Thurstone’s „primary mental abilities“). Factor analyse was developed to settle this issue.

From a different research base, mathematics educators have raised concerns whether tests, especially intelligence tests, are apt to assess mathematical giftedness (see e.g. Meissner et al. 2008). Namely, closed tasks as they occur typically in intelligence tests admit only one correct solution, thus leaving too little room for originality and creativity and representing a too narrow view of mathematics as a finished well-defined topic (Käpnick 1998, Wagner & Zimmermann (1986).). Consequently these researchers developed specific tests for their respective enrichment projects. We draw on the work of the Käpnick group, since their test is completely published in Käpnick (1998). Indeed, Käpnick developed a system of indicative tasks for mathematical giftedness, starting from the interpretation that mathematical giftedness means giftedness for specifically mathematical activities and drawing on the system of indicators developed by Krutetski (1976) as well as on own observations.

Käpnick’s system of indicators for the detection of potentially gifted third- and fourth-graders focuses on features of mathematical giftedness like originality and fantasy during mathematical activities, retentiveness for mathematical content, the capability to structure , the capability to change representations, reversibility and transfer of operations as well as supportive personality traits like a high level of mental activity, a high level of commitment, enthusiasm for problem solving and perseverance.

STUDY 1

To clarify which of the three possible explanations for interindividual in mathematical achievement applies to the MALU sample, we administered both a general as a mathematics giftedness test. Our interest lay in determining the “natural occurrence” of mathematical giftedness. Therefore we tested whole classes of fifth-graders. To ensure comparability, we confined ourselves to grammar schools.

DESIGN

In August 2008 and in August 2009 we administered in 23 classes of grade 5 in Hannover, Germany, a general giftedness test (CFT-20R) and a mathematical giftedness test (a sample of Käpnick’s indicative tasks). 684 fifth graders completed both tests, each of which lasted for one school period.

The rationale for choosing the tests was as follows: Besides Käpnick’s indicative tasks (henceforth designated as Käpnick test) there is only one mathematical giftedness test in Germany, namely the HTMB by Kießwetter, which is unfortunately unpublished (but see Kießwetter (1985) and Wagner & Zimmermann (1986)). For organisational reasons, it was necessary to shorten the test so it could be administered in 45min instead of 90min. To that end, we performed a stepwise regression analysis via SPSS, including the test items that yielded maximal multiple validity for the total test score. Referring to the published test results of the 154 study participants of Käpnick (1998), we thereby obtained a multiple validity of .959.

As intelligence test we choose the CFT-20R (Weiß 2006), tracing back to the Culture Fair Test by Raymond B. Cattell. Like Raven's Progressive Matrices test, the CFT-20R is independent of speech comprehension. The Culture Fair Test scores loade higher on the "General Intelligence" factor than on the "Achievement" factor, which is consistent with the concept of the test being a measure of "fluid" rather than "crystallized" intelligence (Cattell, Krug & Barton (1973)). According to the German manual, the abilities measured by the CFT-20R include the problem comprehension in novel situations (Weiß 2006, p.16). In contrast to the Raven test, current standards for fifth-graders are available for the CFT-20R. Furthermore, this test contains a shortened version that can be administered in a school period.

RESULTS

Within the given sample, the test score of CFT 20-R and Käpnick test correlate .374 ($p<0.01$), so we can assume a weakly positive association of the underlying constructs (as one would expect): to put it differently, 14 % of the variance of one test score can be explained by the variance of the other one.

Since theory excludes a direct causal connection between the test scores, according to the three possibilities above one has to consider the following connections between the latent variables general giftedness (g) and mathematical giftedness (m):

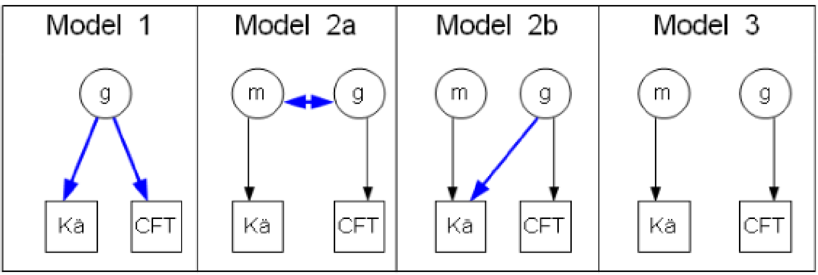


Fig. 1: Theoretically possible models including latent and manifest variables

To find out, which of the models fits best given data, the corresponding structure equation models where subjected to a confirmatory factor analysis (Bollen & Long 1993). The utilized software was AMOS 17 (Byrne 2001). The global model fit indices are given in table 1:

model	χ^2	p-value	χ^2/df	CFI	RMSEA	SRMR
model 1	55.724**	.000	2.786	.935	.051	.0400
model 2a	37.106**	.008	1.953	.967	.037	.0324
model 2b	9,532	.890	0.596	1.000	.000	.0153
model 3	173.524**	.000	8.676	.722	.106	.1258

Table 1: Selected model fit indices for the four theoretically possible models

All indices suggest that model 2b can explain best the empirical relation between the manifest variables (see e.g. Bühner 2006, pp. 252ff).

This result assorts well with the contingency table in table 2 in which we distinguish high achievers according to both measures from the rest of the sample. As usual (Rost 2009), high achievement in the IQ test is defined by a cut-off rate of 130, (i.e. two standard deviations above the mean). For the Käpnick test, no cut-off value for high giftedness has been published, since Käpnick (1998) is of the opinion that it is unsuitable to determine giftedness only by the use of a test score. But for the sake of comparability as well as for statistical reasons, the same proportion of the sample should be considered high achieving in the CFT 20-R and in the Käpnick test[1]. Furthermore we set a cut-off value by only analysing the tasks. Both cut-off values match. High achievement was coded as 2, others as 1. Based on these preliminaries, we can conclude that the result of confirmatory factor analysis is in good accordance with the fact that the entries in position 21 and 12 of table 2 are nonempty, but significantly lower than one would expect for independent variables (χ^2=30,618, 2-sided, p<0,01):

			Käpnick test	
			1	2
IQ test	1	number	580	46
		expected number	568.3	57.7
	2	number	41	17
		expected number	52.7	5.3

Table 2: Contingency table for high achievement (label 2) in both tests

If we would have to choose one of these tests, we would prefer the Käpnick test

1 There is a slight deviation in the actual number, since the cut-off score in the Käpnick test was attained by several students.

because of the results of the confirmatory factor analysis: The Käpnick test loads on the g-factor too. But if we would like also to select the students with a good score in the intelligence test but with an average score in the mathematical giftedness test (position 21, table 2), we have to administer both tests. Otherwise these 41 students, that means 39% of the possibly gifted students, were lost.

STUDY 2

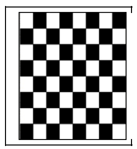
Since we are interested in the contribution of cooperation to the problem solving outcome, we had first to determine the influence of the individual giftedness variables. Since a pilot study revealed that the prediction of a pair's achievement by individual cognitive capability characteristics was rather low, we wanted also to determine in comparison the prediction of individual problem solving outcome.

DESIGN

In four classes of grade six four MALU tasks were processed individually and four other tasks were processed in pairs. Also, the mathematics and German marks of the students as well as the CFT-20R and Käpnick scores were collected. We have the complete data for 108 sixth graders. The individual and pair scores constitute the criterion variables for problem solving, the test scores and marks form the predictor variables. Figure 2 depicts a typical MALU task.

Oh yes the chessboard

Peter loves playing chess. He likes playing chess so much that he keeps thinking about it even when he isn't playing.



Recently he asked himself how many squares there are on a chessboard. Try to answer Peter's question!

Fig. 2: The chessboard problem (idea: Mason et al. 2006)

RESULTS

To assess the predictive validity of both tests, we first correlated bivariately the test scores with the criterion variables as well as the other two predictors:

		German	maths	individual scores
N =130 grade 6	Käpnick test	r=0.350**	r=0.411**	r=0.290**
	CFT	r=0.199**	r=0.394**	r=0.305**

Table3: prediction of marks and individual test scores

DISCUSSION

The intelligence test as well as the mathematical giftedness test are valid predictors (according to the literature) of mathematical achievement, but they explain only 16% of the variance regarding mathematical of mathematical performance at school (if we regard the mark as a criterion rather than a predictor) and even less, namely 9% of the variance regarding mathematical problem solving in the sense of MALU. Regarding the theoretical conception of the Käpnick test, it is especially surprising that the latter value is even less than the former.

Thus questions regarding the reliability of the test administration are in order. We can cope with them as follows: the correlations of the test scores with the marks fall in the ordinary realm – e.g. CFT-20R correlates with .40 to the maths mark (Weiß 2006, 87). For the Käpnick test, no comparison values are published.

The intercorrelation of the tests is lower than e.g. in the TIMSS study, where the – maths score correlates to basic cognitive skills with .49 (figural) resp. .59 (verbal), see Baumert et al. 1997). This suggests that the measured construct by the Käpnick test is farer from general giftedness and might be more specific for mathematics. Alas, one has to account also for the contradictory evidence that the explained variance for mathematical problem solving is even lower than for school marks.

By combining the tests, the predictive validity can be augmented considerably, but it still remains unsatisfactory:

		maths mark	individual score
N =130 grade 6	Käpnick test+CFT	R=.469	R=.347
	Käpnick+CFT+maths mark	--	R=.399
	Käpnick+CFT+ maths mark +German mark	--	R=.415

Table 4: multiple correlations of test scores and marks

The multiple correlation of the maths score with Käpnick and CFT test is R=.469, which gives an explained variance of R^2=.220. For problem solving the values of R=.347 and R^2=.120 are even lower. An obvious conjecture is that a significant factor of problem solving was not assessed – however it remains unclear, which factor that could be. It is *not* the mathematical foreknowledge, since the multiple correlation rises only to R=.399, if the mathematics, score is not regarded as criterion, but is included as an extra predictor measuring prior knowledge of mathematics. The German results of the PISA study would suggest to consider language competence, but this can also be excluded, since the multiple correlation rises only by .016 to R=0.415, if one includes the German mark as a predictor. All in all, both tests and both marks together explain only 17% of the variance within the individual MALU scores. Since these results are far from satisfactory, we postpone the analysis of MALU scores which were obtained by pairs, since it is known that groupwork adds an extra difficulty in predicting problem solving performance (Kunter et al. 2005).

To explain our results, one might reckon that the format of the test tasks differs significantly from the MALU tasks: A typical intelligence test tasks can be processed in around a minute and requires only to mark the unique correct solution with a cross. In contrast to this, a MALU task as in figure 2 typically takes the children about 20min to 30min to perform. For a correct solution, a written answer out of contiguous arguments is required, and several solutions are well possible.

However, it remains unclear, to what extent this plausible distinction applies also to school marks. It is known (Jordan et al. 2008) that the cognitive potential of exam tasks is rather low in German secondary schools. This would support the view that the marks are comparable to intelligence test scores – and a similar argument has been put forward by Rindermann (2006) concerning the mathematics tasks of the PISA study which where deemed curricularly valid also for the German classroom.

Insofar it remains an open question how problem solving performance (either individual or even more in pairs, as was also observed by Kunter et al.(2005), when analyzing the outcome of the German extension concerning problem solving of the PISA study) can be validly predicted by test tasks. This is in accordance with the view of the mathematics education community that test scores can only be one part of the selection process for an enrichment project. A holistic selection procedure, which could be based on interviews and/or process observations, is certainly more satisfying, but of course also considerably more costly than any paper and pencil test.

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THE ROLE OF DIGITAL TECHNOLOGIES IN NUMERACY

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This paper presents a model for numeracy that integrates the use of digital technologies among other elements of teaching and learning mathematics. Drawing on data from two school based projects which include records of classroom observations, semi-structured teacher interviews and artefacts such as student work samples, two vignettes are presented which illustrate possibilities for technology integration into classroom practice. While positive examples of the use of digital tools are outlined, we ask how a greater number of teachers could develop an orientation towards recognising and taking advantage of opportunities to create holistic numeracy tasks that seamlessly incorporate the use of digital tools.

The notion of numeracy (which in some international contexts is also known as mathematical literacy) as the capacity to make use of mathematics to accommodate the demands of the lived worlds of private and public life, has been an issue of discussion within mathematics education from at least the time of the Crowther Report (e.g., Ministry of Education, 1959). Subsequent reports and influential literature (see for example, Cockcroft, 1982; Steen, 1999) have since emphasised the importance of numeracy as a focus for schooling. More recently, the importance of numeracy was recognised internationally through the OECD's Program for International Student Assessment (PISA). According to PISA’s definition mathematical literacy is:

> an individual’s capacity to identify and understand the role mathematics plays in the world, to make well-founded judgments, and to use and engage with mathematics in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen. (OECD, 2004, p.15)

This interpretation of numeracy is also consistent with the widely accepted definition within Australia: “To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life” (Australian Association of Mathematics Teachers, 1997, p.15).

While this may be interpreted as a call for vocational specific approaches to teaching mathematics, Straesser (2007) warns against narrow approaches to mathematics education and training. He views mathematics as a strategic tool that can be adapted for a range of contexts and settings. In particular, he signals a concern for the “black box” view of mathematics in the workplace where the underpinning features and functions of mathematics are subsumed into simple routinised practice. Straesser goes on to suggest that the type of mathematics that spans the gap between school mathematics and the workplace is “no longer part of mathematics or ‘the rest of the

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world' alone, but are a new type of knowledge bridging the divide between mathematics and the rest of the world" (p. 169).

These statements imply that the purpose of learning mathematics in schools should have a broader reach than singly gaining proficiency within the discipline of mathematics itself. While traditionally mathematics curricula have placed little emphasis on the use of mathematics in the beyond school world (Damlamian & Straesser, 2009) there are developing areas of research and practice which focus on the integration of the teaching and learning of mathematical knowledge and the utilisation of this knowledge in real world contexts. This includes research into the increasingly emerging issue of the shape of numeracy education in an age where digital technologies have an impact on nearly all aspects of life; where young people are growing up in what Steen (1999) describes as "data drenched" worlds. Jorgensen Zevenbergen (2011) has argued that young people have already begun to accommodate their information saturated environments through the development of more holistic approaches to solving problems by making use of all available tools – especially digital technologies.

This paper explores how teaching and learning in schools can best support young learners to develop technology integrated mathematical capacities that will prepare them for the beyond schools worlds of work and active citizenship.

Digital Tools and New Numeracies

Noss (1998) points out that the valuing of mathematics for its utility in the workplace and in civic life stems from the Cockcroft (1982) report. While the balance between teaching of mathematics from a purely mathematical versus utilitarian perspective is still a matter of debate for curriculum authorities, syllabus writers and teachers, it now seems accepted that applications of mathematics must have some place in students' mathematics educations. However, as Noss (1998) further argues, the nature of these applications must be connected to current practices in working, private and civic life and that these practices are now tied inextricably to the capabilities offered by digital technologies.

In a further exploration of this theme (Zevenbergen, 2004) notes that there is intergeneration difference in the numeracy expectations of workplaces she investigated. She observed that younger workers were happy to defer cognitive labour (e.g., mental arithmetic) to digital tools which enabled them to take on the more strategic aspects of their work more effectively. These young adults were very good at estimating, problem solving and holistic thinking. Zevenbergen concludes from this and subsequent studies (Jorgensen Zevenbergen, 2011) that the influence of technology in schools and the workplace, and by implication other aspects of the lived in world, has shaped the habitus of young people who, as a result, are reshaping the various structuring practices that serve to recognize and validate particular dispositions and skills within their workplaces. This new generation of workers also seem to have the capacities to make use of their personal mathematical knowledge and their confidence and capabilities with ICTs (Information and Communication Technologies) to solve on the job problems in more inventive ways than their more experienced co-workers.

These commentaries imply that mathematical knowledge alone will not be sufficient to meet the demands of the ever changing workplace and that the capacity to think adaptably, a disposition to continue to learn new approaches to solving problems as they arise, and the capacity to embrace the use of technological tools are as important as the type of mathematical knowledge traditionally taught in schools. These are elements of Straesser's "in between worlds". But what is this new type of knowledge and what would it look like if we were to see it in a mathematics classroom?

Theoretical framework

Increasing interest in the role of ICTs in enhancing the learning and teaching of mathematics in context-laden situations has led to the development of ICT inclusive models of mathematical inquiry. Confrey and Maloney (2007), for example, draw on Dewyian principles of inquiry learning to develop a framework in which technology is assigned a vital role in the application of mathematics to contextualised situations by coordinating the inquiry, reasoning, and systematising processes that lead to a final outcome. In another attempt to describe the role ICTs play in applying mathematics to real world situations via a mathematical modelling approach, Geiger, Faragher and Goos (2010) found that digital technologies can influence all aspects of the solution process. While both of these studies provide insight into the potential for digital tools to influence how we use mathematics to solve problems in real world contexts, neither attempts to address the broader issue of how this potential can be harnessed in concert with other important aspects of teaching and learning mathematics.

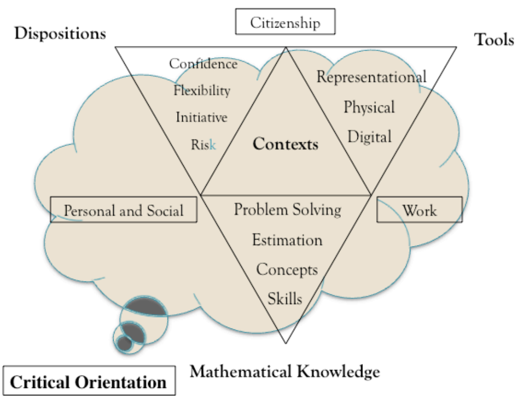

Figure 1. A model for numeracy in the 21st century (Goos, 2007)

In a development of the concept of numeracy to accommodate the changing nature of knowledge, work and technology, Goos (2007) developed a model that incorporates attention to real-life contexts, the deployment of mathematical knowledge, the use of physical and digital tools, and consideration of students' dispositions towards the use of mathematics. The development of a critical orientation was also emphasised in relation to numeracy practice, for example, the capacity to evaluate quantitative, spatial or probabilistic information used to support claims made in the media or other contexts (Figure 1). This model offers a broader interpretation of the role of mathematics and ICTs in bridging the gap between school mathematics and the wider world and has been used as a framework to audit mathematics curriculum designs (Goos, Geiger & Dole, 2010) and for analysis of teachers attempts to design for the teaching of numeracy across the curriculum (Goos, Dole & Geiger, 2010). The use of this model to examine classroom practice will now be illustrated through two vignettes drawn from two primary school classrooms.

METHODOLOGY

The data presented in this paper are drawn from two numeracy projects which were conducted independently – one in each of two Australian states. The aim in each case was to empower teachers to work with numeracy across all curriculum areas. Pairs of teachers were selected from schools that placed expressions of interest in the relevant projects with their school system. In the first project 10 pairs of middle school teachers (Years 6 to 9) were selected from schools across South Australia during 2009 (Goos, Geiger & Dole, 2010). In the second project 12 pairs of primary and secondary school teachers (Years 1 to 12) were chosen from schools across the south-east corner of the state of Queensland during 2010. The Loucks-Horsley, Love, Stiles, Mundry and Hewson (2003) framework for professional development underpinned the design of both projects.

In both projects, teachers came together for an initial meeting to become familiar with the ideas embedded in the numeracy model and to work through investigations that allowed for the elaboration and clarification of the ideas embedded in the model. After this initial meeting teachers were asked to adapt activities presented in the workshop to their own classroom contexts, or to develop new ideas based around the elements of the numeracy model and trial these in their classrooms. After a number of months, teachers were brought together again to present examples of activities they had trialled and to engage in further curriculum planning while being supported by teachers from other schools. The project concluded with another cycle of trialling activities, visits from the research team and a final presentation to the whole project group. Between each of the whole project meetings a research team consisting of the authors of this paper and representatives of the sponsoring system authorities visited teachers to discuss the success of the activities they were trialling and to provide further input and support as was necessary. The data used in this paper are drawn from field notes of classroom observations, records of semi-structured interviews which took place when the research team visited teachers and artefacts such as student work samples and computer files collected during school visits. The quality of classroom learning experiences were analysed in relation to how they related holistically to the numeracy model.

VIGNETTE 1

This example is drawn from the 2009 study in South Australia. As part of the project, one teacher developed an activity within her Year 6 Physical Education (PE) program where students investigated the level of their physical activity through the use of a pedometer that they wore during all waking hours over one week. The collected data, that is, the number of paces walked or run, were entered into a shared Excel spreadsheet every day. Students were asked to analyse their own data by using facilities within Excel, for example, the graphing tool, and then to compare their results with those of other students (see Figure 2).

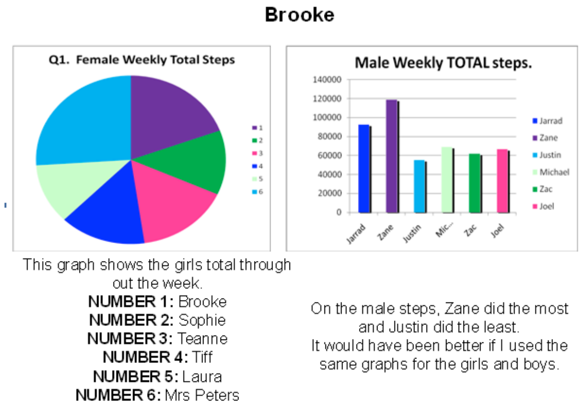

Figure 2: A comparison of males' and females' weekly total steps.

As part of this analysis, students were asked to convert their total daily and total weekly paces into kilometres to gain a sense of how far they typically walked in the course of a day or a week. The task was also designed to help students realize that the distance they walked was not determined by the number of paces alone as an individual's pace length was also a factor. In order to make this conversion, students were required to design a process for determining the length of their own pace. After some discussion, which was guided by the teacher, students negotiated an approach which was acceptable to all members of the class. This involved marking out a distance of 100 metres along the footpath which bordered the school against which students counted the number of paces they each took to walk this distance. After demonstrating the procedure for obtaining the length of her pace and the converting paces in a day to kilometres from her own personal data, the teacher asked students to complete conversions of their own pace totals to kilometres. She also suggested that students compare their kilometric distances with each other and to discuss why they were different. The teacher indicated the next session would include a further investigation of the number of paces Usain Bolt takes during a 100 metre sprint.

Tools were used throughout the lesson, physical tools such as tape measures and digital tools. Digital tools included pedometers, electronic calculators and Excel spreadsheets. Technology in this investigation provided the capability to collect data (pedometer), perform initial calculations (electronic calculators) and record, analyse and represent data (Excel spreadsheet). These tools also mediated discussion between students in relation to differences they observed as they critically compared their own results to those of others and attempted to explain the differences. Thus, technology was connected to all other elements of the numeracy model: mathematical knowledge (measurement, estimation, ratio, collection, organizing and representing data); context (use of a pedometer to collect personal data in an outdoor investigation); dispositions (challenging students to think flexibly about the representation of their personal details so these could be compared with others); and critical orientation (comparing their own results with others and speculating on the reasons for differences).

VIGNETTE 2

The second vignette comes from the study situated in Queensland during 2010. In this example the teacher endeavoured to promote the mathematics learning of her Year 5 students by engaging them in an international web based activity in which whole classes of school students were required to document their steps per day as recorded on a pedometer. Students entered the number of steps they recorded each day over a two month period (October to November) into a spreadsheet provided by the teacher. The total for each day was then calculated and entered by the teacher into the website interface. After entering data the website could be interrogated for: a record of daily entries represented graphically (Figure 3); a progressive class average by week and month represented both numerically and graphically; position rank in comparison with other schools participating in the Challenge.

Figure 3: Daily class step totals

The site also included a facility that mapped how far the class step total had taken them along a predetermined route across the globe beginning in North America and then passing through South America and Africa before finishing in Europe. Information about each country visited on this route was available from the website as each new location was reached. The teacher reported that students were very engaged in this challenge. Their desire to improve their position against other classes of students in their school and across the globe promoted extensive discussion of what was meant by average, how to improve their class average in a targeted fashion and how to interpret and use the information available via the web based tool to understand and further promote their position.

Digital tools were central to the activity. Technologies included pedometers, Excel spreadsheets and web based tools. Again, technology provided the capability to collect data (pedometer), but the Excel spreadsheet here was more a means of banking and collating data before it was entered into the web based analysis tool. Once entered into the web interface the teacher, with her students, was able to complete comparative analyses of data from within the class, across the school and internationally. This provided opportunity to engage students in a discussion about how they might contribute in order to improve their position in comparison to other classes in the school and in other countries or what it would take to progress their class to the next destination on the global journey. Technology was connected to other elements of the numeracy model: mathematical knowledge (measurement, estimation, mean, graphical representation); context (use of a pedometer to collect personal data in an outdoor investigation); dispositions (motivation through using mathematics to improve relative position in a gentle competition); and critical orientation (strategies to improve the class average).

