When visual and verbal representations meet - The case of geometrical figures

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OF THE 37TH CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

» Mathematics learning across the life span «

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PROTO-ALGEBRAIC LEVELS OF MATHEMATICAL THINKING

Lilia P. Aké[1], Juan D. Godino[1], Margherita Gonzato[1], Miguel R. Wilhelmi[2]

[1]University of Granada (Spain), [2] Public University of Navarra (Spain)

Researches on the nature and development of algebraic reasoning in early grades of primary education have been inconclusive about the boundaries between mathematical practices of algebraic nature and those not algebraic. In this report we define primary levels of algebraization in school mathematics activity and prototypical examples of answers to a task for each level, based on the type of objects and processes proposed by the onto-semiotic approach of mathematical knowledge. This model can be useful to develop the meaning of algebra in elementary school teachers and empower them to promote algebraic thinking in primary education.

Key words: elementary algebra, mathematical practice, reasoning level, teacher's training, onto-semiotic approach.

INTRODUCTION

The complex issue of making advances to clarify the nature of algebraic thinking is necessary from the point of view of education. As Radford says (2000, 238): "To go further, we want to add, we need to deepen our own understanding of the nature of algebraic thinking and the way it relates to generalization". The development of a comprehensive model of elementary algebra could facilitate the design of instructional activities that promote the emergence and progressive consolidation of algebraic reasoning.

In this report we address this problem by using some theoretical tools of the *Onto-semiotic approach* to research in mathematics education (Godino, Batanero and Font, 2007). We believe, together with various authors (Mason and Pimm, 1984; Carraher, Martinez and Schliemann, 2008; Cooper and Warren, 2008), that generalization and also the means to symbolize both generalization situations and modelling (in particular, using equations) are key features of algebraic reasoning.

First we summarize the vision of elementary algebra according to the onto-semiotic approach developed in Godino, Castro, Ake and Wilhelmi (2012); then we define two levels of proto-algebraic reasoning framed between two other levels: one, in which the reasoning is purely arithmetic (level 0 of algebraization), another in which the algebraic features are consolidated (level 3). Finally we highlight some implications of the model for the training of primary school teachers.

ONTO-SEMIOTIC APPROACH TO ELEMENTARY ALGEBRA

The pragmatic, anthropological and semiotics perspective of the onto-semiotic approachto research in mathematics education (OSA) (Godino, Batanero and Font, 2007; Godino, Font, Wilhelmi and Lurduy, 2011) provides theoretical tools that can help to characterize algebraic reasoning in terms of types of objects and processes involved in mathematical practice. Considering a mathematical practice as intrinsically algebraic can be based on the presence of certain types of objects and processes, usually considered in the literature as algebraic.

Prototypical algebraic objects

In the framework of elementary algebra the following are considered as prototypical algebraic objects:

1) Binary relations ─equivalence or order─ and their respective properties (reflexive, transitive and symmetric or antisymmetric). These relationships are used to define new mathematical concepts.

2) Operations and their properties, performed over elements of various sets of objects (numbers, geometric transformations, etc.). The algebraic calculation is characterized by the application of properties. Concepts like equation, inequality, and procedures such as elimination, factorization, etc. can also intervene.

3) Algebraic functions, generated by addition, subtraction, multiplication, division, potentiation and root extraction of the independent variable. It is necessary to consider different types of functions (polynomial, rational, radical) and its associated algebra (operations and properties).

4) Structures and their types (semigroup, group, ring,…) studied in abstract algebra.

Prototypical algebraic processes

Particularization and generalization processes are particularly importantfor algebraic activity, given the role of generalization as one of the key features of algebraic reasoning. Thus, for analysing algebraization levels of mathematical activity it is useful to focus attention on the objects resulting from the generalization and particularization processes. As a result of a generalization process we obtain a type of mathematical object we call *intensive object*, which becomes the rule that generates the class (collection or set) of generalised objects and that enables the identification of particular element as representative of the class (Godino et al., 2011). Through particularization processes new objects are obtained that we call *extensive* (particular) objects. A finite set or collection of particular objects simply listed should not be considered as an intensive until the subject shows the rule applied to delimit the constituent elements of the set. Then the set becomes something new, different from the constituent elements, as a unitary entity emerging from the set. Therefore, besides the generalization process giving rise to the set, there is a process of *unitization*.

Moreover, the new unitary entity has to be made ostensive or materialized by a name, icon, gesture or symbol. The *ostensive object* embodying the unitary object emerging from generalization is another object that refers to the new intensive entity, so there is a process of *representation* accompanying to the generalization and materialization processes. Finally, the symbol is released from the object which represents to become the object upon which actions are performed (*reification* process).

The different types of algebraic objects and processes can be expressed with different languages, preferably alphanumeric at higher levels of algebraization. Nevertheless, primary school pupils might also use other means of expression to represent objects and processes of algebraic nature (Radford, 2003).

In the next section, we describe the boundary between arithmetic and algebra in terms of the dualities and processes described. This boundary is not objective or platonically established, since these dualities and processes are relative to the context where mathematical practice is developed. In fact, the algebraic character is essentially linked to the subject's recognition of the rule that shapes the intensive, the consideration of the generality as a new unitary entity and its enactment by any semiotic register for subsequent analytical treatment. This threefold process (recognition or inference of generality, unitization and materialization) allowsdefining two primary levels of algebraic thinking, distinguishable from a more advanced level in which the intensive object is seen as a new entity represented with alphanumeric language.

ALGEBRAIZATION LEVELS

In this section we describe the characteristics of the practices to solve mathematical tasks, affordable in primary education, which allow to define different levels of algebraization. We propose to distinguish two *proto-algebraic* levels of primary algebraization. These levels are framed between a 0 level of algebraization and a third level in which mathematical activity can be considered as properly algebraic. This level is assigned, not to the task itself but to the mathematical activity that is performed. To explain the features of the algebraization levels we use examples of student teachers' responses to a task on geometric patterns. The description of such teaching experience is not the aim of this report due to space restrictions.

The problem posed to a sample of 52 student teachers is as follows:
See the following figure, and answers:

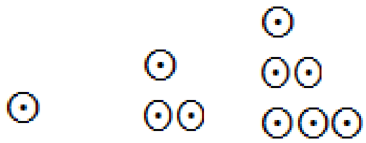


Fig. 1   Fig. 2   Fig. 3   ⋯

a) How many balls are there in figures on fourth and fifth position?

b) How many balls are there in figure 100?

Level 0 of algebraization

If we want to train primary school teachers so they can help their pupils to develope algebraic reasoning, we need to describe the mathematical practices of level 0, that is, those that do not include algebraic features. This is an unclear issue in the literature on early algebra (Carraher and Schliemann, 2007). We propose the following rule to assign level 0 of algebraization to a mathematical practice:

Extensive objects, expressed by natural, numerical, iconic or gestural language, are involved. Symbols that refer to an unknown value can also intervene, but that value is obtained as a result of operations on particular objects.

Figure 1 shows an example of mathematical activity we consider indicative of absence of algebraic thinking.

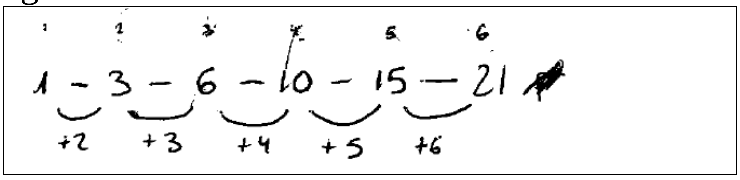


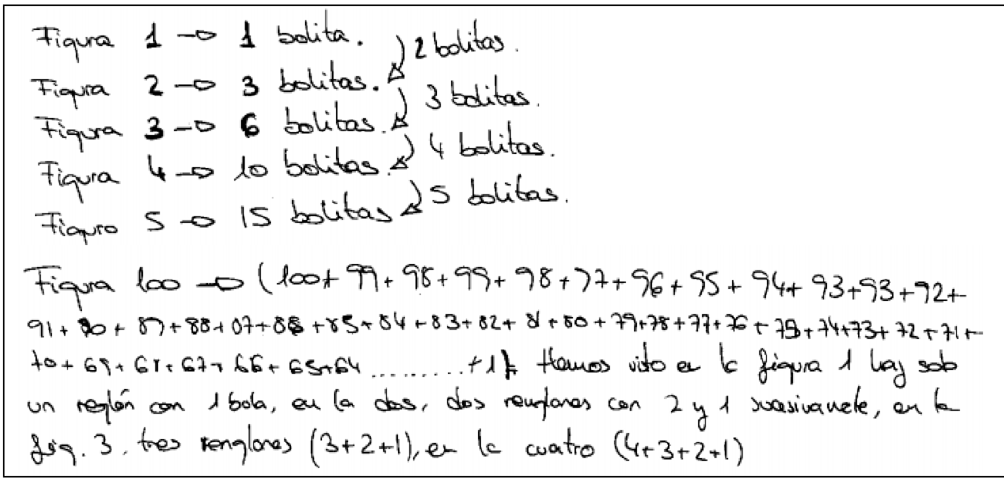
Figure 1. Level 0 response

The student writes the first six values of the independent variable of the function (order number of the figure) and below the number of balls that corresponds to each value, along with the criteria for obtaining these values (sum of successive natural numbers). He uses a numerical and visual language to express particular values, and makes no attempt to generalize the assignment criteria, or the initial and final sets of the correspondence. It is true that for the first six terms the student writes a formation rule, which extrapolated to any subsequent term would be indicative of the kind of factual generalization that Radford (2003) describes, but in this student's case such generalization does not occur.

Level 1 of algebraization

Intensive objects, whose generality is explicitly recognized by natural, numerical, iconic or gestural languages, are involved. Symbols that refer to the recognized intensive objectsare used, but there is no operation with those objects. In structural tasks relationships and properties of operations are applied and symbolically expressed unknown data may be involved.

Figure 2 shows a student's response that exemplifies this proto-algebraic level of thinking.



Translation: We have seen in Figure 1 there is a row with only one ball, in the second, two rows with 2 and 1 successively, in Fig. 3, three rows (3 +2 +1), in the fourth (4 +3 +2 +1)

Figure 2. Level 1 response

This student finds a general rule (*intensive object*) that allows him find the value of the function for any value of the independent variable (figure position) and that explicitly define with a sum of consecutive numbers. He uses ordinary language (to explain the formation rule) and arithmetic language (natural numbers and the sum of the first 100 natural numbers), but he has not been able to find a symbolic expression for this sum. The student can find the number of balls in figure 100, without forming this figure and without, therefore, explicitly count the beads, but operating with the sequence of particular numbers. It is a *factual generalization* (Radford, 2003). The operational scheme is limited to the concrete level, which however would allowhim to deal successfully with virtually any term.

Level 2 of algebraization

Indeterminate or variables expressed in literal-symbolic language to refer the intensive objects recognizedare involved, but they are linked to the spatial or temporal information of the context. In structural tasks the equations have the form Ax ± B = C. In functional tasks the generality is recognized, but there is no operation with variables to obtain canonical forms of expression.

An example of this algebraization level is shown in figure 3.

Al multiplicar una fila de bolitas por otra (a la que restamos 1 para no contar varias veces las mismas bolitas) obtenemos un cuadrado de bolitas ( ). Dividiéndolo entre 2 obtenemos un triángulo, pero aún habría que añadirle la fila nueva de esa serie para obtener la cantidad correcta.

Para este patrón se añaden tantas bolitas como indique el ordinal de la figura. Así, para la Fig 11, habrá que añadirle 11 bolitas a la cantidad que tuviese la Fig. 10.

$\boxed{x = \frac{n\cdot(n-1)}{2}+n}$ (n = ordinal de la figura)

Fig. 4 = $\frac{4\cdot 3}{2} + 4 = 6+4 =$ 10 bolitas

Fig 5 = $\frac{5\cdot 4}{2} + 5 =$ 15 bolitas

Fig. 100 = $\frac{100\cdot 99}{2} + 100 = \frac{9900}{2} + 100 = 4950+100 =$ 5050 bolitas

Translation:
Multiplying a row of balls by other (to which we subtract 1 not to count several times the same balls) we get a square of balls. … Dividing it by 2 we get a triangle, but still the new row of that series should be added to get the right amount.
For this pattern as many balls as those indicated by the ordinal of the figure are added. Thus, for Fig. 11, 11 balls will be added to the amount that Fig. 10 had.

Figure 3. Level 2 response

The student finds a correct formula for calculating the number of balls on the figure in any position, expressed with alphanumeric language. The justification of the formula is based on visual reasoning, expressed with confuse and not entirely correct natural language, since the visual inference of the formula requires forming a rectangle of sides *n*(*n*–1), and not a square. He does not operate with variables to get a canonical expression of the correspondence criterion. The student's reasoning includes aspects of contextual and symbolic generalizations (Radford, 2003). There is an explicit use of generic elements for the figure position and the corresponding number of balls, expressed in contextual terms and also symbolically. However, the mere use of literal symbols in a general expression is not enough to recognize the presence of aproperly algebraic practice.

Level 3 of algebraization

Intensive objects are generated which are literal-symbolically represented, and operations are carried out with them; transformations are made in form of symbolic expressions preserving equivalence. Operations are performed on the unknowns to solve equations of the form Ax ±B = Cx ± D, and symbolic and decontextualized canonical rules of expression of patterns and functions are formulated.

Level 3 of algebraization supposes, in our proposal, operate with the intensive objects symbolically represented, and therefore those objects have any contextual connotations. On the student's response (Figure 3) the symbolic expression of the proposed formula, $x = \frac{n(n-1)}{2} + n$, is related to the visual arrangement of the beads. Any attempt that the student could perform, operating with this expression to obtain alternative forms, for example, $f(n) = \frac{n(n+1)}{2}$, would be indicative of a more consolidated algebraic activity (level 3).

SUMMARY AND IMPLICATIONS FOR TEACHER TRAINING

We can identify more advanced levels of algebraic reasoning, such as those involving recognition, statement and justification of structural properties of mathematical objects involved. However, our approach focuses on identifying "what is algebraic" regarding "what is non-algebraic" in mathematical practice. In order to achieve this identification, we consider useful to introduce two intermediate levels of proto-algebraic activity.

We should recognize that boundaries between levels might sometimes be blurred and that within each level we can make distinctions that could lead to propose new levels of algebraization. However, our approach can be useful to guide the action of an elementary school teacher who tries to stimulate the progression of his/her pupils' mathematical thinking into progressive levels of generalization, representation and operative efficiency.

In figure 4 we summarize the main features of the proto-algebraic reasoning model we have described. In summary we propose to use three criteria to distinguish levels of elementary algebraic reasoning:

1. The presence of intensive algebraic objects (i.e., entities which have a character of generality, or indeterminacy).
2. Type of language used.
3. The treatment that is applied to these objects (operations, transformations) based on the application of structural properties.

The algebraization levels we propose are related to two aspects that Kaput (2008) identifies as characteristic of algebra and algebraic reasoning, namely algebra as:

a) Systematic symbolization of generalizations of regularities and constraints.
b) Syntactically guided reasoning and actions on generalizations expressed in conventional symbolic systems.

Aspect a) is specified in our model in levels 1 and 2 of proto-algebraic reasoning, while b) is associated with level 3, where algebra is already consolidated. Our requirement of using literal-symbolic language to assign a properly algebraic level (level 3) to mathematical practice, and the requirement of operate analytically/ syntactically with this language is concordant with other authors interested in defining "the algebraic" (e.g., Puig and Rojano, 2004).

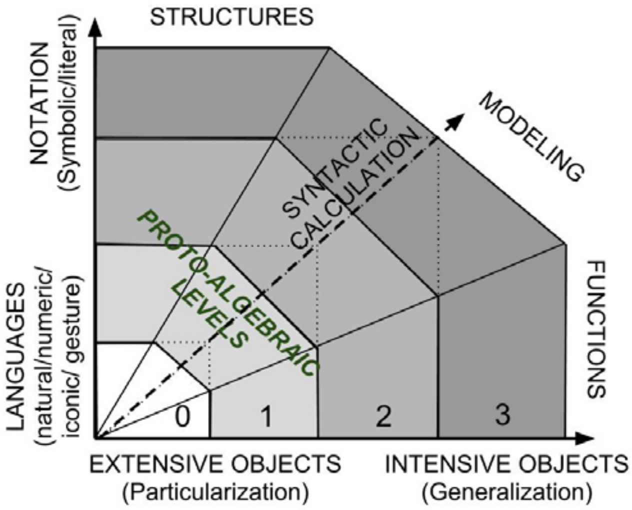


Figure 4. Levels of proto-algebraic mathematical thinking

In line with the proposals of the authors researching in the field known as "early algebra" (Carraher and Schliemann, 2007), we proposed to distinguish two primary levels of proto-algebraic reasoning to distinguish them from other forms stable or consolidated of algebraic reasoning. The key idea is to "make explicit the generality", of relations (equivalence or order), structures, rules, functions or on modelling mathematical or extra-mathematical situations, while operating with such generality.

The analysis of the nature of algebraic thinking has implications for teacher education. It is not enough to develop curriculum proposals (NCTM, 2000) that include algebra from the earliest levels of education; the teacher is required to act as the main agent of change in the introduction and development of algebraic thinking in elementary classrooms. The characterization model of "early algebra" that is proposed on this report, including the distinction of levels 1 and 2 of proto-algebraic reasoning, can be useful in training primary school teachers.

Acknowledgment

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MATHEMATICAL MODELING THROUGH CREATIVITY LENSES: CREATIVE PROCESS AND OUTCOMES

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The aim of this paper is to demonstrate the process of mathematical modeling development and its significant outcomes through creativity lenses, as part of a more inclusive intervention study. The intervention included engaging students in three modeling workshops involving authentic, hands-on mathematical situations. An analysis of students' modeling process and outcomes revealed their creative thinking skills. The participants were mathematically talented primary school students who were members of "Kidumatica" math club. This "visual" analysis gave us a better and clearer view of students' creative skills as manifested in the diversity of their significant mathematical ideas and the variety of approaches leading them to create, invent and discover significant conceptual tools.

INTRODUCTION

Creativity and innovation are becoming increasingly important to development and progress in the 21st century, because of their contribution to economic prosperity, the promotion of individual and societal welfare, and rapid scientific and technological growth. Within the context of core knowledge learning (e.g. mathematics), students must also acquire essential skills such as problem solving, creativity and systems skills for success in today's rapidly changing world (OECD, 2012). Model-eliciting activities (MEAs) not only provide students with the opportunity to apply their creative skills, they also encourage skill development and improvement (Lesh & Doerr, 2003). The development of students' creative competencies (Guilford, 1967; Sriraman, 2009) is crucial, although sometimes students' creativity may not be easily observable. In this paper, we demonstrate how analysis of students' modeling processes and their outcomes through the lens of creativity can lead to a better understanding and recognition of students' creative thinking during the mathematical modeling process.

MATHEMATICAL MODELING AND CREATIVITY

MEAs are non-routine, ambiguous, structured and complex. According to Sriraman (2009), these characteristics are required for the emergence of students' creative thinking. These activities are intended for middle-school students, specifically in groups of three to five. MEAs are designed according to six principles: reality, model construction, self-evaluation, documentation, sharability and reusability, and the effective prototype (Lesh, Amit, & Schorr, 1997). Model-development processes usually involve a series of recursive cycles consisting of interpretation, development, testing, and revision. This process encourages students to discover, invent or develop different mathematical patterns and rules using different pathways and representations, thereby increasing their tendency to produce original ideas (Guilford, 1967;

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Sriraman, 2009). Guilford saw the potential for creativity as a dynamic ability that can be developed in students, and defined creative ability as divergent thinking, characterized by fluency, flexibility, originality, and elaboration.

METHODOLOGY

The study described herein is part of a larger, inclusive intervention study aimed at revealing the implications of MEAs on students' creative mathematical thinking. The intervention program included three workshops based on different MEAs, which were worked on by small groups (3–4 students). Each MEA workshop had three parts: a warm-up activity, a modeling task and a student presentation session. The first MEA was based on the "Bigfoot" modeling task (Lesh & Doerr, 2003), in which students were asked to help a scout group discover who fixed their fountain. The only clue was “huge” footprints left in the mud. Students had to develop a conceptual mathematical tool that would enable estimating the height of this “giant” man. Each group of students received a cardboard with an image of an authentic large footprint's stride, and measuring tapes and calculators. The second MEA, called "Pocket Money", was based on the “The Sears Catalogue Problem” (Lesh et al., 2000). Students were asked to help a boy named Shmulik. Shmulik’s parents want to give him the same amount of pocket money that his sister got 10 years ago and he needs help in persuading them to increase the amount of money they are giving him today. Each group of students received two cards that included pictures and prices from 2001 and 2011 of the things Shmulik wishes to buy with his pocket money. The third MEA, called “Relay Race”, was designed especially for the intervention study according to the six aforelisted principles (Lesh et al., 2000), based on English and Watters' modeling activity “Volleyball”(Lesh & Doerr, 2003). In this task, the students were asked to help a sports committee select 4 runners (out of 8) to participate in an upcoming relay race. Each group of students received two tables of data on the 4 boys and 4 girls who had won gold or silver medals in the 80 and 100 meter races held in the previous autumn, winter, spring and summer seasons. The first table contained their records in those prior races. The second included their medals in the prior races, as well as a descriptive evaluation by the sports teacher as follows: MK—accelerates toward the end of the race, AK—good at short- and long-distance races, and SK—excels in group sports. For each of the above modeling tasks, the students were asked to write down a mathematical justification of their solution and an explanation of how to use this conceptual tool.

Participants

Participants in this study included 85 "high-ability" and mathematically gifted students in the 5th through 7th grades who are members of the "Kidumatica" math club. The "Kidumatica" program provides a framework for the cultivation and promotion of exceptional mathematical abilities in youth from varied socioeconomic and ethnic backgrounds (Amit & Neria, 2008).

Data

The data consisted of students’ documents written during the MEA, classroom observations, and video-recordings of the students' presentations of their models. The written data included students' modeling drafts, conceptual tools and written presentations. It should be emphasized that the students were asked to write down everything, so that drafts, sketches and final solutions could be collected. The video-recording included students' oral presentations of their models, researcher interviews and class discussions. Transcripts of these videotapes were used along with students' written data to assist researchers in the analysis.

Instruments

WTS – Ways of Thinking Sheets

Students' mathematical strategies for solving the modeling were documented according to WTS (Chamberlin, 2004). This instrument was designed by Chamberlin for teacher investigations of students' work in modeling activities. In our research, we used this instrument to document students' unique and significant strategies.

QAG – Quality Assurance Guide

The QAG (Lesh et al., 2000) was used to assess the appropriateness of the model and identify the strengths and weaknesses of different results produced by the students. This instrument provides guidelines for determining how well the client's needs are met by the students' solution, thus quantifying the quality of the solution by dividing it into five different levels of performance. This quantification then enables statistical comparisons using the standardized test scores.

FINDINGS AND RESULTS

MEAs engage students in hands-on exploration; the model-eliciting process requires students to pass through several cycles. Each group went through different cycles of interpretation, development and testing, refinement, improvement and elaboration. These cycles demonstrated their creative thinking abilities, consisting of fluency, flexibility, originality (of the appropriate outcome) and elaboration (including refinement and generalization).

Fluency

Figure1: Students engaging in different phases of the "Bigfoot" modeling task

Developing a conceptual tool requires multiple modeling cycles that involve different ways of thinking about the goals, facts and problem situation (Lesh & Doerr, 2003). This process requires students' fluent thinking ability which is demonstrated through the variety of ideas and conjectures they raise, and their consideration of different perspectives and interpretations that could lead them to the discovery of significant patterns and regularities (Guilford,1967). The following shows the development of the students' modeling process during the “Bigfoot” MEA. The first phases were premature and naïve, with some students exhibiting difficulties coping with the complexity and ambiguity of how to use the data to create a meaningful model. However, as the process progressed, they improved their interpretations, and discovered repetitive behavior in the data which led them to mathematize the situation and develop diverse mathematical responses, some of which could be innovative and original. In Images 1 and 2 (Figure 1), we see one of the first interpretations made by two boys: they took one of the boy's shoes, and tried to estimate how many times they could place it along his body. In Image 3 (Figure 1), they have moved to the next phase, using the measuring tape to measure their height, which they later use as part of their data. These students moved from accumulating the number of times they could place their shoe to measuring their height, which helped them discover a different construct of the data.

Other groups went through different phases. For example, the following was recorded from one of the groups (of 6th graders) in their advanced phases:

Y': "We need to find the ratio between the height and the shoe length…"

N': "and the width, we need to measure the shoe width.”

Y’: "But it [the width] differs along the shoe."

Y': "So we will measure it at the thinnest part."

In this group, the students found that they could obtain different patterns, as affected by the diversity of data and their sources.

Flexibility

During each of the modeling processes, students demonstrated the capacity to consider a variety of approaches to, and perspectives on a particular problem, reflecting their ease in switching from one mental operation to another (Kruteskii, 1976; Guilford, 1967), retrieving information, knowledge (including concrete and intuitive knowledge) and ideas from a variety of disciplines. These were then used to find several different perspectives and approaches to describing both the dataset and its behavior via different types of representations (verbal, figural, algebraic and graphs, for example) (Gilat & Amit, 2012). The following results demonstrate 5th-grade students’ approach to the “Pocket Money” MEA. There were two phases to the students’ modeling development; in both, the students carefully chose five products: a bowling ticket, a movie ticket, a kid's meal, a skateboard and a snack. The students began by identifying the relationship between product prices and calculating the ratio between each of two equivalent products from 2001 and 2011, respectively (Figure 2, phase I). Then they calculated the average of all five ratios and wrote "2.16 times more" referring to how much 2011 prices had increased. Though at their presentation, the students demonstrated a different pathway, where they calculated the ratio between the sum of the five products’ prices in 2001 and 20 NIS, the amount of pocket money in 2001. This ratio allowed them to find the amount of pocket money Shmulik should get from his parents in 2011 by dividing the sum of the five products' prices for 2011 by the former ratio, as illustrated in their explanation in Figure 2. The students demonstrated flexible thinking, switching from different strategies and pathways to a global, more general, easy-to-use conceptual tool.

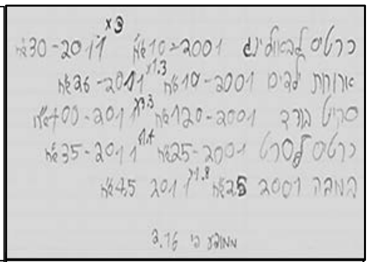
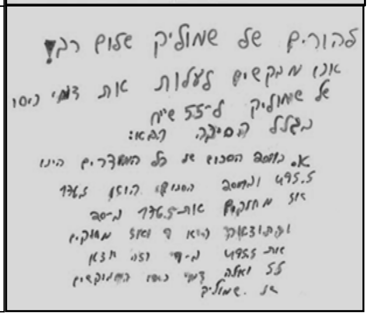
Phase I	Bowling ticket 2001 - 10 **NIS** x 3 2011- 30 **NIS** Kids' meal 2001 - 19 **NIS** x 1.3 2011- 26 **NIS** Skateboard 2001 - 120 **NIS** x 3.3 2011- 400 **NIS** Movie ticket 2001 - 25 **NIS** x 1.4 2011- 35 **NIS** Snack 2001 - 2.5 **NIS** x 1.8 2011- 4.5 **NIS** Average 2.16 times more	
Phase II	Hello Shmulik's parents! We ask to raise the amount of Shmulik's pocket money from 20 NIS to 55 NIS because in 2011, the sum of all products was 495.5 NIS and in 2001 it was 176.5 NIS; so divide 176.5 by 20 to get 9; then divide 495.5 by 9 to obtain 55, the amount of pocket money Shmulik should be getting.	

Figure 2: 5th grade outcomes of "Pocket Money" MEA

Originality and Appropriateness

	Strategy description	# of groups (31)
1	Shoe length * ratio	14
2	Shoe size * averaged ratio + constant	10
3	Two linear equations whose variable depends on the age of the person	2
4	Shoe width or (width + length)* ratio	2
5	Ratio between several parts of the body	2
6	Two linear equations whose variable depends on the ratio between shoe width and length	1

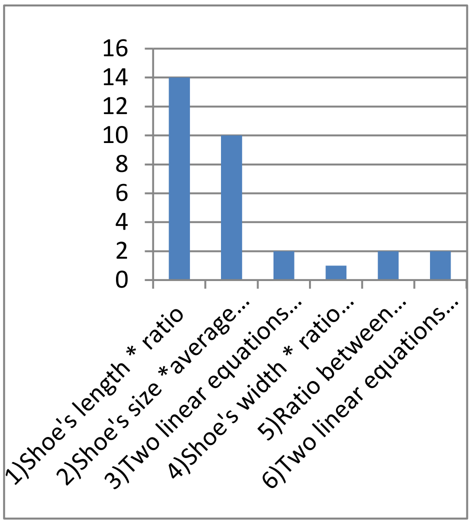


Figure 3: Frequencies of student strategies in the Bigfoot modeling task

Originality was quantified according to the statistical rarity of the responses (Guilford, 1967). The following results demonstrate an analysis of the modeling responses of 31 groups working on the "Bigfoot" MEA. The responses were first documented using WTS (Chamberlin, 2004), which assists in documenting students' strategies and following up on their thinking process, their reasoning, their sources of knowledge (mathematical and general), the patterns and rules they found and the mathematical presentation used to represent them. The QAG (Lesh et al., 2000) with five performance levels was used to evaluate the appropriateness of the products documented by WTS. Appropriateness refers to the extent to which a proposed conceptual tool (e.g. mathematical model) is sharable, manipulatable, modifiable, and reusable for constructing, explaining, predicting and controlling mathematically significant systems. Only appropriate strategies (QAG’s score $\geq$ 3) were used to evaluate the originality of the students’ responses. In the "Bigfoot" modeling task, the students demonstrated six different clusters of strategies (Figure 3). The 6th cluster was the least frequently used strategy as illustrated in Figure 3, and it therefore scored high on originality, while strategies belonging to first cluster scored low on originality.

Elaboration

Elaboration becomes apparent in the students' refinement, generalization and abstracting abilities which, according to Lesh, Amit, & Schorr (1997), Lesh et al. (2000) and Guilford (1967) represent students' extending, refining, or integrating their ideas to develop a new level of more abstract or formal understanding.

The key to this procedure is averages:

We calculate the average time for each of the children and reduce scores based on the teacher's evaluation:
SK - minus 1 point
MK - minus 2 points
AK - minus 3 points
In addition to this summation, we add the child’s result in the autumn and round it to an integer.

<u>Example:</u> <u>Miriam</u>
[41+40.5+39.7+39]:4-2+41=79.05≈79
Note: the square brackets was added by the researcher

The two with the lowest score will participate in the relay race.

Figure 4: 6th grade students' conceptual tool for the “Relay Race” MEA

Figure 4 illustrates the conceptual tools developed by 6th grade students for the "Relay Race" MEA. The foundation of their conceptual tool or, as the students described it, “*The key to this procedure is averages,”* referring to the average of the 200 m results for each runner in all four competitions. Here the students used averaged time as the key element of their model, while their conceptual tool was based on comparison, as demonstrated in the dashed rectangle in Figure 4. The team defined the comparison as *“the person who has the lowest score [is the best] will participate in the relay race."* Though the average was the key element, students continued to mathematically elaborate on the developed model, engaging other essential elements such as the season in which the relay race took place and the teacher evaluations for each runner; they considered the data relations and mathematized them according to their interpretation; teacher evaluation was ranked from 1 to 3 (the "best") and was subtracted from the average time, and the results of the 200 m race in autumn were added.

Generalization

The following model was developed by a team of three 6th graders for the provided data and problem situation and then generalized so that it could support other similar situations. The students' conceptual tool was based on a comparison of averaged velocity in the 200 m and 80 m competitions which took place in the autumn, as illustrated in Figure 5. They used the <u>"average velocity"— the ratio</u> between the summation for the 200 m and 80 m races and the summation of the autumn results for the 80 m and 200 m runs. This group elaborated upon and refined their model by adding a ranking score for medals (10 for gold medal and 5 for silver). They also offered a ranking for teacher evaluations but did not proceed any further with this. In the last part, the students offered the "members of the sports committee" a "general" procedure that could be used to select players in other sports based on <u>an "average achievements" ratio</u>, and gave the example of selecting a football player using the ratio of all goals scored in all games divided by the number of games.

To the members of the sports committee,
This is our method and who we chose:
We chose Gil, Ali, Mica and Liri.
We chose them according to this method:
-First we summed everyone's results for the 80 m and 200 m races (run in the autumn).
Then, we performed the following division:

280 : results = velocity
Summing 80+200

-After obtaining the results, you choose the 4 best (those whose results are highest). If results are equal, consider medals:
*more medals = a higher level is given
*Evaluations can be used as well:
-recommended to choose AK and SK first.
-Add 10 points to the final score for a gold medal and 5 points for a silver one.
You can also use this method for other sports, such as football:

opposite
Ratio=Number of games: Number of goals at all of the games

We hope you will use this method

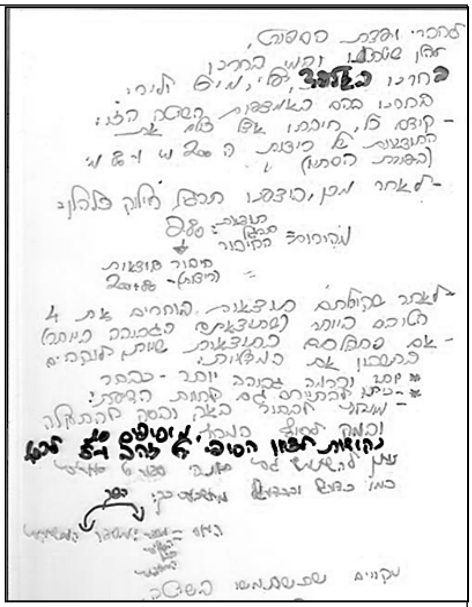


Figure 5: 6th grade students' "Relay Race" conceptual tool

CONCLUSIONS

Through the use of creativity lenses, students' creative thinking features are clearly revealed as they manifest themselves in students' modeling process and outcomes. The data were obtained from an inclusive intervention study in which the students participated in three workshops with different MEAs. The creativity lenses were derived from the four characteristics of divergent thinking: originality and appropriateness, fluency, flexibility and elaboration (Guilford, 1967). The modeling process involved multiple cycles of exploration in which new ideas, responses and pathways were generated and alternative solutions were invented or discovered, tested and revised (Guilford, 1967; Sriraman, 2009), encouraging students to utilize their fluent thinking skills. Moreover, the results demonstrated how this recursive process stimulates students' elaboration skills, including refinement and generalization. The results revealed students' flexible thinking as they shifted between various pathways with various levels of correctness, depending on their interpretations, mathematical abilities, general knowledge and skills (Lesh & Doerr, 2003). The results suggested that various responses may be appropriate, thus increasing the possibility of creating an inventive and original conceptual tool (Guilford, 1967; Lesh et.al, 2000). This "visual" analysis gives us a better and clearer vision of students' creative thinking which manifests itself through the creative modeling process and its outcomes.

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MEASURING IMMEASURABLE VALUES

Annica Andersson, Lisa Österling

This paper critically explores research on values in mathematics education from a methodological perspective. In the contexts of conducting large-scale international collaborations and comparisons we problematize the interpretation of learning activities as indicators of a certain value. Interviews with students supported our work, and we argue that a learning activity can be interpreted out of different categories of values, depending on the context.

Keywords: mathematics, mathematical values, cultural values, contexts, methodology.

INTRODUCTION

Values like the prize of gold on the market are values that are easy to measure and compare. Values that guide students when they decide what is important when learning mathematics are difficult to measure, and even more difficult to compare. Still, this is the aim of the Third Wave project.

The Third Wave Project (Seah & Wong, 2012) was initiated 2008 in Monash University in Melbourne, Australia. It is an international research project investigating teachers' and students' values in mathematics learning in different cultures. The relation between values and learning activities can help us understand why lessons are different in different cultures. A second aim is to develop a survey tool to continue investigating values, independently of culture (see Seah, this volume). This paper concerns Study 3 within the project: "What I find important (when learning mathematics)" (WiFi). WiFi is a survey study, conducted in countries as Australia, Brazil, China, Hong Kong SAR, Malaysia, Singapore, Sweden, Taiwan, Turkey and the US. This large-scale investigation consists of a Web Based questionnaire with 89 questions, some multiple choice and some open questions. It is to be distributed to 11 and 15 year old students in the different countries. Stockholm University is coordinating the Swedish part of the study.

Our task as the Swedish team was to translate the quantitative questionnaire, developed in an Australian-Asian context, into Swedish with possibilities to, first, research Swedish students' values and, second, to be able to make international comparisons. The aim of this paper is to problematize the interpretation from the posed questions, as a value indicator, to a certain value.

THEORETICAL BACKGROUND

Values in mathematics education are "the deep affective qualities which education fosters through the school subject of mathematics" (Bishop, 1999, p. 2). However, according to Hannula (2012), there is a terminological ambiguity in the research field of mathematics-related affect. Hannula describes the ambiguity if values researched are values by the individual or the community. Seah & Wong (2012) take the stance

2013. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 17-24. Kiel, Germany: PME.

that "values are regarded in [the Third Wave project] from a sociocultural perspective rather than as affective factors." To give one example, Andersson and Seah (2012) demonstrated, with a socio cultural theoretical perspective the complex interplays amongst learning contexts, the valuing involved, and student agency when analysing the Swedish student Sandra's narratives over a year's participation in mathematics class. While changes in learning contexts lead to variations in student agency with regards to engagement, Sandra's story demonstrated the interplays between what these contexts value and whether these values are aligned (or not) with what Sandra values as a learner.

The diversity of values has meant a need to differentiate amongst the many values that are portrayed in the classroom. Bishop (1996) emphasised three categories of values in the (Western) numeracy classroom, namely, mathematical, mathematics educational, and general educational. To investigate cultural values, the project uses the theoretical framework of Hofstede and Hofstede (2005).

In the WIFI-study, the three categories (mathematical values, mathematic educational values and cultural values) all have sub-dimensions of values, and the study deals with a set of 24 different values. Children responding to the questionnaire cannot be expected to relate directly to a value; hence, the questions posed are about different learning activities, regarded as value indicators. Seah and Peng (2012) conducted a scoping study in Sweden and Australia, where students were asked to write down or take photos when they found themselves learning mathematics well. The learning activities pictured were treated as value indicators, and the results allowed the researchers to reflect on the problem of making a difference between a value and a value indicator.

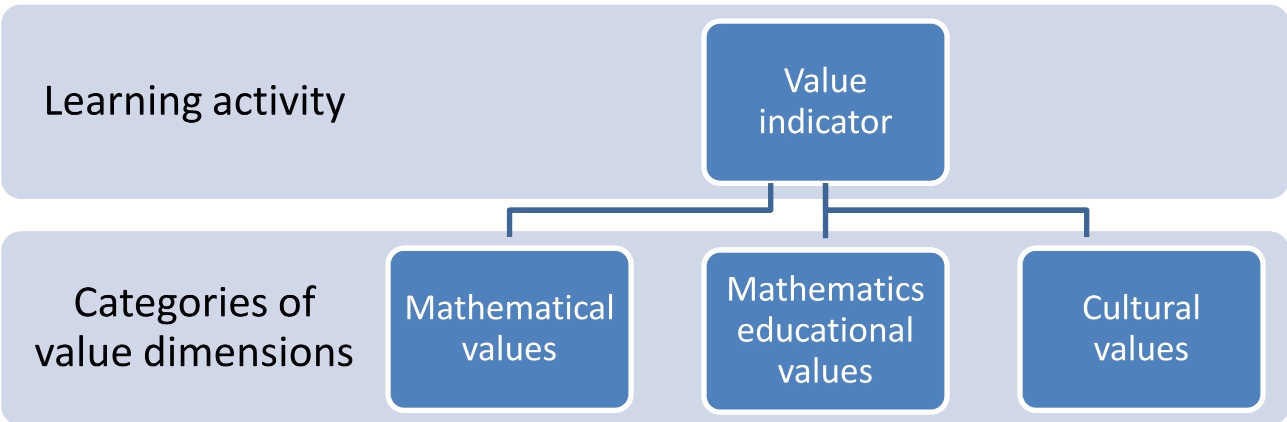

Figure 1: Categorisation of values from value indicators

Here, the indicator is analysed within three categories of values. In every category, there are several value dimensions.

An example of how figure 1 is used may be useful. In the designing stage of the WIFI questionnaire, the learning activity "Learning the proofs" is categorized as an indicator of the mathematical value of rationalism (see Bishop, 1988), and "Doing mathematics by myself" is categorized as an indicator of the cultural value of individualism (see Hofstede 2005).

In this paper, the mathematical value dimensions used are the mathematical values of rationalism, objectism and control. Rationalism is central in mathematics; it is about valuing reasoning and proof. Objectism emphasizes that mathematics is constructed from objects by an axiomatic system, and can be applied and concretised. The value control emphasizes procedures and mastery of rules. The Mathematical Educational value dimensions used in this paper are application, computation, recalling and effort.

The cultural values dimensions have an impact on several areas in society, from family to companies, and Hofstede and Hofstede (2005) also ascribe its impact on education. In this paper, we discuss the cultural value dimensions of individualism, as opposed to collectivism, and uncertainty avoidance. In an individualist culture, knowledge is valued differently from in a collectivist culture. The purpose of learning in an individualist culture is less to know how to do than to know *how to learn*. The assumption is that learning in life never ends; even after school and university learning will continue (e.g. through post academic courses). The individualist society in its schools tries to provide the competencies necessary for the "modern man" (Hofstede & Hofstede 2005, p. 98). A collectivist society in it schools values knowledge that is beneficial for the society. A diploma is also valued differently, in an individualist society, a diploma gives the holder a better economic status but it also improves his/her self-respect. In a collectivist society, a diploma provides entry to higher-status groups.

Valuing uncertainty avoidance in school is about wanting structure and right-answer-questions rather than open-ended questions (Hofstede & Hofstede, 2005). Students do not question teachers or textbooks; they demand them to be correct. Hence, their own results are being attributed to circumstances or luck. The opposite position, a culture with weak uncertainty avoidance, is one in which students are expected to be rewarded for originality; and one in which results are attributed to a person's own ability.

To be able to analyse the relation between value indicators and values, a few earlier Swedish studies were consulted. The Swedish School Inspectorate (2009) made an assessment on mathematics teaching in Sweden. It concluded that Swedish teachers were still relying then on the textbook when teaching mathematics. Instead of relying on the curriculum, they trust the textbook to address all mathematics needed. The focus is often the practicing of calculation procedures. The historical development might have influenced this. Lundin (2008) concludes that historically, the focus has been on learning calculation procedures. He writes that "This need led to the promotion of schoolbooks filled with a large number of relatively simple mathematical problems,arranged in such a way that they (ideally) could keep any student, regardlessof ability, busy – and thus quiet – for any time span necessary." (p. 376). These students' experiences are reinforced in the context of Swedish mathematics education; according to Lindqvist, Emanuelsson, Lindström and Rönnberg (2003), textbooks in Swedish mathematics education seemed to define the essence of school mathematics. This way of organizing mathematics education is believed to support teachers in managing non-homogeneous group of students so that each student could work according to his/her previous learning and needs, as well as following curriculum and reform concerns (Johansson, 2006).

Taking another viewpoint, Björklund Boistrup (2010) showed four assessment discourses in Swedish classrooms when researching students' semiotic resources and assessment acts. She labelled one assessment discourse "Do it quick and do it right". Within such a discourse, we can expect teachers and students to value activities that allow them to practice calculating procedures. However, she also describes an assessment discourse labelled "Reasoning takes time", where the teacher assess mathematical reasoning. Another Swedish example is the different learning contexts Sandra, in the example above, participated in.

The aim of the Swedish mathematics curriculum emphasises the school subject of mathematics as problem solving, application, methods, reasoning and concepts. Problem solving has a status not as an application of mathematics, but as a part of mathematics. ”Teaching in mathematics should essentially give students the opportunities to develop their ability to:/…/ formulate and solve problems using mathematics and also assess selected strategies and methods" (Ministry of Education, 2011).

Despite those different aims and learning contexts, is there a way of describing common values in Swedish mathematics education?

METHODOLOGY

In order to better relate the value indicators to an appropriate value, we conducted short scoping interviews (Kvale & Brinkman, 2010) with eleven Swedish students, aged 10-15 years old. The students were asked to elaborate on two open questions: "What do you find important when learning mathematics?" (The name and aim of the questionnaire) and "How would you design maths lessons if you were to decide yourself?" The students' responses were then categorised to match the questions in the questionnaire with the purpose to indicate the correspondences between indicators and values in the Swedish context.

Analysis

This is an example to describe the analysis process.

Interviewer: What do you find important when learning mathematics?

Student: I calculate in my textbook and I do homework. (Jag räknar i matteboken och jag gör läxor)

First, the interview answer is regarded as our value indicator. Second, we analyzed the correspondence between the student's interview answers and the questions in the questionnaire. Question 57 in the WIFI-questionnaire says "Homework", so there is a corresponding question to one part of the students answer. Question 36 says "Practicing with a lot of questions". There is a certain correspondence to "calculate in my textbook". Third, the questions that appeared most frequently in the interviews were chosen for a categorization out of all three value categories (mathematical values, mathematics educational values and cultural values) and the underlying value dimensions. In this analysis process, we use the motivations expressed in interviews by the students, as well the theoretical frameworks described for values, as well as research about traits in Swedish mathematics education.

RESULTS

When comparing the answers students gave in the interviews to the questions in the questionnaire, four questions matched several answers. Those questions are: A) “Problem solving” (six students), B) “Knowing the times tables” (multiplication tables), (six students), C) “Practicing with lots of questions” (seven students) and D) “Connecting maths to real life” (three students).

Questions B) and C): “Knowing the times tables” and “Practicing with lots of questions”

Students mentioned different calculation abilities, “knowing the times tables” was the most common, but addition (“tiokamrater”, “additionstabellen”) was also mentioned. We related all those answers to the question “Knowing the times-tables”. These are examples of activities where it is important “to do it quick and do it right” (Björklund Boistrup, 2010). Five students, 10-13 years old, gave answers that we related to the question “practicing with lots of questions”, even though what they said was “working in the textbook”. Four students, 13-15 years old, said that they found it not rewarding or discouraging to work in textbooks, and they wanted mathematics teaching to contain more problem-solving activities, implying that problem solving tasks were missing in the textbook.

In the WIFI Research Guidelines, the question “Knowing the times tables” is categorised as an indicator of the mathematics educational value of recalling, and “Practicing with lots of questions” is categorised as an indicator of the valuing of effort. What the Swedish students actually said was “working in the textbook”, not “practicing with a lot of questions”. The Swedish School Inspectorate (Rapport, 2009, p. 5) found that working in the textbook is practicing procedural calculations. We argue, from the Swedish learning context, that is also an indicator of the mathematical value of control, concerned with the mastery of rules and procedures.

The question formulates “Getting the right answer”. We interpret both “knowing the times tables” and “Practicing with a lot of questions” as similar indicators, you are likely to get the right answer if you know the times tables or practice with a lot of questions. For this reason, we argue that these questions are also indicators of the cultural value of uncertainty avoidance. In the uncertainty avoidance-dimension, Sweden ranks 48/49 out of 53 countries (Hofstede & Hofstede, 2005). This means that there is a weak uncertainty avoidance in Sweden. In school, uncertainty avoidance is about wanting structure and right-answer-questions rather than open-ended questions. Students do not question teachers or textbooks, they demand them to be correct, and their own results are being attributed to circumstances or luck. The opposite position, which goes for Sweden, is students expected to be rewarded for originality; results are attributed to a person’s own ability. The younger students’ answer, that it is important to do procedural activities, contradicts the common Swedish value. But when the older students express that they want less work in the textbooks and more problem solving, this can be interpreted as an indicator of weak uncertainty avoidance, and of the students socialising themselves into the Swedish society.

Questions A) and D): Problem solving and Connecting maths to real life

Five of the older students mentioned problem-solving, mostly in the context that they liked problem-solving and wanted more problem solving activities, rather than working in a textbook. In the research guidelines, problem solving is categorised as an indicator of Mathematical Educational Value of Application.

It is not obvious what students are valuing when they say problem solving. It might be a way of them to express “doing something else than working in the textbook”, as they do not have the vocabulary to express any alternative but problem solving. They gave a variety of explanations why they prefer problem solving, like working together, more variation, teacher solves problems, more fun, learn differently, working in pairs and share ideas.

If problem solving is considered as a part of mathematics rather than a tool for learning mathematics, as it is described in Swedish curriculum, it is more relevant to categorise it as one of Bishops (1996) Mathematical values, the mathematical value of objectism, where applying mathematical ideas is emphasized. From our interviews it is hard to determine whether students view problem solving as a mathematical content or a tool for learning.

Concerning cultural value dimensions, Hofstede & Hofstede (2005) describe their impact on education, and in the description of the individualist cultural dimension, there are findings relevant to problem solving. Sweden ranks nr 10/11 out of 53 nations in the individualism/collectivism cultural dimension which means that Sweden is an individualist rather than collectivist society. The purpose of learning in an individualist society is less to know how to do than to know how to learn. An individualist society rather tries to provide the competencies necessary for lifelong learning. (Hofstede & Hofstede, 2005). The question in the WIFI-questionnaire related to this dimension is formulated “Working out the maths by myself”. This is often a part of problem solving in the Swedish context, application can be a part of problem solving, but not always. This is not only a linguistic difference, rather a different practice.

From the discussion above, we argue that problem solving is not only a value indicator of the mathematics educational value of application. In the Swedish learning context it can also be categorised as an indicator of mathematical value of objectism, as well as a cultural value of individualism.

Five of the older students mentioned that mathematics was important for finding a job, or to get a good grade or good education in the future. They value mathematics as an important competence in life. Three answers could be related to the question in the WiFi-questionnaire about “Connecting maths to real life “. In the research guidelines, this is categorised as an indicator of mathematics educational value of application. But from the motivations we got, we argue that this rather indicates a cultural value in the individualism – dimension, in the same way as for the problem solving question. We also argue that these answers are indicators of the mathematical value of objectism, where students value knowledge of mathematical objects for giving an explanation of real world phenomena.

As a result, we can argue that these four questions can be regarded as value indicators for one value in each one of the value categories proposed, mathematical values, mathematics educational values and cultural values.

CONCLUDING DISCUSSION

In the analysis section above, we have showed that the different categories of mathematical, cultural, and mathematics educational values are related to different value indicators. They can overlap, that is, a particular value indicator may suggest the valuing of one or more categories of values in the mathematics classroom. The individual students' values are assumed to be influenced by mathematics, mathematics education, culture and probably more at the same time. This means we have to take more into consideration than one check in the "important"-box to determine what value a certain answer indicates. The interpretation will probably vary between cultures, so the WIFI study will give us the distribution of value indicators rather than values. Value indicators can be measured, compared and analysed. Values still seem immeasurable.

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INTUITIVE THINKING IN A CONTEXT OF LEARNING

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Traditional studies approach students' intuitions giving a task and observing their reasoning. In Mathematics Education, it would be interesting to provide an understanding of intuitions specifically in learning processes. This is the aim of this paper. To accomplish it, we provide both a theoretical framework to encompass the specificity and the complexity of intuition in learning mathematics, and an example, from classroom activities in a longitudinal study, that shows intuitive thinking as emerging within a socially shared activity, and interrelating with other ways of thinking throughout the students' objectification process. We conclude that intuitions may come from the individual's insight, but it is in the socio-cultural activity that they are part of mathematics learning processes.

Traditionally, from different perspectives (Fischbein, 1987; Gigerenzer & Selten, 2001; Kahneman, Slovic, & Tversky, 1982), intuitions have been regarded as a way of thinking that is in contrast with the mathematical deductive one, identified as mediated, analytical and justification-requiring, in accordance with the rules of formal logic (Fischbein, 1987). The intuitive form of thinking, indeed, establishes a necessity that does not follow the logical necessity criterion.

The *theory of knowledge objectification* (TKO) is specifically interested on *how people think when they learn*, rather than generally *how they think* (Radford, 2008). Looking at intuitions from a teaching/learning point of view requires both a theoretical and a methodological shift. On the one hand, given this change of theoretical focus from *thinking* to *thinking in learning*, instead of looking at cognitive functioning, we need to provide theoretical tools that encompass the *consciousness movement* (Radford, 2012) as reflecting a cultural and historical dimension that transcends it. Therefore, the aim of this study is to frame intuitions within an Activity Theory strand (see Roth & Radford, 2011). On the other hand, an experimental investigation cannot look at the response of the subject exposed to a specific task or problem to scrutinize his cognitive structure, but it is necessary to provide data that show the complexity of the consciousness's movement towards a mathematical generalization.

LITERATURE REVIEW ON INTUITION

Fischbein (1987) underlines that intuitions are neither a source, nor a method, but a type of cognition. He distinguishes between: *perception*, a form of immediate cognition, and *intuition*, which exceeds the given facts. Perceptual knowledge is immediate, while intuitive knowledge is also extrapolative (Fischbein, 1987). Immediacy is the widely acknowledged basic feature of intuition. Furthermore, intuitions refer to self-evident statements that exceed the observable facts. Being apparently self-evident, intuitions appear generally as absolute and unchangeable–they

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possess a coercive character. A certain statement that is accepted as self-evident is also accepted globally as a structured, meaningful, unitary representation. The unity between the particular, the specific, the directly convincing example, and the general principle derived through similarity and proportionality from the particular case, *needs to be established* by the subject in order to have intuitive knowledge. The globality of intuitions – based on tacit, perceptual elaborations – is generally expressed in a selection process which tends to eliminate the discordant clues and to organize the others so as to present a unitary, compact meaning (Fischbein, 1987). In a sense, globalization in intuitive knowledge can be regarded as a form of generalization. This is in line with Radford's (2013) understanding of sensuous cognition as a feature of living material bodies which have responsive sensations. Perception is, according to this view, the substratum of mind and it is culturally shaped. Perception is a sensing form of action and reflection, which can pave the road to culturally-historically forms of mathematical generalization.

Gigerenzer & Selten (2001) consider intuition the most effective form of thinking compared to rational and deductive thinking. Intuition for Gigerenzer is good adaptation, namely both copying prestigious individuals, and conforming to the most common behavior in the population. Fischbein (1987) suggests that intuitions do not disappear from intellectual (mathematical) endeavors, because they are an integral part of any intellectually productive activity. In fact, as a consequence of Goedel's incompleteness theorem, formally, any mathematical system cannot be absolutely closed, it cannot possess in itself all the necessary formal prerequisites for deciding about the validity of all its theorems (see Fischbein, 1987). Psychologically, no productive mathematical reasoning (solving problems, producing theorems and proofs, etc.) is possible by resorting only to formal means (Fischbein, 1987). This is reported by Liljedahl (in press) in a study on famous mathematicians' "AHA!" experiences: the aspects of illumination that sets this occurrence apart from other mathematical experiences are affective in nature. The cognitive components are not absent, but mathematicians comment on attributes such as *a sense of* certainty, *a sense of* significance, *a sense of* simplicity. The "AHA!" would be ascertained through verification, but this "sense of" is the very real aspect of illumination. Following Roth & Radford (2011), within the Vygotsky-Leont'ev strand of cultural-historical activity we stem from in this work, cognition cannot be understood independently of emotions. Emotions are not a static, trait-like feature of the subject, but they constitute a holistic expression of the subject's current state with respect to the object and the subject's *sense of likelihood of* success. Emotions mediate the movement of the activity itself. Hence, we can see intuitive knowledge as the expression of the sensuous-valuational and volitional character of activity.

Existing studies on intuition share some limitations. Both Fischbein's and Gigerenzer's understanding of intuition lack a precise explanation of what intuitions in mathematics really are, beyond a definition that casts them in opposition to logical thinking. If, according to Fischbein, globalization, as well as immediacy and self-evidence, does not lead necessarily to an intuitive acceptance, how does the individual establish the unity between the particular, the specific, the directly convincing example, and the general principle derived through similarity and proportionality from the particular case? Furthermore, we believe that it is necessary to look at the emotional and sensuous dimension of the individual not as an element that hinders or enhances thinking, or as a need to care of for a successful cognition, but as a constitutive part of thinking itself. The view of intuitions as an effective form of adaptation (Gigerenzer & Selten, 2001), beyond pure rationality, of the individual to the constraints and challenges of its environment is based on the illusion of an *autonomous self-determined individual* that constructs his knowledge (Radford, 2012). In our view cognition is not only the individual's adaptation but it is a *mediated reflexive activity* (Radford, 2008). The cultural and historical dimension embodied in socially shared activity (Roth & Radford, 2011) is the true substance of the individual's self-determination and cognition.

INTUITIVE THINKING IN A CONTEXT OF LEARNING

We are not disregarding the importance of the previous results, both in psychology and in mathematics education. In fact, we are aware that there exist mathematical concepts that are more intuitive than others (Fischbein, 1987) – or that there are ways of framing mathematical tasks that foster intuitive thinking more than others, but the fact that we recognize concepts that are more intuitive than others is not *absolute*: it is *culturally determined*. And for a certain subject a concept, which is culturally recognized to be intuitive, may be not intuitive – or viceversa, regardless of his incorrect or correct answer to a task. Hence, it is important to consider how the subject relates himself to the concept, and not only the concept itself. Similarly, we are aware that there exist individuals who are more intuitive – and researches studying their behaviour are worth considering, but this approach leads to accounting for learning as adaptation (Gigerenzer & Selten, 2001). Learning is *also* adaptation to a physical/social/cultural environment, but it is *more* than mere adaptation.

In our view, cognition is a mediated reflexive activity (Radford, 2008). The teacher plays a crucial role in learning, since he is the only one who knows *where the activity should lead to.* In this view, intuitive learning is a determinate way to intertwine the subject, with its material and ideal components, a reified cultural and historical activity (the so called mathematical object or mathematical content), and a set of semiotic means (ideal and material) that allow the individual to become part of, re-enact and make sense of such an activity. The intuitive relationship between the individual and the content of knowledge is a reflexive activity mediated at an embodied, perceptual and sensuous level. Thinking is not purely sensorial, nor purely conceptual. Intuition can be seen as the sensuous side of intellectual-emotional activity when the activity is mediated mainly through objects, artefacts, gestures, bodily movements, deictic and generative use of natural language (Andrà & Santi, 2011). Intuitions are a relationship between the subject and a content of knowledge that allows sensibilities to notice, to think, to become in proximity and synchrony with generality. Intuitions are a way of being and becoming of the consciousness in its movement towards the generality of mathematical knowledge, with the feeling you are close to and re-enacting what culturally transcends you. Intuitive thinking allows the students moving towards a more accepted concept, relying on previous knowledge both cultural and historical, and that is being objectified in the reflexive activity. This mode of existence is dialectically entangled with the logical and discursive one. The dialectics between the two modes of existence accounts for the consciousness trajectory towards the recognition of the general that is the essence of mathematics.

AN EXAMPLE FROM CLASSROOM ACTIVITY

Data come from a longitudinal study that observes the development of algebraic thinking in students of a same classroom from grade 2 to grade 6. In particular, this paper focuses on a cycle started in the school year 2010/11. We discuss a teaching/learning sequence involving grade 3 students in the school year 2011/12.

The experimentation has been designed according to activity theory methodology (Roth & Radford, 2011): (a) presentation and discussion of the activity to the whole class, (b) work in small groups of students, with the support of the teacher who goes around and discusses with each group, (c) general discussion and a new cycle begins.

The mathematical content of the activity is part of grade 3 curriculum of Ontario: the search for regularity in number sequences. Data are collected both from videotaping and written material produced by the students.

The first task asks the students to find the regularity of the series: 25, 22, 19, 16. The students also know that they should underline the important words on the sheet. In a group of four, Estela proposes “find”, “regularity”, “this” and “series”, and James suggests to reduce to “regularity” and “series”. After having agreed about the most important words to underline, Estela proposes to use the number table, and she goes taking one of them from the teacher’s desk. In the meanwhile, Mike tells to James that he already knows the answer.

1. James: Seriously. [*he gazes Mike’s eyes*]
2. Mike: Yes, I know the answer: one subtracts 3 at any time. You subtract 3 at any time. [*Mike makes no gesture, he stands firmly in front of James*]

Estela comes back with the number table and tries to make sense of the task, addressing James. Alone, Mike counts with his fingers “1,2,3 (*Figure 1-a,b,c*). 1,2,3” (*Figure 1-d*), then he talks to the group:

3. Mike: I know what it is, I know what it is. [*At this point Mike looks at his mates*]
4. Estela: What is it? [*Estela addresses Mike, changing her posture*]
5. Mike: one subtracts 3 any time. [*The students take the number table and Mike counts on it (figure 1-d)*]. 1,2,3. 1,2,3.
6. Mike: 25 minus 3, 1,2,3. 1,2,3. [*James follows Mike’s pointing on the table*]
7. Mike: Look, 45
8. James: 25.
9. Mike: then –3, 1,2,3. 1,2,3. [*points on the table, follows numbers in reverse order*]
10. Estela: 25. So 25, 22

11. Mike: 1,2,3. 1,2,3. [*still points on the table*]
12. Estela: 19.
13. Mike: 1,2,3. [*continues to point on the table*]
14. Estela: Oh, yes!

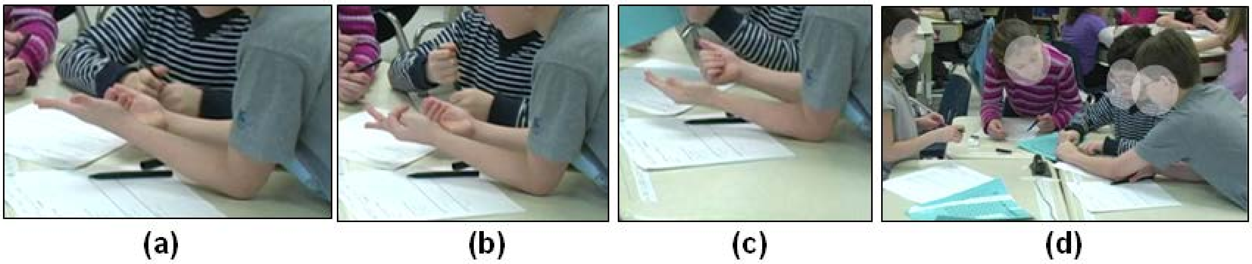

Figure 1: Mike's intuition on his hand and on the table (Mike is the boy on the right).

We notice that Mike has a starting intuition about the regularity of the series. He tells it to James, but immediately after this, Estela comes back to the group with the table and she catches the group's attention wondering how to deal with the task. Alone, Mike tries his intuition by counting with his hands (*Figure 1-a,b,c*), and then comes back to the group: resorting to the number table, he shows the solution to his mates, repeating several times "1,2,3" and pointing on the number table (*Figure 1-d*). Mike's first intuition can be accounted as an "AHA!" experience, in that he lives a sense of certainty, a sense of significance, about his solution. The nature of Mike's intuition seems to be affective rather than cognitive at this stage. In fact, we can see a sense of proximity with the general rule. In a second moment, he tries the correctness of his solution by counting with his hands, while the other three students wonder for a response. This is a private, more cognitive, moment for Mike. Emotions are still part of his thinking, giving him a sense of likelihood of success about his starting idea. This moment is immediately followed by a public one: Mike shares his intuition. In the rest of the excerpt, Mike repeats again and again the numbers "1,2,3", pointing with his fingers on the number table. Finally, Estela intuits the general rule ("Oh yes!"). Mike's intuition starts to become shared. In Estela's voice we can perceive a sense of disclosing, accompanied with positive emotions. The students' behaviour highlights the need for the intuitive part of mathematical thinking in terms of their space-time and tactile experience, bodily movements, rhythm. This intuitive thinking pivots around the number table as a semiotic means of objectification that allows the synchronic use of gestures and language, through which the students develop their space-time experience. We remark that at this point, in their movement towards mathematical generalization, the students have not yet fully objectified the generality behind the sequence of numbers, they are becoming part and re-enacting what culturally transcends them. Now the group activity goes on: the students have to write their answer on the sheet.

15. Mike: Ok, one does 25 – 3
16. Estela: Yes, yes, yes, yes 25 – 3, so 25 – 3 = 22
17. Maria: Are you sure? Because…

18. Estela: Yes. 22 –, we have already counted on my number table.
19. James: 22 – 3 =
20. Estela: 3 = 19, right? [*echoes James, but addresses Mike*]
21. James: Yes, I believe that it is so.
22. Estela: 19, 19 – 3
23. Mike: I have already done, look.
24. Estela: = 16.
25. Mike: Look, that I have already done.
26. Estela: We should make a circle around all the threes. Circle all the threes, like circle 3, circle 3, circle the third 3. Is there a statement like an idea of a statement which we can write such as: one counts, one subtracts always three, or something similar.
27. Mike: One subtracts three at any number
28. Estela: (*contemporary to writing*) One subtracts, always…

The students are in the process of agreeing about the written answer to report on the sheet. Firstly, they write down the computations, accounting of what they have done on the number table. Again, the number table is a SMO the students resort to in order to have a sense of likelihood about what they are expressing in written words. Maria's doubt ("are you sure?", 17) comes from her emotional sense of likelihood, given that she is still struggling to intuit the general rule. The reference to the number table is made explicit in 18 by Estela, who replies to Maria. Then, the students discuss about the statement to be written, and a new word arises in the discourse: "always" (26). This can be taken as a movement towards mathematical generalization, but also allows us to infer that intuitive thinking does not inform only an initial moment, nor it should be disregarded as if there is a moment in the students' trajectory in which they think "truly mathematically". In fact, this intuitive thinking both supports and triggers the need for a formal and discursive objectification of the general rule. The students, even if it isn't required in the task, try to express the general rule: you *always* subtract three, at *any* number. There is also a redundancy in the use of both "always" and "any" in their shared written solution.

In the two sequences already shown, it is evident how intuitive thinking, in our socio-cultural approach, sometimes can be an autonomous stroke of genius of a student, but it is in the communitarian self (Radford, 2012) –the shared activity between the members of the group– that it becomes objectified, as it is testified in the students' evolving use of language: already in 2, Mike uses the expression "at any time", and he repeats it in 5. But it is Estela who, in 26, suggests to use "always". Mike echoes her words, saying "at any number" (27), and recognizing the emerging generality. The communitarian self resorts to the territory of artifactual thought: artifacts, in fact, constitute what we are, feel, think, etcetera. The number table is a good example of a semiotic means belonging to the territory of the artifactual thought: it is not just a representation of the first 50 natural numbers, but it culturally determines the way the students make sense of the sequence. James, for example, in 6-13, follows with his fingers Mike's pointing on the table, and without saying a word he objectifies the rule intuited by Mike. Both in 8 and in 19, we have a clue to infer that James can recognize such a rule, since he contributes to the discussion, correcting Mike in 8 and suggesting how to go on in 19. And the number table belongs to the culture, both in general as part of the mathematical knowledge, and in particular as part of the classroom culture. Also the list of computations (15-24) is a semiotic means of objectification, which (differently from the number table that is given) is created and shared within the group, and allows a further leap towards the general rule: the students, in fact, firstly underline the threes on their sheets (26), then they write a closing statement where they underline "One subtracts, always" (28). The consciousness' movement of the students is constantly culturally and historically determined both by their previous knowledge and the semiotic means of objectification to which they are exposed. The interplay between intuitive (e.g. –3) and more abstract (e.g. "always") forms of thinking, and the communitarian consciousness' movement, is expressed throughout the dialogue, especially when the students invite each other to pay attention ("look").

DISCUSSION AND CONCLUSION

We have argued that our understanding of intuition doesn't allow us to cast them in the individual's cognitive and psychological behaviour, nor in the structure of the mathematical object: we must look at the dialectical relationship—culturally mediated and transcended—between the individual consciousness and the content of knowledge, between the particular (–3) and the general ("One subtracts, always"), that gives subjective activity objective reality and bestows objective reality with the subject's determinations. Intuitive thinking can be seen as the sensuous side of intellectual activity when the activity is mediated mainly through objects, artefacts, gestures, bodily movements, deictic and generative use of natural language. In intuitive thinking the mathematical content belongs to the student's space-time experience in terms of emotions (Mike's sense of certainty in 1), feelings, perception, movement, rhythm (Mike's counting 1-2-3 on his hand and then on the number table, sharing his intuition), manipulation of objects (the number table itself), which account for the sense of proximity with, and enactment of, the general ("any", "always") that transcends the individual. This act of mediated recognition determines both the content and the subject. It is a process of being and becoming that could not take place without the intuitive part of this double-sided activity that we call thinking.

From an educational standpoint, this student-content relationship we termed as intuition is a way of thinking, or rather, a mode of existence that is always present in the consciousness' trajectory towards the objectification of mathematical knowledge both in learning-teaching processes and in the cultural and historical development of mathematics. It is necessary to further scrutinize the nature of this special type of mediated reflexive activity that we bound to intuitive thinking both to better understand this phenomenon and to design suitable instruction in the classroom.

Finally, this work can be taken as an attempt to show that intuition may belong to a private, individual sphere, but it is in the *communitarian self* that it becomes part of the mathematical activity we call learning. In that, we are also addressing an issue on the political, as insightfully pointed out by Pais and Valero (2012). We remark that our understanding of politics follows Milani's (1967) words: "through teaching I have learned that the problem of the others is the same as mine: coming out alone is avarice, coming out together is politics".

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STRAIGHT ON THE SPHERE: MEANINGS AND ARTEFACTS

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This report is drawn from a teaching intervention which aims at introducing spherical geometry in a primary school. Assuming a perspective according to which learning should involve students in processes leading to the explicit formulation and elaboration of meanings, this report is meant to explore the dynamics between students' elaboration of meanings emerging from an activity with artefacts and students' explicit reflection on the use of the artefacts themselves.

CONCEPTUAL FRAMEWORK AND RATIONALE

The use of technologies for teaching and learning has raised several issues concerning, for instance, the role of digital and non-digital artefacts in mathematics teaching and learning, the potential and critical aspects related to their use in the classroom, the professional needs of teachers (for an overview, see Hoyles and Lagrange, 2010).

Generally speaking, the learning potential of artefacts is supposed to rest on their *mediating function*, that is on the potential for artefacts to play as intermediary entities able of establishing links between the artefact user and the object towards which the artefact's use is directed (Rabardel, 1995; Meira, 1995; Trouche, 2000; Borba and Villarreal, 2005). More specifically, Rabardel (1995) assumes the existence of two directions of mediation: the *pragmatic mediation*, oriented towards the action on the object and its transformation, and the *epistemic mediation*, oriented towards the user's awareness of the object, its properties and its changes after the user's action.

With that respect, Mariotti and Maracci (2012) noticed that the study of the *mediating function* of artefacts often focuses on the analysis of its role in relation to the accomplishment of tasks, while the complexity of the relationship between use of artefacts and meanings-making risks to remain concealed. Assuming a Vygotskian perspective (Vygotsky, 1978), we firmly believe that the objectives of education entail the conscious development of meanings. This raises the need to explicitly address the issue regarding students' awareness of the meanings developed in relation to the use of artefacts.

Another crucial epistemological issue concerns the relationship between the *personal meanings* which the individual develops through the use of artefacts and the *mathematical meanings* at stake, the appropriation of which is the objective of an educational intervention (Bartolini Bussi & Mariotti, 2008). Bartolini Bussi and Mariotti (2008) remark that the *personal meanings* cannot be assumed to evolve spontaneously towards the desired *mathematical meanings*, which for their very nature are general, de-contextualized, and not-negotiable. In fact, the meanings emerging in relation to the use of artefacts develop with respect to a complex system of meanings (that includes stable and unstable meanings, meanings elaborated in depth and still

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developing ones). The evolution of students' meanings needs to be mediated by an expert (the teacher) aware of all this complexity.

Summarizing we think that there is the need of deepening the study of the meanings which the students develop in relation to the use of artefacts and the conditions under which these meanings can evolve towards mathematical meanings. More specifically, our study *aims at investigating the process of meanings-making in relation to an activity involving the use of artefacts with a specific focus on the dynamics among:*

- students' explicit reflection on the meanings emerging from the activity,
- their elaboration of the use of artefacts, and
- their explicit reflection on such use,

in the context of a specific teaching intervention at primary level.

METHODOLOGY

The teaching intervention is part of a pilot project which is still in progress. Currently, two 5th grade classes (with 20 and 16 pupils) of a primary school are involved. The intervention is structured in 10 sessions of one hour each (two sessions for week), held during the normal school-time. It includes both group activities - in which students are involved in the exploration of concrete models of plane geometry (floor, board, sheet) and spherical geometry (plexiglass sphere [1], see Lénárt, 1993) - and collective discussions during which the outcomes of the previous group activities are shared and discussed. Depending on the specific task, pupils can use several artefacts: adhesive tapes, cords, rubber bands, paper tapes...

All the sessions are videotaped. The analysis is developed on the basis of the videos realized and on the verbatim transcripts of the verbal interactions among students and between them and the teacher. The data analysed are drawn from the activity within the larger class.

THE TEACHING INTERVENTION: PRINCIPLES AND AIMS

We present synthetically the main principles which inspired the design and realization of the teaching intervention, in order to frame the activity which we analyse in the next section (a discussion of these ideas is out of the scope of this report):

- the importance of the epistemological analysis of the mathematical content, object of the didactical intervention (Arzarello & Bartolini-Bussi, 1998);
- the idea of "mathematics laboratory" as the phenomenological space of mathematics teaching and learning, structured by the use of specific technological artefacts and negotiation processes (Chiappini, 2007);
- the role of the use of artefacts (Bartolini Bussi & Mariotti, 2008);
- the role of kinesthetic and perceptual experience in the process of meanings-making (Nemirovsky, 2003; Radford, 2003) ;

[1] Due to how the plexiglass sphere is built, a maximum circumference and two antipodal points are clearly marked on its surface. The pupils will spontaneously refer to them as "equator" and "poles" respectively.

- the role of verbalization processes in the process of meaning-making (Leont'ev, 1964/1976; Bartolini Bussi & Mariotti, 2008);
- the teacher's role in setting up the activities, supporting pupils' exploration and managing the classroom discussion (Bartolini Bussi & Mariotti, 2008).

The teaching intervention is centred on the exploration of concrete models of plane geometry and spherical geometry, and on the comparison between the outcomes of these exploration. The leading hypothesis is that such kind of activity fosters pupils' consciousness-raising of the geometrical relationships characterizing the geometrical objects of plane (they are already familiar with) and sphere. The epistemological analysis led us to take the notion of geodesic as the starting point of our intervention, and design activities centred on the realization and exploration of straight paths (see also the analysis of Arzarello et al., 2012). As Hilbert and Cohn-Vossen state (1932/1990, p. 220, italics in original):

> "The geodesic lines, or geodesics, of a surface are a generalization of the straight lines of the plane. Like the straight lines, they are endowed with several important properties distinguishing them from all other curves on the surface. Hence they may be defined in various ways […] as *shortest* lines, as *frontal* lines, and as *straightest* lines."

DATA ANALYSIS AND RESULTS

We analyze the first two sessions devoted to the exploration of straight and non-straight paths on the sphere. For the sake of readability, we split the description of the sessions into episodes which correspond to different leading tasks posed by the teacher. Before these sessions, pupils were involved in activities concerning plane geometry. Specifically, they were asked to realize straight and non-straight paths walking on the floor of the classroom and of the hallway, and using adhesive tape and cord. This activity was meant to let pupils reflect on the kinesthetic experience of "going straight" on the plane and to explicitly notice that "*it's easy to lay down the tape on the straight paths, while on the non-straight ones it becomes tangled, spoiled*" (here and in the following, pupils abbreviate "adhesive tape" with "tape"). That is a general property: if a piece of "strict enough" tape is carefully laid down on a surface so as to adhere on it without wrinkles, then the tape "approximates" a geodesic.

First session - episode 1. The teacher launches to pupils the request of drawing straight and non-straight paths on the sphere. The class is split into 4 groups of 4 pupils each; each group has a plexiglass sphere and marking pens. In a few minutes, all the groups draw "straight" paths (i.e. maximum circumferences on the sphere) and "non-straight" ones. When comparing the outcomes of their work, the pupils immediately show a clear agreement on which paths are straight and which are not. Their spoken reports, instead, reveal different levels of elaboration of this experience. Some pupils only describe the non-straight paths as "haphazardly realized", and the straight ones as "difficult to realize". Other ones (e.g. P) try to describe the paths making reference to spatial properties, and identifying the procedure they followed:

P: the non-straight path, as we can see, you can go everywhere, turn and then go back, do what you want more or less. In the straight path, you can move straight […] from one pole to the other one, [...]continue, continue, and […] the equator.

Episode 2. After the comparison of the outcomes of the group work, the teacher poses the task: "*how can we ascertain that a path is straight or not on the sphere?*" The task is decoded in two radically different ways by P and F. The former refers to the drawing procedure and to the spatial properties which in his view characterize the straight paths. The latter expresses the need to introduce an artefact for checking the paths:

P: from your departure, you fasten something on it and <u>go ever straight</u>, if when you can back you are again in that point, it means that you have turned around (*he sketches circles on air with a finger*)… <u>you've gone straight and you've done all the rotation of the sphere</u> (*he sketches with a finger a larger circle*)

F: in my opinion, to manage to make a straight path […] <u>using the tape,</u> because the tape […] shows if you manage to put the path in straight way, and on the contrary if you don't manage, it means that it isn't straight, it is curved.

These interventions already intimate the different roles that use of an artefact, spatial and graphical properties, and kinesthetic experience, play in the development of meanings related to "straight path on a sphere" for the two pupils. This difference will emerge more and more clearly in the following.

Episode 3. The teacher re-launches F's idea to the class, asking pupils to try and lay the adhesive tape on their paths, and observe what happens. Then, the pupils share and discuss the outcomes. That leads pupils to explicitly elaborate on the use of the artefact for accomplishing the task, and to reach a general consensus regarding the fact that: the tape adheres well on the sphere along the straight paths, while it is difficult to make it adhere along the non-straight paths – "*it folds up*", "*it is very complicated*", "*it is impossible*". The analogy with the experience on the floor is recalled by the teacher who provides a synthesis of the discussion and concludes: "*the tape is an instrument for checking*", (later on) "*the best one*". Given the consensus which has been apparently reached, the teacher intervenes to provide a first "official" characterization of the "straights paths": a straight path on the sphere is a closed line which divides the sphere in two equal parts. At the very beginning this characterization is agreed upon by all pupils. Then, at the very end of the session, unexpectedly, P intervenes proposing a new "straight path" (a non-maximum circumference, that we will call the "P's path"):

P: […] not ever it cuts the sphere into perfect halves […]. If you start from here and go straight straight […] turn here (*points out a vertical non-maximum circle*) here is a larger part and here is a smaller, then it doesn't cut perfectly.

This path is immediately refused in a peremptory way by most pupils.

Second session - episode 4. The teacher recalls the path proposed by P and asks pupils to express their point of view: is the P's path straight? The following discussion reveals that the meanings related to "straight path on the sphere" were not so shared and stable as appeared at the end of the previous session:

G: it can be both [straight and non-straight] because, when you draw a path on the sphere, the sphere is rounded […] then if you make a circle (*she points out a non maximum circle on the sphere*) […] it can seem straight but, to me, it isn't.

F: to me it isn't straight […], because you have done the circle […] that means to turn

P: […] here I haven't curved […] you go ever straight, look (*his finger goes through the non-maximum circle he has drawn before*)

It is here evident an emergent tension between an intrinsic point of view (according to which there are straight paths on the sphere) and an extrinsic one (no path is really straight on the sphere). Later on, P adds:

P: you go ever straight, […] the sphere is rounded and then you shape a circle […] <u>because the sphere is rounded, you don't make a circle</u>, because you walk, you go ever straight.

No pupil evokes the use of artefacts to test the P's path, notwithstanding the apparent solid consensus on the use of the adhesive tape as a decisive instrument for checking.

Episode 5. Then the teacher suggests P and all the groups to draw this kind of path on their sphere and try and stick the tape on it. P and all the pupils experience that the tape, when laid on non-maximum circumferences, folds up and form several wrinkles. But this experience does not lead all of them to refuse the P's path. On the contrary, some pupils begin to question the use of the adhesive tape itself. Three different points of view emerge little by little: (a) it is difficult to use the tape accurately; (b) a certain tolerance is allowed, that is the presence of few folds is not decisive; (c) one can use the tape in different ways, for instance one can cut and stick small pieces of tape instead of a long continuous one. P is the first proponent and the most strenuous defender of the first two points of view:

P: it doesn't come very well… but... <u>with the tape it is difficult</u> because here [in the equator] it is easier […]

P: also in my opinion with the tape is more difficult but it is straight as well, because the tape should turn or fold up, <u>here it comes just few folds</u>.