Conclusion

In order to prepare students for the types of worlds Steen (1999) and Jorgensen Zevenbergen (2011) have described, more holistic approaches to teaching and learning numeracy are necessary. It is also apparent that a model of numeracy in which digital tools are seamlessly integrated with other elements of mathematics use in context bound situations, is required in order for students to move more readily into these “data drenched” worlds. The cases presented in this paper demonstrate that such integration is possible if teachers have a model for teaching which draws their focus to additional elements of numeracy other than mathematical knowledge alone. Digital tools in these cases have been used to collect, analyse and represent data. The results of these processes provided material for students to critically examine the situations they are investigating and to speculate on what measures are necessary to change outcomes in their favour. It has to be acknowledged, however, that these are two outstanding cases and that not all teachers in the project developed tasks that challenged students to use all elements of the numeracy model in such an integrated way. Further research is necessary into how to assist teachers to develop a habitus which orientates their thinking towards recognising and taking advantage of opportunities to create tasks relevant to the lived in worlds of their students and beyond.

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STUDENTS' JUSTIFICATION STRATEGIES ON THE EQUIVALENCE OF QUASI-ALGEBRAIC EXPRESSIONS

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This paper describes how 11-13 year-old students justify the equivalence or non-equivalence of models and rules for figural linear patterns constructed in a specially designed computer microworld, the eXpresser. Students were asked to build their own models of a figural pattern and associated rules and to reflect on model and rule before sharing them with a peer to discuss equivalence. Findings point to the diverse justification strategies students articulated to support equivalence and their potential for fostering the development of students' algebraic ways of thinking.

INTRODUCTION

Introducing algebra to secondary school students is widely known to be problematic (see for example, Küchemann & Hoyles, 2009; Kaput *et al.* 2008). Pattern activities are widely adopted (Mason, 1996; Lee, 1996) although, as Dörfler (2008) and Küchemann (2010) suggest, they are rarely presented in ways that encourage an awareness of the structure underlying the patterns.

In the MiGen[2] project, we are designing a technical and pedagogical environment to support students in expressing algebraic generalisations arising from how they visualize the structure of a figural pattern. Our goal is to introduce pattern-based activities in ways that promote algebraic ways of thinking; that is, to identify the commonalities that form a structure and express relationships that represent this structure (Noss et al, 2009; Geraniou et al, 2009). It is relatively straightforward to show generality using numbers and gestures, but expressing it in words or in an algebraic form has proved consistently problematic (Radford, 2010). Frequently, the algebraic rule is disconnected from the problem and simply added as an end point.

In the MiGen project, we are trying to bridge this gap between showing and saying by providing students with a set of tools and scaffolds to express generality, to provide an alternative representational infrastructure that can play the expressive role of algebraic symbols and their grammar. At the core of the system is *eXpresser*, a microworld designed for students to construct patterns from repeated building blocks of square tiles. Accompanying the software is a set of designed activity sequences that comprise figural pattern tasks alongside reflective and collaborative activities, all focussed on fostering students' engagement with and expression of justification and generalisation.

1 This research was conducted while Eirini Geraniou was at the London Knowledge Lab, IOE, University of London.

2 The MiGen project is funded by the ESRC/EPSRC Teaching and Learning Research Programme (Technology Enhanced Learning; Award no: RES-139-25-0381). For more details about the project see http://www.migen.org

This paper presents data from several studies in UK schools, focusing on the collaborative phase and in particular how 11-13 year old students justified to their peers the equivalence (or otherwise) of their quasi-algebraic rules in eXpresser.

THEORETICAL BACKGROUND

It is widely known that even though students are capable of generalising a pattern or a rule, few are able to explain why the rule is valid (Coe & Ruthven, 1994; Ellis, 2007). Most rely on empirical examples to justify the truth of statements: it would hardly be surprising if a student who generalises based solely on specific cases, were to use one or more examples as a form of justification. Relevant research in fact suggests that a student who generalises by attending to the structure of a pattern and relating every algebraic expression to the corresponding part of the pattern-model-construction has a better chance of understanding the generality of their expression and possibly produce a general argument to justify the equivalence of rules (Ellis, 2007; Küchemann, 2010).

Thinking algebraically is more than thinking about structures and the general: it is the 'use and availability of symbolism to reason about and express generalisations' (Kieran, 1989). Research documents different strategies students employ when constructing the algebraic rules that underpin patterns. For example, Rivera and Becker (2008) differentiate between constructive and deconstructive generalisation depending on whether or not students perceive the figural pattern as having overlapping components. Chua and Hoyles (2010) also refer to reconstructive generalisation, where components of the pattern are rearranged to reveal the pattern structure. All these strategies are used as ways students support their chosen method for deriving the general rule.

We conjecture that focusing on figural pattern activities and orchestrating students' discussions on equivalence of models and related quasi-algebraic expressions in the context of a dynamic computational system, could prove to be a powerful approach to fostering algebraic thinking (see also Tall & Thomas, 1991; Kieran & Sfard, 1999). Such discussions should therefore stimulate students *explicitly* to relate the symbolic representation to the relevant parts of the pattern, give meaning to symbols and allow justifications to be formed in a quasi-algebraic manner.

METHODOLOGY

The data presented in this paper are from twenty-eight 11-13 year old students from three schools in England. Students were familiarised with the MiGen[1] system in two lessons through a number of introductory activities, asking students to construct figural models. In eXpresser, an initial figure is presented dynamically drawing students' attention to the general problem, rather than the inevitably static (and therefore specific) problem that could otherwise be posed on paper. Figure 1(A) shows the "Train - Track" model : it is animated randomly from left to right with the

1 The eXpresser is a part of a much broader computer environment that we refer to as the MiGen "system".

value of the model number changing accordingly[1]. Students were asked to construct the Train-Track model in eXpresser using different patterns and combinations of patterns (examples are shown in Figure 1B), depending on their perceptions of the Train-Track's structure.

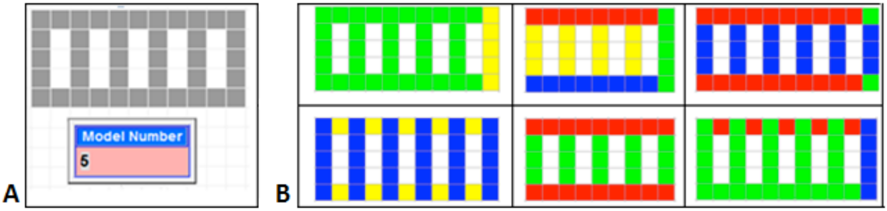


Figure 1: The Train-Track task (A) and different students' perceptions of it (B).

After constructing their model, students were asked to reflect on it by answering the following questions: *(1) Use your model to find the number of tiles for Model Numbers: 6, 12, 1 and 100. (2) Is your rule correct or not? In the next task, you will discuss with another student. Make some notes here to explain why your rule is correct or not to prepare for this group activity*. Based on the dissimilarity of their models[2], students were paired to work on a collaborative task to discuss the correctness and equivalence of their rules. In this paper, we focus on students' reactions to the question: *Can you explain why the rules look different but are equivalent? Discuss and write down your explanations*.

The students' discussions were orchestrated by a teacher or researcher and audio-recorded, transcribed and analysed qualitatively. A number of justification strategies were identified. Once a first set of categories had been established, the raw data was revisited to assess their validity and evaluate whether all the justification strategies used by students were adequately captured. When this was not the case, a new category was incorporated and validated against the data. This iterative cycle was repeated a number of times.

JUSTIFICATION FOR EQUIVALENCE

The data analysis revealed three main categories: *structural, symbolic* and *empirical.* These are described below along with their subcategories[3].

(A) Structural Justification for Equivalence

Justifications in this category all focused on the structural aspect of the pattern by, for example, comparing the building blocks used in the different patterns and making arguments as to their equivalence (based on this comparison) with little if any reference to the symbolic rule. We distinguished three subcategories illustrated below with data from the study.

1 For this and other design priorities and rationales, the reader is referred to Noss *et al.* (2009).
2 An intelligent component suggests to teachers possible groupings based on the dissimilarity of students' models. This is outside of the scope of this paper (see http://www.migen.org for the system's functionalities and other publications).
3 Note that sixteen out of the twenty-eight students used more than one strategy.

A1) Reconstructive Justification (number of responses, 20). In this subcategory, different building blocks are compared or reconfigured as illustrated by the case of Janet and Nancy (see Figure 2).

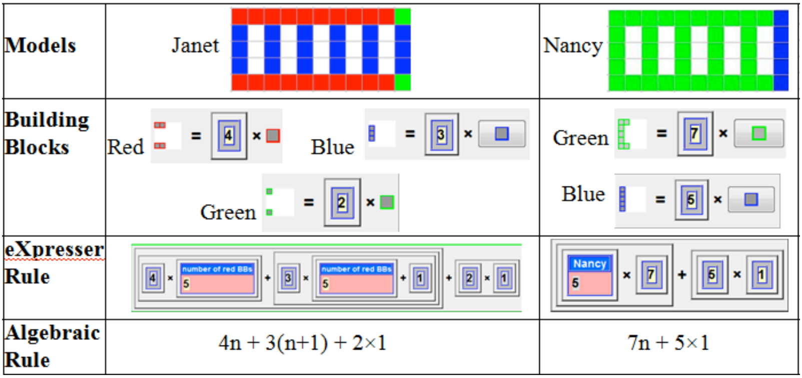


Figure 2: Janet and Nancy's model, building blocks and general rules[1]

Nancy compared her building block with that of Janet's:

Nancy: Yeah it's one red building block plus one blue building block so that would actually kind of make the…

Janet: yeah, it would make the same shape...

Nancy: because one red building block added to one blue building block…

Janet: and that's the same as one of my green building blocks.

Students complemented each other's arguments and concluded that their building blocks were in fact the same. Neither explicitly related the models to their rules or linked the number of tiles in each block to the coefficients in the algebraic expressions. Rather, they simply compared the building blocks underlying the patterns used.

A2) Experimental Justification (number of responses, 7). In this subcategory, students choose a specific case and compare their two models and rules for this case, as illustrated by Alex and Anne.

1 Students are encouraged by the design of the system to "name" their numbers (Noss et al., 2009, Geraniou et al. 2009).

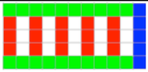
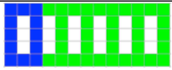
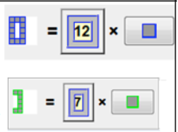
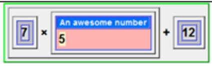
Models	Alex 	Anne 
Building Blocks	Green Red Blue 	Blue Green 
eXpresser Rule		
Algebraic Rule	(2n×2) × 2 + 3n + 5	7n + 12

Figure 3: Alex's and Anne's model, building blocks and general rules

Alex: I kind of got a C, but coloured them in different ways so I mean the 5 is only added at the end...

Anne: then there are just 7 tiles in one model.

Alex: Yes, but your first model has 12 tiles and your second model has 7 tiles. For 5 red blocks I have 5 blue extra tiles, but you have 12 blue extra tiles.

Anne was able to read Alex's rule and recognised the configuration of tiles that formed a similar building block to hers. Yet it was evident that both students considered each building block as a separate model. At first, Alex chose to change the number of red blocks in her model to 5 to match Anne's model, but then realised that it was just not possible to match: the two models, in fact, had different constant terms. Alex then decided to compare the two models for the *same* model number and then justified the non-equivalence of the two rules.

A3) Justification by Contradiction (number of responses, 5). Students use the same model number and calculate the number of tiles used, and notice that they obtain the same - or in this case, different - answers, as illustrated by Amy and Nick (Figure 4). They had to go back to comparing their models structurally.

Amy: I think for model number 5, I've got 43 and Nick's got 40.

Nick: Oh, I think I might know why. Hers is 7 blocks high. Mine is 5 blocks high. So if it was, if she had 5 blocks high it would be the same.

Nick noticed that for the same value of the independent variable, their models could *never* be the same. His justification was based on a contradiction.

(B) Symbolic Justification (number of responses, 12)

This category comprised student justifications focused on their eXresser rules and justified their equivalence by adding the constants and variables in each rule and comparing them as illustrated by Leo and Penny's case (Figure 5).

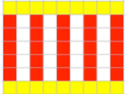
Models	Amy 	Nick 
Building Blocks	Red Yellow 	Green Yellow 
eXpresser Rule	5 × 5 + 2 × 9	5 × 7 + 5
Algebraic Rule	$5n + 2{\times}9$	$7n + 5$

Figure 4: Amy's and Nick's model, building blocks and general rules

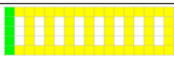
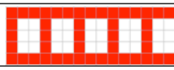
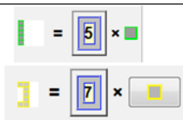
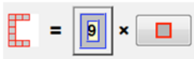
Models	Penny 	Leo 
Building Blocks	Green Yellow 	Red 
eXpresser Rule	5 × 1 + 8 × 7	5 × 9
Algebraic Rule	$5{\times}1 + 7n$	$n + 9$

Figure 5: Penny's and Leo's model, building blocks and general rules

When paired, Leo realised that his rule was incorrect, but was able to derive a correct general rule that he wrote on paper as [5] × 9 – [5] × 2 + 5. This is what they both compared with Penny's rule.

Leo: I had 5 times 9 because I had 9 things but I have to take away 2 of my red building block, so I have to take away 10 tiles because I need to have 5 sevens. I had that many on the end of each one [pointing at his model]. That is why I have to take away 2 and then plus 5 because I need an extra line at the end. The 9 minus 2 is equal to plus 7 and the 5 is the same and then the 5 is the same so they're the same rule but written differently.

Penny: Mine is 5 times 1 plus 8 times 7. These 8 times 7 because we've got 8 of the 7 blocks and so 8 times 9 minus times 2 is 8 times 7.

They concluded that Leo's second rule on paper was equivalent to Penny's rule.

(C) Empirical Justification for Equivalence

Some students focused solely on the numerical aspect of the rules, avoiding any reference to the structure of their model constructions. Two subcategories were distinguished.

C1) Matching-Terms Justification (number of responses, 12). In this category students pick a constant or a variable and compare with the equivalent term in the other students' rules. Here is Alex at an early stage of her collaboration with Anne:

Alex: They both have 7 in them plus something to make the end of the pattern.

She picked a constant in her rule and identified it in Anne's rule too (see Figure 3). She noticed the similarities in the algebraic expressions, but also the difference in the added constant term (5 in Alex's rule, but 12 in Anne's rule).

C2) Evaluating-Terms Justification (number of responses, 2). In this category, students compared the number of tiles for different model numbers. Later in their discussion, Alex chose a value for the independent variable and compared the answers for the two rules:

Alex: Model number 1 is blue blocks and it's got 12 tiles in total. The backwards C is model number 2. So, we have 12 plus 7…19 tiles.

Anne: No, model number 2 is 2 backward Cs plus the blue block. So, 2 times 7 plus 12… 26 tiles.

Anne's answer included 7 more tiles because of the blue block she had added to her model (see Figure 4). The students were confused at this point as to what the model was and what the model number was.

CONCLUDING REMARKS

The collaborative task challenged students to read, deconstruct and match their rule with their own and their partner's model. In their justification efforts, students revisited their generalising actions, built on them, and took new actions that were more powerful and meaningful. Their investment in building their own models supported them in deriving generalisations by directing their focus towards relationships between quantities, and the quasi-algebraic discourse of eXpresser - the grammar of objects and relationships between them - gave students a means to express generalisation without the machinery of algebra. The findings point to the students' preference for referring to the structure of their models to justify equivalence of their rules, since most students (20 in total) used the reconstructive justification strategy. The second most common strategy (12 students) was symbolic justification (B). This result supports the usefulness of eXpresser for students' introduction to algebra and possibly proof (as the next step from justification). We are currently working on elaborating the collaborative phase to assist in bridging from arithmetic to algebraic expression.

In summary, we can claim that students' engagement in acts of justifying through collaboration seemed to support further their generalisation skills in a number of ways: (a) recognise the importance of seeing structure, (b) find the invariants and variants (constants and variables) in models and rules, (c) express relationships using an independent variable to link patterns within models and (d) see the rationale for and recognise the power of mathematical generalisation.

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EXPLORING THE MYSTERY OF CHILDREN WHO READ, WRITE AND ORDER 2-DIGIT NUMBERS, BUT CANNOT LOCATE 50 ON A NUMBER LINE

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Carole Livesey, and Kathie Turkenburg

Australian Catholic University

Two interpretive Place Value tasks were added to the Early Numeracy Interview *in 2010 to gain further insight about 761 Grade 2 and Grade 3 students' construction of conceptual knowledge associated with 2-digit numbers. Previously, the researchers had noticed that most students were successful at reading, writing and ordering 2-digit numbers, but that interpreting these numbers for problem solving remained a struggle for many. Analyses of students' responses showed that the new tasks distinguished students who previously were assessed as understanding 2-digit numbers, but who could not identify 50 on a number line or state the total of a collection (36) that was reduced by ten. The new tasks assist teachers to identify students who need further instruction to fully understand 2-digit numbers.*

INTRODUCTION

Research during the *Early Numeracy Research Project* in Australia (ENRP, Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002) found that being able to read, write, order and interpret 2-digit numbers was a difficult growth point for young children to reach. It seems that most children learn to read and write 2-digit numbers fairly easily, but that interpreting the cardinal value of these numbers is the greater challenge. In a later study involving over 7000 Australian primary students, Gervasoni, Turkenburg, & Hadden (2007) were also concerned by the number of students they identified in Grades 2-4 who were yet to fully understand 2-digit numbers. If we are to improve young children's whole number learning then it is important to understand the challenges children face in coming to understand 2-digit numbers. This is the key issue explored in this paper that reports on the refinement of the ENRP Early Numeracy Interview (ENI) and framework of growth points (Clarke et al., 2002) as part of the *Bridging the Numeracy Gap* Project (Gervasoni, Parish, Upton, Hadden, Turkenburg, Bevan, Livesey, Thompson, Croswell, & Southwell, 2010). The research team aimed to refine and extend the ENI and growth points, originally designed for use in the first three years of schooling, so that they were more appropriate for assessing students across the primary school. The aspect of the research reported here is the refinement of the assessment tasks for Growth Point 2 (GP2) – reading, writing, interpreting and ordering 2-digit numbers.

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THE CHALLENGE OF UNDERSTANDING 2-DIGIT NUMBERS

Many studies have provided insight about the challenges involved in understanding and using 2-digit numbers. One important finding is that children who have not constructed grouping and place value concepts often have difficulty working with multi-digit numbers (Baroody, 2004). Also, being able to interpret numerals to order them from smallest to largest is another difficulty for some children. Griffin, Case, and Siegler (1994) observed that this involves integrating the ability to generate number tags for collections, and make numerical judgments of quantity based on the construction of a mental number line (Griffin & Case, 1997; Griffin et al., 1994).

Other studies have found that successful problem solving with two-digit numbers depends on children's ability to construct a concept of ten that is both a collection of ones and a single unit of ten that can be counted, decomposed, traded and exchanged for units of different value (e.g., Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989; Steffe et al., 1988). Cobb and Wheatley (1988) found that some children develop a concept of ten that is a single unit that cannot be decomposed, and proposed that this type of concept is constructed when children learn by rote to recognise the number of tens and ones in a numeral, but do not recognise that the face value of a numeral represents the cardinal value of a group.

Fuson et al., (1997) identified five different correct conceptions of 2-digit numbers and one incorrect conception that children use, several of which may be available to a given child at a particular moment and used in different situations. These six conceptions provide researchers with a detailed model to analyse children's use of 2-digit numbers and was considered by researchers when developing the ENRP Place Value framework of growth points and the associated Early Numeracy Interview (ENI). However, for the ENRP, researchers opted for a less complex model than the Fuson et al. model that they hoped would be more user-friendly for teachers. Ten years on, in refining the ENRP assessment interview and framework of growth points as part of the research reported in this paper, it will be important to consider whether the Fuson et al. model better explains the difficulties that some children experience in coming to understand 2-digit numbers. The six conceptions of 2-digit numbers are explained in detail in Fuson et al. (1997). They are the: Unitary Multi-Digit Conception; Decade and Ones Multi-digit Conception (noticing word parts); Sequence of tens and ones conception (noticing the advantage of counting by tens associated with partitioning in tens); Separate Tens and Ones conception (noticing the number of tens and the number of ones); Integrated sequence-separate tens conception (noticing that the number of tens is linked to the number name); and the Incorrect Single-Digits Conception (viewing each digit as representing ones).

Fuson et al. (1997) contend that for full understanding of number words and their written symbols, children need to construct all five of the correct multi-digit conceptions, and that this requires considerable experience and time. Thus, we believe that the refinement of the Early Numeracy Interview needs to ensure that teachers can identify students who use the integrated sequence/separate tens conception of 2-digit numbers. Indeed, we are interested to learn whether or not students who previously reached GP2 in Place Value are successful with this task.

Constructing a Mental Number Line

Another important characteristic of number learning is forming a mental number line. This requires the ability to visualise and abstract a number line so that students can locate any given number, order numbers by quantity, and generate any portion of the number line that may be required for problem solving.

Griffin, Case and Siegler (1994) proposed that success in early arithmetic depends on the formation of a mental number line in association with understanding the generative rule that relates adjacent cardinal values (i.e., each adjacent number in the number line is one more or one less than its neighbour); and understanding the consequence of the previous idea: that each successive number represents a set which contains more objects, and thus has a greater value along any particular dimension.

One way to help children develop a mental number line for use in problem solving is to engage them in activities involving an empty number line. This is a strategy widely used in the Netherlands and aims to link early mathematics activities to children's own informal counting and structuring strategies. "The choice of the empty number line as a linear model of number representation up to 100 (instead of grouping models like arithmetic blocks) reflects the priority given to mental counting strategies as informal knowledge base" (Beishuizen & Anghileri, 1998, p. 525). This emphasis in the research literature on the importance of the mental number line and empty number line as a means of interpreting numbers is not reflected in the ENI until GP5. When refining the ENI it may be useful to include a 2-digit number line task earlier in the interview to determine whether students who reach GP2 are able to interpret numbers on a number line.

ENRP Assessment and Growth Points

The Early Numeracy Interview developed as part of the Early Numeracy Research Project (Clarke, Sullivan, & McDonough, 2002), is a clinical interview with an associated research-based framework of growth points that describe key stages in the learning of nine mathematics domains. The data examined in this paper were drawn from the ENI and Growth Point Framework, so it needs to be understood.

The principles underlying the construction of the growth points were to: describe the development of mathematical knowledge and understanding in the first three years of school in a form and language that was useful for teachers; reflect the findings of relevant international and local research in mathematics (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; Wright, Martland, & Stafford, 2000); reflect, where possible, the structure of mathematics; allow the mathematical knowledge of individuals and groups to be described; and enable a consideration of students who may be mathematically vulnerable. The processes for validating the growth points, the interview items and the comparative achievement of students are described in full in Clarke et al. (2002). The following are the growth points for Place Value.

1. Reading, writing, interpreting and ordering single-digit numbers.
2. Reading, writing, interpreting and ordering two-digit numbers.
3. Reading, writing, interpreting and ordering three-digit numbers.
4. Reading, writing, interpreting and ordering numbers beyond 1000.
5. Extending and applying place value knowledge.

Each growth point represents substantial expansion in knowledge along paths to mathematical understanding (Clarke, 2001). The number tasks in the interview take about 20 minutes for each student and are administered by the classroom teacher. There are about 40 tasks in total, and given success with a task, the teacher continues in a domain (e.g., Place Value) for as long as a child is successful. Teachers report that the ENI provided insights that might otherwise remain hidden (Clarke, 2001).

Refining Assessment Tasks for 2-digit Numbers - Growth Point 2 (GP2)

This paper examines students' Place Value Knowledge and the effect of two new tasks designed to identify students who were assessed at GP2, but who may not interpret successfully the quantitative value of 2-digit numbers. The data examined is drawn from the 2010 assessment interviews of nearly 3000 Reception (R) to Grade 3 students (5-8 years old) from 42 low SES school communities in Victoria and Western Australia who are part of the *Bridging the Numeracy Gap Project* (Gervasoni, Parish et al., 2010). This is a Federal Government funded Project aiming to close the education gap for low SES and Aboriginal and Torres Strait Islander students, and is a collaboration between 42 school communities, Catholic Education Offices in the regions of Ballarat, Sandhurst, Sale, and Western Australia, and Australian Catholic University. The new tasks are shown in Figure 1 below.

Pop-Sticks Bundling Task

Ask the child to unpack the icy pole sticks. Here are some icy pole sticks in bundles of ten (*offer the chance to check a bundle if it seems appropriate).* Here are some more loose ones. *Show white card for 36.*

a) Get me this many (icy pol e) sticks. *(If the child starts to count all in ones, interrupt and ask them if they can do it a quicker way with the bundles. If they can't,* Tell me how you worked that out.

b) Please put one bundle back. How many sticks are there now? How do you kno w that?

2-Digit Number Line Task

Show the child the mauve 2 -digit number line card.

Look at this number line. Please tell me the largest number. *(100)*

Point to the little mark . What number would go here? *(acceptable number range is 45 -55)*. b) Please explain.

Figure 1: New Growth Point 2 tasks. Students' Place Value Knowledge.

Part b of the Bundling Task was designed to distinguish those students who use the integrated sequence/separated tens strategy when interpreting a collection of 36 pop-sticks. Inclusion of the number line task reflects the emphasis in the research literature of the importance of students developing a mental number line to interpret quantities when problem solving.

A key issue for the research reported in this paper was to determine students' Place Value Growth Points, and whether the new GP2 tasks identified students who were not successfully interpreting the quantitative value of 2-digit numbers. Figure 2 shows the distributions of Growth Points at the beginning of the 2010 school year for nearly 3000 Reception-Grade 3 students.