Only for few pupils this test is decisive: the P's path cannot be regarded as a straight path because the tape forms a lot of wrinkles, see M and F's joint argument:

M: [...] the circle [the P's path], to me

F: that P said to be straight

M: to me is not straight, because <u>we have tried [...] and it [the tape] becomes ever tangled</u>

F: and <u>then it means that it is a non-straight path</u>

The interventions of F and P polarize the following discussion on two opposite arguments: those based on the use of the artefact (by F), and those based on the spatial properties of the paths and on the drawing experience (by P). No agreement between them seems possible. And the consideration of the spatial properties of the paths gain a greater and greater consensus among pupils:

G: this path [the equator] is straight, but if you make it [the path] smaller […] of course it is more difficult [to lay the tape down] but it's the same stuff, it's equal.

Episode 6. Notwithstanding the numerous interventions of the teacher aiming at regaining a consensus among pupils on the use of the artefact, no conciliation is reached. The observation that the tape is tangled originates a tension between two opposite interpretations; the tape is tangled because: (a) the path is not straight, (b) the tape has not been accurately stuck. Such tension is evident, for instance, in the use of terms "*tangled*", "*crooked*", and "*badly put*" as nearly synonymous, and in the prompt and inflamed reaction of F (the pupils are describing the tape laid down by F along a circle, drawn in the blackboard by F in order to recall the use of the tape on the plane):

Some pupils: [the tape is] "tangled", "crooked", "badly put"

F: no! badly put, no! It is put completely tangled because it is not straight!

Later on, the same tension emerges again, describing the tape on the sphere (P's path):

F: on a sphere it comes tangled […], look […] that it comes ever taaaaaaaaaaaangled!!

P: because […] you must put it well

The questioning of the artefact becomes more and more radical:

B: the tape […] of course you've to tangle, but if you had another thing, you don't tangle

P: […] I don't mean that it [the tape] has this problem, [but] it has this characteristic. On the sphere, if you haven't put central (*near the equator*), it is tangled.

SYNTHESIS AND CONCLUSIONS

The analysis developed in the previous section highlights the complexity of the dynamics involving the explicit reflection on the meaning emerging from the activity, the elaboration of the use of artefacts and the explicit reflection on such use. Several elements, many of which we could not analyse here, play a key role in the process of meaning-making: the use of artefacts; the semiotic activities - both verbal and non-verbal - in which the students were involved; the kinesthetic activities, and in particular the modalities of drawing lines on the sphere; the teacher, in setting the activity and managing the discussions. All these can contribute to the evolution of different and, sometimes, contradictory meanings, which coexist in the classroom and even in single pupils (e.g. G in thinking that the P's path is both straight and not straight).

The common experience with the artefacts, while playing a crucial role in meanings-making, cannot suffice in itself to assure that meanings are shared among pupils and evolve as desired. However, what makes our analysis interesting to us is a more complex aspect. In fact, the tension between the different emerging meanings stimulates pupils' reflection not only on such meanings, but, in particular, on the use of the artefact, and led pupils to adjust, doubt, or even refuse the artefact for its "inadequacy". Hence it is the co-emergence of contrasting meanings which led pupils to question the artefact. With that respect, an interesting issue to us is whether specific constraints of artefacts and activities can avoid a so radical questioning, or conversely, how the possibility of a radical questioning of the mode of use of specific artefacts can be exploited for educational purposes.

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CONFLICTING DISCOURSES THAT SHAPE MATHEMATICS TEACHERS' PROFESSIONAL IDENTITY

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The ongoing research reported in this paper concerns mathematics teachers identity and the tensions within it due to contradictions between numerous discourses struggling to shape it. The concept of professional identity, *defined in discursive terms, allows for empirical inquiry that, unlike the traditional research evolving around the notion of* beliefs, *leaves much space for the sociopolitical context. Qualitative and quantitative techniques of critical discourse analysis are used in analyzing written and spoken texts from different sources, representing different communities: policy documents, newspaper articles, various education-related websites and interviews with mathematics teachers.*

THE THEORETICAL FRAMEWORK

The notion of beliefs has been borrowed from psychology to explain the active, transforming role of individual mathematics teachers in applying proposed reforms. Because of reported inconsistencies between professed beliefs and actual decisions, researchers tend to infer beliefs from observations on the teacher's practice rather than just from what she or he says in an interview. This cognitivist approach has been criticized as non-operational (Sfard & Prusak, 2005) and as suffering from logical fallacy of circularity (Lerman, 2002), whereas the cause that seeks to explain the action is derived from the action itself.

In addition, the traditional approach seems to be overlooking the fact that the real motives for how modern mathematics education is designed and functions reside in political and economic realm, rather than in the domains of mathematics or education in the narrow sense (e.g., Keitel & Vithal, 2008). Within this context, modern math ed, instead of caring for development of mathematics thinking and understanding, is often cultivating “prescription readiness” among the students, thus serving the neo-liberal market demands (Skovsmose, 2011).

The problems of the cognitivist approach arise from the Cartesian split between mental and “bodily” (material) phenomena. Activity theory, based on the concept of activity as “purposeful changing of natural and social reality” (Davydov, 1999, p. 39) is a promising alternative conceptualization of mental capabilities. Activity is mediated by physical (tools) and semiotic (communicational) means. Taking as a point of departure the related “participationist” vision, according to which “patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual”, Sfard (2008, p. 78) arrived at the conclusion that thinking can be defined as “individualized version of (interpersonal) communicating” (p. 81). Following Sfard and her colleagues (Heyd-Metzuyanim & Sfard, 2012; Sfard & Prusak, 2005), I define teachers' professional identity as those reifying narratives about mathematics and

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mathematics teachers that answer the basic questions underlying teachers' decision-making: “What is mathematics? Why, what and how should we teach?” It is worth noting that those questions and the related narratives are “fractal” in the sense that they can be asked/told with regard to any area of math ed, even to the narrowest one. The identifying narratives are coming from numerous discourses and, in result, they are often contradictory.

In order to be able to deal in the research with sociopolitical issues of math ed , I adopt the approach known as *critical discourse analysis* (CDA; Fairclough, 1995). Among the distinctive features of CDA one can count (1) viewing discourses as a practice of using language; (2) assuming dialectical relationships between discourses and social structures; (3) studying issues of power distribution, exploitation and other forms of social distortions; (4) denying the neutrality of social research, i.e., arguing for the necessity of considering social implications of the research.

The majority of social fields is afflicted with interdiscursive conflicts—conflicts between discourses struggling for imposing their view of reality. The winning, hegemonic discourse is perceived as “common sense”, thus serves as the ideology. In educational research, a few attempts have been made to identify discourses struggling for impacting teachers’ identity—see e.g., Sachs’ (2001) work on the conflict between “managerial” and “democratic” discourses. This area of research, however, is rather scarce and there is still much need for thorough empirical studies in which mathematics teachers' professional identity is investigated from discursive, especially that of CDA, point of view.

RESEARCH OVERVIEW

In spite of the comprehensive body of research on collisions in mathematics teachers' decision-making on the one hand, and about sociopolitical contexts of math ed on the other hand, there is still much need for theoretically sound, empirically grounded research that would link these two areas. I argue that the present study may contribute to providing what is missing.

The main goal of my research is to explain Israeli mathematics teachers’ decision making by considering the broad sociopolitical context of their practices. The first necessary step in dealing with the issue is to describe different, possibly conflicting discourses that feed into the teacher's professional identity. Each such discourse is characterized by its representation of mathematics and mathematics teachers, its origin (community) and purpose, and the means with which it struggles for domination.

I use two kinds of data: interviews with mathematics teachers and written texts from various sources, some of them to be found on the web (that is, I am using techniques known as “Web As Corpus”, WAC).

The interviewees are secondary school mathematics teachers and the interviews are designed in such a way as to minimize the influence of either the interview setting or the interviewer’s discourse. The analysis of interview transcripts is mostly quantitative, and they include looking for *nodal points* (concepts around which the discourse is built) and *floating signifiers* (nodal points that conflicting discourse define in different ways; Laclau & Mouffe, 2001), identifying argumentation strategies and metaphors, etc. In two pilot interviews I found two conflicting discourses regarding *mathematics teaching* (floating signifier), built around the concepts of *exam* and of *understanding* (nodal points). The discourse of *exam* uses metaphors of avoidance, such as "*to cover* (which in Hebrew means, literally, to conceal) things that can be asked in exam" and resorts to *organizational* argumentation, rested on time management. For example, fostering the student's intuition is rejected as activity that takes time that could be used for a more direct preparation to the exam (e.g., for solving routine textbook problems). In contrast, the discourse of *understanding* uses metaphors of struggle, such as "*to attack* the problem from several directions" and has recourse to *cognitive* argumentation. Here, teaching and learning are described as "timeless" processes involving relationships between such entities as *sense of success*, *understanding* and *love*. It seems that today, the discourse of *exam* dominates over that of *understanding*, and the question that needs to be answered is that of the power source(s) of its hegemony. It is also interesting to ask why some teachers nevertheless choose the discourse of *understanding*.

WAC techniques involve retrieving texts from the Internet, choosing appropriate sources—those that represent relevant communities, and thus discourses. The chosen corpora are analysed mostly programmatically, e.g., by searching for patterns in frequencies and co-frequencies of key words. Some of these techniques are exemplified in the following sections.

GOOGLE BOOKS N-GRAMS

The term *n-gram* signifies contiguous sequence of *n* words. Google books project provides the database containing all n-grams, $1 \leq n \leq 5$, that appear in at least 40 digitized books. The database shows number of appearances of each n-gram for each year. From 36,499 Hebrew books included in the project, 31,309 were printed after the establishment of the state of Israel. My analysis was restricted to those latter years. There is no distinction in Hebrew database between scientific literature and fiction, thus the following data can be described broadly as representing the literary discourse on mathematics and math ed.

Relative frequency of unigram (1-gram) "mathematics" shows steady growth from $3.85 \cdot 10^{-6}$ in 1948 to $6.55 \cdot 10^{-6}$ in 2008 (increase of 70%). At the first glance, this result corroborates the claim about the significance of mathematics nowadays. And yet, the subsequent analysis of bigrams (2-grams) demonstrates that this assumption may be wrong. Of all bigrams featuring the word "mathematics", 44.0% are related to math ed, with "mathematics teacher" the most frequent of them (22.8%). It is notable that in other school subjects this proportion is quite different. For instance, only 5.2% of bigrams that include the word "physics" are related to physics education. For biology and chemistry, the results are 14.6% and 26.0%, respectively. Finally, the use of "*mathematics* teacher", which for five decades had been lagging behind the use of "*Hebrew* teacher", "*English* teacher", "*history* teacher" and "*literature* teacher", in the last few years has run past all of those. These latter findings—the fact that in almost half of the cases the word “mathematics” appears in the context of math ed and the rapid growth in the use of “mathematics teacher"—account for most, if not for all, of the increase in the relative frequency of the word “mathematics”.

The disproportionally extensive use of the word “mathematics” in educational context raises a question of whether the modern math ed is really about learning *mathematics*. This finding seems to corroborate one of the principal claims of critical math ed research, according to which the main purpose of math ed is to serve specific sociopolitical and economical interests.

COMPARATIVE COLLOCATES ANALYSIS

Most of CDA quantitative techniques of corpus analysis are based on “the comparison of frequencies and the analysis of the syntagmatic environment of key words” (Orpin, 2005, p. 39). I will demonstrate one of such techniques—collocates analysis.

Corpora

Five corpora were compiled from different sources on the Internet, with each corpus representing a particular community:

1. “*Teachers*”. Source: The biggest Israeli forum of teachers—“Tapuz morim”[1]. The broadness of its area of discussion and the anonymity of most participants support a free exchange of opinions concerning all aspects of teachers' identity. The corpus, therefore, may be considered as bringing the authentic voice of the teachers.

2. “*Journals*”. This corpus includes two sources: *The Bulletin for Mathematics Teachers*—semi-annual journal meant for high school mathematics teachers issued currently by the National Center for High School Mathematics Teachers at Haifa University, and privately owned *Educational Connection* [*Kav le-Chinuch*]—weekly journal “read today by the majority of educational decision makers [of Israel]"[2]. The corpus represents the discourse of the educational establishment, directed to the educators.

3. “*Governmental*”. Source: Protocols of The Knesset Committee on Education, Culture and Sport meetings. *Governmental* represents the political-governmental discourse.

4. “*Newspapers*”. The sources are two popular Israeli daily newspapers: *Israel Today* (Israel ha-Yom) and *NRG Maariv*—online edition of Maariv newspaper. Obviously, this corpus represents the media (more specifically, the press).

5. “*Alternative*”. Source: *The Natural Way* (*be-Ofen Tiv'i*)—a well established Israeli Internet community, devoted to issues of “education, natural parenthood and green life"[3]. The community is made distinct by its strong preference for alternative forms of

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[1] http://www.tapuz.co.il/Forums2008/forumpage.aspx?forumid=352.
[2] http://www.kav-lahinuch.co.il/?CategoryID=576.
[3] http://beofen-tv.co.il/cgi-bin/chiq.pl?%E1%E0%E5%F4%EF_%E8%E1%F2%E9.

education, such as homeschooling or open schools. The corpus represents the alternative discourse on (mathematics) education, attracting people with sceptical attitudes toward the state education system.

The corpora were processed in the following way: the units of text (threads of forum, articles of newspapers etc.) that contain a given sequence of letters (e.g., “mathemat"[4]), were extracted and, by removing non-Hebrew characters and some additional cleaning, has been transformed to one long string of Hebrew words, separated by spaces. Total words count of the processed corpora: *Teachers* – 888,106; *Journals* – 959,478; *Governmental* – 6,007,382; *Newspapers* – 737,221; *Alternative* – 9,131,985.

Collocates analysis

This method is based on retrieving a list of words (collocates) that appear in a text within a given distance (span[5]) from a given “node word”. The idea behind the collocates analysis is that words that are significantly “attracted” to the specific node word (tend to appear in a close vicinity of this word) in the specific corpus give a good idea about the meaning of that word within the given corpus and, consequently, in the discourse that a given corpus represents.

The analysis was performed for three node words—"mathematics”, “teacher” and “teach” (plural verb). For each node word in each corpus, and for each collocate, two values were calculated: “observed frequency” *O*—number of a collocate’s occurrences—and “expected frequency” *E*—frequency in the (hypothetical) case of random reciprocal appearance of the node word and the collocate. The simplest measure of attraction, *mutual information* (MI), is $\log_2(O/E)$, which gives positive scores for “attraction” ($O>E$) and negative scores for “repulsion” ($O<E$). As Evert (2008) notes, MI measure is biased towards infrequent words. I thus used a somewhat more complicated formula, accounting for *statistical significance* as well.

The collocations then were categorized as belonging to several semantic fields. For example, the category of “cognition” included words “knowledge”, “understand”, “think” etc. The score of category was calculated as the sum of all its members’ scores in a given corpus. For a given node word, the categories[6] can be thought of as axes of coordinate system, in which each corpus is represented by point, according to categories’ scores in that corpus. To transform the data to more convenient and intuitive form, *principal components analysis* (PCA) was implemented. The PCA reduces the number of dimensions, keeping most of the original data set information. This goal is achieved by creating appropriate new coordinate system, when new axes (called principal component, PC) are special linear combinations of the original ones, so as the corpora coordinates in the first three[7] PCs, taken together, account for about 90% of original data information.

[4] To incorporate words as "mathematics", "mathematical" etc.
[5] I use symmetric span of 4, i.e. looking for collocates within the 4-words range from both sides of the node word.
[6] Their number varies around 15-20, depending on the node word.
[7] In the present case. For other data this number can be different.

The convenient way of PCA results examination is provided by biplots of the points (corpora) coordinates in PCs together with the coefficients that the original axes (categories) have in PCs construction. Then *the proximity of a category to certain corpus indicates that this corpus is distinguished by attracting that category*. Fig.1 shows biplot of the first two PCs for the node word "mathematics". For example,

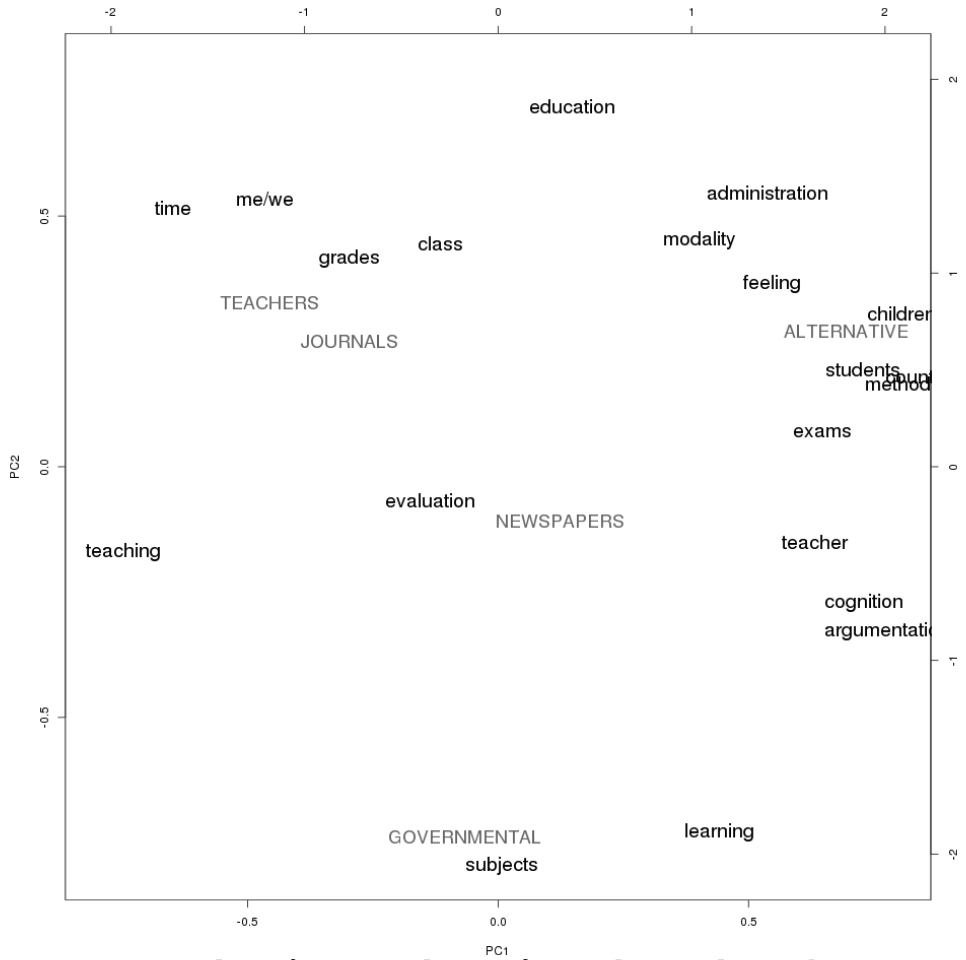

Figure 1: Biplot of PC1 and PC2 for node word “mathematics”.
Corpora are in grey capital letters, categories are in black, lowercase.

categories *time* (words like "before", "after" etc.) and *me/we* (first person pronouns), have negative coefficient in PC1 and positive coefficient in PC2, and corpora *Teachers* and *Journals* have negative PC1 coordinate and positive PC2 coordinate. Thus those categories are especially attracted to "mathematics" in the two corpora. For the complete analysis the PC3 should be considered, but the complementing biplot, as well as biplots for other node words are not presented here due to space shortage.

Interpretation of the results of collocates analysis, as recorded in the biplots

1. *Node word “Mathematics*". In the *Governmental* texts, the word *mathematics* signifies mainly a school subject (*subjects*[8]), whereas in *Journal* and *Teachers* corpora it is to be understood as a part of a personal (*me/we*) story (*time*) of the text authors.

[8] Here and below, categories names (italicized) mentioned in a corpus description mean that the corpus is distinguished by attracting those categories.

The stage of the story is the classrooms (*grades* and *class*), the theme is *teaching*. Surprisingly (although probably not for those who hold the critical view on math ed), the *Alternative* corpus is the one that deals with the cognitive aspects (*cognition*) of mathematics and its teaching *methods*, deliberating extensively (*argumentation*) about those issues. Also, *children* has more central place here than formal positions of *students*, indicating that the main concern is with the needs of a future citizen (child) rather than just with the “technical” needs of a person who tries to function within the educational system (student).

2. *Node word “Teacher*". Again, *Governmental* discourse relates the word *teachers* mostly to school *subjects*. For *Journals*, *time* is still central, indicating the preoccupation with stories about processes, however the stories are not personal here, but rather refer to how the topic is or should be taught (*methods*). For the *Teachers*, the issue is still personal (*me/we*), but this time, it is related to the teacher’s profession. Teachers emerge from these texts as concerned about the *terms* of their employment and as having debates, and possibly disagreements, about it (*argumentation*). *Newspapers* speaks about what should be done (*modality*) in the light of *exams* outcomes, with regard to teachers’ (deficient?) *training*. *Alternative* is concerned mainly with assessment (*evaluation*) of teachers as professionals (*workforce*) within the education system (*administration*) and with their impact on *students*.

3. *Node word “Teach” (plural).* This node word, although close in its meaning to the previous one, was chosen in view of the difference between “teacher” as profession and “teaching” as the action that not only teachers can perform. This assumption can explain the striking opposition between *Journals* and *Alternative*. While the first identifies teaching with *teachers*, explaining again what they should do (*modality*), when the issue of *exams* is at stake, the latter speaks about the teaching as such, possibly performed by other agents (e.g., parents). Again, all this is strongly related to the cognitive (*cognition*) side of the process and is addressing *children* directly.

Conclusion

To summarize the relationships between the corpora with regard to the three node words findings, *Teachers* is closest to *Governmental*, and then to *Journals*. The most distant (and the most independent overall) is the corpus *Alternative*. Thus, the tentative conclusion can be made that the teachers' identity discourse is influenced most strongly by the governmental discourse, and also, at least to some degree, by conceptual-ideological discourse of educational establishment. Of the five discourses, it is the *Alternative* that brings to the fore the all-important issue of mathematics education, and especially its cognitive side and its importance for children.

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WHAT CHILDREN KNOW ABOUT MULTIPLICATIVE REASONING BEFORE BEING TAUGHT

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We investigated children's informal knowledge of multiplicative reasoning. Data were collected at the end of first grade, before this mathematical domain was explicitly taught. A large sample of children (n = 1176) was assessed in a relatively formal test setting, through an online test containing 28 multiplicative problems with and without contexts. On average, the children correctly answered more than half (58%) of the problems, indicating that before formal instruction on multiplicative reasoning, children already have a considerable amount of knowledge available in this domain, which teachers can build on when teaching them formal multiplication and division. We found that multiplicative problems with an equal groups semantic structure and context problems with a picture involving countable objects were easiest to solve.

INTRODUCTION

Children have usually already built up a considerable amount of informal knowledge in a mathematics domain before this domain is taught (e.g., Baroody, 1987; Ginsburg, Klein, & Starkey, 1998). Characteristic of this knowledge is that it is constructed in response to situations that children encounter in daily life, and that it is developed largely independent of explicit instruction (Leinhardt, 1988). Many mathematics educators have stated the importance of building on children's informal mathematical knowledge when teaching them mathematics (e.g., Baroody, 1987; Leinhardt, 1988). They argue that through their informal knowledge children can give meaning to the formal symbols and procedures of mathematics (e.g., Baroody, 1987). Despite the importance of connecting the formal mathematics to the informal mathematics children bring to school, researchers found that teachers often fail to make these connections (e.g., Leinhardt, 1988). An explanation for this might be that teachers underestimate children's pre-instructional knowledge (e.g., Lee & Ginsburg, 2009; Van den Heuvel-Panhuizen, 1996). For teachers to be aware of this knowledge, it is crucial that the informal knowledge of children is revealed. In the study described here, we aimed to map children's informal knowledge in the domain of multiplicative reasoning just before they start receiving formal instruction on this domain.

Multiplicative reasoning

The mathematics domain of multiplicative reasoning, comprising multiplication and division, is clearly distinguished from the domain of additive reasoning, including addition and subtraction (e.g., Schwartz, 1988; Vergnaud, 1983). In contrast to additive reasoning, in which quantities of the same type are added or subtracted (e.g., 2 cookies and 3 cookies are 5 cookies altogether), multiplicative reasoning involves quantities of different types (e.g., 3 boxes with 4 cookies per box means 12 cookies altogether). A multiplicative situation is, in its most elementary form, characterized by a group structure which involves sets (groups, e.g., boxes) of items with the same number of items (e.g., cookies) in each set (see Greer, 1992).

Previous research on children's informal multiplicative knowledge

Most research on informal mathematical knowledge has focused on counting, addition and subtraction (e.g., Ginsburg et al., 1998), but also in the domain of multiplicative reasoning, which comes later in the curriculum, studies have shown that young children already have some informal understanding of the domain before it is formally introduced in school (e.g., Anghileri, 1989; Kouba, 1989; Mulligan & Mitchelmore, 1997; Nunes & Bryant, 1996). Kouba (1989), for example, found that around 25% of first graders could already solve simple multiplication and division word problems. Furthermore, in a longitudinal study by Mulligan and Mitchelmore's (1997), Australian children at the beginning of grade 2 correctly solved an average of 31% of the multiplicative word problems, increasing to 48% at the end of grade 2 and 55% at the beginning of grade 3 (all these measurements were before formal instruction on multiplicative reasoning). In all earlier studies on informal multiplicative knowledge, the problems were either presented in a physical context (e.g., Anghileri, 1989) or the children were allowed and encouraged to use physical materials, such as counters and blocks, to construct a physical representation for themselves (e.g., Kouba, 1989; Mulligan & Mitchelmore, 1997). From these studies it is not known whether children can also show this informal knowledge when assessed in a more formal setting in which they have te work on their own, do not have physical representations available, and are presented not only context problems but also bare number problems.

Problem characteristics

Several studies have shown that the mathematics performance children display may be related to characteristics of the problems offered to the children.

Problem format. Research has shown that, for students who have had no or only limited formal instruction on a particular mathematics domain, context problems in that domain are often easier to solve than bare number problems (e.g., Koedinger & Nathan, 2004; Van den Heuvel-Panhuizen, 2005). For young children, contexts presented with manipulative materials are particularly helpful in solving arithmetic problems (e.g., Ibarra & Lindvall, 1982). But also without the use of physical materials, context problems may be easier than bare number problems, especially when context problems include a picture involving countable objects.

Semantic structure. Based on earlier literature (Greer, 1992; Mulligan & Mitchelmore, 1997), three semantic structures can be considered relevant for first-graders' multiplicative reasoning: equal groups (e.g., 3 boxes with 4 cookies each), rate (e.g., 1 cookie costs 3 euros, how much do 4 cookies cost?), and rectangular array (e.g., 3 rows of 4 chairs). Of these semantic structures, equal groups problems have generally been found to be easiest (e.g., Christou & Philippou, 1999). For the case of informal multiplicative knowledge, though, Mulligan and Mitchelmore (1997) did not find differences in difficulty level between the three abovementioned semantic structures.

Operation. In children who have received formal instruction on multiplication and division, it has generally been found that multiplication problems are easier than division problems (e.g., Christou & Philippou, 1999). This may be a result of the school curriculum, in which multiplication commonly is formally introduced before division. In contrast, Mulligan and Mitchelmore (1997), in their study on children's informal (i.e., pre-instructional) or early abilities in solving multiplicative problems, found approximately equal difficulties for multiplication and division problems. They explain this by arguing that young children intuitively connect multiplication and division and can use the same strategies for both.

Research question

In our study, we aimed to extend previous research by collecting data about children's informal multiplicative knowledge in a relatively formal setting, which closely matches the school practice as children progress in their school career. This will give us more information about the actual size of the assumed gap between children's informal knowledge and the formal mathematics they have to learn in school. If we could show that children's informal knowledge can be applied in or transferred to a more formal setting, then this would imply possibilities for teachers to draw more on children's informal knowledge when introducing formal multiplication and division.

The following research questions were specified:

1. To what extent are children, just before they start receiving formal instruction on multiplication and division, able to solve multiplicative problems in a relatively formal setting (in a formal test procedure, without physical objects provided, and including context problems as well as bare number problems)?
2. How is the informal knowledge children display influenced by characteristics of the problems offered to them?

METHOD

We carried out a large-scale survey in the Netherlands. Since in the Netherlands multiplication is formally introduced at the beginning of grade 2, we decided to assess children's informal knowledge of multiplicative reasoning at the end of grade 1.

Participants

In total, 53 first-grade classes from 53 different primary schools in the Netherlands were involved in the analysis, comprising a total of 1176 students (580 boys, 596 girls; M = 7.2 years, SD = 0.4 years).

Test of informal multiplicative knowledge

Students' informal multiplicative knowledge was measured by an online test with 28 multiplicative items. The use of an online test ensured a relatively formal, standardized test setting, and it facilitated our large scale data collection. As explained before, physical objects were not provided to the students as aids in solving the test problems. However, the students were not forbidden to use their fingers as manipulatives.

Composition of the test. The 28 multiplicative items varied according to problem format, semantic structure, operation, and countability level. The numbers used in the test items were 1, 2, 3, 4, 5, 6 and 10. Regarding problem format, the test contained 14 context problems (see Figure 1a-b) and 10 bare number problems (see Figure 1c-d). Additionally, 4 "groups-of problems" were included to specifically assess students' understanding of the groups-of structure typical of multiplicative situations (see Figure 1e). The bare number problems included 6 problems with "times" instead of the × symbol (see Figure 1c), and 4 doubling problems (see Figure 1d).

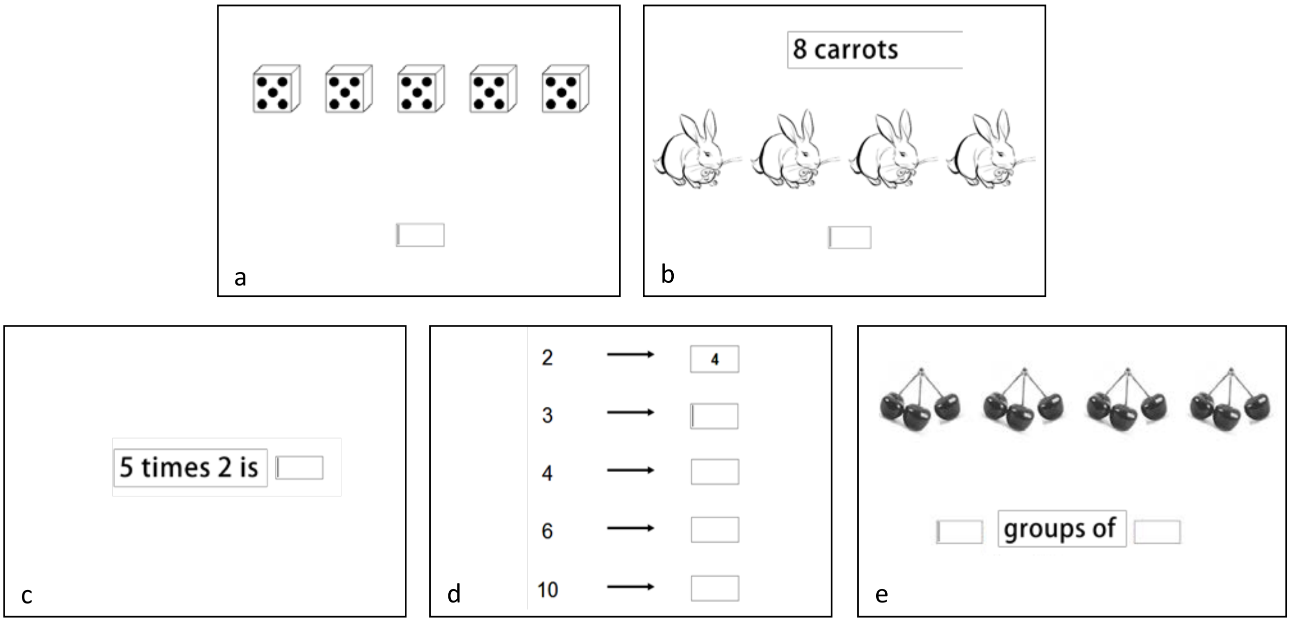

Figure 1. Examples of test items and questions: a. "How many points together?" b. "Eight carrots. How many carrots does each rabbit get?" c. "Five times two is…" d. "Make it double. Each time fill in the answer." e. "What sentence fits the picture?"

The context problems varied by their semantic structure: 9 were equal groups problems, 3 were rate problems, and 2 were rectangular array problems. Furthermore, regarding the operation involved, 10 of the context problems were multiplication problems, whereas 4 were division problems. Since we aimed to measure the informal multiplicative knowledge that is available to build on when formal multiplication and division are introduced, we decided the majority of the problems to be of the equal groups structure and the multiplication operation, which are most common in early formal instruction of multiplicative reasoning.

Finally, the test items differed with respect to their countability, that is, the extent to which the picture presented in the problem could be used to find a solution by counting. For example, in the problem in Figure 1b, the rabbits are countable, but the carrots are not. We distinguished four levels of countability.

Test administration. The online test was administered at the end of the grade 1 school year, in June/July 2010. To control for order effects, four different versions of the test, containing the items in different orders, were randomly assigned to the students. Each test item was individually displayed on the screen (except for the doubling problems, see Figure 2b) and the accompanying question was read aloud by the computer.

Data processing. Since the text boxes in which the students had to type their answers accepted all kinds of input, not all responses were in the form of a number. Input errors were corrected when it was clear which number was meant by the student, such as “4’0” or “4o” instead of “40”. For 0.59% of the item responses this resulted in a change to a correct answer.

Psychometric properties of the test. The reliability of the test consisting of 28 multiplicative items was sufficiently high (Cronbach’s alpha of .89). An exploratory factor analysis for dichotomous data (Revelle, 2012) indicated a 4-dimensional factor structure. The four factors can be interpreted as follows: 1) context problems (14 items), 2) bare number times problems (6 items), 3) bare number doubling problems (4 items), and 4) groups-of problems (4 items). Apart from this 4-factor structure, there is also some evidence that the test can be well represented by a unidimensional summary, for it appeared that about 37.7% of the total variance can be attributed to the first dimension, and there was a large ratio of 4.00 of the first and second eigenvalue. Thus, next to looking at factor subscores, it also makes sense to regard the test as a whole.

RESULTS

On average, the students ($n = 1176$) answered more than half of the total of 28 items correctly. We found a mean proportion correct of .58 ($SD = .23$), with 2 students (0.2%) having no answers correct and 19 students (1.6%) having all items correct. When zooming in on the four groups of items identified through factor analysis, we found mean proportions correct of .63 ($SD = .23$) for context problems, .52 ($SD = .35$) for bare number times problems, and .63 ($SD = .44$) for bare number doubling problems, and .47 ($SD = .38$) for groups-of problems.

To study the influence of problem characteristics on problem difficulty, we performed a Wald Chi-square test for each characteristic (e.g., operation), comparing the mean proportion correct of items belonging to the different categories of the characteristic (e.g., multiplication vs. division items). The nested structure of the data (children within schools) was accounted for by using the TYPE = COMPLEX option in Mplus (Muthén & Muthén, 1998-2010). As an effect size measure we used the ω^2 estimate of explained variance, for which a value of .010 can be interpreted as a small effect, .059 as a medium sized effect, and .138 as a large effect (e.g., Kirk, 1996).

For all problem characteristics, results were highly significant ($p < .001$). For the characteristic problem format ($\chi^2(3) = 1229.14$, $\omega^2 = .038$), it appeared that context problems and bare number doubling problems were easier to solve than bare number times problems and groups-of problems (for descriptives see above). Regarding semantic structure ($\chi^2(2) = 1374.23$, $\omega^2 = .041$), equal groups problems appeared to be easier ($M = .68$, $SD = .23$) than rate problems ($M = .54$, $SD = .37$) and rectangular array problems ($M = .53$, $SD = .37$). For operation ($\chi^2(1) = 178.58$, $\omega^2 = .007$), although the difference between multiplication ($M = .61$, $SD = .24$) and division problems (M = .56, SD = .32) was significant, the effect size was trivial ($\omega^2 < .01$), meaning that the effect was so small that it can be considered irrelevant. Thus, in our study multiplication and division problems can be considered equally difficult. Finally, for countability level ($\chi^2(3) = 128.71$, $\omega^2 = .090$), it appeared that problems offering pictures with more opportunities for counting were easier to solve (no terms countable: M = .56, SD = .30; 1 term countable: M = .50, SD = .32; both terms countable: M = .69, SD = .27; both terms and solution countable: M = .73, SD = .26).

CONCLUSIONS AND DISCUSSION

Our results show that first-graders, even when assessed in a relatively formal setting, display a substantial knowledge of multiplicative reasoning before being taught. This finding extends earlier findings (e.g., Kouba, 1989; Mulligan & Mitchelmore, 1997), of young children being able to solve several multiplicative problems in individual interviews, with the help of physical objects. Interestingly, in our more formal setting we found a higher percentage correct (58%) than did Kouba (29%, first-graders), and Mulligan and Mitchelmore (48%, second-graders).

Further extending previous research, we found that in addition to the ability to solve context problems, pre-instructional multiplicative knowledge for many children also included the ability to solve bare number multiplication problems, in the form of doubling or with the × symbol replaced by the word *times*. In accordance with previous research (Koedinger & Nathan, 2004; Van den Heuvel-Panhuizen, 2005), context problems appeared easier than bare number problems with *times*. The groups-of problems in our study appeared the hardest, indicating that, although many children can solve multiplicative problems, this does not necessarily mean that they have an explicit understanding of the groups structure typical of multiplicative situations.

Regarding the semantic structure of the problems, we found that equal groups problems were easier to solve than rate and rectangular array problems, which is in contrast with Mulligan and Mitchelmore's (1997) finding that before formal instruction, there is no difference in difficulty between these semantic structures. Our finding that the equal groups semantic structure was easier may partly be explained by the fact that informal and preparatory multiplicative activities that occur in the Dutch first-grade curriculum primarily focus on equal groups situations.

With respect to operation, from our results it appeared that, before formal instruction on multiplicative reasoning, multiplication and division can be considered equally difficult. This finding calls into question the usual approach in the Netherlands of introducing division later than, and separated from, multiplication. Mulligan and Mitchelmore (1997) found that young children use the same strategies for both multiplication and division, indicating that children intuitively see connections between the two operations. Possibly, simultaneous introduction of multiplication and division would better exploit these informal insights, but further research is needed.

Finally, we found a significant effect of countability level, indicating that multiplicative problems offering pictures with more opportunities for counting were easier to solve. This indicates that, in addition to physical materials (Ibarra & Lindvall, 1982), also pictures can act as manipulatives and assist in solving problems.

The above results should be taking with caution. Due to testing time restrictions our test had a limited number of items and was not counterbalanced for all problem characteristics. In order to more thoroughly study the effects of the different problem characteristics, an item set is needed in which all characteristics are combined with all other characteristics. Another shortcoming, which is rather insurmountable in a large-scale study like ours, is that we do not really know the conditions in which the children made the test. Although the teachers were not told to give the students physical objects, we cannot be sure that such objects were indeed did not employed.

In conclusion, we found that, when instruction of multiplicative reasoning starts in grade 2, students have a lot of informal knowledge to build on, which can even be tapped in a relatively formal setting. For teachers, next to knowing what informal knowledge children generally bring with them, it is also important to be aware of the knowledge base their individual students have available. Our study showed that a computer-based test can be a useful way of assessing this knowledge.

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INTEGRATING STATISTICS IN MATHEMATICS TEACHING: TEACHERS' UNDERSTANDINGS RELATED TO SAMPLE AND SAMPLING NOTIONS

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This study is a part of an ongoing research which aims to support teachers to integrate statistics in their mathematics teaching and to investigate the development of the way they perceive this integration in the context of learning and teaching statistics. A study group has been formed where ten secondary school teachers and two researchers work together in various activities for a period of an academic year. In this paper we analyse three meetings where the discussion focuses on sample and sampling. Some initial results indicate the development of teachers' awareness of the complexity of the notions of randomness and representativeness and the importance of conducting experiments in the teaching of statistics.

INTRODUCTION

In recent years statistics has gained increased attention in mathematics education. Nevertheless, statistics is still in its infancy (Lajoie & Romberg, 1998). Research shows that many students find statistics difficult to learn and understand in both formal and everyday contexts and that learning and understanding may be influenced by ideas and intuitions developed in early years (Gal & Garfield, 1997). On the other hand mathematics teachers seem to have many difficulties in helping their students to acquire a deep understanding of statistical concepts and techniques (Watson, 2001). In Greece, a recent curriculum emphasises statistical and probabilistic concepts and reasoning posing new challenges for the mathematics teacher.

A difficulty that teachers face for adopting a reform-oriented teaching of statistics is related to the epistemological status of stochastics, which is contradictory to deterministic mathematics (Steinbring, 1986). This requires different way of thinking and different means to deal with it than those that mathematics teachers use for the other mathematics areas. One way to tackle the above problem is to promote teacher awareness of the relationship between statistics and mathematics. In spite of a notable amount of studies (e.g. Gattuso, 2008) which focus on a fruitful collaboration between statistics and mathematics, there is little in the literature to show how the integration between statistics and other mathematics areas can be promoted in teacher education.

This is an ongoing study that aims to support teachers in the direction of the smooth integration of mathematics and statistics in their teaching, and it is guided from the following research questions: a) How can the conceptual connections between statistics and mathematics be promoted in professional development and b) What kind of understandings teachers develop in making such connections in relation to teaching and learning?

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THEORETICAL BACKGROUND

In this study we see teacher collaboration in a study group as a means for teacher professional learning (Arbaugh, 2003). This context supports self-action and reflection, negotiation and interplay. Moreover, it makes it feasible for the researchers to get insight in how teachers' awareness, understandings and practices develop.

The work of this study group is based on an inquiry approach. The inquiry is incorporated in two forms. First, teachers have the opportunity to engage with statistical inquiry. According to a number of studies (e.g. Makar, 2008), by engaging teachers in an investigate cycle (Problem-Plan-Data-Analysis-Conclusion), teachers are supported to develop or deepen understandings of statistical concepts, develop a statistical way of thinking, and link to other areas of mathematics. Moreover, providing teachers with opportunities to address inquiry-based problems themselves, can act as a catalyst to develop their teaching and adopt more investigative and conceptual approaches. Second, teachers have the opportunity to inquire in the teaching of statistics. Jaworski (2006) points out that inquiry enables teachers to engage critically with key questions and issues in practice, "such practice can involve addressing mathematical tasks in classrooms, developing approaches to mathematics teaching or finding ways of working with teachers to promote teaching development" (p. 187).

In our attempt to encourage the development of teachers' awareness for the relationship between statistics and mathematics we prompt teachers to identify sources of meanings, differences and commonalities of mathematical concepts under different contexts, namely the deterministic and the stochastic context (Biehler, 2005). Biehler calls this process reconstruction of meaning and suggests it as an approach that can broaden teachers' picture for mathematical concepts and support the improvement of teaching.

In this paper we concentrate on the concepts of sample and sampling. Tversky & Kahnemann (1983) document the persistent errors and "misconceptions" that people make when making probabilistic "judgments under uncertainty". They theorize that people's mental resources are too limited to be able to generate probabilistically accurate judgments. People are forced to fall back on computationally simpler heuristics such as the representativeness heuristic they describe. On the other hand, many studies (e.g. Abrahamson, 2009) demonstrated how the most obvious traps described by Tversky and Kahneman (1983) can, under the appropriate conditions, be circumvented. Pratt (2005) suggests four pedagogic implications on how teachers foster students' understanding of probability and chance: purpose and utility, testing personal conjectures, large-scale experiments and systematic variation of context.

Moreover, students find it difficult to integrate expectation and variation (uncertainty) into the sampling construct. Other difficulties are related to the different meanings that are attributed by students to the term "sample" in different contexts (Watson & Moritz, 2000).

RESEARCH DESIGN

The study group and its characteristics

The study group consists of ten teachers and two researchers. Eight of the teachers have a degree in mathematics and two in applied mathematics and physics. All of them recently completed or are about to complete the same postgraduate program in mathematics education. In this program they attended one course in statistics. According to their teaching experience, five of the members are novice (T1-T5) and five with varied teaching experience (T6-T10). In spite of their extensive teaching experience their experience regarding to the teaching of statistics is very little in all cases except T8. T7 and T9 are quite familiar with the content of statistics. All teachers agreed to work voluntarily in this study group. The researchers participate in all discussions, often challenging teachers to reflect on emerging issues and giving them particular tasks and materials and resources (eg. selective bibliography, textbooks and statistical software).

Professional tasks

The suggested tasks are usually developed further in the process of the group interaction. The topics discussed in the meetings are: producing data, data exploration and statistical inference. In this paper we will be restricted in the study group's work related to the topic of producing data with main focus on the concepts of sample and sampling. The teachers were engaged in the following three tasks:

Task 1. Personal involvement in statistical tasks: Teachers were asked to work in groups to define a problem, develop a sampling method and explain their choice.

Task 2. Reading research papers: Teachers were asked to read one of five research papers chosen by the researchers and discuss them within the group (the papers of Kahneman & Tversky (1983) and Fischbein & Schnarch (1997) were discussed in the data analysed in this paper).

Task 3. Analysing and transforming tasks: Teachers were given a task from a textbook (see Figure1). They were asked to work in groups to identify the main statistical ideas as well as teaching and learning goals. After the initial discussion, they were asked to transform this task to be more appropriate for the classroom.

In the first task the main focus was to engage teachers in statistical inquiry while in the other two in the inquiry in learning and teaching statistics. All tasks aimed to promote the relation between statistics and probability implicitly (tasks 1 & 3) or explicitly (task 2).

Data collection and analysis

This study is part of an ongoing research. The group of teachers meets twice a month for a whole academic year. Each meeting lasts for about two hours. Individual semi-structured interviews with all the participants were conducted before the first meeting, focusing on their teaching experience, academic background and attitudes upon statistics and teaching statistics.

Randomization and Its Effect

What you'll need: a box, equal-size slips of paper for a random drawing, a coin

The students in your class are to be the subjects in an experiment with two treatments, A and B. The task is to find the best of three ways to assign the treatments to the subjects.

1. Choose two leaders from the class, one for Treatment A and one for Treatment B. The leaders should flip a coin to decide who goes first, then alternately choose class members for their teams (much like choosing sides for a soft ball game).
2. For each treatment group, record in a table the number of subjects, the proportion of females, the proportion who have brothers or sisters, and the proportion who like to read novels in their spare time. Do the two groups look quite similar or quite different?
3. Next, divide the class by writing the names on the pieces of paper and putting them in a box. Randomly draw out half of them (one by one) to be assigned Treatment A. The names remaining are assigned Treatment B.
4. Repeat step 2 for these groups.
5. Finally, divide the class by having each person flip a coin. Those getting heads are assigned Treatment A, and those getting tails are assigned Treatment B.
6. Repeat step 2 for these new groups.
7. What are the strengths and weaknesses of each method of assigning treatments to subjects? Which method is least random?

Figure 1: Problem taken from Watkins, Scheaffer, & Cobb (2008) (p. 249)

All the meetings are videotaped. At the end of each meeting the teachers report their reflections on the work in the group. Moreover, the principal researcher (the first author) writes a report summarising the main themes and emerging issues of the meeting. This report is notified to all members for further thought and reflection. For the analysis regarding to sample and sampling notions, based on the principles of grounded theory, we focused on critical episodes among discussions which reveal how teachers conceptualise the relationship between probability and statistics. In this report we trace teachers' perceptions through the three main tasks and study emerging shifts in teachers' understandings.

INITIAL RESULTS

Simple random sample· a random choice or a free choice

Through the first meeting there was a debate whether a simple random sample constitutes a representative subset of the survey population. This issue was dominant in the group's discussions and different views about the notion of randomness revealed. From the analysis of the group discussion, it seems that some teachers use the word 'random' to refer to a sample that was selected 'by chance'. For example, in the group discussion about the choice of T3's, T7's and T6's subgroup to select a simple random sample (SRS) of students to address the question 'Which is students' favourite course?' T5 claimed that we cannot trust a SRS even if the sample size is sufficient:

T5: A random sample is not necessarily representative. I mean, it could be random (meaning fortunate in this instance) but unfortunate. I mean that the students that are chosen would be only the 'good' students, those who prefer mathematics or ancient Greek courses.

T9: Supposedly that the sample size is sufficient. I don't think so.

In a similar way, T8 suggested stratified sampling method as the only way to achieve representativeness which cannot be achieved through a SRS:

T8: The main issue is to choose an appropriate sample.

T9: We said that the sample size will be large enough.

T8: Not only the size, to be appropriate. I mean the proportion according to the Grade level.

T9: But we don't care about students' grade level. …

T8: You can't just choose from a list. You need to define the proportions with respect to each grade level.

T9: If you choose randomly, the proportions will be close to the actual. Aren't they? … I mean is it possible that in your population the 60% are 8th Graders and from a random choice you choose for example 10%?

T8: Why this cannot be? If your choice was free then it can be.

T9 challenged T8's position by implying that a random sample follows the laws of probability. T10 made explicit the role of probability and mathematics in general in sampling situations.

> 'Randomly means that every individual in the population have equal chance of being selected. If I say I want this number of boys, that number of girls, so many 8th Graders etc, that is another thing … If we refer to a random sample and talking about predetermined proportions, this is not random. We need to know what it is a random sample and how it can be produced. There is power in the random... There is mathematics behind'.

Understanding the role of availability and intuition

In the second meeting teachers discussed on the papers they read. They linked the findings of the papers to their personal understandings and intuitions. They were also involved in solving the tasks presented in the papers by themselves and talk about their solutions. Through the discussion, a growing awareness of teachers' conceptualizations regarding statistical and probabilistic ideas seemed to emerge. For example, in the discussion about availability heuristic (Kahneman & Tversky 1973b), T10 identified that the meaning we attribute to statistical terms is affected by our own experiences and images we have from the everyday use of the terms:

> 'It is very important how we use the word sample in our everyday language. When we refer to sample for example in blond tests, we know that the sample has exactly the same characteristics with the population. By keeping this in mind, we believe that whenever we get a sample we can generalize to the population neglecting the sample methods'.

T7 recognized his own false intuitions and appreciated the need for statistical inquiry as a means for developing understanding: 'You need to experiment in order to understand this. I have taught it in school, I have also been taught it as a student but I still can't understand it. This is not convincing. I mean internally'.

In the following episode a strong connection between statistics and probability seems to emerge regarding the teaching of these areas:

T7: I think we don't have the right intuitions to build on such concepts (he means probabilistic).

T9: I agree with that. Actually, we don't have the right experiences.

[…]

T7: From my point of view, the right intuitions are not directly accessible but only indirectly through statistics and combinatorics.

Researcher: What could be helpful for the students?

T9: The question is what could be helpful for us! (laugh)

T7: There has to be space for discussion and negotiation in the classroom. The school context is not very supportive to this direction due to many limitations that exist.

T10: You don't actually need to discuss anything. You just give an open problem to students and the other things come alone.

T9: I think that for statistics and probability, experiments and simulations are very important.

T6: It is very difficult for us.

T10: There are simple things that you can do.

T7: One difficulty is that you can't verify the answers. … I insist that the development of the right images can be only through statistics and the analysis of real data. How else?

In the above dialogue the teachers realized the need for developing their own understanding while the need for an experimentation-based teaching approach and the constraints that the school context imposes were also addressed.

Thinking with respect to teaching and learning

In the third meeting, the teachers also initially attempted to solve the problem (fig.1) and discuss their solutions. In the discussion the notions such as randomness and representativeness were reconsidered. In this case it seemed that there was an agreement for what is a random sample, although the uncertainty about the meaning of representativeness of a random sample still existed. In the following dialogue in a subgroup, T8 expresses a different meaning for the random choice than the one he had expressed in the first meeting. T8 refers explicitly to the concept of probability when he talks about randomness and seems that he trusts now the SRS as sampling method:

> "It is random (he means the method with the small papers). Every piece has an equal chance, namely 1/30, if you have 30 students in the classroom. … The best method is the one with the pieces of paper".

In relation to the representativeness all the subgroups, who worked on this activity gave different answers with respect to the best-method choice. One subgroup (T4-T6-T8) suggested the second method as the best one, explaining that they prefer a random method and the second is better than the third as it produces groups of the same size. The subgroup of (T1-T7-T10) chose the first one as the best to produce homogeneous and representative groups. In this decision aspects such as sample size and the fact that students' choice criteria will be subjective are considered. Finally, the last subgroup (T2-T3-T9) rejected the first method and claimed that the other two methods were equivalent.

In the discussion regarding how a student can be benefited from such an activity and how the task can be transformed to support students' learning T10, T9 and T8 summarized the work of their subgroup:

> "It is important that the students make conjectures and then try the methods. I mean that, regarding to the 3rd method, the students probably guess that by flipping a coin the two groups would have approximately the same size. If they do it, they will come to realize that this conjecture is not correct, that there is variability". (T10)

> "It would be interesting to do such an activity for a big population, such as to the whole school. In this case, the student would realize that the second and the third method produce equivalent groups". (T9)

> "It is important for the students to have a specific goal. If I could transform it, I would firstly make clear which specific characteristic will be studied and what the goal of this activity is, and then I would ask them to discuss the three methods". (T8)

T10 pointed out again the students' active involvement with the statistical experimentation as a central teaching goal. T9 extended the tasks in the direction of helping students to understand the effect of sample size to determine the appropriateness of a sampling method. T8 emphasized that for the students it is important to have a clear goal when decide about the effectiveness of a method.

CONCLUSION

In this paper, we focused on various ways for integrating statistics and other areas of mathematics in teacher education. The tasks that were designed by the researchers and developed in the group meetings supported teachers to consider epistemological and didactical issues related to the teaching of statistics as this emerged from the group discussions. The nature of the tasks as well as the collaboration among teachers with different backgrounds in statistics and its teaching seem to have a positive impact on the development of teachers' awareness of the complexity of statistical content regarding to the learning and teaching. By connecting sample and sampling with the concept of probability, teachers start to reconstruct their meaning of statistical notions in general as Biehler (2005) points out. In the group discussion pedagogical implications suggested by Pratt (2005) for fostering students' understanding of probability and chance were addressed by the teachers themselves in the process of negotiation. Testing personal conjectures, purpose and utility and large-scale experiments were identified in the process of transforming a statistical task to a students' classroom activity. Moreover, the teachers considered students' involvement in experimentation as a process of creating meaning. We expect that in the next phases of the work more insights will emerge related to the effectiveness of integrating statistics and mathematics in teaching and teacher education.

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THE INFLUENCE OF VERBAL LANGUAGE IN A MATHEMATICAL CONTEXT: A STUDY ON NEGATION

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This paper deals with the "negation" of verbal statements and addresses this topic by means of a functional linguistic approach. The negation is investigated as an effect between a sentence and the subjects' coding of abstract diagrams. The study, carried out with about three hundred Italian science undergraduates, shows that implicatures occurring in everyday communication heavily affect the interpretation of a variety of sentences.

INTRODUCTION

The importance of language in the learning of mathematics has grown up in the last decades and the central role played by the language in the learning of mathematics has become the subject of a rapidly increasing investigation (see e.g. Ferrari 2004, Sfard 2001). The centrality of language in the interpretation of obstacles in doing mathematics can be found, for example, in Sfard's claim that "learning mathematics may now be defined as an initiation to mathematical discourse …" (Sfard 2001, p.28), and so languages are to be regarded not just as carriers of pre-existing meanings, but as builders of the meanings themselves.

In this framework this paper is aimed at investigating the influence of (Italian) verbal language in a mathematical context. In particular, the focus is on the "negation" of verbal statements addressing it by means of a functional linguistic approach, with the hope of giving a new insight differing from previous studies on this topic which do not take into account the pragmatic aspects of language, i.e. the complex interactions between language and context (Lin *et al.* 2003, Antonini 2001, Barnard 1996, etc.). Negation is a fundamental concept for the construction of meaning in general, and, in particular, for the construction of meaning in mathematical context. Difficulties concerning this topic can significantly influence the learning of other mathematical concepts and, in particular, the understanding of the links among the mathematical concepts. In particular, difficulties occurring in the negation of statements are often related to the use of quantifiers. In literature, it is shared the idea that negation may cause troubles, for example, in proofs by contradiction (Lin et al. 2003), in the use of syllogisms and in particular of modus tollens (Wason and Johnson-Laird, 1972), etc. Despite the recognized relevance of negation, few studies have been carried out in the field of the learning of mathematics but, on the contrary, a lot of research has been developed about negation in numerous other fields such as logic, philosophy, linguistics, psychology, etc. (some references are Aristotle, Grice 1975, Wason and Johnson-Laird 1972, Miestamo 1985, Horn 1989, etc.).

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In particular, the investigation is focused on the ability at negating verbal statements of the kind "All A is B", that is to interpret a sentence of the kind "Not all A is B". The present study is the sequel of a study on the same issue described in Bardelle (2011) and in Ferrari (2004) where the interpretation of negation was investigated as an effect between two sentences. In that case the subjects had to recognize whether two abstract sentences had the same, or different meaning (equivalence of statements). In the experiment described here, negation has been investigated as an effect between a sentence and the subjects' coding of a physical state of affairs. In particular, abstract diagrams (ranks of circles) have been exploited to investigate students' interpretation of the negative sentence of the kind "Not all A is B". The aim is to explore the role of context (purposes of the communication, relationships among participants, mode, subject of the communication, etc.) in the mathematical interpretation of such statements and moreover attention is paid to investigate students' awareness of the role of mathematical language when facing negation of statements.

THEORETICAL BACKGROUND

The application of functional linguistics (Grice 1975, Halliday 1985) to the learning of mathematics (cf. Ferrari 2004) has been remarkably developed in the last years and it has assigned to the "functions of languages", rather than to semantic and syntactic aspects of the standard linguistics, a role of primary importance for the understanding of difficulties related to the learning of mathematics. According to this approach, the context influences the construction of meaning. Context refers to several aspects as the relationships among participants, the purposes of communication, the mode of communication, the subject of communication, etc. The context influences the register (Halliday 1985) that is a linguistic variety based on use (linguistic resources used by an individual to express meaning related to some context and goals of a communication). Many difficulties in mathematics are due to improper use of registers or to a their improper understanding. The analyses of the responses of students by taking into account the registers adopted, allows to explain the interference between the technical language of mathematics and other languages, first of all the everyday one. The overlapping of colloquial registers with technical ones has been for a long time subject of research for the understanding of difficulties in the learning of mathematics (Mason and Pimm 1984, Ferrari 2004, Kim et al. 2005, Bardelle 2010) but, in most of these studies such interference usually has not been sufficiently analysed or explained. This work is based on the Ferrari's idea that "the registers customarily adopted in advanced mathematics share a number of features with literate registers and may be regarded as extreme forms of them" (Ferrari, 2004, p. 387). Literate registers violate the cooperation principle (Grice, 1975) that usually occur among participants in everyday communication or who share a common context. The difficulties arise both from using technical terms and from the organization of texts. For example, in a mathematical register "some" means "at least one", but in a colloquial register "some" is interpreted as "more than one but not all". The use of "some" as "more than one but not all" is an example of an *implicature*, that, in the frame of pragmatics, is the portion of the information provided by the text that follows from the assumption that it is adequate to the context rather than from its propositional content (Ferrari, 2004). This example, typical of colloquial speech at least in Italian language (some evidence on the same issue is anyway reported also in English, see e.g. Mason and Pimm 1984 and references therein) is due, in particular, to the violation of the Maxim of Quantity (Grice 1975, p. 47) according to which a communication has to be as informative as required (for the current purposes of the exchange).

THE EXPERIMENT

Subjects

The experiment has been carried out with 294 Italian science (biology, chemistry, computer science, environmental science) freshman students at the University of Eastern Piedmont in Italy. The results come from a set of questions administered to students in a written placement test and subsequent interviews. The test was administered after a two-week precalculus bridging course in order to verify students' initial knowledge. A two-hour unit of the course has been devoted to illustrate some aspects of mathematical language (such as connectives and quantifiers) within the setting of naive set theory. Both the course and the test were not compulsory, but warmly recommended. Moreover, students could achieve a bonus according to the results of the test. There were no negative consequences if students failed the test, but in that case, they were then strongly recommended to attend tutoring sessions in the first semester.

Tasks

Four types of questions were developed for the study. Students were asked to recognize the truth or the falsehood of the negation of a statement with a universal quantifier trough some diagrams. The negative sentences used in the experiment were of the kind "Not all A is B". The questions were grouped into two categories: 1-questions involving diagrams with two circles only and 2-questions involving diagrams with more than two circles (four circles). The questions have a multiple choice format with the possibility of multiple responses. The following tables present an English translation of the questions.

Which of the following diagrams (multiple choices are allowed) make the statement "Not all the circles are black" true?		
A) 	B) 	C) 

Table 1: Question 1t (**Q1t**).

Which of the following diagrams (multiple choices are allowed) make the statement "Not all the circles are black" false?		
A) 	B) 	C) 

Table 2: Question 1f (**Q1f**).

Which of the following diagrams (multiple choices are allowed) make the statement “Not all the circles are black” true?		
A) 	B) 	C) 

Table 3: Question 2t (**Q2t**).

Which of the following diagrams (multiple choices are allowed) make the statement “Not all the circles are black” false?		
A) 	B) 	C) 

Table 4: Question 2f (**Q2f**).

The sample was split into four groups of about 74 students each. Each of the four questions was assigned to one group.

Interviews

After the analyses of the written responses 14 students were individually interviewed in order to explore their understanding. The students were chosen according to the factor analyses of the written responses in order to investigate all the patterns of answers with more than 10% of frequency.

The interviews were not compulsory and explanations about the experiment were given to students before starting the interview. The interviews were semi-structured. All students were asked explanation about their answer to the written question they had to face in the admission test. Moreover, they were asked a question 1t or 2t (where the subjects had to recognize the truth of the statement), if they answered a question 1f or 2f (where the subjects had to recognize the falsehood of the statements), in the entrance test and viceversa. They were asked whether they attended the precalculus course and whether they used logical concepts while they were answering that kind of question in the entrance test. Such questions were accompanied by personalized ones aimed at explaining students’ reasoning.

RESULTS

Table 5 shows students’ written responses to the four questions. We recall that the sample was split into four groups. The groups had about the same number of students and each group had to face one of the four questions respectively. In the following tables A, B, C denote the options A), B), C) of the questions respectively, AB denotes that students had chosen both option A) and B) and so on. As a first result, questions 1t and 2t proved to be more difficult than questions 1f and 2f (34,24% versus 41,89% in 1-questions and 22,22% versus 48% in 2-questions). Secondly, different percentages are due to questions of type 1 or 2, that is, with two circles or more than two circles respectively. In Q1t and Q2t proper answers (AC) are more numerous in 1-questions that in 2-questions (34,25% versus 22,22%), while in Q1f and Q2f proper answers (B) are less numerous in 1-questions than 2-questions (41,89% versus 48%).

Item	Q1t	Q1f	Q2t	Q2f
A	1,37%	1,35%	0,00%	0,00%
B	1,37%	41,89%	1,39%	48,00%
C	60,27%	21,62%	76,39%	20,00%
AB	1,37%	28,38%	0,00%	25,33%
AC	34,25%	6,76%	22,22%	5,33%
BC	1,37%	0,00%	0,00%	1,33%
ABC	0,00%	0,00%	0,00%	0,00%
Total	100,00%	100,00%	100,00%	100,00%

Table 5: Responses by percentage to written test.

In Q1t and Q2t the most common behaviour is to answer just C, that is the diagram with both black and white circles. This behaviour is due to an implicature occurring in an everyday communication, according to which "Not all the circle are black " means that "just some circles are black and some do not". This phenomenon is well described in this interview

1 I: Why did you choose only C [question Q1t] and not also A?

2 S1: "Not all circles are black" means that there are white circles but it does not rule out that there are black circles. I would have chosen A if it was written "All the circles are not black"

This student clearly explained that the sentence "Not all the circle are black" is not appropriate i.e. cooperative to the state of affair of diagram A with all white circles.
The same implicature is also responsible of response (AB) in Q1f and Q2f. In this case the students chose diagram (A) and diagram (B) as the complementary of diagram C, in the sense that they chose (C) as the diagram that makes true the statement "Not all circle are black" and hence they ruled out C since they had to mark diagrams that makes the statement false. This behaviour is described in the interview of another student

1 I: Can you explain the reason why you chose both A and B [question Q1f]?

2 S2: "Not all circles are black" means that there are both white and black circles ... hence C makes true while A and B makes false.

It seems that the misreading of the text of the question followed by an implicature as above is the cause of response (C) to questions Q1f and Q2f. In this case all the students interviewed, who answered (C) to Q1f or Q2f in the entrance written test, declared that they misread the question. They did not read the requirement about the falsity of the statement but they answered looking for the diagrams that corresponded to the statement "Not all circle are black", i.e. that made the statements true. An example of this behaviour is given in the following interview.

1 I: Can you explain the reason why you chose C [question Q2f]?

2 S3: "Not all circles are black" means that there must be at least one white circle besides black circles.

3 I: What does the question require?

4 S3: It requires to find the diagrams that correspond to "Not all circles are black".

5 I: Can you please read again the question?

6 S3: I'm sorry…. I didn't read false…I answered to another requirement…

Probably, also the answer (AC) is due to the improper reading of the text of the question which lead them to look for diagrams that made the statement true instead of false. Unfortunately, there are not interviews supporting this interpretation, since the answer (AC) was not investigated by interviews.

Another purpose of the interviews was to understand why the questions 1t and 2t proved to be more difficult than questions 1f and 2f (34,24% versus 41,89% in 1-questions and 22,22% versus 48% in 2-questions). The following interview is interesting about this phenomenon. The interviewed student S4 answered (B) to question Q2f.

1 I: Can you explain the reason why you chose B [question Q2f]?

2 S4: Saying that "Not all circles are black" is false means that all circles are black.

3 I: Can you explain better?

4 S4: "Not all circles are black" means that the circles are white or some white and some black. The contrary is all circles are black.

5 I: Can you tell which diagrams make "Not all circles are black" true [Q2t]?

6 S4: C

7 I: So you are saying that diagram B makes the sentence false while diagram C makes the sentence true? That's right?

8 S4: Yes!

9 I: And what about the diagram A? How does it make the sentence, true or false?

7 S4: It is not relevant to the question.

From this interview it arises that the syntactic and semantic aspects of "Not all circles are black", concerning its truth or falsity, are overcome by the pragmatic one. Actually, in everyday communication sentences are often assessed related to their adequacy rather than their truth. This holds for negative sentences too, as their goals often overcome their truth. It is intended to stress how the formulation of the text of questions i.e. in terms of truth and falsity did not sufficiently draw the attention to a more mathematical context.

From the factor analysis of responses to written questions a discrepancy between 1-questions and 2-questions emerged. It seems that diagram C of 2-questions (3 black circles and 1 white circle) evoked more the use of a conversational implicature than diagram C of 1-questions (1 black circle and 1 white circle). This should be explain the higher percentage of answers C in Q2t (76,39%) than answers C in Q1t (60,27%). According to this pragmatic view the white circle of diagram C of 2-questions would be coded as an exceptional item with reference to the residual class of the black circles, and then, the statement “Not all circles are black” is more cooperative when referring to diagram C of 2-questions rather than to diagram C of 1-questions. The statement “Not all circles are black” in the context of diagram C of 2-questions seems to advice the interlocutors that but there are an exception to the fact that all the circles are black. A sentence of the kind “Not all A is B” is pointless regarding diagrams C of 1-questions. Therefore 1-questions could be evoked the use of a less colloquial register but more suitable in a mathematical context.

Finally, it is important to stress that all the students interviewed declared that they answered the questions without thinking to special aspects of mathematical language (such as connectives and quantifiers) encountered, even if briefly, in the precalculus course or described in the notes of the course. The students were not aware of the importance of the role that mathematical language plays in solving these tasks, but they settled for answering in an empiric way only.

This experiment, conducted using diagrams, in order to investigate the interpretation of negative sentences of the kind “Not all A is B”, confirmed the results obtained in the previous studies of Bardelle (2011) and Ferrari (2004), in a quite similar context but where the interpretation of negation was investigated as an effect between two sentences, that is the students were asked to recognize equivalent sentences. The percentage of the results are very similar and the analyses of all these experiments identified the overlapping of everyday registers with the mathematical ones as a typical students’ behaviour. In particular, this experiment has confirmed that verbal component of statements of the form “Not all A is B” heavily affects their interpretation (meaning) according to their conventional use in the colloquial register, that is “not all” is conventionally used to say “some do” and “some don’t” that is “not all A is B” conveys the implicature “There are some A that surely is B”. This behaviour is related in particular to the violation of the maxim of Quantity (Grice 1975); student S4 showed clearly this fact arguing that diagram A (all white circles) is not informative about the truth or falsehood of the statement “Not all circles are black”; also student S1 showed it arguing that “Not all circles are black” is not related to diagram A but that a sentence of the kind “All the circles are not black” would have been relevant.

This experiment has confirmed that the interpretation of verbal statements in a mathematical setting may happen to be based on everyday context and not on a mathematical one and, since some verbal statements, as those presented here, seems to evoke meanings conflicting with the mathematical ones, particular attention has to be paid in their use in the teaching of mathematics.

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FORMAL AND INFORMAL LANGUAGE IN MATHEMATICS CLASSROOM INTERACTION: A DIALOGIC PERSPECTIVE

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A perennial concern with the issue of informal and formal language in mathematics classrooms has led to an assumption that students must move from informal to formal mathematical expression as they learn mathematics. In this report, I draw on a Bakhtinian, dialogic perspective to examine formal and informal language in an elementary school mathematics classroom in Québec, Canada. The students, who are second language learners, are learning about polygons. I argue that informal and formal language are both necessary, and are always in tension.

INTRODUCTION

Many mathematics curricula now include an explicit focus on the development of mathematical language. Such development is generally conceptualised as a shift or transition from students' informal expressions of mathematical thinking, to communication using more standardised mathematical language. The Ontario elementary school mathematics curriculum, for example, suggests that in Grade 4, students should "communicate mathematical thinking […] using everyday language, a basic mathematical vocabulary, and a variety of representations" (OME, 2005, p. 65). By Grade 8, students are expected to use "mathematical vocabulary and a variety of appropriate representations, and observ[e] mathematical conventions" (p. 110). The use of 'everyday' language and 'basic' mathematical vocabulary has been replaced by 'mathematical vocabulary', implying a direction for development from the former to the latter. While this kind of approach seems reasonable, human language use is rarely so straightforward. In this report, I consider the nature of the relationship between informal and formal language in mathematics classroom interaction. To do so, I focus specifically on interaction in a classroom featuring second language learners of mathematics. The role of informal and formal language in such classrooms becomes particularly salient and significant.

FORMAL AND INFORMAL LANGUAGE IN MATHEMATICS CLASSROOMS

It is clear from the literature that the relationship between formal and informal language in mathematics is not entirely straightforward. For example, Pimm (1987) suggests that the appropriation of everyday language within the mathematics register may be a source of confusion for students. A word like 'difference', for example, has a common everyday meaning ('not the same') and a more specific mathematical meaning ('the result of a subtraction'). This observation suggests that there is a degree of overlap between informal, everyday language and mathematical language as it is commonly used. At the same time, research on mathematical discourse has increasingly highlighted the need to broaden the scope of inquiry to include a variety of

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meaning-making resources, including symbols, gestures and a variety of languages and language practices (e.g., Moschkovich, 2008). This work also shows that there are no clear-cut boundaries between mathematical language and everyday language; mathematical language is never entirely formal.

Research in classrooms featuring second language learners or bilingual or multilingual learners has also highlighted the issue of the development of formal mathematical language. In Adler's (2001) study in South Africa, for example, teachers reported a dilemma they experienced about whether to allow their multilingual students to express their mathematical ideas using informal language, or whether they should interrupt students' explanations to teach them formal, standardised mathematical language. This dilemma reflects a more widespread tension that has been observed in several other contexts. In multilingual classrooms, this tension interacts with tensions between students' home languages and the language of instruction and between language policy and classroom practice (see Barwell, 2012).

Both Setati and Adler (2001) and Clarkson (2009) have suggested that in multilingual or second language classrooms, students need to move along three different dimensions: informal to formal mathematical language; spoken to written mathematics; and home language to the language of instruction. This kind of approach is prevalent in the literature more generally and is apparent in the Ontario mathematics curriculum, as discussed above. This approach is productive and has generated valuable suggestions for classroom practice. It has not, however, interrogated sufficiently the nature of the *relationship* between informal, everyday expressions of mathematical thinking and more formal mathematical language. While there is recognition that the relationship is complex and that there are no clear boundaries, research is dominated by a model of transition from one to the other.

THEORETICAL FRAMEWORK: BAKHTIN'S DIALOGIC THEORY OF LANGUAGE

Although Bakhtin is primarily known as a literary theorist, his work includes a highly developed theory of language (see, in particular, Bakhtin, 1981). His theoretical perspective is wide-ranging and difficult to reduce to a list of simple tenets. For this report, I will summarise a few key ideas that are particularly relevant to the issue of formal vs. informal language. First, Bakhtin's theory of language is *dialogic*. This means that language use is dynamic and situated. In particular, any utterance is understood to be a response, one turn in an ever-unfolding chain of utterances, which "cannot fail to be oriented toward the 'already uttered,' the 'already known,' the 'common opinion' and so forth" (Bakhtin, 1981, p. 279). Moreover, each utterance is in dialogic relation with myriad alternatives, whether in terms of alternative pronunciation, choice of words, formulations, choice of language and so on. In this sense, dialogicality is a relational perspective on language. Meaning is made through the relations between sounds, words and utterances, not through these things 'in themselves'. Furthermore, the relationality of language is always towards what Bakhtin often calls 'an alien word', that is, difference or otherness.

Second, language precedes us. This means that we must always use the words of others, alien words, to express ourselves. But these words continue to express much that we may not intend:

> Language is not a neutral medium that passes freely and easily into the private property of the speaker's intentions; it is populated – overpopulated – with the intentions of others. Expropriating it, forcing it to submit to one's own intentions and accents, is a difficult and complicated process. (p. 294)

Hence any utterance is not simply an expression of an individual's idea; it expresses a host of 'other' ideas that derive from preceding usage and must be understood in the light of preceding utterances. Furthermore, since we can never escape from being in relation to the language that precedes us, this language in some sense defines who we are. For Wegerif (2008), "for each participant in a dialogue, the voice of the other is an outside perspective that contains them within it" (p. 353) (see also Radford, 2012).

Third, Bakhtin's theory of language includes a continual tension between a centripetal force towards uniformity ("unitary language") and a centrifugal force towards *heteroglossia*, which refers to the tremendous variety of language-in-use. This variety is related to social differences: "languages of social groups, 'professional' and 'generic' languages, languages of generations and so forth" (Bakhtin, 1981, p. 272), including the languages of mathematics, as well as the languages of social class, race, region and so on. The tension between informal and formal language, observed in many mathematics classrooms, is an instance of Bakhtin's more general tension. Formal mathematical language amounts to a unitary language, the idea of which exists in tension with the diverse forms of expression that students may use. Heteroglossia is an important aspect of dialogicality, since it is variation that leads to the continual interplay of different ideas, perspectives and meanings. As Holquist emphasises, however, "heteroglossia is a plurality of relations, not just a cacophony of different voices" (p. 89).

Finally, in the context of education, Wegerif (2008) contrasts a dialogic perspective with a dialectic perspective (which he associates with a neo-Vygotskian perspective on learning). From a dialectic perspective, differences must be overcome, synthesised into something new (and so, arguably, tending towards uniformity). It is, arguably, this perspective that informs the idea of a uni-directional process from informal to formal language. From a dialogic perspective, by contrast, differences open up possibilities for making meaning; the process is no longer necessarily uni-directional.

RESEARCH CONTEXT: LA CLASSE D'ACCUEIL

The work reported in this paper is from a project designed to examine mathematics learning in different second language settings in Canada, a country with two official languages, English and French. In this report, I refer to one of these settings: a sheltered class for new immigrant learners of French, known as a *classe d'accueil*. In the province of Québec, new immigrant children must attend school in French. If they do not speak French, they attend a *classe d'accueil* for up to a year to learn enough French to join mainstream classes.

I visited a Grade 5-6 class (10-12 years), along with a research assistant, towards the end of the school year, by which time the students had acquired a degree of basic French. Between us, we attended all the students' mathematics lessons for a period of three weeks. We made audio-recordings of whole-class interaction, as well as some small-group interaction. We also collected examples of students' written work and interviewed small groups of students and the teacher. The class comprised 18 students with a variety of origins and language backgrounds, including South Asian, West African and South American. The teacher reported that the main aim of the class was to prepare the students for school life in Québec and to learn to speak and think in French. In mathematics, she focused on vocabulary. All mathematics texts used in class were in French and the teacher insisted on the use of French at all times.

The analysis reported below examines the first of a sequence of three lessons introducing some language and concepts in geometry. During these lessons, the main emphasis was on learning the concepts and words for polygon, non-polygon, convex and non-convex. For this report, I focus particularly on the introduction and application of the words for polygon and non-polygon. Students were also introduced or reintroduced to the words for: straight, curved, quadrilateral, open, closed. Many of these words have broader everyday connotations. The names of basic shapes (e.g., circle, rectangle, triangle) seemed to be familiar to students already. Using principles from conversation analysis, participants' orientations were used to language that they themselves saw as formal or informal.

POLYGON OR NON-POLYGON?

Prior to introducing the terms for polygon and non-polygon, the teacher took the students through two activities. First, she worked with the class on different ways of classifying the students in the room, including, for example, students wearing jeans vs. those not wearing jeans. The teacher referred to the resulting two groups as "les jeans" and "les non-jeans", with emphasis on "non". The discussion therefore introduces the students to the use of the prefix "non". Second, the teacher handed out packets of regular and irregular shapes to students and asked them to work in small groups to sort the shapes into two distinct sets. She invited different groups to show how they had separated their shapes and to explain how they had divided them. Students often struggled to explain their thinking in a way that the rest of the class and the teacher could make sense of, such as when they said (in French) "*I don't know how to say it in French*" or "*how do you call those*" or pointed, traced straight or curved lines with their hands or used words like "*this*" or "*like that*".

Next, the teacher introduced the terms "polygone" and "non-polygone" and drew examples of each on the blackboard. She asked the students to examine her drawings and deduce what a polygon is and what a non-polygon is. She said [1]:

> ok donc il y avait plusieurs façons de classer les figures […] une des façons (.) la plupart l'ont trouvée (.) il y a des figures qu'on appelle les (.) polygones (2) et les autres (.) qu'on appelle les (.) non (.) polygones […] ok c'était bon c'était très bon votre façon aussi on va en reparler plus tard (.) aujourd'hui on va plus voir les autres (.) donc polygone je vais te dessiner des exemples de polygone […] dans les non polygones il y a ça ça ça (6) avec mes

dessins (.) et tu capable de m'expliquer qu'est-ce que ça veut dire (.) polygone (.) et qu'est-ce que ça veut dire non polygone? (.) explique-moi la différence (.)

ok so there are lots of ways to sort shapes […] one of these ways (.) most of you found it (.) there are shapes that we call (.) polygons (2) and the others (.) that we call (.) non (.) polygons […] ok your way was good it was very good as well we're going to come back to it (.) today we're going to look more at the others (.) so polygon I'm going to draw you some examples of polygons (…) for non polygons there's this this this (6) with my drawings (.) can you explain what it means polygon (.) and what does non polygon mean? (.) explain the difference for me (.)

The teacher's introduction features several sets of differences: between the students' distinctions and the teacher's; between the two groups of shapes that the teacher draws on the blackboard; and between polygons and non-polygons. These differences are in dialogue with each other; making sense of the word and the concept "polygon" arises through the differences between the two groups of shapes on the blackboard, between the different ways of classifying shapes that preceded this moment, and so on. This approach captures a little of the tension between formal (unitary) and informal language (heteroglossia) to which Bakhtin refers. The teacher acknowledges the students' ways of classifying shapes, saying they were "very good", but sets them to one side in order to focus on the more formal terms of polygon and non-polygon. As such, she implicitly constructs the students' classifications and language they use to express them as less formal.

In the next few turns, the teacher elaborated on the meaning of polygon and non-polygon, with reference to the examples on the blackboard. She emphasised, in particular, the need for the sides to be straight and the shapes to be closed, pointing to examples on the blackboard as she spoke:

les polygones c'est une ligne (.) droite (.) des lignes droites une ligne qu'on appelle brisée (.) ^ça veut dire quand elle est brisée comme ça c'est quand il y a plusieurs côtés^ et fermée (2.5) s'il y a les lignes courbes ou si la ligne elle n'est pas fermée (.) automatiquement c'est un non polygone (.)

polygons it's a straight line (.) straight lines a line that's called broken (.) ^that means when it's broken like that it's when there are several sides^ and closed (2.5) if there are curved lines or if the line isn't closed (.) automatically it's a non polygon (.)