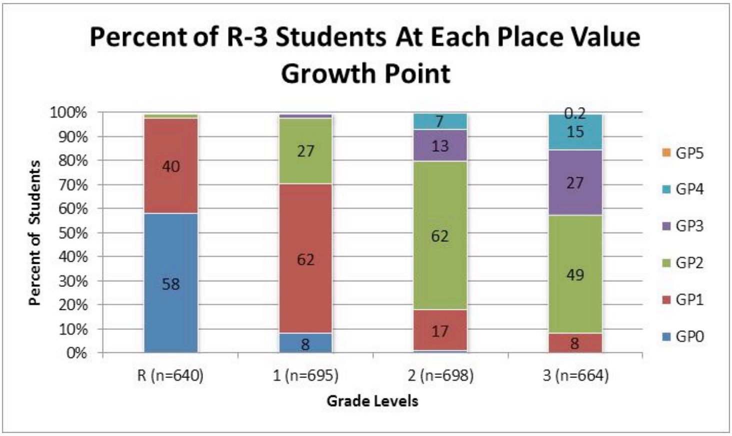

Figure 2: Place value growth point distribution for R-Gr 3 students.

Each student was assessed by their classroom teacher, and the growth points were calculated independently by trained coders to increase the trustworthiness of the data.

An issue highlighted in Figure 2 is the spread of growth points at each level, particularly from Grade 1 onwards. This has been noted elsewhere (e.g. Gervasoni & Sullivan, 2007; Bobis et al., 2005) and confirms the complexity of the teaching process and the importance of teachers identifying each student's current knowledge and knowing ways to customise learning to meet each student's needs.

These data indicate that about two-thirds of Grade 1 students have reached GP1, and therefore the initial focus for Place Value instruction is GP2 – 2-digit numbers. By the beginning of Grade 2, most students reach GP2. However, by Grade 3, half the students remain on GP2. Examination of the assessment tasks for GP3 and GP4 indicate that students cannot reach these growth points if they do not interpret the quantitative value of numbers. We also noted that students could reach GP2 successfully using only procedural knowledge to read, write and order numbers, and collect 36 pop-sticks. The original tasks did not actually require the interpretation of quantity, although conceptual knowledge was assumed.

Next we examined the data to determine the effect of the new GP2 tasks to determine whether these tasks identified any students who were not interpreting the quantitative value of numbers. The first new task required students to identify the value of a quantity that was reduced by ten. Students strategy for achieving this was observed and recorded by teachers on the assessment record sheet, and only students who were judged to be using Fuson et al.'s (1997) integrated sequence/separated tens strategy were deemed to be at GP2. This provided confidence that students were able to use all five correct conceptions of 2-digit numbers. The second task required students to interpret a number line. Students were asked to identify the number that was half way between 0 and 100 on the number line. Students who stated a number between 45 and 55 were deemed to be successful. As most students in Grades 2 and 3 had reached GP2, students in these grades who were assessed at GP2 were selected for further examination, and their responses to the two new tasks analysed.

The data presented in Figure 3 demonstrate that these tasks did identify some students who were assessed at GP2, but who did not successfully interpret 2-digit numbers in the Bundling and Number Line tasks. More than half of the Grade 2 students and one-third of the Grade 3 students on GP2 were not able to solve the two new tasks. This highlights that interpreting 2-digit quantities is an issue for many students. The number line task was the more difficult of the new tasks. The most common incorrect response was 10, with students counting by ones along the number line until they reached the half-way mark. Of the remaining students who were successful, analysis of their responses to the 3-digit assessment tasks showed that none of these students were successful with the interpretive tasks, although most could read, write and order 3-digit numbers. This inability to interpret quantities was the reason why all these students did not progress to GP3.

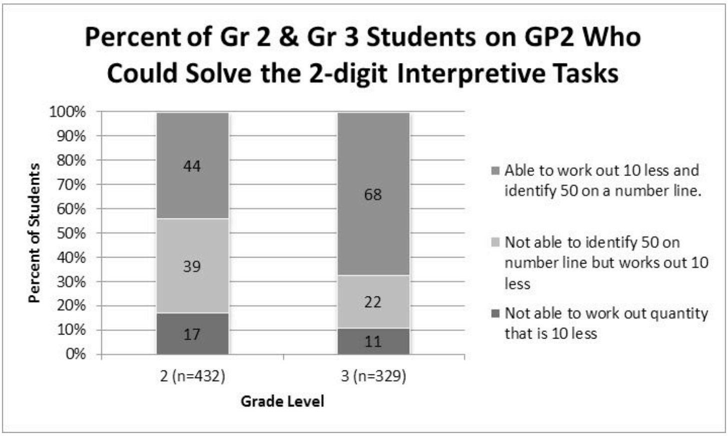

Figure 3: Percent of Gr 2 & 3 students on GP2 who could solve the 2-digit tasks.

CONCLUSION

Analysis of 761 Grade 2 and Grade 3 students' responses to new tasks in the ENI showed that these tasks distinguished students who were assessed as understanding 2-digit numbers, but who in fact could not identify 50 on a number line or state the total of a collection of bundled pop sticks (3 tens and 6 ones) that was reduced by ten. These additional tasks assist teachers to identify students who need further experience with 2-digit numbers to construct full conceptual understanding, and highlight the importance of teachers focusing instruction on interpreting quantities, and not simply reading, writing and ordering numerals. Most children learn to read and write 2-digit numbers quite easily, but interpreting the cardinal value of these numbers is the greater challenge. However, it is this interpretation of quantity that is essential for problem solving and conceptual understanding. Perhaps the fact that the ENI has not included tasks that identify students who do not fully interpret 2-digit quantities has given teachers an inflated impression of some GP2 students' understanding. We argue that some of these students need further instruction focused on their development of 2-digit number conceptions and a mental number line.

One implication of the findings is that learning trajectories associated with Place Value and the development of whole number concepts need to adequately account for students' interpretations of quantities. We believe that the ENRP Place Value growth points and the associated assessment interview needs to be modified accordingly. Such a refinement will give teachers more certainty about students' current knowledge and assist them to design more precise instruction.

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DEVELOPING EFFECTIVE SOCIOMATHEMATICAL NORMS IN CLASSROOMS TO SUPPORT MATHEMATICAL DISCOURSE

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This paper reports on the findings of a three-year case study project situated in ten Pacific coastal public middle schools. We extend the generally accepted construct of sociomathematical norms to include the affect of cultural context on discourse and advancing students' mathematical learning. We posit a teacher's ability to negotiate a set of sociomathematical norms that successfully support student learning may depend, in part, on the teacher's understanding of students' cultural context. In particular, we note that project teachers who engaged students in the oral tradition of "talk-story" were better able to initiate and sustain a level of discourse that extended student learning.

INTRODUCTION

This paper reports on the findings of a three-year case study project situated in ten Pacific island public middle schools. The study extends the generally accepted construct of sociomathematical norms (Yackel & Cobb, 1996) to include the affect of cultural context on classroom discourse and potential to advance students' mathematical learning. Observation data across the three years of the study suggest a teacher's ability to negotiate a set of sociomathematical norms that successfully support student learning may depend, in part, on the teacher's understanding of students' cultural context. In particular, we propose that project teachers who engage students in the oral tradition of "talk-story" are better able to initiate and sustain a level of classroom discourse that facilitated student learning.

The project's research focus and corresponding professional learning activities focus on investigating and deepening teachers' Content Knowledge for Teaching (CKT). We recognize that teaching effectiveness is influenced by the mathematical knowledge that guides the specific tasks of teaching, including the knowledge needed to (a) interpret and enact state standards and curricula; (b) assess student work (Adler & Davis, 2006); (c) respond appropriately to student questions; and (d) choose and/or create questions and problems that correctly target specific mathematical concepts (Ball, 2003). In fact, studies at the elementary level have found that improving teachers' mathematical knowledge for teaching significantly affected students' learning of mathematics (e.g. Hill, Rowan, & Ball, 2005). Understanding the potentially positive implications of advancing teachers' CKT, the research team was intrigued when early in the process of collecting and analysing the observational data, we noted an unexpected and significant cultural factor that appeared to negatively impact student learning.

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We suggest that although advancing teachers' content knowledge for teaching may advance one's ability to support student learning of mathematics; it is equally important to advance teachers' understanding of culturally relevant practices that serve to engage student participation in the mathematics classroom. This paper reports on Keoni, a Native Hawai'ian teacher, whose practice incorporates the oral tradition of "talk-story," a form of discourse practiced in students' homes and communities. Students' cultural understanding of "talk-story" encourages them: to be keen listeners when others communicate ideas; to be compassionate, and support others who may need help to communicate their thinking; and to remain open and not dispute another's ideas without justification (Affonso et al., 2007). Importantly, talk-story is embedded in the practice of *kuleana*, an individual's responsibility for their actions that require them to do their best work in support of the community.

THEORETICAL BACKGROUND

For the purposes of this study, we have adopted the framework of *sociomathematical norms*. We recognize that each classroom develops its own set of norms regarding mathematics. "Sociomathematical norms . . . are established in all classrooms regardless of instructional tradition" (Yackel & Cobb, 1996, p. 462). Yackel and Cobb (1996) argue that these norms are established in stages, as components of the development of a classroom culture. During the first stage, the teacher and students engage in discourse about the *description of a procedure* (e.g. instructing students in the necessary procedures to do a mathematical task). The discourse that occurs during the second stage *describes an action* on a real (mathematical) object. During the third stage, the teacher and students accept or reject the second stage as a valid *object of reflection* and decide if it is valid for others [italics added for emphasis]. These three stages can be interpreted as stages of computation, conceptual explanation and reflective action to other instances or cases. Importantly, as students begin to consider the adequacy of an explanation as it pertains to and can be used and understood by others in the classroom community rather than simply for themselves, the explanation itself becomes the explicit object of discourse (Feldman, 1987). In particular, this project characterizes sociomathematical norms through the lens of students' *intellectual autonomy*, defined in respect to students' contribution to the routines of discourse in the classroom community. This autonomy is co-constructed between the teacher and students. "[W]hat constitutes an acceptable mathematical reason is interactively constituted by the students and the teacher in the course of classroom activity" (Yackel & Cobb, 1996, p. 469). This process of sensemaking is further complicated in a multicultural setting, as evidenced throughout the project observations.

The research team was curious to understand the significant disconnect that seemed to exist between teaching practice and the students' cultural reality when students were not allowed to talk-story in class. Project teachers' ability to hear and interpret students' mathematical thinking through the filter of talk-story was compromised when that thinking and its expression were markedly different than their own

(Thames, 2006). In reciprocal fashion, students struggled to understand and respond to a classroom culture that was markedly different than that found in their home and community. The resulting cultural disconnect between classroom practice and students' need to talk-story limited teachers' abilities to effectively negotiate an appropriate set of sociomathematical norms needed to advance student learning.

Native Hawaiian education, the practice of *Aó*, is based on building the necessary skills to survive and thrive in an ocean-based community.

> The elders well knew that, *I ka nānā no a 'ike,* by observing, one learns. *I ka ho'olohe no a ho'omaopopo,* by listening, one commits to memory. *I ka hana no a 'ike,* by practice, one masters the skill. . . To this a final directive was added: Never interrupt. Wait until the lesson is over and the elder gives you permission. Then—and only then, *nīnau*. Ask questions. (Pukui, Haertig, and Lee, 1979, p. 48 as cited in Chun, 2006, p. 4)

Thus, the five components of Aó, require students to: observe, listen, reflect, do, and question. This deliberate order "form[s] an approach that is different from the methods of inquiry we find in education today" (Chun, 2006); and frames the cultural norm Hawai'an students expect when they enter the classroom.

METHODOLOGY

The professional development component of this project was designed to increase middle school mathematics teachers' ability to: 1.) develop discourse communities within their classrooms; 2.) identify and create questions that promote student discussion; and 3.) focus on student understanding from a formative assessment perspective. This design was based on the understanding that effective questioning and formative assessment strategies extend from CKT and serve as a model for teachers' and students' emerging development of sociomathematical norms in the classroom. The project was situated in racially and ethnically diverse public school classrooms. The Hawaiian population by ethnicity includes Asian (42%), White (24%), Native Hawaiian or Other Pacific Island (9.4%). The picture of diversity becomes even more complex when mixed-race students are also considered, with 58% of Asian, 39% of White, and 23% of Native Hawaiian and Other Pacific Islanders reporting as one race alone or in combination with one or more other races (United States Census Bureau, 2000). Within this culturally rich setting, educators are challenged to understand and implement appropriate cultural practices that better support student learning. Specifically, the vast majority of the project population derives from a long-standing oral tradition for group work called "talk-story" (Affonso, Shibuya, & Frueh, 2007). "Talk-story is respected for its value of behaviors that resemble ground rules," (p. 403) which serve as the basis for connecting school work to students' cultural understandings.

The study examined teachers' capacity to identify and create questions leading to student discussions that have been linked to promoting student understanding, such as funnelling vs. focusing, (Wood, 1998) and generating discussion. Ten middle school mathematics teachers were observed over the course of a three year professional development project. Two researchers conducted each observation, with one person charged with following the progress of the discussion and tracking the questions asked by the teacher and the students, the number of teacher/student and student/student interchanges, and the mathematical content of the interchanges. The other researcher was charged with tracking the mathematical trajectory of the class and the emerging sociomathematical norms that framed classroom discussions. Immediately following each observation, the two researchers met to debrief, compare notes, and to create a single document to authenticate the pedagogical and mathematical path of each lesson.

The resulting data was then organized into a map that emerged from the analysis of the discourse between the teacher and students. Interactions were classified as teacher-generated or student-generated according to who asked the initial question that began the discussion. Questions which elicited a limited set of specific and correct answers were categorized separately from questions that led to discussion of underlying mathematical concepts. Questions were also categorized by direction (teacher to student, student to teacher, or student to student).

RESULTS

This study was initially conceived to identify teachers' implementation of formative assessment practices as a result of their participation in a professional development project. The working hypothesis was that providing teachers with ongoing opportunities to learn about funnelling, focusing, and discussion generating questions would support students' engagement with mathematics and would assist them in identifying and creating appropriate questions and discussions. In addition to the case study observations, data for this study came from a pre-post test of CKT utilizing the University of Michigan's *Learning Mathematics for Teaching* instrument, project generated student pre-and post-tests, teacher reflective logs, and field notes and reports from the professional development facilitators. The teacher pre-and post-tests were used to determine changes in teachers' CKT, with particular attention paid to the items that focused on interpreting student work and the ways to promote student understanding of mathematics; and questions that asked teachers to consider possible responses to student misconceptions. Data from the teacher and student assessments were analysed quantitatively to determine the extent to which any changes occurred and if these changes were significant. The case study observations were analysed qualitatively to illustrate possible links between the professional development experiences and any change in teachers' use of formative assessment strategies, paying particular attention to questioning strategies of the three types and the resulting classroom discussions.

For the purposes of this paper, we report on Keoni. We noted early on that the lesson structure and classroom norms established by Keoni differed significantly from that of other project participants. Keoni began each lesson by modelling a concept so that students had an opportunity to observe expert practice. During each lesson we noted that he would often tell students “Stop what you’re doing, look up here, and listen to me.” Many of Keoni’s lessons incorporated a project to be completed in small groups. Over time we observed a reoccurring lesson structure in which students would *observe* the task being done by the teacher, *listen* to his explanation, have opportunities to *reflect* and talk in groups as they *practiced* the new skill. Only then were students encouraged to *ask questions*.

For example, in one lesson (a scaling activity), Keoni put a map of the students’ island under the document camera and called their attention to the map scale. He showed them how to measure the scale in centimetres and set up the activity that followed (observe) They decided to use 6.6 centimetres as equivalent to 6 miles. Having observed Keoni’s use of the scale, students were told to work in groups and use the map and scale to find the actual distance between two island locations.

> *Ok, boys and girls, stop, look and listen.*(listen) *Please do a little self evaluation and think about if you are on task. …What strategy can we use to figure out what centimetres would be in miles? Think back to the strategy we could use to do this*. (reflect)

After the teacher explained the process and set up the activity, he allowed students to work in groups at their own pace. (do) Stopping at one table, he asked students what information they needed to know to calculate the ratio. He scaffolded this question by reminding students of a similar activity that they did earlier in the week. Periodically, he shared strategies with the whole group. This was a safe time for students to practice their understanding and make mistakes with fear of embarrassment. (question) We observed much “on task” talk during this block of time as students asked questions of each other and of the teacher. As Keoni observed each group, he monitored their mathematical progress while simultaneously reinforcing the concept of kuleana (a sense of responsibility for oneself and others). He commented to one group,

> *What I’m noticing is that you’re not working as a team. We need to figure out how we can work this out together so that we figure it out as a team …together.*

After noticing that two students had lagged behind, he admonished one group,

> *You need to figure out what your teammates need ...to catch up to you and answer the remainder of the questions.*

Keoni established a set of classroom norms in his mathematics classroom that reinforced students home culture, and which socialized “students to the often implicit cultural expectations of the classroom such as turn-taking, participation rules, and established routines” (Echevarria & Short, 1999, p. 5). Additionally, when 1700 project students were given a project-created algebra test (29 question pre and post), Keoni’s students on average, were able to correctly answer two and one half more questions on the post test than they did on the pre-test. This rate student growth exceeded the project class average of two more correct answers on the post test.

DISCUSSION

After observing multiple classrooms the first year of the project, we were curious to understand teachers' reoccurring admonitions to students to *not* talk-story. Realizing our "outsider" status, we spent time with the researchers in the Hawaiian Studies program at the University of Hawai'i, exploring native cultural traditions. In particular, we appreciate the help of Dr. Morris Lai and the Pihana Na Mamo project staff for helping us better interpret what we were seeing in classrooms.

We examined different levels of participation by students who are intellectually autonomous in mathematics contrasted with those who rely on the pronouncements of an authority to know how to act appropriately. The practice of structuring classrooms around student-student interactions, while potentially valuable, does not guarantee that the interactions will be purposeful and effective (Lobato, Clark, & Ellis, 2005). The link between the growth of intellectual autonomy and the development of an inquiry based classroom environment becomes apparent when we note that, in such classrooms, the intent is for the teacher to guide the development of a community of inquiry, which encourages a transference of responsibility from the teacher as sole source of mathematical knowledge to a shared responsibility of sense making and collaboration. "Many teachers find it easy to pose questions and ask students to describe their strategies; it is more challenging pedagogically to engage students in genuine mathematical inquiry and push them to go beyond what might come easily for them" (Kazemi & Stipek, 2001, p. 60). The purpose of establishing sociomathematic norms is to "help students to clarify their statements, focus carefully on problem conditions and mathematical explanations, and refine their ideas" (NCTM, 2000).

Theoretical arguments for the discourse-learning connection are based on social-constructivist and social-cognitive perspectives (Cobb, Yackel, & Wood, 1992; Hatano, 1988; Pimm, 1987). It is commonly accepted by researchers, teachers, and teacher education programs that discourse benefits student learning; if students are talking about mathematics, they must be learning about mathematics (Piccolo et al, 2008). Based on the findings, the project observation team noted a significant lack of examples of teachers and students participating in mathematical discourse that could be considered "doing mathematics." We attribute this gap in the educational process in two ways: (1) teachers lacked significant content knowledge for teaching to support the level of mathematical discourse needed for student learning, and (2) there was a significant disconnect between what constitutes teacher-allowed discourse and students' cultural understanding of community-based discourse.

CONCLUSION

Paradoxically, project teachers consistently warned students that talk-story was not allowed in the mathematics classroom. –Teachers considered talk-story to be "a waste of students' time," and so its practice was actively and consistently discouraged. It is our contention that limiting students in this manner effectively stops the potential to develop a positive set of sociomathematical norms that encourage mathematical discourse and concomitantly, student learning. Rather, the majority of students whose cultural apprenticeship for learning stems from an oral tradition are disenfranchised from practicing learning strategies that they have learned at home and in their community. "No talk-story" effectively silenced student discourse and thus, reduced learning. In the case of Keoni, students were encouraged to do mathematics in a culturally appropriate manner that supported essential classroom discourse and advanced student learning.

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EXAMINING THE CONNECTION BETWEEN TEACHER CONTENT KNOWLEDGE AND CLASSROOM PRACTICE

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This paper extends existing research regarding content knowledge for teaching (CKT) and the role it plays in advancing student learning. In this report, two teachers (one with high and one with low measured CKT) are observed on the same day teaching the same content. Many studies have recently been published linking student achievement to teacher's CKT, and in the U.S., many schools are beginning to include CKT measures in teacher hiring and retention decisions. The classroom teaching observed for this study illustrates that content can effectively be taught by teachers across the spectrum of CKT levels, but that the observable and significant differences in their teaching leads to important questions for inservice and preservice teacher educators.

INTRODUCTION

The single greatest factor determining student achievement is the quality of the teaching[1]. This paper extends current research into teacher effectiveness by extending existing research into content knowledge for teaching (CKT) and the role it plays in advancing student learning. Specifically, we examine the connections that exist between teacher CKT and classroom practice. The research reported here was part of a larger case study of participants in a National Science Foundation project that investigates feasible models of implementing formative assessment in mathematics.

The paper researches and extends the construct of *Content Knowledge for Teaching* (CKT), the mathematical knowledge and skill unique to teaching (Ball, Thames, & Phelps, 2008). This construct is based on the understanding that just as many professions require effective practitioners to possess skills that are distinctive to their work, effective teaching requires not only a deep understanding of mathematical procedures and concepts, but also of the learning trajectories and emerging knowledge of students in schools. Mathematics teachers use CKT to identify how mathematical tasks relate to and build upon one another, recognize salient features of tasks, and includes understanding how a shift of features of a task can aid (or hinder) the development of additional ideas, concepts, or procedures. We seek to further our understanding of the way that teachers' knowledge of CKT influences teaching practice and resulting ability to teach effectively.

CONCEPTUAL FRAMEWORK

It is generally accepted that mathematics teachers' effectiveness is influenced by the mathematical knowledge they possess. For example, when teachers differentiate problems to challenge and/or provide additional scaffolds for students, their

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understanding of mathematics allows them to: 1.) listen to students' explanations of unconventional solution strategies and quickly determine whether or not they are likely to lead to generalizable approaches, 2.) press student thinking through appropriate questioning, and 3.) create or select formative and summative assessment problems that are mathematically appropriate for the class.

Over the past two decades, research studies suggest that while individuals with bachelor's degrees in mathematics may have a specific kind of knowledge, they often lack what Liping Ma (1999) described as a *profound understanding of fundamental mathematics*, a deep understanding of basic mathematical ideas. And yet, a major factor in increased student achievement is a knowledgeable, skillful teacher (NCTA, 1996). In fact, Darling-Hammond and Ball (1998) conclude that teacher quality accounts for 40% of the variation in student achievement. Knowing how to respond appropriately to students' questions and develop the ability to choose or create questions and problems targeting specific mathematical concepts is at the centre of the content knowledge needed for teaching (Ball, 2003). Studies involving teachers of elementary students have found that improving their mathematical knowledge for teaching significantly affects students' learning of mathematics (e.g. Hill, Rowan, & Ball, 2005). At question is how best to conceptualize and implement appropriate components of mathematics content, pedagogy, and other aspects of teaching to pre and inservice teacher education.

METHODOLOGY

This study is a comparative case study of two teacher participants in a professional development project. The project included thirty-two teachers from 15 schools in a Pacific coastal district. Both of the teachers reported on in this study taught in the same school. Overall, the project teachers participated in five days of full-cohort professional development in June 2008 and four days in June, 2009; five half-day follow up sessions during the 2008-2009 and 2009-2010 school years, and at least three coaching visits per year from project staff.

Data collection included the University of Michigan's *Learning Mathematics for Teaching* (LMT) instrument to measure any change in participants' CKT. This test was administered at the beginning of the summer institute in year one, after one year of participation, and again at the end of the project. The content strands of this test include items intended to assess teacher's fluency with determining and interpreting patterns, functions, expressions, equations, and representations. The instrument consisted of 29 responses in the form of multiple-choice questions. Project-created student pre and post tests were administered to all of the participating teachers' students in September and May of both years. Analysis of the second year's student data has not been completed and is not included in this analysis.

The two teachers involved in this case study, Elina and Keoni, were chosen because although they worked closely together (they both taught seventh grade and met daily to plan their lessons), they represented the upper and lower quartiles of scores on the

LMT. Both Elina and Keoni were observed a total of five times each year over a two year period. Their preservice coursework was similar, and completed at the state university. They are both relatively new teachers, with five and three years of teaching experience; and on the teacher pre-survey they both reported a high level of satisfaction with their ability to work with technology.

Two researchers conducted each observation, with one person charged with following the progress of the discussion and tracking the questions asked by the teacher and the students, the number of teacher/student and student/student interchanges, and the mathematical content of the interchanges. The other researcher tracked the mathematical trajectory of the class. Immediately following each observation, the two researchers met to debrief, compare notes, and to create a single document to authenticate the pedagogical and mathematical path of each lesson. Interactions were classified as teacher-generated or student-generated. Questions which elicited a limited set of specific and correct answers were categorized separately from questions that led to discussion of underlying mathematical concepts. Questions were also categorized by direction (teacher to student, student to teacher, or student to student).