Again, the interaction between informal and formal language is apparent. For example, the teacher uses the formulation "une ligne brisée" (literally 'a broken line' but meaning rather something like a 'jointed' line). This formulation appears on the worksheet she gave out just after the above quotation. The formal definition on the worksheet reads "Un polygone est une ligne brisée fermée, tracée sur une surface plane." (*A polygon is a broken, closed line, drawn on a plane surface*). The teacher also explains 'brisée', using more informal language ("*when it's broken like that it's when there are several sides*"). Both formal and informal language are marked: the word "brisée" is preceded by "*that's called*" indicating a new term; the subsequent clarification opens with "*that means*", which signals a more informal formulation.

The worksheet also featured a set of shapes, labelled A-L. The students were asked to list each shape in a table, in columns labelled "POLYGONE" and

"NON-POLYGONE". After the students had worked on this task, the teacher went through the shapes with the whole class. For each shape, the exchange had a similar structure: the teacher nominated a student and stated which shape; the student stated if they thought it was a polygon or a non-polygon; the teacher asked why; the student provided a reason; the teacher revoiced or clarified with the student, in some cases leading to a new classification. For example:

Teacher	le *C* [E38] le *C* tu l'as mis dans quelle colonne?	C *[E38] which column did you put* C *in?*
E38	non polygone	*non polygon*
Teacher	non polygone (.) pourquoi?	*non polygon (.) why?*
E38	parce que c'est (.) pas ligne droit	*because it's not a straight line*
Teacher	ce n'est pas ligne droite c'est ligne (1.5) cou:rbe (.) garde je vais t'écrire ici (.) une ligne ça peut être droite (.) ou cou:rbe (2) k? bravo	*it's not a straight line it's line (1.5) look I'll write it here for you (.) a line can be straight (.) or curved (2) okay? well done*

The teacher's revoicings make small adjustments to the students' formulations. These adjustments are at different times to grammar, syntax, pronunciation, word choice or to mathematical distinctions. In the above exchange, E38 says "ligne droit" (the final t is silent). "Ligne", however, is feminine, so the standard adjectival form that follows would be "droite" (the t is sounded). The teacher's revoicing adjusts this and then elaborates, adding in "curved" as the contrast with straight. She then writes the two words on the blackboard. Over the course of 12 exchanges (for shapes A-L), some abbreviation occurred: students accounted for their classification without being prompted, for example. One student explained that her shape is polygon "*because the lines are straight and closed*" to which the teacher replied "*perfect*".

The lesson moved on to look at other attributes of geometric shapes. At the end of the period, however, the teacher asked a student to define polygon:

E53	un polygone c'est comme forme qu'il a des lignes droites (.) et il n'y a pas de trous	a polygon is like shape that it has straight lines (.) and there aren't any holes
Teacher	ok lignes droites fermées (.) si je te demande c'est quoi un non-polygone: [E54]	ok straight lines closed (.) if I ask you what's a non polygon [E54]
E54	non polygone c'est comme ah il y a comme il y a un trou dans le carré ou les lignes sont c-courbes	non polygon is like ah there's like there's a hole in the square or the lines are curved
Teacher	donc il y a une ligne courbe ou une ligne qui n'est pas fermée	so there's a curved line or a line that isn't closed

In these brief exchanges, two students give an account of their interpretation of the meaning of polygon and non-polygon. Again, the interaction between formal and informal language is apparent. Both students use the informal word "*hole*", for example, while the teacher revoices each time using the more formal word "closed".

DISCUSSION

In the lesson described above, the language of mathematics precedes the teacher and her students. They must grapple with it in an attempt to make it submit to their intentions. For the students, this struggle includes their encounter with the otherness of new words, new distinctions or new ways of using language. The teacher also encounters otherness, in the students' diversity, of accents, pronunciations, non-standard French and their informal expressions of mathematics. According to Wegerif (2008), this otherness "contains them within it". In the above lesson, the students must try to see things as the teacher does. When she asks "*polygon or non-polygon?*", they must use her formal terms and her distinction to respond. The question contains them; it reflects a centripetal force towards a particular way of seeing shapes and doing mathematics. By the same token, however, when the students reply, using a more informal language of holes and lines and gestures and pointing, the teacher must try to see things as they do. Their utterances, then, reflect a centrifugal force that contains the teacher. Throughout, there is a dialogue between the two. Through this dialogue, the students and the teacher come to use language in new ways: the language of both changes through the lesson in response to the utterances of the other.

A dialogic perspective on formal and informal language in mathematics classrooms highlights a relationship between formal and informal that is not uni-directional. Rather than steady progress from informal to formal, these students work at both. The teacher, too, must make skilful use of varying degrees of formality. Of course, students need to learn formal mathematical language as part of learning mathematics, but this does not mean that informal language disappears; nor is it simply a scaffold to reach more formal language. Both are necessary; they will always be in tension.

Note

1. Transcription: (.) or (2) for pauses, ^ ^ for whispering, […] for omitted parts. The translations are my own.

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THE IMPACT OF PARTICIPATION IN A COMMUNITY OF PRACTICE ON TEACHERS' PROFESSIONAL DEVELOPMENT CONCERNING THE USE OF ICT IN THE CLASSROOM

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Abstract *Although technology can be used as an important tool in teaching mathematics, its integration into teaching methods lags behind. To support teachers in their use of Information and Communication Technologies (ICT) in the mathematics classroom, a Community of Practice has been set up which consisted of the researchers and twelve teachers teaching eight grade students. The influence of the community on teachers' professional development has been evaluated. Analysis shows that throughout the project teachers have become more confident in their use of ICT and more aware of the importance of teacher guidance when ICT is used to support student learning. Evaluation of the enterprise shows that teachers' development has not been optimally supported by the community.*

INTRODUCTION

In the last ten to twenty years, digital technology has evolved from being solemnly a gadget to being an essential part of everyday life. This development greatly influenced education, specifically mathematics education, which becomes apparent in the growing use of smartboards and graphic calculators in the classroom. The National Council of Teachers of Mathematics' position statement claims, "Technology is an essential tool for learning mathematics in the 21st century, and all schools must ensure that all their students have access to technology," (NCTM, 2008, p. 1). Central to this use of technology in the classroom is the guidance by the teacher. Teacher practice significantly affects student learning (Ely, 1996), and teachers "...play [an] important role in [determining] the time, place, and manner for technology to be engaged in the classroom" (Brown & Cato, 2008, p. viii).

Although technology can be used as an important tool in teaching mathematics, its integration into teaching methods lags behind. According to Sabra and Trouche (2013), new technology creates new needs and complicates the work of teachers. Therefore, it is necessary to support teachers' professional development concerning the use of ICT in the classroom. Wenger states that Communities of Practice can greatly support learning of both the individuals and the community. To explore this support, a Community of Practice was formed to support teachers in their use of ICT in the classroom.

THEORETICAL FRAMEWORK

Wenger (1998) advocates the emphasis on collective learning as a substantial part of adult – and non-adult – learning. This collective learning results in "…practices that reflect both the pursuit of our enterprises and the attendant social relations," (Wenger,

2013. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 81-88. Kiel, Germany: PME.

1998, p. 45). A community in which these practices are central is defined as a Community of Practice. Communities of Practice can be described using three dimensions: mutual engagement, a joint enterprise, and a shared repertoire (Wenger, 1998). These three dimensions formed the thread of this study and will be explicated below, where Wengers' definitions are extended with notions from more recent research.

Mutual engagement is an important source of coherence within the community. Participants need to feel included in what matters, giving them a sense of belonging. Besides this individual need, community engagement needs to be fostered by diversity and partiality, because mutual engagement involves not only our own competence, but also the competence of others. In time, this engagement will connect participants to each other in ways that are diverse and complex, forming relationships which will reflect the complexity of the group's collective actions.

The second dimension, a joint enterprise, gives participants a shared purpose, enlarging the sense of coherence within the community. The goal of the enterprise should be the result of a collective process of negotiation, reflecting the full complexity of the mutual engagement. During the realization of this goal, the connection between the community and the 'real world' is made by the production of boundary objects. These are products made by, or within, the community which can be used outside the community. It is important that all participants are able to equally contribute to the realization of the goal. In doing so, two aspects have to be taken into account: participants should consider more than their own perspective, and they should feel mutual accountability (Kisiel, 2009).

The third dimension, a shared repertoire, is the result of the different activities which are carried out to pursue the enterprise's goal. The origination and development of this shared repertoire can be described by a process called "Community Documentational Genesis" (Gueudet & Trouche, 2012). This is an extension of the process of Individual Documentational Genesis (Gueudet & Trouche, 2008). Documentational Genesis represents the process through which an individual uses a certain resource within his or her scheme of utilization and so turns it into a document. This is a dynamic and cyclic process. Community Documentational Genesis arises when Documentational Genesis is considered within a Community of Practice. The result of this process is Community: a repertoire of shared documents including resources, knowledge, and practices (Sabra & Trouche, 2013; Wenger, 1998).

METHODS

This study was part of a larger research project called the DPICT project (Drijvers, Tacoma, Besamusca, Doorman, & Boon, 2013a, 2013b). During the school year 2011-2012, six pairs of teachers were asked to use three pre-designed mathematics modules in their eighth grade classrooms. The modules were designed on a Digital Mathematics Environment (DME). During this period, a Community of Practice was set up to support the teachers. This community consisted of the six pairs of teachers, four researchers, and two master's students. Interaction took place through five face-to-face meetings and communication on a digital platform called Moodle.

To foster mutual engagement, teachers were included in everything that mattered for the project. They implemented and tested the Modules, indicated design errors (which were consequently fixed by the designer), and led the discussions in the meetings with their findings and opinions. Diversity and partiality were established by choosing the teacher-pairs from six different schools throughout The Netherlands and letting teachers choose their own approach when using each module. Several teacher activities were analysed to evaluate the teachers' mutual engagement. First, blog writing activity was analysed: teachers were asked to post a blog on Moodle for every lesson they taught in which they used the DME. A count was kept of the blogs written per module. Second, Moodle activity was analysed: teachers could visit a forum, post additional documents, and read documents posted by either the researchers or other teachers. A count was kept of the different pages which the teachers visited in Moodle. Finally, teachers' opinions were evaluated. Teachers were asked to give their opinion on the activities within the community in a questionnaire at the end of the project. These opinions were analysed and linked to their activity.

The quality of the joint enterprise was analysed by evaluating the enterprise's goal and the related individual and communal activities, including the production of boundary objects. In the questionnaire at the end of the project, teachers were asked to give their view on the enterprise. These views were analysed and related to the evaluation described above.

The analysis of the shared repertoire was focused on the processes of Individual and Community Documentational Genesis, specifically the development of knowledge and attitudes. A list of topics of discussion was extracted from recordings of the meetings and from the written blogs. Subsequently, the topics judged as most relevant by the researchers have been explored in depth.

The analysis of the Community Documentational Genesis was based on the development of the five chosen topics in the meetings. The teachers sparsely used the forum on Moodle. Therefore, this data-source was not included in this analysis. The analysis of Individual Documentational Genesis was based on the development of the five chosen topics in the blogs. The data from the blogs were supported by an ICT questionnaire which focused on teachers' attitude towards ICT and interviews focused on what teachers encountered when using ICT in the classroom. To complete and verify the resulting picture, the teachers were asked to complete a final questionnaire at the end of the project. The Community Documentational Genesis has been linked to the Individual Documentational Genesis, similar to research done by Sabra and Trouche (2013). The connection between the Genesis and actual teacher practices lay outside the focus of this article. The interested reader is referred to Drijvers et al. (2013a) and Drijvers et al. (2013b).

RESULTS

The results are listed below according to Wenger's (1998) three dimensions.

Mutual Engagement

During the project, teachers were asked to write a blog for every lesson they taught in which they used the DME. This should have led to a total of about 100 blogs per module. The amount of written blogs however, is far lower, ranging from 86 blogs for the first module, to 57 for the second, and 58 for the third. Teachers lost interest in writing their blogs after the first part of the project. This is supported by the teachers' evaluation of the blogs in the questionnaire, which showed a relatively low opinion of the added value of the blogs.

Teachers' activity on Moodle also lessened as the project progressed. The visits in the months of September-November were nearly twice as frequent as the visits in the months of December-June. Of the different aspects of Moodle, teachers' visits of the blogs were most frequent. Apparently, although teachers did not value the blogs much, this only impaired their writing activity and did not keep the teachers from reading them frequently. This can be explained by time-constraints, an impairing factor which teachers mentioned more than once during the meetings with regard to their blog writing. Teachers only sparsely visited the Moodle forum, and almost never visited the additional documents posted by the researchers and other teachers. Their opinions in the questionnaire support this fact, showing a relatively low appreciation of the forum and the additional documents.

Joint Enterprise

For the teachers, the goal of the enterprise was to learn how to use ICT – or more specifically the three modules designed in the DME – in the classroom. A secondary goal for the teachers was to investigate the added value of the use of ICT in the classroom. Individual activities related to the teachers' primary goal include the preparation of lessons in the blogs, exploring the different features of the DME, and using the modules in the classroom. The writing of blogs can also be considered a communal activity, dependent on the degree to which teachers keep their peers' perspectives in mind while writing their blogs. Other communal activities include participation on the forum, reading peer blogs, and participating in the face to face meetings.

During the project, the relations between the members of the community gradually shifted. At the start of the project, the researchers intended for authority between members to be equally divided. As the project progressed, however, the power shifted partially, making the researchers the authority figures. This change was unintended and likely due to the members settling into their basic roles. In other words, the researcher, who initiated and guided the project, was the natural authority figure, while the teachers, who applied for the project, naturally followed his lead.

Boundary objects were a missing element in the community. During the project, teachers could read and post documents on Moodle. These documents ranged from articles on the theoretical framework supporting the research to actual lesson plans and study guides. Teachers only sparsely read and posted these documents, which indicates their lack of feeling of mutual accountability. Emphasizing this point, only two teachers took the opportunity to post documents on Moodle. During the meetings, most discussions lingered on ideas and opinions on the use of ICT in the classroom, not making the step to concrete lesson-plans. This, again, points to a lack of boundary objects, which normally form the connection between the community and the 'outside world' (Wenger, 1998). Tasks associated with generating these objects were missing, although these are an important part of the community (Gardner, 1994). A more thorough description of the content of the discourse is given in the paragraph below on the *Shared Repertoire*.

In the questionnaires, teachers indicated that they felt supported by the community during the project. They were most positive about the contact with colleagues, the opportunity to share experiences, the technical support of the researchers in using the DME, and the opportunity to use the ICT-modules which the project offered. As stated before they did not fully appreciate the added value of the blogs and documents, which showed in their use of these resources.

Shared Repertoire

The analysis of the shared repertoire focused on the development of the knowledge and attitudes of the teachers on five topics: computer versus paper, feedback, tests, DME-technical, and technical facilities.

The topic *computer versus paper* has been prominent during all the meetings, having been discussed almost thrice as much as other topics. It concerns the balance which teachers have to make between letting the students work on the computer, letting the students work out of their books, and guiding the students in their work on the computer. At the start of the project, teachers were undecided on how they would make this balance, even considering letting students work independently on the computer. In both the Individual and Community Documentational Genesis, it becomes apparent that towards the end of the project teachers could better enunciate the balance they chose. In the final questionnaire, they emphasized the need for teacher guidance when working with computers, leaving their original idea where students' working independently on the computer was possible. This development has also been found in the research by Abboud-Blanchard and Vandebrouck (2012).

The topic *Feedback* concerns the feedback which the DME offers on student answers. The Community and Individual Documentational Geneses show that, during the project, teachers became more sceptical about the value of this feedback, stating that it limits students' independence of thought and understanding more than expected. The final questionnaire shows that they still appreciated the feedback, stating that it motivates students and lets them work independently.

The topic *Tests* considers the choice which teachers have to make between using either a digital or paper test. During the course of the project, half of the teachers chose to use a digital test at least once. Both the meetings and the blogs show that, when choosing between paper and digital tests, teachers consider the way students have practiced and how they will be tested in their final exams. After use of the tests, teachers were sceptical on the grading done by the DME. Often they did not agree with the points assigned, increased their total revision time. This discovery resulted in discussions on the form of the digital tests, for which a more deterministic form, which can be graded better by the DME, might be better suited. In the final questionnaire, two teachers stated that they learned that the choice for either using digital or paper tests is dependent on what you want to know. Well performed digital testing is more deterministic of nature than paper testing, which gives the teacher more insight into student understanding.

DME-technical represents the technical issues concerning the DME, including activities such as logging in and creating accounts. This topic was only discussed in the initial meeting, which inhibits the analysis of the topic development describing the Community Documentational Genesis. Analysis of the blogs, however, shows that during the project teachers became more confident in their use of the DME, solving problems easier and faster. This Individual Documentational Genesis is confirmed by the results from the questionnaires and interviews.

The last topic, *Technical facilities*, concerns the technical facilities which the school offers. Analysis of the Individual and Community Documentational Genesis shows that teachers became more and more confident in their use of the facilities, solving problems easily even when facilities were lacking. The only impairing factor which teachers could not overcome was the infrastructure of the classroom, the location and formation of computers in the classroom sometimes greatly influenced their lessons. The increase of confidence and capability to solve problems has also been found in the research by Abboud-Blanchard and Vandebrouck (2012)

CONCLUSION AND DISCUSSION

The goal of this study was to evaluate the use of a Community of Practice to support teachers' professional development. Analysis of teachers' engagement within the community showed that as the project progressed, they did not fully utilize the available methods for support. This could be due to many factors, of which some follow from the analysis of the joint enterprise. Boundary objects were sparse, as neither the teachers nor the researchers fully recognised the value of these documents. A possible reason for this is that the teachers did not have enough feeling of ownership over the project, a result from potentially unevenly distributed authority. Without full responsibility, teachers did not feel fully accountable for the different tasks performed within the community.

Analysis of the development of knowledge and attitudes showed that the Individual Documentational Genesis was in accordance with Community Documentational Genesis. To evaluate the influence of the Community of Practice on teachers' development, however, a causal relation is needed: a connection which shows that the community discourse directly influences the knowledge and attitudes of individuals. Such a connection was not found in this study. In contrast, evidence for such influence was found in similar research done by Sabra & Trouche (2013), a project with a greater emphasis on boundary objects. In that project, research instruments were more directed at exploring the influence of the Community of Practice, as for example reflections by teachers on all the communal activities (Sabra & Trouche, 2013).

When broadening the search to an overall influence of community activities on individual thinking, more examples are found. The theoretical evaluation of articles by Voogt et al. (2011) is most relevant in the context of this research. A causal relationship was found between community activities and teacher change, which is defined by knowledge, beliefs and attitude (Clarke & Hollingsworth, 2002). The main difference between the articles researched by Voogt and the study presented here is the clear existence of boundary objects in Voogt's research, formed in that case by the curriculum.

With respect to this project, two improvements could be made which may make it possible to find a causal relation focused on teachers' professional development. First, with regard to the data, more should have been gathered on teacher practices, such that a development of their practices could be thoroughly mapped and linked to their Individual Documentational Genesis. Second, with regard to the setup of the intervention, the most communal aspects of the Community of Practice (the meetings) could have focused more on the actual practices, the boundary objects. By this, the content of the community practices and individual practices would be more congruent, and links between the Documentational Genesis and practices would be more apparent.

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A TRIPARTITE COOPERATION? THE CHALLENGES OF SCHOOL-UNIVERSITY COLLABORATION IN MATHEMATICS TEACHER EDUCATION IN NORWAY

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One goal of Norway's new primary teacher education programme of 2010 was improved school placement: the relationship between the teacher education institution, practice schools and pre-service teachers was to be formalized as a tripartite cooperation. However, in the area of mathematics education, cooperation is not straightforward: tensions arise because of pre-service teachers' prior experience and beliefs, and differences between university college training and school practice. This paper reports on questionnaire data and focus group interviews with first-year pre-service teachers and their mentors following school placement. It illustrates the complexity of the partnership and its impact on pre-service teachers' professional development in the area of mathematics.

BACKGROUND: THE SCHOOL-UNIVERSITY-PRE-SERVICE TEACHER PARTNERSHIP

As mathematics teacher educators in Norway, we are obliged to focus on supporting an idealised tripartite cooperation between teacher mentors, pre-service teachers and our university college (hereafter HiOA). Based on national guidelines, HiOA developed a plan for in-school placement, focusing on how to share responsibility for pre-service teacher education between educators at HiOA and teacher mentors in partner schools. This shared responsibility is underlined by the joint development of the pre-service placement plan by teacher educators, mentors and pre-service teachers.

During the first year of the 4-year programme, the overall focus is on the teacher's role. However, students' personal epistemologies of mathematics – what mathematics is, and how it is developed in teaching and learning – frequently associate it with memorized facts and rules, solution speed as an indicator of ability which is fixed and which cannot be acquired/improved through effort, and the equation of mathematical truth with teacher approval (see de Corte, Op't Eynde & Verschaffel, 2002; Schoenfeld, 1989; Smestad et al, 2012). Such beliefs are associated with 'transmissionist' rather than 'connectionist' styles of teaching (Pampaka et al, 2012).

While the university attempts to challenge such beliefs, the impact of school placement can force a return to earlier embedded ideas, particularly when assessment, testing and accountability are high on the agenda. Both pre-service teachers and teacher educators can experience a number of tensions between school practice and university theory/practice. Nolan (2012) reported on conflict between support for inquiry-based pedagogies at university level, and instrumentalism in practice schools. She argues that

2013. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 89-96. Kiel, Germany: PME.

this is not just due to the role of accountability and assessment in schools, but also to pre-service teachers' educational habitus and cultural routines associated with teaching: 'every adult knows what teaching and learning should look like because he or she has spent thousands of hours as a student in school' (Bullock & Russell, 2010, p. 93, cited in Nolan 2012). Allen (2009) also found that beginning teachers privileged what they had learned on placement rather than university theory.

Goos (2009) analyses the gap between what pre-service teachers are taught at university and what they actually do when they teach, focusing on the need to understand how they *interpret* their teacher education programs, how (and why) they *appropriate* certain aspects of those programs, and the nature of the different *influences* on the execution of their teaching plans. So, for example, Arvold (2005), like Nolan, uses the idea of *habitus* as an explanatory device, but in this case to argue that pre-service teachers attend to different aspects of their teacher education programs and make sense of them differently, through the lens of their prior experience of being taught mathematics. Bednarz and Proulx (2005) also suggest that pre-service teachers appropriate different things from their teacher education courses, resulting in different views of what they about, which are in turn reflected in their own teaching practice.

In this paper, we examine the relationship between theory and practice held by the different partners involved in the practicum. We focus on the tripartite cooperation in the early stages of the project, addressing the following research question: How do pre-service teachers and their mentors perceive the connection between what pre-service teachers are taught about mathematics education in University College and their learning from practice within the school placement?

Our analysis discusses the challenges of school placement, from the points of view of both pre-service teachers and their mentors. We will suggest that pre-service teachers do not necessarily take on the intended messages of their university teaching, partly because these are filtered through their prior experience, but also because of the difficulties of translating theory into practice when faced with diverse classroom demands. We also explore how school placement experience plays a role in pre-service teachers' development as they reflect on these tensions.

METHOD

Two hundred and eight first-year pre-service teachers at HiOA completed questionnaires after their school placement in 1st–4th grade. Information was gathered on the influences of school and HiOA training on their teaching practice, and their perceptions of mathematics and mathematics teaching and learning. For comparison, their 46 teacher mentors completed questionnaires covering their experiences as mentors, and their perceptions of mathematics teaching and learning, and their mentees' performance as teachers. Questionnaires comprised a number of statements requiring 5-point Likert scale responses, and also 3 free-text questions. In these, pre-service teachers were asked to describe a practice situation where (1) they benefitted from learning on their mathematics course at HiOA, and (2) they benefitted from learning from their teacher mentor. Mentors were asked 2 parallel questions about mentees' use of learning from HiOA and from themselves. Question 3 asked both pre-service teachers and mentors to describe the challenges for pre-service teachers of using HiOA learning in practice.

Fourteen teacher mentors also participated in one of 2 focus groups, in which they were asked to reflect on the teacher mentor role. Similarly, 25 pre-service teachers formed five focus groups, in which they were asked to reflect on the challenges of their school placement, on their own development as a teacher of mathematics and on the role of their teacher mentors. Including focus groups in the methodology enabled a broadening of the analysis to extended reflections about participants' experiences in the placement partnership.

Analysis

The Likert-scale data were coded on a 5-point scale ("strongly disagree" = 1 and "strongly agree" = 5), and comparisons between pre-service teacher and teacher mentor responses analyzed using Mann-Whitney U tests. The free text data and the focus group data were analyzed thematically, in order to identify the discourses of mathematics learning and teaching which participants drew on, and their perceptions of connections between theory and practice. We blend our analysis of the quantitative and qualitative data in the following sections.

Teaching and learning in university college and school

Following on from the suggestion that pre-service teachers do not necessarily take what teacher educators intend from their courses, we were interested to understand whether pre-service teachers attributed what they learned and did during their placement to their HiOA experience or to their workplace learning with their teacher mentor. We were also interested to explore the university-school partnership link by comparing their responses with those given by the teacher mentors to parallel questions. Analysis of these free text responses and related Likert-scale scores identified some interesting mismatches, two of which we describe here.

The first of these involved mismatches regarding the use of manipulatives (physical models) in teaching. Forty-nine per cent of pre-service teachers recorded this as a technique learned from their HiOA course, and 15% said they had learned it from their teacher mentor. However, teacher mentors took a different view: only 15% reported use of manipulatives as something their pre-service teachers had learned at HiOA, versus 39% who reported that this was something they had taught the pre-service teachers themselves. These mismatches are fleshed out in the focus group data, where teacher mentors commented on the 'gap' as resulting from pre-service teachers' failure to understand how to translate what they learn at HiOA into practice:

> I think they [pre-service teachers] need to be better at thinking/using manipulatives when they explain... But they don't even think of it. ... You do work with manipulatives here [at HiOA] but they don't see the usefulness...

Aware that the HiOA educators stress the importance of manipulatives, the teacher mentors felt, however, that they had not managed to teach the pre-service teachers how, when and why they should use them in their lessons, and that this was something that they themselves made clearer:

> I know that [HiOA] operates with manipulatives a lot but not with the transfer...

> I think they learn from [...] tying the practical contexts to the theoretical. It is no use [in learning maths] to just bake buns with your pupils, you also have to actually write it down, convert between units of measurement, specify the units.

Pre-service teachers also commented that they were limited in their experience of the practical use of manipulatives:

> I think the challenge was the materials [we] worked with, because we were trained to work with [manipulatives], in the introduction of a topic, and there was very little to work with.

The second issue concerned the central role of understanding pupil reasoning. As a major focus of the HiOA course, we had expected that pre-service teachers would be likely to cite this as a beneficial piece of learning from their course. However, only 13% of them (and 7% of teacher mentors) did so, and a further 4% of pre-service teachers (and 22% of teacher mentors) said this was learned from the teacher mentor. This pattern may be related to a series of findings from the Likert-scale data on pre-service teachers' perceptions of mathematics teaching, which indicated a conservatism about teaching and learning and pupils' roles which was not reflective of the HiOA programme intentions. More 'traditional' personal epistemologies of mathematics were reflected in 50% of pre-service teachers' agreement or strong agreement that "Mathematics is a subject for rote learning". They were also more conservative than the teacher mentors in response to completions of the opening statement "When pupils are to learn mathematics, it is important that…..". For example, teacher mentors agreed significantly more strongly than pre-service teachers with the completion statements "…they use their own algorithms", " …they take what they know as a starting point", "…they have to explain what they think" and "…they can use fantasy and creativity in their work" ($p<0.01$).

These issues are followed through in the focus groups, where teacher mentors frequently reflected on pre-service teachers' difficulties with adjusting their teaching plans to fit pupils' needs. They saw this as something that they needed to model in their role as mentors:

> … the [pre-service teachers] must try [...] different methods, and it is paramount that they see us as role models. And also […] see that there are many ways forward, and while they are with us they can find out how pupils think, that they can linger on some things. I think that linger is the right word; for the most part they just go directly on, doing what they have planned. And then they are not so good at assessing afterwards.

Here the same issue is raised but also connected to a perception that pre-service teachers lack subject knowledge:

> ...when pupils explain how they think, I often feel that [pre-service teachers] fail to follow the pupil and it is certainly a matter of training but I also think it's about their basic understanding of numbers and mathematics [...]. Then I have to get involved, to say "I think I understand how you think", because they [the pre-service teachers] stand there

> perplexed, and also the pupil sits there thinking "what did I say wrong?" and often it isn't wrong.

A slightly different angle notes the effect of pre-service teachers' assumptions about the nature of mathematics and related previous experience:

> I think maths is also a subject where students are very afraid of doing something wrong, because they think like "oh! it must be done correctly" so that they get hung up on some boring methods sometimes and they don't dare to take a wider view as they do in other subjects.

While mentors alone focus on the need to understand pupils, they share a common concern with pre-service teachers regarding the need to make oneself understood as a teacher. Here a teacher mentor talks about the need to be careful about terminology:

> In most mathematical topics you must be extremely careful what terms you use with the pupils, because in front of the class, as soon as you start fumbling, or you let the pupils make a mess of it for each other, it is going to be a problem.

In the following quotation, a pre-service teacher expressed a parallel concern with explanation of her own understanding:

> We must try to explain things as simply as possible. This is a challenge because it always goes through a filter, namely the teacher, who understands it.

Returning to the questionnaire data, teacher mentors were less likely to agree that "To become good at mathematics, you need to do lots of exercises" and that "The solution of a mathematics exercise is either right or wrong" ($p<0.05$), but in the focus groups some nevertheless described their classes in such terms, showing the influence of national testing:

> We have been working on [national] assessment tests in mathematics - so very much practicing for the test.

Here a pre-service teacher notices an emphasis on exercises in school placement:

> Going through the problems ... on the blackboard. Then we ask the pupils how they would solve this task, we talk a little about the solution. Then the pupils do the work individually.

Although these comments were few, they indicate a potential source of affirmation for deeply embedded traditional views about the nature of mathematics, as well as a further source of potential conflict for pre-service teachers regarding their experience of putting HiOA theory into practice.

The relationship between theory and practice

These results indicate the presence of various mismatches between school and university experience, and between university input and pre-service teachers' attitudes. As we have seen, they revolve around the issue of putting theory into practice, the focus of the third free-text question, which asked about the challenges for pre-service teachers in using learning from their HiOA course in practice. Only 8% of pre-service teachers replied that there were none, while 24% responded that it was difficult to translate theory into practice, and 12% that it was difficult to find the right language.

Several themes emerged, including a perceived absence of HiOA teaching on particular school topics:

> It can be difficult to draw connections and parallels between theory and practice. Especially considering that the topics we have used in school practice have not been particularly emphasized at HiOA.

But it was recognised that this could be a translation issue:

> Not many situations have come up that can be linked to the topics we've had. And if they have, I haven't thought about them in a way that relates to what I've learned at HiOA.

Thirty-four per cent of students said that mathematics at HiOA was too difficult for them, or was irrelevant for their teaching. Many comments were clearly illustrative of the problems of applying pedagogic principles noted above:

> It's not easy to connect what I have learned with [my practice] in the school placement because I feel that much of the curriculum isn't linked to the teaching of first grade, but to further grades.

Teacher mentors recorded fewer barriers, but also cited difficulty in translating theory into practice (20%), difficulty/irrelevance of mathematics at HiOA (13%) and insufficient mathematics at HiOA (9%). In free text responses, 13% said that they did not know what pre-service teachers learned at HiOA. Focus groups also included criticism of pre-service teachers' subject knowledge:

> Some have poor background knowledge when they come, I think. I had students in practice [in.] ... fourth grade, and then it was elementary things they did not know, I was quite surprised.

Some comments blamed lack of enthusiasm for uninspired teaching, but others were more indicative of the problem of application of theory into practice:

> I had a student who could not explain to the pupils what she intended, she became more and more frustrated.

This could include not having the confidence to depart from the lesson plan:

> … they think it's hard to meet the challenge [when] they get a lot of input from pupils [...] to use the input for further teaching ... it seems that they do not dare to do so […], "What I have written, I'll execute!".

In terms of the partnership itself, the questionnaire data showed that 91.6% of pre-service teachers agreed that "Experiences from practice have been important in the rest of the programme". Indeed, a number of them were critical of the HiOA course in their focus groups:

> ... there's nothing wrong with theory, but we must learn how to combine it with practical methods. It needs to be explained to us, why, how and when. … It is the practical work that I remember best.

While these and other comments suggest that many pre-service teachers see university college and school placement as very separate, others were more reflective about how the two together contributed to their development as professionals:

> I've heard several of the class who talked about what they have done in practice, but said they had not used what they had learned here [at HiOA], but it was exactly what we learned here that they had. I think you don't quite connect, I think that reflection days are very good for becoming more aware of that.

Some reflected on the difficulties of this stage of their development, and the need to learn from HiOAs' aim to teach pedagogic principles as opposed to 'recipe-following' teaching tips:

> Math teaching at HiOA focuses on our awareness of how we think when we do various calculations. I find that difficult, and have not come so far in the process yet that I feel I can take advantage of this when teaching.

Others noted the difficulties of being a novice but also the importance of reflection:

> It's easy to forget to use one's knowledge in some situations. But in retrospect, one thinks of what was done and finds that there was a much better option.

> One has to reflect along the way to learn by experience.

DISCUSSION

Previous research indicates that pre-service teachers will inevitably draw selectively from university programmes, through the lens of their own experience and beliefs. This is an effect which can be reinforced in school placement. In addressing our research question, this study has illustrated the complexity of the tripartite partnership involved in school practice. We have found that many of our pre-service teachers had missed the point of much of HiOA's input, and that their experience of the school placement is one of learning concrete practice from their mentors which they see as more informing than their university programme. For their part, mentors are often critical of their mentees' subject knowledge, but see themselves as acting as important translators of theory into practice. Additionally, pre-service teachers' learning in both institutions is mediated by their prior experience and perceptions of the nature of school mathematics.

These findings indicate some ways forward in enabling pre-service teachers to make the most of their school placement and for the University College-school partnership to be strengthened, including better communication with mentors, and more opportunities for reflection on the nature of mathematics and on the relationship between course content and placement experience.

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CROSSING THE BORDERS BETWEEN MATHEMATICAL DOMAINS: A CONTRIBUTION TO FRAME THE CHOICE OF SUITABLE TASKS IN TEACHER EDUCATION

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In mathematics teaching, closed boundaries between mathematics domains may convey to students a "fossilized" image of mathematics, and, in turn, cause difficulties in problem solving. Teachers should promote a "crossing" perspective in their teaching. In order to make teachers able to cross the borders, teacher education must encompass suitable tasks to be experienced and discussed. This paper reports a study aimed at framing the choice of such tasks, and the analysis of related problem solving behaviors. A contribution to framing comes from an adaptation of Habermas' construct of rational behavior. An experimental situation is planned and analyzed according to the resulting framework. Some educational implications are derived.

INTRODUCTION

Usually, mathematics is taught in secondary school as a discipline divided into separate domains (algebra, geometry, analytic geometry, etc.), each of them with specific theories, problems to solve and tools to solve them. This situation conveys a "fossilized" image of school mathematics and is one of the causes of students' difficulties in complex pure and applied mathematical problem solving. The traditional organization of mathematics teaching at university level, with reference to curricula designed for prospective mathematics teachers, may reinforce such kind of teaching and even justify it at the teachers' eyes - against present trends in the development of pure and applied mathematics, where frequently problems in a domain are tackled by using tools and strategies belonging to other domains.

Two questions arise: how to promote student-teachers' awareness of the nature of mathematics (in particular, the potential inherent in crossing the boundaries between different domains, according to the needs of problem solving activities); and at the same time: how to prepare them to teach mathematics avoiding the usual closed compartments of teaching? According to a position shared by most mathematics educators (see Watson & Sullivan, 2008), prospective teachers should experience meaningful activities that allow them to develop awareness about crucial aspects of mathematics, and at the same time suggest them suitable tasks/methodologies for teaching, focused on those aspects. In our case a natural, initial step might be to allow student-teachers to experience, compare and discuss different ways (different tools and strategies, belonging to different domains of mathematics) of dealing with a task in pure or applied mathematics.

In this perspective, in order to plan and analyze suitable didactical situations we think that it would be useful to elaborate a framework including:

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- a description of the specific competencies (concerning the specificities of the different areas of mathematics and possible links to establish between them) prospective teachers should develop;

- an analytical tool allowing to characterize and compare specific validation criteria of statements and inferences, specific problem solving strategies and specific ways of communicating in different domains of mathematics;

- an appropriate methodology to create and satisfy the need for detecting and comparing the specificities of the different domains of mathematics, in particular when involved in the solution of the same problem.

As concerns the first point, an initial contribution by the first two authors concerns the Cultural Analysis of the Content to be taught as a competence of epistemological, historical and anthropological analysis to be gradually acquired by teachers (see Boero & Guala, 2008).

As concerns the second point (related to the subject of this report), we can rely upon the previous work by the first and the third author who adapted Habermas' construct of rational behavior in discursive practices as a tool to plan and analyze some mathematical classroom activities: conjecturing and proving (see Boero, Douek, Morselli & Pedemonte, 2010); and the use of algebraic language in proving and mathematical modeling (see Morselli & Boero, 2011).

The aims of the study reported in this paper are: to further develop the adaptation of Habermas' construct of rational behavior, as a tool to characterize and compare the "rationalities" of different domains of mathematics; and to ascertain if the adapted construct can be exploited to plan and analyze suitable classroom activities aimed at preparing teachers to cross the closed borders of mathematical domains, thus contributing to the third point.

THEORETICAL FRAMEWORK

The interest of adapting Habermas' construct of rational behavior in discursive activities (as an analytical tool to characterize and compare the "rationality" of different domains of mathematics) depends on the fact that it consists of three inter-related components, which can be referred to three crucial and deeply interconnected aspects of mathematical activities:

- epistemic rationality (ER), consisting in the conscious validation of statements according to premises, true statements and inference rules that are shared in the reference community;

- teleological rationality (TR), consisting in the conscious choice of strategies and tools to achieve the aims of the activity;

- communicational rationality (CR), consisting in the conscious choice of suitable means to share ideas, problems, solutions in the reference community.

In Italy and in other countries two main subjects of secondary school mathematics are: synthetic (in particular, Euclidean) geometry; and analytic geometry. Using the adapted construct of rational behavior, we can distinguish between:

- a rational behaviour of synthetic type (briefly: synthetic rationality), according to the model of Euclidean geometry (but it can be transferred to other geometries too), based on strategies (TR) referring to visual evidence, aimed at proving the truth of statements and the validity of geometrical constructions (ER) through the construction of deductive chains based on axioms and previously proved statements; natural language not only plays a crucial communicational (CR) and reflective role, but also a treatment role to validate statements (ER: deductive chains mostly consist of verbal statements enchained through verbal links);

- a rational behavior of analytic type (analytic rationality), rooted in Greek mathematics (Menaecmus and Apollonius) and firstly made explicit in general by Descartes (1637; 1979), which consists (TR) in the construction of appropriate equations expressing the links between the relevant variables of the problem to be solved as if it would have been already solved, and in the solution of those equations - the resulting values of the unknowns allow to identify the solutions of the problem. We can further extend the scope of analytic rationality from the use of algebraic equations to include the use of calculus tools (like the use of the derivative to deal with tangent lines). In the case of analytic rationality natural language in its mathematical register works as a communicational (CR) and reflective tool, while algebraic language plays the major role in treatment, and epistemic control (ER) is exercised on the construction, transformations and interpretation of algebraic expressions (for details on ER in analytic rationality, see Morselli & Boero, 2011).

The aforementioned distinction guided us in planning a specific teacher education task; we have tested it in a selection process, thus in a situation far from an educational perspective (but, as we will see in the last section, the discussion of the task was an occasion to stimulate a reflective and learning process for some candidates). With reference to the chosen task, the aims of this research report are: to illustrate potential, specific features and limitations of synthetic and analytic methods, corresponding to specific aspects of synthetic rationality and analytic rationality; to describe and interpret people's behaviors according to the adapted Habermas' analytical tool; and to derive some implications as concerns teacher education (in the perspective of a more flexible teaching of main subjects of secondary school mathematics curricula).

While the adapted Habermas' construct plays a major role in the a-priori cultural analysis of the task and in the a-posteriori interpretation of behaviors, for the evaluation of the distance between people's behaviors and the requirements of epistemic and teleological rationalities we will exploit other theoretical tools:

- the construct of figural concept (Fischbein, 1993), which may account for the difficulties met in the mastery of figures without relating them to properties and definitions, or in the mastery of formal definitions and representations without the support of figural evidence;

- the construct of procept (Gray & Tall, 1994), which in particular accounts for the different ways of dealing with equations like $y=ax^2+bx+c$ as processes or as symbolic objects representing geometric entities - and the related difficulties.

METHOD

The task at issue was administered to 35 candidates to become mathematics teachers in secondary schools; most of them had a master degree in mathematics; the others had a master degree in Engineering or in Physics, with a strong curriculum in Mathematics; some candidates had also a Ph. D. in Mathematics or in Physics. The selection had to result in the choice of 15 candidates, who will enter one-year intensive professional preparation (including courses of mathematics education and stages in the schools) to become teachers.

Candidates had already passed a national test (35 candidates at the Genoa University had been successful in that test, out of 76) based on 60 multiple-choice questions. The further steps in the selection process (in each university) included a written test based on open problems, and an oral discussion with the local Commission "starting from the discussion of the written test" (in the case of the Genoa university).

The following task was prepared as one of the three tasks for the Genoa university written test (the other two tasks concerned calculus and probability issues; three hours was the whole available time):

> To characterize analytically the set ***P*** of (non degenerated) parabolas with symmetry axis parallel to the ordinate axis, and tangent to the straight line $y=x+1$ in the point (1,2).
>
> To establish for which points of the plane does it exist one and only one parabola belonging to the set ***P.***
>
> To find straight lines that are parallel to the ordinate axis and are not symmetry axes of parabolas belonging to the set ***P.***

The formulation of the task, as well as the a-priori analysis, was guided by the aforementioned theoretical framework concerning different rationalities in different mathematical domains. In our intention, the formulation of the questions would have encouraged the adoption of analytic methods without preventing candidates from using synthetic considerations with heuristic/teleological and control/epistemic functions, or even to get the solution for the third question. Moreover the formulation of the first and third questions would have encouraged the use of the language and methods of analytic geometry without preventing candidates from using calculus tools (an alternative choice more oriented towards calculus would have been to use calculus terminology: "quadratic functions", "graphs", "graph slope", and so on).

In the a-priori analysis of the task, we had imagined that:

- candidates could have answered the first question by intersecting the straight line $y=x+1$ with a generic parabola of equation $y=ax^2+bx+c$ passing through the point (1,2), and imposing that the intersection points collapse in that point; but also calculus notions could have been used by considering the quadratic function $f(x)= ax^2+bx+c$ with two conditions: $f(1)=2$, $f'(1)=2$;

- candidates could have answered the second question by choosing a generic parabola of the set ***P*** and imposing the condition of passing through a generic point (x_0, y_0), and then finding algebraically for which points the coefficients of the parabola are determined in an unique way; synthetic geometry might have provided them with the conjecture that lines x=1 and y=x+1 had to be excluded, and/or with a possibility of checking the correctness of their algebraic solutions. Note that synthetic geometry was not suitable to easily answer the second question, because proof (that, given a point S out of the excluded lines, it belongs to one and only one parabola of the set ***P***) requires the use of geometric properties of parabola, which are usually neglected in Italy in the teaching of conic sections;

- candidates could have answered the third question: either by analytic methods (once answered the first question, they could have written down the equation of the symmetry axis, or imposed that the first derivative is zero, given the equation of the parabola); or (once answered the second question) by synthetic geometric considerations bringing to the exclusion of the line x=1 due to the fact that the tangent line in the vertex (1,2) should be perpendicular to the symmetry axis, against the given condition of tangency in that point to the line y=x+1.

Possible limitations inherent in the didactical contract (concerning the legitimacy of the use of methods not alluded to in the text of the task, a serious problem in the case of a selective task) were at least partly overcome by the comment of one member of the commission, who under request of a candidate made explicit the fact that "different methods may be used to answer each of the three questions".

After the written test, the discussion with candidates (the further, final step of the local selection process) concerned this task for a representative sample of about one half of them (the other candidates discussed the other tasks of the test). The discussion of the work done by them in the written test was organized according to a semi-structured interview, around one or two of the following issues:

- difference between analytic and synthetic methods to deal with the questions (in terms of strategies and criteria of validation), in particular as concerns the exclusion of the lines x=1 and y=x+1 for the second question, and the line x=1 for the third one;

- heuristic and control potential of synthetic methods;

- relationships between the method of collapsing the points of intersection straight line/parabola in the tangency point, and the method which exploits the derivative of the quadratic function.

For the present research, data at disposal are: candidates' written solutions; and the records of the discussions of the candidates with the Commission.

SOME RESULTS

In order to give an idea of the preliminary analysis of the solutions, the following table summarizes some crucial features of the solutions of the first 11 candidates. AnGeo, Calc, SynthGeo are for the respective methods, with brackets indicating traces and/or trials, not the main adopted method; A added to AnGeo means purely algebraic calculations, with no substantial reference to geometric properties of parabolas, tangency, etc; PF and F mean, respectively, partial failure (when only one part of the answer is provided and is correct) or total failure. -- means: question not dealt with by the candidate.

Candidate	Degree	First question	Second question	Third question
1	Eng	AnGeo; (Calc)	--	AnGeo, A; F
2	Ph.D.Math	AnGeo	AnGeo	AnGeo
3	Math	AnGeo	AnGeo; PF	AnGeo; PF
4	Math	AnGeo; (Calc)	--	AnGeo
5	Math	AnGeo; (Calc)	AnGeo; PF	AnGeo
6	Math	AnGeo; (Calc)	--	AnGeo
7	Math	AnGeo	SynthGeo	AnGeo
8	Phys	AnGeo; (Calc)	AnGeo	AnGeo
9	Math	AnGeo, A	AnGeo	SynthGeo
10	Ph.D.Math	AnGeo; (Calc)	AnGeo	AnGeo
11	Ph.D.Math	AnGeo; (Calc)	AnGeo	AnGeo; (SynthGeo)

Some results emerging from further qualitative analyses of available data are:

a) the very limited use of the synthetic method; only 6 candidates out of 35 proposed some very short arguments of synthetic type; we observe that the formulation of the problem does not encourage it, thus using it requires consciousness of its potential and limitations. Moreover almost all those who engaged in synthetic geometry activities were unable to develop a rational behavior on the sides of TR (parabolas are only sketched, with no relation with the algebraic expressions representing them and very weak traces of some properties of parabolas of the set ***P***) and ER (drawings are very poor, with no comment, and sometimes do not include the parabolas under the line y=x+1, thus they cannot be used to check the validity of results derived through analytic methods). In terms of figural concepts (Fishbein, 1993), we may say that the figural aspect prevails on the conceptual one, with lack of epistemic control on the drawings and the related geometric figures;

b) the lack of functional connections between ER and TR; once engaged in the analytic geometry method, the algebraic calculation of the solution brings to a result which is neither checked by coming back to parabolas and their geometric properties, nor referred to the initial aim of calculations and (in the case of questions 2 and 3) to previous results;

c) for most students, CR works well only on the side of analytic geometry and of calculus; some students produce sequences of algebraic calculations with very few and not always appropriate words to present their solutions;

d) in some cases, the purely algebraic management of the analytic geometry method (11 candidates out of 35 performed only algebraic treatment) prevents students from discovering mistakes or lacks in their conclusions (ER); in terms of procepts the symbolic expression: $y= ax^2+bx+c$ is only a formal expression, with no reference to the process of generation of a line in the Cartesian plane and to its result.

e) many students, during the a posteriori discussions, had difficulty in connecting a typical feature of TR in the case of the analytic geometry approach to the first question (collapsing the intersection points parabola/line $y=x+1$ in (1,2)) with the limit process encapsulated in the expression $f'(1)$. During the discussions, the fact that the secant line does not pivot around the point (1,2) seems to prevent candidates from seeing that the analytic geometry process results in the approach of the secant to the tangent line in the point (1,2), and thus in an alternative way to access the derivative $f'(1)$;

F) during the discussion, the authors noticed positive learning reactions by the candidates (in spite of the psychological stress, due to the selective character of the discussion); most of them were able to realize (even with evident surprise!), under the commission members' guide, that:

- the method of collapsing the intersection points between the line $y=x+1$ and the parabola into the point (1,2) is another way of generating the derivative of the quadratic function for $x=1$;

- synthetic geometry can work as a tool for conjecturing and for checking results of analytic methods for questions 2) and 3);

- synthetic geometry can also allow to answer question 3), once question 2) has been solved.

CONCLUSIONS AND EDUCATIONAL IMPLICATIONS

The first data analysis shows the potential of the adapted Habermas' construct to produce suitable tasks for putting into question the rigidity of the separation between different mathematical domains, and to analyze people's behaviors in terms of their distance from rational behavior. The rigidity inherited from secondary and university teaching of mathematics is revealed, in terms of ER, TR and CR components, through the difficulties of moving forwards and backwards between synthetic and analytic geometry considerations, but also of identifying relationships between processes in analytic geometry and in calculus. The only language used by most candidates at an enough satisfactory and precise level is the language of analytic geometry. Thus, the task might be a starting point for an activity of teacher education aimed at putting into evidence the negative consequences of the rigid separation (in mathematics teaching) between different mathematical domains, and the opportunities offered by crossing the boundaries between them through suitable tasks. A further development concerns the conception of a teacher education experience, starting from the experimented task. The task, indeed, might be suitable (as revealed during the discussion) to start a program of Cultural Analysis of the Content to be taught (on the epistemological side), in particular in terms of critical consideration of nature, potential and limitations of analytic and synthetic methods, and features of the related rationalities. With reference to this possibility, an open problem concerns the opportunity that the adapted Habermas' construct , introduced as a tool for the researcher, becomes also a tool for teachers to identify and compare specific features of activities in synthetic geometry and in analytic geometry (and in other domains too), and of synthetic and analytic methods to deal with problems like the one considered in this report.

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THE PMA: AN EARLY MATHEMATICS SCREENER AND PROGRESS MONITORING TOOL

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There is a critical need for early identification of students who are experiencing difficulties in mathematics and, then, the provision of immediate and targeted intervention in order to build foundational skills and knowledge. The purpose of this study was demonstrate the effectiveness of an early mathematics screener -- the Primary Mathematics Assessment. The results demonstrate that the PMA can assess four comprehensive areas (number, relationships, measurement, and space) within 8 minutes per student, has strong internal reliability withing these four subconstructs, and predicts well to a state standardized test two year later.

BACKGROUND

Based on the poor performance of fourth grade students on national and international tests of mathematics, it is evident that U.S. students in the early grades are not adequately prepared in mathematics (Clements, Xiufeng, & Sarama, 2008; NRC, 2009). Using large data sets or nationally representative samples, several researchers have also demonstrated that students who complete kindergarten with an inadequate knowledge of basic mathematics concepts and skills will continue to experience difficulties with mathematics throughout their elementary and secondary years (Duncan et al., 2007; Jordan, Kaplan, Ramineni, & Locuniak, 2009). This research points to the critical need for early identification of students who are experiencing difficulties in mathematics and, then, the provision of immediate and targeted intervention in order to build foundational skills and knowledge (Ginsburg, Lee, & Boyd, 2008).

There is a great need and demand for reliable, efficient, and valid primary level mathematics screening and diagnostic tools to identify students with mathematics deficiencies so teachers can intervene with differentiated lessons in order to remediate student deficiencies. Most current tools provide inadequate diagnostic information or are too time consuming to administer on a large scale to an entire classroom of young children. The purpose of this study is to examine the Primary Mathematics Assessment (PMA) system as a viable tool for assessing students age 5 to 8.

Response to Intervention (RtI) in mathematics

There has been a conscious effort over the last decade to create and implement a Response to Intervention (RtI) framework in elementary schools across the U.S. This model requires schools to analyse student learning in the context of instruction (Gersten, 2009). A major tenet of the RtI model is to initially screen the entire class for early identification of those students who have specific weaknesses. This framework hinges on one critical tool – a comprehensive universal screener. The development of a quick screener is paramount for grades K-2 as these students are too young to reliably and independently take tests and, therefore, must be administered the test individually by a trained administrator.

This paradigm of improving instruction through assessment hinges on accurate and early identification of students' mathematical deficiencies. However, using a screener for early identification of deficiencies is only the first step of the process. Once identified, these students must undergo a more thorough evaluation with a more comprehensive targeted assessment so effective instructional interventions can be designed for remediating their deficiencies. This process of monitoring the success of instructional interventions through continuous evaluation is embedded in the (RtI) model (Fuchs et al., 2007). More specifically, monitoring students' academic progress through formative assessments coupled with early and immediate feedback has shown increases in student achievement by 0.7 standard deviation units (Hattie, 2009).

Mathematics screeners at the primary level

The screening and diagnostic instruments for K-2 mathematics currently in use have not demonstrated adequate predictive validity against standardized achievement tests (Clements et al., 2008; Fuchs et al., 2007). Clements and colleagues have demonstrated that most early childhood diagnostic instruments in mathematics have been limited to number concepts and do not include other important domains that are predictive of later success in mathematics. While these screeners are quick to administer, they produce an insufficient profile of student deficiencies, which results in ineffective interventions. And screening for only number concepts does not evaluate other important mathematical skills necessary to succeed in these recommended standards for K-6 mathematics. The National Research Council (2009) has called for better quality instruments to diagnose students' level of competence in different areas of mathematics. The review of the extant research on mathematics skills in the primary grades supports four key areas that predict students' later success in mathematics: concepts of number, relationships, measurement, and spatial reasoning. Each of these areas is critical to the development of mathematical competencies and should be evaluated in any early mathematics screener or diagnostic instrument (Clements et al., 2008; Clements & Sarama, 2004).

Current comprehensive multi-item mathematics assessments require thirty minutes to an hour to administer (NCRTI, 2011). These longer forms provide more detailed information about student deficiencies, but are difficult to administer to a large number of young students in a timely fashion. The expenditure of resources with the longer form tests at the K-2 level is great. A 30 - 45 minute screener administered to a class of just 20 K-2 students will require 10 - 15 hours to complete as opposed to a 10 minute screener, which would require 3.3 hours. Also, a large number of students who are tested with comprehensive instruments will be found not to have any mathematical deficiencies, resulting in an unnecessary use of valuable time and resources. What is needed is a tool that utilizes a brief stage 1 screener, to quickly identify students who need a further stage 2 comprehensive evaluation.

The Primary Mathematics Assessment (PMA)

The Primary Mathematics Assessment (PMA) – Screener is used to quickly identify those students who have mathematical deficits. It is a research-validated, universal screening tool that assesses domains beyond simple number concepts (Brendefur, Thiede, & Strother, 2011). The conceptual framework incorporates concepts of number, relationships, measurement, and spatial reasoning.

Early childhood researchers describe the importance of early **number** skills within the domain of number such as number recognition, number sequencing, and fluency and flexibility (Clements et al., 2008; Clements & Sarama, 2004; Ginsburg & Baroody, 2003; Lee, Lembke, Moore, Ginsburg, & Pappas, 2007). Three mathematical skills: number knowledge, ordinality, and quantitative reasoning have been demonstrated to have an average effect size of 0.34 on later academic success (Duncan et al., 2007). Fluency and flexibility are intimately linked. Students are 'fluent' with whole numbers when they can solve problems, answer questions, and extend patterns in a quick and efficient way (Baroody & Dowker, 2003). When dealing with fluency, speed is important. By quickly recalling a basic addition fact, a student has demonstrated fluency. But fluency is often the by-product of flexibility (Beishuizen & Anghileri, 1998; Thompson, 1997). Flexibility is the ability to solve problems in a variety of ways, use information already known to solve unknown problems, and the capability to determine the most efficient method to use when confronted with a challenging problem (Star & Madnani, 2004). Flexible mathematical thinkers have been shown to develop faster recall of basic facts and to be more successful in classroom settings (Beishuizen & Anghileri, 1998).

Understanding **equality** and the relationship of numbers and solving **contextualized problems** form the basis of algebraic understanding (Van Amerom, 2003). Hiebert and Carpenter (1992) demonstrate that young students are capable of using operation properties (commutative, inverse, identity, etc) when solving arithmetical problems and naturally transfer informal knowledge of these operation properties to new situations. However, Demby (1997) and Lee and Wheeler (1989) provide evidence that by the time students reach high-school algebra they are reluctant or unable to use these operation properties when solving problems. Realizing this issue, other countries built curricular opportunities in grades 4 – 6 to assist students in transitioning from solving contextualized problems and informal approaches to formalized symbolism and algebraic reasoning and notation (Anghileri, Beishuizen, & Van Putten, 2002; Van Amerom, 2003). Accurately solving contextualized problems (e.g. word problems) is a key factor in early mathematics achievement.

Measurement of length has a direct link to knowledge of fractions and decimals because measurements often do not use complete units (Cramer, Post, & del Mas, 2002; Watanabe, 2002). A table can be 3 ½ feet wide. Students must make sense of that 'part' of the unit left over after the 3 complete units are counted. This is different than just counting discrete objects like fingers or cubes (Kamii & Clark, 1997). When counting units of length, the student begins to develop a model for the continuous nature of rational numbers (e.g. fractions, decimals, percents). This supports students learning about fractions and ratios in later grades (McClain, Cobb, Gravemeijer, & Estes, 1999). Many nations that use informal measurement and measurement estimation as a way to introduce fractions perform at a much higher level than the U.S. on rational number items found on standardized tests (Stephan & Clements, 2003; Watanabe, 2002). Students in these countries have an understanding of the meaning of rational numbers connected to measurement (Mullis et al., 1997).

Researchers have demonstrated that **spatial reasoning** has a very high predictive value for mathematics achievement (Gustafsson & Undheim, 1996). Two categories of spatial reasoning are spatial visualization, the ability to mentally transform objects, and spatial orientation, the ability to remain unconfused when the object's positioning changes. Spatial reasoning also helps develop fluency with flexible operations in arithmetic and strengthens and supports students' ability in measurement (Tartre, 1990) and, as with measurement, builds concepts of proportional reasoning (NRC, 2006).

METHODS

Instrument

The PMA is built on four areas. First, Number includes items in Number Identification, Number Recognition, Number Sequences, and Fact Fluency. Second, Relationships examines Relational Thinking and Interpreting Context. Third, measurement included Iteration and Partitioning. Fourth, spatial reasoning includes Decomposing and Composing shapes.

Context: Setting and Participants

The participants in this study were from multiple districts across one state in the U.S. They included Kindergarten, first and second grade students in over 34 schools, which were from rural, suburban, and urban areas. Students from these schools were from low, middle, and high SES and had varied ethnicities (Table 1).

	Kindergarten n = 5405	Grade 1 n = 5673	Grade 2 n = 4629	Average Percentage
Gender				
	53.3%	51.1%	51.9%	52.1%
	46.7%	48.9%	48.1%	47.9%
Ethnicity				
	68.2%	64.5%	61.3%	64.7%
	19.9%	24.4%	25.9%	23.4%
	11.9%	11.2%	12.9%	12.0%

Table 1: Demographic characteristics of student participants.

Measures

The PMA progress monitoring tool and the PMA-Screener were used to test students at the beginning of middle of the school year for kindergarten, first and second grades. All students were screened using the PMA-S and, then, a random sample of students at each grade level was tested using the progress monitoring tool.

RESULTS

We conducted Rasch analyses on the items measuring each domain across the PMA to verify the assumption of unidimensionality had been satisfied. In Rasch analysis, the assumption of unidimensionality is satisfied when a set of items accounts for at least 20% of the variance and when no other contrast (set of items) can explain 5% of the variance (Reckase, 1997). As seen in Table 2, the dimensionality assumption was met for all domains. Scales within constructs were also tested to verify unidimensionality.

We examined the reliability of the scales. Two of the PMA scales were in the excellent range ($\alpha > .90$) – number and spatial relationships – while relationships were considered in the good range ($.80 < \alpha < .90$) and measurement in the acceptable range ($.70 < \alpha < .80$) (Nunnally, 1978). Table 2 highlights the reliabilities.

Construct	Number of Items	Variance Explained	Reliabilities
Number	25	32.0	.93
Relationships	17	41.2	.86
Measurement	14	36.2	.77
Spatial Relationships	11	32.1	.92

Table 2: PMA scale dimensionality and reliabilities.

For each domain, we constructed a small set of screener items that assess the skills within the larger domain. This screener, the PMA-S contained a small set of items from the PMA progress monitoring tool, which from the Spearman–Brown prophecy formula can decrease the reliability of the tests (Allen & Yen, 1979). As seen in Table 3 below, we were able to produce highly reliable screener for the four domains.

Construct	Number of Items	Correlations (n = 110)
Number	6	.94**
Relationships	6	.91**
Measurement	4	.89**
Spatial Relationships	4	.68**

Table 3: PMA Screener and Targeted Assessment correlations (**p <.001)

DISCUSSION

The PMA Screener is the short form and predicts well to the PMA progress monitoring tool. The PMA system is intended to meet the needs of classroom teachers by providing a series of items from all four of the predictive constructs that can still be given to individual students in less than 10 minutes. As teachers and schools have the ability to compare individual student screener results to class, school, and state-wide results, the PMA-S can assist teachers in giving special emphasis to certain mathematical topics during instruction and to provide more targeted support for students demonstrating deficiencies in specific areas. The PMA-S offers teachers data indicating their students' responses to items that are mathematically important from a predictive standpoint yet do not require teachers to adhere to any particular curricula or textbook.

Because the screener predicts well to the progress monitoring tool, teachers only have to assess students performing in the bottom quartile (for example) with the longer PMA. The PMA system as a whole becomes an effective formative assessment that can be used to guide instruction and support teachers' efforts to assist students in grades K-2 learn mathematics that will support their long-term mathematics achievement.

The purpose of the study was to examine the reliability and validity of a screener and progress monitoring tool. The PMA system can consistently and reliably be used to identify students who need additional work in a particular mathematical topic. This detailed information to teachers regarding their students' strengths and weaknesses will be an asset to making timely decisions on interventions.

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A DEFINITION FOR EFFECTIVE ASSESSMENT AND IMPLICATIONS ON COMPUTER-AIDED ASSESSMENT PRACTICE

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For a decade, computer-aided assessment (CAA) has been used extensively with first-year mathematics and engineering undergraduates studying mathematics modules at the institution under investigation. This project sought to evaluate the effectiveness of CAA. Using assessment literature and activity theory to frame the study, this paper explores the aims of assessment and what it means for assessment to be "effective": it proposes a definition for effective assessment and discusses whether CAA can be considered effective assessment by this definition.

BACKGROUND

This project seeks to evaluate the use of a computer-aided assessment (CAA) system at a higher education establishment in the United Kingdom. The CAA system is used to test mathematics learning in first year mathematics and engineering mathematics modules. It asks questions that are mainly procedural in nature, in multiple choice and numerical input forms. Although lecturers employ the facilities that CAA offers in different ways (Robinson et al., 2012), most students have the opportunity to practice similar questions to the ones they receive in the summative test.

Performing this evaluation required a comparison of the CAA system against an accepted standard for assessment; however, the trend for evaluations of CAA conducted hitherto has been in the form of self-report commentaries of practice. These evaluations lack an objective standard upon which to compare; hence, no precedent has been made and there are calls in the literature for a rigorous review (for example, Bull & McKenna, 2001 and Sangwin, 2003).

Formative assessment appeared to offer such a standard: it has been widely discussed in the literature; it has been argued to be effective (Black & William 1998, for example); and proponents of CAA suggested that it could be adopted with formative intentions (Bull & McKenna, 2001). However, there remain two concerns with using formative assessment as a standard for evaluation.

First, the definition of "formative assessment" is disputed and the term is used inconsistently. For example, there is disagreement whether it is necessary for students to demonstrate improvement as a consequence of the feedback they receive from formative assessment. Sadler (1989 pp.120-121) believed that improvement is necessary, citing Ramaprasad's (1983 p.4) distinction between information about performance and feedback, which requires the student to act upon information about performance. This distinction is not maintained in the most recent definition of formative assessment (Black & Wiliam, 2009).

Black and Wiliam (2009 p.10) believed this requirement was too stringent: determining whether an assessment is formative would require establishing that the assessment has caused an improvement that would not have occurred otherwise. Furthermore, they added, “even the best designed interventions will not always result in better learning for all students” (Black & Wiliam, 2009 p.10, emphasis in original), making it difficult to confidently declare an assessment “formative”.

Determining whether practices are formative assessment or not depends on the definition of formative assessment used.

Second, there are gaps in the theoretical underpinning of formative assessment. After Black and Wiliam (2009) developed the theory of formative assessment further, using cultural-historical activity theory for some aspects, Bennett (2011) and Taras (2010) believed that there was scope to develop the theoretical framework of formative assessment still further.

Ultimately, we decided that formative assessment did not provide the standard that we wished to evaluate CAA with. Instead, we would develop a definition for effective assessment that describes the process of assessment, the criteria for success, the roles of the actors in the activity of effective assessment and the outcomes.

THEORETICAL FRAMEWORK

Cultural-historical activity theory (CHAT) offers three ideas in particular that would be necessary in a definition for effective assessment that were not developed fully for formative assessment.

First, there is the role of the community on activity. In CHAT, the community provides resources, shares the activity and imposes rules. For a student, teachers and peers have an impact on the activity of learning; there are rules of the classroom and there are shared responsibilities. For formative assessment, Taras (2010 p.3017) noted, “It is never quite clear who is involved in … the assessment cycle”. Black and Wiliam (2009 p.9) suggested they used a framework derived from CHAT to develop themes for their theory of formative assessment; however, while Black and Wiliam discussed the role of the teacher in setting assessment and regulating learning, there remains a desire to explore the roles of peers and individual learners in more detail.

Second, activity theory maintains that all human activity is purposeful — “the expression ‘objectless activity’ is devoid of any meaning” (Leont’ev, 1978). Therefore, the way in which a student interacts in a learning activity is shaped by his/her own learning goals. When discussing the notion of object-oriented activity in expansive learning, Engeström and Sannino (2010 p.4) pointed out that “motives cannot be taught, they can only be nurtured” over time with students. That is, teachers and peers can influence or dictate students’ goals.

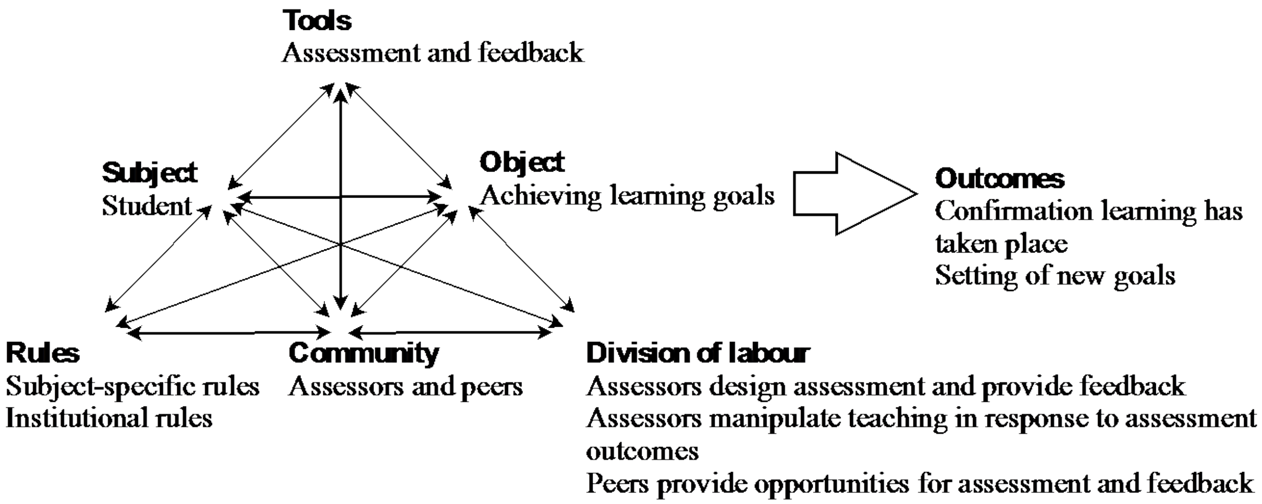

Figure 2: An Engeström (2000) representation of the activity of effective assessment

Black and Wiliam (2009 p.22) believed teachers must have learning intentions and engineer opportunities for learning. There are two problems with this construction. First, the teacher's learning intentions and the student's learning goals may not always cohere; the formative assessment literature does not consider the outcomes of such a contradiction. Second, many teachers have little, if any, influence on self-assessment, peer assessment or peer support that happens outside the classroom, in which students may set their own learning goals. The outcomes of these assessments are not discussed in the formative assessment literature.

The third CHAT construct that we consider missing from formative assessment arises from the model of expansive learning proposed by Engeström and Sannino (2010 pp.8-9): that learning occurs in cycles through the development of the object of learning activity. That is, the objects of future learning activities are evolved from previously achieved learning goals. This is particularly true for mathematics learning, where learning more advanced concepts quite often demands requisite knowledge from simpler concepts. It is this evolution that permits viewing learning as a cycle and assessment as a process within it: successful learning warrants the setting of new learning goals and assessing new learning.

We propose the following definition of "effective assessment" that incorporates the advice from assessment literature and the constructs of CHAT.

A DEFINITION FOR EFFECTIVE ASSESSMENT

We consider assessment and feedback as tools in the activity of learning, with learning goals forming the object of learning. The student is viewed as the subject of the learning activity (fig. 1): it is the student that performs the assessment and, most often, the aims of the assessment relate to the student's learning and development.

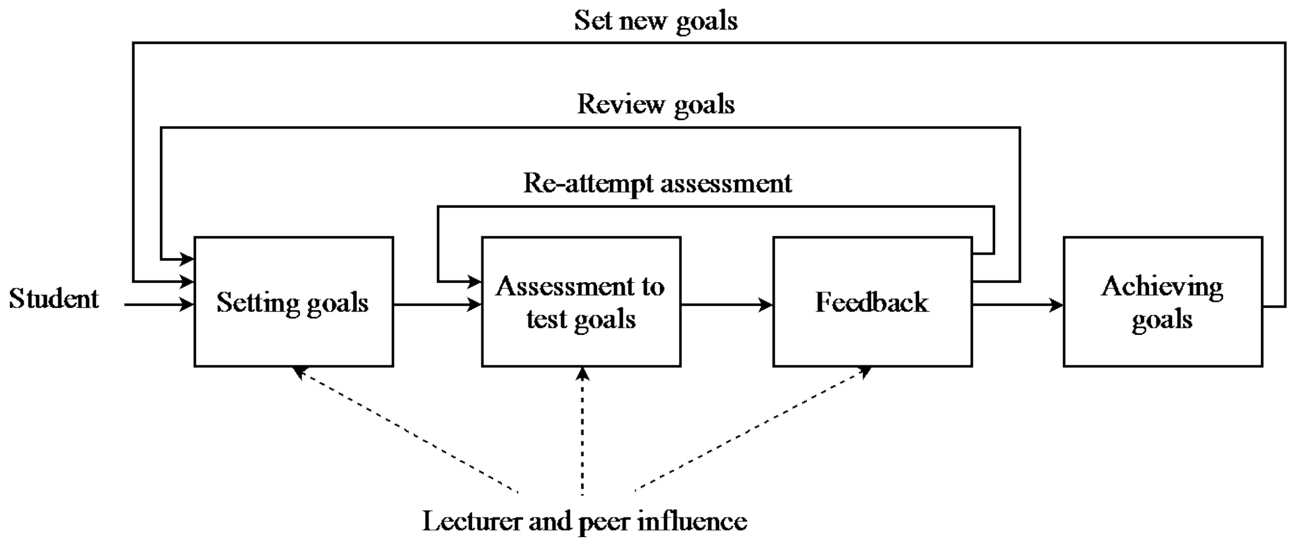

Figure 3: Effective assessment in a model learning cycle and the influence of lecturers and peers on the processes in this cycle

Effective assessment is, in part, defined as an assessment that enables the student to achieve his/her learning goals. However, the student does not act alone in this activity. Assessors and peers have roles as members of the learning community. They provide opportunities to assess learning and provide feedback. They also have an influence on the student when setting learning goals (fig. 2).

As Yorke (2003) suggested for formative assessment, effective assessment should be both a process and part of a cycle. In this cycle, the student possesses initial learning goals. He/she undergoes assessment to test whether those learning goals have been met and receives feedback. On receiving feedback, the student might re-attempt the assessment; or revise his/her learning goals; or, if the student has achieved his/her learning goals, he/she can set more challenging learning goals.

Lecturers and peers have an influence in setting goals, setting assessments and providing feedback. Initially, a novice student may be completely directed by the lecturer: goals may be set (explicitly or implicitly) on behalf of the student; and the assessment and feedback are managed entirely by the lecturer. In effective assessment, the student gains experience, knowledge and understanding so that he/she can take more responsibility for these stages of the learning process.

It could be argued the ultimate aim of learning is that the student "should be able to do unaided what previously needed knowledgeable support" (Yorke, 2003 p.496); an effective assessment should support students in developing self-regulation skills and in setting new personal learning goals autonomously. Therefore, for an assessment to be effective, the impact of lecturers and peers on the activity of learning is reduced.

We define "effective assessment" in the following way:

- An effective assessment must be a purposeful assessment with the aim to test whether the student has achieved his/her learning goals.

- An effective assessment must be part of a wider learning cycle in which the student sets more challenging learning goals with diminishing influence and input from lecturers and peers.
- An effective assessment must give opportunities for the student to receive feedback that is related to his/her performance in relation to his/her learning goals and opportunities for the student to demonstrate that he/she has developed sufficiently to achieve his/her learning goals.

Since each statement of the definition refers to the individual student, it is not possible to separate the effectiveness of an assessment from the individual. That is, if an assessment has been particularly effective for one student, one cannot conclude that it is effective for all.

With these criteria, we can describe the extent to which CAA is effective for each student. Evaluating the effectiveness of CAA with several students according to these criteria yields common strengths and weaknesses.

METHODOLOGY

The definition of effective assessment demands the study of individual students to evaluate effectiveness for his/her circumstances. We adopted a case study approach — limiting the study to one CAA system at one institution — to identify strengths and weaknesses particular to this system. Nine self-selecting first-year undergraduates from four disciplines (four mathematicians, three aeronautical and automotive engineers, one materials engineer and one sports technology engineer) and six lecturers teaching first-year mathematics modules (three in mathematics, two in materials engineering and one in mechanical engineering) were interviewed.

The interview questions related to how students and lecturers use CAA, the influence of peers and lecturers on CAA, and how CAA has helped both students and lecturers. A professional transcriptionist transcribed the audio files and the first author coded the participants' responses according to parts of the definition for effective assessment and other parts of the learning process (fig. 2).

ANALYSIS

The students

In terms of setting learning goals for themselves, many of the students set seemingly superficial short terms goals that related to their long-term aims of career success. Consequently, many of the students' goals when using CAA were expressed in terms of a percentage of marks.

Many of those students believed that achieving high marks in CAA was a demonstration that they had developed the required knowledge and understanding that was expected of them. These students set high percentages as their goals for CAA, with some not happy unless they achieved 100%. To that end, the practice test facility was used extensively to fully prepare for the summative test. They were confident that CAA helped them to improve in this respect and the feedback was detailed and appropriate enough.

Other students felt that CAA did not test their conceptual knowledge as well as other assessments had. It seems for these students, CAA related poorly to the perceived implicit learning goals. One such student was interested in pursuing an academic career in mathematics research; however, her learning goals were also expressed in terms of marks and did not lead to more challenging learning goals beyond the CAA summative test. Even for these students, there is a dependence on the lecturer to provide implicit learning goals.

These students had developed procedures for approaching and completing the CAA tests; largely comprising a regime of practice test attempts and learning the method offered in the feedback. Although this aided the students in achieving their goals, it appears that it did not inspire them towards further learning.

Although the students were satisfied that they had achieved their learning goals — insomuch that they had achieved the marks they had set for themselves — it would appear that they felt the primary benefit of doing so was the accumulation of marks that contributed to their module grade. Some went further: they wished to accumulate the "easy marks" that CAA offered to lessen the burden of the exam for passing the module. From these comments, CAA has not been effective for these students in terms of encouraging further learning with more challenging learning goals.

Peers have an important role in CAA, with many of the students reporting that they had engaged in collaboration at some point during their first year for CAA. Most of the students had clear, though not always correct, views on when collaboration becomes plagiarism; other students had less clear views and were prepared to engage in practices that could be interpreted as plagiarism. For example, one student believed that helping others in a summative test was satisfactory, since the purpose of the first year is to ensure that all students have developed a common foundation of understanding for subsequent study.

Over the course of the first year, the students became more willing to engage with their peers by offering mutual support. The culture of these student cohorts appears to have had a role in this, since one student commented that she was aware that others on her course were collaborating and later did the same. Another student expressed an initial reservation to collaborate on CAA, but he had recently started to collaborate with others prior to the interview. While the students found such support helpful, it appears to come at the expense of developing self-regulation skills, since the students became less likely to attempt the assessment by themselves first.

Few students referred to the influence of lecturers, particularly in terms of support during assessment. It would appear that they were content with the assessment and feedback they were offered and they possessed little desire to self-assess beyond the compulsory assessment content. It could be argued that this is a culture that they have become accustomed to: that studying beyond the provided content for which credit can be received has insufficient value.

The lecturers

Although most of the lecturers would have liked CAA to test students' knowledge and understanding more deeply than it currently does, the primary reason for using CAA is the need for regular assessment — for both formative and summative purposes — and CAA provided a means to offer regular assessment to large cohorts without a significant marking burden. Other forms of assessment addressed the need to test students' conceptual knowledge.

The lecturers believed that the students would only be sufficiently motivated to engage with CAA if they were offered module credit. However, they were concerned about the impact of collaboration and the impossibility of eliminating the plagiarism that arises from group-work, since, in some cases, it is not possible to invigilate the entire cohort in one computer laboratory. Hence, lecturers typically offered between 2.5% and 5% of module credit for each CAA summative test.

Some lecturers reported that many students develop a mechanical approach to answering CAA questions. As a result, some students have become quite adept at performing a mathematical procedure without having the flexibility to adapt to different contexts. Often, this problem is not identified until the final exam.

DISCUSSION AND CONCLUSIONS

In terms of satisfying learning goals, CAA is effective to a point: the students were content that the feedback enabled them to demonstrate an improvement; the lecturers are content that students have the opportunity to practise what they have learned in lectures and receive immediate feedback.

A concerning issue is that these students perceived high marks in CAA to be an indication of satisfying implicit learning goals set by the lecturer. The lecturers indicated that CAA might not always be an appropriate test of the knowledge and understanding that they wish to test of students.

The students did not express their aims in terms of the learning that is required; their learning goals were stated in terms of percentages and they often achieved those goals. On the one hand, achieving these learning goals gave the students confidence and reassurance that they had learned the material. On the other hand, students had set superficial goals that did not indicate what had been learned, and perhaps this explains in part why more challenging goals were not set.

Our earlier work with a similar cohort revealed similar findings: students face a contradiction between pursuing more challenging learning goals and concentrating efforts on pursuing marks (Broughton et al., 2011). The culture and history to which these students belong weighs in favour of pursuing marks: past and future examinations are perceived to determine success; and since no marks are offered for learning beyond summative assessment material, it is perceived to have little value. Hence the learning cycle is broken.

The implication is that CAA is effective for low-level goals, where the depth or breadth of understanding is not important for the student. However, CAA did not inspire the students to continue the learning cycle and explore new learning goals, so there is a point where CAA is no longer effective. The challenge for CAA is to expose students to the value of pursuing further learning.

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MATHEMATICAL KNOWLEDGE BUILDING OF LOW ACHIEVERS IN A RICH LEARNING ENVIRONMENT – A CASE STUDY

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We traced the thinking processes and interaction patterns of low-achieving students studying subtraction with decimal numbers, as they worked in small groups within a rich learning environment involving a computerized game, play money, peer interactions and various scaffoldings and meta-scaffoldings. We used videotaped sessions, worksheets, observations and pre- and post-program teacher evaluations to describe and characterize their complicated, nonlinear knowledge construction process, and their shifts from old behavioral and cognitive habitudes to new ones. A case study of one such student is presented herein, demonstrating the potential of this innovative pedagogy.

INTRODUCTION AND THEORETICAL BACKGROUND

The question of how to increase the impact and effectiveness of teaching and learning processes presents an important challenge to researchers and teachers alike. This challenge is particularly significant when it comes to low-achieving students in mathematics, for whom many of the existing teaching practices appear to be of little benefit (Haylock, 1991).