RESULTS

The LMT scores are shown in Table 1 below. The test was administered three times. The same form was given at the beginning of the project and again after one year of participation. A post-test was given at the end of the project.

	Pretest 1 June 2008	IRT Scale Score	Pretest 2 May 2009	IRT Scale Score	Posttest May 2010	IRT Scale Score
Elina	76%	0.381756	83%	0.723952	93%	1.517683
Keoni	48%	-0.73437	48%	-0.73437	74%	0.348509
Project totals	70%		78%		78%	

Table 1: LMT scores

We recognize the limitations of reporting percentage scores for individuals and the small number of participants in this study. The figures are reported only for purposes of comparison within the data set. Keoni's score was unchanged on the second test, although he did change eight answers on the second test iteration. Elina's score increased with each test, and she had among the highest scores for each test.

The importance of the growth in both of their scores is seen as a predictor of student achievement. For each one point gain on all project teacher's post-test scores in year 1, their students achieved 0.448 higher points on the student post-test after accounting for the influence from the other teacher variables (Olson, Im, Slovin, Olson, Gilbert, Brandon, Yin, 2010). The results of the student test are shown in Table 2 below.

	Average number of correct responses Fall 2008	Average number of correct responses Spring 2009	Difference Post – Pre
Elina	13.9	19.2	5.3
Keoni	14.2	16.7	2.5
Project Total	14.0	16.0	2.0

Table 2: Student pre and post test results

Please note that these student scores are from the first study year, the year in which Keoni's CKT score did not change. Also notice that the increase in the number of correct responses in both cases are above the project average. In particular, Elina's students' averaged a gain of over five correct answers on the post test (among the greatest improvement of all participant teachers). Although Keoni, with one of the lower scores on the first two test administrations, also showed improvement. We were curious to investigate the circumstances behind the fact that in spite of low CKT, Keoni was an effective teacher who improved student learning. This motivated us to carefully review our observations of Elina and Keoni.

The case study findings reported here are from two observations (one each for Elina and Keoni) done on a single day late in the Fall of the second year. The observation from this day was very representative of all of our observations of them, and the results we report could easily have come from other observations. An additional statistic that should be reported is the number of mathematical errors made while teaching (an error was coded as a mathematically incorrect statement made to the class). Overall, Keoni coded an average of 2.6 mathematical errors per class, while Elina made 1.4 errors. As was their usual schedule, Elina and Keoni had met daily during their planning time to jointly discuss and plan instruction. Given the space limitations for this paper, we will restrict our discussion to the explication of one activity for each teacher. In Elina's case we will examine the focus problem she did at the beginning of the class. With Keoni, we will look at the discussion of a homework problem from the previous day that was reviewed in class.

Elina's teaching style is to move through classroom work very quickly. Students have to attend very carefully to keep up. There is no "catch up" time built into her class. If students fall behind at any point, they may miss critical information. In this class, the focus problem asked students to find the true statement about ΔXYZ from a list that related to ΔABC (Figure 1). The scale factor from ΔABC to ΔXYZ is given as 4.

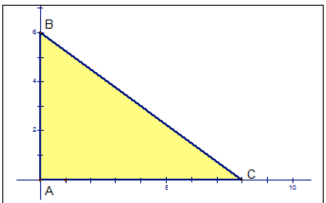

Figure 1: Triangle ABC

1. The area of ΔABC is 16 times the area of ΔXYZ
2. The area of ΔABC is ¼ the area of ΔXYZ
3. The area of ΔABC is 4 times the area of ΔXYZ
4. The area of ΔABC is 1/16 the area of ΔXYZ

Elina begins the activity by displaying the task using the document camera. She is very comfortable using technology to project student responses for all to see. The text says "Using the following similar figures identify the true statements. Hint: find the areas of both triangles." She wants the students to input answers on their calculators and send them to her.

1	E	Send me the area of triangle ABC. *She counts down* 10, 9, 8, ... Send
2	E	Send me the area for triangle XYZ. *Counts down from 10.* . . . . . Send
3	E	Last question . . . Send me the numbers of the questions that you thought were true. *(students appear a bit confused)* Send me the ones that you thought were true. 10, 9, 8, …….. Send
4		*After looking at the submitted responses (she has not displayed them on the screen for the rest of the class), Elina recognizes that many students are confused.*
5		Can somebody remind us how to find the area of a triangle?
6	S1	Base times height
7	E	One half base times height. … *Pauses* . . .Does ½ make a big difference?
8	S2	Yes
9		*Elina now shows the student responses for the area of triangle ABC. 11 out of 19 responses show 24, the correct answer.*
10		*Looking at the screen, Elina notes that there are 3 responses of 48.*
11	E	What did these 3 people forget to do? *(pointing at 48)*
12	S3	Divide
13	E	Some people forgot to divide by two.
14	E	Let's take a look at this one. ….11. *Five responses of 11.* How many of you just added? *(no one responds)* …
15	E	That's something we may have to review. ….huh? …the area of triangles.
16	E	On #1, is ABC being multiplied by 4? *Several student respond with "No"*
17		*Elina works through the solution aloud and determines that it is True.*
18	E	What is this question asking? *#1, which stated that the area was 16 times larger.*
19	S2	You can fit 16 ABC triangles into XYZ.
20	E	No, but close. *(This is true, but not what the question asked. Elina did not clarify this point.)*
21	S4	ABC is bigger than XYZ. *(This is not true.)*
22	E	Is this true or false? *Several students say "False"*

In Keoni's class, there was very little full class discussion and his primary teaching method was to provide instructions to the full group, circulate between individual tables, and answer student questions. The task for the day involved scaling a rectangle on a grid by a scale factor of ¼. The rectangle is shown in Figure 2. As

Keoni begins the activity, he goes over the instructions. He tells them that they will need to answer numbers 1 and 2 before they can answer the rest of the questions. He carefully tells students that for some of them this is a review of how to draw the figures on a grid, but that because some students don't know how to do this, they will revisit how to draw the figure. Keoni is giving students very detailed instructions about how they are to proceed. The first instruction asked the students to plot the points. The second asked students to dilate the original figure by ¼ using the point A as the origin. The rest of the worksheet asked the students to describe what needed to be done to perform the dilation. Students are required to get Keoni's approval for the first portion of the task before they can move on to the next step. He sets a timer for 10 minutes and tells students to draw the figures. As Keoni goes around the room and checks students' answers, he asks a couple of students if they mind moving to other tables to share their process and thinking with other students.

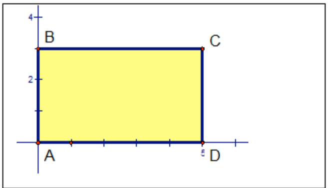

Figure 2: Rectangle ABCD

When the timer rings, most of the students are still struggling to identify the points on the grid. Keoni goes to each table and makes sure that they are able to draw the figure correctly before he allows them to proceed to the next step. As he has students check their points, they catch that several individuals have mislabeled the figures. Once they correct their labels, Keoni lets them check off that problem.

Next, he asks the class if dividing by ¼ is the same as multiplying by .25. Several students say "Yes." Keoni gives students 20 minutes to do the rest of the questions. After three minutes, he stops the work and tells the class that everyone is having some difficulties, and they're going to go over the problem step-by-step.

> *What I'm noticing that you're not working as a team. We need to figure out how we can work this out together so that we can figure it out as a team …together.*

After Keoni says this, two girls who had correctly solved the problem got up from their seats and went to the other side of the table to help their table partners find the solution. Several students at another table also began helping a table partner who was struggling. Keoni allowed students "an extension of the time," telling the class that they should "figure out what your teammates need to catch up to you and answer the remainder of the questions." Two girls at a front table were persistent in their effort to help a girl who was clearly struggling. Several groups had huddled together and were working on the problem. Students actively responded to Keoni's call to work together and were engaged in finding a solution.

DISCUSSION

The difference between the two classrooms is striking. Elina was relentlessly efficient. She had very specific classroom procedures and rules that she expected students to follow without deviation. Several times she told students, "Let's not waste time ….yeah?" As seen in Lines 1 and 2 above, she followed many tasks with a countdown from 10 to keep the class moving forward. Her style in responding to incorrect student answers was similarly direct (see Line 7). Her instruction followed a characteristic pattern, in which she quickly reviewed responses, comments or made corrections, and then moves on. Although she constantly asked questions and listed to student feedback, it was clear that Elina was the focal point of this classroom. This is clearly seen in Lines 19 – 21, and although the student's response was accurate, Elina chose to keep students focused on the side lengths being multiplied by 4.

Elina's interactions with individual students tended to be brief and to the point. Her CKT was evident in both her interactions with students and with the mathematics. Questions were largely funnelling (Wood, 1998), and seem intended to move students in a set direction. She did not need to mask her understanding of the content by making broad, general statements. Rather, her comments were driven by a predetermined solution strategy. While Elina maintained a strict focus, students did feel comfortable teasing her (the boys in particular).

Keoni was also quite intentional about each step in the process of instruction, but his focus was more on the social nature of the learning community. He consistently reminded students that they had a responsibility to their groups. Keoni reinforced a culturally appropriate community dynamic. In contrast to Elina, Keoni often gave students "an extension of time" so that they might complete their work. His major press was to create a collaborative community of students engaged in the mathematics. Unfortunately, Keoni's lack of appropriate CKT allowed many students to leave the classroom unsure of how to solve this particular problem. A major source of misunderstanding for students when scaling is to understand the difference between relative change (multiplicative) and absolute change (additive). This activity led students to think additively and will likely cause them to have misconceptions later. Keoni was unable to resolve this situation. Also, several times when discussing the scaling activity he referred to sides as congruent, not corresponding, a major mistake that may also lead to later confusion for the students.

CONCLUSION

The challenge for this study is to derive conclusions from two dimensions of data, CKT and pedagogical practice, which seriously compound traditional comparison methods. Although CKT has been qualified as a valid predictor of teaching effectiveness and student achievement, there remain other factors that also positively influence teaching effectiveness. In this study, Elina's CKT was measurably greater, as was her student's achievement. But higher student scores may also be the result of Elina's pedagogical style, which was demonstrably different from Keoni's. Conversely, the classroom environment developed in Keoni's class did result in student learning, in spite of a lack of specific mathematical direction and a greater number of mathematical mistakes.

This preliminary study was undertaken to investigate an interesting discrepant case, and to define parameters for future research. Recognizing CKT is closely linked to classroom practice, how do we increase the CKT of inservice teachers with relatively high effectiveness but low content knowledge? Further, since preservice teacher course work largely concentrates on pedagogy with limited CKT focus, we continue to question what may be done to improve CKT of preservice teachers? We posit that improvements for preservice and inservice teacher education lie in our ability to understand (a) how CKT is supported by pedagogical practices, (b) how pedagogy can advance CKT, and (c) possible connections that will result in more effective practice. We believe that future research should study more than teacher content knowledge or pedagogical practice in isolation. Without attempting to mandate a course of study and practice that devalues either CKT or supportive pedagogy, our continued challenge is to learn enough about the intersection of CKT and pedagogical practice to support teacher learning from both perspectives.

[1] National Comprehensive Centre for Teaching Quality, 2005

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A CONSTRUCTION OF A MATHEMATICAL DEFINITION – THE CASE OF PARABOLA

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Although definition is a cornerstone in the building of mathematics, it is neglected in the school syllabi of many countries. Research studies in mathematics education report some difficulties students experience with definitions: Students have a limited grasp of the role of definition and don't always distinguish between descriptions and definitions. The result is a poor understanding of concepts and consequences thereof. The parabola is an example of a concept that students grasp as the graph of a function rather than as an independent geometrical shape. We present a research study in which students were engaged in the process of constructing the definition of a parabola as a locus. We argue that by doing so the students are likely to improve their understanding of parabola and may overcome inappropriate concept images.

THEORETICAL BACKGROUND

Mathematical definition

The origin of the word definition is the Latin word *finis,* which means end, boundary, border. "When you define something you put boundaries around what it can mean. A good definition puts an end to confusion about what a term means" (Schwartzman, 1994, p. 68). Pimm (1993) calls the notions of definition, theorem, proof, and proposition *meta-mathematical marker terms*: "terms which serve to indicate the purported status and function of various elements of written mathematics" (p. 261).

A famous example for the importance of a definition and its role is Cauchy's "mistake" as presented by Lakatos (1978): Cauchy has been regarded by historians of mathematics as the person who gave the calculus its final foundation and put it on solid ground. In his Course d'Analyse (1821), Cauchy proved that the limit of an everywhere convergent sequence of continuous functions is continuous, a claim that is today considered false. Was it carelessness? Oversight? According to Lakatos, Cauchy's "mistake" stemmed from using a different definition of continuity from the one we use today. While this famous example emphasizes the arbitrariness character of definition our study emphasizes the conceptual understanding aspect of definition.

Definitions in mathematics education

Vinner (1991) states that "the ability to construct a formal definition is a possible indication of deep understanding" (p. 97), and the NCTM Standards (2000) recommend "to give students experiences that help them appreciate the power and precision of mathematical language" (p. 63). In spite of that, Borasi (1992) claims that "an analysis of the most popular syllabi and textbooks, as well as conversations

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with several mathematics teachers, soon made it clear that despite its importance, the notion of mathematical definition is rarely, if ever explicitly examined in precollege mathematics instruction" (p. 7).

In the research literature there is evidence for various difficulties students experience with mathematical definitions. Students don't perceive the nature and the functions of mathematical definitions (e.g. Edwards & Ward, 2004; Zaslavsky & Shir, 2005), and have difficulties using definitions. A difficulty documented in many studies is that students use concept images that are not compatible with the concept definition (Tall and Vinner, 1981) to solve problems (e.g. Alcock & Simpson, 2009; Vinner, 1991).

Researchers argue that putting students in a situation in which they feel the need for definition will promote the acquisition of the definition and the understanding of the concept (e.g. Kidron, 2008, in press; Nachlieli, 2004; Ouvrier-Buffet, 2006).

In this paper we present a study in which students were engaged in a process of constructing the definition of a parabola as a locus, an independent geometrical shape made up of points with a common property rather than just any curve or even a curve that is a product of a specific functional rule. We argue that by doing so the students grasp a better understanding of this concept and overcome some wrong concept images concerning the parabola.

Parabola

The origin of the word parabola is the word *παραβολή*, which means parallelism and refers to the angle of a conic section. Menaechmus was the first one to study the conic sections in the middle of the 4th century BC (Knorr, 1982). He sectioned the cone at different angles. The intersection of a cone and a plane parallel to a generator of the cone is a parabola. Another planar of a parabola is the locus of points in the plane that are equidistant from a given point, the focus and a given line, the directrix.

The graph of the quadratic function $f(x)=ax^2$ is a parabola with focus $(0,\frac{1}{4a})$ and directrix $y=-\frac{1}{4a}$. Since the graph of any quadratic function $f(x)=ax^2+bx+c$ is a shift of the graph of $f(x)=ax^2$, the graphs of any quadratic function is a parabola. The opposite is true, too: for any parabola there is a Cartesian coordinate system in which this parabola represents the graph of $f(x)=ax^2$.

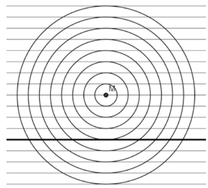
Students usually come to know the parabola at first as the graph of a quadratic function. This is probably the reason why many students grasp the parabola as a graph of a function (sometimes any curve that is not a straight line) and not as an independent geometrical shape like, for example, a circle.

THE ACTIVITY

In this section we describe the activity used in the research. It was designed to raise the need for a definition of a parabola as a locus and to construct this definition. The activity has 4 parts. In the first part the students deal with the notion of locus, in the context of a circle as well as in the context of a perpendicular bisector; they already met the notion of locus in class when defining a circle.

The second part of the activity is represented in Figure 1.

In the drawing there are circles with a common center M and parallel lines. The distance between any two neighboring lines is 1 unit. Every line except the one through M is tangent to a circle.



a. Number the circles from 1 to 8, the inner one being 1.
b. Denote the bold line by L and number the lines above it from 1 to 12.
c. Mark the intersection points between line n and circle n, n=1…8.
d. Which shape do you think passes through the points you marked?
e. Are you sure?

Figure 1: The second part of the activity.

In the third part the students are asked to find a common property of all the points they marked, add more points with this property, and to define the resulting shape using this property. In the fourth part the students are led to realize why the graph of a quadratic function is called parabola. Hence, in the third and the fourth parts the students construct the definition of a parabola as a locus.

THEORETICAL FRAMEWORK

Since the activity offers students opportunities for constructing abstract mathematical knowledge, Abstraction in Context (AiC; Hershkowitz, Schwarz & Dreyfus, 2001) provides a suitable theoretical framework and methodology for the analysis of learning with the activity. In AiC, abstraction is defined "as an activity of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner" (Schwarz, Dreyfus & Hershkowitz, 2009, p. 24). According to AiC, a process of abstraction has three stages: the need for a new construct, the emergence of the new construct, and the consolidation of the new construct. Abstraction will not occur without the need for a new construct; this need may stem from an intrinsic motivation to overcome obstacles such as contradictions, surprises, or uncertainty. The second stage is the central stage during which the new construct emerges. Consolidation is a long-term process, discussed further below.

The RBC model

Abstraction is a mental process and as such it is not observable. For analyzing the second stage, AiC suggests three observable epistemic actions: Recognizing (R) - the learner recognizes that a specific previous construct is relevant to the problem he or she is dealing with; Building-with (B) - the learner acts on or with the recognized constructs in order to achieve a goal like understanding a situation or solving a problem; Constructing (C) - using B-actions to assemble and integrate previous constructs by vertical mathematization to produce a new construct. Hence R-actions are nested within B-actions, and B-actions are nested within C-actions. C-actions may be nested in higher level C-actions.

Constructing refers to the first time the learner uses or mentions a construct. Later uses may be part of consolidation. Consolidation is characterized by self-evidence, confidence, immediacy, flexibility and awareness when dealing with the construct (Dreyfus & Tsamir, 2004), as well as by language becoming more and more precise (Hershkowitz, Schwarz & Dreyfus, 2001) a characteristic of consolidation, which is especially appropriate for the case of definition. We will argue below that this is also a characteristic of the construction of a definition itself.

The match between the design of the activity and the stages of AiC

The aim of the activity is constructing the definition of parabola as a locus. The design of the second part (Figure 1) is expected to raise the need for this construct. By connecting points, students obtain a shape that looks like one they know from another context - quadratic functions. They might ask: Is this the parabola we know? What actually is a parabola? This question expresses the need for a definition. In the third part of the activity, students construct the definition of a parabola as a locus using previous constructs (locus, the property of the points). In the fourth part they realize that a graph of a quadratic function is a parabola according to this definition. This is their first opportunity for using the new construct, and hence for consolidation.

A priori analysis

Constructing can be a long and complex process. In order to focus the analysis of student protocols, we carried out an a priori analysis of the knowledge elements that students might act upon by R-, B- or C-actions. We also operationally define these knowledge elements. Our analysis then looks for these knowledge elements in the protocols in order to follow the constructing process, keeping in mind that students might also use alternative constructs (Ron, Dreyfus & Hershkowitz, 2010). Here we only discuss the operational definition of the knowledge element expected to be constructed during the activity - parabola as a locus - ignoring knowledge elements from the students' previous knowledge like: function graphs or locus.

We shall say that students have constructed the parabola as a locus if they say, in their own words, one of the following: a. for a given straight line and a given point, every point in the plane that is equidistant from the line and the point is on a curve called parabola; or b. for a curve called parabola there exist a straight line and a point from which every point on the curve is equidistant. Part 3 of the activity encourages formulation a, but we assume that students who use a also mean b. Way b might also arise in the rest of the activity, when students discover that a graph of a quadratic function has the same property as the shape they just defined.

One of the characteristics of a definition is precision, as was demonstrated in "Cauchy's mistake" and as expected by the NCTM recommendations cited above. Kidron (2008) reports a learning experience aimed to develop students' understanding of the need for a formal definition. During the construction process of the concept she observed verbalization changes, which found expression in a more and more precise language. To have a better view of the construction process of the definition we also looked for such verbalization changes.

CONSTRUCTING THE PARABOLA DEFINITION AS A LOCUS

Here we report on an interview with two grade 11 students, Noa and Gal, which was audio-taped, transcribed, and analyzed using the nested epistemic actions model for abstraction in context. In this section we present excerpts from the transcript and our interpretation of the analysis as a process of constructing the definition of a parabola as a locus.

At the end of the third part of the activity, Noa and Gal were asked to complete the sentence: "A parabola is the collection of all points in the plane that __________". The following excerpts stem from the part of the interview where they dealt with this question:

185	Noa	[reading] A parabola is the collection of all points in the plane that
186	Gal	are at a straight distance
187	Noa	at the same distance
		…
194	Noa	The collection of all points in the plane that are at the same distance… you understand? I'm trying to write
195	Gal	No this definition is wrong. This point is not at the same distance from this point.
		…
210	Gal	We have one symmetric point that is exactly opposite… it is
211	Noa	Whose distance… the collection of all points in the plane that, like, if their symmetry is at the same distance, somehow. Is it related to symmetry?
212	Gal	Of course
213	Noa	Good. The collection of all points in the plane that are… that the point and its symmetry are equidistant from the ends of the segment? You should also somewhere…
		…
270	Noa	If we, like, chose a line if we choose a point they will have the same distance
		…
286	Gal	They are not equidistant from this thing
287	Noa	Why? If you are looking at the point

288	Gal	They are not equidistant from the point and the line. This point and this point have no connection
289	Noa	Why? But if each one you look at it by itself it is at the same distance from L and at the same distance from M
290	Gal	OK
291	Noa	Like, you see, it isn't like they all equal the same thing. But if you take a point and draw two lines the two lines will be equal
292	Gal	So there will be the same distance also…
293	Noa	Yes. All the points in the plane which everyone by itself is equidistant from a given point and a given line. It is not as though this will be equal to that [presumably referring to the distance of two points from the same object] but this will be equal to that [the distance of a single point from M and from L].
		…
299	Noa	So we'll choose a length, draw it from this direction to that… like, from the focus and from the directrix, we'll intersect them and we'll go on with any ratio we want and we'll get a parabola.

The students recognized that equidistance of points is relevant to the definition. They began to build-with it intending to find the common property, but they were influenced by the locus of a circle which they had considered at the beginning of the activity: all points equidistant from a given point. Hence they looked for a common property in which all points are equidistant from a fixed object (lines 186-187, 194). Gal realizes that this doesn't work (line 195). A long search for the common property follows, during which they recognize symmetry as a knowledge element potentially relevant to their task but fail to build with it the definition (lines 210-213). Somewhat later, Noa finds the common property, which they need to complete the definition (line 270). This is a component of the main construct. Gal is still caught in the wrong conception that all the points must have the same distance from something and Noa directs her not to look at all the points at the same time but at a single point every time, which helps her see the common property (lines 286-292). Here they build with the property that Noa found and construct the definition of a parabola as a locus (line 293). Noa completes the task by expressing the other direction of the definition (mentioned in the a priori analysis above): starting from a given point and a given line one can build a parabola (line 299). Now, Noa and Gal grasp the parabola as a geometrical shape in the plane. In the fourth part of the activity they realize that a graph of a quadratic function is a parabola because it satisfies the definition they have constructed.

Paying attention to the utterances formulating versions of the definition along the constructing process, we observe the progressively more precise language the students use (see lines 186, 194, 211, 213, 270, 289, 293 in this order). We observe, that like in Kidron (2008), in the case of constructing a definition, language becoming more precise is a characteristic of the constructing process itself and not only of the consolidation stage.

CONCLUSIONS

Since definition is a fundamental component in the mathematical world, it is important to make students understand the need for definitions and let them participate in the process of defining; this includes realizing the need for precise language that forms the difference between a definition and a description. Activities designed to put students in a situation of need for a definition and to let them experience the process of defining are a tool of choice for achieving this aim. Constructing a definition by themselves is likely to let students achieve a better understanding of the concept they defined and overcome concept images, which are not compatible with the concept definition.

In this paper we presented a research study, in which students were engaged in an activity designed to raise the need for a definition of parabola as a locus and construct this definition. As a consequence of the process of constructing the definition, the students had the opportunity to coordinate different registers: the parabola as a function with an algebraic representation, its graphical representation and of course the independent geometrical shape made up of points with a common property. It is characteristic of the definition that it permits to have all these representations incorporated in a single term, and we argue that the process of constructing the definition has the potential to help students realize that the concept comprises not only more than any one of its representations but also more than the union of its representations. We see this as the central characteristic of students' participation in the process of defining and we see the parabola activity described and analyzed in this paper as one example of a design that affords students an opportunity for constructing a definition, thus participating in a process of defining.

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STUDENT DIAGRAMMING MOTION: A DISCURSIVE ANALYSIS OF VISUAL IMAGES

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This paper examines pre-service elementary teachers diagramming for a given story. The Systematic Functional framework (O'Halloran, 2005) for visual images is used to analyse the content and representational features of the diagrams. We analyse the strategies that the pre-service teachers used to produce visual images within the framework to identify the different meanings (representational and interpersonal) of visual images. In particular, given the central importance of time and motion in the story, we focus on the strategies used to express time and motion in their diagrams.