Although the population of low-achieving students is heterogenic, some cognitive difficulties and behavioral characteristics are common. For example, such students find it difficult to retrieve basic mathematic facts from their memory (Geary, 2004) and to use effective computation strategies based on meta-cognitive skills (Goldman, 1989). They are sensitive to the learning context (e.g., written and oral arithmetic practices or everyday and formal mathematics), and find it much harder than other students to solve simple and complex addition and subtraction problems (Linchevski & Teubal, 1993). These difficulties may lead them to use less sophisticated strategies, and thus commit more errors. As they repeatedly experience failure and cannot keep up with the class, their motivation and self-esteem decrease. Therefore they might have a weak sense of internal responsibility, be passive and/or rely on external authority (Geary, 2004; Linchevski & Tuval, 1993; Haylock, 1991).

Some teachers believe that low-achieving students are unable to deal with high-order thinking-skills tasks (Karsenty & Arcavi, 2003). Such teachers typically conclude that the most effective way of promoting mathematical performance in low-achieving students is to ‘drill and kill’ (Anderson, Reder & Simon, 2000), that is to focus more on the mathematical algorithms than on the mathematical meaning.

Conversely, it is our belief that these students need an intervention aimed at improving their understanding of mathematical procedures, concepts and terminology. With the

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appropriate support, children may be able to advance in what Vygotsky (e.g., 1978, in Wertsch & Stone, 1985) termed the Zone of Proximal Development, the metaphorical area between their current cognitive performance on their own and their potential cognitive performance. The Learning in Context approach, namely presenting mathematical concepts and procedures in a context relevant to the child's day-to-day life (Gravenmeijer & Doorman, 1999) and with appropriate scaffoldings, may be a key to promoting meaningful learning for low-achieving students (Ben-David Kolikant & Broza, 2010).

This premise led us to design an environment in which studying subtraction's strategies involves playing a computer game enacting the managing of an ice cream shop. Games have the potential to engage and motivate students in becoming active rather than passive, by enabling experiments and explorations without fear of failing in front of the entire class (Squire, 2008; Gee, 2003). The use of games for teaching may thus be particularly beneficial for low-achieving students. Moreover, we hoped that through active participation in a meaningful and authentic learning environment, subtraction strategies for calculating change will develop naturally, as the concrete context of working with play money would serve as a cognitive scaffolding (Wood, Bruner, and Ross, 1976).

We were aware that in the context of learning mathematics with tools, meaning construction requires the guidance of a teacher, to mediate the use of tools and to orchestrate the students' activities (Mariotti, 2000). Hence, meta-scaffoldings are required. These "scaffolds for the scaffolds" (Pea, 2004) involve the teacher's support and directions to students who are using a tool or working with other scaffoldings.

The goal of the study is to examine whether and how students construct and use new knowledge and strategies within the environment. Here we focus on one fifth grade student, Tom, in a case study that demonstrates the complicatedness and non-linearity of the learning process in the environment, as well as the positive potential of such interventions.

METHODOLOGY

Twenty six low-achieving fifth grade students took part in the above-mentioned extracurricular program, for one weekly hour, for the duration of 8 weeks. They studied subtraction with decimal fractions prior to the topic being studied in their parent math classes, learning in small groups (up to 4 students), with a teacher trained by the first author. The instruction framework emphasized a delicate transition from the realistic environment to formal math. For this reason, for example, in the first four lessons, subtraction was presented only through monetary simulations and problems, with no formal exercises. From the fifth lesson onward, the formal representation of operations was interwoven into the learning situations, while maintaining the focus on authentic contexts.

When playing the learning environment's "ice cream shop" game (http://kids.gov.il/money_he/glideriya.html), the students acted as sellers: they received orders, prepared ice-cream, and then calculated and gave change. In addition, students were asked to work in supplementary online study units, which concerned the transition between money and formal representations, as well as change calculations. Students also enacted game-like situations with mock Israeli money (*shekels* and *agorot*).

While students engaged in computerized activities, the teacher stayed in the background, observing their work and difficulties, taking notes for the following discussion and intervening when needed. Much of class time was devoted to pair and group discussions. The teacher's interventions did not include direct corrections of students' strategies, but rather meta-scaffolding questions that encouraged the students to use the tools in the environment in order to build their own strategies.

Our primary data source was the transcripts of 8 videotaped, 45-minute-long learning sessions, accompanied by 8 screencaptured computer sessions video screenshots (about 20 minutes each). Other tools included pre-program student interviews focusing on mental computation strategies, observation of the parent mathematics classes, student evaluations filled in by their parent math class teachers' pre-and post-program, and individual worksheets each student filled in during the extracurricular lessons.

The transcripts were coded twice by two researchers. In the first step, we segmented utterances into episodes, so that each episode began with the presentation of a new task (Ben-David Kolikant & Broza, 2010). We then classified each episode, according to the problem it deals with, and examined: (1) who participated in it; (2) the tools that were involved; (3) the knowledge pieces that emerged; and (4) the difficulties that arose, including whether they were solved, and if so how and by whom.

In order to understand more deeply the knowledge building processes involved in working with the environment, an integrative analysis of the transcripts was conducted as a second step. For this purpose, we adapted the microanalysis epistemic actions model (RBC) developed by Hershkowitz, Schwarz & Dreyfus (2001). According to them, *Recognize* (R) means "identifying cases in which a student makes use of a construct or structure that has been constructed earlier" (p. 212). *Building with* (B) occurs "when students are engaged in achieving a goal such as solving a problem, understanding and explaining a situation, or reflecting on processes" (p.215). *Constructing* (C) refers to "finding a new phenomenon and reflecting on it, on its internal structure and on its external relationship to things they [students] know already" (p.216).

After identifying the episodes in which *constructing* occurred, we searched for historical evidence, i.e. indications in previous episodes, that could hint about the specific ways this new piece of knowledge could have been constructed (i.e. *recognizing* and *building with* actions). This integrative analysis enabled us to focus on the developmental changes in the student's thinking and behavior chronologically, as well as to examine it with respect to the literature of low-achieving students.

In the following section we present a case study of one of the low-achieving students who best demonstrates this process, tracking important episodes and aspects of his learning process within the environment.

TOM: A CASE STUDY

Our pre-program data regarding Tom indicated a passive, unmotivated student, with no solid mathematical strategies. During his interview, Tom used his fingers in order to calculate the result of exercises that require "breaking up the ten" like: 11-8 or 31-7 and wrongly calculated 50-28 and 100-18. He succeeded in adding exercises like 95+10 or 395+10. His teacher reported that Tom was unmotivated, used partial strategies, which led him to wrong answers, and had difficulties in providing explanations for his calculations. His math score was 60.

During the first half of the program (lessons 1-4), Tom was silent and seemed unconfident. During the computer assignments (game simulations and other online study units), he was more engaged and active, but still lacked confidence, allowing his peers to make the final decisions. Yet, whenever the teacher asked him to, he provided his own answers.

Tom constructs a new strategy

A shift in Tom's performance emerged in the fifth lesson. In fact, he *constructed* a new strategy for the first time. The task was to calculate verbally the change for a client who paid 20 *shekels* for an ice cream cone that cost 7.70 *shekels*. Tom's first answer was erroneous: 13.30. When asked by the teacher how he reached this result, he told her that he performed the formal subtraction exercise 20-7.70. When prompted for further explanations, Tom moved to the context of using money and expressed himself using addition, an operation he was more comfortable with (as indicated in his pre-program interview), rather than subtraction:

1. Tom: I add 30 *agorot* to the 70 *agorot*, it becomes 100 *agorot*.
2. Teacher: [Do you want me] to write 30 *agorot* plus 70 *agorot*? [Tom nods]
3. Teacher: O.k. so how do I write it? Tell me exactly how to write it.
4. Tom: Seventy plus thirty.
5. Teacher: [Writes on the board: 70+30] … this is how it should be written?
6. Tom: Yes.
7. Teacher: Yes? OK.
8. Tom: Equals 100.
9. Teacher: Yes. 100 what?
10. Tom: *Agorot*.
11. Tom: Now, we add this (30 *agorot*) to the 7.70, it becomes 7.100. Then while adding this….
12. Teacher: Wait, wait, what are we doing now? I am writing exactly all the exercises you tell me to write. So tell me what to write.

Tom’s shift to talking in monetary terms (line 1) could be viewed as a result of the teacher's demand for explanation, apparently a meta-scaffolding that forced him to reflect about his attempts at calculation. This situation reflects the group’s discussion norm, established earlier by the teacher, when she accepted procedures with the money model as satisfactory. This is probably the reason why, when the teacher insisted on formal writing (line 3), a task less familiar to Tom, his next explanation was still focused on the monetary context.

Tom's thinking in monetary terms was made even more evident when he reached the result of 7.100, which in formal terms is wrong. The teacher's hint ("what are we doing now?") was perhaps aimed at this. However, for Tom, this representation was apparently appropriate for his purposes: it had *shekels* on the left (of the "decimal" point) and *agorot* on the right. He thus continued with his mental money model:

13.	Tom:	And then 100 *agorot* are a *shekel*.
14.	Teacher:	Again? [seems that it is difficult for her to follow]
15.	Tom:	I see that 100 *agorot* is *shekel*.
16.	Teacher:	100 *agorot* are 1 shekel [writes as an equation]

The rule 100 *agorot* = 1 *shekel* was elaborated in lesson 3 by another student, in a different context (when converting *shekel* coins to *agorot*). Tom *recognized* the usefulness of this rule. In fact, he used it to *build* his own strategy. Tom's representation of 7.100 (utterance 11) is a unique scaffold that he *built* for himself, when converting 7 *shekels* and 100 *agorot* to 8 *shekels*.

Moreover, in this discursive move, Tom disregarded the teacher's repeated hints (utterances 2,3,5), aimed at moving his informal representation to a more formal representation in writing. Tom preferred staying with the familiar and meaningful context of using monetary terms, which was given legitimacy by the teacher up to that lesson, showing confidence and persistence for the first time. Nonetheless, as he moved on with his explanation, he did move to formal symbols. This time the teacher kept reminding him of the money context:

17.	Tom:	And then we add it to seven and it becomes…
18.	Teacher:	We add the *shekel* to seven? So, let's write seven *shekels*.
19.	Tom:	Plus.
20.	Teacher:	Plus one *shekel*, it becomes?
21.	Tom:	Eight.

At the end of the episode, Tom subtracted these 8 *shekels* from 20 *shekels*, which gave him 12. And then, to get to the final result he added the 30 *agorot* to the 12, coming up with the correct answer of 12.30. This was the first time he had used such a strategy of mixed operations, addition for the decimals and subtraction for the whole numbers.

Tom's use of new strategies across contexts: Progress and Regression

Tom didn't use the same strategy during the following lesson 6 and the beginning of lesson 7, although there were opportunities for this. Instead, he kept making calculation errors, as if regressing to his old habits.

A progressive move was noted later in lesson 7, during the group discussions and the meta-scaffoldings led by the teacher (e.g., creating conflict between two different strategies and using a checking procedure to find out whose strategy is correct). Tom elaborated another correct strategy (as can be seen in Figure 1, reproduced from his worksheets), feeling enough confidence to use only subtractions, both for the decimals and for the whole numbers.

Moreover, Tom returned to the rule that 100 *agorot* = 1 shekel in lesson 8, this time as an anchor to transfer his new mental subtraction strategy to the written context, while solving the exercise 10-3.99. Despite the relative dependence of low-achieving students on context, reported in the literature, Tom could express his strategy in writing intuitively (Figure 1).

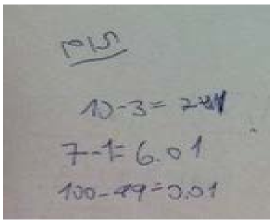

Figure 1: Tom's written representation

He then initiated a checking procedure, where he used the *shekels/agorot* conversion rule, again for converting "9.100" to 10 (Figure 2), similar to what he did in lesson 5. We concluded that this rule served as a scaffold for Tom to deal with subtraction problems.

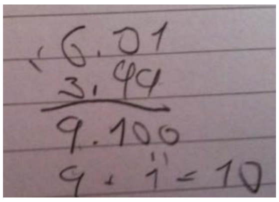

Figure 2: Tom's checking strategy

Tom's overall progress was also reflected in his parent math class teacher's post-program evaluation, which emphasized his increased flexibility in using various strategies during calculations, his reduced passivity and his improved motivation and explanation ability. Similar progress was observed in about a third of the students taking part in the program, and further examples will be presented at the conference.

DISCUSSION AND CONCLUSIONS

Through his participation in the program, Tom *constructed* new knowledge using scaffoldings such as the money model and the rules it entails (e.g., 100 *agorot* equal 1 *shekel*), as well as more abstract yet informal representations (e.g., 7.100). Although these informal representations did not align with the accepted and even more abstracted formal representation, they helped him perform correct calculations, use subtraction strategies for decimals as well as whole numbers, engage in mathematical discussions, and adopt mathematical habits (such as checking himself). He was able to apply his newly constructed knowledge to new tasks and situations, albeit inconsistently (using it in lessons 5, 7 and 8, but not 6, even when appropriate). Furthermore, he seemed more confident in his knowledge and abilities (as demonstrated by his disregard of some of the teacher's hints and prompts), more motivated and less passive (as reported by his parent math class teacher).

According to Yackel & Cobb (1996), children's thinking displays socio-mathematical norms that were shaped during interaction. Tom's cognitive and behavioral changes were made possible through the affordances of a specific combination of tools (e.g., the ice cream shop game), peer interaction, and the meta-scaffoldings and socio-mathematical norms initiated by the teacher's mediation (e.g., urging for explanations).

Tom's case study highlights the complex knowledge construction of low-achievers, characterized by progression and regression, and fragile and localized consolidation. Considering the methodological limitations, the indications of his progress might not be easily generalized to other contexts. However, as a micro example of the main findings (to be reported at the conference), Tom's story indicates the positive potential of our pedagogy's meaning-focused, teacher-mediated, context-driven instruction of mathematics for low-achieving students.

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INCONSISTENCIES IN STUDENTS' UNDERSTANDING OF PROOF AND REFUTATION OF MATHEMATICAL STATEMENTS

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The study reported herein is part of a larger study[1] that examined high-school students' understanding of the roles of examples in proving. Data is based on a series of students' interactions with specially designed mathematical tasks that elicit their thinking. The findings provide a complex account of students' conceptions and reveal inconsistences in their understanding. In particular, all students in our study exhibited indicators of understanding that for a universal statement to be true it has to hold for all cases. At the same time, some of these students remained convinced that a statement can be 'proven' through examination of several confirming examples.

BACKGROUND

NCTM (2000) Standards and Common Core State Standards for Mathematics (CCSSI, 2010) state that reasoning and proving are an integral part of school mathematics. In order for students to engage in proving they need to develop an understanding of the status of empirical evidence in proving and refuting mathematical statements (e.g., Harel and Sowder, 2007). However, studies consistently show that students at all grades and levels tend to rely on examples that satisfy a given statement as sufficient evidence for proving it. This phenomenon is referred to as *empirical proof scheme* (Harel and Sowder, 2007), *naïve empiricism* (Balacheff, 1988), or *example-based proof* (Healy and Hoyles, 2000). In addition, students often hold incorrect views with respect to counterexamples: they reject them or treat them as exceptions (Balacheff, 1988). These studies suggest that developing an understanding of the status of examples in proving (and refuting) is a non-trivial process.

Although, the Standards define instructional goals and outcomes, they do not specify the methods for achieving them (CCSSI, 2010). Stylianides and Stylianides (2009) maintain that there has not been enough research into the ways of supporting students in developing coherent understanding of the role of empirical evidence in proving. Thus, students are often left to develop this understanding on their own, with insufficient direct instructional support.

The goals of our study were to explore high-school students' understanding of the status of empirical evidence in proving and refuting mathematical statements, along with ways in which this understanding can be diagnosed and enhanced. To address these goals we developed a framework (Buchbinder and Zaslavsky, 2009) that captures our conceptualization of *what it means to understand the status of examples in*

[1] The study was supported by Israeli Ministry of Education.

determining the validity of mathematical statements. The framework provided a basis for constructing special tasks that elicit students' conceptions and for analyzing students' conceptions.

What does it mean to understand the status of examples in proving?

Every mathematical statement can be characterized by a domain *D* of mathematical objects to which it refers (e.g., *'all integers ending with 7'*) and a proposition that describes a certain property *P* (e.g., *'multiple of 7'*). A universal statement that is based on domain *D* and property *P* states that every *x* in *D* has the property *P* (e.g., the false universal statement: '*every integer ending with* 7 *is a multiple of* 7'). An existential statement that is based on domain *D* and property *P* states that there exists *x* in *D* that has the property *P* (e.g., the true universal statement: '*there exists an integer ending with* 7 that *is a multiple of* 7'). With respect to a given domain *D* and a property *P*, four types of mathematical objects can be defined, based on whether or not an object *x* belongs to the domain *D* or not, and whether it satisfies the given property *P* or not: 1. An object that belongs to *D* and has the property *P* (e.g. *x=77*). This is a *confirming* example, for both universal and existential statements; 2. An object that belongs to *D* and does not have the property *P* (e.g. *x=17*). This is *counterexample* or a *contradicting* example for the universal statement and a *non-confirming* example for the existential statement; 3. An object that does not belong to *D*, and has the property *P* (e.g. *x=70*); 4. An object that does not belong domain *D* and does not have the property *P* (e.g. *x=71*). Objects of types 3 and 4 are *irrelevant* to both kinds of statements (universal and existential). We separate them as they may be interpreted differently in terms of their logical status. Our framework describes the logical status of each type of example with respect to the two types of statements (Buchbinder and Zaslavsky, 2009). Thus, one confirming example is insufficient for proving a universal statement, but is sufficient for proving an existential statement. One counterexample is sufficient for refuting a (false) universal statement, but a non-confirming example is insufficient for refuting an existential statement. Irrelevant examples have no logical status in the sense that they do not support any proof or refutation of a statement.

In the spirit of Borgen and Manu (2002) we conceptualize 'understanding' of the roles of examples in determining the validity of mathematical statements in operational terms as becoming fluent with types of inferences that can and cannot be drawn based on the four types of examples with respect to two types of statements. In this paper we focus on students' understanding of the status of confirming and contradicting examples in proving or refuting of *universal* statements. According to the conceptual framework such understanding entails: (1) recognizing the type of the statement (universal); (2) realizing that in order for it to be true the proposition has to hold for all the elements in the domain; (3) realizing that confirming examples are insufficient for proving; and (4) understanding that a single counterexample is sufficient for refuting a false universal statement.

THE STUDY

Instruments

Based on the conceptual framework presented above, we constructed a collection of 6 types of tasks that aim at revealing and enhancing students' understanding of the roles of examples in proving. Each type of task addressed various aspects of the framework, and the collection as a whole covered all aspects of the framework[2] (Buchbinder & Zaslavsky, in press).

The tasks drew on topics from the regular 9th and 10th grade mathematics curriculum in Israel. While we wanted to ensure that students have the relevant content knowledge to cope with the tasks, we tried to confront them with statements that were unfamiliar to them, and which had a potential to evoke uncertainty regarding their truth-value. Uncertainty is widely recognized as a powerful trigger for creating situations that promote students' intellectual need for proof (e.g., Zaslavsky, 2005). The process of resolving the uncertainty can both reveal and enhance students' understanding. One type of task, inspired by Healy and Hoyles (2000) and by Zaslavsky and Ron (1998), which we term *"Who is right?"*, creates uncertainty by confronting students with a false universal statement followed by arguments of five hypothetical students stating their opinion on its truth-value. Student A uses multiple confirming examples to “prove” the statement; Student B refutes the statement with a single counterexample; Student C maintains that multiple counterexamples are needed; Student D maintains that the statement is false but does not accept counterexamples as sufficient, and requires a general argument; Student E maintains that since both confirming and contradicting examples exist, the truth value of the statement cannot be determined.

> Five students worked independently on determining whether the following statement is true or false: *For every natural number* n, n^2+n+17 *is a prime.*
> For each of the arguments raised by the students below, decide whether it is correct or not, and justify your decision.
>
> > *Tali:*
> > I checked the value of the expression for 10 different natural numbers (odd, even, prime) and in all cases the result was a prime. For example:
> > For $n=2$, I got *23*, which is a prime. For $n=3$, I got *29*, which is a prime.
> > For $n=11$, I got *149*, which is a prime. Thus the statement is **true**.
>
> > *Yael:*
> > I tried $n=16$ and got: $16^2+16+17=289$. *289* is not a prime since $17\cdot17=289$.
> > Thus, the statement is **false**.

Figure 1: Two parts of the algebraic version of the task ‘*Who is right?*’

[2] For discussion of types of tasks and the underlying design principles, see Buchbinder & Zaslavsky (2012).

For each argument, participants were asked to determine whether it is correct or not and to justify their decision. Figure 1 shows 2 parts (Students A & B) of the algebraic version of the task.

Data Collection

Two parallel versions of the tasks (algebraic and geometric) were implemented with six pairs of top-level 10th grade students from two distinct schools in the northern area of Israel. The group included 7 girls and 5 boys who volunteered for the study. Each pair of students participated in a series of six, one hour long, task-based interviews. Across all task types, each pair responded to 11 tasks involving universal statements. During the sessions, students coped with the different tasks with minimal intervention from the interviewer. There were no time constrains, so students could discuss the task with each other as much as they needed. Data collection included video recordings of the interviews, students' written work and researcher field notes.

Data Analysis

The data were analyzed using qualitative research methodology. Students' written work and utterances consistent with the framework were coded as 'indicators of understanding' (IOU). E.g., expressions stating that confirming examples are insufficient for proving. Students' responses inconsistent with the framework were coded as 'non-normative responses' (NNR). E.g., explicit acceptance of an example-based 'proof' as valid. Note that only *explicit* indicators of understanding (or mis-understanding) were coded.

Each task was chosen as a unit of analysis, even though multiple IOUs and NNRs could occur in it. Also, since students worked on the tasks in pairs, and it was not possible to distinguish between individual contributions, both types of indicators (IOU and NNR) were assigned to pairs, not to individuals.

FINDINGS

The findings provide a complex account of students' understanding. All students exhibited IOUs in each one of the aspects outlined by the framework. Note that each pair received 11 tasks involving universal statements, thus, there were 66 possibilities to exhibit IOUs, NNRs, or both.

With respect to *confirming* examples, we recorded 16 IOUs (Table 1). This relatively low rate (only 24%) can be related to the fact that only *explicit* indicators of understanding were recorded. As shown in Table 1, all pairs provided at least one explicit IOU that confirming examples are insufficient for proving. At the same time, *all* pairs also exhibited at least one NNR, such as justifying a statement by checking several confirming examples, or accepting such justifications, made by others, as valid. Overall, the same number of IOUs and NNRs was documented for understanding the status of confirming examples, with only two pairs exhibiting more IOUs than NNRs.

	Understanding the status of Confirming examples in proving		Understanding the status of Counterexamples in refuting	
Student pairs	No' of IOU	No' of NNR	No' of IOU	No' of NNR
Neta and Ronit	2	3	8	3
Nurit and Limor	6	4	18	1
Omer and Yaron	1	1	11	0
Tami and Natalie	2	4	6	3
Keren and Ben	2	3	11	2
Paz and Ronen	3	1	8	0
Total	16	16	62	9

Table 1: Distribution of indicators of understanding (IOU) and non-normative responses (NNR) with respect to the status of examples and counterexamples in proving and refuting universal statements.

All students provided multiple evidence of understanding of the role of *counterexamples*. Overall, 62 such IOUs were documented. In other words, in 94% of tasks involving false universal statements, students provided explicit indicators of understanding that a single counterexample refutes a universal statement. The 9 cases of NNRs reflect the instances in which students required multiple counterexamples for refuting a false universal statement.

We illustrate our findings through the case of one pair of students' encounters with the parts of the task illustrated in Figure 1.

The case of Neta and Ronit

Neta and Ronit started by checking some small values of *n*, which appeared to confirm the statement. Then they turned to examine the hypothetical students' arguments:

Ronit: Is Tali's response correct? Yes. Why? ….According to her results…

Neta: [While writing] In addition to Tali, we tried several numbers and every time the result was a prime. Thus, Tali is right.

Ronit: Wait! Look at the response of Yael. [Reads it aloud]. *289* is not a prime…

Neta: She is right, what can I tell you…

…

Ronit: So, first of all, Tali is right. It [the statement] is true but not for all natural numbers. Because here, Yael proved that if we take *n=16*…..It's not that the statement is false…. It's like…. this statement is false. It's not for every natural n. So here, Yael is right and Tali not. Because she [Tali] didn't check all natural numbers. Perhaps some of them do not [satisfy the statement].

Neta: The statement is false.

Ronit: So, Tali says that the statement is true, because she tried different numbers and the resulting numbers are primes. She is right, like, in her way, but she is not right in that…. the statement is false.

Neta: So, both Tali and Yael are right.

Ronit: Yael is right. It is not "for every natural *n*".

Neta: Yes. [While writing] Yael is right because she found a proof that not every natural number that we substitute for *n* gives us a prime number.

Neta and Ronit did not change their written justification for Tali's utterance. They moved on with the task but later returned to Tal's response. It seems that they realized that their acceptance of both Tali's and Yael's arguments constitutes a contradiction. Following is their attempt to resolve the conflict:

Ronit: OK. Now we have to go back to Tali. [Reads Tali's response aloud]. She is right!

Neta: Definitely. She is right. We can tell that Tali is right since we do not know what happened earlier.

Interviewer: What do you mean?

Neta: We have met her [Tali] earlier. And she is right. For example, we meet Tali on Sunday, and she proves to us that the statement is true. She gives us examples, gives us the whole investigation that she made, and she shows us that she got it right. We read her report, and we see that she is right. The next day, we meet someone else - Yael, and she shows us that the statement is false. So the first girl was right, but the second girl is also right. Afterwards.

Ronit: We can say that it [the statement] is false based on what Yael did. It is false because we saw what Yael did and we found out that not for every natural number that we substitute for *n*, the result will be a prime.

Interviewer: Do I understand correctly, that if you would not have met Yael, you would say that Tali's response is correct?

Ronit: Exactly.

Neta: Yes.

DISCUSSION

Applying our framework to analyse Ronit and Neta's case we can see that they correctly identified the statement as universal and explained that it has to hold for all natural numbers. They accepted Yale's counterexample as refutation and used it to justify why the statement is false. At the same time, Neta and Ronit referred to Tali's example-based argument as valid, even after direct prompting. Their line of reasoning can be described as "the statement is true, unless shown otherwise". Outside mathematics it is common to regard repeating evidence as true unless contradicting evidence is presented; which, in turn, does not necessarily overthrow previous results. It is possible that Neta and Ront's reliance on confirming examples for justifying universal statements stems from such 'every-day logic'. This is consistent with Leron and Hazan (2009) who maintain that in case of conflict between mathematical reasoning and every-day logic, students often resolve the conflict in favour of the latter.

Our findings outline a complex picture of students' understanding of the roles of examples in determining the validity of mathematical statements. Specifically, we identified two types of inconsistencies. The first type of inconsistency is manifested as discrepancies between students' responses to different tasks. In particular, with respect to the status of *confirming* examples in proving, the students, as a group, exhibited the same number of non-normative responses as the number of indicators of understanding (Table 1). This means that while on some tasks the students stated explicitly that confirming examples are insufficient for proving, on other occasions (or even on the same task) they used confirming examples to justify that a certain universal statement is true.

Some students justified the use of confirming examples by maintaining that they have been chosen in a specific way - systematically or by random. Balacheff (1988) terms this type of reasoning - *crucial example*. Neta and Ronit justified their reliance on confirming examples by referring to the timing of occurrence of a counterexample. Though their reasoning was unique for our group of students, we hypothesise that it can occur with other students outside our group. Thus, our findings concur with the literature on students' difficulties to accept the limitation of empirical evidence as means for proving (Harel & Sowder, 2007, Healy and Hoyles, 2000).

Contrary to the literature on counterexamples (Balacheff, 1988, Zaslavsly and Ron, 1998) the students in our study exhibited strong understanding of the status of *counterexamples,* accepting them as refutations. The data in Table 1 and Neta and Ronit excerpts from Neta and Ronit's discussion illustrate this finding.

The second type of inconsistency in students' understanding of the roles of examples in determining the validity of universal statements is their apparent lack of connection between the roles of examples and counterexamples in this process. From a logical point of view, to understand that in order for a universal statement to be true it must hold for all elements in the statement’s domain and that a single counterexample is sufficient for refuting a false statement, implies that confirming examples are insufficient for proving and that a general justification is needed (Harel and Sowder, 2007, Stylianides and Stylianides, 2009). Our findings, and specifically the case of Neta and Ronit, suggest that students held two conceptions that logically are contradicting.

Implications for education

Supporting the development of students' understanding of proving, is a non-trivial task for mathematics educators. One approach to that involves designing instructional tasks that highlight limitations of empirical evidence by emphasizing the role of counterexamples (Buchbinder and Zaslavsky, 2012, Stylianides and Stylianides, 2009). The type of task *Who is right?* proved successful in evoking uncertainty, and in promoting students' awareness of their own conceptions. In most cases this led to enhanced understanding of the roles of examples in proving. However, as our data show, some students did not resolve the uncertainty in mathematically correct way. More research is needed to determine the types of tasks and instructional scaffolding needed to promote students' understanding of the roles of examples in proving.

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LONGITUDINAL INVESTIGATION OF THE EFFECT OF MIDDLE SCHOOL CURRICULUM ON LEARNING IN HIGH SCHOOL

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This paper presents initial findings from a longitudinal study, which investigates the effect of the Connected Mathematics Program (CMP), a middle school reform curriculum on student learning in high school. The findings of this study showed that students who used the reform curriculum in middle school performed significantly better than or equally well to those students who did not use reform curriculum in their middle school. The findings of this study not only show the necessity of examining the curriculum effect beyond the grade levels, but also suggest possible effective ways to investigate the curriculum effect beyond the grade levels. In addition, the findings of this study suggest the potential long-term effect of problem-based mathematics curriculum on student learning.

In the past decade, a number of studies have been conducted to understand the impact of mathematics education reform in general and standards-based curriculum in specific in the United States (e.g., Cai et al., 2011; Harwell et al., 2007; Post et al., 2008; Reys et al., 2003). So far, researchers have focused mainly on examining curricular effects within a grade band. There has been no study that examines the relationship between using Standards-based or traditional curricula and students' learning *across the middle and high school grade bands*. This paper reports findings from a large project, *Longitudinal Investigation of the Effect of Curriculum on Algebra Learning* (LieCal). LieCal Project investigates how the use of different types of middle school curricula affects the learning of high school mathematics for a large sample of students from ten high schools in an urban school district.

BACKGROUND AND THEORETICAL BASIS

Standards Movement and Reform Curriculum

In the late 1980s and early 1990s, the National Council of Teachers of Mathematics (NCTM) published its first round of *Standards* documents (e.g. NCTM, 1989), which provided recommendations for reforming and improving K-12 school mathematics. These *Standards* documents not only specified new goals for school mathematics, but also specified major shifts in teaching mathematics. With extensive support from the National Science Foundation (NSF), a number of school mathematics curricula were developed and implemented to align with the recommendations of the *Standards*. The Connected Mathematics Program (CMP) is one of the *Standards*-based middle school curricula developed with funding from NSF (Lappan et al., 2002). The CMP curriculum was designed to build students' understanding of important mathematics through explorations of real-world problems. Students using the CMP curriculum are

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guided to investigate important mathematical ideas and develop robust ways of thinking as they try to make sense of problems based on real-world situations.

LieCal-Middle School Project

LieCal Project consists of middle school project and high school project. In the LieCal-Middle School Project, we have been investigating the differential effects of the CMP and more traditional (non-CMP) curricula on middle school students' learning of algebra. It was found that CMP and non-CMP curricula are very different. For example, in the non-CMP curricula, equation solving is introduced symbolically by using the additive and multiplicative properties of equality (equality is maintained if the same quantity is added to, subtracted from, multiplied by, or divided into both sides of an equation). On the other hand, in the CMP curriculum, equation solving is introduced using real-life contexts that are incorporated into contextually based justifications of the equation solving steps. We also conducted an analysis of the algebra problems in the CMP and non-CMP curricula (Cai, et al., 2010). The results strikingly illustrate the extent of the difference between the types of problems posed in the two curricula. We classified the mathematics problems in the CMP curriculum and one of the non-CMP curricula into four increasingly demanding categories of cognition: memorization, procedures without connections, procedures with connections, and doing mathematics. It was found that a significantly greater percentage of the tasks in the CMP curriculum (71%) than in the non-CMP curriculum (21%) are cognitively higher-level tasks (procedures with connections and doing mathematics) ($\chi^2(3, N = 3311) = 759.52, p < .0001$).

Besides conducting analyses of the CMP and non-CMP curricula, we also analyzed the instruction of the classroom teachers who implemented the two types of curricula (Moyer et al., 2011). We found that CMP teachers emphasized the conceptual aspects of learning significantly more often than the non-CMP teachers ($t = 12.40$, $p < .001$). On the other hand, non-CMP teachers emphasized the procedural aspects of learning significantly more often than the CMP teachers ($t = 10.43$, $p < .001$).

Regarding student performance, we found that on the open-ended tasks (assessing conceptual understanding and problem solving), the growth rate for CMP students over the three middle school years was significantly greater than that for non-CMP students (Cai et al., 2011). In particular, our analysis using Growth Curve Modelling showed that over the three years, the CMP students' scores on the open-ended tasks increased significantly more than the non-CMP students' scores ($t = 2.79$, $p < .01$). An additional analysis using Growth Curve Modelling showed that the CMP students' growth rate remained significantly higher than non-CMP students on open-ended tasks even when students' ethnicity was controlled ($t = 3.61$, $p < .01$). On the other hand, CMP and non-CMP students showed similar growth over the three middle school years on the multiple-choice tasks assessing computation and equation solving skills. These findings suggest that, regardless of ethnicity, the use of the CMP curriculum is associated with a significantly greater gain in conceptual understanding and problem solving than is associated with the use of the non-CMP curricula. However, those relatively greater conceptual gains do not come at the cost of lower basic skills, as evidenced by the comparable results attained by CMP and non-CMP students on the computation and equation solving tasks.

LieCal-High School Project

Our previous findings with middle school students are similar to the findings from research studies of the effectiveness of Problem-Based Learning (PBL) on the performance of medical students (Dochy et al., 2003; Hmelo-Silver, 2004). That is, using a PBL approach to train medical students, researchers found that PBL students performed better than non-PBL (e.g., lecturing) students on clinical components in which conceptual understanding and problem solving ability were assessed. However, PBL and non-PBL students performed similarly on measures of factual knowledge. When these same medical students were assessed again 6 months or a few years later, it was found that the PBL students not only performed better than the non-PBL students on clinical components, but also on measures of factual knowledge (Vernon & Blake, 1993). This result may imply that the conceptual understanding and problem solving abilities learned in the context of Problem-Based Learning facilitate the retention and acquisition of factual knowledge over longer time intervals. The CMP curriculum can be characterized as a problem-based curriculum (Cai et al., 2010). Analogous to the results of research on the learning of medical students in the PBL research, we found that CMP students outperformed non-CMP students on measures of conceptual understanding and problem solving during middle school. Also analogously, CMP and non-CMP students performed similarly on measures of computation and equation solving. Therefore, it is reasonable to hypothesize that the superior conceptual understanding and problem solving abilities gained by CMP students in middle school may result in better performance on a delayed assessment of manipulation skills such as equation solving, in addition to better performance on tasks assessing conceptual understanding and problem solving. The purpose of the LieCal-High School Project is to test the hypothesis. In this paper, we want to investigate how CMP and non-CMP students perform in high school on different learning outcome measures.

METHODOLOGICAL CONSIDERATIONS

Participants

In the previous middle school study, a quasi-experimental design with mixed methods has been used. We have followed more than 1300 students (650 using CMP and 650 using Non-CMP curricula) from a school district in the United States for three years as they progressed through grades 6-8. In the 2008-2009 school year, most of these 1300 CMP and non-CMP students from the middle school study entered high schools and became high school freshmen. In the LieCal-High School Project, we followed students enrolled in the 10 high schools that have the largest numbers of 1300 CMP and non-CMP students.

High School Curriculum and Instruction

All high schools in the district are required to use the same district-adopted mathematics curriculum. CMP and non-CMP students were mixed into each class in each of the ten high schools. Thus, all of these CMP and non-CMP students used the same curriculum and taught by the same teachers in their high school.

Student Learning Outcome Measures

In the LieCal-High School Project, we also used a quasi-experimental design with statistical controls to examine longitudinally the relationship between students' high school learning and their curricular experiences from their middle grades. We used various student learning outcome measures to assess student learning in high school. For example, we have developed open-ended problem solving and problem posing tasks to assess student conceptual understanding and problem solving. We have developed multiple-choice tasks to assess students' basic skills in algebra. We have also collected state assessment data, course grades, enrolments in advanced math courses, and SAT/ACT registrations and scores to assess student learning.

INITIAL FINDINGS

As we noted above, the results of the LieCal-Middle School Project presented parallels to the results of research on the learning of medical students using the PBL approach. CMP students outperformed non-CMP students on measures of conceptual understanding and problem solving during middle school. In addition, CMP and non-CMP students performed similarly on measures of computation and equation solving. Thus, we hypothesized that the superior conceptual understanding and problem-solving abilities gained by CMP students in middle school could result in better performance on a delayed assessment of manipulation skills, such as equation solving, in addition to better performance on tasks assessing conceptual understanding and problem solving in high school. So far, we have collected all of the achievement data for the LieCal-High School Project. While we are still conducting the data analysis from various aspects, the initial findings show evidence to support the hypothesis. That is, on all student learning outcome measures, CMP students performed better than or as well as non-CMP students in high school. In this paper, we present evidence from three learning outcome measures.

Ninth Grade Results

In the school district, Classroom Assessments Based on Standards (CABS) was administered to the 9^{th} graders every 6 weeks. Each CABS task typically consists of a single open-ended mathematics problem that students are asked to solve and explain. In the 2008-2009 school year, we provided the school district with field-tested CABS open-ended problems that aligned with the adopted high school curriculum. Every six weeks, the participating teachers administered one of the LieCal-provided CABS assessments to the 9th grade students in the 10 LieCal high schools. An Analysis of Covariance (middle school achievement as covariate) showed that on some tasks, the 9^{th} graders who used CMP in middle school performed significantly better than those

9^{th} graders who used non-CMP in middle school ($F = 4.69$, $p < .05$). On the rest of the tasks, CMP students performed equally well as tnon-CMP students.

Tenth Grade Results

Students in the school district were required to participate in the state test, which is a standardized test. It is composed of items specifically designed for the state to assess basic mathematical skills. The purpose of the state test is to provide information about student attainment of mathematical proficiency to students, parents, and teachers, information to support curriculum and instructional planning; and a measure of accountability for schools and districts. For the high school students, only tenth graders were required to take the state test. We have collected the data to see how CMP and non-CMP students perform on this state test.

As mentioned above, in the LieCal Middle School Project, we used both open-ended tasks to measure student conceptual understanding and problem solving and multiple-choice tasks to measure students' basic mathematical skills. We conducted an Analysis of Covariance (ANCOVA) using students' 6^{th} grade based line data on both open-ended tasks and multiple-choice tasks as covariates and 10^{th} grade state math test scale score as the dependent variable. As shown in Table 1 below, CMP students have significantly higher 10^{th} grade scaled mean score than the non-CMP students ($F(1, n= 492) = 7.76$, $p < 01$). In particular, the adjusted mean for CMP students on the 10^{th} grade state math test is 533.5 and 525.9 for non-CMP students. When we used students' 6^{th} grade based line data on open-ended tasks and multiple-choice tasks separately in the ANCOVA, the findings are similar. That is, CMP students have significantly higher 10^{th} grade scaled score than the non-CMP students when using open-ended tasks as a covariate ($F(1, n= 500) = 3.90$, $p < 05$) and using multiple-choice tasks as a covariate ($F(1, n= 502) = 5.13$, $p < 05$).

Covariate	F-Value	Significant Level
6^{th} Grade Project Multiple-Choice (MC) Tasks	5.13	< .05
6^{th} Grade Project Open-ended (OE) Tasks	3.90	< .05
Both PI Developed 6^{th} Grade MC and OE tasks	7.76	< .01
6^{th} grade State math scaled score	9.58	< .01
7^{th} grade State math scaled score	9.57	< .01
8^{th} grade State math scaled score	11.79	< .001

Table 1: Analysis of Covariance on 10^{th} Grade State Math Scaled Score

We also used 6^{th} grade state math test scaled score, 7^{th} grade state math test scaled score, and 8^{th} grade state math test scaled score, respectively, as the co-variable in the ANCOVA analysis, and we found that CMP students have significantly higher 10^{th} grade scaled score than the non-CMP students, as shown in Table 1. For example, the adjusted mean for CMP students on the 10^{th} grade state math test is 531, but 523 for non-CMP students, using the 8^{th} grade state test score as the covariate.

Eleventh Grade Results

In the eleventh-grade, we have administered 13 open-ended tasks to assess the impact of middle school curriculum on students' high school learning. Two of the tasks were problem-posing tasks. Students were given graphs or equations and then they were asked to pose mathematical problems based on the graphs or equations. Problem posing can be a feasible, liable, and valid measure of the effect of middle-school curriculum on students' learning in high school (Cai et al., in press).

A total of 390 11^{th} graders were included in the analysis (243 former CMP and 147 former non-CMP students). In order to compare the high school performance of those students who had used the CMP curriculum in middle school to that of students who had used more traditional curricula, we divided their scores from the baseline examination taken in the 6^{th} grade into thirds. Generally, when comparing the problem posing performance of the CMP students in each third to the non-CMP students in the same third, the CMP students performed as well or better than the non-CMP students in the same third. For example, when grouped into thirds using the baseline equation solving scores, the CMP students in the top third were more likely ($z = 2.01$, $p < .05$) to generate a problem situation that matched at least one of the graph conditions (slope and intercept). Similarly, the CMP students in the top third were more likely to generate a problem situation that reflected the linearity of the graph ($z = 2.40$, $p < .05$).

DISCUSSION

Curriculum reform is often seen as holding great promise for the improvement of mathematics teaching and learning. The findings of this study extended findings from earlier investigations that the effect of reform curriculum on student learning went beyond the grade band. In particular, the findings from this study showed that students who used CMP curriculum in middle school performed significantly better than or equal to those students who did not use CMP in their middle school.

The contribution of this study can be discussed from two aspects. First, the findings of this study not only show the necessity of examining the curriculum effect beyond the grade band, but also suggest possible effective ways to investigate the curriculum effect beyond the grade levels. In the past there is no study that has examined curriculum effect beyond grade levels. This study breaks the new ground in curriculum studies. Second, the findings of this study suggest the potential long-term effect of problem-based mathematics curriculum/instruction on student learning. In mathematics education, there is a growing consensus that problem-based mathematics instruction offers considerable promise. Theoretically, this approach makes sense. As students solve problems, they can use any approach they can think of, draw on any piece of knowledge they have learned, and justify their ideas in ways they feel are convincing. The learning environment of teaching through problem solving provides a natural setting for students to present various solutions to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding. Empirically, there are needs for more data confirming the promise of problem-based mathematics instruction. The CMP curriculum can be classified as a problem-based curriculum. The use of CMP curriculum in middle school not only has the positive effect on students' high school performance on open-ended problem solving (9th grade results) and problem posing (11th grade results), but also on basic mathematical skills assessed by the state test (10th grade results). Thus, the findings of this study suggest the potential long-term effect of problem-based mathematics curriculum on student learning.

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PRE-SERVICE PRIMARY TEACHERS' KNOWLEDGE FOR TEACHING OF QUOTITIVE DIVISION WORD PROBLEMS

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The purpose of this study is to analyse pre-service teachers' specialized content knowledge of quotitive division word problems with fractions. Two tasks were solved by 84 pre-service teachers integrating both the resolution of the problems and the interpretation of primary school students' answers. Results suggest that having a common content knowledge it is not enough to analyse and interpret students' answers and errors since the activity of interpreting need the specialized content knowledge. Furthermore, results indicate that the majority of pre-service teachers' interpretations were focused on the validity of the method and that there were more pre-service teachers that identified the procedural errors than the conceptual ones. Finally, some implications for teacher training programs are given.

THEORETICAL BACKGROUND AND OBJECTIVES

In this study we explore pre-service teachers' specialized content knowledge for teaching mathematics. This is a kind of content knowledge that previous research has shown that is seemed to be critical for effective teaching and could be acquired during teacher preparation programs (Morris, Hiebert, & Spitzer, 2009).

Since Shulman's (1986) initial research, many other studies have continued to specify the different types of mathematical knowledge needed for teaching. Ball, Thames, and Phelps (2008) proposed a more detailed classification of the mathematical knowledge for teaching (MKT) identifying three subcategories inside the content knowledge: the *common content knowledge* is the knowledge of mathematics that most educated people acquire; the *specialized content knowledge* is the mathematical knowledge that is unique and essential for teaching mathematics; and the *horizon content knowledge* is the knowledge needed to link the mathematical concepts and relate them with the curricula.

Specialized content knowledge is the knowledge that allows teachers to imply themselves in common teaching tasks such as representing the mathematical ideas to the students, evaluating whether student responses show an understanding of the key concepts and examining and understanding no common students' strategies. We focus on the specialized content knowledge since this is the knowledge that pre-service teachers can develop during their teacher education programs and because this knowledge is required to the interpretation of the students' understanding. This task has been highlighted as one of the professional tasks in mathematics teaching (Fernández, Llinares, & Valls, 2012; Son, 2010).

- In this study, we examine pre-service teacher content knowledge related to quotitive division word problems. Problems with a multiplicative structure have been broadly studied, particularly, quotitive division word problems, identifying students' strategies and students' difficulties and errors; some of them related with the interpretation of the remainder (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Downton, 2009; Vergnaud, 1994). However, it is necessary more studies focus on the teachers' knowledge for teaching of this type of multiplicative structure problems. This information is relevant for the teachers' training programs.

Pre-service teachers' knowledge of problems with a multiplicative structure

Research has shown that pre-service teachers have clear weaknesses in understanding quotitive division word problems. Graeber, Tirosh, and Glover (1986) pointed out that pre-service teachers tend to interpret the division only as a partitive that involves sharing a quantity (dividend) between a given number (divisor) of equal-sized groups and have difficulties in situations where they have to interpret the division as quotitive that involves finding how many groups of a given size (divisor) will go into a given quantity (dividend). Ball (1990) found that pre-service teachers have difficulties with the meaning of a division of fractions since they are able to interpret only the division as partitive. However, this interpretation is not valid in situations where the divisor is a fraction (quotitive division). Nillas (2003) described pre-service teachers strategies used when they solve problems with fraction divisions and showed that pre-service teachers' ability for solving does not imply that they have a good conceptual comprehension of the topic. Pre-service teachers of this study failed in problem posing.

Osana, and Royea (2011) designed a teaching experiment focused on fractions. Their results revealed a better conceptual understanding but pre-service teachers also failed in the ability of problem posing. Furthermore, these authors identified some cognitive obstacles in pre-service teachers when they tried to think about solutions and represent them symbolically. Tirosh (2000) also designed a course to improve pre-service teachers' knowledge about the division of fractions. Before the course, the majority of pre-service teachers were able to divide fractions. However, anyone knew how to explain the algorithm, and their interpretations of incorrect students' answers were based on the application of the algorithm or on difficulties related to the reading and comprehension. At the end of the course, pre-service teachers identified other causes of students' common errors, for instance, the interpretation of the division as partitive or a limited conception of the fraction concept or of the operations properties.

Most of the previous research is related to pre-service teachers' resolution of problems that required the division of fractions. However, there is not too much information about how pre-service teachers interpret students' answers in this type of problems (related to pre-service teachers' specialized content knowledge).

Objectives

The aim of the study is to examine pre-service primary teachers' specialized content knowledge related to quotitive division word problems. More specifically, the research questions were formulated as follows:

- How do pre-service primary teachers solve quotitive division word problems?
- How do pre-service primary teachers interpret students' strategies and errors in quotitive division word problems?

METHOD

The participants were 84 pre-service primary teachers (PPTs) in their first year of their degree. These pre-service teachers replied to two tasks (task 1 and task 2). In task 1, PPTs had to solve four quotitive division word problems. These problems were chosen and modified from previous research that had shown that pre-service teachers have difficulties in solving them (Verschaffel, De Corte, & Bogart, 1997). In task 2, PPTs had to analyse four primary school pupils' answers to each problem from task 1. In this study, we present the results of one of the problems that implies the use of a fraction, a continuous magnitude and it is asked for the quotient and the remainder of the division.

> You have four cakes. You would like to give three fifths to each child.
>
> - How many children can you give cake?
> - What part of the cake left?

Primary school students' answers (Figure 1) included in task 2 were selected attending to students' strategies and errors identified in previous research (Bulgar, 2003; Tirosh, 2000). In the answer A, the student makes each cake in 5/5 using a unit of measurement equal to the piece to be given to each child (3/5). He/she places this unit in the sliced rectangles that represent the cakes and then counts how many times do 3/5. The child commits a procedural error of measurement (only two pieces are in black color) and does not express the excess "3 pieces" as a fraction. Moreover, "3 pieces" cannot be the part of the cake left because it is equal to the divisor (conceptual error). In the answer B, the student uses fractions and divides the number of cakes between the fraction of the cake that you have to give each child. He/she uses correctly the algorithm but does not know to interpret the terms of the division 20:3 since the divisor means how many fifths you have to give each child and not the parts in which the unit is divided (conceptual error). So, the student expresses the remainder as 2/3 (and not as 2/5). In the answer C, the student uses also fractions but inverts the terms of the division as if this operation had the commutative property (conceptual error). Furthermore, he/she applies the algorithm incorrectly (procedural error). Both errors are neutralized and this is why the result is correct. In the answer D, the student also makes each cake in 5/5 and enumerates the pieces that give to each child. The answer is correct. PPTs had to grade students' answers with 0, 0.5 or 1 point and justified their punctuation. The justifications provided gave information about PPTs' interpretations of the method used by the students and whether they identified the errors made by students.

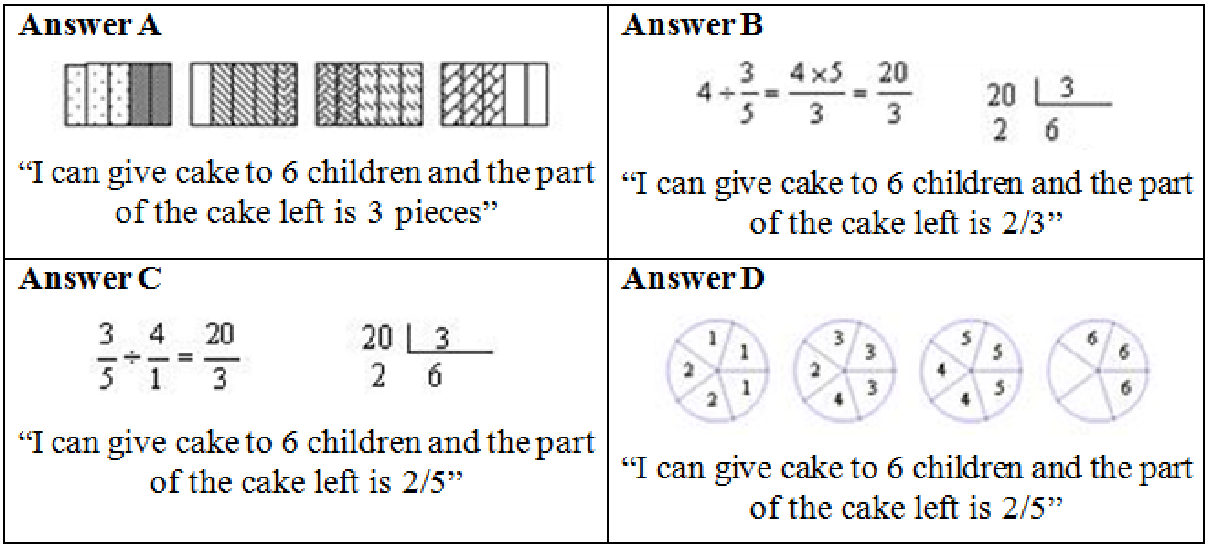


Figure 1: Students’ answers included in task 2

Firstly, pre-service teachers solved task 1 (problem solving) and 15 days later, they solved task 2 (interpreting the primary students’ answers).

Pre-service teachers’ answers to each problem in task 1 were classified as correct when the method was correct. Answers were classified as almost correct when pre-service teachers applied a correct method and indicated how many children can be given cake but they said “two pieces” or “2/3 of cake” (is that, they did not interpret the remainder correctly). Answers without sense were classified as incorrect. Furthermore, the methods used by PPTs were classified as the use of natural numbers, measurement or the use of fractions (Bulgar, 2003). Answers were classified as “the use of natural numbers” when PPTs converted the 4 cakes in 20 fifths, reasoning and justifying the response using natural numbers. Answers were classified as “measurement” when PPTs created a new unit of measurement equal to the piece of cake that you have to give to each child (3/5 of a rectangle or a circle) and took the measure on the 4 cakes. Answers were classified as “the use of fractions” when PPTs used operations with fractions (for instance, repetitive additions or subtractions or a division). Figure 2 shows an example of each method.

The PPTs’ justifications to the primary school students’ answers in task 2 were analysed individually by three researchers. The agreements and disagreements were discussed in an attempt to share categories in order to classify PPTs arguments. Once we shared these categories, we applied these categories to all the data. With regard to the interpretation of the method used by students we generated three categories: interpretations that are based on the validity of the method (if pre-service teachers considered whether the strategy works or not in the given problem situation), interpretations that are based on the generalizability of the method (if they considered whether the strategy works for any problem), and interpretations that are based on the clarity of the method to identify the remainder of the division (if they considered whether the strategy was presented in a clear way). If PPTs only considered the correctness of the result, we classified this answers as others. For example, the next

PPT interpretation of the answer D refers to the validity *"the student knows the strategy and develops it correctly; therefore he/she also obtains a correct result"*. The following PPT interpretation refers to the validity-generalizability *"It is a correct strategy but it is not the most adequate one. If the student had bigger numbers, this strategy did not work"*. And finally, the next PPT interpretation is related to the validity-clarity *"The strategy used to solve the problem (a drawing) led to see clearly that the student understands the problem and knows how to solve it. In this strategy, we can observe clearly the 2/5 of the cake"*.

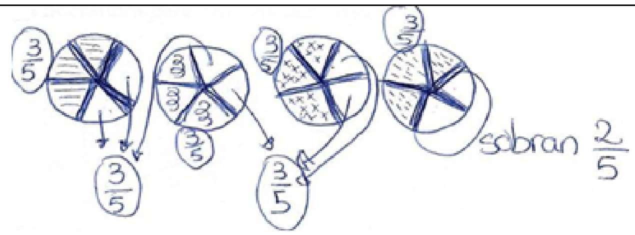
Method	Example
"the use of natural numbers" This answer was classified as almost correct since the pre-service teacher did not interpret the remainder	"The first cake is divided in 5 pieces. The second cake in 5 pieces. The third and fourth cake also in 5 pieces. So, we have 20 pieces ($5 \cdot 4 = 20$) If we give to each child 3 pieces, we divide the number of pieces between the number of pieces that we have to give to each child" 20 | 3 2 6 "We can give cake to 6 children and the part of the cake left is 2 pieces"
"measurement" This answer was classified as correct	 "You can give 3/5 of the cake to 6 children. The part of the cake left is 2/5"
"the use of fractions" This answer was classified as correct	1 pastel $= \frac{5}{5}$ 4 pasteles $= 4 \times \frac{5}{5} = \frac{20}{5}$ If you want to give 3/5 of the cake (pastel) to each child $\frac{20}{5} : \frac{3}{5} = \frac{20 \times \cancel{5}}{\cancel{5} \times 3} = \frac{100}{15} = \frac{20}{3} = 6'67.$ 20 | 3 2 6 "You can give cake to 6 children and the part of the cake left is 2/5"

Figure 2: Examples of each method

In relation to the identification of students' errors we examined whether PPTs identified the conceptual errors, the procedural errors, or both type of errors.

RESULTS

How do pre-service primary teachers solve quotitive division word problems?

Only 54.8% of pre-service primary teachers provided a correct answer to the quotitive division word problem, and 16.7% gave an almost correct answer, is that, these pre-service teachers used a correct method but failed in the interpretation of the remainder of the division (Table 1).

Method	Correct	Almost correct	Incorrect	Total
Natural numbers	11.9	8.3	0.0	20.2
Measurement	27.4	2.4	14.3	44.1
Fractions	15.5	4.8	7.1	27.4
Others	0.0	1.2	7.1	8.3
Total	54.8	16.7	28.5	

Table 1: Percentages of pre-service teachers who gave a correct, almost correct or incorrect answer and the method used

On the other hand, 44.1% of pre-service teachers used a measurement method, using a graphical representation to solve the problem, and 27.4% of pre-service teachers used a fractions method. 52.2% of the pre-service teachers who used a fraction method, they used the division algorithm. Furthermore, 56.5% of the pre-service teachers who used a fraction method gave a correct answer, is that, almost half of the pre-service teachers that used the fraction division algorithm or the repetitive subtraction of fractions did not interpret the remainder or gave an incorrect answer.

These results indicate that few pre-service teachers solved the problem correctly, and most of them struggled in the interpretation of the remainder. With regard to the method used, the majority of them used a measurement method implying a graphical representation.

How do pre-service primary teachers interpret students' strategies and errors on quotitive division word problems?

Table 2 shows the percentages of the type of interpretation that pre-service primary teachers gave to each primary school students' answers. Most of the pre-service teachers' interpretations were focused on the validity of the method (71.4%). However, some interpretations of pre-service teachers in answers A and D (graphical method) were related to validity-generalizability (7.2% and 17.8% respectively) and in answer D to validity-clarity (16.7%).

	Answer A	Answer B	Answer C	Answer D	Total
Validity	73.8	79.8	76.2	55.9	71.4
Validity-generalizability	7.2	0.0	2.4	17.8	6.9
Validity-clarity	1.2	2.4	3.6	16.7	6.0
Others	17.8	17.8	17.8	9.6	15.7

Table 2: Percentages of the type of interpretation that pre-service primary teachers gave to each primary school students' answers

Table 3 shows the percentages of PPTs who identified the conceptual and procedural errors of students' answers. For example the next pre-service teacher identified the conceptual error in answer B *"the method is correct but the result is incorrect. The remaining portions must be a multiple of 1/5 and you obtain 2/3. The student commits a* *mistake"* and the next, he/she identified the procedural error in answer A *"The method is correct but the result incorrect because the student left a piece unpainted"*.

Type of error	Answer A	Answer B	Answer C
Conceptual	4.7	23.8	4.8
Procedural	32.5	-	7.2
Conceptual-Procedural	0.0	-	1.2

Table 3: Percentages of pre-service teachers who identified the type of error in each answer

Any PPT identified all errors in student's answers. We underline the few pre-service teachers who identified the errors in answer C since they observed that the result was correct. Furthermore, there were more pre-service teachers who identified the procedural errors in answer A and C (32.5% and 7.2%, respectively) than the conceptual errors (4.7% and 4.8%, respectively).

CONCLUSIONS AND DISCUSSION

The aim of this study is to analyse pre-service primary teachers' specialized content knowledge of quotitive division word problems examining how they solve quotitive division word problems and how they interpret students' strategies and errors on these problems.

The majority of pre-service teachers who solved correctly the problem used a measurement method (graphical representation) (common content knowledge). Few pre-service teachers used fractions (division or subtraction). Furthermore, few pre-service teachers identified the students' errors (conceptual or procedural) and most of their interpretations of students' answers were focused on the validity of the method (few pre-service teachers focus on the generalizability of the method). These findings suggest that having a common content knowledge it is not enough to analyse and interpret students' answers that require specialized content knowledge.

This study also shows that there were more pre-service primary teachers that identified the procedural errors than the conceptual ones. This result is consistent with other studies such as Son (2010). Furthermore, if the result was correct (answer C) but not the procedure, pre-service teachers did not identify the error since they only focused on the result. On the contrary, if the result was incorrect (answer B), pre-service teachers analysed deeply the answer identifying the error.

Finally, there were pre-service teachers who did not solve the problem correctly but they identified the validity of the method used by students. We think that this type of task could be useful to help pre-service primary teachers to develop specialized content knowledge in teacher training programs.

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FACILITATING PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' LEARNING OF PROBLEM SOLVING

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This study explored two "self-study of personal experience" approaches to determine their effectiveness in helping prospective secondary mathematics teachers to develop mathematical problem-solving knowledge for teaching. Participants solved non-routine problems and recorded their processes using two different approaches. Both approaches were found to be helpful for their learning, but one was more effective in highlighting taken-for-granted cognitive aspects of the process, thus producing a more realistic view and model of problem solving as a way of thinking mathematically. The study offers support for the use of a combined approach in teacher education to provide a rich basis of personal experience to make sense of problem solving.

INTRODUCTION

Problem solving is central to the mathematics curriculum and an integral part of all mathematics learning. This is highlighted in the National Council of Teachers of Mathematics [NCTM] (2000, p. 52) problem-solving standard that states:

> Instructional programs from prekindergarten through grade 12 should enable all students to build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; and monitor and reflect on the process of mathematical problem solving.

This focus on problem solving is a shift from how it is treated in traditional mathematics classrooms and conceptualized by teachers whose knowledge of it is based on experiences in such classrooms as students of mathematics. This makes it a challenge for such teachers to engage students in genuine problem solving, that is, solving a task for which the solution method is not predetermined. This justifies the importance of ongoing research to identify ways to work with teachers to help them to transform their thinking and teaching of problem solving to meaningfully support students' learning of it and mathematics. This study contributes to this by exploring two "self-study of personal experience" approaches to determine their effectiveness in helping prospective teachers to develop problem-solving knowledge for teaching.

RELATED LITERATURE

Studies on problem solving in mathematics tend to focus more on the learner or problem solver. So the literature offers us insights into the nature of problem-solving expertise (Schoenfeld, 1985; Silver & Marshal, 1990); problem-solving strategies or heuristic processes (Polya, 1954; Schoenfeld, 1985); effective teaching of problem solving (Schoenfeld, 1985; Silver & Marshal, 1990); and belief systems regarding problem solving (Callejo & Vila, 2009). Teachers have been studied as problem solvers both in terms of understanding their approaches to problem solving and in

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helping them to become better problem solvers. As Thompson (1985) argued, teachers need to attain experience in "mathematical problem solving from the perspective of the problem solver before they can adequately deal with its teaching" (p. 292).

Some studies that focus on prospective mathematics teachers’ learning have investigated interventions to improve their problem solving. For example, Szydlik, Szydlik, and Benson (2003) studied an approach to help prospective teachers to become autonomous problem solvers by promoting community autonomy rather than autonomy of individuals. The participants worked on “demanding problems” in small groups, then discussed their findings, strategies, solutions and arguments. The authors found that the participants experienced a broadening in the acceptable methods of solving problems. They concluded that a classroom focusing on problem solving using a variety of strategies, reflection on the process of problem solving, and engagement in the process of exploration, conjecture, and argument can help prospective teachers develop mathematical beliefs that are consistent with autonomous behavior.

With a different focus, Guberman and Leikin (2013) studied the development of prospective teachers' problem-solving competencies through the use of multiple-solution tasks. They identified the participants’ strategies used in solving the multiple-solution elementary mathematics problems and their ability to produce multiple solutions to the problems they solved. They found that the multiple-solution tasks were effective in helping the participants, whether high or low achievers in mathematics, to significantly improve their problem-solving competencies. There was also a significant shift in the participants’ problem-solving strategies for multiple- solution tasks from mainly trial and error strategies used in the pre-test towards systematic strategies in the post-test. By the end of the course, the participants were solving problems flexibly, changing representations used, and employing more advanced problem-solving strategies.

In this study the focus is not on improving the prospective teachers’ problem solving but their understanding of the process for teaching. The emphasis is on a self-study process in which the prospective teachers record and learn from their own experiences solving mathematical problems through individual self-reflection and collaboration with peers. Thus the study offers an approach that highlights self-study and personal experience as a basis of prospective teachers’ learning.

THEORETICAL PERSPECTIVE

As in Mason, Burton, and Stacey (1982), problem solving is being considered here in relation to mathematical thinking. This study is also framed in Mason et al.’s view of the importance of self-study of personal experience as a basis of improving mathematical thinking or problem-solving ability. They suggest that problem solving “can be improved by tackling questions conscientiously; reflecting on this experience; linking feelings with action; studying the process of resolving problems and noticing how what you learn fits in with your own experience” (p. ix).

Mason et al. (1982) encourage the writing of one’s thinking to help one notice and thereby to learn from one’s experience. There are several things worth noting, particularly: key ideas; key moments that stand out in one's memory; and positively what one can learn from this experience. To facilitate this process, Mason et al. suggest four key words to use in making notes and in one's thinking: Stuck!, Aha!, Check, and Reflect. Whenever one realizes one is stuck, one writes down *Stuck* and why one is stuck. "For example: I do not understand . . .; I do not know what to do about . . .; I cannot see how to . . .; I cannot see why . . ." (p. 16). Whenever one gets an *Aha*, i.e., an idea or thinks one sees something, write it down. For example, "write down Aha and follow it with Try . . .; Maybe . . .; But why . . ." (p. 16). One then *Checks* any calculations or reasoning; any insight on some examples; that the resolution does in fact resolve the original question and *Reflects* on what happened. These key words provide a scaffold around which a resolution is built, and encourages checking and reflecting on one's resolution, an essential ingredient for improving one's mathematical thinking.

While this process is intended to improve one's problem-solving or mathematical-thinking ability, in this study it is being adapted to improve mathematical problem-solving knowledge for teaching [MPSKT]. The focus is on one aspect of MPSKT identified in Chapman (2012); *knowledge of problem solving*, i.e., "teachers should have conceptual and procedural knowledge of mathematical problem solving. This includes understanding the stages problem solvers often pass through in the process of reaching a solution" (p. 108). In particular, the goal is to check the effectiveness of a self-study of experience with non-routine problems using only the key words Stuck and Aha as a basis of doing this compared to one without them.

METHODOLOGY

The participants were 20 prospective secondary mathematics teachers [PSTs] in the second semester of their two-year post-degree education program. This was their first course in mathematics education that included a focus on learning through and about problem solving. The study focused on the following intervention intended to help them to understand the problem-solving experience to develop this aspect of MPSKT.

The intervention consisted of two approaches. In **Approach 1**, the PSTs were required to solve three problems; describe in detail narrative form the processes they went through in solving them; develop a model of problem solving based on the processes; share and discuss their processes and models in small groups; develop a group model of problem solving and represent it as a flow chart. In **Approach 2**, they were required to solve three different problems and describe their solution processes by recording every time they were stuck and got an idea (an Aha!); develop a model of problem solving; share and discuss their processes and models in small groups; develop a group model of problem solving and represent it as a flow chart. Following this, they wrote journals reflecting on what they learned from Approach 2 that they did not from Approach 1. All of the problems were taken from Bolt (1989) so they are of similar nature. Three were used in each approach to allow the PSTs to see a pattern to develop their model of problem solving. Examples of these problems:

Approach 1: Emma was always looking for ways to save money. While in the remnant shop she came across just the material she wanted to make a table-cloth. Unfortunately the piece of material was in the form of a 2m x 5m rectangle and her table was 3m square. She bought it however having decided that the area was more than enough to cover the table. When she got home however she decided she had been a fool because she couldn't see how to cut up the material to make a square. But just as she despaired she had a brainwave, and with 3 straight cuts, in no time at all, she had 5 pieces which fitted neatly together in a symmetric pattern to form a square using all the material. How did she do it? [p. 23]

Approach 2, Problem 1 [A2P1]: For the end of season squash tournament there were 27 entries. The tournament was arranged on a knockout basis with the loser of each match being eliminated. A number of players received a bye in the first round so that from the second round onwards the number of players going forward at each stage was halved. Norman and Theresa, the squash captains, met to arrange the draw. Their first problem was to decide how many matches would be needed in the first round and hence how many players should have byes. Norman was worried, he didn't really know how to begin, but Theresa with experience of organising tennis tournaments on similar lines was very quickly able to say how many rounds would be needed, how many byes to give and how many matches there would be in the whole tournament. What are the numbers involved? How many matches would need to be played in a tournament with N players? [p. 17]

Approach 2, Problem 2 [A2P2]: The micro millionaire studied his balance sheet at the end of the year with great interest. The total income from the sale of the very popular Domomicro model came to ₤1,000,000,000. What aroused his interest was not so much the total as that neither the number of micros sold nor the cost of an individual micro contained a single zero digit. How many micros were sold? [p. 23]

Data consisted of copies of all of the PSTs' written work required for the activities. There were also field notes of their small-groups and whole-class discussions. **Data analysis** used an emergent approach to identify (i) the nature of the participants' description of the their problem-solving processes for the two approaches; e.g., what they considered to be Stuck and Aha, how they represented them, what they highlighted in the process; (ii) the nature of their problem-solving models in relation to Polya's four-stage model; (iii) what Approach 2 offered them over Approach 1 in terms of their learning about problem solving and teaching it. This information was summarized for each participant and compared for similarities and differences in their processes and thinking. The findings reported here regarding the effectiveness of the approaches are based on what was common conceptually in their processes.

FINDINGS

There were significant differences in nature of the PSTs' descriptions of their processes in Approach 1 versus Approach 2. Representative descriptions for Approach 1, problem 1 is in Table 1 and for Approach 2, problem 1 in Table 2 for PSTs 1 and 2.

PST 1	PST 2
I read the problem and immediately thought I would draw rectangles measuring 5” by 2”. ... After a couple of unsuccessful drawing, I realized that drawing wasn’t helping me much and that the rectangle may need to be folded and then cut to make the 5pieces necessary. I cut out some rectangles and began folding them. … At this point, I thought of the calculation of area of the square and something didn’t feel right when I was folding the rectangles. … In my initial folding attempts, I kept folding the short side of the rectangle over to form a 2x2 square and a 3x2 rectangle. After realizing that each side of the resulting square would have to be √10 or 3.162, I stopped doing this and tried another strategy of folding the rectangle lengthwise to determine whether I could get 5 pieces that way….	I read the problem and pick out all the facts so I can start looking for a pattern. … If I have 3 cuts, how can I make 5 pieces? Try a few and see how the pieces look. … So I will need to make 4 pieces with 2 of the cuts and then cut one of those 4 pieces to create the fifth piece. Now that I know how that works, I need to try doing this on the 2 by 5 rectangle…. No, this doesn’t make sense… this makes a 3x3 shape but this only uses 4 pieces instead of 5 and one of the pieces doesn’t fit in. Okay, let’s try again. … Ok, but this only has 4 pieces, so it doesn’t make sense, I don’t want to build this with all the decimals. Besides, I don’t think breaking it into decimals lengths will help. Maybe the square is 4 by 4? … No, that doesn’t fit properly. Maybe I am going about this completely wrong. I’m going to reread the problem...

Table 1: Samples of PSTs’ description of process in Approach 1, Problem 1

PST 1	PST 2
S [Stuck!]: I don’t see how to ensure all evens, because a number like 14 will give 7 remaining, then a bye is required A [Aha!]: The numbers should be powers of 2 (2^n) S: I’m not sure that this doesn’t eliminate some legitimate possibilities. S: I don’t see how to check with-out just picking numbers at random to test that are even but not 2^n A: Trying with random even numbers smaller than 27 ... S: Starting with 16 players means 11 byes in first round, then where do they go? S: I don’t know … if we are not to have any byes after the first round, it’s ambiguous. If we can, changes the	S: I’m not sure what to do with the odd number of players A: Try to divide 26 by 2 since half of the players have to get byes S: I cannot see how that is going to work since the 2nd round will have 10 matches which isn’t half of 7 matches in the 1st A: Try having all the players playing in round 1 except one. …. S: I’m not sure what to do with this extra person in the 3rd round. It makes sense to give the person a bye to the 4th round but the problem only mentioned byes in the 1st round. A: Maybe the first round isn’t the only place that players received a bye. … Therefore there are 5 rounds 2 byes and 26 matches. So in a tournament with N

problem completely A: Will allow byes in any round ... Each round that finished with an odd number also had a bye … S: I can't be sure that will always be the case A: Try testing numbers near 27 to see what pattern develops [… figuring out …] A: Number of matches is N-1 always. … Number of rounds is given by the power of the next highest power of 2 ... S: I can't nail down the pattern for the byes, although I can see one developing …	players, there will be N-1 matches S: looking back on the problem, I have a few problems with the odd numbers which don't divide evenly. … A: Maybe that is how we can figure out how many byes we'll have all together. Every time we get an odd number of winners in a round we'll get a bye …. S: I'm not sure if the question with N players is using the same method as the problem uses A: Try some other tournaments with different numbers of players and see how many matches they get …

Table 2: Samples of PSTs' process description in Approach 2, Problem 1

For Approach 1, there was limited attention to Stuck in the PSTs' descriptions of their process. For most of them, the focus was on the strategies they thought would work to resolve the problem. They recognized when something did not work or make sense or feel right, as in the case of PST 1 and PST2, when considered a major block to their progress, but these often were framed in a way that seemed to be less important to the process. The result was a process with the possibility of barriers to one's strategy. The general orientation of their problem-solving model was to read problem, try a strategy, if does not work try something else, if out of ideas start over (re-read question) or quit.

For Approach 2, without exposure to theory on the nature of Stuck and Aha, the PSTs' seemed to have a very good understanding of what they involved when considered in relation to Mason et al. (1982). They were very detailed in identifying them with equal attention throughout the process. For example, they all identified some level of Stuck after reading the problem, as in the case of PST 1 and 2, Approach 2, problem 2 [A2P2]

> S: I do not understand the question. How can you figure out how many he sold if you only know the total sales value? [A2P2, PST 1]
>
> S: What to do? The numbers are too big. I don't see where to begin. [A2P2 PST 2]

This started a process that involved a series of Stuck-Aha-Check cycle. Thus the general orientation of the PSTs' problem-solving model was now a complex path with problems nested within problems. For example, each Stuck was a problem of various level of challenge within the problem. Each Aha was a plan to get out of Stuck followed by trying the plan and reflecting on the outcome. Thus, instead of, for example, one four-stage process to a solution as associated with Polya (1954), the problem-solving process emerged as a nested series of a cycle of four stages where each stage could have a Stuck-Aha cycle.

The PSTs highlighted better understanding from Approach 2 of the meaning of heuristics, having a plan, being stuck, and how to support their students' problem solving. For example, based on their shared experience, they understood "having a plan" more meaningfully in terms of the Ahas. One PST explained, "it is really an Aha; an idea that you think will work but does not necessarily work out as intended and could lead to another stuck." Based on this perspective, their process reflected several ways of approaching "a plan". For example, it could be a conjecture.

> A: I still think it has something to do with the fact that neither N (the number of units sold), nor C (the unit cost) contains a single zero digit. [A2P2, PST 1]

> A: maybe the interpretation of this does not matter, because all we need to be concerned with is the fact that his are lighter than normal [A2P3, PST 1]

It could be recalling something.

> A: I remember any number in base ten can be represented as …

It could be thinking of a simpler situation (what Mason et al (1982) call "specializing")

> A: Let me try and simplify the problem ... Maybe I should try another simplified version of the problem, but this time let V=1000 and odd power of 10. ... [A2P2, PST 1]

> A: Break it into smaller numbers … which are powers of ten ...[A2P2, PST 2]

It could be looking for patterns.

> A: I see a pattern. There are two factors for all powers of ten ...[A2P2, PST 2]

Their process also reflected different ways of thinking about being stuck. They saw it as anything that impeded their progress, whether small or large; e.g., [A2P2, PST 1]

> S: But what does that really mean? What does it mean to not contain a single zero digit?

> S: The problem asks for how many were sold. … Does that mean the problem is looking for a unique answer? I do not know. [A2P2, PST 2]

Approach 2 allowed the PSTs to notice how being Stuck allowed them to reflect, to think about what to do, to think about what they know and didn't know, to be challenged, and to think mathematically. Thus they were necessary to learn to think when solving problems and to work on such problems. Recording Stuck allowed them to persevere because they knew what they needed to overcome and think through to continue. They quit only when they kept cycling to the same stuck and needed help to shift their thinking out of it. They needed someone to intervene. One explained, "I got to a point where I needed someone to give me a hint. … I see the importance of working with others." This understanding allowed them to make sense of when a student is stuck and how to intervene with a question or prompt that would make sense to the student. They realized the importance of Stuck and allowing students to figure out why and not intervening too soon.

CONCLUSION

Both approaches were helpful for the PSTs' learning about problem solving [PS]. However, Approach 2 highlighted more of the taken-for-granted cognitive aspects of the PS process, thus producing a more realistic view and model of PS as a way of thinking mathematically. In Approach 1, the PSTs included affective aspects in describing the PS experience (e.g., feelings of frustration) while Approach 2 focused mainly on the cognitive aspects. Thus an approach that combines the two will likely be more powerful than each separately in capturing the lived experience of PS. The study offers support for the use of such an approach in PST education to provide a rich basis of personal experience for a meaningful self-study to make sense of PS. It will allow the PSTs to go beyond describing the steps leading to a solution which gives the appearance of a linear process and hides the thinking aspect of it. The peer collaboration and discussions, creation of problem-solving models are also key aspects of the approaches to allow them to validate and extend their processes and what PS means. However, such approaches are not sufficient by themselves to help their overall development of MPSKT, which involves other factors not dealt with in this study.

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MATERIALS USE FOR TEACHING GEOMETRY IN TAIWAN

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This is a study focused on materials use for teaching Year 6 geometry. The results from classroom observations and interviews of ten Year 6 teachers and a national survey show that primary school teachers in Taiwan tend to use traditional materials, such as diagrams on wall charts or drawings and concrete materials. They think that manipulations and observations of concrete materials are the best ways for teaching geometry but they tend not to notice their limitations (three examples are given) or notice the advantages of using IT materials. Taiwanese teachers need to learn how IT materials can be used for teaching specific geometry topics, and their practical concerns of easy access to computers and projection in classrooms must be addressed.

INTRODUCTION

Traditional geometry teaching usually shows theoretical properties and principles with diagrams and students solve geometrical problems from diagrams (Berthelot & Salin, 1998). After Piaget's emphasis on individual cognitive processes and activity for learning, kinaesthetic activities, such as drawing and folding shapes, became prominent in geometry teaching and learning. Thus, activities with concrete materials, such as sticks, mosaic puzzles, tangrams and cubes, are nowadays generally used by teachers. Other researchers (e.g., Clements & Battista, 2001; Hannibal, 1999) note that children already have some basic geometric concepts about shapes before they enter school and stress that materials for teaching geometry can also be from the students' everyday world such as toys, books or even TV programs.

More recently, many IT materials for teaching geometry have been developed, including Logo driven Turtle Geometry, computer software 'apps' displayed with interactive whiteboards, and multiple Dynamic Geometry Environments (DGE), including the programs of Cabri-3D, Geogebra and the Geometer's Sketchpad (GSP). Their use has received a great deal of research attention. Much research has shown dynamic geometry software can have positive effects on geometry learning for students (Chen, Lai, Tsai, & Huang, 2007; De Lisi & Wolford, 2002; Laborde, Kynigos, Hollebrands, & Stresser, 2006; Leung, 2008; Vincent, 2003). They have researched how computers can show properties of shapes dynamically, support manipulations and interactions between the geometric figures by learners, stimulate the need for proof, and provide the best solution for visual representations.

However, although academic research has paid much attention to IT materials use and shown advantages for using these dynamic materials, this may not have yet penetrated into common practice. This paper investigates whether the current primary teachers in

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Taiwan understand the benefits of IT materials use for geometric learning, and whether they use these materials in their teaching.

METHODOLOGY

This paper reports results on just one aspect from a large study of Taiwanese teaching of geometry (see Chiang, 2013). The main part of this study is a set of case studies of the teaching practices of ten current Year 6 primary school teachers. There is also a national survey which was intended to provide data on whether the in-depth findings from the case study were likely to be representative of all teachers in Taiwan. Year 6 is an important year for learning geometry in Taiwan, and so was a good choice for a study of primary teachers.

Participants and their schools

In the case studies, all ten participant teachers (called T1 to T10) were qualified, current Year 6 teachers with at least five years of primary teaching experience in Taiwan. The backgrounds of these teachers showed diversity of gender (8 women, 2 men), ages (from 28 to 50), type of teachers' training institutes, years of teaching experience (from 5 to 30 years), and having undertaken postgraduate study or not. The teachers were from four schools with varying characteristics and they used different textbooks. Thus, the results from these ten teachers are likely to represent typical geometry teaching in Taiwanese primary schools.

The survey used stratified random sampling to select 30 primary schools across Taiwan and all Year 6 teachers in these schools were invited to participate. The aim was to sample around 1% of the 10,671 Year 6 teachers distributed in 2,563 primary schools across Taiwan, Further details are given by Chiang (2013). All 30 selected schools participated, and 152 teachers completed the survey during the 2008 school year (72% response rate). There were small omission rates on individual items - up to 7 missing responses in the items reported here - which were considered in calculating average responses.

Data collection

Because this study aims to investigate what materials Taiwanese Year 6 teachers use for teaching geometry and why they use them in the classroom, it is important to see what the teachers do in practice and talk to teachers to understand their reasons. Thus, in the case studies, the research instruments are a classroom observation framework, and a semi-structured interview schedule. Several types of documents were also collected and reviewed including diaries and planning notes kept by the teachers throughout the observation period. For the specific focus of this paper, observations supplied the main data on what kinds of materials the teachers selected to use for teaching; the interviews provided the main information about why they made these choices. The survey used a questionnaire with Likert items designed by the first author. One item is relevant to this paper. All instruments and interviews were established in Mandarin, the official language in Taiwan, and then translated to English for the purpose of this report. Pilot testing of the questionnaire, interview schedules and the classroom observation framework for the case study by volunteer teachers in Taiwan was followed by refining the schedules and framework before implementation.

Each case study teacher was observed when he/she taught the lessons of one geometry topic in school year 2008/2009. On average about 9 consecutive lessons were observed per teacher. The observed topics included "The properties of two-dimensional shapes", 'The area formulas of (rectilinear) shapes", "Circumferences" and "The area formula of circles". After finishing all the lessons of the topic, each teacher was interviewed face-to-face and one-to-one by the researcher, and the interview included questions about the teachers' choice of materials. The interview questions were (1) "Why did you use this [material] for showing this [some geometric concept] ?", (2) "Why (not) use IT materials for representing this [some geometric concept]?" and (3) "Have you ever used IT materials for showing geometric concepts? Why?"

The questionnaire for the survey contained one item on materials use: "Which materials do you use to teach geometry lessons?" Participants were provided with a list of materials as shown in Table 1 and there was also space for participants to add other materials not listed. For each material, participants selected responses from "always", "often", "usually", "seldom", and "never" which were scored sequentially 5 (always) to 1 (never) for analysis.

The strength of the data collection arises from the range of teachers in the study, the variety of textbooks used and topics taught, the number of lessons observed, the in-depth observations. Lessons were closely observed by the researcher and video-taped for later analysis, and teachers were able to explain reasons for their practices in the subsequent interview. The national survey also provides the broad picture around Taiwan and enables the researcher to test the generality of the findings.

RESULTS

The results from both parts of the study shows the teachers generally ignored IT materials for teaching geometry, and thought concrete materials or real-world objects are the best materials for teaching geometry. In doing this, they overlooked some of he inadequacies of traditional materials. The following sections discuss the details.

The materials used by the teachers

Firstly in the case study, the classroom observations showed that these ten teachers mainly used the concrete materials provided by the textbook publishers. These come ready-made with most textbook series in Taiwan. They include wall charts, large geometric tools for the blackboard, concrete materials for teacher demonstrations and accessory books for each student with pre-marked card for constructing and cutting, folding, transforming and comparing. In the observed classes, teachers often used the wall charts for showing diagrams and they often drew accurately constructed diagrams using the large blackboard set-squares and compasses for explaining the concepts or answers. In addition, the majority of the teachers used concrete materials from the supplied students' accessory books or the materials box for transforming or combining shapes. Some of the ten teachers drew students' attention to real-world objects which illustrated geometric concepts, such as the frame of the window for parallel lines, and a single-wheeled cycle for the circumference of a circle. Some teachers even made classroom materials by themselves, for example to show the area of overlapping shapes and sectors. However, only one teacher used the instructional DVD that was provided with the textbooks and none of them used other IT materials.

The result from the survey item on materials use x also shows the teachers (N=152) tended to use the materials provided by the textbook publishers and rarely used IT materials. The average and mode ratings shown in Table 1 give measures of the popularity of each material. Using the textbook and the wall charts and drawings was almost universal. IT materials use is far less than use of the other materials except materials personally made by the teacher.

Materials	Average rating	Mode rating
Textbook and its exercise book(s)	4.7	5 (always)
Concrete materials supplied with textbook	4.2	4 (often)
Wall charts or blackboard drawings	4.7	5 (always)
Your personally created materials	3.0	3 (usually)
IT materials	2.8	2 (seldom)
Real-world objects	3.7	4 (often)

Table 1: Summary of ratings for use of geometric materials (N= 152)

The results of both the case studies and the survey are clear. Taiwanese Year 6 teachers mostly use traditional materials for teaching geometry, such as diagrams displayed on the wall chart or the blackboard, and manipulation of concrete materials (including real-world objects). The textbook publishers also tended to prepare these traditional materials for the teachers.

The reason why the teachers choose these materials

The interviews probed why the teachers made the choices of materials that they did. The teachers generally explained that they thought using concrete materials is the best or most appropriate method for teaching geometric concepts. The majority of the ten teachers said that they would not generally use IT materials for teaching geometry. Only two of the teachers were able to describe any IT materials that would be helpful for teaching specific topics. One teacher mentioned IT materials for teaching angles and another knew about materials for teaching the area formula of circles.

Four of the case study teachers replied that they had used IT materials for teaching geometry in the past and six teachers replied that they had not. Two of these six teachers believed they would disadvantage students. T4 said “The materials of technology [e.g., computer software] - I won’t use them because I feel it only wastes time to set up everything. I also need to darken the classroom. It is inconvenient for me”. T10 showed similar feelings when saying “I think these IT materials, like a computer, projection TV and cables, will make the students distracted from their study. Geometric concepts are much more important than the other mathematics topics so I won't do that in class".

Weaknesses of using traditional materials for geometry

According to the classroom observations and interviews, the Taiwanese Year 6 teachers generally thought that manipulating and observing concrete materials would clearly show the geometric concepts and assist in constructing the students' geometric thinking. However, the classroom observations revealed several weaknesses that appeared in the teachers' geometry teaching because of the use of hand-drawn diagrams or manipulated concrete materials for showing some specific geometric concepts. Three examples are presented.

The first example is that accurately constructed drawings or diagrams on the blackboard or wall chart could not show the difference between lines and segments, and therefore could not clearly represent the concept of parallel lines. For example, in T2's classroom, the teacher drew two segments of equal lengths to demonstrate parallel lines. Then, when T2 called one student [S1] to draw a line parallel with a line on the blackboard, another student [S2] claimed S1's drawing was wrong because S1 had drawn a segment of length different to that of T2's drawn line (actually, of course, a segment). T2 and the majority of the students all agreed that parallel lines should be the same length without noticing this is a misconception. However, dynamic geometric software such as 'Geogebra' shows apparently infinite straight lines (a line still appears no matter how far out the window goes) and so can clearly distinguish lines from line segments. This can be used to explain the concept of parallel lines clearly.

The second example highlights a pedagogical need for dragging figures. T1's students were persistently confused by the differences and inclusive relationships among parallelograms, rectangles, squares and rhombuses. T1 had tried her best to repeat all properties of these by drawing diagrams. He/she also drew a Venn diagram on the blackboard for explaining that rectangles are a special kind of parallelogram and squares are special kinds of rectangles and rhombuses. Even so, one student [S3], speaking for many, still argued squares are not rhombuses or rectangles because squares have four equal sides and right angles but rhombuses and rectangles have not. If T1 could have used dynamic geometric software to demonstrate how squares deform to rhombuses etc, by transforming the segments and angles of the quadrilaterals (as suggested by Leung, 2008) it seems likely that the students would have more easily understood these inclusive relationships.

A third example arose when teachers T9 and T10 derived the formula for the area of a circle by using concrete materials. These are cardboard circles separately cut into 8 and 16 sectors, which are reassembled to form an approximate rectangle. Figure 2 from (a) to (g) shows T9's procedures of manipulating these concrete materials. The known area of the rectangle gives the unknown area of the circle. T10's manipulation was similar to that of T9 and both are similar to the textbook.

(a) (b) (c) (d)

(e) (f) (g)

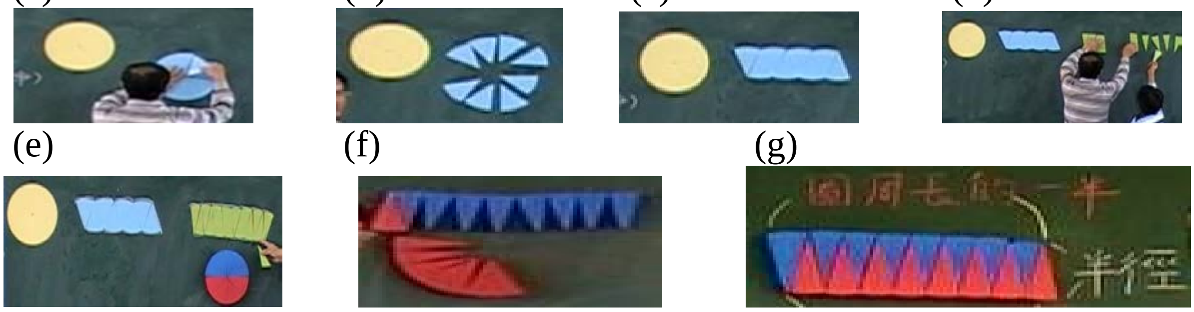

Figure 2. T9 derives the area formula of circles by rearranging cut-out sectors.

In the classes of both T9 and T10, most students believed the 'rectangle' shape in figure 2 (g), was (tending to) a trapezium or parallelogram instead. In fact, it has one pair of parallel straight sides (the short ones), and a pair of parallel multiply curved sides. Both teachers asserted that this was a rectangle when the number of sectors is much larger. However, it was clear to the observer that the students had difficulty imagining that the shape was going to a rectangle in the limit, rather than a parallelogram or trapezium. Some students seemed to doubt this conclusion. Even though both T9 and T10 had spent more than ten minutes for arranging these concrete sectors together on the blackboard, the representation of these materials still could not convince the students. In fact, if the teachers could use dynamic computer software such as a Flash 'app', even 100 sectors could be easily assembled together in an animation and the 'rectangle' appears on the computer screen within a few seconds. This is shown in Figure 3 (a) to (e) from the website "Mathematics Field".

(a) (b) (c)

(d) (e)

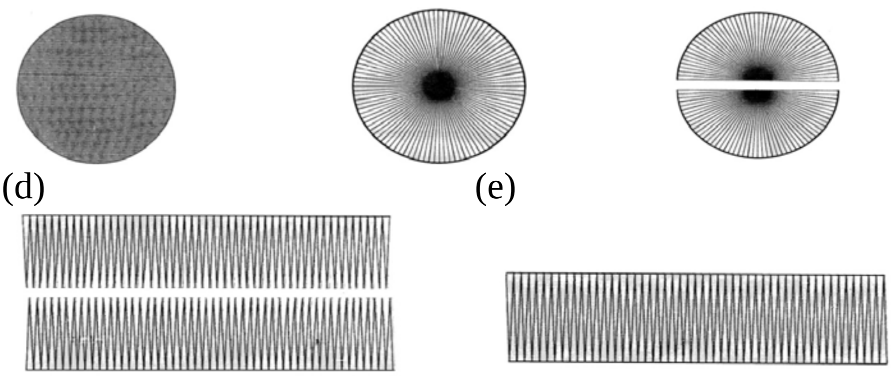

Figure 4. Screen shot of a circle cut into 100 pieces and reassembled, from "area of the circles" in Mathematics Field (http://www.paps.kh.edu.tw/aspx/math_menu/math_source.aspx)

CONCLUSION

This study had strength because it used in-depth analysis of nearly 100 lessons from 10 teachers, teaching four different geometric topics in Year 6 with follow-up interviews and the representative nature or the findings being confirmed by a national survey with careful sampling. The study gave substantial insight into the close detail of how materials are used in teaching geometry. The consistency of the findings makes it likely that advice and training for Taiwanese teachers about new possibilities for material use will be beneficial.

The study found low use and low awareness of IT materials for teaching geometry. In the case studies, a few teachers mentioned the detail and benefits of using IT materials for teaching geometry, but none of them used it in the observed lessons. Moreover, the majority of the teachers appeared not to know the advantages of using dynamic geometry software or animated figures from the Internet (e.g., Flash applets). A few teachers even actively rejected using IT materials for geometry teaching.

Taiwanese Year 6 teachers should be aware that traditional geometric materials cannot fully represent the whole of geometric concepts (such as parallel lines, reasoning the area formula of circles with many sectors or the inclusive relationships among quadrilaterals), and that dynamic geometry software or special purpose ‘apps’ can assist.

According to the case study teachers’ responses in the interviews, limited lesson time and inconvenient facilities in the classroom may be major obstacles to the use of IT materials in class. So addressing practical issues for IT use seems essential. Convenient access to computers in the classroom and good projection facilities is a pre-requisite. Cost need not be a great issue since there is good dynamic geometric software which can be freely obtained where licences for proprietary products are not possible. IT materials also have other practical advantages: teachers do not need to return them to the storage room or the material boxes as they do with concrete materials and there are no heavy class sets to carry. The school teachers can obtain these IT materials for teaching geometry easily.

Eventually, textbooks could also use IT materials within their textbooks and save the budgets, space and resources for making concrete materials for some specific geometric concepts. However, IT materials initially take more time and budget to create and maintain for the schools and textbook publishers. That may be the reason why the textbook publishers still tend to provide only traditional materials. Also it is possible that if the educational authority do not actively approve of and promote IT skills for the teachers, the teachers might quickly lose interest in using software for teaching geometry. This is a risk in a system with strong central authority such as Taiwan.

The most important advantage is using these IT materials lies in the new insights that the new dynamic environment can give students, including when used in conjunction with traditional materials. Dragging in dynamic geometry has been shown in the literature to have many advantages - this study showed its potential use in understanding definitions and inclusion relationships. The area formula example showed the value of the IT capacity to show multiple cases (different numbers of sectors of a circle) quickly and to demonstrate cases beyond by-hand manipulative skill. The case study also highlighted how showing specific geometric concepts, such as segments and lines, might be better done or at least profitably supplemented, in a virtual environment.

To sum up, all Taiwanese primary teachers should be aware that there are many advantages for using IT materials for geometry. Diagrams, drawings or concrete materials alone cannot represent all geometric concepts. Practical limitations currently seem large, but in reality may be relatively easily overcome in most schools in Taiwan. Teachers will also need to be shown how IT materials can help students achieve current learning goals more readily.

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UNFOLDING THE MULTIFACETED NOTION OF ALGEBRAIC THINKING

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The present paper aimed to empirically test a theoretical model which analyses secondary school students' algebraic thinking. The model is based on Kaput's theoretical assumption about the two core aspects that serve as lenses for interpreting the nature of algebraic thinking: (i) making generalizations and expressing those generalizations through increasingly symbolic forms, and (ii) syntactically guided reasoning and actions on generalizations. Data were collected from 190 8th, 10th and 12th grade students through a written test. The analysis indicated that a variety of algebraic tasks could be categorized according to the aspect of algebraic thinking they required for being processed. The findings of the study also suggest that the model involving the analysis of algebraic thinking into core aspects was the same among the three age-groups.

INTRODUCTION

It has been well documented that algebraic thinking is a wide conceptual field which does not merely coincide with what we know as traditional algebra. Algebraic thinking is considered to be within the conceptual reach of all students and vital for their participation in society (Mason, Graham & Johnston-Wilder, 2005). For this reason, great importance is given to the development of algebraic thinking, which "moves across the grades" instead of being taught through traditional courses of algebra in the middle school (NCTM, 2000). This idea raised the important issue of which are the aspects of algebraic thinking both in the primary and secondary education. Yet, the nature and components of algebraic thinking have not been coherently defined. Researchers' approaches to describe algebraic thinking are characterized by diversity and few of them made an attempt for depicting a thorough picture of the field (Carraher & Schliemann, 2007).

A considerable number of research studies described the kinds of meaning secondary students make when they are engaged with algebraic tasks either through constructivist / cognitive or social / cultural frameworks (Kieren, 2007). More recent research focused on the development of young learners' algebraic thinking (e.g., Irwin & Britt, 2005; Warren & Cooper, 2008; Zaskis & Liljedahl, 2002). While both of these bodies of research provided important advances to the field, it still remains unclear how they are related to each other. As Carraher and Schliemann argued (2007), it has not yet been clarified whether early algebraic thinking represents a distinct domain of study or it is better to be integrated into a more general algebraic terrain that captures the teaching and learning of algebra for both younger and older students.

The present study aimed to address this issue. More specifically, the purpose of the study was to investigate whether students' performance in algebraic tasks can be

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analysed in specific components. The analysis was based on the theoretical assumptions proposed by Kaput (2008) about the core aspects that characterize algebraic thinking. Although Kaput's perspective on algebraic thinking is used extensively in algebra studies, it has not been empirically tested. Moreover, Kaput's theoretical framework offers a scarce opportunity for investigating the evolution of algebraic thinking from K-12 grades. While Kaput's main thrust was to define early algebra, his ideas also respond to the algebra of middle and high school. This study aimed to empirically test the way in which this perspective might be adapted in secondary education.

In the following, there is an overview of research concerning the nature and components of algebraic thinking, including a description of Kaput's theoretical assumptions that were used for analyzing the data in this study. Then, there is a description of the empirical study conducted and its results.

THEORETICAL BACKGROUND

Kieran (1996) was among the first who tried to conceptualize algebra as a multifaceted mathematical activity. Employing this idea, Kieran (1996) developed a model which encompasses three types of activities; generational activities where equations and expressions are generated from various situations; transformational activities where expressions are simplified according to rules; and global, meta-level activities in which quantitative situations are not strictly represented in a symbolic way but they can be understood relationally. Early algebraic thinking is linked to the global meta-level of algebraic activity. According to Kieran (2004), the global meta-level activities involve more general mathematical processes such as the analysis of relationships between quantities, the identification of structure, the study of change, generalization, problem solving, modeling, justification, proof, and prediction. They are considered as appropriate for for the introduction of young learners to algebraic thinking, since they do not require the use of letter-symbolic forms.

Similarly, Radford (2000) suggested that algebraic thinking entails efforts of the individual to represent generality in certain ways. This process does not necessarily involve mathematical symbols. In this perspective, Radford (2004) added to the field by clarifying the importance of "semiotic mathematical and non-mathematical" systems in students' production of meaning when they encounter algebraic tasks. In particular, there are three sources of meaning in algebraic activities: (a) the algebraic "structure itself" (e.g. the letter-symbolic representations), (b) the problem context (e.g. word problems, modeling activities) and (c) the exterior of the problem context (e.g. social and cultural features, such as language, body movements, and experience). Kieran (2007) reflected on Radford's conceptualization of meaning in algebraic activity, by suggesting that the first source also involves mathematical representations, such as graphs and tables; students could draw on multiple representations in conjunction with letter-symbolic representations for producing meaning in algebraic tasks.

Kaput, for many years, sought ways for organizing algebraic thinking (Carraher & Schliemann, 2007). His perspective is slightly different from that offered by Kieran (Kieran, 2011). While Kieran (2004) stressed out that younger students could be engaged in global meta-level activities without the use of the letter-symbolic form, Blanton and Kaput (2005) placed an emphasis on the process of establishing, systematically expressing and justifying generalizations in increasingly formal ways. They highlighted that expressing generalizations with symbols depends on students' age and level. Kaput (2008) further specified that there are two core aspects of algebraic thinking: (i) making generalizations and expressing those generalizations in increasingly systematic, conventional symbol systems, and (ii) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. In the case of the first aspect, generalizations are produced, justified and expressed in various ways. The second aspect refers to the association of meanings to symbols and to the treatment of symbols independently of their meaning. Kaput (2008) asserted that these two aspects of algebraic thinking denote reasoning processes that are considered to flow in varying degrees throughout three strands of algebraic activity: (i) generalized arithmetic, (ii) functional thinking, and (iii) the application of modeling languages for describing generalizations.

Relying on Kaput's theoretical framework, this study aimed to analyze secondary school students' performance in algebraic tasks. Specifically, the purpose of the study was: (a) To investigate whether different types of algebraic tasks could be used to explore the core aspects of algebraic thinking, and (b) To investigate the extent to which different aged-groups of middle and high school students reflect these aspects.

METHODOLOGY

Participants

A total of 190 secondary school students participated in the study. 48 were students of Grade 8 (13 years old), 56 were students of Grade 10 (15 years old), 53 were students of Grade 12 taking courses of basic mathematics and 33 were students of Grade 12 majoring in mathematics (17 years old).

The test

Drawing on theoretical and empirical evidence from existing research studies, a test on algebraic thinking was constructed. Furthermore, the test was aligned with the content of the mathematics textbooks used in middle and high schools in Cyprus. More specifically, the test included 9 tasks which reflected the three strands of algebra as described by Kaput (2008). Assuming that these 9 tasks required different aspects of algebraic thinking to be processed, they were accordingly categorized into two groups.

The first three tasks (FT1, FT2 and FT3) investigated the participants' functional thinking. Two of them required finding the n^{th} term in geometrical patterns and expressing this generalization in a verbal, symbolic or any other form. The third one required the operation on a symbolized expression for investigating the correspondence among the temperature degrees in Celsius and in Fahrenheit. It was assumed that for responding to these tasks the aspect of generalization and expression of generalization was required, since they entailed the exploration and expression of regularities. The next three items (GA1, GA2, and GA3) intended to capture the strand of generalized arithmetic. The first two involved solving equations while the third one involved the solution of an inequality. In these tasks, the participants had to treat equations as objects that expressed quantitative relationships. More specifically, they had to treat an organized system of symbols without any reference to the meaning of the symbols. Thus, it was assumed that syntactically guided action was required. The last three tasks addressed mathematical modeling (MM1, MM2, and MM3). In particular, the first one required the generalization of regularities by observing the relationships represented by tabular data. For this reason, it was assumed that generalization and expression of generalization was required. The last two tasks engaged the participants with the analysis of information that were presented symbolically, graphically or diagrammatically. The first one was about encoding information represented graphically in respect to the services offered by a phone company and calculating the total cost of phone calls. The second task involved encoding information that was represented diagrammatically for calculating the volume of packages. Because both tasks required associating meanings extracted from the modeling situation to symbols, it was assumed that they required syntactically guided action.

Scoring and Analysis

The marking of the test was based on the scale 0-2, where 0 was given for an incorrect answer, 1 for a partially correct answer and 2 for a correct answer.

The quantitative analysis of the data was carried out using an electronic structural equation modelling program, MPLUS (Muthén & Muthén, 1998). Confirmatory Factor Analysis (CFA) was used for investigating whether the theoretical assumptions of the model about the core aspects of algebraic thinking fitted the data. Goodness of fit of the data was evaluated by using three indices: chi-square to its degree of freedom ratio (x^2/df), Comparative Fit index (CFI), and Root Mean-Square Error of Approximation (RMSEA). The observed values of x^2/df should be less than 2, the values for CFI should be higher than 0.9, and the RMSEA values should be close to zero.

In the present study, the hypothesized model consisted of two first-order factors, the aspect of generalization and expression of generalization and the aspect of syntactically guided action. The first-order factors comprised a second order factor which reflected secondary school students' algebraic thinking. Generalization and the expression of generalization are related to the tasks within the strands of functional thinking (FT1, FT2 and FT3) and modelling (MM1). Syntactically guided action is related to five tasks , two of them within the strand of generalized arithmetic (GA1, GA2, and GA3) and two of them within the strand of the application of modeling languages (MM2 and MM3).

RESULTS

The theoretical assumptions of the model were tested using CFA. The construct validity of the model was evaluated by examining whether the tasks employed in the present study loaded adequately on each of the two factors. The results indicated that the data did not fit the model well (CFI=0.919, x^2=58.782, df=26, x^2/df=2.26, RMSEA=0.081).

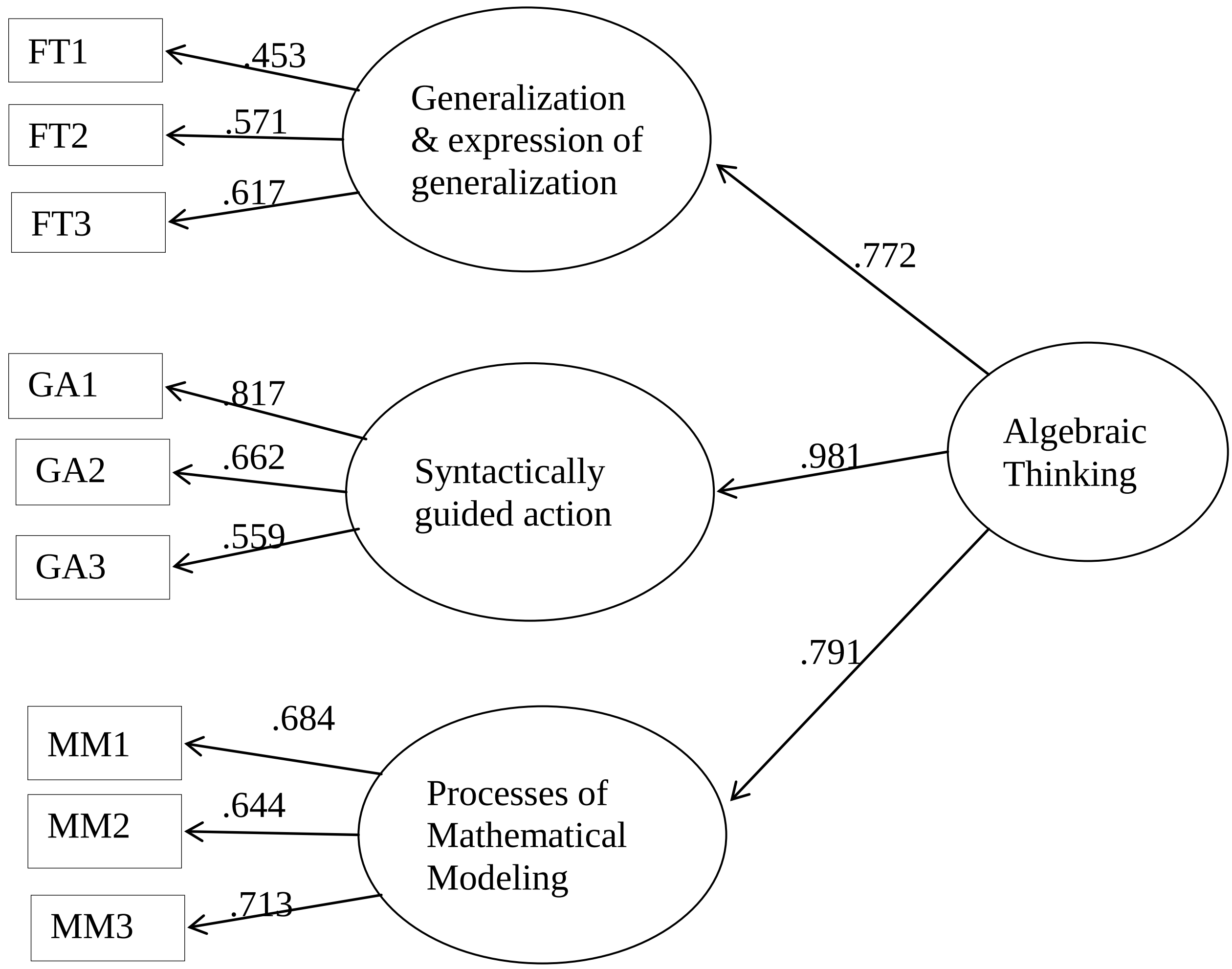

Fig. 1 The model of algebraic thinking. FT1-FT3: generalizing numerical patterns to describe functional relationships (Functional Thinking); GA1 – GA3: using arithmetic as a domain for expressing and formalizing generalizations (Generalized Arithmetic); MM1-MM2: modeling as a domain for expressing and formalizing generalizations (Mathematical Modeling).

Subsequent model tests revealed that the model fit indices could be improved by modifying the model. More specifically, the items which were addressing modelling situations (MM1, MM2, and MM3) were linked to a third factor which could represent a third aspect of algebraic thinking. The three model factor that emerged after this modification had a very good fit to the data (CFI=0.983, x2=30.829, df=36, x2/df=1.28, RMSEA=0.039). Figure 1 presents the model that emerged from the analysis. The observations that arise from Figure 1 indicate that each of the tasks loaded adequately on each factor, as they were statistically significant, with z values greater than 1.96. The models' goodness of fit demonstrate that the tasks used in the written test were grouped into three distinct components.

The second aim of the study concerned the examination of possible differences between the three age-groups in the structure of the model. To this end, one more test was conducted in order to investigate whether the model in Figure 1, which analyses algebraic thinking into core aspects, is independent of the age of the individuals. Multiple group analysis was applied, where the CFA model was fitted separately for the three age-groups. The multiple groups CFA model with the factor loading constrained to be equal across the three age-groups provided good fit to the data (CFI=0.975, x^2=60.600, df=61, x^2/df=1.40, RMSEA=0.047). This result provides support for the invariance of this structure between the three age-groups. Students of 13^{th}, 15^{th} and 17^{th} years old dealt in a similar way with the algebraic tasks included in the test.

DISCUSSION

The purpose of this study was to investigate whether algebraic tasks could be used to explore aspects of secondary school students' algebraic activity. The findings of the study indicated that there are three distinct aspects of algebraic thinking. The tasks that were designed within the context of functional thinking were linked to the aspect of generalization and the expression of generalization in increasingly systematic, conventional systems. The tasks that were designed within the context of generalized arithmetic were linked to the process of syntactically guided action. Hence, the two core aspects described by Kaput (2008) indeed do represent distinct components of algebraic thinking. Nevertheless, the modeling activities were linked to a third discrete factor, signifying that they require different reasoning processes in relation to the other two groups of tasks.

Two reasons seem to offer possible explanations for this finding. First, modeling activities are described in literature to be of highly cognitive demand (English & Doerr, 2003). In this perspective, such activities might require the integration of a more complex spectrum of reasoning processes compared to those needed for manipulating the activities of generalized arithmetic and functional thinking. According to Kieran (2007), modeling is involved in *global-meta level* activity and addresses not only algebraic thinking but also more general mathematical processes. Similarly, Carraher and Schliemann (2007) when describing the dimensions characterizing algebraic reasoning declare that modeling is central in any kind of mathematical activity. Thus, modeling is not directly associated with any of the dimensions of algebraic reasoning.

The observation that modeling activities required an aspect of thinking other than the one embedded in generalized arithmetic or functional thinking, might also be explained by the fact that these activities involved representations such as tables and graphs. This provides support to Kieran's (2007) argument for adding multiple representations in the synthesis proposed by Radford (2004) about the sources from which students produce meaning when they are engaged in algebraic tasks. The treatment of multiple representations in conjunction to letter-symbolic representations in order to extract specific algebraic meanings might be associated to the application of a differentiated assortment of reasoning processes. Kaput (2008) also stressed that representations such as algebraic notation, graphs and tables both are used for expressing generalizations in conventional forms and act as mediators of individuals' algebraic thinking.

The secondary aim of the study was to examine whether different age-groups of secondary school students are able to integrate the various aspects of algebraic thinking. The multiple groups CFA analysis showed that the model of algebraic thinking was the same among the three age-groups. The independence of the model construction from the age factor supports that the core aspects of algebraic thinking are within the conceptual reach of students of different ages, despite the differences in the duration and content of algebra teaching they received. This finding provides empirical validity to the assertion that algebraic thinking is a broader conceptual field.

Moreover, the findings indicate that Kaput's (2008) ideas, which were developed in the context of early algebra, respond to secondary school students' algebraic thinking. It seems then that early algebra and secondary education algebra might not constitute two distinct domains of study.

Drawing on the results of the present study, future research could follow different directions. For instance, the way by which pivotal themes on early algebra include the various and diverse aspects of algebraic thinking needs to be examined. Research needs to address the issue of algebraic thinking from a cognitive perspective in order to articulate the nature of the reasoning processes that are inherent in each of the aspects of algebraic thinking. From an instructional perspective, research needs more systematically to reflect on the routes by which algebraic thinking growth can be encouraged within the classroom settings.

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A POST-HUMANIST PERSPECTIVE ON A GEOMETRIC LEARNING SITUATION

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This research report presents a post-humanist approach to analysing a geometrical activity involving grade 9 students. In looking at students' practices in using mathematical tools in different contexts, this study considers the range of components involved in a learning situation, rather than focusing only on the learner, taking into consideration the student, the tool (the Geometer's Sketchpad) and mathematics, all of which can be considered to have influence or agency in such a learning environment. I use the construct of intra-acting agency to examine the relation between the components of the situation.

INTRODUCTION

Traditional perspectives on human practice are being challenged by researchers within a post-humanist paradigm (Barad, 2007; Sorensen, 2009; Malafouris, 2008). Post-humanists view the individual as important but not as the only "participant" or "agent." In contrast, many learning theories, like constructivism, focus on the individual as the main source of action and agency. Socio-cultural theories acknowledge the role of others in shaping an individual's actions, but are still principally about the human. Technology-based theories like instrumental genesis aim to understand the way in which tools affect human action, but still subordinate the tool to the epistemic subject. These anthropocentric perspectives position the subject as an external author; a post-humanist perspective adopts the idea that non-human elements can "participate" in various forms of practice.

In this study, the mathematical practice of a classroom of students will be considered. The focus will not be solely on the students, but on the intra-actions between subject, their tools and the mathematics. Agency will be granted to the non-human elements of this environment to help identify forms of activity. This is not a study of individual parts collected together but one of a mutual co-constitution of emerging agencies. The ultimate goal of this study is to show how this intra-action might look in a mathematics setting.

THEORETICAL FOUNDATION AND FRAMEWORK

A post-humanist perspective does not view learning as an individual achievement (Sorensen, 2009, p. 5). This challenges an anthropocentric perspective, which can be limiting in that it dismisses the physical world around us and how it shapes us. In her studies on quantum physics, philosopher and physicist, Karen Barad (2007), describes her observations of Bohr's work on particle physics indicating that actants become defined in the emergence of activity: "Objects are not already there; they emerge through specific practices" (p. 157). She uses the term intra-action, as opposed to

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interaction, so that the focus is on things emerging and not on capacities or attributes of things before they come together. I contend that there is a tendency to think that individuals are fully formed and stable but this perspective can lead to a focus on individual capabilities. But instead of focusing on what an individual is bringing to an interaction, I suggest that the question should be what kind of distributed activity occurs across the human and the non-human actants. The focus is not on pre-action nor on post-action, but on action itself.

In addition, Barad (2007) challenges the idea of analysing individuals or things outside of context. The very notion of identifying individuals or things distinctly involves creating divisions or boundaries. According to Barad, these cuts are arbitrary, subjective and continually shifting. For example, traditionally, to speak of an individual would typically include a person bounded by their skin. But if a blind person is using a walking cane to help navigate an environment, their "self" is clearly extended. The tip of the cane might be considered the extent of their "touch".

In her analysis of Bohr, she describes how concepts are dependent upon apparatus, or modes of observation: "Concepts, in Bohr's account, are not mere ideations but specific physical arrangements" (p. 54). Although Barad is using Bohr's model of observation in a context of quantum mechanics, I contend that the context is analogous to a learning environment for a mathematics student. In any educational context there are different arrangements of mathematical tools. I propose each has its own emerging outcomes and corresponding concepts, such that where an apparatus begins or ends is a matter of subjectivity. Certain arrangements bring forth different features, ways of looking at, or constraints of observation or action. The thinker or rational being needs to be redefined, not as an individual but as a subject immersed in activity intra-acting with other things: "Knowing is a matter of intra acting" (Barad, p. 149). Therefore, mathematical activity is considered to be an assemblage of human and non-human agencies.

Using Barad, agency is operationalized as a construct to identify methodologically what emerges from the intra-action of a student with a mathematical tool or concept. Agency has traditionally been conceptualized as a human capacity but many researchers now see it as emerging from intra-action, thereby granting non-humans the ability to act (Malafouris, 2008). Barad states that agency is not an attribute but the ongoing reconfigurings of the world (p. 141). Agency can be thought of as an action or a doing. Intention is not synonymous with agency, for otherwise it becomes a human-centered construct.

Although Pickering's model is based within a humanist paradigm, his perfomative idiom is helpful in identifying assemblages of agency. In his study of practitioners in science studies, Pickering highlights the cycle of resistance and accommodation, which occurs in scientific work with machines. Within an education setting, this model may be analogous to a student using, say, a dynamic geometry software (DGS). In any task, the DGS may provide a resistance or a challenge, and the student will then need to accommodate their action to overcome this challenge. Although Pickering focuses on the individual, we can name the resistance as a material agency. The material, non-human element imposes a restriction upon the user. Further, a DGS may extend possibilities or distribute activity of the person, as the walking stick had done for the blind individual.

As Sorenson posits, "…to decenter we can still emphasize the individual" (Sorensen, p. 57). Given the setting of this study, I have chosen to emphasise the individual by introducing the notion of self-agency. The human is an exceptional figure and how she acts can be acknowledged so as to keep analysis clear. Self-agency is the degree of agency a person has, when using an "I" voice such as "I am driving this car" they are enacting a self-agency. According to Knox (2011), a developmental psychologist, self-agency is necessary to development; I contend this development of self-agency parallels Barad's idea of becoming.

Providing students opportunities to act, they come to see themselves as participants, which may lead them to experience self-agency. Opportunities for self-agency do not necessarily evoke self-agency, nor is self-agency guaranteed or even linear. What is important here is that what results from exercising self-agency is a "sense" of agency. An individual may or may not have a sense of agency in a particular context. This is an important feature of this study because one must have a sense of agency in order to participate in a performative idiom (Pickering, 1995).

The question of this study is based on a change of physical arrangement. A geometry activity is observed in two different contexts. The first involves a traditional classroom, the second includes a newly-introduced digital tool. In observing the two contexts, I identify significant changes in the students (their self- and sense of agency), their practices (actions) and the resulting mathematics. These actants are in the process of becoming. The mathematics adopted in this study is a discipline of negotiation, conjectures and exploration, not one of infallibility. In this study, I have chosen to focus more on the co-constitution of student and the tool, leaving their co-constitution with mathematics for another study.

METHODOLOGY

The theoretical framing of this study demands close attention to the back and forth and integrated intra-action of the student using tools in a mathematical activity. Attention to discourse, written or verbal, provides the means by which I identify activity. I use James' (1983) distinction of the "I" voice as expressions of self-agency and his distinction of the "me" voice as the objective self, as that which is being acted upon. I will use these distinctions of voice to identify resistance and extensions. These will be examples of material agency. Student discourse will be a major source of identifying intra-action between themselves and the software.

RESEARCH ACTIVITY AND PARTICIPANTS

The data for this study was collected in a Vancouver high school in a grade 9 (14 years old) classroom during a geometry unit. Mathematics 9 in British Columbia has an extensive geometry component that involves rotations, symmetries, circle properties as well as coordinate geometry. Students had last worked explicitly with triangles and squares in grade 6.

The teacher introduced a two-phase activity based on what he had done in previous years. I requested a third phase. The teacher's two phases of instruction corresponded to my interest in looking at different practices and using different tools in different environments. In phase one, the teacher drew (freehand) what looked like a triangle and a square on the whiteboard for all of the class to see. He requested that students try to identify how they might determine whether these geometrical figures were, in fact, as claimed, a triangle and a square. Students worked in pairs to encourage discussion and wrote their responses. For phase two, the teacher took all the students to a computer lab, sat them in pairs and requested the students use *The Geometer's Sketchpad* (GSP) (Jackiw, 1988) to construct both a triangle and a square. During phase two, the teacher allowed students to explore the software's environment, as this was the first time the students had used the program. He also went around and gave guidance and support by approaching pairs of students who seemed to be having difficulty or who were asking questions. In addition, he challenged student "constructions" to see if dragging would break them. Although the triangle was constructed by almost all students, the square provided more of a challenge. Students most commonly "fit" four segments together, but when the teacher dragged one of the vertices of the "almost-square" (Figure 1), the "square" would morph into another shape. Students were given more time to try to construct the square over the course of the 80-minute class in the computer lab. For phase three, the teacher brought all the students back to the classroom and requested that they again write, in pairs, how they would determine whether a given figure is a square. The researcher was present during all three phases; he also interacted with students, lent support and "challenged" their constructions.

All written work for phases one and three were collected and analyzed. Data from the computer lab was collected by using SMRecorder, which records all the digital activity on the screen as well as verbal utterances of students.

ANALYSIS

The analysis of the data is based on identifying examples of changes in students' conception of themselves or of the mathematics. The majority of data in this study is based on data from the computer lab because this is where the agency emerged and made itself known. I identify examples of both self-agency and material agency in working with GSP. I also contrast the transition from phase one to phase three identifying significant changes in students' conceptions of geometrical shapes. I then present four examples rich in intra-action and agency.

In phase one, students written work in the classroom, almost exclusively, listed properties of the geometrical shapes. Their conception of these geometrical shapes was based on properties. Although the figures drawn at the front of the room did not have these properties (they were drawn freehand), the students discussed, recalled, using the diagrams to guide their memories of grade 6 geometry. In all of the written work there was no reference to the "I" voice, nor were there references to the shapes as imposing themselves in any way. It is relatively clear, in this activity, where the boundaries were drawn. The mathematics was represented on the whiteboard, and the students were the subjects expected to absorb or recall the knowledge.

In the computer lab, the proposed activity supported a process of exploration which in turn actualized enactments of agency. For example, in constructing an "almost-square" (Figure 1), multiple pairs of student could not get the lengths of the sides to equal. One way to deal with this was to draw one segment and copy and paste three more. This was a good idea (although this still did not "construct" a square), for the segments were all the same length, but the lengths did not remain constant under dragging, as Laura found out.

Laura: ohhhhh, how come it changes length?

This back and forth attempt to make the square is an example of Pickering's model of resistance and accommodation.

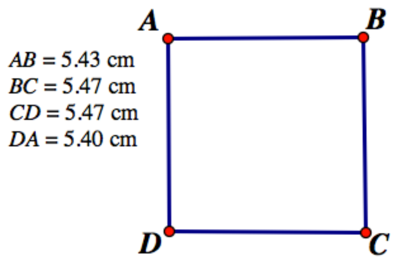

Figure 1: An "almost-square"

There were multiple examples of self agency in the computer lab that were evident as the students worked on the task:

Ricardo: I want to see what moving this will do.

Alice: I want to know what happens when I try this…

There were also examples of resistance, where the software did not do what the student expected:

Mitchel: It won't let me drag the point.

Heather: How come this part is not moving?

The transition between phase one and three is significant. Data from the classroom after the intra-action with GSP, in contrast with the phase one activity, was distinct in that the conceptions of squares and triangles were different. In general, their descriptions of the square from phase three included new vocabulary, new metaphors and new forms of engagement. In their written activity new words were used such as: pull, put all, flip, adjusted, drag, copy and paste, angle and locked.

The following four examples were chosen because they were rich in intra-action and agency. The first three are occurrences from the computer lab. The fourth example was an occurrence in the classroom during phase three.

Justin and David constructed a triangle and then translated it partly off the screen and the question "Is this a triangle?" was posed.

Justin: Is this a triangle?

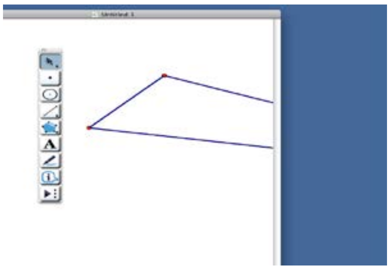

Figure 2: Justin’s triangle

In this particular activity we see an example of students generating new questions; there were new opportunities for negotiation. Unlike a drawing of a triangle on a whiteboard or a sheet of paper, this triangle was initially fully visible and then translated off the screen. The limitation of the screen became negotiable due to the intra-action of the student and screen agencies. The agency of the screen limits visibility but also the tools allows for easy access to translate the triangle back. The boundaries of the triangle are challenged. The students seem to be the ones asking the question, but the screen and the triangle occasion this situation. Justin’s half triangle is an example of the relationship between humans and negotiation, a challenge not available without the tool. Justin challenges the perspective of the student and introduces the question of where the mathematics lives. Does it exist off the screen?

Also in the computer lab, another pair of students, Luna and Michel, described to the teacher how they constructed the square using the grid option in GSP. They thought they had constructed a perfectly good square (Figure 3). Most other students were getting their square pulled apart by the teacher, but Luna and Mishel were confident that their square would hold up since it lined up with the coordinate grid. The teacher, however, changed the scale on the grid and the square became a rectangle (Figure 4). They did not try to figure out another way to construct the square; instead they based their construction on the limitation of not being able to change the scale.

Luna: This created a 1x1 square and no matter how you move the point, it stays a square – unless you change the grid.

As long as someone did not change the scale, the square that they had made was a square. Luna and Michel’s definition of their virtual square illustrates an assemblage of human and non-human forms for they based their definition on a particular situation in GSP which included software, agency and mathematics. The definition held all components together.

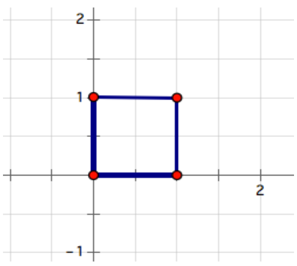

Figure 3: Luna’s square

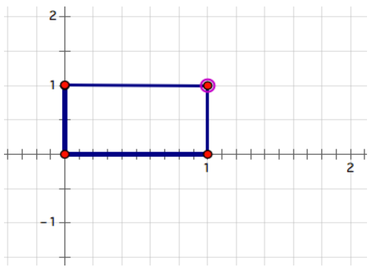

Figure 4: Luna’s rectangle

According to James the diagram in Figure 5 is not a triangle. James discussed, with the teacher, how GSP expected endpoints to be connected properly otherwise segments could be dragged away from each other and the shape did not retain invariant features. James challenged the idea of endpoints and intersections. A new way of categorizing intersections was introduced; intersections did not become "points" without self-agency and the tool.

James: The four sides must be touching but not intersecting.

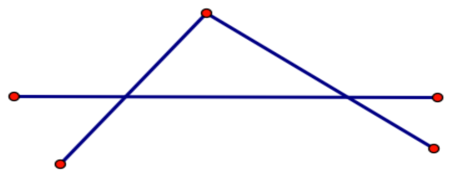

Figure 5: James' non-triangle

The last example draws from phase three. One male student said the following while explaining to the teacher what a square was.

Leo: It has four corners, 90 degree angle, four equal sides, has 360 degrees. You can move it around it is still 360 degrees. The four points are attaching perfectly so you can move it around.

When the teacher asked him what it meant to move it, he moved his hands around in the air as if he was turning a steering wheel. The mathematics was changing because the object had changed definition – it had become accessible and he had developed a sense of agency with it in that he knew he could move a square and it would hold its invariance. With the tool, the square became available for empirical challenge, thus radically affecting student's acceptance to what a square was.

DISCUSSION AND CONCLUSION

In the computer lab, the students used *Sketchpad* to test whether a shape is a square. The shape became a figure to move around, push; an object with hinges. But a student needs a sense of agency to begin the enactment and a self-agency to endorse the square. Without the ability to flip, move, drag, the determination of whether the figure is a square is not possible. Only in the combination of invariance and movement could a square be actualized. The boundaries in such an intra-action are difficult to identify. In the classroom, boundaries were easy to identify but with Sketchpad, possibilities were enhanced, for the students were doing things with squares and triangles that they had not conceived. Dragging the triangle off the screen, challenging its existence outside of perception was something not possible in the classroom. Moving his hands in the air, Leo's sense of agency is actively trying to access the square. The possibilities of engagement were extended for the square did not exist without intra-action. Otherwise there would be no way to determine the difference between an "almost-square" and a proper square. The square depends on the student to act and the student depends on the tool to act and the boundaries of agency continually shift.

If we are to accept Bohr's statement that concepts are physical arrangements we should consider that Sketchpad is such an arrangement. Thus, the concept of a triangle is different than its representation on the whiteboard. The concept of a triangle is not based on properties of a transcendental platonic geometrical figure but an actualized digital form that necessitates student engagement. In phase three when students were describing the triangle in terms of gestures, new words, and new metaphors, the tool did not just draw attention to different aspects of the triangle but reconceptualised the triangle. As de Freitas and Sinclair (2012) write: "A concept of this kind, with logical and ontological functions … resists reification while carving out new mathematical entities and forming new material assemblages with learners" (p. 12).

This study troubles existing, humanist assumptions about the role of tools. If the tool can alter the way we look at simple geometrical figures as well as the way we look at our own involvement in mathematical activities, both the way digital tools are designed as well as the way they are presented can have very important effects on our mathematical experiences.

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EXAMINING 5TH GRADE STUDENTS' ABILITY TO OPERATE ON UNKNOWNS THROUGH THEIR LEVELS OF JUSTIFICATION

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The study examines students' ability to operate on unknowns through students' levels of justification in generalized arithmetic tasks in which algebraic expressions are present. Two tasks about generalization of properties of numbers were administered to 73 fifth-grade elementary school students and then 10 semi-structured interviews were carried out. Results indicate that a respectable percentage of students can operate on one unknown by providing generalizable arguments about the result of an "unknown even number+3" without the need of reducing abstraction. On the contrary, most of the students face difficulties to think at an abstract level when confronted with the operation that involves two unknowns and provide numerical examples as justifications. Implications of these findings are discussed.

INTRODUCTION AND THEORETICAL BACKGROUND

Building generalizations from arithmetic is taken by many educators and researchers as the primary entry to algebra (Kaput, 2008). Traditionally, arithmetic has focused on computational accuracy and efficiency, while much of algebraic problem solving focuses on reasoning about operations and numbers, generalizing their properties and *reasoning* about more general relationships (Blanton, Levi, Crites & Dougherty, 2011; Kaput, 2008). According to Caspi and Sfard (2010), algebraic thinking begins when one starts scrutinizing numerical relations and processes in the search for generalization. The generalizations start as conjectured relations and some work must be done mathematically before a conjecture is accepted as a generalization (Blanton et al., 2011). Classroom practice that promotes reasoning and proof, provides the chance to students to build *arguments* to *justify* these conjectures and through this process conjectures are transformed into generalizations (Blanton et al., 2011).

Therefore, in today's classrooms students must be encouraged to make conjectures, should be given time to search for evidence to prove or disprove them, and should be expected to explain and justify their ideas (NCTM, 2000). A number of chapters and research papers (e.g. Blanton, et al., 2011; Carpenter, Franke & Levi, 2003) focused on elementary school students' ability to engage in generalized arithmetic tasks and build convincing arguments concerning the properties of numbers and operations. For example, Carpenter et al. (2003) examined students' ability to build generalizations about classes of numbers like "An odd number plus an odd number is an even number". According to Blanton et al. (2011) this type of generalization derives from the fundamental properties of numbers and operations since students' work to *justify* their answer focuses attention to the structure that underlies computation. For example, the student could write down "b+b+1+d+d+1=b+b+d+d+2" and explain that if we add a number to itself we get an even number, and 2 is an even number, so the result must be an even number (Blanton et al., 2011).

The abovementioned type of justification falls in the highest level of justification as proposed by Carpenter et al. (2003). More specifically, Carpenter et al., (2003) describe the following different types of justification:(a) appeal to authority where students relate their reasoning to a rule or procedure that was taught or told to them by someone with authority, (b) justification by example where they use numerical examples to test the conjectures and (c) generalizable arguments where students present a logical argument (verbal, symbolic or concrete) that applies to all cases. Research suggests that children's justifications will often use simple empirical arguments based on testing a number of specific cases (Blanton et al., 2011). However, students in grades 3-5 should learn that several examples are not sufficient to establish the truth of a conjecture (NCTM, 2000). As in the studies mentioned above we also use the general notions of justification and argument, rather than proof, since they are more appropriate for the elementary grades (Blanton et al., 2011).

The generalization tasks of the present study, for which students had to justify their answers, were expressed symbolically through the use of literal symbols. For this reason, we took also into consideration students' wrong interpretations of the variable as described in the literature. In some cases students assign one numerical value to the literal symbol (sometimes based on their place in the alphabet) even in cases where it must be seen as a generalized number (Kuchemann, 1981; MacGregor & Stacey, 1997). In addition to this, students might interpret the letters as abbreviations of an object's name (Kuchemann, 1981; MacGregor & Stacey, 1997). Another difficulty identified is when students are confronted with expressions like 3+n, which they rewrite as a single entity with no operation, such as 3n (MacGregor & Stacey, 1997).

However, despite the difficulties, some studies (Carraher, Schliemann & Brizuela, 2001; Hewitt, 2012) provide examples of nine-year-old children using algebraic notation to represent a problem of additive relations using algebraic expressions (e.g. N+5). These students were not only able to operate on unknowns but they also understood the unknowns to stand for all possible values (Carraher et al., 2001). Similarly, a recent study by Hewitt (2012) showed that 5th grade students were able to work with formal notation and more specifically with complicated linear algebraic expressions, after three lessons with a certain computer software.

Based on the above, the purpose of the present study is to examine 5th grade students' ability to operate on unknowns and engage in the process of justifying their generalizations about properties of numbers. The levels of justification are used as a tool that can help us determine students' ability to generalize properties of numbers by operating on unknowns or their need to reduce the level of abstraction and work with numerical examples. Unlike previous studies, in which the use of letters was usually encouraged for the representation of students' generalizations, the algebraic expressions in the present study are provided in the instructions of the tasks in order to enable us to examine students' understanding of the symbolic notation and students' ability to operate on unknowns (from K+K to M+M+3). Also, in order to move one step further and add on previous studies' results (Carraher et al., 2001), we use tasks that involve operations with two unknowns (instead of just one unknown). Therefore, we sought answers to the following questions: (a) Are 5th grade students of the study able to work in generalized arithmetic tasks in which literal symbols are present? (b) Are 5th grade students of the study able to use generalizable arguments as justifications by indicating an ability to operate on unknowns or do they turn to the use of numerical examples? (c) Is the level of justification provided by students the same in both tasks?

METHODOLOGY

Participants and Procedure

The participants were seventy-three 5th grade elementary school students of three different classes. These students had never had (constant) formal instruction about the use of letters, besides the fact that they met the use of literal symbols-unknowns in a few tasks that are included in their mathematics textbooks. Initially, the generalized arithmetic test was administered to the participants. Based on the test results and the taxonomic qualitative method of analysis by Spradley (1980), different levels of justification were formed. Once this was done, 6 semi-structured interviews were carried out with students who adopted the same level of justification in both tasks and 4 semi-structured interviews with students who had changed their initial level of justification (in the first task) to a higher level justification in the second task.

The Generalized Arithmetic Test

The test included two tasks which concerned generalization of properties of numbers (see Table 1). Students had to complete the test in a 30 minute session.

Task 1: Variable K can be any integer number. Indicate whether the sum K+K results in an even or an odd number. Explain your answer.
Task 2: Variable M can be any integer number. Indicate whether the sum M+M+3 results in an even or an odd number. Explain your answer.

Table 1: The algebraic thinking tasks

Coding of the Responses and Analysis of Data

In both tasks, each answer (regardless of the justification-explanation) was coded as correct (success=1) or incorrect (success=0). Correct answer for task 1 was "an even number" whereas for task 2 "an odd number". Then, a second code was given for the type of justification. It must be noted that we examined the level of justification only in correct answers because wrong answers or incomplete work could not reveal any type of justification. In contrast to previous studies (Carpenter et al., 2001), justification by authority was not expressed by students in this study, not even during the interviews. Also, our findings revealed that the previously described general level "justification by example" (Carpenter et al., 2003) should be analyzed further since two sub-levels (1a and 1b) were identified. The four categories of justification that were identified based on the test results and the interviews are the following: (a) no explanation-0: Their explanations are phrases repeating the instructions of the task without really providing an explanation; (b) use of random examples-1a: They use several random examples to justify their answer. They think that some examples are enough to prove a conjecture and they are not able to express generalizable arguments; (c) working with groups of numbers-1b: They work with classes of numbers to justify their answers (e.g. they choose to work with even and odd numbers). They realize somehow the fact that some random numerical examples are not enough and they tend towards a more coherent way of working. They may also have an idea that some numerical examples of odd numbers and even numbers are still not enough however, they are not able to produce another more generalized argument; (d) generalizable arguments-2: They use general statements to justify their answer and they do not need examples to test their answers. At this level students are able to explain if something is true and most importantly they can explain *why* the statements are true, something that is necessary in proof.

For the quantitative analysis of data, descriptive statistics were applied, whereas for the qualitative analysis the taxonomic method (Spradley, 1980) was used.

RESULTS AND DISCUSSION

As shown in Table 2, the majority of students, 72% and 71% were able to provide correct answers in task 1 and task 2, respectively. However, their justifications can provide a clearer picture of students' achievement in these tasks. Table 2 shows that the majority of students used justification by random examples in task 1, whereas only two students were able to justify through generalizable arguments in this task. Twelve students that had used justification with random examples (level 1a) in task 1 and 11 students that had used justification with classes of numbers (level 1b) in task 1, provided higher level of justification in task 2 by using generalizable arguments. This indicates that 25 students were able to build on their previous generalization (even if that generalization occurred from sufficient or insufficient justification) in order to form the generalization that M+M+3 will give an odd number. Therefore, twenty three students were able to "compress" in some way the procedure they had used in task 1 and instead of following that procedure again they realized that the sum of M+M is the same as the sum of K+K. They were then able to "treat" K+K or "M+M" as a sum and not as a procedure (Caspi & Sfard, 2010). The fact that they could then reason about the sum of an "unknown even number+3" reveals their (spontaneous) understanding to operate on unknowns. Their difficulty to operate on unknowns in the first task is probably due to the greater difficulty of the operation that requires adding two unknowns. It must be noted that students' worksheets with correct answers (regardless of their level of justification 0,1a,1b,2) provided evidence that students interpreted the letters K and M as taking all possible values. Only one student who gave correct answers to the tasks (but wrong justifications) and six students who provided wrong answers from the beginning (and thus their justifications were not examined), interpreted the letters K and M as specific unknowns. More specifically, these students assigned the values 10 and 12 to the letters K and M respectively, based on their place in the Greek alphabet. For the remaining 8 students that did not provide correct answers we could not determine how they interpreted the letters since their work was incomplete.

Task	Correct answers		Level of justification for the correct answers								Wrong justification	
			0		1a		1b		2			
	N	%	N	%	N	%	N	%	N	%	N	%
K+K	59	80,8	11	18,6	32	54,2	13	22,0	2	3,4	1	1,7
M+M+3	59	80,8	11	18,6	21	35,6	1	1,7	25	42,4	1	1,7

Table 2: Frequencies and percentages of correct answers and for levels of justification for the generalization tasks

While most students with "1a level of justification" in task 1 remained at the same level of justification in task 2, the majority of students with "1b level of justification" in task 1 adopted "level 2 justification" in task 2 and all students with "level 2 justification" in task 1 remained at the same level in task 2. Due to space limitations, we provide below the transcripts of the interviews with one student of each of these three cases.

1a Level of Justification in both Tasks

At this level students use a few examples to justify their answer. It is obvious that students' decisions that the sum of K+K will be an even number, is based on the few examples they use and they are not able to express a generalizable argument. Their need to reduce the abstraction is evident also in task M+M+3 in which they are not able to build on their previous work and they persist to the use of numerical examples:

Interviewer: You have used the examples 2+2=4, 3+3=6, 4+4=8 to decide that K+K always results in an even number. Why have you used these examples?

Lena: I wanted to try out some numbers and see what was going on.

Interviewer: Are you completely sure that the result will always be an even number?

Lena: Yes, because I tried three examples.

Interviewer: Ok. How can you be sure that is true for other numbers as well?

Lena: If it works for these numbers, then I guess it will work with all numbers. However, the only way to be completely sure is to try other numbers as well, I mean use more examples with more numbers.

Interviewer: Ok. As I can see your explanation in task 2 is the same as in task 1. However, you already knew that K+K results in an even number so you could have used this information in M+M in order to avoid numerical examples.

Lena: No, even if I have used the information that M+M is an even number I would not know what to do afterwards with +3, since I cannot add 3 to an even number that I don't know. I wanted to use numbers and see what is going on.

From 1b Level of Justification in Task 1 to Level 2 Justification in Task 2

Students in this case provide justifications using classes of numbers in task 1, however they provide more generalizable arguments in task 2:

Interviewer: As you have explained in task K+K you chose to work with even numbers first and then with odd numbers. Can you explain why?

Laoura: K could be any number. I did not have so much time or space in my test to start writing down all numbers. So, I figured out that I could work with groups of numbers in order to see what is going on...em....and I found out that the result is an even number regardless of the group of numbers.

Interviewer: Ok. As I can see you have used 5 examples with odd numbers and five examples with even numbers. Are these examples enough to decide?

Laoura: I think yes...because 5 examples of odd numbers and 5 examples of even numbers convinced me that K+K gives an even number...I could try more examples but I had to stop somewhere.

Interviewer: Can you explain why this holds for all numbers?

Laoura: Hm.....This is what we get when we try it with numbers of different groups.

Interviewer: As I can see in task 2 you justified without the use of examples that (M+M)+2+1 will be an odd number. Can you explain a bit more?

Laoura: As I tested with examples…K+K gives an even number, so M+M which is the same thing will also result in an even number. But when we I add an odd number to an even it becomes odd.

Interviewer: How do you know that?

Laoura: As I wrote (in M+M+2+1) we added another pair but when we add one more (resulting in adding 3) it becomes odd since one is left alone.

Level 2 Justification in both Tasks

This level of justification indicates a higher level of abstraction and reveals conceptual understanding regarding properties of numbers. This student is able to operate on abstract symbols by indicating clearly that K+K equals to the formalism 2×K and is also able to use another letter (E) to represent even numbers:

Interviewer: You wrote that you are sure that K+K is going to be an even number because K+K has "a half". What do you mean?

Leo: K+K means we have two times K and it means I have two times "something". If I have two times something (says 2×K in words)...then this means I can divide it by 2 and nothing is left.

Interviewer: And how did this help you decide that the sum of K+K is an even number?

Leo: Only even numbers can be divided by 2. When we divide an odd number by two we have one item left. Since K+K can be divided by 2 and not leave anything, then it is an even number.

Interviewer: Ok. In task 2 you have explained that M+M+3 will be an odd number and you wrote (E)+3 because everything in E will have a pair but this does not hold for 3. Can you explain why you wrote that?

Leo: I wrote E to represent all even numbers since this is the result of M+M as I explained previously for K+K, thus any even number plus 3 gives an odd because everything in E and 2 can be divided by 2….and not leave anything…but one is left alone because we add 3 and not 2.

CONCLUSION

It is encouraging that students of this study were able to engage in tasks that involved literal symbols, even if they did not have previous formal algebraic instruction (Carraher et al., 2001; Hewitt, 2012). The majority of students indicated through their justifications an understanding of the unknown as standing for all the possible values.

Furthermore, the present study indicates that the use of numerical examples as justifications depends on the difficulty of the task, which in this case is related to the number of unknowns in the operation. Most of the 5th graders of the present study were able to use generalizable arguments to justify their generalizations about "an even +3" (in task 2) but turned to the use of numerical examples in the case where they had to generalize operations with two unknowns "K+K" (in task 1). Based on these results, two implications occur. First, the fact that almost all students (except 2 students) were not able to justify through generalizable arguments in the first task, reveals students' superficial understanding (a) of adding a number to itself and the connection of that result to multiplication and (b) of properties of numbers. Their difficulty was due to their need to reduce the level of abstraction in order to examine the behavior of the operation with unknowns and not due to wrong interpretation of the letter. In addition, the fact that school mathematics emphasize the traditional view of arithmetic prevents students from reflecting on the operations and on the properties of numbers. Therefore, classroom practice that encourages reasoning and justification and the use of tasks that focus attention on the structure that underlies operations and numbers are necessary to help these students develop conceptual understanding regarding properties of numbers. Starting even from activities with concrete materials (e.g. blocks) providing the chance to students to "see" how the sum with even and odd numbers works and then helping them represent their observations though the use of symbols, enhance both arithmetic and algebra. Second, the fact that one third of the 5th graders provided evidence that K+K equals M+M and could reason about "an unknown even number+3" without having formal algebraic instruction coincide with previous studies results that younger students are able to operate on unknowns and work with formal algebraic notation (Carraher et al., 2001; Hewitt, 2012). The results are also in line with the results provided by Caspi and Sfard (2010), who found remarkable structural similarities between students' verbal meta-arithmetic (in our case use of general arguments) and formal reified algebra (in our case algebraic expressions K+K, M+M+3). Therefore, the present study provides some further evidence that younger students are able to work with algebraic expressions. More than a decade ago, Carraher et al. (2001) pointed out that "surprisingly little is known about children's ability to work with algebraic notation" (p. 138). The growing amount of evidence that is provided by recent studies should be taken into consideration by curriculum reformers in order to help students invest on their true capabilities that will in turn have paybacks for both algebra and arithmetic. Algebraic understanding will evolve slowly over the course of many years; however we need not await adolescence to help its evolution (Carraher et al., 2001, p. 137). Nevertheless, further research in this direction will shed more light about what younger students *are* able and *not* able to do concerning work with algebraic notation

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RETHINKING AND RESEARCHING TASK DESIGN IN PATTERN GENERALISATION

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This paper describes the design principles behind a test instrument, called the JuStraGen test, that had been specially developed to assess students' ability to generalise figural pattern generalising tasks, as well as to measure the effects of two task features on their rule construction. A discussion of some student responses then follows to shed light on how students dealt with some tasks in the test.

BACKGROUND

Several past studies on pattern generalisation have reported low success rates for figural generalising tasks in which just a single configuration was presented. In a study by Hoyles and Küchemann (2001), nearly 2800 high attaining Year 8 students in the UK were asked in one of the tasks to inspect a single generic case in order to find the number of grey tiles needed to surround a row of 60 white tiles. The border-tiling task had a success rate of 42%, which was considerably low taking into account the students' prior attainment. The same border-tiling task was also used by Cañadas, Castro and Castro (2011) on over 350 Years 9 and 10 students in Spain, this time asking them for the number of grey tiles needed to surround a row of 1320 white tiles. The success rate of about 55% was similarly moderate. Like these researchers, Steele (2008) had rather limited success in getting students to work out a functional rule for predicting the number of blocks in a staircase with *n* steps in a classic *Staircase* task that shows only a four-step-high staircase. Six of the eight students in the US had difficulties constructing the functional rule, which was quadratic. The type of function in this task might well have been a contributing factor. In another "classic" matchstick task that appeared in TIMSS–2007, a single configuration showing a row of four squares made of 13 matchsticks was provided and Year 8 students were asked about the number of squares in a row that could be made using 73 matchsticks. The success rate for Year 8 students internationally was barely 9% compared to about 41% for Singapore students (Foy & Olson, 2009).

The success rates of students in the abovementioned studies clearly indicate that the rule construction process in pattern generalisation is often fraught with difficulties, with many students often failing to navigate this process successfully. Such difficulties could be attributable to several student-related factors, ranging from ignorance of appropriate generalising strategies (Moss & Beatty, 2006) to lack of spatial visualisation techniques (Warren, 2005) and inexperience in using the highly specific mathematical language of algebra to express generality (Hoyles, Noss, Geraniou, & Mavrikis, 2009). But in the light of the earlier paragraph, we posit that student difficulties in rule construction might also be triggered by task features, in particular,

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two features that we categorise as *the format of pattern display* (Chua & Hoyles, 2012) and *the type of function*. Revisiting the *Staircase* task in Steele's (2008) study for example, we wonder if the students' difficulties were influenced by the provision in the task of only a single generic case, or by the quadratic nature of the underlying pattern. Whether Steele realised these potential obstacles in her task is unclear, but Küchemann (2010), however, firmly maintains that the factor contributing to student difficulties in that task was the format of pattern display, and not the type of function.

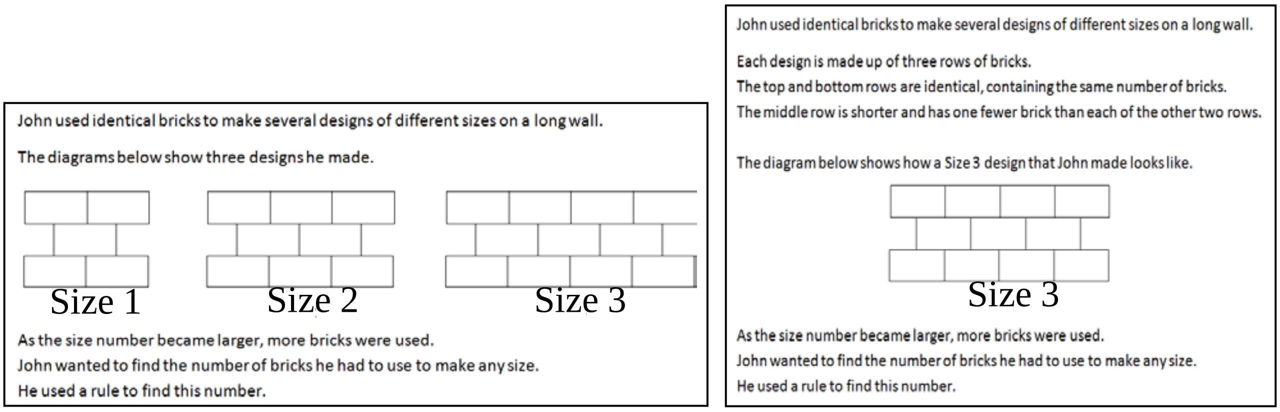
Our present study, therefore, aims to examine systematically the effects of different formats of pattern display and types of function on students' pattern recognition and their ability to generalise. In order to carry out the study, it was first necessary to construct and validate an instrument; in this case a specially-designed paper-and-pencil test. In this paper, we address the following question: What task design considerations were taken into account when devising the test instrument? In what follows, we describe the development of the test instrument, present some test items and highlight some student responses to illustrate its implementation.

DEVELOPMENT OF TEST INSTRUMENT

We were unable to identify from the review of the research literature a test instrument that would characterise the effects of task features on students' pattern recognition and their ability to generalise. We therefore set out to design a new test instrument, which we entitled *Strategies and Justifications in Mathematical Generalisation* (*JuStraGen*). It was developed specifically to achieve the aims of the present study.

The *JuStraGen* test was designed to provide an assessment of students' ability to generalise figural pattern tasks, as well as a measurement of the effects of two task features on their rule construction. It is a paper-and-pencil test consisting of eight generalising tasks designed to investigate how students construct and justify the functional rule for predicting any term of a pattern in the tasks. Of the eight tasks, the underlying pattern structure was linear for four of them and quadratic for the other four. Furthermore, the test was also developed specially to examine systematically the effects of *the format of pattern display* (i.e., successive vs non-successive configurations) and *the type of function* (i.e., linear vs quadratic) on students' ability to construct the functional rule. Figure 1 shows a linear task in the two different formats. Students were required individually to work out a functional rule for the pattern in terms of the size number, and justify how they obtained the rule.

To examine whether different formats of pattern display had any effect on students' rule construction, we chose to use a between-subjects experimental design involving two groups, Group 1 (G1) and Group 2 (G2), of students. As for testing whether different types of function had any effect on students' rule construction, a within-subjects experimental design was adopted. In short, G1 worked on both linear and quadratic generalising tasks with successive configurations whereas G2 was given identical tasks but with non-successive configurations.



(a) Successive format (b) Non-successive format

Figure 1: Bricks

The development of the *JuStraGen* test was guided by the research design above and the following general considerations.

(a) ***Number of generalising tasks*** Deciding on how many tasks to set is a tricky matter: too few tasks may limit the generalisability of the results about the effect of task features on students' success in establishing the functional rule; whilst having too many tasks is simply not practical given the time needed to complete them. After pre-piloting a task to gauge the amount of time students needed to complete it, we decided to set eight tasks. We believed that this number of tasks was a reasonable figure for covering a range of non-successive configurations.

(b) ***Task scenario*** Most figural generalising tasks used in research and in textbooks ask students to consider a sequence of configurations and then make some near and far generalisations, followed by finding the rule underpinning the pattern depicted in the sequence (see Rivera & Becker, 2008). The tasks rarely provide a scenario in which the purpose of representing the pattern with a functional rule might be apparent. For some students, it might, therefore, be difficult to see *why* they have to do what is required of them. To provide some impetus for students, we tried to adopt the notion of *purpose* (Ainley, Pratt, & Hansen, 2006) to make the tasks as meaningful as possible for the students. We framed the generalising tasks in different scenarios, such as making wall designs for *Bricks*, and stated the motive as wanting the students to help the character in the task to find the rule for constructing any size (e.g., John wanted to find the number of bricks he had to use to make any size in *Bricks*. Write down the rule John might have used in terms of the size number).

(c) ***Parallel tasks*** To determine whether the format of pattern display influenced the students' construction of the functional rule, each task was created in two different formats, with its pattern depicted as (1) a sequence of three successive configurations, and (2) a single configuration or a sequence of two or three non-successive configurations. For instance, the *Bricks* task in Figure 1 above shows three configurations (Sizes 1, 2 and 3) for the successive format and a single generic configuration (Size 3) for the non-successive format.

(d) ***Matching tasks*** To determine whether the type of function influenced the students' construction of the functional rule, each linear generalising task had a matching quadratic generalising task. Table 1 below lists the matching linear and quadratic generalising tasks, with details about the format of pattern display. For each pair of tasks, the description of the scenario was kept invariant: for instance, both *Bricks* and *Wall Design* were set in the same scenario of creating wall designs using bricks. Furthermore, the shape of the configuration in each linear task was created to resemble as closely as possible that of the matching quadratic task. Considering the *Birthday Party Decorations* and *Christmas Party Decorations* tasks for example, both sets of configurations look alike except for the blocks in the middle. We believe that careful considerations to such details during the task design process are essential as pre-emptive measures for minimising the possible interference of task scenario on the outcome of the *JuStraGen* test so that more robust conclusions can be drawn about the effect of the type of function on how students construct the functional rule.

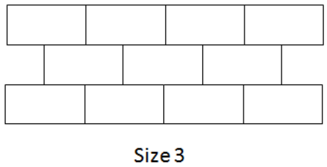
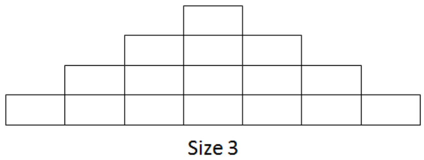
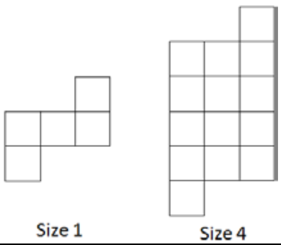
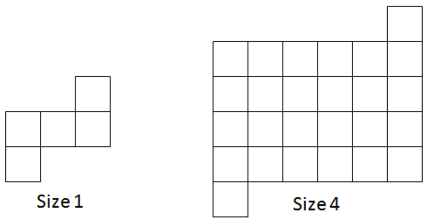
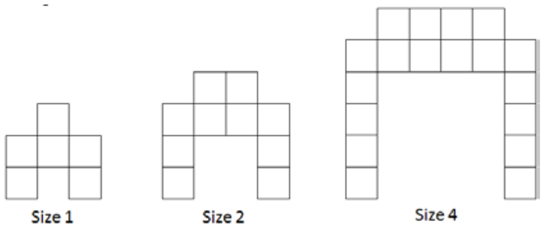
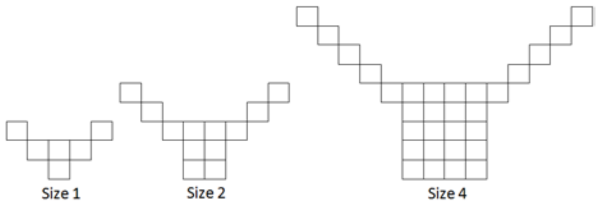
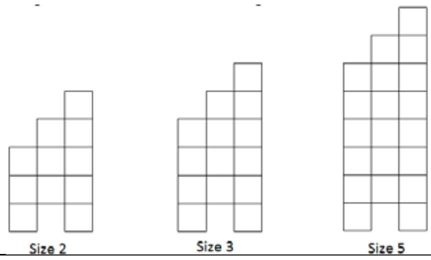
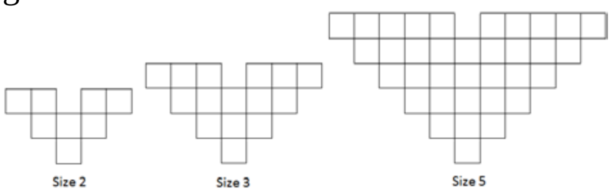
Linear	Quadratic
Bricks For successive format: Sizes 1, 2, 3 were given 	Wall Design For successive format: Sizes 1, 2, 3 were given 
Birthday Party Decorations For successive format: Sizes 1, 2, 3 were given 	Christmas Party Decorations For successive format: Sizes 1, 2, 3 were given 
Towers For successive format: Sizes 2, 3, 4 were given 	Oh Deer! For successive format: Sizes 2, 3, 4 were given 
High Chairs For successive format: Sizes 2, 3, 4 were given 	Tulips For successive format: Sizes 2, 3, 4 were given 

Table 1: Matching linear and quadratic generalising tasks

Bricks, *Birthday Party Decorations*, *Towers*, *High Chairs*, *Oh Deer* and *Tulips* were six new generalising tasks designed specially for the *JuStraGen* test. *Christmas Party Decorations* and *Wall Design* were adapted from studies by Rivera (2007), as well as Smith, Hillen and Catania (2007).

(e) ***Number of non-successive configurations*** In order for students to move to articulating the functional rule underpinning a pattern, we notice from the literature review that there are two common approaches in figural generalising tasks: first, to provide three configurations (see Rivera & Becker, 2008; Smith et al., 2007); and second, to show just a single configuration to represent a generic case of the figural pattern, as we have discussed previously. What is less common in the literature, however, is the use of two configurations. So far, we have found only three studies using it. The *Ladder* problem in Stacey's (1989) study showed two successive configurations whereas Healy and Hoyles (1995), as well as Warren and Cooper (2008), used two non-successive configurations in their studies. All these studies provided little, if any, explanation of the rationale for choosing to use these numbers of configurations. But, nonetheless, these numbers do appear to be sufficient to allow students to detect the pattern and then construct the rule. So we can infer that having more configurations would not make any difference. Guided by the outcome of the literature review, the present study decided to use one, two or three non-successive configurations in the *JuStraGen* test.

One might now ask whether it is really possible to discern the underlying pattern structure from just a single configuration. To address this concern, it was important to offer a *general* description of the single configuration. Although the description provided essential information for students to realise how the pattern would grow, it did not disclose the functional rule underpinning the pattern however. Furthermore, the use of a single configuration was limited to only one pair of generalising tasks – *Bricks* and *Wall Design*.

No description of the configuration was given for the remaining pairs of generalising tasks. Like single configuration, the use of two non-successive configurations was also limited to one pair of tasks – *Birthday Party Decorations* and *Christmas Party Decorations*. Three configurations were provided in *Towers* and *Oh Deer*, as well as in *High Chairs* and *Tulips*.

(f) ***Structure of task*** All the generalising tasks were unstructured in order to allow students scope for exploration so that they could come up with their own interpretations. This would allow us to see how the students came to recognise and perceive the pattern without scaffolding. so there were no part questions asking for near or far generalisations that would gradually lead students to detect and construct the functional rule underpinning the pattern.

(g) ***Size number of configurations*** The size numbers of the three given successive configurations ran from either Size1 to Size 3 or Size 2 to Size 4. As for the non-successive format, any single configuration starting from Size 3 was thought to be a reasonable generic case for representing a pattern. Thus Size 3 was given in *Bricks* and *Wall Design*. Warren and Cooper (2008) used solely odd-numbered sizes (Sizes 1 and 3, and Sizes 1, 3 and 5) in two of their tasks that involved two or three non-successive configurations. Their choice of configurations, we would argue, might be unfortunate because students might think that the even-numbered sizes did not exist in these tasks. So for generalising tasks with two or three non-successive configurations, we believed it was important to include both odd-numbered and even-numbered sizes so as not to mislead any students into thinking that certain sizes did not exist in the pattern. Therefore, we included both Sizes 1 and 4 in *Birthday Party Decorations* and *Christmas Party Decorations*. In a similar vein, Sizes 1, 2 and 4 were shown in *Towers* and *Oh Deer*, and Sizes 2, 3 and 5 in *High Chairs* and *Tulips*.

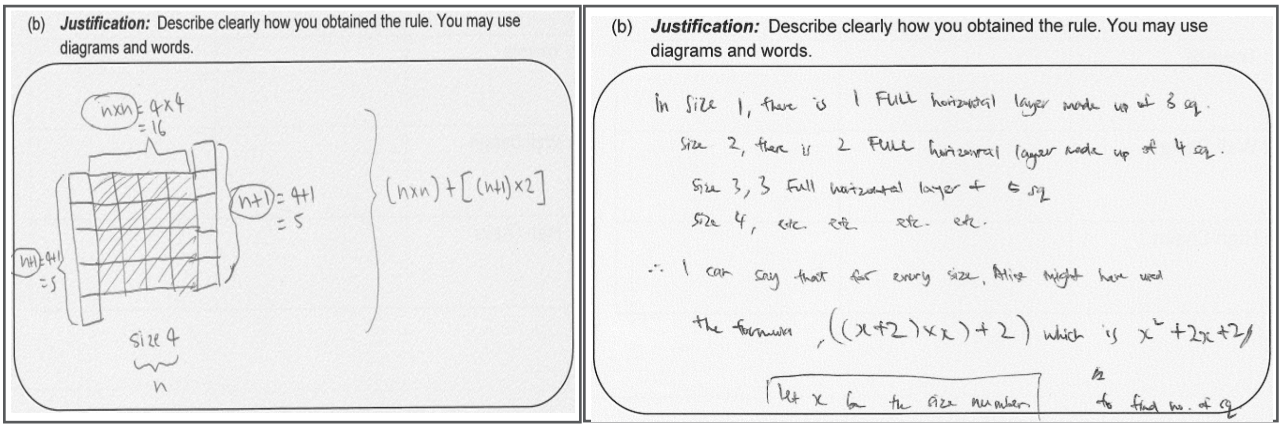
(h) ***Shape of building materials*** Square cards or tiles and rectangular bricks were used to build the configurations. Other shapes such as circles and triangles were omitted in order to eliminate the confounding influence of the shape used to build the configurations on students' ability to generalise.