DIAGRAMS IN MATHEMATICAL THINKING

Diagrams are "the natural accomplice of thought experiment" writes the philosopher of mathematics Gilles Châtelet; they "capture gestures mid-flight" (2000, p. 10). Diagrams have played a central role in the development of new mathematical ideas, as evidenced in Châtelet's historical investigation. Researchers also highlight the important role that diagrams can play in school mathematics problem solving (Diezmann & English, 2001; Polya, 1957; Nunokawa, 2006). However, diagrams appear infrequently in student work (Kress & van Leeuwen, 2006).

The goal of this paper is to examine diagrammatic conventions used by pre-service elementary teachers. In particular, we are interested in the ways they choose to use diagrams to think about time and motion, which are crucial aspects of many mathematical concepts, including functions. This interest is fuelled by the recognition that visual images—and diagrams in particular—are a key component of the mathematical discourse, and that, as such, learning how to read, use, and create them is central to mathematical learning.

THEORETICAL FRAMEWORK

O'Halloran (2005) argues that mathematical discourse shifts through three semiotic resources: grammars of language, mathematical symbolism and visual images. Given that each resource is perceived according to its unique discourse and grammatical systems, she suggests a Systematic Functional (SF) framework for each resource. The SF framework for visual images enables an analysis of the content, and also the representational features of graphs and diagrams. The content analysis includes the analysis of the two major systems—discourse semantics and grammar—through which visual images are organized as a semiotic resource for representational (experiential and logical), interpersonal and compositional meanings.

Representational (experiential and logical) meaning is the main function of a visual image. The experiential meaning is concerned with the construction of experience through a sequence of episodes, figures, relations, and parts, whereas the logical meaning is concerned with spatio-temporal relations. Discourse semantics is not just about the development of mathematical content through a sequence of images, but it is also about the logical and compositional meanings. O'Halloran emphasizes that "our perceptual apparatus permits logical deductions based on spatiality to be performed through visual means rather than depending upon formalized linguistic and symbolic selections" (p.145). The interpersonal meaning of a visual image is realized through the choices of labels, colour, font, line width and so forth that one uses to construct a diagram or graph. The choices that one makes have direct impact on a viewer's engagement with the significant aspects of the representational meaning of the graph or diagram. As we will not be focusing on compositional meaning in this study, we will not elaborate on it here.

The particular choices that are made in terms of these different meanings are highly conventionalised in mathematics. For example, an understanding of the experiential meaning encoded within a graph comes from reading the graph as a set of relations unfolding temporally and spatially as framed through the Cartesian coordinate system. In terms of interpersonal meanings, it is conventional to labels axes and use certain italics for variables. Learning how to read, use and make diagrams thus involves becoming aware of these conventions. However, as newcomers to the discourse, learners will likely employ non-conventional strategies of expressing meanings in their diagrams. Our goal in this study is to examine these strategies, and study how they can be seen to express productive mathematical meanings.

DIAGRAMMING TIME AND MOTION

PARTICIPANTS AND TASKS

Twenty-five pre-service elementary teachers enrolled in a mathematics methodology course in a medium-sized North American University participated. The course covered basic mathematical ideas in number theory and geometry. As an introduction to story graphs and the Cartesian coordinate system, we asked them to create a visual image related to the following story: *MellowYellow decides to walk to the corner store, which is less than a mile away from her house. She gets about halfway there and stops to pick up a penny. She looks at it for a while and then starts walking toward the corner store again, but faster than before, to make up for lost time.*

Our goal was to draw on the participants' non-normative strategies for graphing the story in order to relate them to conventions used in Cartesian graphs—and in so doing, to help them appreciate, interpret and make such graphs. We guided the class discussion by focusing attention on aspects of their diagrams that were mathematically relevant. We used this to introduce a dynamic representation of the story on a Cartesian coordinate system as a way to scaffold their understanding of how the coordinate graph encodes meaning about time, motion, and distance. In this paper, we focus only on an analysis of the diagrams. For a discussion of the transition to Cartesian coordinates, see Sinclair & Armstrong (2011).

ANALYSIS OF DIAGRAMS

O'Halloran uses her SF framework to analyse the visual images that are part of the mathematical discourse. These images employ several norms of the discourse. They are also to be seen as final products of written text as opposed to sketches used to explore or understand a problem. The visual images we will concern thus differ from those of O'Halloran: they are being created as a way of exploring a problem; and, they do not necessarily conform to the norms of the mathematical discourse. In our analysis we thus focus on the strategies that were used to produce visual images that communicate meanings related to time and motion.

Given the central importance of time and motion in the MellowYellow story, we first classified the visual images into three categories: (1) those that do not express temporality or movement, (2) those that express one but not the other, (3) and, those that express both. In terms of the second category, any expression of motion also fundamentally involves time, but this category includes visual images in which time was not expressed explicitly. Given that the prompt required only the production of a visual image (and not of corresponding text), we will focus on the first two meanings of the SF framework: representational and interpersonal. Given the fact that the participants were not fluent in the mathematics discourse, we expect their meanings to be expressed differently than the way they are in formal mathematics; in particular, we expect they would not provide Cartesian coordinate systems to express their experiential meanings, nor the accompanying labels and objects. Nevertheless, they could still express representational and interpersonal meanings that draw on previous mathematical experiences or on everyday discourses of visual images.

Absence of time and motion

This category includes three diagrams (out of 25). The diagrams illustrate a path, the origin, the destination and a penny halfway between the two (as exemplified in Figure 1). The diagram provides a snapshot of the situation, with MellowYellow captured at the halfway point, about to pick up the penny. In order to understand the experiential meaning expressed in the diagram, one needs to assume that the girl in the diagram walks along the path, from the house to the corner, even though she is only represented as being at one particular location. The presence of the line is meant to evoke the meaning of travel along a path—if the line segment was absent, this experiential meaning would be much harder to discern.

As described by O'Halloran, the logical meanings of the diagram are mainly spatial in nature, as they are expressed through the positioning of the important component of the story: the house, the corner store, MellowYellow and the penny. The sun does not contribute to the logical meaning of the visual image. Three markers are used to draw attention to what is important: the two labels for the house and corner store and the tick mark to indicate the position of the penny. The viewer is positioned as being perpendicular to the event, but the girl is shown facing the viewer, thus drawing attention to the subject of the event, rather than the quality of the motion. The presence of the sun, as well as the house and corner store, lead to a much more prolonged interpersonal meaning than what is found in a mathematical visual image.

Figure 1. George's diagram: where both motion and time are absent

Presence of motion, but not time

This category contains two subcategories that we describe as *discrete* (10/25) and *continuous* (8/25). Meanings for motion were expressed using a variety of strategies, with the former involving discrete techniques and the latter continuous ones.

Figure 2 shows Petra's diagram, in which the strategy for expressing motion involves shifting from a single arrow (-->) to a triple one (-->>>). This discrete indication of change of speed is accompanied by the labels "Speed = A" and "Speed = A x 2," which provides interpersonal meaning about the relative speed of each type of arrow. Unlike the first diagram, the penny in this one is much less visually important (indeed, it requires an arrow and text to mark its presence). And while the experiential meaning is mainly communicated through the presence of a path and a character placed on it, it differs from Figure 1 since the character is represented in side view, which contributes to the sense of MellowYellow moving. As with Figure 2 there are also many interpersonal meanings (tree, bicycle, house, corner store). The use of the labels and the arrows draws interpersonal attention to a change in speed.

Figure 2. Petra's diagram: use of arrows to express change in speed

Figure 3 also expresses movement discretely. The steps are metonyms for MellowYellow, who does not actually appear in the diagram. Six out of ten diagrams distinguish faster from slower movement by using symbols such as steps and arrow.

Figure 3. Julia's diagram: use of steps to express change in speed

None of the discrete diagrams are explicit about the passage of time, or about the fact that faster walking results in less elapsed time during the second part of the journey. Using the literal features of the story, these diagrams encode spatial, logical and temporal relations. However, the experiential meanings do not include the dimension of time, nor the relationship between movement and time.

We now consider diagrams that communicate motion using continuous techniques. The diagrams are similar to the discrete ones in terms of illustrating origin, destination and the halfway penny location. Figure 4 exemplifies this category. Like the discrete models, the diagrams do not explicitly express the dimension of time. Jordan's diagram includes three different MellowYellows, which hint at the passing of time, with one MellowYellow used for each of the three major events (walking, stopping, running). The longer curved lines provide an experiential meaning similar to the longer strides of Figure 3. These focus the viewer's attention more than the literal components that provide interpersonal meaning (the fence, swing set, and tree at home). The sharp right angle turn in the road also draw attention to the significant—and perhaps even singular—event of picking up the penny. The diagram expresses the passage of time, but not the relation between time and motion.

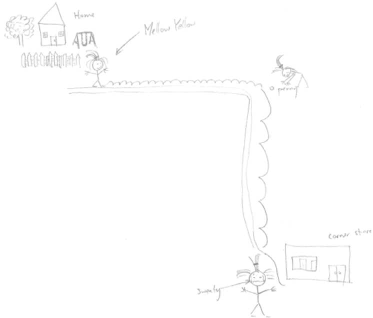

Figure 4. Jordan's diagram: use of larger 'waves' to indicate faster speed

We include one final example, which contains three parts, each using different techniques for expressing continuous motion. This visual image contains both the arrow and character techniques seen in the previous diagrams. However, it also contains a third graph in which distance is plotted again speed. Again, none of the diagrams explicitly evoke time. All are very sparse, with few literal features included. If the graph was drawn first, this lack of literal features in the two other parts may be influenced by the almost entirely logical meanings of the graph.

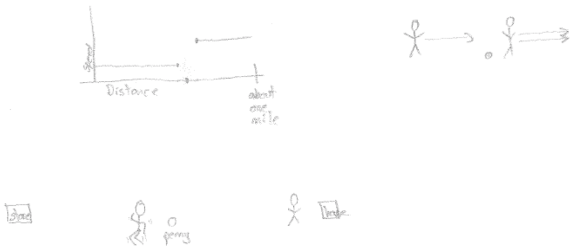

Figure 5. John's diagram: combination of techniques

Presence of motion and time

This category includes four diagrams in which both time and motion were expressed. As with the previous category, we found two types of diagrams in this category, one invoking time and motion in a discrete fashion and the other in a continuous one.

Niki's diagram in Figure 6 tells a three-part, discrete story (labelled numerically). Each part involves different types of movement (walking, resting and running), with the change in speed indicated by the depiction of the MellowYellow character (the running legs and the "I'm late" text both express the meaning of faster speed. The experiential meaning comes in part from the use of three characters, which express three different temporal events, and in part from the depiction of walking versus running. However, the logical meanings are also very strong, both in terms of the spatial arrangement, but also the labelling of the three different types of motion and the "less than 1 mile" indication of total distance.

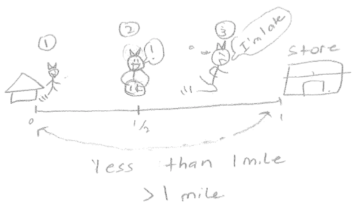

Figure 6. Niki's diagram: three-part sequence of motions

Unlike Niki, Jack expresses time and motion continuously. His diagram includes two parts: the first part is similar to the diagrams in the second category. He uses arrows to express movement. Arrows are labelled "regular speed" and "faster" to qualify the movement, and a circle at the halfway point along the horizontal line segment is labelled "time spent." The second part includes two bulb-like symbols and a dashed line circle. The larger bulb may either indicate slower speed or longer time, while the smaller one would indicate either faster speed or shorter time, respectively. Based on the presence of the question mark at the ½ point, we think that the bulbs indicate time, with the question mark suggesting that the amount of time elapsed while picking up the penny is unknown. The dotted line might thus indicate an event that is not associated with the passage of time, something that is possible in the virtual setting of the diagram, but not the real setting of the story.

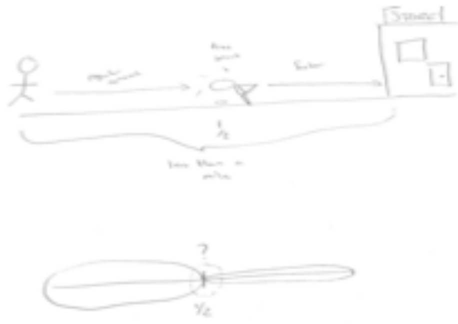

Figure 7. Jack's diagram: explicit expression of time and motion

It is obvious that Jack notices the importance of the dimension of time in the story. If the second diagram is meant to elucidate the dimension of time, then we can infer some compositional meaning in his visual image, with the second one elaborating the first, or providing a parallel representing focused on time rather than motion.

DISCUSSION

We have identified three categories of diagrams in terms of expressing time and motion. Of the twenty-five diagrams twenty-two express motion, but only four explicitly express time. This difficulty of thinking of motion in terms of time is consonant with historical developments; Koyré (1996) finds that in pre-modern scientific thought, it was more difficult to think in terms of time than in terms of space when it came to problems about motion. Indeed, Radford writes that time "remained an implicit notion, embedded in the duration of motion" (p. 47).

On reason for distinguishing discrete from continuous modes of expressing motion relates to the convention of Cartesian graphs, which employ the latter. However, discrete strategies were more frequently used. In addition to the examples shown in Figures 4, 5 and 6, we also saw diagrams drawn in cartoon style, with split frames for each event. Interestingly, Sinclair & Armstrong (2011) found that among grade 8 students engaged in the same task, almost half of them used this strategy. This suggests that a certain kind of visual literacy taken from a non-mathematical context can shape the strategies learners use to create mathematical diagrams. One advantage of inviting learners to create diagrams using discrete strategies such as split-frame cartooning is to emphasize the advantage that the Cartesian coordinate system has in explicitly expressing both quantitative changes in time and, of course, speed.

O'Halloran's framework enabled a detailed analyses of the strategies used to express different meanings related to the stories. While we saw a preponderance of interpersonal meanings that are not conventionally used in the mathematics discourse, we also found several representational meanings that expressed the central mathematical aspects of the story, such as origin, location, distance and speed. Of particular importance were the different strategies used to express motion, including specialised symbolic markers (arrows, footprints, loops, words, etc.). Discrete expressions of time were less explicit, but could be seem in diagrams with multiple appearances of MellowYellow. We suggest that further work with diagramming could help learners identify the difficulty of expressing motion through time and better motivate the use of the conventional discourse of coordinate systems. One problem with the hasty move to Cartesian graphs is that, for non-experts, as static visual representations, they effectively remove the experiential temporal dimension of a phenomenon. But this dimension is central to an understanding of functions.

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TEXTUAL EXPRESSION OF AREA MEASUREMENT IN ELEMENTARY CURRICULA: ILLUMINATING OPPORTUNITIES TO LEARN

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This study explores opportunities to learn area measurement content through an examination of the textual elements in three widely used U.S. elementary curriculum materials. We focus on what the textual elements illuminated in regards to students' access to area measurement concepts and procedures by describing the knowledge expressed and the curricular voice. We then provide a description of the textual expression of elements that are critical to building an understanding of area measurement. By examining the textual elements, we were able to describe the opportunity to learn area measurement. Our analysis indicated that while students were spoken to, they were often asked to "do" rather than "know." While there were instances of underlying concepts, students did not often have direct access to them.

Extensive evidence has shown that U.S. students' grasp of spatial measurement—length, area, and volume—is poor, despite the wealth of spatial experience and knowledge they develop and use outside of school. This evidence includes analyses of the U.S. National Assessment of Educational Progress (NAEP) performance by 4th, 8th, and 12th graders (Blume, Galindo, & Walcott, 2007); cross-national comparisons such as the Third International Mathematics and Science study (TIMSS) (National Center of Education Statistics, 1997), and smaller research studies that have focused on students' patterns of reasoning (Chappell & Thompson, 1999; Woodward & Byrd, 1983).

Empirical research on students' and teachers' knowledge, theoretical work on mathematical language and discourse, and observations of classroom lessons suggest that the poor learning of spatial measurement may be contributed to the interaction of the following six factors: weaknesses in K–8 written curricula (Lehrer, 2003), insufficient content and time devoted to teaching measurement, the predominance of static representations of 2-D and 3-D geometric figures, the nature of classroom discourse about measurement (Sfard & Lavie, 2005) , the "calculational" orientation that dominates classroom instruction and discourse (Thompson, Phillip, Thompson, & Boyd, 1994), and weaknesses in teachers' knowledge of measurement (Simon & Blume, 1994). This study focuses on the first factor, the written curriculum, because it seemed like a worthy factor to investigate as it may affect the other factors. Furthermore, it has been noted that investigating the features of written curriculum is

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an underdeveloped area of research that needs our attention (Stein, Remillard, Smith, 2007).

While performance in all spatial measures is poor, this study specifically focuses on the ways in which area measurement is expressed in curriculum materials. We focus on area measurement because it is mostly seen as a product of two measures, length times width instead of a measurement activity (Lehrer, 2003). Moreover, no simple tools exists for measuring area, thus motivating an algorithmic approach which hides the underlying meaning of procedures (Stephan & Clement, 2003) and makes length times width even harder for students to understand (Kamii & Kysh, 2006). Instructional approaches become crucial in developing a strong understanding of area measurement and with the reliance on curriculum materials by many teachers (Remillard, 2005) it is important to know more about how area measurement is treated in curriculum and how this aligns with what we know about how students' develop an understanding of area measurement.

PURPOSE

In this paper we address one aspect of an NSF-funded study, the purpose of which is to understand the capacity of U.S. written curricula to support students' learning of measurement. We describe using textual elements to code how spatial measurement content is expressed in written curriculum materials and what this type of analysis illuminates. Features of written curriculum, such as textual elements, have received little attention in curriculum analysis (Stein, Remillard & Smith, 2007) and we hypothesize that these elements provide different Opportunities to Learn (OTL). We begin to address some of these questions by examining how knowledge is presented to students through textual elements in curricula for area measurement in grades K-4. The textual expression is important in area measurement because knowing how curricula provide opportunities for engaging with area concepts in addition to skills and procedures is critical as "often the tools and procedures used in measuring area mask the intended conceptual aspects that underlie area measurement" (Stephan & Clements, 2003, p. 10).

METHOD

Data

Evidence of wide use in the U.S. and substantial differences in basic design principles guided our choice of curriculum materials. We analysed The University of Chicago School Mathematics Project's (2007) *Everyday Mathematics* (EM), Scott Foresman-Addison Wesley's (2008) *Michigan Mathematics* (SFAW), and Larson's (2004) *Saxon Math* (Saxon).

Framework & Analysis

Our analysis included: a) locating measurement content and b) coding measurement content. To locate the measurement content, two coders found every lesson, problem, and activity in all curricula that pertained to measurement.

To code measurement content, members of the larger research team developed a Curriculum Coding Scheme, a set of structured knowledge and textual elements. Knowledge elements are divided into three kinds of measurement knowledge (i.e., conceptual, procedural, conventional). Textual elements are divided into five types of expression (i.e., Statements, Demonstrations, Worked Examples, Problems, Questions) and were used to code how knowledge was expressed. Statements, Demonstrations, and Worked Examples have been unproblematic to identify, whereas Questions and Problems (taken collectively as "Queries") have been more difficult. We found it useful to distinguish between these types of queries using three criteria: student autonomy, expectations for responses (i.e., one or all students engage), and cognitive demand. Questions were determined by the expectation that not all students engage in the query, little to no student autonomy, and low cognitive demand, whereas Problems were determined by the expectation that all students engage, student autonomy, and high cognitive demand. If two out of three criteria indicated Problem, the instance was coded as Problem. We also found it important to note whether instances appeared in student materials or only in teacher materials because these different types of access may impact OTL. Therefore, for each textual element, we distinguished voice (whether the text speaks to teacher or student) as other researchers have suggested (Herbel-Eisenmann, 2007). See Table 1 for descriptions and examples of the most frequent textual elements.

	Knowledge Elements		
	Conceptual	Procedural	Conventional
Statements	Assertions of basic principles Ex: "Smaller units produce larger measures."	Directions to complete a set of steps/actions Ex: "To measure a segment, you should..."	Assertions about tools, notations, and systems Ex: "1 foot = 12 inches"
Questions	Query requiring conceptual knowledge and two of the following: a) under teacher direction; b) limited number of students expected to respond; c) low cognitive demand Ex: Whole Class: "Why did we use different numbers of pattern blocks?"	Query requiring procedural knowledge and two of the following: a) under teacher direction; b) limited number of students expected to respond; c) low cognitive demand Ex: Whole Class: "How many cm long is this line segment?"	Query requiring conventional knowledge and two of the following: a) under teacher direction; b) limited number of students expected to respond; c) low cognitive demand Ex: Guided Practice: "How many ft in 1 yd?"

Problems	A query requiring conceptual knowledge and two of the following are true: a) not under teacher direction; b) all students expected to respond; c) high cognitive demand Ex: Small Group Task: "Which will give you a greater measure [paperclip, unit cube]? Explain."	A query requiring procedural knowledge and two of the following are true: a) not under teacher direction; b) all students expected to respond; c) high cognitive demand Ex: Individual Task: "Measure each object to the nearest ½ inch."	A query requiring procedural knowledge and two of the following are true: a) not under teacher direction; b) all students expected to respond; c) high cognitive demand Ex: Individual Task: "How many mm are in 9 cm?

Table 1: Descriptions and examples of most frequent textual elements.

RESULTS & DISCUSSION

We focus on what the textual elements illuminated in regards to students' access to area measurement concepts and procedures by describing the knowledge expressed (i.e., conceptual procedural, conventional), and the curricular voice (i.e., student, teacher). We then provide a more detailed description of the textual expression of the elements recommended by Stephan and Clements (2003) for building students' understandings of area measurement.

Knowledge Expression

The analysis of the textual elements of the three curricula illuminated both similarities and differences. Of the five textual elements all three curricula contained mostly Queries. They accounted for more than 79% of all textual elements in each grade. These Queries were most often of a procedural nature. There were very few instances of conceptual or conventional Queries; never accounting for more than 6.5% for any curricula at a particular grade and more often being below 1%.

Unlike with Queries, we noticed more variety in the ways Demonstrations and Worked Examples were expressed in the curriculum materials. Demonstrations were more prevalent in Saxon. For each grade, Saxon had a higher percentage of Demonstrations, capping at 17.5% in grade 4, while EM and SFAW range from 0% to 6.8%. With respect to Worked Examples, SFAW had a higher percentage, ranging from 2.7% to 7.1% of the total codes in each grade, whereas EM and Saxon each only had one grade above 2%.Similar to Queries, for all three curricula, Demonstrations and Worked Examples were procedurally focused.

Statements accounted for anywhere between 0% and 11.4% of the total codes for a curriculum at a particular grade. In all grades except for K, EM had the largest percentage of Statements and in grades 3 and 4 had over double the number of

Statements as SFAW and Saxon. In all curricula, Statements were used to express all three types of knowledge, one being no more prevalent than the others. However, since conceptual knowledge was expressed so little by the other textual elements it is interesting to note that, Statements were used equally or more often to express conceptual knowledge than other textual elements in over half of the grades..

Curricular Voice

Generally, there was more teacher voice in the earlier grades (i.e., K-1) and as the grades increased (i.e., 2-4) we found more instances of student voice in SFAW and EM. Saxon, on the other hand, was quite different. Saxon had more teacher instances in all grades (over 54% in each grade) except for Grade 1. Looking more closely at the textual elements, we found that most Problems were in student materials, whereas most Questions and Statements were posed or made by the teacher. This result may make sense based on our coding scheme, which involves determining whether a query is teacher directed. If a query is posed by the teacher it may be more likely to be teacher directed than if it is in the student materials. Unlike, Queries, Statements, as defined by our scheme, were equally likely to appear in student materials or be made by teachers. However, in all but four grades (EMK, SFAW1, SFAW3, EM4), Statements were more often in teacher materials than student materials.

Elements for Building Students' Understanding of Area

Research indicates that computationally focused instruction might lead to weak conceptual understanding. Stephan and Clements (2003) provide the following recommendations for developing students' understanding of area measurement:

> (a) construct the idea of measurement units (including measurement sense for standard units); (b) have many experiences covering quantities with appropriate measurement units and counting those units; (c) structure spatially the object they are to measure; and (d) construct the inverse relationship between the size of a unit and the number of units used in the measurement." (Stephan & Clements, 2003, p. 13-14).

In our analysis we looked at the ways area content was expressed in the curricula by using these instructional recommendations. We picked knowledge elements that satisfied each of these recommendations. In some cases we had a one-to-one fit between one of our knowledge elements and one of the recommendations, like unit-measure compensation from our coding scheme and recommendation (d). In other cases, we clustered elements from our coding scheme to satisfy the recommendations, such as using all of our covering and counting knowledge elements to satisfy recommendation (b).

The first recommendation, which we call conception of area measurement was predominantly (over 44 % for all three curricula) expressed as Statements to be spoken by teachers to students. Students did not have access to many of the concepts that might help build a conception of area measurement directly, except for a few instances in SFAW and EM.

The second recommendation, which we call covering and counting, has more of a procedural focus. Each curriculum provided opportunities mostly in the form of Queries (over 90% for all three curricula) with some other textual elements being used in lower frequencies. Of all four recommendations, covering and counting had the most emphasis and variation in terms of the ways content was provided. For example, in addition to Queries and Statements, which were common for the other recommendations, there were also Demonstrations and Worked Examples.

The third recommendation, which we call Spatial Structuring of Space was expressed mostly by Queries. Each curriculum provided Queries both in teacher and student materials. In addition, there were two Statements in EM and two Demonstrations in SFAW.