This section highlighted the key design principles that we had applied to develop carefully crafted generalising tasks for the *JuStraGen* test. Due to limited space here, we will briefly describe what some students actually did when dealing with the non-successive tasks in the next section.

SOME FINDINGS AND CONCLUSION

56 G2 students (aged 14 years) from a secondary school in Singapore were administered the *JuStraGen* test with non-successive configurations. Having learnt the topic of number patterns before sitting the test, the students should be able to continue for a few more terms any pattern presented as a sequence of numbers or configurations, make a near and far generalisation and establish the functional rule in the form of an algebraic expression.

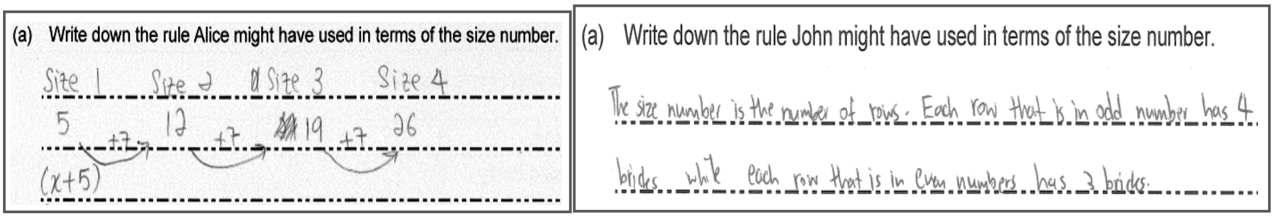
The unstructured nature of the generalising tasks allowed students plenty of scope for developing their answers. When a single or two configurations were given, some students had to work out other configurations before they could see the structural relationship from the geometrical arrangement of tiles or cards (see Figure 2(a)). For some other students, finding additional configurations was not necessary at all as they were able to abstract the structural relationship from the given diagrams by treating them generically (see Figure 2(b)). Therefore, students' ability to derive the functional rule was clearly assisted by their *awareness* of the structure inherent in the pattern, and *not* the format of pattern display. We consider this finding very encouraging, knowing that our decision to design unstructured tasks was appropriate.



(a) with additional configurations (b) without additional configurations

Figure 2: Recognising structure in Christmas Party Decorations

Students' inability to recognise the pattern underpinning a single or two configurations is not totally unexpected and, in particular, two student responses are worth discussing with respect to our design principles. Figure 3(a) shows a student misinterpreting the *Bricks* pattern despite the provision of a description of the configuration. For this student, the number of rows in a configuration corresponds to its size number, and the number of bricks per row alternates between four in odd rows and three in even rows. Although there were only six such cases (11%) in the present study, the frequency of cases could have been higher if the task had not provided the description. We are, therefore, now convinced that the inclusion of a general description of the single configuration in the *Bricks* task was crucial and necessary.



(a) Bricks (b) Christmas Party Decorations

Figure 3: Student errors

The response in Figure 3(b) shows a wrong pattern produced by a student for the *Christmas Party Decorations* task. Somehow the student must have inspected the difference between the given Sizes 1 and 4, then figured out that the difference could be evenly divided over four successive configurations. This discovery eventually led to working out Sizes 2 and 3. The numerical terms {5, 12, 19, 26} did not match the figural pattern even though they formed a linear sequence. Its validity could have been easily verified by drawing out the configurations for Sizes 2 and 3. We want to argue that the student's error is *not* caused by any design flaw in the task but by the student himself or herself for making a wrong assumption about the pattern and using an inappropriate strategy (i.e., finding the common difference).

To conclude, this paper introduced the *JuStraGen* test instrument that was developed from scratch to serve the purposes of the present study. We hope the detailed description of the design of the test instrument will permit other researchers to use the instrument in the same way as we used it and to further develop it so it can serve as a useful tool for the community.

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CHORAL RESPONSE AS A SIGNIFICANT FORM OF VERBAL PARTICIPATION IN MATHEMATICS CLASSROOMS IN SEVEN COUNTRIES

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Choral response is an under-researched aspect of mathematics classroom discourse. We analysed the use of choral response in 22 junior secondary mathematics classrooms from 7 countries. Reporting a categorisation scheme developed from this research, we demonstrate that the function of choral response in many mathematics classrooms goes far beyond the simple recitation and memorisation drills suggested in the literature. Examples are provided of each form of choral response, from approval or agreement to the completion of mathematical propositions and the identification of mathematical procedures. Choral response warrants greater research attention and appears to be most evident in those classrooms where student-student interaction is least frequent, offering a culturally-compatible method to promote student speech.

INTRODUCTION

The research-based advocacy of student engagement in classroom dialogue (eg Walshaw and Anthony, 2008) tends to privilege the voice of individual students. Choral Response (CR) is rarely recognized as a legitimate form of verbal participation with the potential to engage students in classroom dialogue about mathematics. Yet choral response, also known as unison responding, has been shown in primary school and non-mathematical contexts to facilitate a high degree of active student involvement (Carnine, Silbert, Kame'enui, & Tarveer, 2004) and to build confidence in low-achieving students by allowing them to perform well in front of peers, rapidly increasing active student response in group instruction (Heward & Wood, 2009). However, because of the lack of complexity typical of students' choral responses compared to the sophistication possible with elaborated individual responses, choral response is often associated with recitation and memorizing drills, and has been criticized as not conducive to good learning (Doyle, 1986).

It appears that most of the studies of choral response have been conducted in primary schools, particularly with language classes (eg Grow-Mienza, Hahn, & Joo, 2001; Wang, 2010). Few studies have investigated the use of choral response in secondary mathematics classrooms. We have analysed the use of choral response in 22 secondary mathematics classrooms from 7 countries around the world. This study extends our research on spoken mathematics (eg Clarke & Xu, 2008), by examining the mathematics content of choral response, its sophistication, context and purpose.

METHODOLOGY

We analysed video records of 110 lessons from 22 classrooms in Australia (Melbourne), China (Hong Kong and Shanghai), Germany (Berlin), Japan (Tokyo), Korea (Seoul), Singapore, and the USA (San Diego). The lessons were taken from the data corpus collected for the Learner's Perspective Study (LPS). Details of the project methodology are available elsewhere (eg Clarke, 2006). For this analysis, it is important to note that three video cameras were used (teacher camera, student camera and whole class camera) and each provided an audio record from which classroom speech could be analysed. We distinguished three types of public utterances: teacher utterance, choral utterance, and (individual) student utterance. Public utterances were those that occurred in whole-class discussion or during teacher-student interaction.

City	School/ Classroom	Average number of Choral Responses per lesson (average over 5 lessons)	Percentage: CR/AU	Percentage: IU/AU
Shanghai	SH1	75	30%	15%
	SH2	30	12%	26%
	SH3	35	15%	17%
Hong Kong	HK1	7	4%	39%
	HK2	26	9%	15%
	HK3	8	3%	32%
Seoul	KR1	44	16%	2%
	KR2	83	26%	5%
	KR3	70	32%	0%
Tokyo	JP1	5	1%	33%
	JP2	1	0.4%	16%
	JP3	3	1%	28%
Singapore	SG1	34	10%	28%
	SG2	42	8%	34%
	SG3	26	8%	31%
Berlin	GE1	1	0.5%	41%
	GE2	3	1%	42%
San Diego	US1	22	4%	39%
	US2	43	13%	27%
Melbourne	AU1	0	0%	38%
	AU2	1	0.2%	44%
	AU3	0	0%	37%

Table 1. Average number of choral responses per lesson for each classroom

Table 1 displays the average number of choral responses (CR) per lesson for each of the 22 junior secondary mathematics classrooms studied, expressed as a percentage of all public utterances (CR/AU), and compared with the individual student utterances

(IU) expressed as a percentage of all public utterances (IU/AU). All other public utterances were spoken by the teacher.

Given that the average was calculated over five lessons for each classroom, the entries in table 1 can be taken as indicative of the level of usage of choral response in each classroom. The results suggest that the classroom frequency of choral utterances varies significantly between cities/school systems and in some cases within the same school system. The number of choral responses as a proportion of total public utterances also differed from classroom to classroom. In this analysis, we sought to compare indicative levels of use of choral response between classrooms, and also to investigate the diversity of forms and the relative sophistication possible through the use of choral response. Table 1 provides an indication of the relative frequency of choral response in the classrooms studied and of the variation in use between school systems. Our second goal required the careful classification of choral response types.

No.	Type of choral response
1	"Yes, No" (select choice) responses - the class is given two (at most three) options and have to choose the correct option
2	Numerical responses - where a numerical value (other than an indexical label referring to a point/option/equation etc) is the intended answer to the teacher's question
3	Mathematical symbolic expressions - consists of a combination of numbers, pronumerals and/or mathematical symbols representing equations, algebraic expressions, ordered pairs, points, vertices, etc.
4	Mathematical terms - word(s) or phrase(s) used in the mathematical discourse relevant to the topic taught
5	Mathematical procedures - step(s) involved in solving a problem or deriving an answer
6	Mathematical propositions - all or part of a mathematically proven or declarative statement affirming that a mathematical fact or relationship is either true or false
7	Non-mathematical responses - a response related to an organisational or social aspect of the task or instruction
8	Unclassified responses - an undecipherable response or an utterance expressing excitement or social ritual.

Table 2. Categories for types of choral response

We used an iterative approach to develop a set of categories for the types of choral response identified. Starting with the lesson with the most choral responses from each classroom, we generated an initial set of categories, which were then augmented by consideration of other lessons, leading to the classification system shown in Table 2.

In this paper, we are interested in the responses that are directly related to mathematics activities, that is, responses that contain mathematics content. Therefore, this paper will only focus on the first six types of choral response. In order to determine the nature and the level of sophistication of the response, each category was further differentiated into two sub-categories: recall and analysis (Table 3).

Recall	When the class recalls a mathematical fact; recognises the answer to a question that is either already included in the question on the board/screen/handout/textbook or can be counted at a glance; or reads aloud information from the board/screen/handout/textbook
Analysis	When the class obtains/derives an answer after reflecting on or solving a problem by working it out mentally or with pen and paper or by using technology

Table 3. Nature of choral response

TYPES OF CHORAL RESPONSE

In the remainder of this paper, we illustrate the different types of choral response by drawing upon examples from the lessons analysed and discuss the value of choral response as a form of verbal participation in mathematics classrooms.

Yes/no (select choice) responses

Among all the lessons studied, one of the most common choral responses required the "yes/no" selection from two or at most three options. Among the responses analysed, more than one-fifth belonged to this category. In each instance, there was only one right answer. The question usually related to a known fact, a concept taught previously, a previous question or the evaluation of a student's oral/written response. Below is an example from one Hong Kong classroom, in which students were given two choices in making judgements about y values and x values.

Example 1: HK2-L02 (00:02:45:14) (*recall*)

T: This is a pair of simultaneous equations: y equals x plus one and y plus two x equals sixteen.

T: We talked about it yesterday. These are two equations, a pair of y and a pair of x, how should the pair of y values be? [Two seconds of silence]

T: Are they equal or not?

Ss: Equal.

T: How about the values of this pair of x?

Ss: Equal.

T: They are equal. When they are solved, their values should be the same. Let me put it the other way around: if I substitute the values, they fit perfectly. That is the values that will satisfy both the first and the second equations.

The first question from the teacher was not immediately answered by the class, possibly because of its ambiguity. The closed question "Are they equal or not?" was proposed after a two-second silence from the class, and the class was able to respond in chorus. This choral response required the recall of information previously learned.

Numerical responses

Another common type of choral response required the students to recognize or provide a numerical answer to a given question (approximately one-fifth of analysed responses). We are not talking here about an indexical reference to a numbered equation or example, but to a question for which the correct answer was a quantity.

Example 2. SH1_L01 (00:03:47:05) (*recall*) [discussing 2x + y = 10]

T: Use the second rule and divide two from both sides of the equation, right? So, we can simplify it to two x plus y equals ten. Then, classmates, now let's see how many unknowns are there in this equation?

Ss: Two.

T: Two unknowns. So what's the index of the unknown?

Ss: One.

Even in an example as simple as this one, the teacher's question is predicated on the assumption that all the students had some knowledge about unknowns and indices.

Mathematical symbolic expressions

Mathematical symbolic expressions provided another common choral response category in the classrooms analyzed. These could consist of a combination of numbers, pronumerals and mathematical symbols representing equations, algebraic expressions, ordered pairs, and so on. In Example 3, the students were required to transform the equation mentally and express it in the general form of a linear equation in two unknowns.

Example 3. HK2_L02 (00:06:53:32) (*analysis*)
Solving a pair of simultaneous equations by substitution: y + x = 3 and 2x + y = 24

T: But is there any here? In the two equations, none of them is expressed in the general form. We don't have anything like x equals something or y equals something.

T: If we face such situation, we have to express one equation of our choice in the general form.

Ss: *y* equals three minus *x*.

T: *y* equals//

Ss: //three minus *x*.

Mathematical terms

In some of the classrooms studied, the students were expected to use specific mathematical terms in responding to the teacher's questions.

Example 4. US2_L05 (00:21:55:25) (*recall*)

T [Writing on board: y = mx + b] That is a special form of a what? What graph?

T Curve or a line - linear or non-linear?

Ss Linear

T Linear. Okay, what are these components? [Pointing to equation]

	What's this? [Circles the M].
Ss	Slope.
T	Slope. [Writes 'slope' on board] What's this?
Chelsea	The Y intercept.
T	[Jumps into air with meter stick over head] The what?
Ss	Y intercept.
T	Y intercept, yeah. Oh heavens. Okay. Y intercept. [Draws arrow on white board pointing to the B in the equation] Okay, put that into your notes.

The rehearsing of mathematical terms by the whole class was a key characteristic of some of the classrooms analysed (Clarke, 2010). The elicitation of mathematical terms in these classrooms could be seen as a purposeful attempt by the teacher to help the students memorize the terms that were regarded as mathematically important.

Mathematical Procedures

Another type of choral response involved the procedures or sub-procedures required to solve a problem. Students were invited to solve a problem together by orally stating the steps involved. This was usually elicited by a series of teacher questions.

Example 5. SH1_L02 (00:27:00) (*analysis*)

T:	Now we have to decide its abscissa. How can we do it [identify the x-coordinate of point P]?
Ss:	Draw a vertical line from point P to the x-axis.

In this case, no explicit information about the form of the choral response was provided and the successful provision by students of the expected choral response depended on the understanding of the students regarding how it should be said based on their previous experience (that is, on their proficiency with the discourse and meta-discursive rules of that mathematics classroom).

Mathematical propositions

Students were sometimes required to provide an elaborated answer as part of a mathematically proven statement or a declarative statement that affirmed that something was either true or false in general. Below is an example in which the students were asked to respond with a mathematical proposition.

Example 6. SH3_L01 (00:06:02:28) (*analysis)*

Identifying linear equations in two unknowns. 1) $2x + 3 = 0$ 2) $x + 2y - 1 = 0$

T:	The first is not a linear equation in two unknowns. What about the second one?
Ss:	It is.
T:	Yes, say together, why?
Ss:	It has two unknowns and the unknowns are of power one.

The students were asked to provide a reason for why the second equation is a linear equation in two unknowns by rehearsing a definition that was learned previously. Rather than a simple recall of the definition, it involves some analysis of the situation before the students can generate the answer in unison. The generation of such sophisticated choral responses is a consequence of repeated and purposeful practice by the students, where the phrases of mathematical justification are rehearsed.

CONCLUSION

It is clear that the function of choral response in many mathematics classrooms goes far beyond the simple recitation and memorizing drills suggested in the literature. The six types of mathematical choral response reported in this paper demonstrate the diversity of ways that each teacher employed this discursive element to serve different instructional purposes. A simple "yes" or "no" could be a response to a very closed question, the main purpose of which is to keep the students on task. But it could also be a sophisticated response that required the evaluation of a solution or of a generalized mathematical statement. A choral response was also elicited as a way to involve students in simple mental calculation or in the process of solving a problem. In some classrooms, students were also required to complete the statement of a mathematical proposition as a whole class. The generation of sophisticated choral responses was limited to only a few of the 22 classrooms analysed. For these classrooms, it is clear that the collective way of responding had become a normative practice in the classroom. Elsewhere (Clarke, Xu & Wan, 2010), we have reported that the encouragement of student-student interaction is a classroom strategy employed extensively in some classrooms and not at all in others. Choral response is most evident in precisely those classrooms in which student-student interaction is least frequent, and conversely (see Table 1), and can be interpreted as a culturally-specific solution to the challenge of stimulating student spoken mathematics. Certainly, the use of choral response in mathematics classrooms warrants more attention than it has received to date. A companion paper will report classroom discourse patterns employing choral response as an integral element.

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TEACHER NOTICING AND GROWTH INDICATORS FOR MATHEMATICS TEACHER DEVELOPMENT

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This report is based on data gathered as part of a UK project looking into tackling underachievement in primary mathematics through a focus on creativity. We initially analyse, within the framework of noticing (Mason, 2002), if in the discussions of in-service primary school teachers on the project, there is evidence of the shifts in teachers' noticing, proposed by Jacobs, Lamb, and Philipp (2010) as growth indicators. Results show evidence of these shifts. However, we go on to analyse the data further and find a significant shift that is not captured by Jacobs et al's indicators. We conclude by arguing for a re-conceptualisation of the idea of growth indicators, towards a more cyclical sense of movement and development.

INTRODUCTION

We report on the analysis of four transcripts of teacher meetings that took place over the academic year 2011-12, in the context of a project aimed at tackling underachievement in primary mathematics through creativity. The funded project[1] is a collaboration between the University of Bristol and the charity "5x5x5=creativity"[2]. For the purposes of the project, we define creativity within mathematics to be indicated by students noticing patterns; asking their own questions; and making their own conjectures. In the first year, which we report on here, three primary/infant schools (A, B and C) in the South West region of the UK were involved. One teacher from each of the three schools joined a project group that met five times over the academic year. These were twilight meetings that generally lasted just over an hour. Alf convened this group and, in between meetings, visited the schools to observe and then lead sessions with the teachers' classes, with a focus on running activities and class discussion in a way that allowed and supported student creativity. Alf made on average 10 visits to each school. The focus of the group meetings was on teachers sharing the work they had been doing, which included strategies for developing creativity and tackling underachievement. The ages of the children in the focus classrooms were 6-7 years old, in schools A and B and 7 to 9 years old in school C.

THEORETICAL FRAMEWORK AND OBJECTIVES

The development of the skill of noticing

Learning to notice is part of the development of expertise. Noticing what is happening in a classroom is an important skill for teachers. However, noticing effectively is both complex and challenging (Jacobs et al., 2010). Mason (2002) characterized noticing as: keeping and using a record; developing sensitivities; recognizing choices; preparing to notice at the right moment; and validating with others. Using Mason's work, van Es

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and Sherin have conducted extensive work on noticing in mathematics education (van Es & Sherin, 2002) often using “video clubs” in which teachers watch and discuss video from each others’ classrooms. These authors proposed a noticing framework that includes identifying noteworthy aspects of a classroom situation; using knowledge about the context to reason about the classroom interactions; and, making connections between specific classroom events and broader principles of teaching and learning (van Es & Sherin, 2008). They found that teachers can improve their noticing by being supported in moving from a focus on teachers’ actions to students’ conceptualizations and by moving from evaluative comments to interpretative comments based on evidence. This improvement in noticing, going from descriptive and evaluative noticing towards a more interpretative one has been reported in other studies such as Crespo (2000).

Recent studies have provided other contexts for the development of the skill of noticing. For example, Coles (2012) proposed aspects of the role of the facilitator of teachers discussing what they notice from video clips of teaching. One of these aspects is moving to interpretation. Having had a period of time sharing accounts *of* (Mason, 2002) what was observed on a video clip (the task being to reconstruct the precise words or actions and their chronology), in focusing on the detail of what was noticed or observed, it is possible to then move to accounts *for* (interpretations of what occurred and why) avoiding judgmental comments. Noticing is supported by having a period of time describing the episode in all its detail and re-watching the clip when needed. Santagata, Zannoni, and Strigler (2007) offered a lesson-analysis framework (the identification of learning goals; the student learning in relation to those goals; and alternative teaching strategies to accomplish those goals) to help prospective teachers gain expertise in observing and reasoning about classroom events. Lundeberg, Cooper, Fritzen, and Terpstra (2008) suggested that video-supported reflection enabled elementary prospective teachers to write more specific (versus general) comments about their teaching and focus less on themselves and more on children when they reflected on video clips of their teaching and, therefore, this facilitates noticing. Star, and Strickland (2008) indicated that, after a teaching methods course where improving observation skills through videos was an explicit goal, prospective service teachers’ observation skills increased, particularly in teachers’ ability to notice: features of the classroom environment; the mathematical content of a lesson; and teacher and student communication during a lesson. Scherrer, and Stein (2012) pointed out improvements in teachers’ ability to notice interactions between teachers and students when analyzing classroom discussions. In this report, we are interested in the development of in-service teachers’ noticing and use a framework linked to children’s mathematical thinking skills.

The development of teachers’ noticing of children’s mathematical thinking

Jacobs et al. (2010) conceptualize the expertise of professional noticing of children’s mathematical thinking as a set of three interrelated skills: attending to children’s strategies; interpreting children’s understanding; and deciding how to respond on the basis of children’s understanding. This conceptualization is focused on in-the-moment decisions, based on what the teacher notices. Teachers have to make such decisions on a daily basis in the classroom when children offer strategies or explanations. In the study, findings also indicate that this skill could be developed, providing growth indicators that can help professional developers identify and celebrate shifts in teachers' professional noticing of children's mathematical thinking (p. 196, numbering added). Specifically,

A shift from general strategy descriptions to descriptions that include the mathematically important details.

A shift from general comments about teaching and learning to comments specifically addressing the children's understanding.

A shift from overgeneralizing children's understandings to carefully linking interpretations to specific details of the situation.

A shift from considering children only as a group to considering individual children, both in terms of their understandings and what follow-up problems will extend those understandings.

A shift from reasoning about next steps in the abstract to reasoning that includes consideration of children's existing understandings and anticipation of their future strategies.

A shift from providing suggestions for next problems that are general to specific problems with careful attention to number selection.

There are six indicators but we focus on the first four as the last two are linked to future instructional decisions. In the meetings that we analyse, teachers are reflecting on their work with their classes (describing students' strategies and interpreting understanding) and so they did not talk about what they were going to do next.

Recently, research has shown evidence of prospective teachers' professional noticing of children's mathematical thinking development in relation to this framework. Fernández, Llinares, and Valls (2011; 2012) show that participation in on-line debates supports this development in the specific domain of proportional reasoning. Text produced by prospective teachers in on-line debates helped some of the teachers attend to the mathematical elements of proportional and non-proportional situations and link these elements with characteristics of students' understandings. In Fernández et al. (2012), there was evidence of such shifts from general strategy descriptions (before the participation in the on-line debate) to descriptions that included the mathematically important details (after the participation). However, more studies, focusing on the different contexts that could improve this skill, are needed.

So, our objective is to analyze if, in the discussions of in-service primary school teachers who participated in the project introduced above, there is evidence of any shifts in relation to the first four indicators. In reflecting on this analysis, we go on to offer a critique of the indicators themselves.

METHOD

In this report, we focus on one of the three in-service teachers: Anna (a pseudonym). Anna is an in-service teacher in school B, which is an infant school in an area with high levels of social deprivation. We chose Anna as the focus since there is the clearest evidence of growth against the indicators in her case, although there is evidence of growth across all the teachers in the study.

We transcribed audio data of four of the five meetings between staff. The first teacher meeting was not audio-recorded to allow for an ethical discussion about participation in the project. For the analysis, three researchers (the authors of this report) analysed, individually, the transcript of the first audio-recorded meeting, looking for evidence of the aforementioned shifts (Jacobs et al., 2010). We discussed agreements and disagreements as we shared what we saw as evidence for shifts. Through these discussions we identified common filters to use in looking at the data. For example, one filter we used was to look for when teachers spoke about a group of children and when they spoke about individuals (this filter is linked to indicator 4). Once we had shared this evidence and come to an agreement about what constituted evidence, we applied these filters to the rest of the teacher meeting data. We briefly explain, below, what we consider to be evidence for each of the four shifts.

A teacher gives a general strategy description (indicator 1) when he/she identifies a tool or mentions that the problem was solved successfully but omits details of how the problem was solved. If, later on, for example thinking about whole-number operations, the same teacher comments how children counted, used tools or drawings to represent quantities, or decomposed numbers to make them easier to manipulate, we would see a shift into the consideration of "mathematically important details". Teachers may give general comments about teaching and learning (indicator 2), such as, *"I learned that it's important to allow students to use different tools to come up with mathematical problem solution"* (Jacobs et al., 2010, p. 186). If, later, they make sense of the details of a student strategy and note how these details reflected what the children did understand, for example recognizing the ability to count by 2s or the ability to switch between counting by 2s and 1s we could identify a shift into giving comments specifically addressing the children's understanding. A teacher overgeneralizes children's understandings (indicator 3) when they go beyond the evidence provided. For instance, saying, *"children understand subtraction and addition — and which to choose when presented with a problem…"* (Jacobs et al., 2010, p. 186). This broad conclusion is difficult to justify on the basis of the children's performance on a single problem on which many may have used different strategies. If, later on, teachers make sense of the details of a student strategy and note how these details reflected what the children did understand in specific situations, we would say that there is a shift into linking interpretations to specific details of the situation. Finally, considering children as a group (indicator 4) is another characteristic of overgeneralising children's understanding and a shift is indicated by discussion of anything linked to individual understanding.

RESULTS

In this section, we present evidence of shifts in Anna's ways of talking. We report on meeting 2 and 3 (of the 4 audio recordings) as there was the clearest evidence of change in between these meetings. We observe that Anna, in all meetings, focused on individual children (indicator 4). In both meeting 2 and 3, Anna spent some time discussing child "M". We have used excerpts of her talking about the same child in order to analyse changes in how she talks.

Anna (meeting 2). So, we've got this boy who actually I don't know if you remember M on the first session and he sat, one of the first times when you came in, when he copied and he sat next to A who records really neatly. He didn't know what was going on but he copied how she recorded, as in one number in each box. So, I was, *he's copied, he hasn't done anything*. But actually from that he's recording his own and recording in that way which is really nice. So here it was, they could each choose, they chose their own number and practising how many different ways they could make that number using the Cuisenaire, so he picked up the yellow. So, we worked out what number that was and it was 'five'. So, then he started building his five wall and recording it and for him this is amazing. So, he is knowing that it all equals five. He is beginning to see well he's adding them together even though it's not in the 1 plus 2 plus 3.

Anna (meeting 3). And then M. He tried this with Cuisenaire and realized he couldn't really work it out so he moved onto a hundred square when he was doing his finding out about the five times table and so then spotted the pattern that he is going and circling on the hundred square, so he could just carry it on. And that was the first step in January of him being able to notice a pattern that he could then use.

Anna has given comments addressing the children's understanding, and is not in the realm of giving general comments about teaching and learning (indicator 2). For example, in meeting 2, she says "*he picked up the yellow. So, we worked out what number that was and it was 'five'. So, then he started building his five wall and recording it…he's adding them together even though it's not in the 1 plus 2 plus 3"*. And in meeting 3, she says, "*he tried this with Cuisenaire and realized he couldn't really work it out so he moved onto a hundred square when he was doing his finding out about the five times table and so then spotted the pattern that he is going and circling on the hundred square"*. We see, in both these instances, a focus on M's understanding (indicator 2). However, we can observe a shift from overgeneralizing children's understanding in meeting 2, to linking interpretation to specific details of the situation in meeting 3 (indicator 3). The evidence is that in meeting 2 she says, "*So, he is knowing that it all equals five. He is beginning to see well he's adding them together even though it's not in the 1 plus 2 plus 3"*. Although there is attention paid here to the child's understanding, we read an overgeneralisation in the comment "*he is beginning to see well he's adding*", which is not something it is possible to observe directly. This kind of comment is also around in meeting 3, for example "*he … realized he couldn't really work it out so …*"; here, again, Anna cannot know whether M realised something or not. However, also in meeting 3, Anna says "*And that was the first step in January of* *him being able to notice a pattern that he could then use"*. Here, in contrast, the comment is a careful interpretation of specific details – M has noticed a pattern that he was able to continue and it was the first time he had done this during the year. It would have been possible to observe this new behavour of M's and the attribution of his noticing is closely linked to what Anna saw and hence is justified. In relation to indicator 1, across both meetings we see evidence of Anna considering mathematically important details although perhaps, as ever, there are more mathematical issues that could be raised.

We also considered whether there are significant changes that took place over the meetings not captured by the growth indicators - the data on Anna indicates there are. Indicators 1 and 2 denote a shift from general descriptions to the particular of classroom events. While we agree with Jacobs et al. (2010) that this is an important shift, we do not see it as the end of the story. Anna, as we have shown, talked in the meetings about individual children and events. As well as, on occasion, generalising from these observations, for example in the except below we read her articulating a general move in her teaching to "letting children speak":

Anna (meeting 2). And I think it goes back to that very first session we did when you let J read those numbers because at that very beginning, it's her trying to spot something and other children are spotting and to us it didn't really make any sense. And it is like letting children like M for example going 'I used a pattern, I did two, two, two, two, two' because he's added two every time and just allowing them to say that out and then gradually you see actually through this that they've then actually begun to spot patterns that they can use that are helpful.

Anna, here, is noticing a similarity in an incident involving J and one involving M. In both cases there is a "letting" of the children talk about what they notice. Articulating this kind of more general label, we see as significant in teacher learning (see Brown & Coles, 2012). The movement is in the opposite direction to what Jacobs et al. (2010) see as "growth", yet we believe the articulation of a label such as "letting children speak" supports future noticing. The label is an example of a "purpose" (Brown, 2005) that supports the development of new actions in the classroom, linked to the label. This kind of articulation is perhaps also an example of what van Es and Sherin label: "making connections between specific classroom events and broader principles of teaching and learning" (2008, p. 245). Like van Es and Sherin, we also see this kind of connection as one of the marks of expertise (Brown & Coles, 2011) and so in supporting teachers to reach such articulations we support them in developing expertise in their classrooms.

DISCUSSION

This study analyses if, in the discussions of in-service primary school teachers who participated in a project, there is evidence of any shifts in relation to the first four indicators proposed by Jacobs et al. (2010) indicating a development in their skill of noticing. Results indicate that in-service primary school teachers who participated in the project showed evidence of shifts related to the way that they notice children's understandings. Therefore, teachers show evidence of development of the skill of noticing. So, the meetings of the project where in-service teachers shared the work they had been doing to tackle underachievement and develop creativity and the visits at the school by one of the researchers running activities and supporting student creativity seem to support this development.

We have found the growth indicators used in the analysis above have helped us to see more in Anna's words. The indicators have helped tease out subtleties and complexities that we did not notice ourselves initially. However, we argue that there is a significant shift not captured by the indicators, when Anna, from the detail of talk about students, is able to articulate a more general statement about principles guiding her interactions and teaching. Articulation of such general statements supports future noticing and the accrual of strategies in the classroom linked to the statement (Brown & Coles, 2012). One issue we find with the idea of a "growth indicator" is an assumption of a one-way direction of development. When it comes to making general comments about teaching and learning, compared to focusing on individuals, we see our work with teachers more as a cycle than an upward gradient. We do not believe it likely that anyone can arrive directly at general statements of principle, if discussion begins at a general level. There is a need to support discussion of the detail of classroom events, as Jacobs at al. (2010) suggest. However, we suggest that growth is also supported by a cycling back, from the particular to the more general, in order to arrive at the succinct articulation of principles or "purposes" (Brown & Coles, 2012) that can be kept in mind, to inform future noticing and future actions.

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DO STUDENTS ATTEND TO AND PROFIT FROM REPRESENTATIONAL ILLUSTRATIONS OF NON-STANDARD MATHEMATICAL WORD PROBLEMS?

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Verschaffel et al. (1994) presented upper primary school children word problems that were problematic from a realistic modelling point of view (so-called P-items). They found that pupils in general did not use their everyday knowledge when confronted with such P-items, and thus solved them unrealistically. In this paper we report two related studies that investigated whether and how illustrations that represent the problematic situation described in a P-item help higher education students to imagine the problem situation and thereby solve the problem more realistically. We found that participants barely look at these representational illustrations and when they do look, there is no effect of the illustrations on the realistic nature of their solutions.

INTRODUCTION

Students encounter various kinds of illustrations in their mathematics textbooks. Some of these illustrations just decorate the textbook page, while others are more or less directly linked to the mathematical content. A lot of theoretical and empirical research has already investigated the relation between text and illustrations, but up to now little or no research has been done about the influence of illustrations on students' approaches to mathematical word problems that are problematic from a realistic point of view (so-called P-items). Previous research shows that students exclude their knowledge of everyday life when solving such word problems (Greer, 1993; Verschaffel, De Corte, & Lasure, 1994). In the present paper we report two closely related studies that investigated whether and how students attend to and use the illustrations that accompany P-items and that are, as we would expect, helpful to solve these problems realistically. Before presenting these studies we will briefly review the research about students' approaches to and solutions of P-items.

THEORETICAL AND EMPIRICAL BACKGROUND

It is generally claimed that, as a result of their year-long participation in traditional mathematical word problem solving lessons, many students approach these problems in a superficial and artificial way. They just search for the mathematical operation(s) to perform with the given numbers, with little or no attention to the meaningfulness of their solution (Lave, 1992; Reusser & Stebler, 1997; Schoenfeld, 1991; Verschaffel, Greer, & De Corte, 2000). In an attempt to provide empirical evidence for this claim, Verschaffel, et al. (1994) presented 75 elementary school pupils 20 word problems to solve. Ten of these problems were standard items or S-items, i.e., "items that ask for the straightforward application of one or more arithmetic operations with the given

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numbers" (Verschaffel et al., p. 275). For example: "A man cuts a clothesline of 12m into pieces of 1.5m each. How many pieces does he get?" The other ten items were problematic items or P-items, i.e., items "in which the mathematical modelling assumptions are problematic, at least if one seriously takes into account the realities of the context called up by the problem statement" (Verschaffel et al., p. 275). For example: "A man wants to have a rope long enough to stretch between two poles 12m apart, but he has only pieces op rope 1.5m long. How many of these pieces would he need to tie together to stretch between the poles?" Only 17.0% of the reactions to these P-items were realistic reactions (RRs), which led the authors to the conclusion that upper elementary school children have a strong tendency to neglect their everyday knowledge when solving mathematical word items.

During the past 15 years several researchers have replicated this finding while others have tried various manipulations and interventions to better understand the origin and development of this tendency, and to counter it (see Verschaffel et al., 2000; Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009). Examples are studies in which pupils are alerted at the beginning of the test that some problems need careful consideration (Verschaffel et al., 2000; 2009; Yoshida, Verschaffel, & De Corte, 1997) or are confronted with more authentic versions of the P-items (Palm, 2008). The results of the studies in which pupils were alerted were not or only moderately positive, whereas the studies in which the problems were made more authentic showed significant positive effects.

In a previous study (Dewolf, Van Dooren, Ev Cimen, & Verschaffel, in press) we investigated whether presenting illustrations that represent the problem situation (i.e., representational illustrations according to the categorisation of Elia & Philippou, 2004) increased the number of RRs on P-items. The study consisted of two parallel data collections in Turkey and Belgium. Respectively 402 and 233 pupils from the 5th or 6th grade from elementary school in Turkey and Belgium were confronted with a subset of P-items and S-items from the study of Verschaffel et al. (1994). One fourth of the pupils received the problems without any manipulation, another fourth with an illustration that represented the problematic situation, another fourth with a warning about the tricky nature of some of the problems in the test, and a last fourth with both an illustration and a warning. The expectation was that there would be a positive effect of the illustrations and of the warning on the number of RRs on the P-items, and especially when these illustrations were combined with a warning. However, contrary to our expectations, neither the presence of an illustration, nor the presence of the warning, and not even the combination of both, resulted in an increase of RRs.

In this paper, we will build further upon this study by reporting two closely related studies in which we will further investigate whether problem solvers pay attention to these representational illustrations and use them to solve P-items more realistically.

STUDY 1

In the first study we investigated why complementing P-items with representational illustrations (that as we expect help the problem solver to build a richer mental representation of the problem situation and thereby to evoke real world knowledge about the situation) does not result in more RRs. To investigate this we collected and analysed students' response data, reaction times (RTs), eye movements, and confidence scores on a subset of word problems from the study of Verschaffel et al. (1994).

Method

Thirty higher education students were equally divided in three conditions: 1) the Representational Illustration condition (RI-condition), in which the word problems were presented together with representational illustrations (i.e., illustrations that provide an overall depiction of the problematic situation), 2) the No Illustrations condition (NI-condition), in which the problems were presented in their original form without illustrations, and 3) the Decorative Illustration condition (DI-condition) in which the word problems were presented together with decorative illustrations (i.e., illustrations that have no connection whatsoever to the content of the problem). For an example of the Rope P-item and the planks P-item together with their representational or decorative illustration see Table 1.

Each student was tested individually. Sixteen word problems (eight S-items and eight P-items) from the study of Verschaffel et al. (1994) were modified so that all parallel S-items and P-items were comparable in number of words, linguistic complexity, number of lines of text, and required mathematical operation(s). These 16 items were presented one by one on the computer screen. Students were asked to solve them and give their answer and possible additional comments orally. Depending on the condition, the word problems were presented with or without an illustration. The word problem was presented on the left side of the screen (text area), and the illustration – representational, decorative, or a blank space – was presented on the right side of the screen (illustration area).

While solving the problems, students' eye movements were recorded with the Eyelink II. Afterwards, the eye movement device was turned off and the students received a paper-and-pencil questionnaire with the same 16 word problems, presented again with a representational or decorative illustration or without an illustration (depending on the condition). In this questionnaire students were asked to indicate, for each item, to what extent they had hesitated about their answer (by responding: a) not at all, b) almost not, c) a little bit, and d) very much), and if so, why.

Analysis and Results

Students' responses on the items were analysed with a logistic regression analysis. First of all, we found that, just like elementary school pupils, higher education students tend to exclude real world knowledge when solving P-items. Only 27.9% of the reactions to the eight P-items were realistic. As in our previous study (***), no effect of representational illustrations was found; the number of RRs in the RI-condition (31.3%) did not differ significantly from the number of RRs in the DI-condition (31.3%) or the NI-condition (21.3%), $X^2(2,240) = 4.38$, $p = .112$.

Rope P-item	
a) A man wants to have a rope long enough to stretch between two poles 12m apart, but he has only pieces of rope 1.5m long. How many of these pieces would he need to tie together to stretch between the poles?	
b) A man wants to have a rope long enough to stretch between two poles 12m apart, but he has only pieces of rope 1.5m long. How many of these pieces would he need to tie together to stretch between the poles?	
Planks P-item	
a) Steve has bought 4 planks of 2.5m each. How many planks of 1m can he get out of these planks?	
b) Steve has bought 4 planks of 2.5m each. How many planks of 1m can he get out of these planks?	

Table 1: The rope P-item and the planks P-item with a representational illustration (a) and a decorative illustration (b).

Second, there were no significant differences between conditions in RT, $X^2(2, 480) = 0.29$, $p = .865$. There was however a significant difference between solving P-items and S-items; P-items were solved significantly slower than S-items, $X^2(1, 480) = 40.42$, $p < .001$. There was no interaction between condition and item type, $X^2(2, 480) = 0.77$, $p = .680$, so, P-items were processed more deeply than S-items, irrespectively of the presence or nature of illustrations.

Third, students' eye movements on the text also showed no significant effect of condition on the number of fixations, $X^2(2, 480) = 1.36$, $p = .507$. However, in line with the RT data, significantly more fixations on the text were needed to solve P-items than S-items, $X^2(1, 480) = 26.23$, $p < .001$, while there was no interaction between condition and item type, $X^2(2, 480) = 0.62$, $p = .732$. Concerning mean duration of the fixations, there was no effect of condition, $X^2(2, 19207) = 0.99$, $p = .610$, item type, $X^2 (1,19207) = 2.80$, $p = .094$, nor an interaction effect, $X^2(2, 19207) = 0.77$, $p = .681$.

Fourth, students' eye movements on the illustration area revealed that they barely looked at the illustrations. There were however significantly more fixations on the illustration area in the RI-condition (1.5% of all fixations), than in the DI-condition (0.4%), where the illustrations were not linked to the problem, or the NI-condition (0.0%), where the illustration area was blank, $X^2(1, 13230) = 34.48$, $p < .001$. To calculate how many illustrations in total were at least minimally processed with a minimum fixation of 150ms (Rayner, Smith, Malcolm, & Henderson, 2009), the longest fixation on each illustration, for each item per student was identified. In the RI-condition 26.3% (16.9% P-items and 9.4% S-items) of the illustrations were processed, versus only 8.8% (6.3% P-items and 2.5% S-items) in the DI-condition. This difference between conditions was significant ($X^2(1, 320) = 7.93$, $p = .005$), leading to the conclusion that students' attention was captured more by the representational than by the decorative illustrations. We also looked at the relation between looking at the illustration and giving a RR, but neither in the RI-condition nor in the DI-condition there was a significant relation.

Fifth, students' responses on the questionnaire showed that hesitations about the correctness of the answer occurred significantly more on P-items than on S-items but the amount of hesitation did not differ significantly between the conditions. When looking at the P-items that were solved non-realistically, we see that students tended to hesitate more about the problematic nature of the P-items in the RI-condition than in the other two conditions, suggesting again some impact of the representational illustrations on students' solution processes of the P-items.

Conclusion

We can conclude that students fixated the illustration area only very rarely, even when being confronted with representational illustrations of P-items. So we have strong evidence that in the vast majority of cases, the representational illustration was not helpful because students simply did not look at them. Two additional explanations that may account for a much smaller number of cases wherein the presentation of a representational illustration did not led to a realistic response, are that sometimes students may have looked at the representational illustrations but without noticing the realistic modelling complexity, or that students' beliefs about solving school word problems may have prevented them from giving a RR as their final answer (even though they may have processed the illustration and noticed the realistic modelling complexity).

STUDY 2

Study 2 departed from the main finding of Study 1, namely that students barely look at the provided representational illustrations. This time we manipulated the presentation of the illustrations to maximize the chance that students would actually attend to them and process them. We hypothesised that, when actually processed, the representational illustrations would help students to think more realistically about the P-items and thus to generate more RRs to them.

Method

Hunderd-and-fourty-two higher education students were randomly assigned to three conditions; 1) a DI-condition in which the word problems were presented together with decorative illustrations, 2) a RI-condition with representational illustrations, and 3) a RIW-condition with representational illustrations and an additional warning.

The experiment was conducted in seven sessions with groups of approximately 20 students. The same eight S-items and eight P-items as in Study 1 were presented to each student individually on the computer. In each condition the illustration was presented for five seconds before the problem text to guarantee that students would actually process them. After these five seconds, the illustration was presented again together with the problem text (as in Study 1). Depending on the condition the illustrations were decorative or representational. In the RIW-condition an additional warning was given above each illustration that stated that they can be helpful to solve the problem. Different from Study 1 (in which students had to state their answer orally), students were asked for each word problem to write down their answer and possible additional comments. Afterwards, students received the same paper-and-pencil questionnaire as in Study 1, in which they had to indicate how much they had hesitated for each item, and if so, why.

Results

Students' responses on the items were, as in Study 1, analysed with a logistic regression analysis. First of all, the percentage of RRs on the P-items, across all conditions, was, although higher than in Study 1, still quite low. Only 53.0% of the reactions was considered realistic. There was no significant difference between the three conditions, $X^2(2, 1129) = 0.01$, $p = .995$. There were 53.1% RRs in the DI-condition, 52.7% in the RI-condition and 53.1% in the RIW-condition.

Second, while students hesitated significantly more about their solutions of the P-items than of the S-items, there was no significant difference in amount of hesitation between the three conditions, and also no differences between the three conditions were found for reasons why students had hesitated. Also when looking at the P-items that were solved non-realistically, there was no difference in the reasons why they had hesitated between the three conditions. These results with respect to students' hesitations are generally in line with what we found in Study 1.

Conclusion

From Study 2 we can conclude that forcing students' to look at the representational illustrations of P-items, even in combination with an explicit warning about the usefulness of these illustrations for solving the problems, did not yield a positive effect on the number of RRs on these items.

GENERAL CONCLUSION AND DISCUSSION

Based on these two related studies we first of all can conclude that representational illustrations do not help to solve P-items realistically, above all, because students simply do not look at these illustrations (Study 1). Second, even when students are experimentally forced to actually attend to these illustrations, and even when they are extra motivated to use them in their solution endeavours, still no positive effect on the number of RRs was found (Study 2).

It must be noted that the percentage of RRs of Study 2 was higher than in Study 1 (53.0% versus 27.9%), but the percentage remained quite low and within the range of percentages found in other previous studies with higher education students (Verschaffel, De Corte, & Borghart, 1997). We have no clear explanation for the difference in percentage RRs between both studies. Possibly, the small sample size of Study 1, or slight differences in the test administration may explain the difference in RRs between both studies.

The findings of both studies need more research to yield a more fine-grained and deeper account of why the representational illustrations did not bear the intended effect and what can be done to those illustrations to make them more effective. For example, it has to be investigated whether simple line drawings or more realistic real life photographs would be more effective than the elaborated coloured drawings used in the present studies. Also modifying the representational illustrations so that the problematic nature of the item is more salient would be interesting, because it is possible that in the illustrations that were used the realistic modelling difficulty (e.g., the fact that, in the rope item, the pieces of rope had to be knotted together) was not sufficiently prominent. Finally, the present studies should also be replicated with elementary school children, because it can be argued that those children will be more attracted by illustrations accompanying word problems than educated adults.

These disappointing findings concerning the use of representational illustrations do not allow us to make some conclusions concerning education. To come to a meaningful and relevant conclusion concerning education, and to make recommendations concerning the use of illustrations together with non-standard word problems in textbooks and mathematics classes, we need to wait for the results of our future studies.

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PRE-SERVICE PRIMARY TEACHERS' EMOTIONS: THE MATH-REDEMPTION PHENOMENON

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The relationship with mathematics of future primary teachers is very often built on negative experiences with mathematics as students and characterized by strong negative emotions towards mathematics. This phenomenon is alarming because of its consequences on teachers' development and practice. Nevertheless, many future primary teachers reveal a desire for "redeeming" themselves from negative past experiences in order to become "good mathematics teachers". In this perspective, we conducted a narrative study aimed at deepening the knowledge of this "math-redemption phenomenon", and trying to identify its roots and features.

INTRODUCTION AND THEORETICAL BACKGROUND

In her very famous book "Do you panic about maths?", Laurie Buxton (1981) describes how, for many pre and in service primary teachers, the relationship with mathematics is often built on several negative experiences with school mathematics and it is characterized by negative emotions. In particular, pre-service primary teachers' negative emotions towards mathematics are confirmed by more recent studies (Di Martino & Sabena, 2011). This phenomenon is worrisome because, on the one hand, emotions towards mathematics influence teachers' practice, and therefore strongly affect the quality of students' learning in mathematics (Hodgen & Askew, 2011). On the other hand, they can seriously interfere with pre-service primary teachers becoming good mathematics teachers (Hannula et. al, 2007). Teachers' emotions are a crucial factor that influence also teachers as decision-makers:

> Teacher knowledge is located in 'the lived lives of teachers, in the values, beliefs, and deep convictions enacted in practice, in the social context that encloses such practices, and in the social relationship that enliven the teaching and learning encounter'. These values, beliefs and *emotions* come into play as teachers make decisions, act and reflect on the different purposes, methods and meanings of teaching. (Zembylas, 2005; p. 467, emphasis as in the original)

Nevertheless, there is not much literature about teachers' emotions: whereas research on teachers' beliefs has been extensive and subsumed into almost all areas of research on mathematics education, the study of teachers' affect has not (Philipp, 2007).

Moreover, the emotions evoked by mathematics are largely a product of the lived experience with mathematics as *students* (Brady and Brown, 2005), but:

> limited research, however, was located that examined the relationship between pre-service teacher education students' experiences with formal mathematics instruction, and their future professional practice. Specifically, more needs to be known concerning the manner

> in which past experiences at school may have influenced both attitudes towards the subject as well as confidence in teaching it. (Brady & Brown, ibidem, p. 37)

Within this framework, three years ago we began a study to investigate about primary pre-service teachers' emotions towards mathematics and their links with their past experiences (as math-students) and their future perspectives (i.e. the emotions evoked by the idea of having to teach mathematics in the future). The results of the study, based on an open-ended questionnaire, confirmed the connection between emotions towards mathematics and past experiences as math-students. Moreover they also highlighted that many future teachers, among those who declare strong negative past relationship with mathematics, express the desire to reconstruct a relation with mathematics (Coppola et al., to appear). This desire, that we call the *desire for math-redemption*, appears to be a very promising phenomenon for teacher education: it is the desire to face the "challenge" of teaching mathematics, starting from a personal reconstruction of the relationship with the discipline. As teacher educators, we have the chance of leveraging this desire, in order to break the chain connecting the negative past school experiences with the negative feelings towards mathematics of many primary pre-service teachers. Therefore, we have recently conducted a new study with the aim of deepening the knowledge of the math-redemption phenomenon. The study is guided by the following research question: what are the cognitive and emotional roots of the desire for math-redemption of future primary teachers? We use a narrative approach to trace these roots in future teachers' mathematical stories.

METHODOLOGY

Procedure and population. Our study developed through two phases:

i) The first phase involved a group of 90 future primary teachers, enrolled in the compulsory course on Mathematics and its Teaching of the University degree for primary school teachers of a relatively small Italian public University. In the first lesson of the course, we administrated the open-ended questionnaire developed in the previous study (Coppola et al., to appear). Respondents were asked to answer in anonymous way (choosing a nickname), within 45 minutes. The questionnaire is composed by 12 questions, investigating emotional disposition, beliefs and perceived competence in mathematics. In this paper, we focus primarily on the answers to the questions related to emotional disposition towards mathematics and towards the idea of having to teach it. They are: Q1: "Write 3 emotions you associate to the word *mathematics*" and Q2: "Which emotions do you feel in knowing that you will have to teach mathematics? Why?"

ii) In the second phase, we conducted 11 semi-structured interviews. This phase involved 11 volunteer students: 6 of them had participated to the first phase and, in answering to the questionnaire, had declared some negative emotions towards mathematics. The other 5 students had been enrolled in the course on Mathematics and its Teaching (and they had filled the same questionnaire in their first lesson of the course) two years before, when the course was not compulsory in order to obtain the degree. So, they had chosen to follow the course, despite the declared problematic past relationship with mathematics.

The interviews were based on the explicitation interview method (Vermersch, 1994). This method is based on particular techniques for the formulations of the re-launchings (questions, reformulations, silences) aimed at facilitating and attending the a posteriori *verbalization* (in the sense of putting into words) of a particular experience. In our case, the initial hints were prepared on the basis of the analysis of the open-questionnaire used in the first phase. They regarded: an episode of the mathematical experience that the interviewee considered particularly significant, a mathematics teacher that has influenced (positively or negatively) the personal relationship with mathematics, the eventual turning points in this relationship, the idea of having to teach mathematics. Because of their nature, interviews had not a settled time: in our case, it varied in a range from 20 to 45 minutes. The interviews were audio-recorded and then fully transcribed.

Rationale. The choice of the research instruments is not neutral: the choice of the open-ended questionnaire reflects our conviction that the variety of possible answers coming from open questions is an irreplaceable value for the purpose of our study. According to Cohen et al. (2007, p. 249):

> It is open-ended responses that might contain the 'germs' of information that otherwise might not have been caught in the questionnaire (…) An open-ended question can catch the authenticity, richness, depth of response, honesty and candor which are the hallmarks of qualitative data.

The data gathered by the questionnaire were analyzed through an inductive content analysis (Patton, 2002). In particular, for a first rough classification of emotions into positive/negative emotions, we referred to the theory of cognitive origin of emotions (Ortony et al., 1988), that describes emotions as "valenced reactions" to consequences of events, action of agents, or aspects of objects, and classify the reactions to events in being pleased and displeased, the reactions to agents in approving and disapproving, and those to objects in liking and disliking. These dichotomies permitted a first classification of emotions into positive and negative. On the other hand, we are aware that open questions too have their limitations: they are still one-way, when compared with interviews. Then, in line with Bruner (1990), that describes narrative as a strong means to interpret human actual thoughts, we completed our survey through the use of interviews. In particular, Kaasila (2007) has highlighted the potential of narrative interviews for the study of pre-service teachers' emotions towards mathematics. Regarding the analysis of this kind of narrative data, Lieblich et al. (1998) recognize two main independent dimensions: holistic vs categorical and content vs form. The former refers to the chosen unit of analysis, which can be the narrative as a whole, or specific utterances singled out from the complete narrative; the latter refers to the traditional dichotomy made in literature between the content and the form of a narrative. Our approach is mainly content-categorical oriented, being considered particularly suitable to study a phenomenon common to a group of people (Kaasila, 2007).

RESULTS AND DISCUSSION

The analysis of the answers to the questionnaire confirms that mathematics evokes negative emotions in many primary pre-service teachers. Looking at the answers to Q1 "Write 3 emotions you associate to the word *mathematics*", we find two alarming results: the 28,9% of the participants writes only negative emotions; three over four (the 75,5%) of the participants to our survey uses at least one of the following terms: *fear, anxiety, stress, distress, tension, anger, anguish, affliction, dread, boredom, panic, discouragement, depression, repulsion, revulsion, frustration, unease*. However, answering to the question Q2 ("Which emotions do you feel in knowing that you will have to teach mathematics? Why?"), the students show more positive emotions about their eventual future enterprise of teaching mathematics: the 43,3% declares positive emotions towards this eventuality, compared with 41,1% that declares negative feelings (10% does not provide an answer to this question, and 5,6% replies with mixed - positive/negative - emotions: for example fear and excitement). These data seem to be related to math-redemption, and this impression is confirmed by reading the motivations written by those respondents indicating *negative* emotions towards mathematics, and *positive* emotions towards the idea of having to teach it. For example, Shirly writes: "*Since I am a person more inclined towards humanities, seeing myself in the role of mathematics teacher is very gratifying*"; and Maggiolina: "*I'm convinced that using a good method, I will be able to get my pupils to love mathematics. I can get my redemption*".

In Table 1 we report the percentages of Positive (P), Negative (N), Mixed (M) emotions evoked by the idea of having to teach mathematics in relation with the four groups identified by the answers to the item "Write 3 emotions you associate to the word *mathematics*" (NE_0, NE_1, NE_2, NE_3 indicate respectively the group of respondents that have indicated 0, 1, 2 or 3 negative emotions):

	P	**N**	**M**	**No reply**
NE_0	63,2%	15,8%	10,5%	10,5%
NE_1	53,6%	42,9%	3,5%	0%
NE_2	36,4%	40,9%	13,6%	9,1%
NE_3	11,1%	66,7%	16,7%	5,5%

Table 1: cross-analysis of answers to Q1 and Q2.

The quantitative cross-analysis summarized in Table 1 shows that a subgroup with positive emotions towards the idea of having to teach mathematics is present in all the NE groups. On the other hand, the different consistence of these subgroups within the whole groups seems to indicate that the strength for pursuing a math-redemption decreases dramatically when the emotions towards mathematics are too negative.

The narrative data, gained through the interviews, provide many cues to understand the differences between emotions towards mathematics and towards the idea of having to teach it, highlighting the math-redemption phenomenon. Content-categorical analysis of these data allows us to identify some features of the phenomenon. All the 11 respondents speak about their serious difficulties in the relationship with mathematics, and they identify a clear turning point in school-experience, a real crisis' moment:

> Angela: During grade 12 I wanted to change school (…) It was a real crisis of rejection, during grade 12. Now I remember! Exactly a crisis of rejection (…) because I was not able to sustain the charge, especially for what concerns maths (…) And then I said to myself "I finish this year, and after that I don't want have nothing to do with math".

In these turning points, the role of math-teachers is always recognized as crucial. Almost all the narratives of the students describe one or more school episodes featuring a math teacher that is disrespectful of the students' needs, sometimes a teacher with whom it is impossible for the students to establish any relationship:

> MariaTeresa: During grade 10 there was a change, I have had a teacher with which really, I was not able to built a relationship (…) and in that moment I have had…in other words like I was done, I was over mathematics.

> Doriana: During high school I had an old school teacher, detached in the relationship (…) he used to write and write entire blackboards with numbers, and, when he arrived at the end, he used to delete the signs and start again.

This poor consideration for students is recognised as particularly problematic for those who have difficulties in mathematics:

> Piurla: They explain a topic, a theorem, something. If you understood: good! If you did not understand something, it was you that did not understand! They did not use to face the question "why did he/she not understand?" or "perhaps I could try to explain that in a different way", No! That was 'The way'!

On the one hand, the firm awareness of this negative influence of the math teacher on their relationship with math elicits in the pre-service teachers the fear to do the same errors. So, fear becomes the emotion with which they approach the teaching of math:

> Margot: [speaking about her first experience as math substitute teacher] I was afraid of not being able in teaching math. That is, I was afraid to make the same errors that my math teachers made with me.

On the other hand, however, the same awareness is one of the main motivations for trying the reconstruction of the relation with math, the germ of the math-redemption:

> Margot: The incentive to restart with mathematics has been the motivation to be a good teacher.

> Iperurania: It would be great if I had to teach math! Just for my past experience I would do something more. If I could teach math, I'd wake up in the morning with a lot of energy, because I want to transmit what has not been given to me, I don't want that my students think of mathematics as I thought of it!

From the interviews, it emerges that this motivation is shared among all the respondents: they show a strong desire to become a better teacher than their own teachers, and to spare their future students from *math-pain*. But, in some cases (2 over

11 of our respondents), a *feeling of uncontrollability* prevails: it seems impossible to imagine overcoming the difficulties *with* mathematics related to strong negative emotions towards mathematics, and to inadequate math knowledge. This feeling of uncontrollability affects the self-perception as mathematics teachers, and determines negative emotions also towards the idea of having to teach mathematics; hence it appears to preclude any possibilities for math-redemption:

La mente contorta: I associate non-positive emotions towards the idea of having to teach mathematics (…) just because I have had bad experiences during high school education, I don't know how cover my blanks!

Tania Bolena: I hope that I will never have to teach math. Sincerely, I don't like mathematics and then I don't know how I could spread passion for maths to my pupils. I already know that I could ruin them!

Vice versa, for all the other 9 cases, the *feeling of controllability* is the key-element for the math-redemption. These respondents express the conviction that, in order to become a good teacher, the reconstruction of the personal relation with mathematics is needed: it will be a hard challenge, but they will be able to win this challenge. This aspect appears particularly explicit in the words of the 6 students that have chosen the mathematics course when it was not compulsory:

Margot: During my first year at university, I could choose between physics and mathematics course. I had no problem with physics, it would have been the easiest way, but I thought "No, it is the time for facing with math, for understanding whether I'm able to get closer to mathematics or if me and maths are on separate rails".

Angela: It was a challenge that I wanted to do! I chose that one, despite my difficulties with maths (…) Thinking to teach mathematics troubles me to a certain extent, but now I am quiet because I have a different approach: before I used to think "no, I am not able" and I rejected to find a solution to the problems, now I gear up and I try to understand how to find a solution, because I believe that difficulties can be overcome.

Among the 9 narratives reporting a travel towards the math-redemption, it is possible to recognize some further interesting common features. All the narratives are full of emotional charge: *pain* is the label usually used to describe the experience as math students. This pain is also related to the awareness that the relationship with math has strongly influenced important choices in the life, sometimes even impeding the pursuing of some personal ambitions:

Margot: I gave up entering college of architecture because of math: what a pity!

From the narratives, it clearly emerges how school experience of the respondents, i.e. their past as math students, influences the process of redemption in terms of:

i) *motivations*: the desire of math-redemption is often linked with the will to take a sort of 'personal revenge' on teachers.

Angela: Surely I'm going against the image that my primary teachers had of me: yes, this is a little revenge!

Mathe: At the end of the middle school, despite the fact that I got the maximum mark, my teacher said to me "but I don't suggest you to carry on with mathematics": he should not have said to me this! Now, maybe, I will become a *mathematician*, and he will not know that!

ii) *emotions*: the pain experienced in the past makes the ongoing math-redemption process full of positive emotions.

Marika: Despite my past rough relationship with mathematics, now I succeed in having this cohesion with math: it is an incredible satisfaction!

iii) *possibility to become a good teacher*, taking care of students' difficulties and being able to understand them, since they personally experienced those difficulties.

MariaTeresa: it is possible to learn a lot of things: above all, from our past negative experiences.

Mathe: It is just because I felt this hostility towards mathematics as a student that I believe to have those motivations and also that experience useful to understand where pupils could run into problems, or feel hostility.

CONCLUSIONS

The research carried out for several years has confirmed the worrisome spread, among future primary teachers, of strong negative emotions towards mathematics. Nevertheless, it has also highlighted a very interesting phenomenon: what we have called the *desire for math-redemption*. Through the analysis of the narratives, we have outlined some typical features of this meaningful phenomenon: the pain during the experience at school; the key-role of a teacher that constitutes a sort of negative model (the narrator recognizes in this negative model what he/she does not want to become); the need of glimpsing the possibility that the reconstruction of the relationship with mathematics succeeds; and, finally, the deep (positive) consequences on the emotional disposition and on the self-perception of glimpsing this possibility. As teacher educators, we cannot ignore these features, in order to appeal to the desire for a math-redemption and to support future primary teachers in the challenge of reconstructing their relationship with mathematics. Our role as teacher educators is fundamental in the whole path of a math-redemption: first, in spurring future teachers to consider the idea of taking on this challenge, in creating propitious conditions for the overcoming of feelings of uncontrollability, and in persuading the future teacher that s/he can reconstruct a positive relationship with mathematics. The support of teachers' educators is crucial also throughout the math-redemption process. In fact, being a "recovery" process, math-redemption is emotionally hard, as expressed by Chicca: "*the process of recovering my relationship with mathematics has been very hard from an emotionally point of view*". The collected narratives show that, because of negative past experiences with mathematics, who decides to face this challenge proceeds with caution. These future teachers, above all at the beginning of this path, feel insecure and they need some help. With regard to this aspect, it is very significant the metaphor used by Margot: "*During the university degree, when I decided to get back into the game with math, I approached it very cautiously, just like children that are learning to walk*".

Staying in the metaphor, as teachers' educators we have to motivate future teachers to take the first steps, as well as to encourage them after the unavoidable falls, guiding them with the hand, in order to make future teachers as confident as to decide to leave the hand and to walk alone.

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REFLECTIVE PORTFOLIO OF MATHEMATICS AS A TOOL FOR REGULATING ASSESSMENT IN THE LEARNING OF MATHS STUDENTS OF HIGH SCHOOL

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In this paper, we present some results related to the learning of math's of a high school student, when involved in drafting a reflexive portfolio of Math's. The results obtained can support the affirmation that in the drafting of the portfolio the student has engaged in guidance and organization processes to choose and prepare a first approach of entry. It gave rise to setting up analogies and to the process of anchoring the aspects related to the chosen task. The translation, justification, interiorization, verification and registration processes were mandatory requirements that were improving with each version in each entry. This is a way to assess which favours an active, conscious and self-regulated participation from the student in their learning, with the treatment of mathematical' content and processes comprehensively, and in the improvement of the mathematical communication.

INTRODUCTION

The labour requirements imposed by society and others advocate an education directed to the development of competences and a strong regulating assessment (Santos, 2004), and mathematics, for social, cultural, formative and political reasons, is considered a central discipline in the achievement of that goals. The expression *regulating assessment* holds the view that, during the process of teaching and learning, assessment takes into account the implementation of learning tasks without time constraints, during which students have the opportunity to create, think/reflect on what they created, redo and so on until the final presentation of the product; and that the teacher monitors and guides the entire evolutionary process of the students (Santos *et al*., 2010). The assessment is thus to be understood as a communication process (Pinto & Santos, 2006), integrated in the own learning process and incorporated into the day-to-day school activities (Perrenoud, 2004; Santos, 2005, 2008; William, 2009). For authors such as Bryant & Timmins (2002), the use of assessment reflexive portfolios are an example of how it is possible, in Mathematics, implement this form of assessment. It is in this context that aims to understand the extent to which the reflexive portfolio, developed all along one school year, can contribute to learning mathematics in a secondary school student. In particular, it seeks to study the processes that develops in the realization of the reflexive portfolio contributing to learning math.

THEORETICAL FRAMEWORK

It was considered that the student learning activity, which depends on the needs that he/she feels, the reason that drives him/her to participate and even the results that he/she expects to achieve by going through this process, it's developed in a activity system (Engeström, 1999), where action of the subject on the object is mediated by mediators artefacts, from which stands out the portfolio and speech (written) produced in dialogic interactions themselves (teacher-student) or imaginary (with internalized subjects), and that makes a significant transformation of the object in the expected results. In this transformation process is presented the concept of Zone of Proximal Development (ZPD) (Vygotsky, 1934), which is here understood as a symbolic space of interaction and communication, used as the main mediator of cognitive activity, a process that is intended to be of increasing self-regulation. The self-regulation in learning is a multidimensional construct that, according to Zimmerman (2000), includes the metacognitive, motivational and behavioural components. It is considered that a student is self-regulated, in that he/she is active in all these three dimensions.

The mathematical thinking is intrinsically linked to the processes that give rise to mathematical knowledge. The cases presented in the literature are diverse and there isn't an "fixed" order or predetermined between them since they are often interrelated and cyclically arise when working in a mathematical situation, furthermore, a series of processes can also be considered a process (Frobisher, 1994). However, it can be stated that there are processes which relate more closely with mathematics as the testing process (Pirie, 1987; Holding, 1991; Frobisher, 1994; Burton, 1984); there are others that are more independent of the mathematical content but that apply when these are solved problems in mathematics as is the process of reflection (Pirie, 1987) or the communication processes (Frobisher, 1994). Pugalee (2004) defines four groups of metacognitive processes used in problem solving: orientation, organization, execution and verification. With more focus on the first group, you can find the processes of interpretation established by Dias (2005). The author establishes a second group of cases - the development that has points in common with other cases brought by Pugalee. These and other processes, given that they involve higher skills, should be encouraged and developed through explicit instruction, for a sufficiently broad, so that the student use the pass to consciously and judiciously (Mason, Burton & Stacy , 1982) and can be studied through communication in written form when solving problems in Mathematics.

METHODOLOGY

This investigation has followed an interpretative paradigm (Yin, 2002), with recourse to case study (Burns, 2000). Participants were three students from a High School from the county of Lisbon but this paper reports one else. Selection criteria used were availability and willingness to participate in the study, the ease of speaking and as differentiating factor different levels of performance in the discipline. Francisca (student with 16 years old) was considered a student with a good performance in math.

An entry portfolio consists of all versions of the resolution of a task chosen by the student and a final reflection, balance of all the work done in this entry. At the end of each semester, at these entries, joins a final reflection on the development of the portfolio. Delivery dates of the first and second versions of an entry and feedback on the first version, are defined prior to the start of the execution of the portfolio, as well as the subject being treated in each of the six entries and covering the main contents provided in the 11th grade. The delivery of the remaining versions is agreed individually, according to the aspects to be improved and the availability of each of the players (student and teacher). The resolution of the problematic situation chosen by the student must be accompanied by a written explanation, the reasoning applied and the reasons for their decisions. It is on these records that the teacher, through their feedback, seeks to provide structures that guide the student to correct their mistakes and overcome their difficulties.

Data were collected through two semi-structured interviews, one at the end of the third entry, the other at the end of the sixth and final entry portfolio and gathering documentary which brings out all versions of the entries made by the student to the portfolio and the compilation of all emails exchanged between teacher and student related to the versions of entries of the portfolio and respective feedbacks.

The data analysis followed the categorization of metacognitive processes of Pugalee (2004). In enforcement proceedings, observable through local actions (eg calculations), monitoring of progress and change-making, are associated with the reflection processes (Pirie, 1987) and self-monitoring (Zimmerman, 2000). In the interpretation process group, identified by Dias (2005), translate, initial experience, internalize and anchor are considered. As purpose of this latter process, we considered the process of zigzag (Dias, 2005), the establishment of analogies (Holding, 1991), and the process of selecting a strategy (Pirie, 1987). Given the specificities of the assessment instrument under study, the justification process (Mason, Burton & Stacey, 1982; Burton 1984; Pirie 1987; Pugalee, 2004; Holding, 1991) that must accompany any execution of the portfolio, was also considered.

RESULTS

The fact that was Francisca that select the tasks for the entries, made her to engage in processes of orientation, familiarizing yourself with the mathematical topic and analyzing information. Francisca was beyond this part because, on its own initiative, within the mathematical theme of each entry, gathered all the information she deemed to be related to the chosen task, putting it by topic, both in terms of content (formulas, definitions, theorems), but also in terms of process (type of reasoning, base procedures), as illustrated in the following excerpt from the beginning of the second entry:

The exercise itself encompasses many points of theme, interconnected: trigonometry, scalar product, geometry and as such requires a combination of all and focuses primarily on the need to employ concepts such as:

- Reduced equation of a straight line - $y = mx + b$ To reach this expression requires a point belonging to that line and a vector of the same director (or any information that will give us the slope). First obtains the slope (m) from the vector director or from any information given and subsequently learns the value of the ordinate at the origin (b) by substituting the expression of the unknowns x and y, the x and y coordinates of a point belonging this line.

- Concept of scalar product – The scalar product it is an operation which is performed between two vectors, obtaining a numerical value of this operation. One of the expressions of calculating the dot product between the vectors $\vec{u}$ and $\vec{v}$ is: $\vec{u}\vec{v} = \|u\|\|v\| \cos(\vec{u}\,\vec{v})$. However, it is relevant to take into consideration to perform this exercise is that the scalar product between two vectors perpendicular to one another is always zero, whereas $\cos 90° = 0$.
- Equation of a circumference – The equation of a circumference (the concept of 10th grade) is constructed from the point that contains the coordinates that belong to the origin of the circumference - a point C of coordinates C (x, y) - and the respective radius (r). The expression is given by $(x - x_c)^2 + (y - y_c)^2 = r^2$.
- Trigonometric ratios – For these exercises is also necessary to take into account the three trigonometric ratios given by: (...) (2nd entry, 1st version, 05/02/2011)

Naturally, in this preliminary work, Francisca found herself also engaged in organization processes, where she presented the information that she disposal in order to have a common thread and sometimes resorted to some schemes or other graphical representations, some of which were built by her. However, in this part, all mathematical themes were treated in a not integrate way. Only then, Francisca proceed to the treatment of specific problematic situation that was chosen for entry. In this treatment, Francisca was seeking what she had written in the preliminary approach, seeking to establish analogies and/or relating the "new situation" to what she already knew and had written in a anchoring process. It was also in this way that the student chose an initial strategy for the resolution of the task. When the strategy did not produce the desired effect, she turned back and tried another. In fact, it was notorious, throughout the execution of the portfolio, a self-monitoring progress and changing decision by the student. Only when she hadn't more ideas that she sent the first version, but leaving the failed attempts recorded and its justifications. In each version, the delivery was accompanied by a current status that the student has developed by a verification process, which not only became the finding discrepancies between the idealized and achieved, but also the analysis of what was her own action or the reasons for obtaining the response clogged, as can exemplify the following extract on the first entry, in which the student bent over a demonstration of a trigonometric equality:

In my first attempt to solve it, I focused simultaneously in both members, in the first member I solved that the notable case of existing denominator of the fraction (A) and in the second member, based in the relation between the tangent and the other two trigonometric ratios, I replaced the tangent by this ratio, with the intention of facilitating the resolution (B). (...) From here I stopped knowing how to continue. I even thought about cutting $cos^2\alpha$ from the numerator of the first member, with $cos^2\alpha$ from the denominator of the same member and do the same between sins, however I needed that they were a multiplication and in this case it was a subtraction. As such, I decided to pick up a member separately and try to reach the other, and this attempt is based my second resolution [presenting then the second attempt of resolution accompanying with the same kind of explanation]. (...) From here, I don't know how to continue. (1st entry, 1st version, 08/01/2011)

Throughout the work of the portfolio, it was necessary to improve and fix aspects of mathematical communication and reasoning. In fact, the registration processes that the portfolio forced allowed the teacher and the student to realized aspects that, by the way that were originally written, revealed that concepts and/or reasoning and the relationship between them weren't very well understood, nor well structured or related. Is in this record that the student, in a process of translation, makes evident the degree of understanding of the situation and the clarification of ideas and allows the teacher to give appropriate feedback towards the correction or improvement in mathematics learning:

I knew that point B has as reduced equation of the line, the equation constructed in the preceding question and was contained in the circumference equation, then sufficed to do a system of equations with two unknowns to determine x and y (…) $\cos x = \frac{[OR]}{\text{hipotenusa}}$. (2nd entry, 1st version, 05/02/11)

A point is not contained in equation.. (…) $\cos x = \frac{[OR]}{\text{hipotenusa}}$ (Instead of the thread token, you should use the symbol of the length). (Feedback given at the 1st version of 2nd entry, 11/02/11)

It was in the remake of her written works that there was verified a zigzag that allowed her to internalize all aspects involved in each choice. In this context, the student also addressed the meaning of the concepts and results in a internalization process.

Throughout the portfolio, Francisca looked into their personal productions and then compare them with the previous ones, to draft a new version. Doing so led her to reflect and thus to study more focused on understanding instead of memorization:

This trend of decorating steps to follow wouldn't let me open enough to realize that the areas cannot be negative!! Usually I face problems of this kind because as I have already structured the idea I never think beyond it, I consider it even a big obstacle for me in solving math problems, because I never remember having to do anything different or get something else that is not immediately present in my head, because of the different circumstances that exercise can have. (5th entry, 3rd version, 14/05/2011)

Also the justification process, in the cross-holding of the portfolio, as a mandatory requirement, improved and become increasingly precise and explicit. In fact, the student initially held in written productions often unnecessarily long and very poorly structured. These are becoming increasingly more precise and explicit: "Yes, that was what I was saying, but had not thought in $\frac{1}{3}$, only integer numbers" (6th entry 2nd version, 31/05/2011).

All processes identified above arising from the preparation of the portfolio were worked explicitly and continued over a period sufficiently extended, changing some behavioural habits of the student, who usually left the study to focus on the eve of summative assessments:

Due to the existence of these entries, I was always being aware of the mathematical subjects worked at class (Year-End Reflection, 06/06/11).

I didn't realize how the asymptotes work. I did the portfolio and the mathematical subjects entered. Then when I went to study [for the test] and I have just only to revisit it, that's good because it isn't all new. It's not that idea "oh now I have test tomorrow, I have test in two days and I don't understand anything of this." And in that sense it is very good. (2nd interview, 06/06/2011)

FINAL CONSIDERATIONS

In her action, Francisca acted on the problematic situations chosen for the entries about concepts, desires, as one of the mediators artifacts (Cole & Engeström, 1993) the portfolio, which constituted a facilitator of the learning activity in order to give the student power in transforming these objects in outcomes such as new mathematical cognitive and metacognitive learning. The language established in the context of the ZPD and therefore here understood as a process of production and negotiation of meaning (Roth, 2004), was another major mediators element (Vygotsky, 1978). In the portfolio, writing, requiring an inner speech and a deliberate and intentional structuring of a network of meanings (Vygotsky, 1934), was seen as an way for reflection and an awareness of mathematical and metacognitive processes for self-regulation essential for the individual to become independent and to learn to work strategically (Pugalee, 2004) and become increasingly active in the dimension of metacognitive self-regulation. The behavioral dimension of self-regulation was also affected positively because it was modified work habits, and the study left to be concentrated on the eve of summative assessments. The portfolio is an effective way to exercise a differentiated pedagogy, and this was noticed by the student, with positive effects on the motivational dimension of self-regulation.

Any of the processes identified, for more simple, at the outset, appear to be, only are effectively learned and reusable by the student without outside help, if they become the subject of an explicit work, regular, in a sufficiently extended period of time, and headed for a awareness of the activity performed (Mason, Burton & Stacey, 1982; Burton, 1984; Schoenfeld, 1992; Frobisher, 1994). From the foregoing, it is clear that the portfolio was a way to give to Francisca, and individually, this way of working, from entry to entry, from release to release, with the rereading, the remake, the reorganization and improvement ideas awareness about her own work. Note that even in situations where the student thought she had no doubt there was an opportunity for corrections and/or improvement of written communication of the mathematical reasoning and even the mathematical content.

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FOSTERING THE TRANSITION FROM ADDITIVE TO MULTIPLICATIVE THINKING

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In this paper we report on results of one aspect of a study that investigated students' development of multiplicative thinking. The focus here is on factors that influenced students' strategy choice. In a one-to-one interview students were given opportunity to choose the task level of difficulty. The findings suggest that there is a relationship between strategy choice and task level of difficulty: the more difficult the task chosen the more sophisticated the strategy choice, the less difficult the task, the less sophisticated the strategy choice.

INTRODUCTION

One of the goals of teachers of mathematics is to move students from naive to sophisticated strategies for solving tasks that involve multiplicative thinking. From the ways that many texts are constructed, it might be assumed that the authors' approach is to start with simple exercises and make them progressively more complex, with the intention that completion of the easier exercises provides the information needed for the harder ones. This paper presents evidence which challenges this approach. Specifically we found that, given an option, many students will choose a challenging task over a less challenging one. Further, having chosen to work on a complex task they will use a more sophisticated solution strategy than they would have on a simpler one. The paper explains the importance of multiplicative thinking, indicates that there are competing views on task difficulty in the literature, and describes the research and results that led to our conclusion. There are implications for resource design and teacher planning.

THEORETICAL FRAMEWORK

A consistent theme in the literature is that multiplicative thinking is the foundation for higher-level mathematics. Multiplicative reasoning is the basis of proportionality, and a necessary pre-requisite for understanding algebra, ratio and rate, interpreting statistical and probability situations, and understanding and reading scale (e.g., Callingham, 2003; Siemon, Breed, & Virgona, 2005). The development of ratio and proportion concepts is embedded within the development of the multiplicative conceptual fields (Greer, 1988; Vergnaud, 1988).

There are specific studies that argue that the difficulties associated with students' lack of proportional reasoning are directly related to their limited experience with the different multiplicative situations (e.g., Greer, 1988; Vergnaud, 1988). Others attribute the difficulties to students' reliance on additive reasoning when multiplicative reasoning is required (e.g., Lamon, 1993; Singh, 2000).

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The development of multiplicative thinking is more conceptually demanding than additive thinking and takes time to develop (Clark & Kamii, 1996; Steffe, 1994). Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, et al., (2002) found that 51% of Grade 2 students were unable to abstract (simultaneously coordinate two composite units mentally, without the use of perceptual models), when solving multiplication tasks. De Corte and Verschaffel (1996) found that many students remain dependent on modelling beyond the junior primary years. Steffe (1994) describes the demands of multiplicative thinking in the following way:

> For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit. (p. 19)

In order to do this requires a level of abstraction and inclusive relationships that are not required in additive thinking (Clark & Kamii, 1996). Singh (2000) found that when students move from additive thinking to multiplicative reasoning with whole numbers, two important changes occur, the first being a shift from "operating with singleton units to coordinating composite units" (p. 273), and the second a change in the meaning given to a number.

The key issue for teachers is whether it is possible to promote the move from additive to multiplicative thinking in students and how this might be done. On one hand, Sullivan, Clarke, Cheeseman, and Mulligan (2001) suggested that abstracting, characterised by students moving beyond the need to create physical models, to forming mental images to find solutions, is a key stage in the learning of multiplicative concepts. They also argued that one of the reasons why students do not make the transition from a reliance on models to abstraction is teachers' reluctance to engage students in problems that gradually remove physical prompts and encourage students to form mental images of multiplicative situations. Similarly, Greer (1988) suggested incorporating more complex number combinations routinely in word problems so that the appropriate operation cannot be intuitively grasped is one way to overcome a reliance on additive thinking. Greer also suggested the need to provide multi-step word problems, rather than single operation word problems, to push students to think more deeply about which operations to use and to move beyond superficial strategies.

On the other hand, Sherin and Fuson (2005) found that students reverted back to repeated addition, or counting based strategies to solve problems involving large number triples (combination of three numbers, two of which multiply together to give the third). Mulligan and Mitchelmore (1997) indicated that students used direct count strategies for large numbers, as they seemed to experience a "processing overload" (p. 322) when attempted to use the same strategy for larger number triples. A close examination of the number triples used in some of the aforementioned studies indicated that those that focused on Grade 1 to Grade 3 were limited to small numbers such as (3, 4, 12), (3, 5, 15), (3, 6, 18), (3, 8, 24), (4, 5, 20), (4, 6, 24), and (5, 6, 30) (e.g., Anghileri, 1989; Clarke et al. 2002; Kouba, 1989; Mulligan & Mitchelmore, 1997). Studies of Grades 4 to 6 students' solution strategies involved more complex number triples, such as (5, 8, 40), (5, 19, 95), (13, 7, 91), (23, 4, 92) (e.g., Heirdsfield, Cooper, Mulligan, & Irons, 1999). The particular number triples used are central to the argument in this paper. In the data below, number triples like (8, 6, 48; 6, 9, 54) are used in tasks described as medium difficulty, and triples like (7, 8, 56; 16, 8, 128) are used in tasks at the challenge level. Noting that the interviews were near the commencement of the Grade 3 year, even the medium difficulty triples are more complex than those used in most other studies at this level.

METHODOLOGY

This paper draws on one of the findings of a larger study of young children's development of multiplicative thinking. The study involved Grade 3 students (aged eight and nine years) in a primary school located in suburban Melbourne. In terms of mathematical achievement, the spread of students overall is similar to statewide results on the systemic assessment. All the students in the grade were interviewed prior to the commencement of the study using the counting, addition & subtraction strategies, and multiplication & division strategies domains of the Early Numeracy Interview (Clarke et al., 2002). The resultant "growth point" data were used to identify 13 students in the class, four at either end of the scale and five in the middle, to participate in the study. This number of students provided a reasonable cross section of the grade of 27. A one-to-one, task-based interview was administered with the students to gain insights into their understanding of and approaches to multiplicative problems. The findings of a subset of these results are reported in this paper.

The first author developed a multiplication task-based interview which consisted of 15 tasks in the form of word problems across five semantic structures identified by Anghileri (1989) and Greer (1992): three Equal Groups tasks; four Allocation/Rate tasks; four Rectangular Array tasks; three Times-as-Many tasks; and one Cartesian Product task (a decision made following the trialling of the tasks). The Allocation/Rate tasks included two two-step tasks and two one-step tasks to gain a better sense of a student's strategy choice. Each task consisted of three levels of difficulty (easy, medium, challenge). In some instances an extra challenge question was offered if the student appeared to find the challenge task relatively easy. The number triples were deliberately chosen with some repetition both within and across levels of difficulty. This was to ensure that students who always chose a particular level of difficulty had one or two questions that might challenge them. For example, the number triple 8, 6, 48 occurs in an Allocation/ Rate task at the medium level of difficulty and in an Equal Groups task at the challenge level.

Each interview was audio taped and took approximately 30 to 45 minutes, depending on the complexity of students' explanations. The problems were presented orally and students were encouraged to work out the answers mentally. However, paper and pencils were available for students to use at any time. Generous wait time was allowed and the researcher asked the students to explain their thinking and whether they thought they could work the problem out a quicker way. Once a response was given the student was asked to explain his/her thinking and record a number sentence on paper. Responses were recorded and any written responses retained. Students had the option of choosing the level of difficulty to allow them to have some control and to feel at ease during the interview. If a student chose a challenge problem and found it too difficult, there was an option to choose an easier problem.

Method of Analysis

While acknowledging that providing students with a choice contributed to the richness of the findings, it also added to the level of complexity both in the analysis and presentation of data. The data were coded for two purposes, first to ascertain student performance and second to identify student approaches to multiplication tasks. As the researcher was interested in knowing both the approaches students used and components of the task that may influence their strategy choice, an extensive analysis was undertaken of each of these components (e.g., semantic structure, level of difficulty, number triples). The results of the analysis presented in this paper pertain only to the challenge and extra challenge levels of difficulty.

The students' strategies were coded according to the level of abstractness and degree of sophistication, drawing on the categories of earlier studies (Heirdsfield et al., 1999; Kouba, 1989; Mulligan & Mitchelmore, 1997; Sherin & Fuson, 2005). Abstractness used in this context refers to an ability to imagine the individual items as a composite unit and to solve a problem mentally without the use of physical objects (including fingers), drawings or tally marks. The strategies chosen by the students were categorised in the following way. The first category, Building Up, is additive:

> *Building Up*: Visualises the groups and the multiplication fact but relies on skip counting, or a combination of skip counting and doubling to calculate an answer.

The other three categories are considered to be multiplicative:

> *Doubling/Halving:* Derives solution using doubling or halving and estimation, attending to both the multiplier and multiplicand.
>
> *Multiplicative Calculation:* Automatically recalls known multiplication facts, or derives easily known multiplication facts.
>
> *Holistic Thinking:* Treats the numbers as wholes—partitions numbers using distributive property, chunking, and/or use of estimation.

RESULTS AND DISCUSSION

From the analysis of data two findings were evident. First, many more students than we anticipated chose the challenge level of difficulty tasks rather than the medium or easy levels, and they responded correctly. Second, multiplicative strategies were the preferred strategy of choice to solve the problems at the challenge level of difficulty.

Table 1 presents the frequency of tasks responses for the challenge and extra challenge levels of difficulty for each semantic structure. Due to the complexity of the Times-as-Many and Cartesian Product semantic structures no extra challenge tasks were offered. From the data presented in Table 1 it is evident that substantial numbers of students chose the challenging level tasks.

	Equal Groups	Allocation/ Rate	Rectangular Array	Times-as-Many	Cartesian Product
Number of Possible Responses	39	52	52	39	13
Challenge Tasks	23	27	34	22	1
Extra Challenge Tasks	9	5	2	Not applicable	Not applicable
Total challenge /extra tasks	32	32	36	22	1

Table 1: Frequency of Task Level of Difficulty Responses for Each Semantic Structure

Of the 195 responses of the 13 students across the fifteen tasks, 123 were for the challenge and extra challenge levels of difficulty, 66 responses were for the medium level and 6 were for easy level of difficulty. It is worth noting that some of these students who chose the challenging tasks and used sophisticated strategies were not higher performing students on the pre-test interviews. Also of note, it is likely that in a one-on-one interview with an unfamiliar adult the tendency for students would be to opt for correctness.

	Equal Groups	Allocation/Rate	Rectangular Array	Times-as-Many	Cartesian Product
Additive	13	4	9	1	0
Multiplicative	19	28	27	21	1

Table 2: Frequency of Strategy Choice for Each Semantic Structure

Three findings are apparent from the data presented in Table 2. First, multiplicative strategies (doubling/halving, multiplicative calculation, holistic thinking) accounted for 78% (96 out of 123 responses). This suggests that students who consistently use these strategies are thinking multiplicatively rather than additively. It seems that offering students the opportunity to engage with number triples beyond what is commonly posed at this level prompts the use of multiplicative solution strategies.

Second, there was less contrast in the distribution of strategies for the Equal Groups tasks. Given students' familiarity with the Equal Groups semantic structure it was expected that more students would use multiplicative strategies especially as five of these students consistently chose these strategies for all other tasks. However, it could be argued that the number triple for the first Equal Groups challenge task (7, 8, 56) was familiar to these students and so they chose a less sophisticated strategy such as Building Up. Interestingly, four of these five students used multiplicative strategies for the Equal Groups extra challenge task 2 (8, 14, 112) for which the use of a skip counting strategy would have been time consuming and inefficient. The following abridged excerpts from the interviews illustrate the students' use of multiplicative thinking for the Equal Groups extra challenge task.

Equal Groups (task 2): On the table are 8 boxes of crayons with 14 crayons in each box. How many crayons are there altogether?

Annie: I know 7 eights are 56 so I just doubled it to get 14 eights and that's 112.

Mark: I know 12 eights are 96 and two more eights is 16, so 96 and 16 is 14 eights that's 112.

Sandy: I know 8 tens is 80 and 4 eights is 32. I then added 80 and 32 to get 112, so 8 times 14 is 112.

Both Annie and Mark used a Multiplicative Calculation strategy derived from known facts such as 7×8 then doubled it, or 12×8 and then added on another 2 eights. Sandy used Holistic Thinking as she partitioned the 14 into 10 and 4 and separately multiplied these by 8, reflecting her place value understanding and use of the distributive property. These responses indicate the students' awareness of the multiplicative nature of the task and the relationship between the numbers.

Third, Holistic Thinking was the preferred strategy for the Times-as-Many tasks (77% or 17 out of 22). Given that this aspect of multiplication is quite different from the other structures and that the number triples were more challenging, one might have expected students to choose a less sophisticated strategy. One could infer from this that the complexity of the semantic structure and the size of the number triples facilitated this level of thinking. Further evidence to support this was the use of Holistic Thinking on the challenge Times-as-Many task 12 (4, 18, 72) by two of the lower performing students (Marty and Lewis) on the pre-test. In the few other instances where they chose challenging level tasks, a Multiplicative Calculation or Holistic Thinking strategy was chosen, whereas Building Up was the preferred strategy chosen for the medium level of difficulty. Their use of a multiplicative strategy indicates an ability to use multiplicative rather than additive thinking when presented with a task involving number triples outside the factor structure implied by the curriculum at this level. The following responses for the Times-as-Many characterised the type of multiplicative thinking students used for the challenge tasks.

Times-as-Many (task 12): Jamie collected 18 stamps. Jack collected 4 times as many. How many stamps does Jack have?

Mark: I know 12 fours is 48 and 6 fours is 24, then I added 48 and 24 to get 72 and that's 4 times as many as Jamie.

Marty: Ten, 4 times is 40 and eight 4 times is um 32. 40 and 32 is 72.

Lewis: Four times as many as 18? 20, oh umm, so 4 times? 80, take away umm 8, is umm 72. I took away 10 first and added 2 onto 70, cause it's easier.

Both Mark and Marty partitioned the 18. Mark partitioned it into two known facts and added the partial products, whereas Marty partitioned it into ten and eight and operated on each separately, reflecting his place value understanding. Both students showed an understanding of the distributive property. Lewis rounded the 18 to 20, a number that he could calculate mentally and then compensated by subtracting eight.

These three findings highlight the value of allowing students to choose the task level of difficulty and uncover students' untapped mathematical capabilities. These findings together suggest that by enabling students to engage with complex tasks prompts the use of more sophisticated strategies than may normally be the case.

CONCLUDING REMARKS

The findings of this study suggest that giving students opportunities to experience complex number combinations and semantic structures such as Times-as-Many, that require them to think more deeply, will encourage them to move beyond the need for models or a reliance on additive thinking to multiplicative thinking. This supports the recommendation by Greer (1988) that incorporating more complex number combinations routinely in word problems so that the appropriate operation cannot be intuitively grasped can encourage students to move beyond a reliance on additive thinking.

The findings also indicate that not only can Grade 3 students engage with tasks across less familiar semantic structures such as Allocation/Rate and Times-as-Many, but do so using more sophisticated strategies that one might expect. As suggested by Greer (1988) providing multi-step word problems and less familiar situations push students to think more deeply about which operations to use and move beyond superficial strategies.

The implications of these findings for mathematics instruction include engaging students in word problems that incorporate number combinations which cannot be intuitively manipulated using additive thinking. It appears that by doing so can prompt the use of sophisticated strategies. This has implications for both teacher educators and authors of teacher texts and other resources.

Second, engaging students in multiplicative word problems across a range of semantic structures may support their developing understanding of multiplication and their transition from additive to multiplicative thinking. Engaging students in a range of semantic structures also develops a deeper understanding of the nature of multiplication. This also indicates the importance for teachers of students as young as Grade 3 not to delay the development of multiplicative thinking by restricting students to the use of models that oversimplify multiplicative situations.

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AWARENESS OF DEALING WITH MULTIPLE REPRESENTATIONS IN THE MATHEMATICS CLASSROOM - A STUDY WITH TEACHERS IN POLAND AND GERMANY

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Pedagogical content knowledge (PCK) about dealing with multiple representations should encompass the awareness of their key role for mathematical thinking and for designing rich learning opportunities. However, it should also include a certain sensitivity for the complexity of changing between representations and thus the problems that learners may have with such conversions in specific classroom situations. Consequently, this study focuses on PCK regarding the role of representations and their conversions for the learning potential of tasks as well as for students' understanding in classroom situations. Such PCK of Polish and German pre-service teachers is compared, with German in-service teachers as a further reference. The results indicate significant differences between the samples.

INTRODUCTION

It is well-known that representations play a crucial role for the construction processes of learners' mathematical understanding. Hence, knowledge about the role of representation for student's learning and about how to deal with multiple representations in the mathematics classroom is an important part of PCK, which merits closer attention. However, specific empirical research is scarce. Such PCK should be investigated on different levels of globality resp. situatedness: Besides the awareness of the fact that multiple representations are important for fostering mathematical competencies on a global level, in particular the more content- and situation-specific knowledge about how to deal with representations may be decisive for the way teachers use representations in their classrooms. Thus, especially knowledge about how representations can be used to create rich learning opportunities and tasks as well as knowledge about what kind of role they play for the learners' understanding in specific classroom situations should be explored. Consequently, this study focuses on such PCK about dealing with multiple representations. We use a trans-national design with German and Polish pre-service teachers, in order to take into account the possibility that such PCK could be culture-bound. Bringing pre-service teacher into focus and using in-service teachers as a reference affords identifying needs for initial teacher education and teacher professional development.

The following first section gives a brief overview of the theoretical background, which leads to the research interest for this study as presented in the second section. We will then describe the design and methods of the study in the third section, report results in the fourth section and conclude with a discussion in the fifth section.

THEORETICAL BACKGROUND

It's impossible to imagine mathematical understanding without the use of representations, since mathematical objects aren't directly accessible (Duval 1999). In fact, usually it's not enough to have a single representation for a mathematical object, since single representations mostly emphasise only some properties of the corresponding object, so multiple representations have to be integrated in order to develop an appropriate concept image (Ainsworth, 2006; Tall 1988). Hence, flexibility with multiple representations is crucial for successful mathematical thinking and problem solving (e.g. Lesh, Post & Behr, 1987; Ainsworth, 2006). Consequently, the key role of dealing with multiple representations is also acknowledged in the German as well as in the Polish national standards (c.f. KMK, 2003; MEN, 2008). We thus assert that awareness of the relevance of using multiple representations for learning mathematics can have a significant impact on the teacher's ability to design rich learning opportunities. However, multiple representations are not per se beneficial for mathematical understanding, but can also be obstructing: Integration of different representations and changing between them are highly demanding for learners and can pose an obstacle to comprehension (Ainsworth, 2006; Duval, 2006). Thus, for supporting learners teachers constantly have to make decisions which must find a balance between encouraging them to change between representations on the one hand and avoiding excessive demands linked with such conversions on the other hand. Being aware of this ambiguous role of multiple representations for learning mathematics and being able to balance the related dilemma in the mathematics classroom should therefore be part of a mathematics teacher's professional knowledge (Dreher, 2012). For exploring such professional knowledge, this study uses a multi-layer model (Kuntze, 2012), that combines the spectrum between knowledge and beliefs (e.g. Pajares, 1992), the domains by Shulman (1986) with levels of globality, i.e. a spectrum between general and specific knowledge resp. views (cf. Törner, 2002; Kuntze, 2012). According to this model, especially the content-specific and classroom situation-specific knowledge domains are in the centre of interest, such as the content-specific awareness of the learning potential of conversions contained in tasks and situation-specific knowledge about dealing with representations in the student-teacher-interaction.

RESEARCH INTEREST

According to the need for research pointed out in the previous sections the study presented here aims to provide evidence for the following research questions:

To what extent are Polish and German pre-service teachers able to recognize the learning potential of tasks focusing on conversions of representations, in comparison with tasks including rather unhelpful pictorial representations?

Are Polish and German pre-service teachers able to realize a change of representations by a teacher in a given classroom situation and decide reasonably whether or not it was sensible for helping students' understanding?

SAMPLE AND METHODS

In order to find answers to these research questions, a questionnaire was designed in German and was then translated into Polish by a native speaker who is also fluent in German and has worked in mathematics education in Germany for several years. At the beginning of the questionnaire there were explanations of the notions "representation" and "pictorial representation" in a mathematical context given to ensure that all participants had a similar understanding of these key terms for the study. Connecting to our prior research, the questionnaire concentrates on the content domain of fractions (e.g. Dreher & Kuntze, accepted; Dreher, 2012).

The questionnaire was administered to 58 Polish pre-service teachers (49 female, 9 male) and 219 German pre-service teachers preparing to teach at primary and lower-attaining secondary schools ("PLS pre-service teachers") (183 female, 26 male, 10 without data) and also to 58 German in-service teachers at academic track secondary schools ("ATS in-service teachers") (23 female, 32 male). Moreover, a version of the questionnaire which was reduced due to time limitations was answered by 67 German prospective teachers at academic track secondary schools ("ATS pre-service teachers") (34 female, 33 male). The Polish pre-service teachers had an average age of 20.2 years (SD=0.6), the German PLS pre-service teachers were on average 20.7 years (SD=2.5) old and the German ATS pre-service teachers had a mean age of 21.4 years (SD=2.2). The German PLS pre-service teachers were at the beginning of their first year, whereas the Polish and the German ATS pre-service teachers were at the beginning of the second year of their university studies, but they all had in common that they were not taught specific courses in mathematics education so far. The German ATS in-service teachers were on average 41.5 (SD = 12.3) years old and had been teaching mathematics for 13.6 (SD =12.3) years.

Corresponding to the research questions for this study two parts of the questionnaire are focused: One part about task-specific views on the learning potential of multiple representations and one part focusing on situation-specific knowledge about changing representations and related consequences for student's understanding. In the first part the participants were asked to evaluate the learning potential of six fraction problems by means of multiple-choice items. The teachers could express their approval or disagreement concerning these items on a four-point Likert scale. They were told that the problems were designed for an exercise about fractions in school year six.

Make up a situation or a word problem which is suitable for the calculation $3 \div \frac{1}{4}$ and then use it to solve the calculation.	Do you know what $\frac{1}{2} \cdot \frac{1}{4}$ is? You can use the pictures below to help: 

Figure 1: Samples for tasks of type 1 (left) and of type 2 (right)

Three of these tasks are about carrying out a conversion of representations, whereas solving the other three tasks means just calculating an addition or a multiplication of fractions on a numerical-symbolical representational level. The pictorial representations which are given in the problems of the second type are rather not helpful for the solution, since they can't illustrate the operation needed to carry out the calculation. Samples for both kinds of tasks are shown in Figure 1. In the second part of the questionnaire the participants were given the transcript of a fictitious classroom situation (shown in Figure 2). The teacher in this classroom situation makes a critical change of representations: A student wants to know how you can see the addition of two fractions in the given rectangle, but the teacher explains the calculation using a pizza representation, supposedly because it is suited better. But in fact, it is here easier to show the addition with the rectangle, since the necessary subdivision in twelfth is already available. Hence, it is neither necessary nor advisable to force the student to engage with another representation at this point, even if previously the pizza representation was used for adding fraction in this class. In order to realise that the teacher's reaction should be seen critically, because the change of representations in this situation is hardly conductive to the student's understanding, awareness for conversions of representations in mathematics classrooms and a certain sensitivity for their key role for students' understanding is needed. Thus, the participants were asked the following question: "How much does this response help the pupil? Please evaluate the use of representations in this situation and give reasons for your answer.".

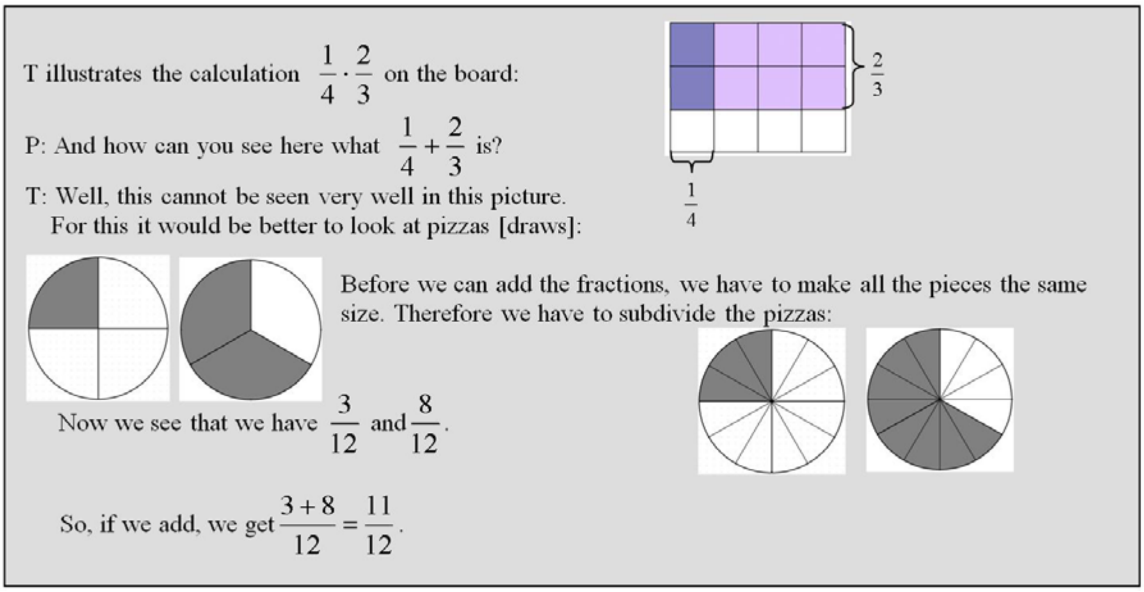

Figure 2: Transcript of the fictive classroom-situation

The answers of the participants were evaluated by coding them under two main aspects: "How was the teacher's response evaluated?" and "Which role does the teacher's use of representations play in the justification for this evaluation?". Possible categories for the first aspect were "no evaluation", "positive evaluation", "negative evaluation" and "balanced/ undecided evaluation". For the second aspect the following categories were used: "no justification given for the evaluation", "justification without referring to representations" (e.g. "The student is confused, since multiplication suddenly turns into addition."), "justification referring to representations in general" (e.g. " The use of diagrams reverts the child back to seeing the question before the challenge.”), “justification referring to the pizza representation only” (e.g. “Especially for adding fractions pizzas are still best suited.”), “justification referring to the pizza and the rectangle representation (comparative), but not to the change of representations” (e.g. “The first drawing is completely confusing for pupils. The pizzas are easier to understand”) and “justification referring to the change of representations” (e.g. “T could and should have shown the addition using the partitioned rectangle. The change hardly helps the pupil.”). All the answers were coded by two raters with high inter-rater reliability: Cohen’s kappa was 0.92 resp. 0.90.

RESULTS

We start with the results concerning the first research question, namely the teachers’ evaluation of the learning potential of the tasks given in the questionnaire. The design of this questionnaire section could be confirmed in two respects: Firstly, a factor analysis including all items in this section yields for each task a single reliable four-item scale (Cronbach’s α range from 0.73 to 0.87) about its learning potential with respect to its use of representations. A sample item of these scales is: “The way in which representations are used in this problem aids students’ understanding.” Secondly, a factor analysis with the six scales about the learning potentials of the six problems yields two “meta-scales” linked to the two types of tasks, namely “conversions of representations” vs. “unhelpful pictorial representations” (cf. Dreher & Kuntze, accepted). Both of these scales are reliable with α = 0.81 resp. α = 0.79.

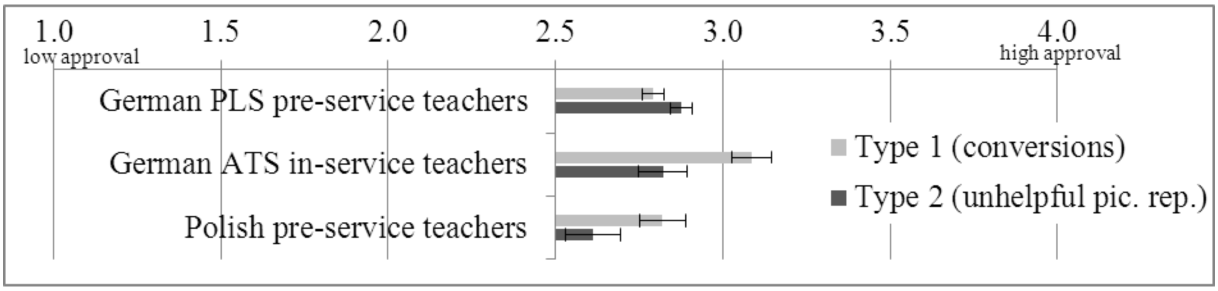

Figure 3: Evaluations of the learning potential regarding the two types of tasks

Figure 3 shows the means and standard errors of the two meta-scales for all the subsamples of this study, except for the German ATS pre-service teachers, since this section was not part of the reduced version of the questionnaire. Comparing the means of the two scales for each subsample seperately yields significant differences: While the German PLS pre-service teachers’ rating of the learning potential was higher for type 2 tasks than for type 1 tasks (T=2.12, df =218, p<.05, d=0.18), the pattern is reversed for the German ATS in-service teachers (T=3.01, df=57, p<.01 d=0.53) and interestingly also for the Polish pre-service teachers (T=2.22, df=57, p<.05, d=0.37). Focusing on the views about the learning potential regarding tasks of the first type, the in-service teachers have given higher ratings than the German PLS pre-service teachers (T=4.221, df=275, p<.001, d=0.63) and than the Polish pre-service teachers (T=2.93, df=114, p<.01, d=0.54). Comparing the subsamples regarding their evaluation of the learning potential of type 2 tasks on the other hand shows that the Polish pre-service teachers have assigned a lower learning potential than the German

PLS pre-service teachers (T=3.61, df=275, p<.001, d=0.53). The Polish pre-service teachers in our sample even tended to give lower ratings than the German in-service teachers, but this difference is not significant.

However, the results related to the classroom situation may show the specific PCK about dealing with multiple representations of the Polish participants in another light. The fictitious classroom situation shown in Figure 2 is designed in a way that a participant being aware of the complexity of changing between representations should realise the change of representations in this situation as being potentially obstructing for the students understanding. Hence, such a participant is expected to criticise the teacher's response and refer to his change from the rectangle to the pizza representation. Consequently in an overview approach, such awareness should be indicated best by the codes "negative evaluation" and "justification referring to the change of representations". It is therefore interesting to look at the proportion of participants in each subsample receiving these codes, respectively the combination of both of them. The teacher's response was evaluated negatively by

65.5% of the German ATS in-service teachers,

34.3% of the German ATS pre-service teachers,

21.5% of the German PLS pre-service teachers and by

13.8% of the Polish pre-service teachers.

And it was referred to the teacher's change of representations by

79.3% of the German ATS in-service teachers,

41.8% of the German ATS pre-service teachers,

33.8% of the German PLS pre-service teachers and by

39.7% of the Polish pre-service teachers.

More answers have received the code "justification referring to the change of representations" than the code "negative evaluation", since there were also participants who declared themselves in favour of the teacher's change of representations. However, for the reasons given above, it is the combination of both of these codes that we assert to be an indicator for the awareness of the situation-specific role of multiple representations for students' understanding. Both codes in combination appeared for

60.3% of the German ATS in-service teachers,

29.9% of the German ATS pre-service teachers,

16.6% of the German PLS pre-service teachers and by

13.8% of the Polish pre-service teachers.

DISCUSSION AND CONCLUSIONS

The key focus of this trans-national study is on the content-specific awareness of the learning potential of conversions contained in tasks and on situation-specific knowledge about dealing with representations in the student-teacher-interaction. The findings suggest certain profiles of PCK related to the teachers' awareness of dealing with multiple representations. Whereas the ATS in-service teachers show on average a relatively high awareness of the learning potential of conversions between representations and of the demands of the unnecessary change of representation in the classroom situation, the samples of pre-service teachers appear to have a less elaborated PCK on the content- and situation-specific levels.

Comparing the samples of the pre-service teachers, the Polish sample shows an interesting pattern compared to the German subsamples: The Polish pre-service teachers preferred the learning potential of the conversion tasks to the potential of the tasks with unhelpful representations, comparable to the in-service teachers. However, only a minority of these pre-service teachers saw the representation change in the classroom situation as a problematic aspect of supporting students' understanding in the classroom interaction. In particular, the relative frequency of answers of German pre-service teachers showing an awareness of this issue was higher. The evidence may hence suggest that the Polish pre-service teacher group does not have a more developed PCK than their counterparts in the German samples, but we can rather conclude different profiles of PCK from the data. The findings could also indicate an influence of classroom culture: Learning situations and types of tasks may be considered according to different underlying criteria according to the socio-mathematical norms in different school cultures.

However, the present findings may challenge the view on PCK and on goals of pre-service teacher education, considered within and across cultures: The dilemma between using multiple representations as a support of insightful learning and the demands of conversions between representations is an established feature which can be considered as valid across cultures. On the level of goals for initial teacher education, solutions and strategies have to be found on a culturally valid level, which can be however informed by comparative studies such as the one presented here.

We would like to recall that the findings of this study should be interpreted with care, as sample size, possible selection effects (e.g. choice of career of university studies) as well as the design of the study constitute clear limitations. The findings hence call for deepening studies which could explore underlying structures in PCK or epistemological beliefs by qualitative methods. Moreover, evaluating views of teachers related to more classroom situations could contribute to explaining the findings presented there. Such analyses are currently being carried out, as the questionnaire comprised of three more open items related to situation-specific PCK. Complementing these analyses and seeking to explain the findings, we also evaluate more general views of the teachers related to dealing with multiple representations.

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KINDERGARTEN TEACHERS' USE OF SEMIOTIC RESOURCES IN PROVIDING EARLY LEARNING EXPERIENCES IN GEOMETRY WITH A PICTURE BOOK AS A DIDACTICAL TOOL

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This study investigates the semiotic resources utilized by kindergarten teachers and their mediating role in young children's geometrical reasoning in different teaching-learning processes based on the use of a picture book as a didactical tool. Data were collected and analysed from two kindergarten classes. The teachers were found to use multiple semiotic resources in different ways. The mediating role of the teachers' use of semiotic resources on children's making sense of geometric shapes was manifested in various ways, but was stronger in one of the two classes.

INTRODUCTION

In early childhood education it is crucial for the learning of mathematics to be connected to children's everyday experiences and make sense to them. Children's literature is considered as a didactical tool which has the potential to provide children with an appealing context in which the problems, situations and questions they encounter are meaningful to them (e.g., Columba, Kim, & Moe, 2005).

A number of studies have provided evidence about the positive role of the characteristics of a picture book itself - pictures and text - on young children's mathematical reasoning (e.g., Elia, Van den Heuvel-Panhuizen, & Georgiou, 2010). Researchers in the field of mathematics education have recently examined the semiotic resources used within classrooms when students work on problems and explorations related to various mathematical concepts (e.g., Bjuland, Cestari, & Borgersen, 2008; Radford, Edwards & Arzazello, 2009). To our knowledge, however, there is no evidence about the effects of various semiotic means that are utilised within classrooms, when mathematics is taught through the use of picture books, on children's learning. In this study, we investigate the semiotic resources used by kindergarten teachers and their mediating role in young children's geometrical reasoning in two different teaching-learning processes based on the use of a picture book as a didactical tool.

THEORETICAL FRAMEWORK

By picture books we mean books consisting of text and pictures, in which pictures have a fundamental role in full communication and understanding (Nicolajeva & Scott 2000). Besides the meaningful context that picture books offer for the learning process in mathematics, through their pictorial representations in conjunction with the text, picture books can also support the initial stages of interpreting and using semiotic representations and in this way contribute to the development of mathematical understanding.

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A main focus of this study is the investigation of the semiotic resources teachers utilize while using picture books in mathematics instruction from a multimodal perspective. A multimodal approach includes ‘‘the range of cognitive, physical, and perceptual resources that people utilize when working with mathematical ideas’’ (Radford et al., 2009, p. 91). A more focused notion proposed by Radford (2003) concerns the semiotic means of objectification, including gestures, speech and inscriptions (such as graphs, formulas, tables and drawings), which have an essential role in the process of sense-making.

This study adopts also a socio-cultural approach to analyse classroom interactions. Mediation is a major term which is used within a socio-cultural perspective. This term is used to illustrate how people interrelate with cultural tools in action (Bjuland, 2012). The term mediation is here applied to identify how the semiotic resources used by kindergarten teachers play a mediating role for children in order to deal with early geometrical reasoning.

Geometry is an indispensable part of contemporary early childhood curricula and educational programs (Sarama & Clements, 2009). In this study the focus is on the perceptual apprehension of two-dimensional shapes (Duval, 1998), including the recognition and naming of figures, through the use of a picture book. Besides playing, walking and looking around, picture book reading could be regarded as another meaningful and natural way by which children can discover spatial relations and geometric concepts (Elia et al., 2010).

The present study addresses the following research questions: (a) Which are the semiotic resources kindergarten teachers use to help children experience geometrical ideas in different teaching-learning processes based on the use of a picture book as a didactical tool? (b) What mediating role does the teachers’ use of semiotic resources play in young children’s geometrical reasoning in different teaching-learning processes based on the use of a picture book as a didactical tool?

METHODS

To collect the data for this case study two classes of two public kindergartens in Cyprus were visited. From each class a lesson was videorecorded. The teachers of the classes were called Melina and Georgia respectively. Melina’s class consisted of 26 children (14 girls and 12 boys), while Georgia’s class consisted of 25 children (12 girls and 13 boys). The children in both classes were from 4 to 5 years of age.

In their lessons, both teachers introduced two-dimensional geometric shapes to the children, by using the picture book ‘Oscar the Button’, which is a familiar book among the kindergarten teachers in Cyprus for teaching geometry. This picture book was not familiar to the children of the two classes, though. It is written by Nagy Eszter and is translated to Greek by Margarita Rega. The book was first published by Siphano Picture Books in 2000 in England. The story is about a button called Oscar who lives on an overcoat. One day he pops off and rolls away to discover the world. But because he is round, nobody wants to know him - doors, roofs and kites all snub him because he is a different shape! Then he meets the Moon, who is round as he is.

In both cases, the story of the book was presented as a full class by the teachers by using a different approach. The first teacher, Melina, followed the typical way to present the book. She read the book to the whole class using a board to place enlarged pictures of the book and then she posed some questions about the picture book's content. The second teacher, Georgia, used the theatre game approach to present the story of the picture book. She took the role of Oscar, holding a puppet of a button (Figure 2b), and during the story telling she urged the children to complete some of the sentences she started or posed some questions which were related mainly to the mathematical components of the book story.

RESULTS

In order to address the research questions we will analyse some dialogues from Melina's class and Georgia's class from a multimodal (Radford et al., 2009) and a socio-cultural perspective (Bjuland, 2012).

Analysing the dialogues in Melina's classroom

The children are sitting in a semicircle, looking at the board with the enlarged pictures of the book. Melina, who is sitting close to the board, produces iconic gestures for all the pictures which have geometric components in order to give more emphasis on the geometric figures which are included in the text and the pictures. Table 1 presents some examples of the semiotic resources that Melina uses, after reading that Oscar was rejected by the triangular roofs of some houses because he did not look like them.

Text/Explanations	Pictures/Gestures	
Text (read by teacher): He (Oscar) met two coloured doors. "Hello", he said. "Can I play with you?" "Well, are you blind?" The doors snap him. "All of us here are rectangles. You are a circle." "Rectangles? Hmm, I've heard somewhere," Oscar said.	She makes a rectangle with her pointing finger.	
Explanation (given by teacher): He was rejected by the triangular roofs as well as the rectangular doors.	She makes a triangle with her pointing finger.	

Table 1: Examples of semiotic resources used by Melina

As shown in Table 1, Melina's words and gestures are coordinated in order to help the children remain focused on the shapes of the figures in the pictures of the book. It must be noted, though, that the mediating role of Melina's semiotic resources on children's mathematical reasoning was not apparent during the reading of the book, but it became obvious later, when she posed some questions about the picture book's content. The extract below illustrates how Melina interacts with the children at a part of this latter phase of the lesson.

Melina: What shape was Oscar?
Children: Circle.
Melina: ...so he (Oscar) decided to leave. Where did he go first? Who did he visit?
Louis: The roofs (he opens his arms and lifts one up).
Melina: What was the shape of the roofs Louis?
Louis: Triangle (he makes a triangle with his pointing finger).
Melina: What other things do you know that have the shape of triangle? Who knows to tell me? Irene, tell me something that looks like a triangle.
Irene: The roofs of the houses.
Melina: Another thing? Let's see... Where did he go then? Fryda?
Fryda: To the rectangle.
Melina: Tell me other things which have this shape, rectangle. Nefeli?
Nefeli: The door.

Melina's iconic gesture and speech at first and her communicative strategy of posing closed (e.g., line 1) and open questions (e.g., lines 20-21) later on encourage the children to introduce the various geometric figures (circle, triangle and rectangle) into the dialogue. The dialogue also illustrates the close relationship between Melina's questions and the iconic gestures she produced before, when referring to the pictures and the text she read from the book. In sum, the semiotic resources in the teaching-learning process here are produced by the teacher-child communication, including linguistic and gestural activity, and the inscriptions of the geometric shapes in the enlarged pictures of the book on the board. A closer look at Melina's presentation with respect to mathematics indicates that she focuses on (a) the iconic inscription of geometric shapes, (b) the names of shapes, and (c) the recognition of shapes in other contexts besides the book story.

As we can see from the above extract, children's activity is influenced by the different semiotic resources the teacher uses. More specifically, when Melina asks them to identify some objects which have the shape of triangle and rectangle, the children express ideas that are obviously affected by the examples presented in the picture book and represented by the iconic gestures of Melina. For example, about the triangle, Irene refers to the roofs (line 16) which have been used in the book for the presentation of the shape of triangle and highlighted by Melina's linguistic and gestural activity. The children do not refer to other objects besides the ones mentioned in the book.

Figure 1: Louis iconic gesture about the triangle.

Interestingly, children also tend to reproduce the teacher's gesture in order to express their ideas. Particularly, when Melina asks children to tell her which is the shape of the roofs that Oscar met at first, Louis (Figure 1) produces the same iconic gesture that had been produced by the teacher previously, when she was reading the book. Specifically, he makes a triangle with his pointing finger on the air (lines 10-11).

Analysing the dialogues in Georgia's classroom

Georgia tells the story to the children, who are sitting in a semicircle, by holding the button puppet or the picture book (Figure 2a, b). The extract that follows illustrates Georgia in action with the children of her class during the presentation of the book, after telling them about the rejection of Oscar by the rectangular doors.

Georgia: He runs quickly and he moves close to a window which has the shape of … (she makes a square with her pointing finger).
Children: A square.
Georgia: A square.
Georgia (As Oscar): Hi, square. Hi square window! Would you like to become friends?
Georgia (As square): No, I don't want to be your friend. You are different. You haven't got straight lines like me (she makes a straight line on air)
Georgia: And what must he do? He cries.
Georgia (As Oscar): Snif. Snif.
Georgia: … Oscar walks alone.
Georgia (As Oscar): Nobody wants to be my friend because I'm different and I don't have straight lines.
Chris: This red shape is a circle.
Georgia: Oh, Chris. Tell me Chris.
Chris: The sun.
Georgia (As Oscar): Oh, I didn't see it before, does it look like me?
Children: Yes it's a circle.
Georgia (As Oscar): Yes it is a circle, oh, and I did not see it before (she makes with her pointing finger a circle on the air).
Chris: The sun.
Georgia (As Oscar): One sun. But now is night, it's dark and I feel so lonely.
Chris: A moon.
Mary: The circle, the moon (she makes a circular motion with the pointing finger of her right hand).
Children: Yes!
Georgia (As Oscar): Does it look like me?
Children: Yes!
Mary: But now it is half, later….

Georgia is telling the story as a narrator or by taking the role of the button, or other objects in the story having different geometric shapes, is posing questions and is giving the opportunity to the children to finish her sentences.

Figure 2: (a) Georgia shows pictures of the book; (b) Georgia holds the puppet of Oscar and Chris's pointing gesture about the sun; (c) Mary's iconic gesture about circle

This multifaceted communicative strategy is synchronized with Georgia's iconic gestures which represent the different geometric shapes, i.e., square and circle (lines 25-26, 57-59), as well as the straight lines, stressing that they are a property of the square, but not of the circle (lines 34-35). Both semiotic resources, speech and gesture, provoke the children to introduce the various geometric shapes into the dialogue. Georgia's speech and gestures in conjunction with the physical/didactical tools she uses (e.g., picture book, puppet) are coordinated in order to help the children stay focused on the inscriptions of geometric figures included in the book. The semiotic resources in the teaching-learning process here are produced by the synchronized linguistic activity of the teacher-child communication, the teacher's and children's gestures, and the inscriptions of geometric figures in the pictures of the picture book.

As far as the mathematics is concerned, Georgia's presentation of the book story focuses on (a) the recognition of geometric shapes through iconic gestures, (b) the names of shapes, and (c) the difference between the circle and other geometric shapes with straight lines (e.g., square), and (d) the identification of objects that have the shape of circle and the similarity between them.

In the last part of the dialogue (after line 36) Georgia, through her words and role playing as Oscar, who is feeling lonely, invites implicitly the children to focus on the latter mathematical strand, that is to explore and find other objects that have the shape of circle. Chris spontaneously shows with a pointing gesture a picture of a sun which is on a board in the classroom and tells Georgia: "This red shape is a circle" (line 43); "The sun" (line 45) (Figure 2b). Then the teacher, without going beyond the story and thus keeping the context meaningful for the children, elaborates on Chris's idea by saying that "now it's night" which implies that there is no sun and Oscar is alone again, in order to give children the opportunity to find other circular objects. Mary, following Chris' response ("the moon"), reproduced (lines 57-59, Figure 2c) the teacher's previous iconic gesture for the shape of circle for the sun (lines 49-52) saying "circle, moon". Then Georgia asks whether the moon looks like Oscar, to encourage the children identify the similarity between these circular items. This question stimulated Mary to find out that in their classroom there was an image of the crescent moon and compare it with the shape of circle. Comparing the crescent moon and the circle, she told the teacher that "now it's half, later...". This last part of the dialogue shows that the semiotic resources used (words, i.e., elaborations and questions based on the book story and children's responses, gesture and inscriptions, i.e., pictures) challenge the children to experience and appropriate the shape of circle, its properties and differences with other shapes (e.g., semicircle). At the same time, this dialogue illustrates a kind of collective activity in which the two children and the teacher are attuned to each other's perspective; they are acting together so as to respond to the implicitly assigned task, to find objects that have the same shape as Oscar.

DISCUSSION

The analyses of the dialogues in two classrooms revealed that the two teachers under study used various semiotic resources in different ways during their interaction with the kindergartners and/or the mathematics involved in the picture book. The most crucial common semiotic resource used by the teachers was linguistic activity which was manifested in two ways by Melina, the first teacher, that is, by reading the text and posing questions, and in multiple ways by Georgia, the second teacher, that is, by telling the story as a narrator, role playing, asking questions, giving explanations and elaborating on children's ideas. Both teachers used gestures in their communication strategies which were mathematics oriented and iconic. That is, gestures were used mainly when the teachers were referring to geometric concepts and they represented visually the geometric figures that were included in the story. For the second teacher, during the whole teacher-child interaction, gestures and speech were coordinated with other semiotic means, that is, the inscriptions of geometric figures in the pictures of the book in order to help the children stay focused on the two-dimensional geometric shapes she intended to introduce. For the first teacher, this coordination took place only while reading the book, during which there was not a teacher-child interaction. During the dialogue with the children the first teacher did not use any gestures or other semiotic resources besides speech (posing questions).

The mediating role of the teachers' use of semiotic resources on children's geometrical reasoning was manifested in various ways, but was stronger in the second class. Children in both classes used in their verbal utterances geometrical terms, and specifically the names of shapes, or examples of objects having a specific shape, which were represented in the pictures and the text of the book and were highlighted by the teachers' words and gestures. Furthermore, the iconic gestures that were used by the teachers in introducing the various geometric shapes were interwoven also in the children's gestural activity when referring to the same or even to different objects. This latter case of reproduction of gesture, which was observed only in the second class, can be considered as an elaboration on the iconic gestures produced by the teacher. In a broader sense, our findings also suggest that the teacher's semiotic resources produced in the teacher-child dialogue in the second class, including words stressing that the circle does not have straight lines, iconic gestures representing the shape of circle, inscriptions of shapes in the book's pictures, and their synchronized coordination with two children's production of semiotic resources, mediated these children experience early geometrical reasoning while comparing the shape of circle with shapes of objects in everyday life. Thus, these semiotic resources contributed to the children's process of objectification for the shape of circle and its differences from other shapes.

The difference in the mediating role of the semiotic resources in children's process of objectification for geometric shapes between the two classes could be also explained by the difference in the mathematics that was addressed through the teacher's use of semiotic resources. The second teacher had a clearer focus with respect to the mathematics she intended to promote through the picture book. She focused on the concept of circle and its comparison with other shapes, while the first teacher addressed the various geometric shapes included in the picture book without concentrating on their characteristics and relationships.

Another explanation for this difference could be the different level of opportunities given to the children for participation and for experiencing the mathematics themselves in the two classes. In the second class children could express their ideas and reasoning, and interact with the teacher about the mathematical content of the book to a greater extent, throughout the whole lesson. Furthermore, the second teacher elaborated on the children's mathematical ideas in an insightful way without going beyond the picture book's story, in order to keep the context meaningful for the children, such that all of the communication partners, teacher and children, could contribute in the production of semiotic resources in order to apply the reasoning strategy of comparing the shape of circle with shapes of objects in everyday life.

In conclusion, this study's findings suggest that the proper use, the coordination and the dynamics of semiotic resources produced in the teacher-child interactions could have a major role in children's active and effective involvement with mathematics in a teaching-learning process based on the use of a picture book. Future research could focus on how various semiotic systems (e.g., speech, gestures, inscriptions) and their connection to different ways of using picture books in the teaching-learning process might enhance children's involvement and learning in mathematics.

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CHOOSING AND USING EXAMPLES: HOW EXAMPLE ACTIVITY CAN SUPPORT PROOF INSIGHT

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This paper presents the results of two studies aimed at identifying the ways in which successful provers (students and mathematicians) engage with examples when exploring and proving conjectures. We offer a framework detailing the participants' actions guiding a) their example choice and b) their example use as they attempt to prove conjectures. The framework describes three categories for example choice (choose examples that test boundaries, emphasize mathematical properties, and build a progression of example types) and three categories of uses (identify commonality, see generality, and anticipate change).

INTRODUCTION: PROOF IN SCHOOL MATHEMATICS

Proof in school mathematics plays an important role in students' mathematical reasoning abilities, with researchers arguing that proof should be a central part of students' education at all levels (e.g., Ball et al., 2002). Yet despite a strong emphasis on proof in school mathematics, students struggle both to produce and understand mathematically valid proofs (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000). Researchers suggest that a critical source underlying students' difficulties is their treatment of examples, particularly the tendency to rely on example-based arguments as justification that a universal statement is true (e.g., Healy & Hoyles, 2000; Knuth, Chopin, & Bieda, 2009).

Although it is important to help students understand the limitations of example-based arguments, we propose that it is equally important to avoid situating example-based reasoning solely as an obstacle to overcome. Given the essential role examples can play in exploring conjectures and developing proofs, we suggest that providing students with opportunities to carefully analyse examples may contribute to their abilities to develop and understand conjectures and proofs. This paper presents the results of two studies aimed at identifying the ways in which successful provers (students and mathematicians) engage with examples when exploring, understanding, and proving conjectures. We offer an initial framework detailing the characteristics of participants' example choices and their example usage as they explore conjectures and develop deductive proofs. By studying the thinking of those who are successful at proving, our aim is to gain insight into the nature of the type of example-related activity that could ultimately support students' proof development.

BACKGROUND AND THEORETICAL FRAMEWORK

The Role of Examples in Conjecturing, Generalizing, and Proving

Examples play an important role in mathematical reasoning, and the time spent analysing examples can provide both a better understanding of a conjecture and insight

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into the development of its proof (Epstein & Levy, 1995). Thinking with examples can help students make sense of conjectures and can support the development of conceptual understanding more generally (e.g., Alcock & Inglis, 2008). Example use has also been found to support students' acts of generalizing (e.g., Goldenberg & Mason, 2008; Naftaliev & Yerushalmy, 2011), and analyzing structural similarities and variation across examples can support proof development (Goldenberg & Mason, 2008; Pedemonte & Buchbinder, 2011).

Research on mathematicians' thinking similarly shows that examples play a critical role in both mathematicians' development of conjectures and in their subsequent construction of proofs (Alcock & Inglis, 2008). Epstein and Levy (1995) contend that mathematicians spend considerable time thinking with examples, noting, "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples." (p. 6) This current study builds on prior work (Lockwood et al., 2012) in which mathematicians described using examples specifically to gain insight into proof. While the initial research on example use shows promise, more nuance is needed in understanding how to best support students' thinking with examples in order to promote proof. The findings presented in this paper shed light on the specific mechanisms through which example exploration provides insight into proof development for both students and mathematicians.

Proof and Proof Activities

We refer to proof and justification interchangeably to mean the activity of ascertaining (convincing oneself) and persuading (convincing others) (Harel & Sowder, 1998). An individual's proof scheme consists of what constitutes ascertaining and persuading for that person. We rely on Harel and Sowder's (1998) proof schemes framework – recently updated (Harel, 2007) – for classifying students' proof schemes. The framework establishes three main classes of proof schemes: (a) External Conviction class, (b) Empirical class, and (c) Deductive class. Proof schemes in the first two classes rely on external authority, the appearance of an argument, manipulation of symbols without a coherent system of referents, or evidence from examples. In contrast, the deductive class of proof schemes represents schemes dependent on generality, operational thought, and logical inference.

METHODS

Student Study: Participants and Instrument

Participants were 20 students aged 12-14, each who participated in a videotaped 1-hour interview. Eleven students were female and 9 students were male. The interview instrument presented students with seven conjectures and students were asked to examine the conjectures, develop examples to test them, and then provide a justification. The conjectures addressed ideas in number theory and geometry. A sample conjecture is as follows: "Kathryn thinks this property is true for every whole number. First, pick any whole number. Second, multiply this number by 2. Your answer will always be divisible by 4." After the students worked with examples for each of the conjectures, they were asked why they chose the examples they did.

Mathematician Study: Participants and Instrument

Participants were 6 male faculty members from two university mathematics departments who participated in 1-hour videotaped interviews. Five participants hold PhDs in mathematics and one in mathematics education. During the interviews, mathematicians were asked to explore three mathematical conjectures and to think aloud as they worked. After each conjecture, the participants were asked clarifying questions about their work, including their example-related activity. Sample conjectures are shown in Table 1.

1	Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s,n)=1$ or $\gcd(s,n)=s$. Show that there exist $s,t \in S$ such that $\gcd(s,t)$ is prime.
2	All the numbers should be assumed to be positive integers. An abundant number is an integer n whose divisors add up to more than $2n$. A perfect number is an integer n whose divisors add up to exactly $2n$. A deficient number is an integer n whose divisors add up to less than $2n$. Conjecture 2a. A number is abundant if and only if it is a multiple of 6. Conjecture 2b. If n is deficient, then every divisor of n is deficient.

Table 1: Sample conjectures given to mathematicians

Analysis

The justifications that the student participants produced were coded according to Harel and Sowder's (1998; Harel, 2007) proof schemes taxonomy and identified as representing proofs from the deductive class, the empirical class, or the external conviction class. Five of the 20 student participants produced no proofs from the deductive class, and 11 participants produced some proofs from the deductive class. Because the paper focuses on successful provers, the student analysis was restricted to the remaining 4 students who produced proofs that were all from the deductive class, and all of the mathematician interviews were analysed. Each of these 4 students was able to produce a deductive proof for every conjecture he or she encountered.

Both the mathematicians' and the students' examples were coded into a pre-existing framework of example *types* and *uses* developed by the authors (Ellis et al., 2012; Lockwood et al., 2012). We then re-analysed the data by coding the participants' responses to each conjecture in order to characterize common themes across their actions. This open coding process led to the development of two major categories of themes, example choice and example use. The research group discussed the codes and clarified uncertainties as emergent codes solidified. A given response could be coded in multiple categories simultaneously, both within and across choices or uses.

RESULTS: EXAMPLE CHOICE AND EXAMPLE USE

We identified two major actions with examples that supported the participants' proof activities: Deliberate and strategic *choice* of examples, and insightful *use* of examples. The categories of example choice and example use are shown in Table 2. Although every category occurred in both the mathematician data and the student data, due to space constraints we limit our discussion to the most salient examples.

Example Choice	*Test boundaries*: Selecting examples that target the boundaries of the hypothesis or conjecture, including counterexamples.
	Emphasize properties: Purposefully choosing examples with particular properties or features relevant to the conjecture in question.
	Build a progression: Building a deliberate progression of specific examples that may range in type or role.
Example Use	*Identify commonality*: Attending to common features or characteristics across multiple examples in order to identify a broader mathematical structure.
	See generality: Identifying a general or representative structure embedded in one example that may provide insight into the structure of a general argument.
	Anticipate and imagine change: Envisioning an example as a dynamic, changing representation.

Table 2: Categories of example choice and example use

Example Choice

The participants demonstrated a dispositional orientation towards choosing examples in a deliberate, strategic manner. The first category of example choice is *testing boundaries*, in which one purposely attempts to find examples that could potentially break the conjecture, or could provide insight into the conjecture's limitations. For instance, in the mathematician's Conjecture 2b, Professor Lowry specifically chose to explore examples that included 6 as a factor because 6 is a perfect number: "We know 6 is perfect…so actually it's a good choice for a potential counterexample, because it's not deficient, but it's not far from being deficient." He further clarified his motivation for this choice by saying it is "likely that if something interesting is going to happen with an example, a boundary case is usually where it would be interesting." Professor Lowry indicated that by examining boundary cases and looking for counterexamples, he suspected that he might gain some insight about the conjecture and about a possible proof. The student participants also selected examples with boundary testing in mind. For instance, Genna examined Conjecture A, the conjecture that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. Genna believed that this must be true for equilateral and isosceles triangles because she could imagine that "two of the sides added together are obviously bigger than the third side." Genna then tried to think of "a triangle that wouldn't work", and drew a scalene triangle. Genna's explorations with different scalene triangles led to a general argument for all triangles. In general, successful provers recognized what they could gain from boundary cases and specifically sought out those types of examples.

Participants also chose examples based on their mathematical *properties*, the second category of example choice. When doing so, participants sought out examples with specific types of properties that they viewed as relevant to the conjecture at hand. This was evident in Genna's selection of a scalene triangle, as she thought that triangles with three different side lengths might be more likely to break the conjecture than equilateral or isosceles triangles. An attention to properties emerged frequently for the mathematicians. In his work on Conjecture 1, Professor Parker drew upon the specific mathematical property of relative primeness in ascertaining the truth of the conjecture: "See I was definitely using relatively prime to this [circles a 4], relatively prime to this [circles a 6], giving me the existence or non-existence of a two and a three." In his work, mathematical properties such as prime numbers, greatest common divisors, and relative primeness were readily available to him, and he referred to them often in selecting examples.

The final category of example choice is *progression*, which describes an attempt to build a set of examples that either varies across different types or roles, or that together contributes to a more complete picture of what is happening. For instance, for Conjecture 2 Professor Lowry chose examples with 6 as a factor. He first specifically chose the example 12, which is 6•2, but he recognized the fact that because 6 was half of 12 he would never attain a counterexample. He thus proceeded to choose an example that did not have a perfect factor that was exactly half of it, selecting 6•3. As he worked through this example, though, he realized that many other factors were being generated. He did not want to have "much stuff between six and the whole number," and this led him to choose 6•11. In the end, it was his work with the 6•11 example that led him to a proof of the conjecture. Professor Lowry's strategic and carefully chosen progression of 6•2, 6•3, and 6•11 provided him with a number of insights and ultimately contributed to his successfully proving the conjecture.

Example Use

The participants' dispositional orientation towards using examples reflected the belief that the purpose of example exploration is not merely to check a conjecture's truth, but to try to understand a conjecture's logic through the example. In the first category of use, *identify commonality*, participants paid careful attention to the variation across multiple examples, attending to what changed and what remained the same as they shifted from one example to the next. For instance, Professor Larkin noted that he wanted to be aware of patterns emerging in multiple examples, in particular "what pattern it's creating for me. So that, if in fact, the [conjecture] is true, I have some sense of the pattern I can create to prove it." In his work on Conjecture 1, Professor Willis also mentioned seeking a pattern across examples, suggesting that he was looking to identify a common structure among his chosen examples that would shed light on why the conjecture might be true.

In the second category of example use, *see generality*, participants were able to develop insight into the mathematical structure of a potential argument through exploration with just one example. This way of thinking is evident in Reed's exploration of Conjecture B, the conjecture that the sum of three consecutive whole numbers is equal to three times the middle number. Reed tested with the triple 33, 34, 35. When the interviewer asked if that example worked, he replied, “Yes it did. That’ll do. Well, of course it’ll always work.” Reed underlined the middle number, 34, and appeared to experience an insight into why the conjecture must always work. When the interviewer asked him to explain, Reed used a new triple, 6, 7, and 8, to explain his insight (Figure 1):

Figure 1: Reed’s example to demonstrate the truth of Conjecture 1

Reed: Eight minus 1 equals 7, and 6 plus 1 equals 7. So take 1 off this (gestures to the 8) and put it on there (gestures to the 6). And it comes out 7 plus 7 plus 7.

The final category of use, *anticipate and imagine change*, refers to an ability to imagine one example as dynamic rather than static, changing the boundaries or features of the example in order to mentally anticipate and test multiple cases at once. In some cases this can also enable participants to deliberately manipulate the example in a way that can assist with insight into a proof. This is seen in Reed’s work with Conjecture A. Reed constructed several triangles to try to imagine whether it would be possible to have two sides longer than the third, which he began to suspect might be impossible. In order to confirm his suspicions, Reed created a final triangle with two sides each of length 4 and the third side of length 8, explaining what would happen if he “straightened” the two sides of length 4, by which he means flattening the sides so the triangle became more and more obtuse until it approximated a line:

Reed: Because if these sides were straightened out to make a line, it’d be this long (gestures a length from A to C, a length longer than the 3rd side) so this line right here from point A to point B is not the same as – the same or longer than points A to C.

The participants’ orientation towards example use was one that cast examples as a way to better understand the conjecture. In contrast, students who were unable to develop deductive proofs viewed examples primarily as a way to test a conjecture’s truth. By moving beyond testing activities, the successful provers in this study were able to leverage the power of examples to provide meaningful insight into the conjectures and their potential proofs.

DISCUSSION

Although the students and the mathematicians differed in the sophistication of the arguments they were able to construct, there were a striking number of similarities in the ways in which each participant group thought strategically about their example choice and made deliberate use of examples in order to think about broader mathematical structures. Their ways of choosing and using examples differed from what occurred in the work of the student participants who did not ultimately produce deductive arguments (see Ellis et al., 2012). These differences suggest that exposure to examples is not sufficient for fostering proof insights; instead, learners must engage with examples in particular ways in order to benefit from their utility as a way to gain understanding and inform the development of deductive arguments.

In addition, there were some important differences across the two participant groups; for instance, the mathematicians were more apt to recognize the potential power of a specific example before choosing it, and they demonstrated an explicit meta-cognitive awareness of the usefulness of examples more generally in providing insight into the nature of a conjecture and its proof. These findings, rather than framing examples as obstacles to overcome, emphasize that students may benefit from instruction in how to strategically choose examples and how to think carefully with the examples they have chosen. Further, instructional practices that encourage students to discuss and justify their choice and use of examples could foster the development of the meta-cognitive awareness demonstrated by mathematicians. A stronger understanding of the strategies successful provers employ as they use examples to create, explore, and prove conjectures could ultimately inform instructional guidelines aimed at more effectively fostering students' abilities to prove.

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AN EXPONENTIAL GROWTH LEARNING TRAJECTORY

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Exponential functions are an important topic in school algebra and in higher mathematics, but research on students' thinking suggests that understanding exponential growth remains an instructional challenge. This paper reports the results of a small-scale teaching experiment with students who explored exponential functions in the context of two continuously covarying quantities, height and time. We present a learning trajectory identifying three major stages of conceptions about exponential growth: pre-functional reasoning, covariational reasoning, and correspondence reasoning. The learning trajectory identifies relationships between these conceptions and the nature of the tasks that supported their development.

INTRODUCTION

Exponential functions represent an important transition from middle-grades mathematics to the more complex ideas students encounter in secondary instruction and beyond. However, the instruction of exponential functions has proved challenging given students' documented difficulties in understanding exponential growth. Students struggle to transition from linear representations to exponential representations, to identify what makes data exponential, and to explain what a function such as $f(x) = b^x$ means (Alagic & Palenz, 2006; Weber, 2002). These challenges suggest a need to better understand how to foster students' learning about exponential growth. As Simon and Tzur (2004) argue:

> The most important use of the elaborated HLT would be for teaching concepts whose learning is problematic generally or for particular students. In such cases, greater understanding of learning processes and how they can be supported…is essential for developing a theoretical basis for dealing with difficult pedagogical problems. (p. 101)

This paper reports on the results of a teaching experiment introducing exponential growth in the context of two continuously covarying quantities. We present a learning trajectory specifying students' understanding over time, identifying connections between students' conceptions and the tasks promoting the development of those conceptions. Findings suggest that situating an exploration of exponential growth in a model of covarying quantities can support both students' understanding of what it means for data to grow exponentially and how to express exponential relationships algebraically.

BACKGROUND AND THEORETICAL FRAMEWORK

Hypothetical Learning Trajectories

Simon (1995) defined a *hypothetical learning trajectory* as "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Clements and Samara (2004) elaborate on this definition, characterizing a learning trajectory as a description of children's thinking and learning in a specific domain connected to a conjectured route through a set of tasks designed to support that thinking. We offer a learning trajectory presenting a model of students' concepts about exponential growth and an account of how those concepts changed as students interacted with mathematical tasks (Confrey et al., 2009; Steffe, 2004). In contrast to learning trajectories emphasizing strategies or skills, we focus on conceptual understanding and its development. Our aim is to contribute to the field's understanding of learning about exponential growth, an area known to be challenging for students.

Quantitative Reasoning and the Rate-of-Change Perspective

A popular approach to function relies on the correspondence perspective (Smith, 2003), in which a function is viewed as the fixed relationship between the members of two sets. Smith and Confrey (Smith & Confrey, 1994) offer an alternative to the correspondence view, which they call the covariation approach. Here one examines a function in terms of a coordinated change of *x*- and *y*-values, moving operationally from y_m to y_{m+1} coordinated with movement from x_m to x_{m+1}. Relying on situations that involve quantities that students can manipulate can foster their abilities to reason flexibly about dynamically changing events (Castillo-Garsow, 2012). This approach may be especially useful in helping students understand exponential growth, given its connection to contexts involving multiplicative relationships.

METHODS

The study was situated at a public middle school and consisted of a 12-day teaching experiment with 3 female eighth-grade students (ages 13-14) in which the first author was the teacher-researcher. Each 1-hour session focused on the relationship between height and time for an exponentially growing Jactus plant; students were able to manipulate the plant's growth using an interactive computer program called Geogebra. Although this scenario is not realistic, the context is realizable (Gravemeijer, 1994) in that students could imagine, visualize, and mathematize the relevant quantities. All sessions were videotaped and transcribed.

We assumed that any understanding students might have about exponentiation before entering a teaching experiment would be dependent on an image of repeated multiplication. Building on that conception, our primary goal was to foster students' understanding of the following set of concepts for an exponential function $y = a \cdot b^x$:

The period of time *x* for the *y*-value to increase by the growth factor b is constant, regardless of the value of a or b.

There is a constant ratio change in *y*-values for each constant additive change in corresponding *x*-values.

The ratio of the change in *y*, which can be expressed as $\frac{y_1}{y_2} = b^{x_1 - x_2}$, is always the same for any same $\Lambda x$and the value of $f(x + \Delta x)/f(x)$ is dependent on Λx.

Data analysis relied on retrospective analysis techniques to characterize students' changing conceptions. Project team members developed preliminary codes for concepts on the trajectory based on students' talk, gestures, and task responses as evidence of understanding at different stages. The first round of analysis yielded an initial learning trajectory, which then guided subsequent analysis in which the project team met as a group to refine and adjust the codes. This iterative process continued until no new codes emerged. Once coding was complete we chose 20% of the data corpus for independent coding, which yielded an inter-rater reliability rate of 92%.

RESULTS

The progression of the students' conceptual development occurred in three major stages, which we call *pre-functional reasoning*, *covariation reasoning,* and *correspondence reasoning* (Figure 1). Although pre-functional reasoning did precede the development of both the covariation and the correspondence views, the latter two ways of thinking did not occur in a sequential nature. Rather, students constructed an early covariation understanding, and then began to simultaneously develop both a more sophisticated covariation perspective and a correspondence understanding. Students were able to flexibly shift between these perspectives as needed.

Pre-functional reasoning

The students entered the teaching experiment with only a qualitative understanding of exponential growth, and were unable to quantify the manner in which the plants grew. Exploration with the Geogebra program enabled the students to then develop a repeated-multiplication understanding of exponentiation. For instance, students encountered a Jactus that grew by quadrupling its height every week. Uditi described its growth: "They're all going up by like times 4, like, 16 times 4 is 64 and then 64 times 4…that's 1024." Absent from Uditi's language is any attention to how much time it takes for the plant's height to quadruple.

Through comparing plants with different growth rates, students began to realize that the growth factor determined the nature of the growth, identifying plants with larger growth factors as those that would grow taller over time. For instance, given a comparison problem with a plant that doubled each week, Laura could determine that the quadrupling plant grew faster than the others, "because it's growing 4 times and it's more than 2 times and 3 times." However, at this stage the students did not attend

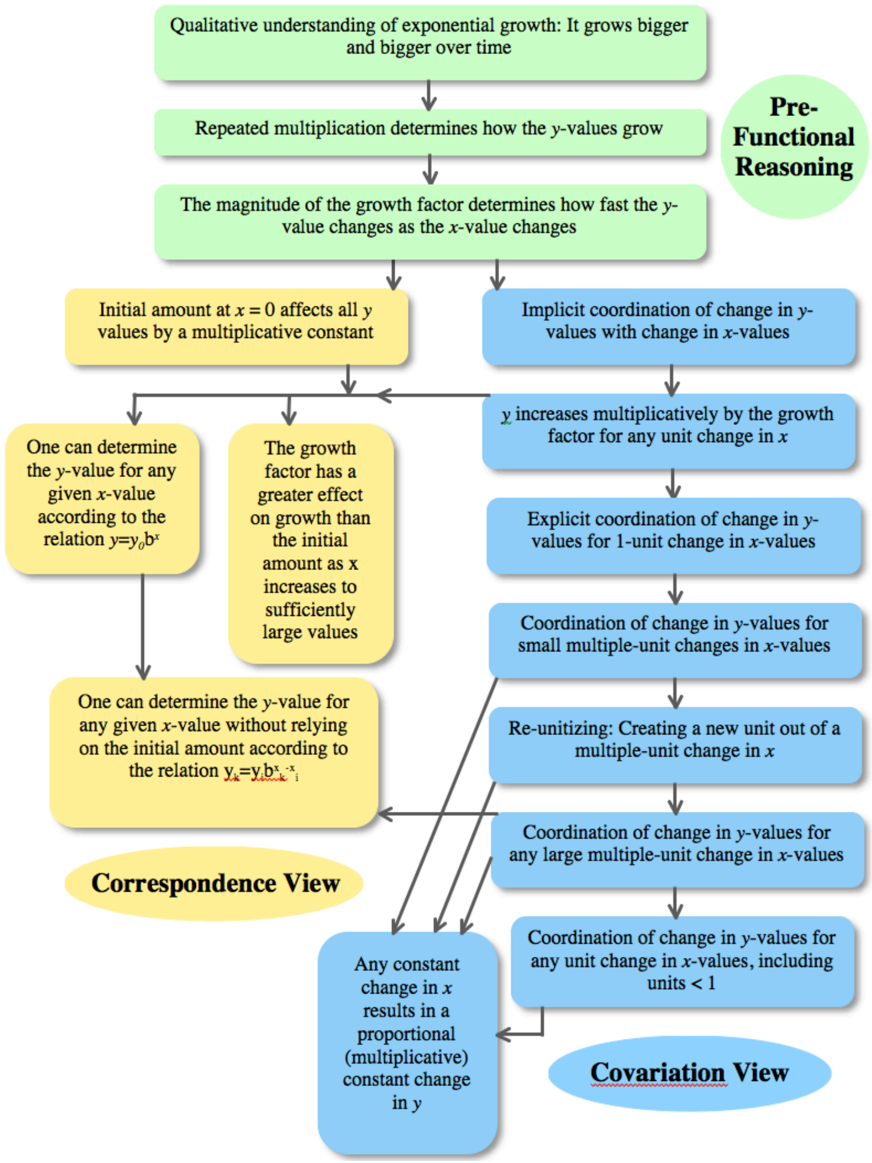

Figure 1: Graphical Representation of Exponential Functions Trajectory

to the connection between the growth of *y*-values and the growth of *x*-values; thus their understanding remained pre-functional. Table 1 summarizes the pre-functional reasoning portion of the learning trajectory:

Concept Definition	Sample Tasks and Data Examples
Qualitative understanding: Understanding that *y*-values grow larger at an increasing rate over time, but manner of increase unspecified.	Manipulate GeoGebra Jactus (via sliders). "It would start, like, small and then it will get, like, really big in a short amount of time."
Repeated multiplication: Understanding that repeated multiplication determines how height (*y*) grows without attending to time (*x*).	Find missing values in a table; Complete far prediction problems for large *x*-values. "They're all going up by like times 4, like, 16 times 4 is 64 and then 64 times 4…that's 1024."
Magnitude of growth factor determines height change: Understanding that the growth factor determines how growth occurs.	Compare plants with different growth rates to determine which grows the fastest. "The tropical [is fastest], because it's growing 4 times and it's more than 2 times and 3 times."

Table 1: Pre-functional reasoning

Early Covariation Reasoning

In an attempt to encourage coordination of the plant's height with the number of weeks it had been growing, we introduced a task to draw the plant's height after 1 week and after 3 weeks if it doubled every week. When asked to predict what would happen for weeks 4 and 5, the students began to coordinate growth in height with weeks, explaining that the height for Week 4 "would be double the last week," (Jill) and the height for Week 5 "would double the week before" (Laura).

As students encountered non-uniform tables of data, they began to coordinate the change in height with the change in weeks for small multiple-week spans. For a table with a gap between Week 15 and Week 18, Uditi divided the plant's height at Week 18 by the plant's height at week 15 and got 64. She then wrote, "___ × ___ × ___ = 64" and determined that the growth factor must be 4. At this stage, students' abilities to coordinate height change with the change in time were reliant on an image of repeated multiplication. Table 2 summarizes the early covariation reasoning portion of the learning trajectory.

Concept Definition	Sample Tasks and Data Examples
Implicit coordination: Understanding that the plant's height grows by a constant multiplicative factor "each time", but the time values are not explicitly quantified.	Non-well-ordered tables containing jumps in week values; Tables in which Δx is not 1. "So here's 8, and then the next is 16 and…it goes up, if that's the rate…"
Multiplicative growth for any unit change: Understanding that for any 1-week time change, the plant will grow by (b).	Predict current height relative to what it was any number of weeks in the past or future. "For each week it goes…times the same number. Times 3, times 3, and the *value* between each week increases as the weeks go on."
Explicit coordination: Coordination of multiplicative growth in height with explicit time changes; time has now been quantified.	Draw the plant each week or with skipped weeks. "In 1 week it's going to be 1 inch and in 2 weeks it's going to be 2 inches, then in 3 weeks it's going to be 4 and 4 weeks it's going to be 8."
Coordination of change in y for small changes in x: Coordination of the ratio of two height values for multiple-week time periods.	Complete non-uniform tables; Determine growth rate given only two point values. "I did 20,615,843,020 ÷ 322,122,547.2 and I got 64. So I tried to figure out what number times itself times itself = 64 and it was 4."

Table 2: Early covariation reasoning

Development of the Correspondence Reasoning

Once the students understood that the growth factor represents the multiplicative change in height per week, they began to express this relationship algebraically. The students also began to shift from thinking about the initial height, a, as simply the unquantified "starting value" to understanding that the initial height could also be viewed as a multiplicative constant (Table 3). The students eventually encountered tasks in which, given two points, they had to determine whether a third point was accurate. If the *x*-values were sufficiently large, this motivated expressing the height of the plant based on an existing height value. For instance, the students worked with a plant given two height values: 956,593.8 inches at 14 weeks, and 8,609,344.2 inches at 16 weeks, and were asked whether the point (1, 0.6) was correct. Uditi wrote, "It is right because $0.6 \times 3^{13} = 956593.8$." Table 3 summarizes the correspondence view portion of the learning trajectory.

Concept Definition	Sample Tasks and Data Examples
Initial height is a multiplicative constant: Understanding that the initial height changes the height at any given week by the multiplicative constant (a).	Compare plants with different initial heights and same growth rates. "Cause it starts out like times that number, like if it's 4 then it's like 4 times as big as the original one's starting number…"
Growth factor has greater effect than initial height: Recognizing that the plant with the larger growth factor will be taller after a large amount of time, regardless of the initial heights.	Compare plants with different initial heights and growth rates to determine which grows the fastest. "The evergreen…because it triples, so when you keep going in the weeks, it's going to be bigger than the one that doubles."
$y = ab^x$*:* Understanding that the relation between x and y can be expressed generally as $y = ab^x$	Determine the plant's height for any given week. "I kept dividing 19,660.8 divided by 4 each time until I got to 0, and then I took 0.3 times 4 to the 15th power." (Writes "$y = 0.3 \times 4^x$")
$y_k = y_i b^{x_k - x_i}$: Understanding that any height value can function as a multiplicative factor provided Δx is adjusted appropriately.	Determine the growth factor given two point values without an initial height. Given points (8; 19,660.8) and (15; 322,122,547.2), a student determines missing b value by writing: "$19{,}660.8 \cdot b^{(15-8)} = 322{,}122{,}547.2$"

Table 3: Correspondence Reasoning

The Covariation Perspective

Tasks with larger gaps between weeks encouraged students to truncate the repeated multiplication imagery. For instance, when students encountered two height values at 4 weeks and 15 weeks and had to determine the growth factor, Uditi took the ratio of the height values, which was 4,194,304, and wrote "$\_\_\_^{11} = 4{,}194{,}304$". She no longer relied on an image of repeatedly multiplying the height at week 4 by the growth factor 11 times to achieve the height at week 15. In the early stages this coordination sometimes relied on a re-unitizing strategy (see Table 4).

In order to foster the coordination perspective for cases in which was $\Delta x < 1$, we asked students to predict how much a plant would grow for different increases in time. For instance, given a plant that triples each week, students had to consider how much larger it would grow in 1 day. Uditi responded to the question with "$3^{.14} = 1.17$", explaining, "I divided 1 week into 7 parts, which represents 1 day each and it's .14 of a week." In this manner Uditi was able to make sense of a non-whole number exponent. Table 4 summarizes the final section of the learning trajectory for the covariation reasoning.

Concept Definition	Sample Tasks and Data Examples
Re-unitizing: Creating a new chunk of time and using that chunk as a unit for multiplicative growth.	Require comparison of the same plant's growth across different data sets with different time units. *(Student creates a unit out of a 3-week chunk)*: "Like since the 8…it's 8 for 3 weeks, 8 times 8…equals 64 for the 6 weeks."
Coordination of change in y for large changes in x: Coordination of the ratio of height values for any time span >1; no longer relying on repeated multiplication imagery.	Determine the growth factor given two data points with a large Δx. "Well for that one we did 3 to the 5th and that was 5 weeks, so then I just doubled that so I did 3 to the 10th because it was double 5, which is 10, then 10 weeks."
Coordination of change in y for change in x < 1: Coordination of the ratio of height values for any time span, even when Δx is < 1. Determine how much a plant would grow in k days given a growth factor (b) provided in terms of weeks, by multiplying a height by $b^{(k/7)}$.	Scenarios with a growth factor in weeks that require determining how much the plant grows in a portion of a week. *(Given a plant that triples every week, determine how much it would grow in 1 day):* "For one week there are 7 days, and I divided 1 by 7…and I got 0.14, so I did 3 over 0.14." [Wrote $3^{0.14} = 1.17$].
Constant change in x yields proportional multiplicative constant change in y: Understanding that for any Δx, the ratio of two heights y_2 to y_1 will be $b^{(x2 - x1)}$ and does not depend on the individual x_1 or x_2 values.	Predict how much larger the plant will grow from week x_3 to x_4 if one knows how much the plant grew from x_1 to x_2, when $x_2 - x_1 = x_4 - x_3$. *Do you think it will always get 8 times as big for any 3-week jump?* "Yes, because 2 x 2 x 2 = 8, like the difference between each week is 2."

Table 4: Covariation Reasoning

DISCUSSION

Steffe (2004) argues that "The construction of learning trajectories of children is one of the most daunting but urgent problems facing mathematics education today" (p. 130). By offering an empirically based learning trajectory focusing on students' initial understanding of exponential functions, we aim to contribute to the field's knowledge of students' learning processes and how they can be supported (Simon & Tzur, 2004). Our findings suggest that situating an exploration of exponential growth in a scenario in which students can manipulate continuously covarying quantities in a dynamic environment fosters their ability to correctly coordinate multiplicative growth in *y* with additive growth in *x*.

In addition, the students' abilities to coordinate the ratio of height values with the additive difference in time values played a significant role in their development of algebraic representations. In general, the students' early covariational thinking preceded their ability to develop correspondence rules of the form $y = f(x)$, which reflects Smith and Confrey's (1994) assertion that students typically approach functional relationships from a covariational perspective first. This study offer a proof of concept that even with their relative lack of algebraic sophistication, middle-grades students can engage in an impressive degree of coordination of covarying quantities when exploring exponential growth.

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BELIEF SYSTEMS’ CHANGE - FROM PRESERVICE TO TRAINEE HIGH SCHOOL TEACHERS ON CALCULUS

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This report focuses on a part of a research programme concerning (prospective) secondary teachers´ beliefs towards their teaching of mathematics and calculus in particular. First the theoretical framework and methodology will be shortly outlined. Afterwards the focus lies on studying these teachers´ beliefs with a particular concern for how these beliefs might change depending on the stage of their professional development. Results from a qualitative study of preservice teachers and teacher trainees will be discussed centered on the nature of beliefs on calculus, their structure being built up and why and how possible changes in their teaching orientation occur.

INTRODUCTION

Teachers are challenged to consistently enhance their professional knowledge as well as to change or to modify their beliefs beyond a status quo achieved after university studies (Clarke & Hollingsworth, 2002). This individual, everyday and life-long learning aims to achieve the overarching objective for mathematics instruction, i.e. to seek a teaching style that facilitates students’ learning as best as possible.

Subsuming this life-long learning process referring to pre-service and inservice teacher training to the construct of teacher change (Hannula & Sullivan, 2008), this learning process concern both knowledge and beliefs with respect to mathematics as a school discipline and the teaching and learning of mathematics (Oliveira & Hannula, 2008). Research in the field of teacher change has identified a crucial point in time concerning a possible change when teachers acquire their first experiences in the classroom (Huberman, 1989; Oliveira & Hannula, 2008). This change occurring in the interplay of existing beliefs and classroom experience (Zaslavsky & Linchewsky, 2008) refers particularly to the teachers’ knowledge and beliefs relating to teaching and learning mathematics (Oliveira & Hannula, 2008).

For this reason, our focus in this paper is on mathematics beliefs of teachers that we attend from the end of their university studies to their commencing teacher career. In Germany, this time lasts two years in which teacher trainees pass through a special programme involving partly self-dependent teaching practice partly teaching practice that is guided by a mentor. As part of a larger research programme (Erens & Eichler, 2012) aiming to investigate secondary teachers’ beliefs concerning a specific discipline (cf. Franke et al., 2007), we regard teacher trainees referring to their beliefs about the teaching and learning of calculus. For this, we briefly outline our theoretical framework and method that is relevant for this paper. Afterwards we discuss results of our research.

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THEORETICAL FRAMEWORK

Although a teacher change concerns both knowledge and *beliefs*, the focus in this paper is on teachers' beliefs that represent the teacher's individual convictions referring to the teaching and learning of mathematics and that are "inextricably intertwined" with the teachers knowledge in respect to these issues (Pajares, 1992, 311).