We found very few instances of the last recommendation, which we call unit-measure compensation. Most instances were Queries, designated both student and teacher (three in EM and Saxon and four in SFAW). In addition, there was one teacher Statement in EM.

In summary, we found that there was little variety in terms of the ways in which knowledge was expressed. Curricula predominantly used Queries and some Statements. For example, almost all instances of covering and counting were provided in the form of Queries, whereas instances of conception of area measurement were mostly Statements. While we pointed out earlier that most conceptual knowledge was expressed by Statements, looking at the elements related to the Stephan and Clements' (2003) recommendations, we found that the type of knowledge used to express content might be dependent on the particular content itself. For example, unit-measure compensation, a conceptual element, was not expressed primarily as Statements, but was expressed almost exclusively as Queries. A possible explanation for this might be the connection between length measurement and area measurement. The inverse relationship between units and measurement is not a concept specific to area. The same concept also holds for length measurement and curricula might assume that they have described this concept in length measurement and they may not see a need to restate the same definition or description; hence, less Statements.

Implications for OTL

Investigating textual elements and their interaction with types of knowledge and voice shed some light on students OTL area measurement. Our analysis indicated that these curricula do speak to students; in fact SFAW and EM speak more to students than teachers. However, when we examined what students had access to, we found that these instances were more about following procedures than engaging with area concepts. Esmonde (2009) urges us to consider students "access to (positional) identities as knowers and doers of mathematics (Gresalfi & Cobb, 2006)" (p.249). If students are to have OTL area measurement, they must be positioned in ways that allow them to identify as knowers and doers of mathematics. Our analysis indicated that while students were spoken to, they were often asked to "do" rather than "know." This is evidenced by the high number of procedural Queries. Furthermore, we see more emphasis on Stephan and Clements' (2003) second recommendation, covering and counting, a more procedurally focused element, than the other recommendations. Conceptual knowledge, what little there was, was expressed mostly as Statements, not as Queries in which students were expected to engage. Furthermore, most Statements were in teacher materials not directly accessible to students. If curricula do not provide opportunities for students to engage with conceptual knowledge of area, they are not providing adequate OTL area measurement.

SIGNIFICANCE

Written curriculum has the potential to influence transformations between written, intended, and enacted curriculum (Stein, Remillard, & Smith, 2007), yet few studies have investigated how mathematical knowledge is presented to students in text. This study illuminates the importance of analysing the textual elements of written curricula and raises issues for researchers; namely, how do features of written curricula express the types of knowledge that contribute to students OTL.

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REPRESENTATIONS AND TASKS INVOLVING REAL NUMBERS IN SCHOOL TEXTBOOKS

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In this paper we analyse the introduction of the concepts of real and irrational number in school textbooks adopted by Brazilian public schools. Our results indicate that irrational numbers are mostly introduced both on the basis of the decimal representation and on the use of tasks which do not foster conceptualisation; and that the mathematical need for the construction of the field of real numbers remains unclear in the textbooks.

INTRODUCTION

Traditionally, in school, the set of integers is constructed from the algebraic limitations of natural numbers. The motivation for such a construction is based on some "daily life" problems in which it is necessary to find the difference between two natural numbers. Similarly, the extension from integers to rational numbers involves the limitation of the operation of division. Therefore, the learning of different sets of numbers in elementary school entails notable cognitive growth: a progressive extension through the algebraic structure of nested number sets, from the primitive notion of counting to the ideas of comparing and measuring.

The case of the extension from rational to real numbers is particularly dramatic. Unlike the previous extensions, this is not simply an algebraic step, as it requires notions of convergence and completeness. This has proven to be a crucial obstacle, which dates back to the incommensurable magnitudes controversy in Pythagorean mathematics. Moreover, only a discrete set of numbers is enough to deal with the *empirical* problem of measurement, whilst the real numbers system accounts for the construction of a consistent *theory* of measure. Therefore, the need to introduce real numbers can hardly be established upon empirical motivations. As research has shown (see the following section), these epistemological obstacles and theoretical constraints have repercussions in teaching and learning. On the one hand, the theoretical roots of the concept of real number are surely incompatible with elementary and secondary school. On the other hand, the concept cannot be built upon empirical or algebraic motivations. Nevertheless, real numbers are an indispensable topic in school mathematics education, due to their inherent importance and their entangled relation with many other equally important topics, such as the circle length and the Pythagorean Theorem. This poses a key question to textbooks and in syllabi design: the balance between rigour and intuition is particularly delicate in the case of real numbers.

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Contradictorily, the real line structure (including its algebraic, topological and analytical properties) is assumed to be well understood by students when higher mathematical concepts are introduced. In fact, the structure of real numbers, as the subtle distinction between density and completeness, is at the heart of the theoretical grounding of effectively all infinitesimal calculus concepts. Therefore, a weak understanding of basic properties of real numbers at school can be a source of obstacles and misunderstandings in postsecondary and university education.

The approach used to introduce real numbers in school textbooks is the focus of this paper. Taking the difficulties pointed out by literature into consideration, we aim to analyse how these numbers are introduced and what significance is given to them in textbooks. We use empirical data from a selection of textbooks used in Brazilian public schools. The study reported in this paper is a part of a broader research project that addresses the approaches privileged by the institutions (mainly through syllabi, school textbooks, and teacher practice) to teach concepts of algebra and analysis that, despite their key importance, have usually deserved lesser attention. Partial results of this project (focusing the concept of continuity) have been presented at the PME33 conference (Giraldo, González-Martín & Santos, 2009).

LITERATURE REVIEW

Difficulties in understanding the concept of real number or some of its properties by students and even teachers have been addressed by the literature. Surprisingly, we have not found research papers focusing on the introduction of this concept in textbooks. Some papers show that even university students give incorrect definitions for irrational numbers, and are unable to explain the necessity to extend the field of rationals (e.g. Soares, Ferreira & Moreira, 1999). Some authors have even found that many prospective teachers associate irrational numbers exclusively to square roots and π (e.g. Sirotic & Zazkis, 2007). For example, Robinet (1986) establishes that for 17-18 year-old students, real numbers are conceived as the reunion of natural, integer, rational, and decimal numbers, together with some numbers as $\sqrt{2}$ and $\sqrt{3}$. Dias (2002) establishes the hypothesis that school teachers' conceptions about the real line structure and the notion of density (in particular their concept image and concept definition) are the same as those present in their students. She found concept images for real numbers as an almost discrete set, through underlying conceptions of a finitude (or even inexistence) of numbers between two given real numbers. This is consistent with Robinet's (1986) study, who found that 43% of the students who can conceive the straight line model for real numbers hold an atomist model for this line.

Another study focusing on the conceptions of prospective teachers (Sirotic & Zazkis, 2007; Zazkis & Sirotic, 2004) showed the inconsistencies between the formal, algorithmic, and intuitive dimensions. For instance, the authors found the idea that for every rational, there is an irrational, and these numbers are placed in an order that suggests the idea of successor in a discreet set. Some participants also had the idea that the set of rationals is "richer" within [0, 1], because they did not know any irrationals (like π) in this interval. The authors also identified some difficulties in conceiving irrational numbers as $5+\sqrt{2}$ as an object.

Other works (e.g. Robinet, 1986; Fischbein, Jehiam & Cohen, 1995) identify more incorrect conceptions, as the idea that an irrational number is a number with an infinite decimal representation (with no consideration for the presence or absence of a period). There is also an identification between rational numbers and the decimal representation (independently of the presence or not of a period), or the definition of irrational numbers as "numbers which are not exact". Finally, we stress the fact that Robinet (1986) showed that high school students give more importance to the different writings for numbers than to their specific properties.

THEORETICAL FRAMEWORK

Our analysis of how textbooks introduce the concepts of irrational and real number is grounded in two dimensions: institutional choices and their repercussions (Chevallard's anthropological theory); and cognitive activities demanded in the textbooks (Duval's theory of the registers of semiotic representation).

Chevallard's (1999) anthropological theory attempts to achieve a better understanding of the choices made by an *institution* in organising the teaching of a given concept, and the consequences of these choices on the significance given to the concepts taught, as well as the learning achieved by the students. Chevallard recognises that mathematical objects are not absolute objects, but entities which arise from the practices of given institutions. These practices may be described in terms of: tasks (t, being T a type of tasks); techniques (τ) used to complete these tasks; a discourse (*technology,* θ) which both explains and justifies the techniques; and the theory which includes the given discourse (Θ). If we want to understand the meaning attributed by a given *institution* for "knowing a mathematical object", we need to identify and to analyse the practices which bring this object into play within the *institution*. From this perspective, we are interested in analysing the type of tasks the textbooks use most often in introducing irrational and real numbers in secondary education. We will observe the types of mathematical and didactic organisation that the textbooks develop around the tasks concerning the concepts of irrational and real number, as we aim to determine whether textbooks establish a complete praxeologic organisation (which accounts for the quartet $T/\tau/\theta/\Theta$), or just a partial organisation.

According to Duval, the development of mathematical understanding requires the use of different semiotic representations of the mathematical objects being studied. The reason for this is that learners need to distinguish any mathematical object from its representation. In order to achieve this distinction the use of different representations is necessary, since the mathematical object cannot be directly accessed and each representation expresses only a restricted group of its characteristics. Therefore, Duval defines treatment of a representation as an activity within one single register, and conversion of representations which happens between different registers. Thus, a key cognitive activity is the ability to make conversions from one register to another.

METHODOLOGY

Sampling

In Brazil, compulsory education is organised into three slots: fundamental school I (grades 1 to 5, ages 6 to 10), fundamental school II (grades 6 to 9, ages 10 to 14), and middle school (grades 1 to 3, ages 15 to 17). Real numbers are usually introduced during fundamental school, grade 8 (age 12). Textbooks used in public schools are bought by the federal government and distributed for free to the students. The textbooks adopted by each school are chosen by the school, out of a list previously assessed and approved by the Ministry of Education. The assessment process is mainly based on referrals by experts.

To portray how the concepts of irrational and real number are presented in compulsory education, we analysed a sample of textbooks approved by the Ministry of Education in the latest assessment processes: 9 titles for fundamental II (out of 16 approved in 2008) and 5 titles for middle school (out of 8 approved in 2007).We constrained ourselves to approved textbooks for two reasons: 1) these textbooks reach all the public schools in the country, and 2) due to the assessment process, these textbooks have been approved by educators active in compulsory education, so they are assessed with didactic, pedagogical, conceptual, and structural criteria.

Elements of the analysis grid

We discuss in this paper the following categories of our analysis grid: (I) types of definitions, examples, and properties used, (II) types of representations used, (III) types of tasks proposed. The subdivision of each category into dimensions is summarized in the following table. These categories reflect our theoretical approach, which takes into account both institutional issues, and the cognitive activity fostered through the use of different registers of representation.

Categories	Dimensions
(I) types of definitions, examples, and properties used	types of definitions coherence between the definition, examples and tasks examples for the definitions examples which problematise the need for a new kind of numbers examples of properties, types introduced and justification
(II) types of representations used	figural register numerical symbolic register algebraic symbolic register natural language register

(III) types of tasks proposed	classification of a number using the belonging relationship classification between rational or irrational classification in true or false determining an irrational number between two numbers calculation by approximation ordering real numbers representing numbers in the real line numerical intervals

Table 1: Dimensions of the analysis grid.

DATA DISCUSSION

Definitions, properties, examples and representations

The definitions given by the textbooks for irrational numbers were classified in two types: "*irrational number is a number which cannot be written in the form of a fraction*" (D_A), and "*among numbers written in decimal form, there are numbers with infinite non-periodic decimals, called irrational numbers*" (D_B). In our sample, 5 textbooks use definition D_A, and 8 use definition D_B. The remaining textbook does not give a definition for irrational numbers and barely uses any problems to introduce them. None of the textbooks uses both definitions at the same time. Concerning the definition of real number, all the textbooks of our sample define real numbers in the following way: "*any rational or irrational number is a real number*" (D_C).

Among the 14 textbooks of our sample, 9 introduce the concept of irrational number through examples of numbers which are not rational, which are presented before the definition, (allegedly) pointing out to the existence of a "new" type of numbers. In these textbooks, a problematisation for the introduction or the existence of these "new" numbers is absent. Six out of these 9 textbooks use $\sqrt{2}$ as the introductory example, 4 of which through the measurement of the diagonal of a square with a unitary side, and other 2 through the decimal representation. The other 3 textbooks use examples of numbers with infinite non-periodic decimal expansions.

In all the textbooks analysed the examples concerning irrational numbers are mainly used to illustrate definitions and properties. The statement of a definition or a property is usually accompanied by some examples verifying it. The numbers $\sqrt{2}$, $\sqrt{3}$, and π are among the illustrative examples used in all the textbooks. The decimal representations of these numbers are assumed to be known. In general, there is no justification to show that the decimal writings of these numbers do not have a period.

It is important to note that, in order to define irrational numbers, both D_A and D_B suppose the existence of another type of number which is not rational. In fact, both D_A and D_B require, respectively, the existence of numbers that are not fractions and of numbers that have infinite non-periodic decimal representations. Thus, D_A and D_B do not establish the existence of a new kind of number. Rather, they label the class called *irrational* among a set of numbers which is assumed to be previously existent. Such a set cannot be any other than the set of real numbers. However, in these textbooks, the definition of real numbers not only comes after the definition of irrationals, but also depends on it, since D_C presents a real number as one that is either rational or irrational. Therefore, these definitions are inconsistent and mutually dependent. From a mathematical point of view, they could not even be considered as definitions. Moreover, definition D_B is based on the assumption that every real number admits a decimal representation, which is a remarkably non-trivial property, whose verification depends on the familiarity with the notion of convergence. Surprisingly enough, the acceptance of this property by students appears to be taken for granted. Another problem we find is the lack of characterisation concerning the nature of both the numerator and the denominator of the fraction as integers, which can later produce some confusion between the concept of *fraction*, and the more general concept of *rate* (which does not always correspond to rational numbers).

The most prominent properties presented in the textbooks concern mainly the operations (closeness and operations between rationals and irrationals), and the localisation of points on a line. Examples are mainly used to illustrate properties. As none of the properties is formally proved, this could lead students to develop the incorrect idea that a property can be proved through the verification of some examples. Despite the fact that 7 of the textbooks state the density property of rational numbers, the density of real numbers is only referred to in 3 textbooks. None of the textbooks mention the density of irrational numbers.

The numerical-symbolic register (including decimal notation, fractions, roots and combinations of those) is clearly privileged over the figural, algebraic-symbolic and natural language registers. In 11 out of the 14 books in the sample, more than 70% of the representations used are numerical-symbolic. Furthermore, the treatment of representations within the same register is privileged over conversion, so students might develop difficulties to coordinate registers. In fact, the only cases of conversion of registers found occur: 1) within the numerical-symbolic register, when conversion between decimal and fractional representations is made, and 2) when students are asked to represent numbers in the real line (possibly fostering atomistic conceptions). However, the student is not asked to perform the inverse conversion, this is, to represent numbers in the real line in their numerical-symbolic form, affecting the coordination of registers (which requires the ability to convert in both senses from one register to another).

Tasks

If we now consider the tasks proposed by the textbooks, there is an absence of questions aiming to foster a conceptualisation of real and irrational numbers. If we consider that an individual's conception of a mathematical object is strongly shaped by the tasks she or he develops with this object, real numbers seem to be reduced to a list of properties and to a definition which does not question their existence. The textbooks in our sample emphasise tasks involving mainly: 1) classification (assuming that the existence of the two categories is natural), 2) determining an irrational between two numbers (without mentioning their density, which could lead to the false impression that there is a definite number of irrationals between two numbers), 3) approximation (eluding the notion of convergence), and 4) representing numbers in the real line (which could lead to an atomist conception of real numbers in the real line). We can say that most of the tasks (t) we have found in the textbooks favour the reproduction of certain techniques (τ, which are usually previously exemplified), and algorithms to get solutions. However, the justification θ of these techniques is usually out of reach at this school level. The list of properties and techniques given is not proved or justified in any way, with no attempt to develop students' mathematical competences related to argumentation. This characterises what Chevallard (1999) calls an incomplete praxeology, that is, the predominance of types of tasks and techniques (praxis , T/τ) over discourse and theory (logo, θ/Θ). Therefore, we conclude that the textbooks of our sample only offer a partial praxeologic organisation, which can have dramatic effects on the students' introduction to the concepts of irrational and real number.

CONCLUDING REMARKS

Our analysis reveals that the approach used to introduce irrational and real numbers in the textbooks puts little focus on conceptual and theoretical aspects, whilst emphasizes routines and algorithms presented without justifications. These "new" numbers are not given any utility, and the tasks they are reduced to do not lead to the development of a discourse or a theory. Properties of the operations and techniques to solve some routine tasks are highlighted in detriment of aspects regarding the construction of some basic conceptual understanding to later, in postsecondary education, deal with topological and analytical properties of the real line, such as density and completeness.

Moreover, definitions are formally inconsistent and the justification of properties is based exclusively on examples. Therefore, mathematical argumentation is absent from the textbooks. The coordination of registers is also lacking in the textbooks, as almost all the discourse developed is found in the numerical-symbolic register. On the other hand, there is little discussion concerning the theoretical need for the construction of the field of real numbers, such as which kind of mathematical problem rational numbers are unable to solve, reducing the family of tasks (T) which could be used to conceptualise these numbers.

It is important to stress that we do not intend to advocate a formal approach to real numbers, aligned with the criteria of mathematical rigour, to school education. However, the mere presentation of procedures and representations, without *any* (not even informal) kind of argumentation, is not likely to contribute to students' understanding of the concept of real number. This seems to be the choice followed by the textbooks in our sample in order to to avoid the road of mathematical formalism. In our view, another (possibly more effective) instructional choice would to adopt a more *problematic* approach. That is, to focus more on the mathematical problems that engender the concept of real numbers – rather than trying to formalise, or (in the other end of the spectrum) just avoiding them and taking things for granted.

Even if we have analysed a sample of Brazilian textbooks, we believe these textbooks share characteristics of secondary textbooks in other countries. In the light of these considerations, it is reasonable to expect that, if teachers simply follow the approaches proposed in the textbooks, then students will be unlikely to be able to build an adequate structure to deal with real numbers and their properties in undergraduate studies, and consequently, they will not be prepared for the kind of mathematical reasoning required in undergraduate mathematical education.

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TEACHERS' PERSONAL CONCEPTIONS OF NUMERACY

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This paper reports on a project that helped teachers to plan and implement numeracy strategies across the school curriculum. Because teachers need to model the kind of numeracy they want their students to develop, we examined how their personal conceptions of numeracy changed during the project. Teachers came to a richer understanding of numeracy that emphasised effective use of mathematical knowledge and skills, solving problems in everyday contexts, and positive dispositions. Other components of numeracy – use of tools and development of a critical orientation – were less often incorporated into teachers' personal conceptions of numeracy.

The term "numeracy" was first introduced in the UK by the Crowther Report (Ministry of Education, 1959) and was defined as the mirror image of literacy, but involving quantitative thinking. In some parts of the world it is more common to speak of *quantitative literacy* or *mathematical literacy*. In the USA, for example, Steen (2001) described quantitative literacy as the capacity to deal with quantitative aspects of life. The OECD's (2004) PISA program defines mathematical literacy as:

> an individual's capacity to identify and understand the role mathematics plays in the world, to make well-founded judgments, and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (p. 15)

Steen (2001) maintains that numeracy must be learned in multiple contexts and in all school subjects, not just mathematics. A recent review of numeracy education undertaken by the Australian government (Human Capital Working Group, Council of Australian Governments, 2008) concurred, recommending:

> That all systems and schools recognise that, while mathematics can be taught in the context of mathematics lessons, the development of numeracy requires experience in the use of mathematics beyond the mathematics classroom, and hence requires an across the curriculum commitment. (p. 7)

This paper reports on a year long research and development project that investigated approaches to help teachers plan and implement numeracy strategies across the curriculum in the middle years of schooling (Grades 6-9). The project was informed by a rich model of numeracy that was introduced to teachers as an aid for their curriculum and pedagogical planning. One of the challenges in promoting numeracy learning in all curriculum areas is for teachers themselves to model the kind of numeracy they want their students to develop (Hughes-Hallett, 2001). Thus the purpose of this paper is to examine teachers' personal conceptions of numeracy and the extent to which these conceptions changed over the duration of the project.

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THEORETICAL FRAMEWORK

In Australia, educators and policy makers have embraced a broad interpretation of numeracy similar to the OECD definition of mathematical literacy: "To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life" (Australian Association of Mathematics Teachers, 1997, p. 15). This definition became widely accepted in Australia and formed the basis for much numeracy-related research and curriculum development.

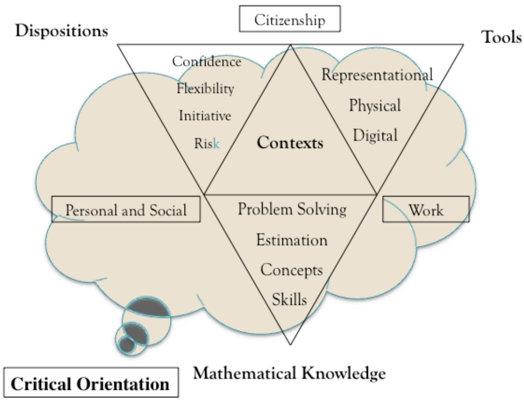

Figure 1. A model for numeracy in the 21st century

Recently, however, Goos (2007) argued that a description of numeracy for new times needs to better acknowledge the rapidly evolving nature of knowledge, work, and technology. She developed the model shown in Figure 1 to represent the multi-faceted nature of numeracy in the 21st century. The model was intended to be readily accessible to teachers as an instrument for planning and reflection; however, its development was also informed by relevant research, as outlined below.

A numerate person requires *mathematical knowledge*. In a numeracy context, mathematical knowledge includes not only concepts and skills, but also problem solving strategies and the ability to make sensible estimations (Zevenbergen, 2004).

A numerate person has positive *dispositions* – a willingness and confidence to engage with tasks and apply their mathematical knowledge flexibly and adaptively. Affective issues have long been held to play a central role in mathematics learning and teaching (McLeod, 1992), and the importance of developing positive attitudes towards mathematics is emphasised in national and international curriculum documents (e.g., National Curriculum Board, 2009; OECD, 2004).

Being numerate involves using *tools*. Sfard and McClain (2002) discuss ways in which symbolic tools and other specially designed artefacts "enable, mediate, and shape mathematical thinking" (p. 154). In school and workplace contexts, tools may be representational (symbol systems, graphs, maps, diagrams, drawings, tables, ready reckoners), physical (models, measuring instruments), and digital (computers, software, calculators, internet) (Noss, Hoyles, & Pozzi, 2000; Zevenbergen, 2004).

People need to be numerate in a range of *contexts* (Steen, 2001). A numerate person can organise their personal and social lives (e.g., finances, health, leisure activities). All kinds of occupations require numeracy, and many examples of work-related numeracy are specific to the particular work context and use mathematics in different ways from how it is taught at school (Noss et al., 2000). Informed citizenship depends on the ability to interpret data, make projections, and engage in the kind of systematic thinking that is at the heart of numeracy. Different curriculum contexts also have distinctive numeracy demands, so that students need to be numerate across the range of contexts in which their learning takes place at school (Steen, 2001).

This model is grounded in a *critical orientation* to numeracy since numerate people not only know and use efficient methods, they also evaluate the reasonableness of the results obtained and are aware of appropriate and inappropriate uses of mathematical thinking. In an increasingly complex and information rich society, numerate citizens need to decide how to evaluate quantitative, spatial or probabilistic information used to support claims made in the media or other contexts. They also need to recognise how mathematical information and practices can be used to persuade, manipulate, disadvantage or shape opinions about social or political issues (Frankenstein, 2001).

METHODOLOGY

Twenty teachers were recruited from ten demographically diverse schools on the basis of their interest in cross-curricular numeracy education. They came from four primary schools (Kindergarten-Grade 7), one secondary school (Grades 8-12), four smaller schools in rural areas (Grades 1-12), and one school that combined middle and secondary grades (Grades 6-12). The focus on teaching numeracy across the whole curriculum meant that it was important to include teachers with varying subject specialisations. Thus participants included generalist primary school teachers as well as secondary teachers qualified to teach particular subjects (mathematics, English, science, social education, health and physical education, design studies).

The project was conducted for a full school year in 2009. There were two elements to the research design. First, three professional development days in March, August and November brought together researchers and teachers to discuss the numeracy model, try out numeracy investigations that drew attention to particular aspects of the model, engage in collaborative planning of numeracy units of work, and reflect on progress. Second, the research team conducted two daylong visits to each school in June and October, between the professional development days, to observe lessons, collect planning documents, interview teachers and students, and provide feedback to teachers on further development of teaching strategies related to the numeracy model.

This paper draws on two types of data collected at the beginning and end of the project. At the start of the first professional development day, immediately before teachers were introduced to the numeracy model in Figure 1, their initial conceptions of numeracy were explored by asking them to complete the following sentence stems:

- Numeracy involves …
- A numerate person knows …
- A numerate person is …
- A numerate person can …

Teachers worked in groups to provide responses. Groups were not required to provide a single response to each sentence stem and so multiple responses were common from each group. This activity was repeated in a slightly different way at the end of the project: before the final meeting we emailed teachers individually and asked them to respond to the same sentence stems again. While each teacher responded to the email, not every teacher responded to all stems, and so the number of responses to each stem varied. Initial and final responses were analysed by matching them to components of the numeracy model and recording frequencies of responses so classified to look for changes over time. For example, *Numeracy involves* … "using mathematics to be successful in everyday life" was matched to the *contexts* component of the model; *A numerate person can* … "use technology effectively" was matched to *tools*; *A numerate person knows* … "when information is misleading" was aligned to *critical orientation*; *A numerate person is* … "confident in their application of mathematical knowledge" was linked to *dispositions*. Responses could be classified in multiple ways if they referred to more than one component of the numeracy model.

The second type of data was collected at the end of the final professional development day, when we provided teachers with copies of the numeracy model and asked them to map their trajectory through the model during the project. Teachers did so by annotating the model to identify the component that represented their main interest or concern when they started the project, and then other components of the model that became progressively more meaningful or significant to them over time. Responses were analysed by listing all the trajectories teachers identified and grouping them according to the starting point and subsequent pathways.

FINDINGS: TEACHERS' PERSONAL CONCEPTIONS OF NUMERACY

Analysis of teachers' responses to the sentence stems and their trajectories through the numeracy model provide insights into their changing conceptions of numeracy.

Numeracy sentence stems

The most frequent responses to the stem *Numeracy involves ...* showed an appreciation of the role of context, and this increased over time: from over half of the initial responses; (8/15; e.g. "everyday connections") to 80% of final responses (8/10; e.g., "application of mathematical processes in everyday practical situations"). Problem solving as an aspect of mathematical knowledge was referred to in nearly half of initial responses (6/15; e.g., "solving problems in life") and half of the final responses (5/10; e.g., "having a repertoire of strategies"), indicating that many teachers maintained a sense of numeracy as involving more than using learned procedures in routine situations.

Initial conceptions of what *A numerate person can …* do were mainly based on the skills aspect of mathematical knowledge: over half of the responses were of this type (10/19; e.g., "use numbers to solve problems"). At the end of the project, nearly three-quarters of responses reflect an understanding of numeracy in which context has priority (11/15; e.g., "sort out how to transfer mathematical knowledge into real life situations"). The next most frequent type of response at the end of the project referred to numerate people being good problems solvers (6/15; e.g., "use problem solving skills to help them to better understand some aspects of numeracy").

In deciding what *A numerate person knows ...*, at the start of the project the most common type of response alluded to mathematical knowledge in the form of specific skills (8/13; e.g., "how to convert currency"), and there was no reference to problem solving. At the end of the project, responses still identified skills worth knowing, but nearly half referred to choosing mathematics that was appropriate to a particular task (7/15; e.g., "how and when to use what skill"). There was also a new emphasis on contexts, with one-third of responses mentioning knowing how to use mathematics in everyday life (5/15; e.g., "understand stock market data; know how to navigate through a map; understand the odds of Melbourne Cup horses"). The importance of positive dispositions was additionally noted in about one-third of responses at the end of the project (4/15; e.g., "not to be scared of numbers").

Comparing initial and final responses to the stem *A numerate person is ...* revealed increasing recognition of contexts as a component of numeracy. While one-third of responses made mention of contexts at the start of the project (3/9; e.g., "someone who uses numeracy in everyday situations"), half did so at the end of the project (7/14; e.g., "a person who can deal with numeracy ideas in their everyday life"). More striking is the increased emphasis on dispositions in responding to this sentence stem: fewer than one-quarter of initial responses (2/9; e.g., "confident and comfortable in making links") compared with more than half of final responses (8/14; e.g., "flexible in their mathematical thinking and confident to take learning risks to test their knowledge and ideas"). Problem solving, an aspect of mathematical knowledge, was not mentioned in any responses at the start of the project but was referred to in nearly half of the final responses (6/14; e.g., "If they cannot deal with the ideas immediately, they are able to use problem solving skills to deal with them").

In summary, this group of teachers came to the project understanding numeracy as involving mainly *contexts* and *knowledge* (skills + some problem solving). By the end of the project, the conception of numeracy displayed by the group had expanded to include an even greater emphasis on *contexts*, a more sophisticated appreciation of *knowledge* (problem solving + judicious use of mathematical skills), and attention to students' mathematical *dispositions*. These are generalisations that represent the most frequent responses. There were also some references at the beginning and end of the project to tools (e.g., "can use a variety of tools", "knows how to access a very full tool box") and a critical orientation (e.g., "knows when data have been manipulated to present bias", "can identify when information is inaccurate"), but such comments were rarer than those that were linked to the other components of the numeracy model. On our school visits we observed increasing use of tools, especially digital technologies such as spreadsheets, but teachers may not have considered this worth mentioning when responding to the sentence stem task. In the case of a critical orientation, classroom observations and teacher interviews confirmed that teachers found this aspect of the numeracy model the most difficult to implement.

Teacher trajectories through the numeracy model

Of the 20 project teachers, 18 completed the mapping task in the way we requested. Figure 2 shows these teachers' starting points and the direction in which they indicated they had developed as the project progressed.

Dispositions (D)	Knowledge (K)	Context (C)
D – C	K – D (2 teachers)	C – K – CO
D – C – T	K – D/C	C – K – D – T
D – C – K (2 teachers)	K – D – T	C – All
D – K/T/C	K – T – D (2 teachers)	
D – K/T – C	K – C – D	
D – K/T – C/CO		
D – K/T/C – CO		

Figure 2. Starting points and trajectories in engaging with the numeracy model

Of the 18 valid responses, 8 teachers indicated that they had entered the project with a concern for students' *dispositions*. Their annotations suggested that they were uneasy with students' negative feelings towards mathematics and wanted to devise numeracy learning experiences that would have a positive impact. Seven teachers indicated that their starting point had been students' *mathematical knowledge* and skills, and their annotations suggested that they believed that if students had appropriate mathematical knowledge and skills, they would be successful in applying these as required in context. Only 3 teachers indicated that they started the project with an emphasis on *contexts*, stating that this approach allowed students to apply their mathematical knowledge in meaningful situations. None of the teachers said they came to the project with a primary interest in *tools* or a *critical orientation*.

Although varied, teachers' trajectories through the model showed some patterns of similarity (see Figure 2). *Knowledge* to *dispositions* (K – D) and *dispositions* to *knowledge* (D – K) were common patterns, possibly indicating teachers' beliefs about the connection between success in using mathematical knowledge and a positive disposition. For the latter pathway, *tools* were linked often with *knowledge*. Only four teachers indicated that they considered the *critical orientation* aspect of the numeracy model, and this was their end point. One teacher, indicated by C – All in Figure 2, put the starting point as *contexts*, but then annotated the model comprehensively to show how integrated and equally important all these elements were.

Although the teachers identified different starting points and trajectories through the numeracy model, at least half of the valid responses to the mapping task indicated they had attended to four of the model's five components during the life of the project: 16 teachers annotated *knowledge*, 16 *dispositions*, 13 *contexts*, and 9 *tools*. These results are somewhat consistent with the analysis of numeracy conceptions revealed in the sentence stem task, where it was found that, by the end of the project, teachers had developed an understanding of numeracy as mainly involving knowledge, contexts, and dispositions. However, it was interesting to observe that teachers' most common starting point in engaging with the model was a concern for student dispositions, when this was not a strong feature of teachers' initial numeracy conceptions as elicited by the sentence stem task. Teachers may have initially paid most attention to components of the model representing student characteristics of concern to them, such as dispositions and mathematical knowledge, and then explored the use of contexts, tools, and, less commonly, a critical orientation as a means of enriching their numeracy teaching.

CONCLUDING COMMENTS

Sharing the responsibility for teaching numeracy in all curriculum areas, in the sense implied by the numeracy model presented in this paper, requires that teachers in primary and secondary schools, whether mathematics specialists or not, have a rich conception of numeracy themselves. The study reported here documented teachers' personal conceptions of numeracy and how these changed throughout a yearlong professional development project. The numeracy model provided a framework for attending to and valuing numeracy in a holistic way. Teachers seemed most comfortable with incorporating the *knowledge*, *dispositions*, and *contexts* components of the model into their thinking about numeracy. Although some teachers reported giving more attention to *tools*, especially digital tools, more professional support is needed for technology integration to develop teachers' confidence and expertise. This seemed to be particularly the case for primary teachers and secondary non-mathematics teachers. Development of a *critical orientation* occurred to a lesser extent. Perhaps teachers still lacked a clear understanding of how this could be incorporated into numeracy teaching, or they may not have felt ready to address this aspect of the model until their understanding of other components was secure. Further research is needed to explore how teachers can be supported in developing personal conceptions of numeracy, as well as numeracy teaching practices, that value a critical orientation, since this perspective is vital to educating informed and aware citizens.

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HISTORICAL JUNCTURES IN THE DEVELOPMENT OF DISCOURSE ON LIMITS

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Existing research on limits documents many difficulties students encounter when learning about the concept. Research also highlights some similarities between mathematicians' realizations of the concept over history and students' realizations of limits in today's classrooms. This theoretical study explores the historical development of limits and identifies some junctures that may also be critical in student learning. The study uses a communicational approach to learning, a framework developed by Sfard (2008), to investigate the development of discourse on limits over history.

INTRODUCTION

The notion of limit presents many challenges to students. Research indicates that students' informal realizations of limits are mainly based on dynamic notion, which can interfere with the representational and formal (static) realizations of the concept (Bagni, 2005; Tall, 1980; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1991). Student difficulties with limits can also be related to difficulties about the underlying notions such as functions, and infinitely small and large quantities (Parameswaran, 2007; Sierpińska, 1987). The dominance of dynamic and procedural aspects of limits in calculus textbooks and teaching as well as students' attitudes towards mathematics are also considered as factors that can contribute to student difficulties about limits (Bezidenhout, 2001; Parameswaran, 2007; Williams, 1991). Additionally, researchers also explored the historical development of limits to uncover some epistemological obstacles related to limits (Cornu, 1991; Sierpińska, 1987) that occurred "because of the nature of mathematical concepts themselves" (Cornu, 1991, p. 158).

Throughout the history of calculus, as the discourse around limits was developed, mathematicians encountered some of the conceptual obstacles students encounter today. For example, "limit as a bound" (Cornu, 1991; Williams, 1991), and "limit as unreachable" (Tall & Schwarzenberger, 1978; Williams, 1991) are among the incorrect student realizations about limit identified by research that were also problematic for mathematicians. Limit as a bound refers to the realization that a limit is a value past which the function cannot go. Limit as unreachable is based on the assumption that a limit is a value that can be approached but never reached. During the historical development of limits, mathematicians also debated "whether a variable can go beyond the limit and whether a variable can definitely reach the limit" (Schubring, 2005, p. 293). Similarly, Bagni (2005) mentioned that the historical

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development of visual, verbal, and symbolic representations of limit may parallel students' development. Williams (1991) also stated that

> Just as students' informal limit models tend to parallel those of the mathematical community prior to Cauchy, it is possible that only by appreciating the sorts of problems that motivated Cauchy's work will students be motivated to understand its implications. (p. 235)

This study explores the historical development of limits and uses an alternative lens to examine whether a communicational approach to learning can provide further insights regarding the nature of conceptual obstacles in the development of discourse on limits. The study addresses the following theoretical questions: (a) What are the historical junctures in the development of limits that resulted in changes in the discourse on limits?, and (b) How, and whether, can those junctures be useful for researchers to gain further information about the teaching and learning of calculus in today's classrooms?

THEORETICAL FRAMEWORK

The study uses the *commognitive* framework (Sfard, 2008), which highlights the close relationship between thinking and communication. This approach assumes that thinking is an individualized form of communication and considers cognitive processes and interpersonal communication as facets of the same phenomenon. Given these assumptions, the term *commognitive* combines the terms *cognitive* and *communicational*. From this perspective, developmental transformations are "the result of two complementary processes, that of *individualization of the collective* and that of *communalization of the individual*" (Sfard, 2008, p. 80, italics in original). Therefore, the study of human development can be considered as the study of the development of discourses, which are constructed and reconstructed through the interplay of individualization and communalization. Sfard (2008) defines the term discourse as the

> different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors (p.93).

This approach considers learning as participation in a discourse, and characterizes mathematics as a specific type of discourse that is distinguishable by its *word use*, *visual mediators*, *routines*, and *narratives*. Word use refers to the ways in which participants use words in their mathematical discourse. Visual mediators refer to the visible objects created and operated upon to enhance mathematical communication. Routines are the collection of metarules characterizing the repetitive patterns in the participants' discourse. Finally, narratives refer to the set of utterances describing mathematical objects and their relationships that are subject to endorsement or rejection. The narratives of a mathematical discourse that are endorsed by the majority of the experts of the community are considered as true.

An important element of mathematical word use is objectification, which results in replacing the talk about processes and actions with states and objects (Sfard, 2008). Through objectification, we identify the commonalities between different processes within a discourse and unify many lower-level phenomena under one name. Objectification increases the effectiveness of mathematical communication and is also a means of formalization. However, it hides the discursive layers that constitute mathematical objects.

Unlike object-level rules, which "take the form of narratives on the objects of the discourse", metarules "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p. 201). Metarules are often tacit due to the metaphorical nature of mathematical objects, which is amplified by objectification and symbol use. The mechanism of metaphor is "the action of 'transplanting' words from one discourse to another" (Sfard, 2008, p. 39). Note that the use of a metaphor is a metarule of mathematical discourse. Therefore, although metaphors are crucial mechanisms with which we build and expand discourses, the incorporation of them into an existing discourse requires changes in the metarules of the previously existing discourse. As a result, the exploration of the metaphors that govern different layers of a mathematical discourse becomes a central part of the exploration of the metarules in the development of the discourse. The implicitness of the changing metarules of mathematics is probably one of the reasons why the insiders of the mathematical discourse (e.g., mathematicians, mathematics teachers) "lose the ability to see as different what children cannot see as the same" (Sfard, 2008, p. 59). Therefore, the identification and analysis of junctures in the development of a discourse with respect to the changes in the metarules may give us information regarding the transitions learners need to go through as they participate in the extended discourse.

METHODOLOGY

The work reported here is part of a larger study that investigated the development of discourse on limits in a beginning-level undergraduate calculus classroom. The larger study examines the historical development of infinity, infinitesimals, and limits; one instructor's and his students' discourse on these notions; and comparison of the development of discourse on these concepts over history with that in the classroom. In this document, only the theoretical portion about limits is presented.

When analyzing the development of discourse on limits over history, the initial focus was on word use. Following Sfard's (2008) terminology, word use about limits was classified as *operational* if limit was referred to as a process based on dynamic motion; and *objectified* if it was referred to as a number or a distinct mathematical entity. This was followed by the exploration of the metaphors, and metarules to identify the historical junctures that led to changes in the metarules of the discourse on limits. For the purposes of this study, the term historical junctures refer to the points in the development of limits over history that resulted in changes in the metarules of the previously existing discourse on limits.

RESULTS

Historical development of the limit concept

The historical development of limit is quite intertwined with the development of functions as well as infinitely small and large quantities. It is not possible to provide all the details of such development in this paper. However, it should be highlighted that the dynamic view with an underlying assumption of continuous motion dominated mathematicians' discourse as they worked on these concepts till the 18th century. Continuous motion and geometrical foundations formed the bases of mathematician's formalizations of the notions such as infinity and infinitesimals till the end of the Renaissance period. However, as mathematicians like Viete (1540-1603), Descartes (1596-1650), Fermat (1601-1665), and Wallis (1616-1703) recognized the use of algebra as an aid to geometry, symbolic-algebraic approaches gained popularity and initiated the stage called the arithmetization of geometry.

Using the limit notion as a process is referred to as the *limit method* in the historical documents. Being the founders of calculus, Newton (1643-1727) and Leibniz (1646-1716) both used the limit method and infinitesimals in their theories as they worked on incremental change. By obtaining the tangent line at a point through the use of sequences of secant lines passing from that point (Lakoff & Núñez, 2000), they were both using limit as a process, which is based on the metaphor of dynamic motion. On the other hand, Newton used a geometric approach due to the nature of his problems at hand, whereas Leibniz relied more on arithmetic.

After Newton and Leibniz, mathematicians such as MacLaurin (1698-1746) and d'Alembert (1717-1783) kept on using this method on their problems. Lagrange (1736-1813) opposed to them on their use of the limit method:

> MacLaurin and d'Lambert used the idea of limits; but one can observe the subtangent is not strictly the limit of subsecants, because there is nothing to prevent the subsecants from further increasing when it has become a subtangent. True limits… are quantities which one cannot go beyond, although they can be approached as close as one wishes. (Lagrange, 1799, as cited in Schubring, 2005, p. 293)

Lagrange's arguments were based on

> the lacking of the concept of absolute value…so that it seems as if the variable goes beyond the limit; the criticism is also at the problem, which has always remained controversial, whether a variable can definitely reach the limit or is only allowed to come close to it at any rate (Schubring, 2005, p. 293).

Here, we again see mathematicians' struggles regarding whether limit was a bound or whether it could be reached. Although Lagrange used words like "true limits...are quantities", it was not until Cauchy (1789-1857) that the notion of limit was objectified. Lagrange talked about limit as a "subtangent", which is the "limit of subsecants". So he realized limits through the limit method and did not explicitly define them. Lagrange's operational word use in his utterance "[true limits]...can be approached as close as one wishes" entails infinitesimal increments as well as dynamic motion.

The discourse of calculus went through a fundamental change with Cauchy. He realized the necessity of establishing a theory of limits, which required the explicit definition of the concept. Cauchy defined limit as follows:

> When the values successively attributed to the same variable approach indefinitely a fixed value, eventually differing form it by as little as one could wish, that fixed value is called the limit of all the others (Kitcher, 1983, p.247).

An analysis of Cauchy's word use reveals that he objectified the notion of limit by referring to limit as a “fixed value”, that is, a distinct mathematical object. Note also that he used the words "successively", and “approach”, which are based on the metaphor of continuous motion. Finally, the phrases “approach indefinitely” and “differing from it as little as one could wish” entail the use of infinitely small quantities, namely, infinitesimals. Therefore, Cauchy’s definition of limit was based on infinitesimals and the continuous motion metaphor, which were both problematic for mathematicians of his time. The dynamic interpretation of limit was considered intuitive by the community since terms like *tending to* have a “connotation of desire, of aspiration. Numbers do not tend” (Fischbein, 1994, p. 239).

Weierstrass (1815-1897) and Dedekind (1831-1916) attempted to ‘remedy’ Cauchy’s definition by finding “a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis” (Dedekind, 1963, p.1, as cited in Kleiner, 1991). These mathematicians wanted to replace Cauchy’s kinematic approach with the algebraic-arithmetic approach. The goal was to reconceptualize calculus as arithmetic by eliminating spatial intuition. In order to do this,

> natural continuity had to be eliminated from the concepts of space, planes, lines, curves, and geometric figures. Geometry had to be reconceptualized in terms of sets of discrete points, which were in turn to be conceptualized purely in terms of numbers: points on a line as individual numbers…The idea of a function as a curve in terms of the motion of a point had to be completely replaced. There could be no motion, no direction, no approaching a point. All these ideas had to be reconceptualized in purely static terms using only real numbers. The geometric idea of approaching a limit had to be replaced by static constraints on numbers alone, with no geometry and no motion. This is necessary for characterizing calculus purely in terms of arithmetic. (Lakoff & Núñez, 2000, p. 308)

By considering space as consisting of discrete sets of points, and by reformulating continuity as the preservation of closeness, Weierstrass replaced the metaphor of continuous motion with the metaphor of discreteness. Weierstrass' formal definition of limit was similar to the following:

Let a function f be defined on an open interval containing a, except possibly for a itself, and let L be a real number. Then $\lim_{x \to a} f(x) = L$ if and only if for any number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Having eliminated the metaphor of continuous motion associated with infinitesimals and geometry, and having arithmetic as its foundation, this definition seemed to provide the precision mathematicians were looking for. The definition also explained anomalous cases that violated the geometric and dynamic conceptions of functions as curves. So this definition was generalizable to a broader number of situations. On the other hand, it can be argued that this definition wipes out all the intuitive tools with which to make sense of the concept. Note that the formal definition of limit is not constructive since it does not help us find what the limit of a function is but helps us prove the limit value we initially hypothesize is indeed the limit of the function at a particular point. That may be why the dynamic approach is still widely used both by mathematicians and students as they work on limits.

Historical junctures in the development of the limit concept

Two types of historical junctures in the development of discourse on limits will be highlighted: one led to the objectification of the concept; and the other led to an alternative realization of limits by the elimination of dynamic motion from the previous discourse on limits. Table 1 shows the junctures that transformed the metarules in the discourse on limits over history.

Junctures	Changing metarules
Cauchy's objectification of limit	Realization of limit as a process is changed to the realization of limit as a fixed value obtained as a result of the process. Limits become distinct objects of mathematics that can be defined and operated upon.
Weierstrass' introduction of the formal definition of limit	The metaphor of continuous motion is replaced by the metaphor of discreteness. Continuous motion and infinitesimals are eliminated from the theories of geometry, infinity, functions, and limits. Motion is reformulated as the static distance between discrete points.

Table 1: Historical junctures in the development of discourse on limits

DISCUSSION

By being based on tacit metarules and metaphors, historical junctures eventually require changes in the word use and endorsed narratives of a mathematical discourse. Although the historical processes of object creation can follow a different sequence than students' individualization of those objects, the communal aspect of mathematics contains the words, visual mediators, narratives, and the metarules (in the form of routines) students need to adapt to as they become participants in the mathematical discourse. Sfard's (2008) framework gives researchers the tools with which we can examine the development of mathematical discourse. A second contribution of this approach is to highlight the elements of teachers' and students' mathematical discourse that can remain implicit in the classroom. Teachers' explicit attention to the changes in word use, metarules, and metaphors can enhance their classroom communication.

The findings of the study indicate that objectification of limit and the elimination of dynamic motion were critical in the historical development of limits. A question that remains to be answered is whether those junctures may also be critical in students' development of discourse on limits. Although the work presented here cannot provide an answer for this, there is some evidence in the larger study regarding the question. As suggested by research on student learning about limits, the students in the larger study *only* used the dynamic aspect of limit. In addition, they rarely referred to limit as a number in their discourse. For example, even in the cases where they computed a limit accurately and wrote $\lim_{x \to 4} f(x) = 2$ using the equal sign, when asked to state what the limit was, they said "it is approaching to two" rather than "the limit is equal to two". In other words, the students used words operationally as they realized limit as a process and could not objectify the concept in their discourse at the end of their instruction on limits. Therefore, objectification and coping with the interplay between dynamic and static aspects of limit were also problematic for the students in the study. Further research is needed to examine the validity of the issue for the learning of limits and other mathematical topics.

Some cautionary comments are worth mentioning. First, the study does not suggest the development of limits over history is identical to students' development. Students may have many idiosyncratic obstacles about limits that are not present in the historical development. Moreover, students are presented with limit related ideas in a different order than historical development of those ideas, and do not have as much time to reflect on the concepts due to their curricular load. Second, the study does not suggest teachers should teach the historical development of mathematical concepts in their classrooms. Rather, it points to the features that can remain tacit in teachers' and students' mathematical discourse (e.g., metarules, and use of metaphors) when they talk about limits. Last, the study does not suggest all learners of calculus should learn about the formal definition of limit and limit-related proofs. Instead, it highlights the conceptual challenges surrounding the interplay between the dynamic and static aspects of limit resulting in different realizations of the concept.

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8TH GRADE TURKISH STUDENTS' VAN HIELE LEVELS AND CLASSIFICATION OF QUADRILATERALS

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In this study, we tested the van Hiele levels of the 8th grade Turkish students and examined their classification preferences (hierarchical or partitional) about relationships between some quadrilateral pairs. We also discussed the effectiveness of the materials (geo-stripes, dot paper and geo-boards) used for teaching the topic of quadrilaterals in terms of the students' classification preferences. The results indicated that most of the students were at van Hiele level 2 before starting their high school education and the students generally chose partitional classification. Therefore, we concluded that the materials had some limitations and were not adequate for learning the topic.

INTRODUCTION

The common theory used in geometry was proposed by two mathematics educators Pierre and Dina van Hiele. This theory with its five sequential levels, explains the learners' acquisition of geometric concepts and the development of geometric thought (Fuys, Geddes & Tischler, 1998). The characteristics of these first three levels are characterized by Burger and Shaugnessy (1986, p.31) as follows:

- *Level 1 [Visualization]:* The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.
- *Level 2 [Analysis]:* The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.
- *Level 3 [Abstraction]:* The student logically orders the properties of concepts, forms abstract definitions and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.

Fundamental mental processes like identification and classification of geometrical concepts have an important role for developing high level geometrical thinking abilities. However, many studies indicate that the abilities of identification and classification, which have key roles to reach a higher level, have not developed enough in students. Therefore, it is not possible for students whose mental processes have not developed enough, to be successful in high school and university level courses requiring them to have logical deduction

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(Fuys, Geddes & Tischler, 1998). It is beneficial for students to have abilities of identification and classification of fundamental geometrical concept at their elementary school years to change this negative situation.

As it can be conclude from the properties presented above, to define, classify and realize the hierarchical relationships between geometrical concepts, with their minimum properties are related the van Hiele level 3. Although there have been controversies about the classification of quadrilaterals (Jones, 2001), many researchers prefer to test students' geometrical level by using relationships among them. Classifying the quadrilaterals in a hierarchical manner plays an important role for elementary students to reach logical deductions. Here the word hierarchy (inclusive) refers to consideration of more specific concepts as subsets of more general concepts. In contrast, students may consider the concepts as disjointed or partitioned (exclusive) from each other (de Villiers, Rajendran, Patterson, 2009). On the other hand, in most curricula, students are expected to think about the hierarchical relationships of the quadrilaterals and reach logical deductions among them.

In his research, de Villiers (1994 p.15) points out that the hierarchical definition and classification is superior to partitional ones because

- it leads to more economical definitions of concepts and formulation of theorems
- it simplifies the deductive systematization and derivation of the properties of more special concepts
- it often provides a useful conceptual schema during problem solving
- it sometimes suggests alternative definitions and new propositions
- it provides a useful global perspective

Classification of Quadrilaterals in the Turkish Elementary Curriculum

The new Turkish elementary curriculum, which was applied in 2005, offers learning environments which give students opportunities to explore mathematical relationships use mathematical communication with their peers and use different learning styles which are enriched by learning centred approaches (MEB, 2005). Upon examination of the elementary geometry curriculum, it can be seen that the topic of quadrilaterals is an important component. Although at every elementary level some types of quadrilaterals are in the curriculum, the main focus occurs at the fifth and seventh grade levels. Both fifth and seventh level students study with squares, rectangles, parallelograms, rhombi and trapezoids. At fifth grade level, students generally learn their side, angle and diagonal properties and compare their differences rather than using logical deduction among them. At seventh grade level students are encouraged to consider hierarchical relationships, as shown in Figure 1, and make progress on deductive reasoning. In conclusion, the students are expected to reach van Hiele level 3 before starting high schools.

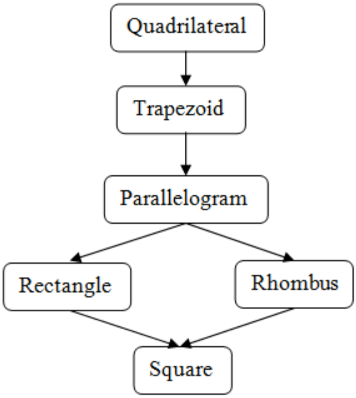
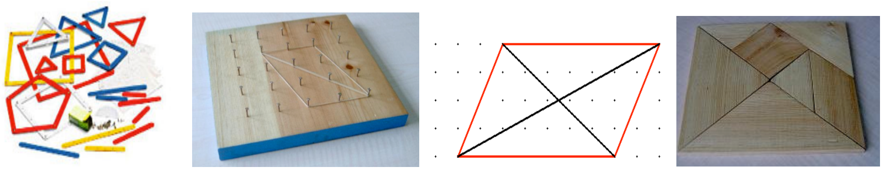


Figure 1: Classification of Quadrilaterals at 7th Grade Level

The Properties of Materials

Geo-stripes, geo-boards, dot paper and tangram are the mainly used materials as shown in Figure 2 for teaching the topic of quadrilaterals.



Geo-stripes Geo-board Dot paper Tangram

Figure 2: The Materials Used for Learning Quadrilaterals

The geo-stripes have more preferred materials than the others. Four stripes are generally used but more than four could be used, as well, to create quadrilaterals. They have flexible structure to change some types of shapes into the other type. The possible conversions with four stripes, after the manipulation, are as follows:

- quadrilateral → trapezoid (sometimes)
- trapezoid → quadrilateral
- parallelogram → rectangle

- rectangle → parallelogram
- rhombus → square
- square → rhombus.

Geo-boards are used together with elastic garters. Students have opportunities to create shapes and examine their properties. Although geo-stripes limit students on convention, students can turn some shapes into others with geo-boards. Dot paper is generally used for exploring the side, angle and diagonal properties of the shapes benefiting from dot observations. The last material, tangram, is generally for creating the quadrilaterals.

The Purpose of the Study

We aim to determine the 8th grade students van Hiele Geometric Thinking Level before starting their high school education. Also, we examined the effectiveness of materials which are used for teaching the topic of quadrilaterals.

METHODS

Participants

The participants for this study 56 (26 boys and 30 girls) 8th grade students were selected from two different classes in an elementary school in the borough of Cayeli, Rize. The school is one of the top schools in this area. The same teacher has taught mathematics to both classes for almost three years. The study was carried out near the end of the spring semester of the 2009/2010 academic year. The ages of students in both groups ranged from 14 to 15.

Instruments

According to van Hieles students at the elementary school levels could reach the third level, so the first 15 questions of the van Hiele Geometric Thinking Test (VHGTT) developed by Usiskin (1982), which consists of 25 multiple choice questions, were taken into account. The researchers also tested students' classification and logical deduction abilities by asking some questions about them. For testing their classification abilities, students were encouraged to identify six different shapes' types hierarchically (Appendix). The students were given ten questions on the relationships between quadrilaterals for testing their logical deduction abilities. For example, for the relationship of a parallelogram and a trapezoid it was asked that *a parallelogram is* __________ *a trapezoid* and students were encouraged to fill the blank using "always," "sometimes," or "never" by choosing one of the words that makes the sentence true.

FINDINGS

Each question was 1 point in the VHGTT. Therefore, in the test, the lowest and the highest score one could get were 0 and 15 respectively. The results of the test are shown in the Table 1. As it can be seen from the table, the scores ranged from 4 to 11 and the average of the test was 6.71. The standard deviation value 1.95 indicates the homogeneity of the scores.

N	Minimum Score	Maximum Score	Mean	S.D.
56	4	11	6.71	1.95

Table 1: Descriptive Statics of VHGTT Results

Table 2 shows the answers given for each shape A to F (Appendix), for example, the figure A was a rectangle: 43.75% (49 of 112) of markings cited it as a rectangle and correctly it was also marked 33.04% (37) as a quadrilateral and 15.18% (17) as a parallelogram. While none of the markings were trapezoid, incorrectly 5.36% (6) were marked as a rhombus and 3% (2.68) were marked as a square. In the table, bold characters represent the correct identifications. When the answers were analysed it was seen that students identified the figures prototypically. Additionally, the shapes were known with their general images. Also, the most preferred second answer was "quadrilateral". Only figure E could be excluded from that view because although it was only a quadrilateral, most of the markings (47 of 81) were trapezoid and it was not considered prototypically like the others.

Figure	Q		T		P		Re		Rh		S	
	N	%	**N**	%	**N**	%	**N**	%	**N**	%	**N**	%
A	**37**	**33.04**	-	-	**17**	**15.18**	**49**	**43.75**	6	5.36	3	2.68
B	**34**	**30.63**	**6**	**5.41**	**47**	**42.34**	11	9.91	11	9.91	2	1.8
C	**33**	**27.27**	-	-	**9**	**7.44**	**7**	**5.79**	**20**	**16.53**	**52**	**42.96**
D	**27**	**27.84**	**43**	**44.33**	22	22.68	2	2.06	3	3.09	-	-
E	**30**	**37.04**	47	58.02	-	-	2	2.47	2	2.47	-	-
F	**31**	**26.72**	**13**	**11.21**	**28**	**24.14**	2	1.72	**32**	**27.59**	**10**	**8.62**

(Q: Quadrilateral, T: Trapezoid, P: Parallelogram, Re: Rectangle, Rh: Rhombus, S: Square)

Table 2: Given Answers with Their Percentages to Each Shape

The results of the analysis shown in the Table 3 indicated that the inclusive properties could not be considered as much as desired. Although none of the correct answers were "never", a number of students marked it. Also, the students were more successful to identify correct relationships going down from the hierarchical chain shown in Figure 1 rather than going up. For example; while

58.93% (33 of 56) marked that a parallelogram is *always* a rectangle, only 39.29% identified correctly that a rectangle is *sometimes* a parallelogram.

The pairs of quadrilaterals	Number of Students' Responses							
	Always		Sometimes		Never		Empty	
	N	%	N	%	N	%	N	%
Q-T	3	5.36	43	82.14	10	17.86	-	-
P-Re	11	19.64	33	58.93	11	19.64	1	1.79
Re-P	22	39.29	18	32.14	14	25	2	3.57
S-Rh	36	64.29	8	14.29	9	16.07	3	5.36
Rh-S	19	33.93	25	44.64	11	19.64	1	1.79
S-Re	22	39.29	27	48.21	7	12.5	-	-
Re-S	6	10.71	11	19.64	38	67.86	1	1.79
P-T	8	14.29	21	37.5	26	46.43	1	1.79
T-P	6	10.71	17	30.36	31	55.36	2	3.57
Rh-P	24	42.86	22	39.29	10	17.86	-	-

Table 3: The Results of the Logical Deduction Part

(Q: Quadrilateral, T: Trapezoid, P: Parallelogram, Re: Rectangle, Rh: Rhombus, S: Square)

DISCUSSIONS and CONCLUSIONS

This study showed that the 8th grade students could not reach van Hiele level 3 before starting their high school education. They were generally at the level of van Hiele 2 so the lack of hierarchical thinking might pose a problem in understanding high school mathematics courses which require logical deduction.

The students had difficulties in ordering figures logically and comparing the interrelationships between them. Also, the students had tendencies to label the shapes by their general images as seen by the classification part. The trapezoid was the most problematic type of all. The students generally did not prefer to associate it with the others; they even entitled a general quadrilateral as a trapezoid without looking at its parallelism. One reason for this might be because the word of "trapezoid" in the Turkish language "yamuk" which means slanty. The students, therefore, were not inclined to label the other types, in particular a square and a rectangle, as a trapezoid.

Although the overall scores were not as high as desired, the logical deduction part results (Table 3) showed that the students were more successful on some pairs. This could be because of the properties of geo-stripes. As it was mentioned before, geo-stripes have limited inclusive conversions between pairs. For example; the students got higher scores for the pairs of rhombus-square and square-rhombus than the rectangle-square and the square-rectangle pairs. Because students could not turn a rectangle into a square with geo-stripes, this limited conversion posed an obstacle for hierarchical thinking. To construct hierarchical thinking, it is important to use transitive materials, but limited transitive materials, like geo-stripes, could be insufficient. So the geo-stripes and the others (Figure 2) could be used for learning the properties of quadrilaterals rather than exploring hierarchical relationships.

To overcome this transitive material problem, it could be more efficient to use the dynamic geometry software. In the dynamic geometry software, the shapes could be constructed with proper construction, so considering shapes hierarchically could be easier. In particular, *the Geometric Supposer* (Schwartz & Yerushalmy, 1985) has rich opportunities for the topic of quadrilaterals, which could make a contribution for developing students' logical deduction abilities.

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Appendix

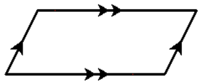
The Questions of the Classification Part

Which words name each shape? Please, circle all that apply.

A

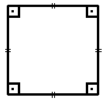
quadrilateral rectangle
rhombus trapezoid
parallelogram square

B



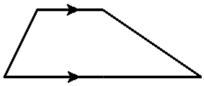
quadrilateral rectangle
rhombus trapezoid
parallelogram square

C



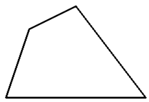
quadrilateral rectangle
rhombus trapezoid
parallelogram square

D



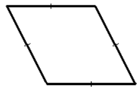
quadrilateral rectangle
rhombus trapezoid
parallelogram square

E



quadrilateral rectangle
rhombus trapezoid
parallelogram square

F



quadrilateral rectangle
rhombus trapezoid
parallelogram square

SIMPLIFICATION OF RATIONAL ALGEBRAIC EXPRESSIONS IN A CAS ENVIRONMENT: A TECHNICAL-THEORETICAL APPROACH

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In this paper we analyze and discuss the influence of CAS technology and an Activity designed with a Technical-Theoretical approach on two 10th grade students' thinking on a Task related to simplifying rational algebraic expressions. The theoretical elements adopted in this study are based on the instrumental approach. Results indicate that CAS and a technical-theoretical-oriented Activity provoked students to theorize on certain aspects of the simplification of rational expressions, thus illustrating the epistemic role of CAS technique and its influence in improving students' learning with respect to specific technical-theoretical components of rational expressions.

INTRODUCTION

Ever since the arrival of Computer Algebra Systems (CAS), many researchers have studied the role of this kind of technology in the learning of algebra (Thomas, Monahan & Pierce, 2004). According to some researchers (e.g., Artigue, 2002; Lagrange 2003) the technical aspect of algebra (i.e., the symbol manipulation) is fundamental in order to promote students' conceptual understanding. Accordingly, Kieran (2004) has pointed out that, due to the fact that conceptual understanding can come with technique, the study of algebraic transformations will be an area of research interest during the years to come. Thus, it is not a coincidence that in the past few years CAS has played a major role, mainly in those studies related to that aspect of algebraic activity that Kieran (2004) has identified as transformational activity.

In this sense, many studies (e.g., Kieran & Damboise, 2007; Kieran & Drijvers, 2006, Hitt & Kieran, 2009) related with the use of CAS and a technical-theoretical approach to algebra, have indicated the potential of this kind of technology in algebra learning. These studies have shown that the use of CAS promotes conceptual understanding if the technical aspect of algebra is taken into account. For instance, Kieran and Damboise (2007) pointed out how weak algebra students can improve both technically and theoretically by means of a CAS experience involving the factoring of algebraic expressions. Kieran and Drijvers (2006) showed that techniques and theory co-emerge in CAS environments where tasks promote the interaction between CAS and paper-and-pencil media.

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According to the reported literature, with respect to CAS studies, little or nothing has been said on the role of CAS technology in students' thinking on the simplification of rational expressions – a task embedded in the transformational activity of algebra. Our interest in studying this domain of school algebra in a CAS environment is based on more than three decades of research that has recognized (e.g., Davis, Jockusch & McKnight, 1978; Matz, 1980) that students have difficulty when they try to manipulate (simplify) rational expressions, making well known errors in tasks of this sort. Thus, the aim of this study is to answer the following research question: Which technical and theoretical aspects are promoted [or emerge] in students' thinking by the use of CAS and an activity designed with a technical-theoretical approach to the simplification of rational algebraic expressions?

THEORETICAL FRAMEWORK OF THE STUDY

The instrumental approach to tool use has been recognized as a framework rich in theoretical elements for analyzing the processes of teaching and learning in a CAS context (e.g., Artigue, 2002; Lagrange 2003). This approach encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). According to Monaghan (2007), one can distinguish two directions within the instrumental approach: one in line with the cognitive ergonomics framework, and the other in line with the anthropological theory of didactics. In the former, the focus is the development of mental schemes within the process of instrumental genesis. Within this approach, an essential point is the distinction between artifact and instrument (for more details see Drijvers & Trouche, 2008).

In line with the anthropological approach, researchers such as Artigue (2002) and Lagrange (2003, 2005) focus on the techniques that students develop while using technology. This approach is grounded in Chevallard's anthropological theory. Chevallard (1999) points out that mathematical objects emerge in a system of practices (praxeologies) that are characterized by four components: *task*, in which the object is embedded (and expressed in terms of verbs); *technique*, used to solve the task; *technology*, the discourse that explains and justifies the technique; and *theory*, the discourse that provides the structural basis for the technology.

Artigue (2002) and her colleagues have reduced Chevallard's four components to three: *Task, Technique,* and *Theory*, where the term *Theory* combines Chevallard's *technology* and *theory* components. Within this (Task-Technique-Theory) theoretical framework a *technique* is a complex assembly of reasoning and routine work and has both pragmatic and epistemic values (Artigue, 2002). According to Lagrange (2003), technique is a way of doing a task and it plays a pragmatic role (in the sense of accomplishing the task) and an epistemic role. With regard to the epistemic value of technique, Lagrange (2003) has argued that:

> Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual

> reflection when compared with other techniques and when discussed with regard to consistency. (p. 271)

According to Lagrange (2005), the consistency and effectiveness of the technique are discussed in the theoretical level; mathematical concepts and properties and a specific language appear. This epistemic value of techniques is crucial in studying students' conceptual reflections within a CAS environment. In our study, this T-T-T framework was taken into account in all aspects: the designing of the Activity related to the task "simplifying rational expressions", the conducting of the interviewer interventions, and the analyzing of the data that were collected.

THE STUDY AND METHODOLOGICAL CONSIDERATIONS

In the present paper we discuss and report the results of the first section of the designed Activity, which is part of a wider research study on the role of CAS and a Technical-Theoretical approach to algebra on the simplification of rational expressions.

Rationale of the Designed Activity.

It is important to mention that in this study we use the term Task as is defined in the T-T-T framework; it refers to a question embedded within the Activity. That is, as Kieran and Saldanha (2008) state, the Activity is a set of questions related to a central Task – in our case, the simplification of rational expressions. Since we have adopted the T-T-T framework for conducting the research study, the Activity was designed so that Technical and Theoretical questions were central and, hence, that students would have the opportunity to reflect on both Technical and Theoretical aspects in both paper-and-pencil and CAS environments. In the present report, only the following parts of the activity are reported: first, students' paper-and-pencil work (with Technical and Theoretical questions); second, their subsequent CAS work (Technical question); and, finally, Theoretical questions related to their work in both environments.

Population.

The participants were eight 10^{th} grade students (15 years old) in a Mexican public school. The selection of the students was made by their mathematics teacher, who believed that they were strong algebra students. None of the students were accustomed to using CAS calculators; consequently, at the outset of the study, all the students received some basic training from the interviewer on how to use the TI-Voyage 200 calculator for basic symbol manipulation.

Implementation of the Study.

The data collection was carried out by means of interviews conducted by the researcher. Students worked in pairs; each work session lasted between two and three hours. Each team of two students had a set of printed Activity sheets as well as a TI-

Voyage 200 calculator. Every interview was audio- and video-recorded so as to register the students' performance during the sessions.

ANALYSIS AND DISCUSSION OF THE DATA

In this report we analyse and discuss only one team's work. This team was chosen for the report because we consider that their work was typical of all participants and represents the role played by both the CAS and the designed Activity (we'll call each member of the team S1 and S2). The analysis, which is qualitative in nature, is based on the team's work sheets, as well as the video-recorded interview. The analysis and discussion of the data is detailed below as follows.

On the paper-and-pencil work related to Technique and Theory.

As per the task design, the first section of the activity helped reveal the students' Technique and Theory related to their paper-and-pencil simplification of rational expressions (see Figure 1). From their work, we confirm that, in this environment, students made the expected errors: they eliminated the 'literal components' that were common to both numerator and denominator, without taking into account whether these 'literal components' were, in fact, a factor of both the numerator and the denominator.

We note too that whenever there were parentheses, the students first tended to expand the expressions of the numerator and denominator (see the first example of Figure 1) before cancelling. This initial expanding, which was not preceded by a first observation in terms of factors, was something that hindered their theoretical reflection and seemed to lead them to make the kinds of errors that are reported in the literature. In their written explanations, they used the terminology of *dividing* (see the second example of Figure 1, where the students wrote, "we divide the same letters").

Ia) Simplifica, usando papel y lápiz, las siguientes expresiones. Muestra todo tu trabajo. Completa la tabla comenzando con la primera fila.

Expresión	Explica tu procedimiento de simplificación
$\frac{x(3+x)}{x}$ $= \frac{3x+x^2}{\cancel{x}}$ $3+x^2$	Primero multiplicamos lo que esta antes del paréntesis por lo de adentro del mismo y después simplificamos x.
$\frac{4x+4y}{x+y}$ $= 4+4$ $= 8$	Al dividir letras iguales los exponentes se restan y como resultan elevadas las variables a la cero ya no se escriben.
$\frac{3x+4y}{x+y}$ $= 3+4$ $= 7$	//

Figure 1. Simplification of expressions: Paper and pencil work.

On the New Technique and Theory, Based on the Use of CAS.

In the context of the designed Activity, the use of CAS led the students to rethink their first techniques and explanations and provoked a theoretical reflection that could explain for them the results given by the CAS. The differences between the two sets of results led them to wonder about their paper-and-pencil techniques and explanations. They began to question the theoretical underpinnings of their work. Figure 2 shows the corresponding students' CAS work.

Ib) Verifica tus respuestas de Ia), para ello, utiliza la calculadora (usa la tecla *enter*). Escribe los resultados dados por la calculadora en la siguiente tabla.

Introduce en la calculadora	Respuesta dada por la calculadora
$\frac{x(3+x)}{x}$	$x+3$
$\frac{4x+4y}{x+y}$	4
$\frac{3x+4y}{x+y}$	$\frac{3x+4y}{x+y}$

Figure 2. Simplification of expressions: CAS work.

For the expressions that involve just one term in the denominator (as in the first example of Figure 2), the students could see that their paper-and-pencil technique was not correct, but could also see how to fix it. As the following extract suggests, they were able to make a quick adjustment to their first technique (adjustment without theoretical justification that called for cancelling each occurrence of the given term in the numerator) so as to eliminate the discrepancy between the results:

1 S1: What is it? [*Asking for the result given by the calculator for the first expression of Figure 2*]

2 S2: *x* plus 3 [*the CAS result for the first expression of Figure 2*]

3 S1: And we wrote 3 plus *x* squared [*She refers to the result which they got by paper and pencil at the time they obtain the CAS result for the first expression of Figure 2*]

4 S2: Yes, We must have taken off only one *x* [*Meaning that they had to eliminate another x*]. No matter. What's next?

However, for the second and third examples of Figure 2, the students could not easily come up with a simple adjustment to their paper-and-pencil technique for simplifying those expressions containing a binomial as the denominator. The following extract illustrates their bewilderment at the CAS results for the last two expressions:

5 S2: Yes, here [*Referring to the first expression of the Figure 2*], it makes sense [*the result given by the calculator*] because the *x's* were taken off, it first multiplied and we missed taking off the two *x's*. [*She states the multiplication procedure that she thinks the calculator did, just as they had expanded the numerator of the first expression of the Figure 2*]. But in here, I'm not quite sure why it's 4, neither the result in here [*Referring*

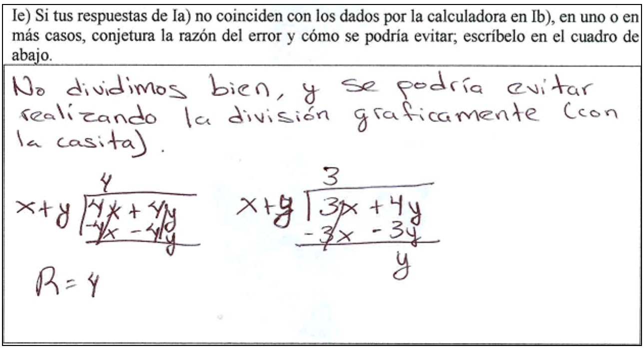
to the last two results (Figure 2) given by the calculator]. Why it is the same [*referring to the 3rd result of Figure 2*], I don't have any idea.

While they could accommodate the result given by the CAS for the first example, the other two examples remained mysterious. They kept asking themselves if there were other ways to think about these simplifications. How might they justify the results given by the CAS? The following extract underlines their dilemma, but then student S1 suddenly had an idea:

6 S2: It's believed that in this case we should've taken off the *x* and the *y*, we take off both [*The repeated terms in the numerator and the denominator of the 2nd expression in Figure 2*]. But why is it 4? [*The result given by CAS*]

7 S1: Let's see [*Pause*]. This is a division of polynomials!

It is clear that the CAS Technique provoked a conceptual change in one of the students (line 7 of the above transcription). This theoretical reflection induced by the discrepant results moved the students from a Technique involving eliminating literal symbols that are repeated in the numerator and the denominator to a Technique involving division of polynomials (the long division of polynomials algorithm) as can be seen in the next Figure 3.


Ie) Si tus respuestas de Ia) no coinciden con los dados por la calculadora en Ib), en uno o en más casos, conjetura la razón del error y cómo se podría evitar; escríbelo en el cuadro de abajo.

No dividimos bien, y se podría evitar realizando la división graficamente (con la casita).

Figure 3. New paper and pencil Technique for simplifying rational expressions.

It is interesting to see how the students came to adapt their new technique and theory so as to make it also fit the case of rational expressions that could not be simplified. They found, on their own, that if the quotient works out exactly, then the rational expression can be simplified – the quotient of the division being the final simplification. But if the division is not exact, then the rational expression can not be simplified and the CAS calculator will give as the result the same expression. For those cases where the denominator is a monomial, the students continued to believe that the technique of cancelling the monomial of the denominator with all of its occurrences in the numerator is workable.

CONCLUSIONS

In this report we have showed the epistemic role of CAS Technique, in the sense that the use of the CAS provoked in students a spontaneous theoretical reflection that allowed them to think of new techniques for simplifying rational expressions. The use of the CAS provoked a change in the students' technique for simplifying rational expressions whose denominator is a binomial (from canceling 'literal components' that were common to both numerator and denominator to using the polynomial division algorithm as the new Technique). This epistemic role played by the CAS was evident also in terms of the students' language, the students' initial language evolving from "canceling and dividing" terms to using terminology involving the division of polynomials.

However, other technical-theoretical aspects did not emerge, such as noticing the structure of the expressions in terms of factors. Thus new questions arise, such as, *How to promote in students' thinking a focus on seeing the expressions in terms of factors?* CAS technology and appropriate tasks may not be sufficient; teacher intervention may be critical in encouraging technical-theoretical noticing of other aspects of this domain on the part of students.

Acknowledgments

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