Following Green (1971) beliefs are organized in *belief systems*, including belief clusters that can coherently be connected, but could also be contradictory (quasi-logicalness). Further, beliefs or belief clusters could be central, i.e. strongly held by a teacher or, by contrast, peripheral. According to the construct of subjective theories, which is defined similar to belief systems (Groeben et al., 1988), belief systems represent teachers' teaching objectives.

Further, teachers' beliefs representing overarching teaching objectives are called teachers' (world) *views* (Grigutsch et al., 1998) involving

a formalist (world) view, i.e. a teacher emphasizes the logical, formal and deductive nature of mathematics;

a process-oriented view, i.e. a teacher emphasizes mathematics as a heuristic and creative activity that allows solving problems using individual ways;

an instrumentalist view, i.e. a teacher emphasizes the "tool box"-aspect of mathematics consisting of formulas and procedures to be memorized;

an application oriented view, i.e. a teacher emphasizes the utility of mathematics for solving real world problems.

On the basis of this theoretical framework, our research question for this paper is how possible belief changes of prospective teachers in the time span from the end of university studies to the beginning as school teachers can be described.

METHOD

In this report we refer to a sample of 10 pre-service teachers, who have just completed their mathematics undergraduate courses at university, and 10 teacher trainees that have participated in the special teacher education programme between university and the professional career in school for about one year and thus have gained their first experience with self-dependent and guided teaching in classrooms. All teachers are secondary teachers that have attended university courses with an emphasis on mathematics and few courses on mathematics education or pedagogy (cf. Oliveira & Hannula, 2008).

Data were collected by in-depth interviews lasting about two hours focusing in particular on calculus and including different topics referring to calculus and mathematics, e.g. instructional content, teaching objectives, reflections on the nature of calculus as a mathematical discipline and as an issue of school mathematics, the students' views, or textbook(s) used by the teachers. Further, we use prompts to provoke teachers' beliefs, e.g. tasks of textbooks, fictive or real statements of teachers or students concerning instructional objectives (cf. Erens & Eichler, 2012).

Data analysis is based on a coding approach that is near to grounded theory (Strauss & Corbin, 1998). In this paper, we report only those beliefs or beliefs clusters that we have identified as, at most temporarily, central for the teachers (Erens & Eichler, 2012).

RESULTS CONCERNING PRESERVICE TEACHERS' BELIEFS

The preservice teachers of our sample, who have just completed their mathematics undergraduate courses at university, have no experience in classroom except of an internship (usually after the initial year at university) that is mainly of observational nature. For this reason, these teachers did not refer to their classroom experience, but refer mostly to an *authority* when they think about a possible classroom practice or teaching objectives.

One authority is an anonymous teacher represented by the preservice teachers' retrospection of their own schooldays.

> Mrs. R: In my calculus course at school we learned procedures to be able to calculate the given tasks – of course it was very schematic, but as a student you were on secure ground. […]

> Mrs. M: When I think back to my own calculus course at school, I can remember doing 20, 30, 40 curve sketchings that we just calculated. Worked perfectly well, but I didn´t really appreciate that – it just made no sense. Making sense of the mathematical methods is something I would like to get across to my students.

Although such retrospection might not necessarily be valid, it gives evidence about the contemporary state of these teachers' beliefs. There are prospective teachers like Mrs. R who combine an instrumentalist view in some sense with a positive attitude represented by the belief that a schematic way of teaching could give students certainty. Although Mrs. M also remembers her calculus course as schematic, she seems to overcome this negatively connoted retrospection. This could serve as evidence that for this teacher a change of perspective has already taken place, i.e. a change from mathematics as schematic tool (instrumentalist view) to something ambiguous that is meaningful, but non schematic. Comparing both teachers, it is striking that Mrs. R acknowledges the *authority* represented in her retrospection whereas Mrs. M does not.

Another aspect of *authority* for prospective teachers concern university mathematics:

> Mr. G: Well, I daresay I could do calculus at school with a more theoretical and formal approach – similar to introducing concepts in algebra and topology. Maybe for some it would make things easier, but this will probably not be possible to implement in most courses.

Like Mr. G some of the prospective teachers hold a strong formalist view of mathematics and suggest teaching according to this view later. However, the scope of answers in the transition from university to prospective teaching ranges from emphasizing a formalist view (formal aspect is most important characteristic of calculus) to a complete rejection of formalism (particularly negative) that is illustrated by the case of Mrs. R.

> Mrs. R: In my university courses it was always hard for me to understand these proofs – especially in calculus. I have always tried to avoid these very formal things and I will probably do that as a math teacher, too. […]

Later in the interview, Mrs. R was challenged with a different statement to the teaching objectives on calculus. Confronted with the assertion: “Formal rigor and precision is a necessary ingredient of calculus” she responded:

> Mrs. R: No, not in my opinion. That is contradictory to my stance on calculus. Of course one has to be precise and to some degree things have to be formally correct, but that is not a prerequisite of good calculus teaching.

The case of Mrs. R shows in some sense a failed attempt of belief change. Thus, she resisted adopting a formalist view in her university studies and seems to retain a view concerning mathematics that was formed in her schooldays (see above).

A mentioned above, *authority* that is represented by a retrospection (*looking back*) referring to an anonymous teacher from schooldays and university studies seem to be a crucial source for pre-service teachers’ beliefs. However, the teachers also make a prediction (*looking ahead*) to their own practice as teachers of calculus (cf. Skott, 2001). For this prediction, the teachers use a *self-referred reflection*, i.e. the teachers seem to make an inference from their own learning to the learning of their future students. For example, Mrs. R mentions her difficulties with university mathematics (representing a formalist view) and her certainty experienced in her classroom that represents an instrumentalist view. Based on this experience, she makes an inference that an instrumentalist teaching orientation would potentially be the best way for her students. In the same way, Mr. G contemporarily favors a formalist view on teaching calculus. The aspect of *self-referred reflection* seems to serve further as a basis to acknowledge or to reject an authority. For example, Mrs. R acknowledges her (anonymous) school teacher representing an instrumentalist view, but rejects university mathematics representing a formalist view as basis for her future practice as teacher of calculus.

Although all teachers refer to both an *authority* and a *self-referred reflection*, in general, it is an expected result that the prospective teachers’ belief systems are quite more inconsistent than the belief systems of experienced teachers that we have regarded elsewhere (Erens & Eichler, 2012). Thus, before gaining first classroom experience and further pedagogical assistance by teacher trainers prospective teachers are particularly unconfident about a number of relevant teaching objectives and also methods to achieve these objectives:

Mrs. M: How I would actually teach derivation rules? Of course we did these things very formally at university- with proofs and so on. Well, to be honest I cannot yet say how I would actually teach that.

Regarding their instructional orientation, i.e. a transmission view or a constructivist view (Staub & Stern, 2002), there is an even greater uncertainty than to other views like the instrumentalist and formalist view. Some prospective teachers value a clearly structured and teacher-centered way in their own lessons as student, others clearly focus on the students and their preferences of effective learning strategies.

RESULTS CONCERNING TRAINEE TEACHERS' BELIEFS

In changing the perspective from a prospective teacher to that of a teacher, our data sample suggests that in the transition from university education to classroom experience there occurs a key change in teachers´ beliefs and a confirmation or rejection of already existing beliefs takes place. Thus, the teacher trainees show a shift from *self-referred reflection* to *reflection* referring to their classroom practice and the learning of their students. In accordance with Cooney (1998), reflection plays an important role in the growth of prospective teachers and can be regarded as a change initiated by own judgment based on first experience (shock of the first practical experience) on the one hand and a composition of skepticism (challenging existing concepts) and desire for change on the other hand:

Mr. C: An example from my lessons? Well, I introduced derivatives just in the way we did it in our own calculus course at school, starting with a secant then approximating it to the tangent and so forth, .., in retrospect I would say in future lessons I won´t emphasize that so deeply; it will be rather more interesting for students to start with an application-oriented example and then spend more time doing exercises.

Starting with existing and known concepts originating from a retrospection regarding his schooldays, Mr. C notices that enacting this conceptualization is not in accordance with his own beliefs. Based on his *reflection* of his first practical experience on teaching calculus, he decides for a change in his (future) classroom practice and, thus, for a possible rejection of an *authority* represented by his retrospection to his schooldays.

Although teacher trainees like Mr. C seem to change their beliefs concerning the teaching and learning of mathematics due to their *reflection*, they also seem to struggle with respect to differences between their beliefs and their classroom performance. This possible cognitive conflict might be seen in connection with the subjective perception of uncertainty and the need to make their lessons manageable:

Mr. C: Accuracy is an essential point in my lessons; there are some lesson phases where I teach according to the principle of direct instruction – not because I think that is the right way, just because at the moment I do sometimes not find another way to get my lessons working.

Throughout the interview Mr. C stated that a dialogic discourse with his students is of central importance to him as he favors math lessons with lots of self-contained student activity and methodological variation, but he experiences many obstacles in developing that kind of discourse in his actual practice. In essence he recognizes that he returns to a transmission approach (Staub & Stern, 2002) in order to structure and control the class. By reporting about this conflict there emerges a conflict between his more constructivist teaching orientation and his uncertainty of teaching practice at this point of his career.

Reflection on and challenging of existing concepts can also be unveiled in connection with the intervention of the teacher training which attempts to form and reshape trainees´ practice:

> Mr. G.: In conceptualizing new content I always use a task-oriented approach, which is a guideline given by our teacher trainers. In my opinion it´s not bad, but I think it´s too stringently guided like our trainers want it to be implemented. […] From time to time I vary a little bit, but at the moment I must keep in mind my demonstrative exam lessons with my students. […] however in doing so the teacher guidance is quite high so I sometimes think I could just demonstrate the tasks myself.

In his mathematical socialization Mr. G can be characterized to favor a rather formalist view of mathematics. Reflecting on his first teaching experience (*reflection*) this formalist view is being challenged and a process of nuanced replacement takes place which is (partly) initiated by theoretical input of his teacher training course (*authority*). Thus, the aspect of *authority* is mingled with the aspect of *reflection* here. The implementation of a specific teaching concept is an instrument of nuanced change: On the one hand Mr. G. expresses alignment with the teacher educator´s idea on the other hand his reflection shows that a cognitive process of individual adaption concerning his teaching orientation is under way.

It is one of the functions of teacher education courses to challenge and correct existing beliefs that contradict the view of appropriate classroom practice of teacher trainers. Thus, when teacher trainees are faced to make pedagogical decisions, their teacher trainers want them to hold certain beliefs and they also want those beliefs to influence practice. Again, it is the question to what degree this intervention of an *authority* effects change in trainee´s beliefs, which is dependent on the individual teachers´ confidence and willingness to identify themselves in the cognitive process of matching intervention and classroom experience. As can be seen with Mr. G, he casts doubts on the method of the task-oriented approach which he is expected to enact in his demonstrative exam lessons. However, these teacher training exams are a decisive criterion for the future career, i.e. whether a permanent teaching position will be offered to him. His personal beliefs, however, are not in accordance with the teacher trainers´ views, and thus his future enacted curriculum will presumably differ from the task-oriented approach. The awareness of the *authority* of teacher trainers can be validated in all the interviews of our sample, as the following example confirms:

> Mr. C.: Quite important to me is that my lessons are well-structured; and that this structure is visible especially to my teacher trainers

DISCUSSION AND CONCLUSION

Beliefs are often robust and “many of teachers´ core beliefs need to be challenged before change can occur” (Sowder 2007, p.160). Research on teacher change has acknowledged that any change or development in teachers´ beliefs is a long-term process (Oliveira & Hannula, 2008). However, looking at possible points of rupture in the education of future teachers in more detail, it is possible to identify deep-seated beliefs in their mathematical and pedagogical socialization as well as some mechanisms of belief changes.

Regarding our sample of teachers, it seems possible to partly ascertain two factors of potential belief changes in the context of the teacher training intervention which are intertwined with each other, i.e. *authority* and *reflection.*

- *Authority* could involve different issues like the (anonymous) teacher of a prospective teacher’s own schooldays, mathematics experienced in university studies, the teacher trainer and also – maybe later – colleagues, textbooks or administrative conditions.
- *Reflection* could be based on a (self-referred) reflection including the experience of a prospective teacher as student in school or university, but in particular the reflection of teaching and learning in the teachers’ first practical experience.

Firstly, a fundamental change involves the shift from self-referred reflection to reflection of the classroom practice. This change also seems to impact on the teachers’ perception of authorities: Whereas prospective teachers at the end of their university studies refer to an authority connected to their own learning (like Mrs. R), teacher trainees take into account several authorities, but adjust the views represented by these authorities with their classroom experience (like Mr. G).

In order to initiate change in teaching mathematics in the course of teacher education and development, it is possible to accentuate two aspects concerning prospective teachers´ beliefs. Firstly the identification and understanding of beliefs (and the belief system) of future teachers can be accomplished by investigating their special perspective and the context of how individual beliefs have been formed. We will investigate the development of the prospective teachers regarded in this paper by subsequent interviews. Examining the generative process in which the existing belief system of pre-service teachers is connected to the new perspective and (teaching) context may facilitate to determine approaches to influence (desired) changes that have an impact on (future) enacted curricula.

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TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING EQUALITY

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This paper has a focus on what can be learned about teachers' knowledge by analyzing their responses and written reflections to items developed to measure mathematical knowledge for teaching (MKT). 30 teachers' responses, as well as written reflections, to one testlet (five items) are analyzed and discussed. The items focus on relational and operational understanding of the equal sign. The results indicate that analyses of teachers' written reflections provide a richer picture of teachers' knowledge than analyses of responses to the multiple-choice items only. It also appears that teaches draw upon different domains of MKT in their written reflections than the items were developed to measure.

INTRODUCTION

Researchers at the University of Michigan have—with their concept "mathematical knowledge for teaching" (MKT) (Ball, Thames, & Phelps, 2008) and measures of teachers' MKT (Hill, Sleep, Lewis, & Ball, 2007)—contributed to the long-lasting discussion about the knowledge needed to teach mathematics. They claim that MKT, as assessed by their measures, makes a difference to the mathematical quality of instruction (Hill et al., 2008; Hill & Charalambous, 2012). Teachers with high MKT elicit, understand and build on student thinking; teachers with low MKT struggle more in their teaching and are unable to follow and build on student thinking (e.g., Hill & Charalambous, 2012).

The MKT measures have been criticized by many, and Schoenfeld (2007), who is one of the critics, has argued that the MKT items should be opened up. Fauskanger and Mosvold (2012) attempted to open up the items by letting the teachers add their written reflections to a selection of multiple-choice items. Their analysis revealed a mismatch between teachers' responses to the multiple-choice items and their written reflections. In this paper, we have taken this one step further when we aim at answering the following research question:

What can be learned about teachers' knowledge of the equal sign by analyzing their responses and written reflections to MKT items?

In order to answer this question, we analyze the responses and written reflections given by 30 teachers to one testlet (including an item stem and five items). If instruction forms the basis for students' lack of sophisticated understanding of the equal sign (e.g., Asquith, Stephens, Knuth, & Alibali, 2007; Behr, Erlwanger, & Nichols, 1980; Kieran, 1981), and most children are capable of a relational understanding of the equal sign if they are given appropriate experience in a supportive context (Seo & Ginsburg, 2003), then it can be argued that teachers' knowledge for teaching meanings related to the equal sign is important.

THEORETICAL FOUNDATION

Several frameworks for teacher knowledge have been developed, and these frameworks differ in important ways. For the purpose of this paper, we use the MKT framework as a theoretical foundation (e.g., Ball et al., 2008). We adopt the broad conception of knowledge that is used by Ball and colleagues (ibid.)—including skills, habits of mind and insights. This corresponds with the view of Beswick, Callingham and Watson (2012), who contend that beliefs and confidence are included in teacher knowledge. The MKT framework builds upon Shulman's (1986) conception of teacher knowledge. One of his concepts—subject matter knowledge—has been divided into three domains. Common content knowledge, which is one of these domains, represents knowledge that is acquired by most educated people. Another domain, horizon content knowledge, involves being cognizant of the large mathematical landscape in which the present instruction is situated; the equal sign's relation to learning algebra (Carpenter, Franke, & Levi, 2003; Knuth, Stephens, McNeil, & Alibali, 2006) can be regarded as part of primary teachers' knowledge at the horizon. The third domain—specialized content knowledge—is defined as "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400). Relational as well as operational understanding of the equal sign—as represented by the testlet used here—might be considered as part of teachers' specialized content knowledge.

Pedagogical content knowledge is divided into three domains where knowledge of content is combined with knowledge of students, teaching and curriculum. Knowledge of content and students is important in order for teachers to be able to predict students' misconceptions related to the equal sign, and knowledge of content and teaching is, among other things, needed to decide how to help students correct these misconceptions. One example of such misconceptions is to view the equal sign as a 'do something signal' (Kieran, 1981).

Prediger (2010) showed that diagnostic competence comprises didactically sensitive mathematical knowledge for teachers. Limited understanding of the meaning of the equal sign is regarded as one of the major stumbling blocks in learning algebra (Carpenter et al., 2003; Knuth et al., 2006), and students do not seem to develop this understanding properly throughout the school years (Knuth et al., 2006). In categorizing different meanings of equality, Kieran (1981) emphasized the important distinction between the following two meanings for equality: the operational (asymmetric) and the relational (symmetric) meaning. These categories were then extended by Prediger (2010), who proposed that relational meaning is divided into four: 1) symmetric arithmetic identity (e.g., "$5 + 7 = 7 + 5$), 2) formal equivalence (e.g., "$x^2 + x - 6 = (x - 2)(x + 3)$), 3) conditional equation (e.g., "Solve $x^2 = -x + 6$") and 4) contextual identities in formulas (e.g., $V = 1/3\pi \cdot r^2 \cdot h$). The items in focus in this paper relate to symmetric arithmetic identity in particular. The testlet also includes operational meaning, since the items are developed to reveal whether or not teachers only regard the equal sign as a “do something signal” without also keeping the relational meaning of the sign in mind.

METHODS

Previous analyses indicate that written reflections related to MKT items give insight into teachers’ knowledge as well as their beliefs (e.g., Fauskanger, 2012; Fauskanger & Mosvold, 2012). In this study, 30 teachers were asked to give their written responses as well as reflections to 28 MKT items (6 testlets and 10 item stems) at home. These teachers were teaching different grade levels, but they were all participating in the same professional development course—the course was lasting one year and a half—and the written work reported on was given as an assignment after their first day in this course. Their teaching experience varied from less than 5 years to more than 20 years, and their formal education in mathematics/mathematics education varied from 0 to 60 ECTS.

The testlet we focus on in this paper includes an item stem presenting the context of a teacher who was supposed to consider whether or not five given statements were mathematically correct. The statements presented were of the following kind: $8 + 15 = \_\_ + 9$ and $14 + 5 = 19 + 5 = 24 + 5 = \_\_$. In addition to the items, the following questions were added for written reflection: 1) Does the testlet reflect a content that is relevant for the grade(s) in which you teach? 2) Do these items reflect knowledge important for you as a teacher? For both follow-up questions, the teachers were asked to elaborate on why they did/did not think so. The teachers were also asked to reflect upon which of the ten items they thought best capture knowledge important for them as teachers.

We began by identifying what was written related to the testlet in focus, followed by a directed/theory driven approach to content analysis (Hsieh & Shannon, 2005). This analysis was based on the literature regarding teachers’ MKT related to the equal sign. For the present analysis, the teachers’ writings were coded on the following dimensions:

Subject matter knowledge

- Common content knowledge
- Specialized content knowledge (operational understanding and relational understanding/symmetric arithmetic identity)
- Horizon content knowledge

Pedagogical content knowledge

- Knowledge of content and curriculum
- Knowledge of content and students (diagnose students’ misconceptions)
- Knowledge of content and teaching

RESULTS AND DISCUSSION

In this section, we first give a presentation and discussion of the teachers' responses to the items in the testlet. Then we present and discuss teachers' written reflections, before we end up in a discussion (and conclusion) of how these two aspects are related.

When looking into the teachers' responses, we learn that 26 out of the 30 teachers gave a correct response to all the five items in this testlet. Jan (5^{th}-7^{th} grade), Frøya (6^{th} grade) and Inge (7^{th} grade) gave a wrong response to one of the items, whereas another teacher, Mons (8^{th}-10^{th} grade), responded incorrectly to three of the five items. Jan, Frøya and Mons responded that the item $14 + 5 = 19 + 5 = 24 + 5 = \_\_$ is not mathematically problematic. It might be that these teachers focus on the operational meaning of the equal sign (e.g., Prediger, 2010) and thus look at the mathematical statement in three parts: $14 + 5 = 19$, $19 + 5 = 24$ and $24 + 5 = \_\_$. When doing that, all three parts—when seen in isolation—can be regarded as mathematically unproblematic. Another explanation could be that the teachers see this as an "equality string" (Knuth et al., 2006); such strings are frequently used by students. Adding the relational meaning of the equal sign, or in particular "symmetric arithmetic identity" (e.g., Prediger, 2010), would have helped these three teachers identify this item as mathematically problematic.

Mons gave a wrong response to the following item: $10 – 7 = 3 + \_\_$. This might be due to operational as well as relational understanding of the equal sign. Operational if $10 – 7 = 3$ is looked upon as "solved" and relational if he does not recognize $10 – 7 = 3 + 0$ as the symmetric solution making this item mathematically unproblematic. The reason why Mons also responded incorrectly to $6 – 2 = \_\_ + 7 = \_\_ + 5 = 16$ might be due to his operational understanding of the equal sign, making him think of the following solution to the item: $6 – 2 = 4 + 7 = 11 + 5 = 16$, not recognizing that such an "equality string" (Knuth et al., 2006) involves an incorrect use of the equal sign. Another possibility is to look at the first part of the item only: $6 – 2 = -3 + 7 = -1 + 5$, and forgetting the $= 16$. When Mons responded incorrectly to three out of the five items in this testlet, it might indicate that his "local MKT" (Hill & Charalambous, 2012) related to equality is relatively low—and in particular his understanding of the relational meaning of the equal sign.

By looking at the teachers' responses to the items only, a conclusion might be that most of them—at least the 26 out of 30 who gave correct responses—have proper understanding of what the items were intended to measure. When analyzing the teachers' written reflections, however, we get a richer picture of the teachers' MKT. The reflections from 25 of the teachers indicate a relational understanding of the equal sign; Dina's (2^{nd} grade) written reflections represent one example of this. When reflecting on which of the ten items that best captured knowledge important for her as a teacher, Dina highlighted the knowledge represented by this testlet. She argued that the five items represent knowledge important at the grade level she teaches, and she wrote:

> The pupils frequently encounter tasks where they are supposed to fill in the correct number on the empty lines. This is a preliminary stage of algebra, and it is important that such tasks

> are introduced early in elementary school. The pupils have to learn about the equal sign. How does one interpret this sign? The equal sign can be interpreted as a sign requiring action—indicating that they should calculate—like in the following examples: 16 + 5 =__, 22 + 19 =__, 146 - 24 = __.
>
> In addition, it is also very important that the pupils get to understand that = means “the same as”. The equal sign means that the value of the expressions on each side should be the same. The pupils then need to experience that the mathematical expression could equally well be placed on the right hand side. The pupils should meet both in tasks that are given to them. They must get the opportunity to explore that 9 can be divided as the sum of 5 and 4. Yesterday, Per walked 9 kilometers. First, he walked 5 kilometers, then he walked 4 kilometers. 9 = 5 + 4.

In these reflections, Dina showed operational as well as relational understanding of the equal sign, and she argued that both are important aspects of her MKT. The way she was highlighting the items’ focus as a preliminary stage of algebra might be related to her knowledge at the mathematical horizon (Ball et al., 2008). It can also be viewed as an indication that she might want to prevent her students from one of the major stumbling blocks in learning algebra—limited understanding of the meaning of the equal sign (Carpenter et al., 2003; Knuth et al., 2006). In their reflections, 12 teachers discussed how the content of these five items can be seen as a precursor to algebra. Referring to the MKT model again, this can be seen as a link to horizon content knowledge (Ball et al., 2008). As an example of this, Oda (4th grade) wrote:

> In elementary school, where I work, these tasks would be relevant. We start by splitting numbers like 7 = 3 + 4 etc. So, already in first grade, we have a language to talk about this. Early on, the pupils need to know what the sign represents. Being conscious about this already from the beginning, we can develop a solid foundation for later use in more advanced equations.

Although the testlet has a focus on content knowledge, teachers also draw upon different aspects of pedagogical content knowledge in their reflections. Some teachers refer to the textbook in their written reflections. Klara (2nd grade) wrote about how the equal sign was represented by a balance scale. The image of a balance scale, she wrote, is probably not something the pupils have previous experiences with, and they might find it rather abstract still. This is an example of how teachers, in their written reflections, argue by drawing upon knowledge of content and curriculum (Ball et al., 2008). Further on in her reflections, Klara also demonstrated experience with pupils and teaching in relation to the equal sign when she wrote that pupils are often familiar with number sentences where the equal sign indicates: “here comes the answer”. She also contended, however, that “the problem often becomes more obvious when we leave out an addend as in 2 +__= 4”. Other teachers presented examples from the classroom and provided reflections that were explicitly related to knowledge of content and teaching (Ball et al., 2008). Carla (1st grade) wrote about the importance of learning the meaning of the equal sign, and she related this to her own experiences with letting the pupils create story problems:

> Per has 3 balloons and two burst. How many is left? You can also do this the other way round. Per had three balloons when he went out, but when he came back home, he only had one left. How many had burst? It is important that the pupils take part in creating their own story problems.

Several teachers drew upon knowledge of content and students in their reflections, and they focused in particular on the importance of getting to know about and identify pupils' misconceptions about the equal sign. In her reflections, Gerd (2nd grade) wrote: "In such tasks, one can see if the pupils have an understanding of the = sign. Many pupils don't distinguish between these signs. In this, they show a lack of knowledge and understanding of what the = sign actually means." Another teacher, Frøya (6th grade), wrote: "Pupils who sit and work together with an 'unknown' talk about what they know. This is something I, as a teacher, can take advantage of, listening to how they get to a solution." In these reflections, she discussed the teachers' ability to listen – and showed that she values diagnostic competence (e.g., Prediger, 2010) – which is important in relation to knowledge of content and students (Ball et al., 2008).

CONCLUSION

Most children are capable of developing a relational understanding of the equal sign if they are given appropriate instruction (Asquith et al., 2007; Behr et al., 1980; Kieran, 1981; Seo & Ginsburg, 2003) and it can thus be argued that investigations of what mathematical knowledge teachers need for teaching this is important. When analyzing teachers' responses to multiple-choice items, like the ones we have in focus in this paper, we learn something about teachers' MKT. By opening up the items and letting teachers add their written reflections, we can learn even more about the teachers' knowledge than through analyses of their responses to the multiple-choice items only.

The items we have in focus in this paper were created to measure aspects of teachers' mathematical knowledge for teaching equality, and analysis of the responses made by the teachers in our study might lead us to the conclusion that most of these teachers had a proper content knowledge in this respect. In a previous study, we found a mismatch between teachers' responses to multiple-choice items and their written reflections (Fauskanger & Mosvold, 2012), but such a mismatch did not emerge from the analysis reported in this paper. Our present analysis did, however, provide us with further insight into how teachers might draw upon different aspects of MKT when responding to such a multiple-choice item—aspects different from the ones the items were developed to measure. This finding is relevant for researchers who are involved in the validation and continued adaptation of items (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012).

Studying teachers' knowledge is a complex enterprise. Our results indicate that even though you can successfully investigate such knowledge by the use of multiple-choice items, a richer picture will be given by adding teachers' written reflections. Further studies of written—and possibly also oral—reflections in relation to the further development, adaptation and use of MKT items are relevant. We also contend that such studies might be important for the further development and refinement of the MKT framework as such.

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DOES THE CONFUSION BETWEEN DIMENSIONALITY AND "DIRECTIONALITY" AFFECT STUDENTS' TENDENCY TOWARDS IMPROPER LINEAR REASONING?

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The aim of this research is to understand the way in which students struggle with the distinction between dimensionality and "directionality" in the context of the relations between length and area of enlarged geometrical figures. 131 third grade secondary school students were confronted with a test consisting of six problems related to the perimeter and the area of an enlarged figure. Results indicate that more than one fifth of the students' answers were directional, suggesting that students struggled with the distinction between dimensionality and "directionality". A single arrow showing one direction (image provided to the students) seemed to help students to see a linear relation for the perimeter problems. Two arrows showing two directions helped students to see a quadratic relation for the area problems.

THEORETICAL AND EMPIRICAL BACKGROUND

Linearity is a powerful tool to model real-life situations, even if these situations are only approximately linear. For that reason, one major goal of mathematics education at all levels is to obtain both procedural and conceptual understanding of linearity in its variety of forms and applications. However, the educational attention that goes to linearity at numerous occasions in students' school careers, along with the intrinsically simple and intuitive nature of the linear model (Rouche, 1989), has a serious drawback: It may lead to a tendency in students to see and apply linearity everywhere, thus also in situations that are not linear at all. Already in 1983, Freudenthal warned for that pitfall: *"Linearly is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear"* (p. 267). Examples of the misuse of linearity can be found at different age levels and in various mathematical and scientific domains (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2012; for a review, see Van Dooren, De Bock, Janssens, & Verschaffel, 2008).

Misuse of linearity: A geometrical context

One of the best-known and most frequently investigated cases of students' misuse of linearity relates to application problems about the effect of an enlargement or reduction of a geometrical figure on its area or volume. The principle governing this type of problems is that an enlargement or reduction with factor k enlarges all lengths (and thus also the perimeter) with factor k, the area with factor k^2, and – for a solid – the volume with factor k^3. A crucial aspect in understanding this principle is the insight that these factors only depend on the dimensions of the magnitudes involved (length, area, and/or volume) and not on the type of figure involved (square, triangle, circle, cube…).

During the last two decades, students' tendency to treat relations between length and area or between length and volume as linear instead of, respectively, quadratic and cubic, has been extensively studied (e.g., De Bock, Verschaffel, & Janssens, 1998; Gagatsis, Modestou, Elia, & Spanoudes, 2009; Modestou, Gagatsis, & Pitta-Pantazi, 2004). In the study by De Bock et al. (1998), for instance, 12- to 16-year-old students were administered paper-and-pencil tests with word problems related to lengths, perimeters, areas and volumes of different types of figures. More than 90% of the 12-year-olds and more than 80% of the 16-year-olds failed on area problems because they applied linear methods.

Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2004) developed and implemented a lesson series with the aim to break 9th graders' tendency to give linear responses in non-linear situations, more specifically in the context of the relationships between the linear measures of a figure and its perimeter, area and volume. It was found that non-linear relations and the effect of enlargements on area and volume remained intrinsically difficult and counterintuitive for many students even after extensive instructional attention. It was also shown that students who, by the end of the lesson series, finally understood that the length-area relationship is quadratic, suddenly started to doubt about the nature of the linear length-perimeter relationship. The authors exemplified this with a striking question raised by a student in the final lesson: *"I really do understand now why the area of a square increases 9 times if the sides are tripled in length, since the enlargement of the area goes in two dimensions. But suddenly I start to wonder why this does not hold for the perimeter. The perimeter also increases in two directions, doesn't it?"* (p. 496). This quote suggests that the student struggled with the distinction between dimensionality and "directionality" (the perimeter of a square is one-dimensional, but it has two "directions"). So we wondered if this type of potential confusion between dimensionality and "directionality" could be an important factor affecting students' tendency towards improper linear reasoning.

Dimensionality

Although dimensionality is crucial to many parts of mathematics and science, research about this concept is scarce. Freudenthal (1983) stated that "dimension is an indispensable tool if magnitudes and their mutual relations are at stake" (p. 266). He pointed out that in measuring magnitudes it is critical to know what kind of magnitudes they are (length, area, volume…), and at this point the dimension has an important role: What dimensions does the object to be measured have? Moreover, it should be stressed for didactical reasons that "the behavior of geometric measures under geometrical multiplication depends on the dimension" (p. 267).

Area measurement is particularly interesting because it involves the coordination of two dimensions. There is extensive evidence that both primary and secondary school students have inadequate understanding of area and area measurement. For example, Carpenter et al. (1988) showed that almost half of a sample of Grade 7 students could calculate the area of a rectangle when given both dimensions; however, only 13% applied their knowledge of the area formula to a square, even when they knew that the sides of a square are equal.

The experiential origin of the area formula is the action of physically covering a rectangle with unit squares. But whereas this action is one-dimensional and involves an additive process, the formula is two-dimensional and multiplicative. Outhred and Michelmore (1996) showed that when students were covering a rectangle with squares (the unit to measure the rectangle), many Grade 1 students did not see the importance of joining the units so that there were no gaps, and drew units individually. Until students began to join the units in two dimensions, they did not usually align rows and columns. Before drawing arrays using only lines, some students drew lines across the width of the rectangle to indicate rows and marked off the units in each row individually while others drew some individual units (usually the top row and the left column) as a guide for drawing the array. So, drawing lines in one dimension appeared to be a precursor to recognise rows as composite units. Such recognition helped students to perceive that squares could be constructed by joining lines in the other direction, and hence realise the two-dimensional structure of an array. In another study, Outhred and Michelmore (2000) focused on understanding the relationship between the size of the array and the linear dimensions of the rectangle in which it is enclosed. They found that, although the fact that the number of units in the array must depend on the measurements of the sides may seem self-evident to adults, it is clearly not obvious to children.

PROBLEM STATEMENT

To sum up, there is quite some evidence showing that students struggle with understanding dimensionality. A major claim underlying the present study is that one reason for that struggle is that students confuse the dimensions of an object or magnitude with its "directions". In this study, we will empirically investigate the existence and impact of this potential confusion. Of course, "directionality" is not a genuine mathematical term, but we use it for referring to the different directions a geometrical (plane) figure has. For example, a triangle has three directions, a square has two directions (if we assume that parallel sides have the same direction), and a regular pentagon has five directions.

So, the aim of this research is to unravel the extent to which and the way in which students struggle with the distinction between dimensionality and "directionality" and how this may affect their tendency towards improper linear reasoning in the context of the relations between length and area of enlarged geometrical figures.

METHOD

131 third grade secondary school students (14-15-years-olds) from four different Spanish schools participated in the study.

These students were confronted with a test consisting of six problems related to the perimeter and the area of an enlarged figure: Two problems were about an equilateral triangle (one related to its perimeter and the other to its area), two problems were about a square (again, one related to its perimeter and the other to its area), and two problems were about a regular pentagon (again, one related to its perimeter and the other to its area). For each problem, students had to choose the correct answer from three given alternatives. Independent of the fact it was an area or a perimeter problem, each problem was accompanied with the same answer alternatives. The first and second alternatives (alternatives a and b in the test) were selected taking into account the common difficulties of students when they relate lengths and areas and the third (alternative c) was selected taking into account the idea of "directionality". Alternative a (linear) was based on the linear reasoning that if the side of a figure is doubled, the perimeter is doubled (correct) and the area is doubled too (incorrect). Alternative b (quadratic) was based on the claim that if the side of a figure is doubled, student may think that the perimeter is multiplied by four (incorrect) and the area too (correct). Alternative c (directional) was based on the idea of directionality. For instance, for an equilateral triangle, if the three sides of the triangle are doubled (three directions), the perimeter or area of the enlarged figure will become $3 \times 2 = 6$ times larger (incorrect). Problems were formulated in a missing-value format, as in previous investigations on students' improper linear reasoning (De Bock et al., 1998; Fernández et al., 2010) and we asked for the perimeter or the area in an indirect way, i.e. by using a variable that is proportionally related to the perimeter or area. Examples of the two items about an equilateral triangle are given in Figure 1.

Participants were randomly divided in three subgroups receiving a different version of the test: D1, D2, and D3. Each test version differed with respect to the images that were shown to the students. In the D1 version, one arrow with two heads was provided (Figure 1). In the D2 version, two double-headed and perpendicularly oriented arrows were provided and, finally, in the D3 version no arrows were given (Figure 2). Our hypotheses were, first, that the single arrow in the D1 version might help students to apply a linear relation for the perimeter problems, but might at the same time strengthen the tendency towards improper linear reasoning for the area problems. Second, we hypothesized that the two arrows in the D2 version might help students to apply the quadratic relation for the area problems, but put them on the wrong track for the perimeter problems. Third, in the D3 version, because of the absence of any extra arrow(s), only the different directions in the figure might lead to responses in which the number referring to these different directions is used. Fourth, we wonder if the type of figure (triangle, square, or pentagon) has an effect on the occurrence of directional answers.

Forty-three participants answered the D1 version of the test, 44 the D2 version, and 44 the D3 version. The order of the problems in each version of the test was varied: In each version problems were put together in six different orders. The three versions of the tests as well as the different orders in each version were randomly provided to the participants. Students received between 10 and 15 minutes to complete the test, which was sufficient for all students.

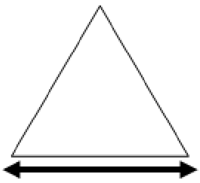
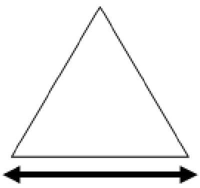
	**Choose the correct option** The weight of an iron fence around a lawn in the form of an equilateral triangle with a side of 8 m is 300 kg. Then the weight of an iron fence of the same type around a lawn in the form of an equilateral triangle with a side of 16 m is approximately a) $300 \times 2 = 600$ kg b) $300 \times 2^2 = 1200$ kg c) $300 \times 3 \times 2 = 1800$ kg
	**Choose the correct option** To fertilise a piece of land in the form of an equilateral triangle with a side of 100 m, a farmer needs 5 kg of fertilizer. If the farmer has to fertilise a piece of land in the form of an equilateral triangle with a side of 200 m, the amount of fertilizer he will need is approximately a) $5 \times 2 = 10$ kg b) $5 \times 2^2 = 20$ kg c) $5 \times 3 \times 2 = 30$ kg

Figure 1: Area and perimeter problems related with the equilateral triangle

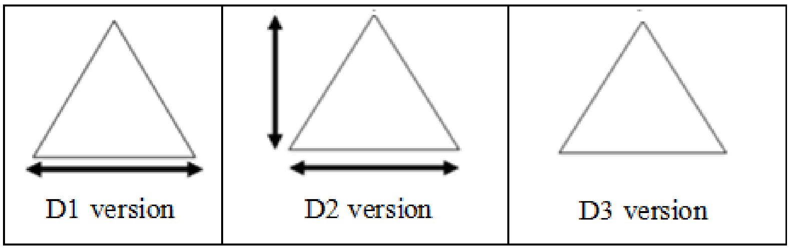


Figure 2: Images given in the different versions of the test

Answers were classified as linear (if a student chose alternative a), as quadratic (if a student chose alternative b), or as directional (if a student chose alternative c). Results were statistically analysed by means of a repeated measures logistic regression analysis using the generalized estimating of equations (GEE).

RESULTS

Table 1 shows the percentages of linear, quadratic, and directional answers for the three versions of the test. Results clearly confirm students' tendency towards improper linear reasoning. As shown in Table 1, 66.7% of the students gave a linear answer on the area problems in the D1 version, 57.6% in the D2 version, and 65.9% in the D3 version. Furthermore, 21.3% of all answers were directional (mean of the three versions). Although, this is not a high percentage, it suggests that in about one fifth of the cases participants may have struggled with the distinction between dimensionality and "directionality". However, against our third hypothesis, this result was independent of test version, since the different ways in which the problems were presented in the three versions did not significantly influence students' tendency to respond directionally.

	D1			D2			D3		
Answer	P	A	Total	P	A	Total	P	A	Total
Linear	**60.5**	66.7	63.6	**62.1**	57.6	59.8	**78.0**	65.9	72.0
Quadratic	17.0	**10.8**	13.9	11.4	**18.2**	14.8	7.6	**16.7**	12.1
Directional	22.5	22.5	22.5	26.5	24.2	25.4	14.4	17.4	15.9

Table 1: Percentages of linear, quadratic and directional answers in each version of the test (A = Area problems, P = Perimeter problems; correct answers are in bold).

On the other hand, as we hypothesized (first hypothesis), the single arrow in the D1 version seemed to strengthen the tendency towards improper linear reasoning: Students tended to give more linear answers on the area problems in the D1 version (66.7%) than on the area problems in the D2 version (57.6%). However, the repeated measures logistic regression analysis showed that this difference was not significant, since the test version and type of problem interaction effect on the choice for the linear answer was not significant. In contrast, the two double-headed arrows in the D2 version seemed to be helpful to find the correct answer on the area problems: Students gave more quadratic answers on the area problems in the D2 version (18.2%) than on the area problems in the D1 version (10.8%) but, again, this difference was not significant (second hypothesis). Finally, the repeated measures logistic regression analysis revealed a significant test version (D1, D2, or D3) and type of problem (area vs. perimeter) interaction effect on the incorrect choice for the quadratic answer, $\chi^2(2, N=131)=19.080$, $p<0.001$, due to the fact that students gave significantly less incorrect quadratic answers in the D2 version (11.4%) than in the D1 version (17.0%).

We also analyzed the effect of type of figure (triangle, square, or pentagon) on the occurrence of directional answers. Table 2 shows students' percentages of linear, quadratic and directional answers for each of the three types of figures.

	Linear	Quadratic	Directional
Triangle	68.32	19.85	17.56
Square	60.31	14.12	25.57
Pentagon	66.79	13.36	19.85

Table 2: Percentages of linear, quadratic and directional answers for each figure

The repeated measures logistic regression analysis showed that the variable type of figure (square, pentagon, or triangle) had a significant effect on the occurrence of directional answers, $\chi^2(2, N=131)=23.301$, $p<0.001$. Pairwise comparisons indicated that both the square-pentagon and the square-triangle difference were significant (while the pentagon-triangle difference was not significant). The table shows that students gave more directional answers for the square figure (25.57%) than for the pentagon or triangle (19.85% and 17.56%, respectively).

CONCLUSIONS AND DISCUSSION

The aim of this research was to unravel the way in which students struggle with the distinction between dimensionality and "directionality" in the context of the relations between length and area of enlarged geometrical figures. Firstly, results confirm a linear tendency in students' answers on problems involving length and area of similar plane figures, as observed in several previous studies (De Bock et al., 1998; Gagatsis et al., 2009; Modestou et al., 2004). Our problems differ from the ones in the other studies since we used a multiple-choice response format. However, this alternative response mode did not radically break students' tendency to give linear responses on area problems.

Secondly, results show that the single arrow with two heads (showing one direction) in the D1 version seemed to help students to apply a linear relation for the perimeter problems. With regard to the D2 version, the two arrows (showing two directions) in this version helped students to apply a quadratic relation for the area problems. However, the two arrows did not put students on the wrong track for the perimeter problems: The D2 version did not foster students to apply a quadratic relation for the perimeter problems.

Finally, the statistical analysis did not show a significant effect of the test version or a significant interaction effect of test version and type of problem on students' choice for the directional answer. So, students gave directional answers independently of the test version. However, about one fifth of the answers were directional suggesting that in a significant number of cases students struggled with the distinction between dimensionality and "directionality". Although the test version did not affect the occurrence of directional answers, a main effect of the type of figure was observed. In fact, students gave more directional answers on problems about a square than on problems about a pentagon or triangle. So, it seems that students struggle more with the distinction between dimensionality and "directionality" in figures where the number of directions and dimensions coincide. This is the case of the square that has two dimensions and also, "two directions".

For future research, it could be interesting to use more open tasks for directional reasoning or to do individual interviews with students with a view to know why they choose a particular answer.

Acknowledgements

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UNIVERSITY STUDENTS AT WORK WITH MATHEMATICAL MACHINES TO TRACE CONICS

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This paper aims to investigate the way past experience with some tools to draw conics becomes part of the experience of designing a new drawer. In particular, it centres on the thinking processes of a group of university students who have the following task: to design a hyperbola drawer. The analysis is carried out using the perspectives of transfer of learning and instrumental approach, and focuses on utilization schemes and the interplay between scientific and technological aspects.

INTRODUCTION

> I love analogies a lot, considering them as my very reliable masters, experts of all the mysteries of nature; in geometry, one has to pay attention to them, especially when they enclose –even if with expressions that seem absurd– infinite cases intermediate between their extremes (and a centre), and thus put before our eyes, in full light, the true essence of an object. Analogy also helped me a lot to draw conic sections. From reading Propositions 51 and 52 [*concerning the metric properties involving the foci*] from Apollonius’s Third Book, one can easily see how to trace ellipses and hyperbolas: these tracings can be made with a thread. […] I regretted that for long I wasn’t able to describe the parabola in the same way. At the end, the analogy revealed to me that to trace this curve is not much more difficult (and the geometric theory does confirm it). (Kepler, 1604, Italian version, pp. 3-5; English translation of the authors)

This brief excerpt from the Italian version of the text *Ad Vitellionem paralipomena* shows how much relevant analogy was for Kepler in geometrical thinking. Kepler’s problem was that of drawing conic sections by means of a thread, from Apollonius’s Propositions. His use of analogy in the case of the parabola is strikinlgy meaningful for us, due to attention deserved to analogy and analogical reasoning by the literature in Mathematics Education (English, 1997). However, we do not want to adopt a specific meaning for analogy over the many considered in the research. Instead, we will refer to it in a naïve manner, as Kepler. We are interested in the spontaneous ways in which elements of situations that have been faced before are recalled in a new situation. This is exactly what Kepler makes. When referring to analogy to think of the construction of a new machine for the parabola, he applies knowledge acquired about hyperbola and ellipse.

To study the spontaneous ways said above, we present here a case study about some university students that are asked to construct a drawer to trace hyperbolas, after they have investigated the functioning of other drawers for conics, using them concretely. Such drawers are for us mathematical machines. A mathematical machine is defined as a tool that forces a point to follow a trajectory or to be transformed on the basis of a

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given law (Maschietto, 2005; Bartolini Bussi & Maschietto, 2008). In this paper, we will centre on how the past experience of the students becomes part of the new experience in which the machine is no longer the starting point but the end point of the task. In so doing, we consider the perspectives about transfer of learning and instrumental approach, and we look at phenomena of transfer in terms of schemes that depend on the type of task.

TRANSFER OF LEARNING AND UTILIZATION SCHEMES

Transfer of learning

The notion of *transfer of learning* has been recently studied in a new perspective that integrates phenomena of cognition, emotion and bodily experience (Nemirovsky, 2011). Drawing on past studies about transfer, Nemirovsky considers transfer of learning as relevant when “it is immersed in the context of common and experiential phenomena of learning”. He defines transfer in terms of experience:

> I see transfer as part of the study of how one experience becomes part of another. People can all sense that experiences do become part of other experiences. It is also clear, I think, that such participation can be lived in numerous ways, some of which I suggest calling “transfer”. (p. 309)

From this point of view, transfer of learning has a dynamic meaning that overcomes any operational definition, depending on the direct and participative engagement of learners. However, since the realm of ways in which an experience becomes part of another is wide, a growing number of studies would furnish information about the features of transfer of learning that characterize it within such realm and about the different ways in which it occurs. The point here is not “to ascertain mechanisms of transfer but to elucidate those experiences that are amenable to being described as transfer of learning.” (*ibid*., p. 334). In this perspective, transfer of learning can be interpreted as strictly related to the subjective feelings of the subjects, instead of being ascribed to something stipulated or secured *a priori*.

In our context, we see the idea of transfer of learning as possibly related to analogy *à la Kepler*. In fact, Kepler uses his previous experience with the ellipse and the hyperbola to solve the problem of finding a way to trace a parabola with a thread. We may think of him as if he were thinking of a machine with tightened thread to obtain the tracing, his use of analogy being reasoning by continuity and extension from past experience. He transfers knowledge acquired about the other conics in the new task, in order to describe the parabola. Regarding our university students that have to face a similar task (thinking of a new machine), we may then ask: How does their past experience with the other conics and drawers become part of the new experience?

Utilization schemes

We adopt transfer of learning as a perspective to analyse how our students solve the design-like problem of thinking of a new machine, after they had concretely used other machines. The presence of artefacts (physical in past activities, potential in the new activity –being its goal) strongly influences the task assigned to the students.

Concerning this influence, we see as interesting the notion of *utilization scheme* as it is studied by cognitive ergonomics research in the analysis of human action mediated by tools (Vérillon & Rabardel, 1995; Rabardel, 2002). The instrumental approach underlines that the use of an artefact to solve a specific task implies to activate certain utilization schemes. Rabardel (2002) defines such schemes as “stable and structured elements in the user’s activities and actions” (p.65). The approach pays attention to the distinction between artefact and instrument: the former is a material or symbolic object, constructed by human beings; the latter is a mixed entity made of the artefact and of associated utilization schemes. The schemes result from a personal construction of the subject or from the appropriation of social schemes already formed outside of him. They are related to accomplishing a specific task on the one hand and, on the other hand, to managing characteristics of the artefact that are strictly related to the given tasks.

A significant element for our context depends on the fact that the task of constructing a tool can be considered between technological and scientific activities, as Weisser (2005) highlights in the field of technology education. In particular, the machine that the students have to think of has the double status of artefact and instrument during the solution phase. Following Rabardel, the process of creating an instrument (that is, the instrumental genesis) has two components: instrumentation, subject-oriented and leading to the emergence and evolution of utilization schemes; instrumentalization, object-oriented and concerning the emergence and evolution of the instrument’s artefact component. Speaking of utilization schemes, Rabardel highlights that they are “the object of more or less formalized transmissions and transfers” (*ibid*., p.84).

With respect to the question of how students’ past experience with the other conics and drawers becomes part of the new experience, we see as fundamental the role of utilization schemes. So, we may ask: Do the students transfer utilization schemes previously formed? How do acquired schemes shape new schemes for the new machine in a new kind of task (to use vs. to construct: the drawer is no longer the starting point but the end point of the activity)?

THE ACTIVITY

The activity is part of a university course on Elementary Mathematics from an Advanced Standpoint. The course can be attended at the second year of the Master’s Degree in Mathematics; its specific topic considers conic sections and their properties, since Greek Mathematics.

Regarding work methodology, the construct of *mathematics laboratory* is the basis of the course’s activities. The mathematics laboratory is meant as a structured set of activities aimed to the construction of meanings for mathematical objects (Anichini *et al*., 2004). It is defined as a space of interaction and collaboration, in which the tasks are addressed and solved using (physical and digital) tools.

Specifically, our students dealt with types of drawers for conics that use a tightened thread. These drawers base on the definition of conic sections as loci of points. Their essential elements are: a wooden flat surface; one pin/two pins for the focus/foci; a thread to materialize distance between each focus and any point it is stretched from. A pen that moves while stretching the thread draws a curve, its point belonging to the curve. For the ellipse, the drawer satisfies the gardener's method. For the parabola, see Figure 1: F is the focus, P is the generic point of the curve.

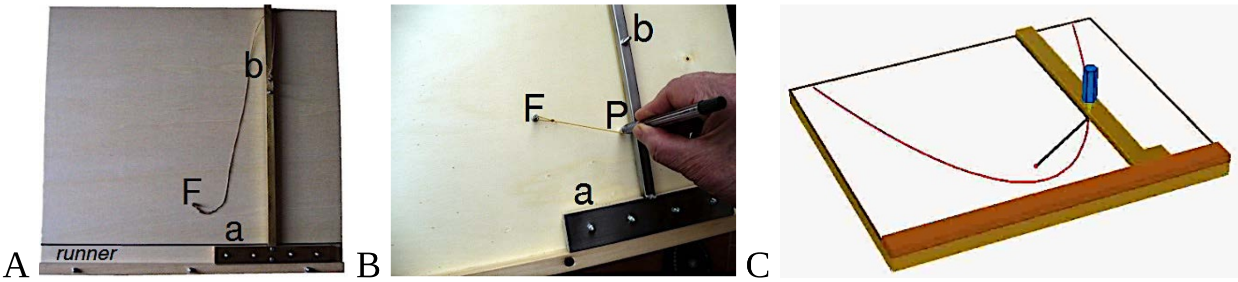

Figure 4. Parabola drawer with tightened thread

In five laboratory sessions the eight university students met five machines. They were divided into two groups, in which one of them had the role of observer. In the first three sessions the students have worked with: the Cavalieri's drawer for parabola, the parabola drawer and the ellipsograph both with tightened thread. The three machines were explored through three phases: to describe their physical structure, parts and spatial relationships; to centre on the product of the machine; to produce conjectures and proofs on that product. An individual report and a collective discussion lead by the teacher (one of the authors) concluded the activity. In this study, we focus on the fourth activity, whose task is completely different from the previous ones, asking the students to imagine how a machine with tightened thread for hyperbola is made. The students have the curve as starting point, but not the machine to trace that curve.

The investigation of how they face this situation is the core of the paper. Data comes from the video-recording of one group, and from the students' written reports and the observer's notes. Our interests are on the way elements from the previous activities with drawers for parabola and ellipse are transferred in the new situation, and on how they originate new ways of writing, new ways of drawing, new ways of thinking.

ANALYSIS AND DISCUSSION

We present here some pieces of the work of one group (we label the group A and the students A1, A2, A3 and A4, the observer). Like for the other group, group B (Ferrara & Maschietto, 2013), four phases can be captured, the first three depending on the means that the students use (paper and pencil; a wooden plan with two pins and one or two threads; a rod). For the sake of space constraints, we only focus on the phases 1 and 2 of group A's work.

In what follows, past experience is usually recalled by linguistic expressions of the kind: "let's think of how we did the other time", "last time". In addition, depending on the moment, technological aspects or scientific aspects can be at play, as well as gestures of usage can be produced (see Ferrara & Maschietto, *ibid*.). We will make explicit reference to utilization schemes and to these other aspects when necessary.

1) Work with paper and pencil on graphical representations

Group A begins its work with the metric definition of hyperbola accompanied by the algebraic expression and the standard graphical representation (with generic point P, foci, vertices, etc.). Immediately, the students recall their previous experience with the ellipsograph to detect first components of the new machine:

A1: Let's think of how we did the other time… we have two foci, two foci that were fixed [*pointing her fingers to the foci*]; maybe we could think that there are two pins

The students transfer the artefact component of the instrument ellipsograph: they look for those components of the machine that materialize elements of the definition of the curve (the pins for the foci, the tightened thread for the distance focus-point of the curve). The students think of the thread as a given part of the new machine, that is, as one of its artefact components. They also search for a link between the length of the thread and the constant *k* in the formula $|PF_1 - PF_2| = k$. However, the relationship (length of the thread, constant *k*) coming from the ellipsograph cannot be transferred as such in this case. In effect, for the ellipse, the length of the thread represents the sum of the two distances focus-point of the curve.

Resting on the formula $|PF_1 - PF_2| = k = 2a$, the students point out the connection between the constant difference and the distance between vertices (apparent on the graphical representation). This marks a new beginning: the established link (scientific aspect) affects the idea of the machine (technological aspect), because it entails the understanding that the components translated from the ellipsograph are not useful for the new machine. So, the parabola drawer is in turn recalled:

A3: You cannot do many things just using the two foci

A1: But for the parabola drawer, we also had the rod (*b* in Figure 1A and 1B)

The parabola drawer intervenes in thinking of the technological aspects of the new machine. A3 recalls through graphical representations utilization schemes for the rod, looking for their application in the present case, without success. Attention is drawn back to the thread as element that has to incorporate the condition about distances. Another utilization scheme associated to the parabola drawer is recovered: when the thread is kept tightened, the equivalence of the distances point-focus and point-directrix is assured ($d(P, F)$ and $d(P, a)$ in Figure 1C). This exploration brings the group to conclude that the length of the thread does not count for the hyperbola and that constant *k* must be looked for in another way (scientific and technological aspects). Reference to the parabola drawer fails to help the students.

Recognizing the presence of material elements in the group discussions, the teacher now furnishes the students with a wooden plan with two pins and a thread. Group A begins to work with one thread, but then asks for a second thread.

2) Work with threads

The idea that guides the group action is to represent the parameter of the curve using the thread. For this reason, the students choose a certain segment of the thread (that corresponds to 2*a* in the formula), they tie one of its ends to a pin and they try to handle it (Figure 2A). A1 tests with the thread segment various configurations that all assure to satisfy the definition of hyperbola. In other terms, A1 begins a sort of process of instrumentation of the thread. The students also try to include in this kind of exploration a ruler that should play the role of the rod.

The observer A4 intervenes in the dialogue to suggest that the students think of the placement of the pencil to draw the curve. The task is brought back by the need for the tracing. The students must pass from a static configuration (test that a chosen point satisfies a certain relationship) to a dynamic one (a movement permits to trace the curve). A4's intervention supports the transfer of a specific utilization scheme of the previous drawers: the pencil guides the movement tightening the thread. As a result of this action, the pencil's point also corresponds to a point of the curve. So far, the students had not transferred this scheme that is instead crucial to draw the curve.

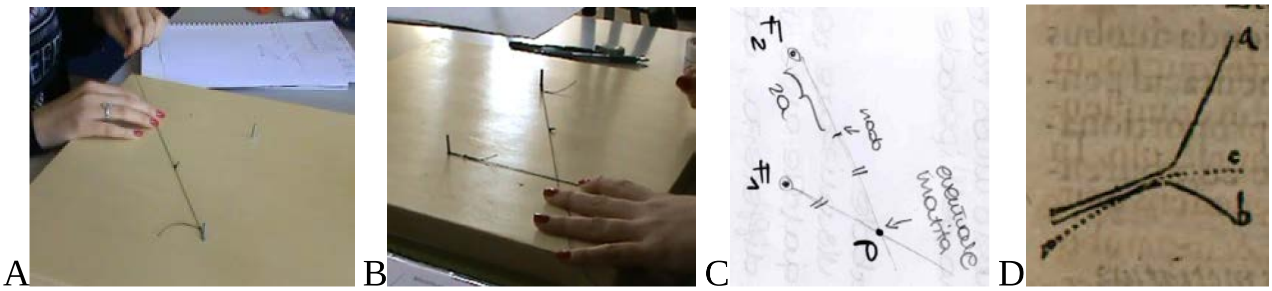

Figure 5

A new intervention of the teacher marks that these explorations do not consider the second focus, pushing the group to produce symmetric actions so to tie another thread to the second pin (Figure 2B). The students find configurations that seem to match sketches by Kepler (1604; e.g. Figure 2D).

The group tries to keep constant the difference PF_2-PF_1 (Figure 2C) and to look for new gestures of usage in an instrumentation of the artefact with two threads. Indeed, moving the threads, the students want to preserve that difference when tracing the curve. At the basis of this attempt, there is the detection of an isosceles triangle (scientific aspect, see Figure 2C). But the students abandon this way as soon as they are faced with a technological issue: "We are not successful in thinking of a tool that can replace my hands to move the two threads as desired" (from A1's written report). In the case of the artefact with two threads, the construction of a new utilization scheme through the placement of the pencil is problematic for the students: they are only able to find discrete points but not the curve by continuous motion. The latter is another element of utilization schemes previously acquired that has to be transferred in the new situation, since it is a fundamental constraint for students' action.

CONCLUSION

In this paper, we focused on how previous experience can become part of a new learning situation. In particular, we centred on the way a small group of university students recalls past experience with some mathematical machines for conic sections in order to face the task of constructing a new machine. This kind of task differs from the previous ones. Before, the students were asked to explore drawers for ellipse and parabola using them concretely (to understand how they are made, how they work, what they trace and why). In the new situation, they are required to think of and design a drawer for hyperbola. So, the machine is no longer the starting point of the activity, but the goal of it. Due to the presence of machines in the tasks, the notion of utilization schemes is interesting for us, especially in terms of their transfer and formation in the new situation.

Considering the perspectives of transfer of learning and of the instrumental approach, we have investigated if the students transfer utilization schemes previously formed, and especially how acquired schemes shape new schemes for the new machine in the new kind of task. Through the analysis of some work of group A, we have observed that the students' process of constructing the instrument for hyperbola bounces between the technological side and the scientific side. The first side regards the fact of having a machine as goal of the activity and investigating its material components (that is, the artefact components). The second side refers to mathematical constraints that have to hold for tracing a given curve (a hyperbola) with that instrument. The relationship between the two sides is complex for the students for at least two reasons: on the one hand, the physical parts that constitute the machine have to materialize mathematical constraints; on the other hand, the curve has to be traced by a continuous motion with the instrument. Utilization schemes do just intervene in the search for such relationship.

Our analysis has shown that the students of group A try to construct a new artefact by transferring artefact components from the instruments to draw ellipse and parabola (e.g. the pins for the foci, the thread). The metric property that defines the hyperbola furnishes the mathematical constraint to be implemented in the machine. This implies that the students look for a condition on the length of the thread recalling utilization schemes activated with the ellipsograph and the parabola drawer. The interventions of the observer and of the teacher help the students focus on other utilization scheme relevant for the machine: the motion of a pencil that keeps the thread tightened serves to trace the curve.

We believe that activities of this kind are relevant for mathematics learning because they encourage the students to make explicit theoretical principles under the machine. Following Koyré (1967), the construction of the new drawer corresponds to "creation of scientific thought or, better yet, the conscious realization of a theory" (p. 106).

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USING FACEBOOK FOR INTERNATIONAL COMPARISONS: WHERE IS MATHEMATICS A MALE DOMAIN?

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Mathematics continues to be an enabling discipline for STEM-based university studies and related careers. Explanatory models for females' underrepresentation in higher level mathematics and STEM-based courses comprise learner-related and environmental variables – including societal beliefs. Using Facebook to recruit participants, we explored if mathematics is (still) viewed as a male domain. Responses were received from 724 people in 84 countries. Findings from nine countries with at least 30 respondents – Canada, China, Egypt, India, Israel, Singapore, UAE, UK, and Australia – are reported. Among those who held gender-stereotyped views (and many did not), mathematics was considered more suitable for males than for females.

INTRODUCTION AND PREVIOUS RESEARCH

In 2011, the United Nation's 55th Commission on the Status of Women was held in New York. The focus was science and technology. At the conclusion of the Commission, a set of 22 conclusions was agreed upon. The importance and benefits of attaining gender equality in science and technology was recognized in conclusion number nine:

> The Commission notes that quality education and full and equal access and participation in science and technology for women of all ages are imperative for achieving gender equality and the empowerment of women, and an economic necessity, and that they provide women with the knowledge, capacity, aptitudes, skills, ethical values and understanding necessary for lifelong learning, employment, better physical and mental health,… as well as for full participation in social and economic and political development. (Commission on the Status of Women, 2011, p. 2)

While women's enrolments in tertiary education internationally have increased dramatically (UNESCO, 2012), they remain under-represented around the world in tertiary computing and engineering courses (UNESCO, 2012), for which studies in mathematics are pre-requisites and/or co-requisites. Hill, Corbett, and Rose (2010) report that, in the USA, the challenge of achieving gender equality in STEM areas is clear when employment figures by gender in STEM-related careers are considered. In the USA, it was reported that "women are vastly underrepresented in STEM jobs and among STEM degree holders despite making up nearly half of the U.S. workforce and half of the college-educated workforce" (Beede, Julian, Langdon, McKittrick, Khan, & Doms, 2011, p. 1).

At the school level in the USA "girls and boys take math and science courses in roughly equal numbers" (Hill et al., 2010, p. 4). This is also the case in Australia,

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except that males outnumber females in the most challenging, calculus-based, mathematics subject on offer at the grade 12 level (Forgasz, 2006).

There is a large body of research indicating that:

> Negative stereotypes about girls' and women's abilities in mathematics and science persist despite girls' and women's considerable gains in participation and performance in these areas during the last few decades. Two stereotypes are prevalent: girls are not as good as boys in math, and scientific work is better suited to boys and men. (Hill et al., 2010, p. 38)

Through the media, gender stereotypes in mathematics and science can be reinforced (for example in print – see Forgasz, Leder, & Taylor, 2007) or challenged (for example on television – see Steinke, 1998). Parents and teachers have been found to hold gender-stereotyped beliefs about and expectations of children's mathematical capabilities; these beliefs are often more strongly held by males (e.g., Tiedemann, 2000).

While several explanatory models for gender differences in mathematics learning outcomes – achievement, participation, and attitudes – include the views of society at large (see Leder, 1992), less often are views related to the gender stereotyping of mathematics gathered from the general public (Leder & Forgasz, 2010), as opposed to stakeholders such as parents and teachers.

The aim of the present study was to gather the views of the general public from around the world about the gendering of mathematics, and to compare views by country. Findings from an earlier stage of this study were reported by Forgasz and Leder (2010). In that study, data were gathered from pedestrians in the street of Victoria, Australia.

THE STUDY: METHOD AND INSTRUMENT

The instrument used by Forgasz and Leder (2010) was used in the present study. However, the items were transferred onto the online survey platform, SurveyMonkey. To obtain views from around the world, Facebook was used as the means of participant recruitment. An advertisement was placed on Facebook inviting potential participants to complete the online survey – see Forgasz, Leder, and Tan (2011) for more details on how this was done. Those who clicked on the advertisement were directed to the online survey instrument.

Instrument

To maximize completion rates, the survey was limited to 15 items which focused on: personal background data; the learning of mathematics at school; perceived changes in the delivery of school mathematics; beliefs about boys and girls and mathematics, and their perceived facilities with calculators and computers; and perceived suitability for careers in science and computing.

In this paper we focus on four questions related to the gendering of mathematics and technology capabilities. The four items analysed and discussed are:

Q1. Who are better at mathematics, girls or boys?

Q2 Is it more important for girls or boys to study mathematics?

Q3 Who are better at using calculators, girls or boys?

Q4 Who are better at using computers, girls or boys?

For each item, participants were required to select from: Boys / Girls / Same / Don't Know/Depends. Respondents also had the option to explain their responses to each question. Space constraints limit the findings presented here to the quantitative data.

RESULTS AND DISCUSSION

Responses were received from 784 participants aged over 18 (an ethical requirement), representing 84 countries. Of these, there were nine countries – Canada, China, Egypt, India, Israel, Singapore, UAE, UK, and Australia – with at least 30 responses from each. The 505 responses from these countries represented 70.2% of all responses received. Frequencies and percentages of the 505 responses from the nine countries are shown in Table 1.

Country	Frequency	Valid %	Country	Frequency	Valid %
Canada	35	6.4	Singapore	35	6.4
China	76	13.8	UAE	46	8.4
Egypt	84	15.3	UK	58	10.5
India	66	12.0	Australia	119	21.6
Israel	31	5.6			

Table 1. Frequency and valid percentage of responses by country

Participants' responses to the four questions listed above were analysed by country. Chi-square tests were conducted to determine if there were statistically significant differences in the frequency distributions to the response options by country. The findings for the survey questions are presented and discussed below. [NB. Sample sizes were too small for analysis by respondent gender.]

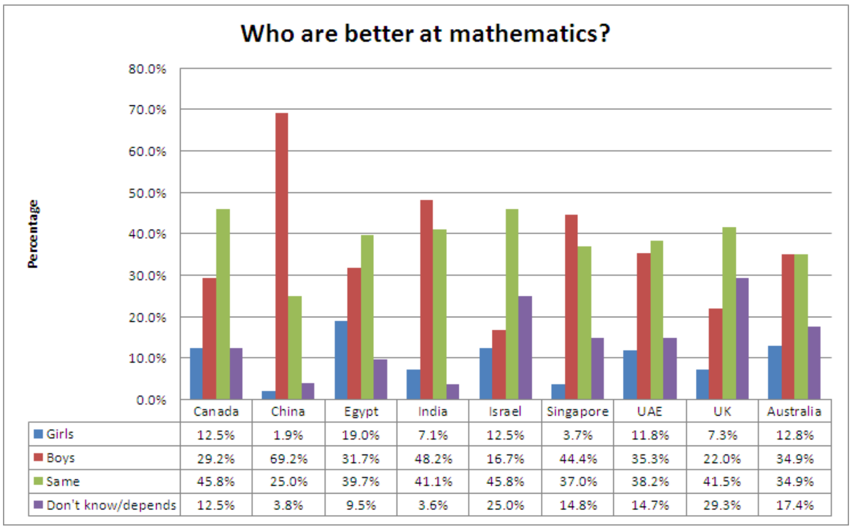
Questions about the gendering of mathematics: Q1 and Q2

The frequency distributions of responses by country for Q1, "Who are better at mathematics, girls or boys?", and Q2, "Is it more important for girls or boys to study mathematics?" are shown in Figures 1 and 2 respectively.

For Q1, "Who are better at mathematics?", the chi-square test revealed that there was a statistically significant difference in the frequency distributions of responses by country (χ^2=56.0, df=24, p<.001). Figure 1 reveals that:

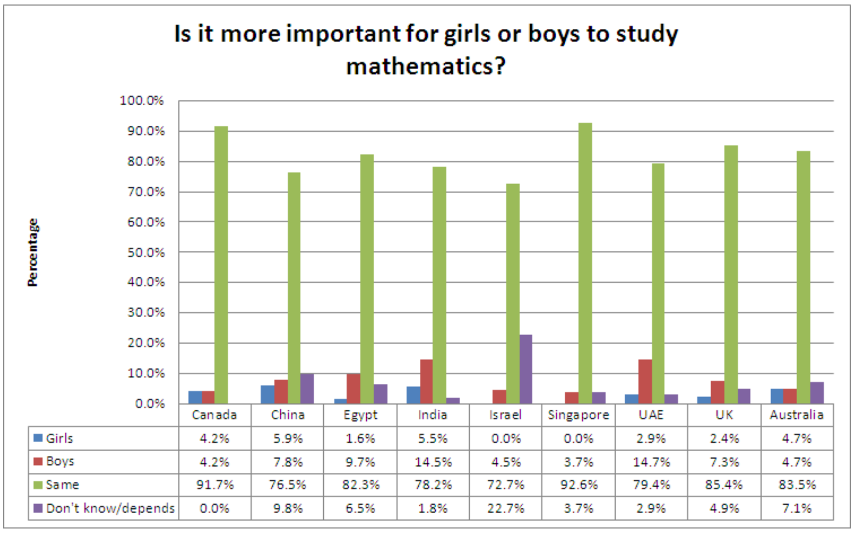
- "Same" (i.e., boys and girls considered equally good at mathematics) was the response with the highest frequency from participants in five of the nine countries – Canada (45.8%), Egypt (39.7%), Israel (45.8%), UAE (38.2%), and UK (41.5%) In these countries, among those whose response was gender-stereotyped (“boys” or “girls”), more respondents identified “boys” than “girls” as being better at mathematics.

- For China (69.2%), India (48.2%), and Singapore (44.4%), “boys” was the most frequent response.
- For Australia the same percentage of people (34.9%) responded “same” and “boys”.



	Canada	China	Egypt	India	Israel	Singapore	UAE	UK	Australia
Girls	12.5%	1.9%	19.0%	7.1%	12.5%	3.7%	11.8%	7.3%	12.8%
Boys	29.2%	69.2%	31.7%	48.2%	16.7%	44.4%	35.3%	22.0%	34.9%
Same	45.8%	25.0%	39.7%	41.1%	45.8%	37.0%	38.2%	41.5%	34.9%
Don't know/depends	12.5%	3.8%	9.5%	3.6%	25.0%	14.8%	14.7%	29.3%	17.4%

Figure 1. Response percentages by country: Who are better at mathematics?



	Canada	China	Egypt	India	Israel	Singapore	UAE	UK	Australia
Girls	4.2%	5.9%	1.6%	5.5%	0.0%	0.0%	2.9%	2.4%	4.7%
Boys	4.2%	7.8%	9.7%	14.5%	4.5%	3.7%	14.7%	7.3%	4.7%
Same	91.7%	76.5%	82.3%	78.2%	72.7%	92.6%	79.4%	85.4%	83.5%
Don't know/depends	0.0%	9.8%	6.5%	1.8%	22.7%	3.7%	2.9%	4.9%	7.1%

Figure 2. Response percentages by country: Is it more important for girls or boys to study mathematics?

In summary, more participants from each country considered “boys” to be better at mathematics than considered “girls” as being better. Respondents from China, with 69.2% saying that “boys” were better at mathematics, held the traditional gender-stereotyped view more strongly than in other countries.

For Q2, “Is it more important for girls or boys to study mathematics?”, there was no statistically significant difference in the response distributions by country. As shown in

Figure 2, an overwhelming majority in each country considered it equally important for girls and boys to study mathematics. Interestingly, among the small minority in each country who held gender-stereotyped views, a slightly higher proportion felt that it was more important for boys than for girls to study mathematics.

Capabilities with Technology: Q3 and Q4

The frequency distributions of responses by country for Q3, "Who are better at using calculators, girls or boys?", and Q4, "Who are better at using computers, girls or boys?", are shown in Figures 3 and 4 respectively.

Chi-square tests for the frequency distributions of responses for Q3 and Q4 were both statistically significant: Q3 (χ^2=63.9, df=24, p<.001) and Q4 (χ^2=38.7, df=24, p<.05).

For Q3, "Who are better at using calculators, girls or boys?", Figure 3 show that:

- In seven countries – Canada (52.2%), China (38.8%), Egypt (57.1%), Israel (56.5%), Singapore (63.0%), UAE (50.0%), and Australia (48.1%) – the most frequent response was "same", that is, that boys and girls are equally capable with calculators. In five of these seven countries, among those with gender-stereotyped views, more respondents considered "boys" than considered "girls" to be better with calculators. The two exceptions were Canada (8.7%) and Egypt (14.3%) where the same percentages of people considered "boys" and "girls" to be better with calculators.
- In India, the same percentage of people (38.9%) replied "same" (ie., boys and girls equally capable with calculators) as replied that "boys" were better than "girls" at using calculators.
- Interestingly, in the UK, the most frequent response was "Don't know/unsure" whether boys or girls were better with calculators (50.0%). Again, however, of those holding gender-stereotyped views, more people indicated that "boys" (7.5%) were better with calculators than said that "girls" were (2.5%).

For Q4, "Who are better at using computers, girls or boys?", Figure 4 reveals that:

- In six countries – China (55.1%), India (52.7%), Israel (52.25), Singapore (48.1%), UAE (48.4%), and UK (40%) – the most frequent response was that "boys" were better than girls at using computers. That in China, India, and Israel, "boys" was the response of the majority (ie., over 50%) indicates that the traditional gender-stereotyped view was very strongly held by participants from those countries.
- In the remaining three countries – Canada (47.8%), Egypt (53.6%), and Australia (40.7%) – the most frequent response was "same", that is that boys and girls were equally capable with computers. However, among those with a gender-stereotyped view, considerably more respondents indicated that "boys" rather than "girls" were better at using computers.
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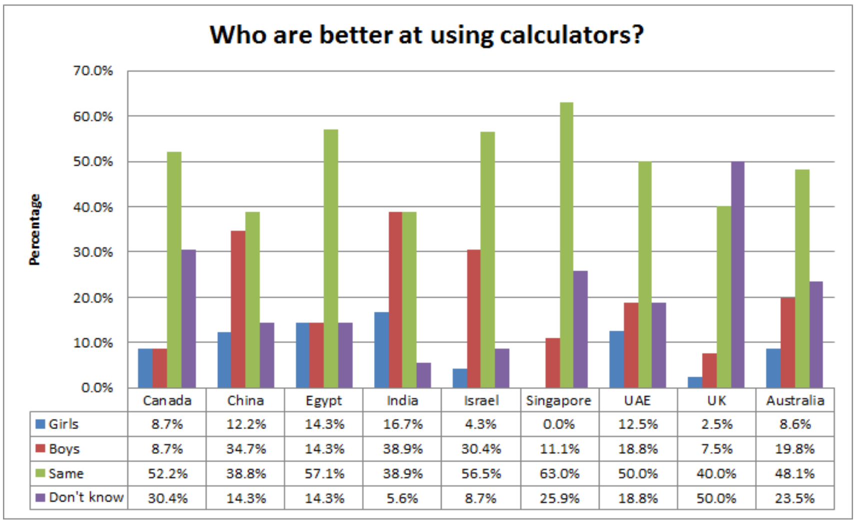

Figure 3. Response percentages by country: Who are better at using calculators, girls or boys?

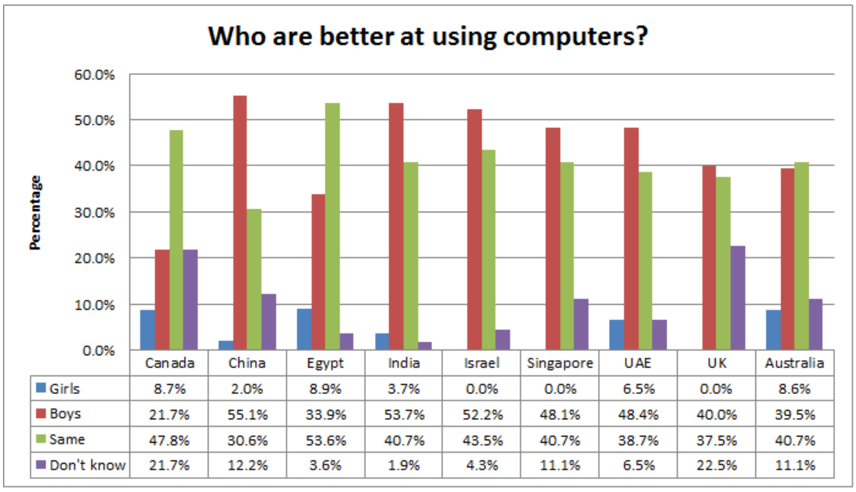

Figure 4. Response percentages by country: Who are better at using computers, girls or boys?

In summary, for both items about capabilities with technology, boys were generally considered more capable than girls. This trend was much stronger with respect to computer use than with calculator use, and the views were more strongly held by participants from some countries than from others.

Conclusions and implications

One very positive outcome of this study was that in all nine countries there was strong endorsement of mathematics as an important study for all students irrespective of gender. Differences by country were evident, however, when it came to perceptions of boys' and girls' capabilities with mathematics (the enabler for higher level STEM studies and career paths), and for boys' and girls' capabilities with technology (calculators and computers).

It is important to note that many people in the nine countries did not distinguish between boys and girls with respect to mathematics or technology capability. However, if a gendered belief was held, it was evident that in all countries it was more likely that the traditional gender-stereotyped view that mathematics is a male domain (i.e., that mathematics studies and related careers are more suited to boys than to girls) prevailed over mathematics being seen as a female domain. There were clear differences by country in the extent to which this traditional view was held. Interestingly, in all nine countries, there was no item for which the response “girls” had a higher percentage response rate than the response “boys”.

Participants from China appeared to hold the strongest traditional beliefs about mathematics as a male domain. Participants from English-speaking countries appeared to be more likely than participants from non-English speaking countries to hold gender-neutral beliefs, that is, they were more likely to respond “same” to the items.

There were limitations to this study. First, the survey was in English and so participants from non-English speaking countries had to be sufficiently fluent in English to respond. This may have skewed the findings to reflect the views of the more highly educated in those countries. Related to this is that the recruitment method limited the potential pool of respondents to Facebook users. This may have skewed findings to reflect the views of those from higher socio-economic backgrounds who have access to the Internet and Facebook. It should also be noted that Facebook may also have introduced an age-related factor, as Facebook users over 18 years of age are more likely to be in the under-35 than in the over-35 age range (Burbary, 2011). Facebook clearly had some advantages as well. Using Facebook allowed data to be gathered from respondents from a range of national backgrounds, not easily obtained through more traditional research approaches.

Clearly further research is needed to obtain data from larger number of respondents from more representative samples of the populations of the countries included here and from elsewhere. However, despite the limitations of the present study, the consistency in the direction of the findings in support of the traditional male stereotype provides strong evidence that gendered perceptions of mathematics remain alive and well in many parts of the world. Changing attitudes and perceptions is a challenge. Yet to maximise human progress, the discipline of mathematics and STEM fields in general cannot afford to sideline half the world’s population, women.

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THE FLOW OF A PROOF – THE EXAMPLE OF THE EUCLIDEAN ALGORITHM

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The notion "flow of a proof" encapsulates various aspects of proof presentation in a classroom. This paper presents a global flow analysis of the proof of the Euclidean algorithm, as presented to a class of prospective teachers. The findings are based on lesson observation, questionnaires and interviews with lecturer and students. The analysis reveals the existence of a gap between the main ideas of the proof as perceived by the lecturer and by the students. This is also reflected in the way the students self-estimated their level of understanding. We explain these findings by relating to the global features of the flow and discuss potential alternative flows.

Learning mathematical proof and justification is considered an essential part of students' mathematical education. Teachers are encouraged to explain and justify mathematical concepts and processes and to use proof as an explanatory tool (Hanna, 2007). Nevertheless, teachers' didactic knowledge regarding strategies for teaching proof is still an emerging research topic. Teachers' lack of such knowledge and strategies has been pointed out (Dreyfus, 2000; Knuth, 2002). Alcock (2010) and Weber (2010) have provided insight into pedagogical considerations used by experienced mathematicians while teaching proof, and found that mathematicians lacked the required strategies to achieve their didactic goals. A few approaches to teaching proof have been suggested but there are still no robust findings regarding the effect of using those approaches in mathematics classrooms. Thus, establishing ways to improve teachers' pedagogical knowledge about proof teaching is a valuable task (Hanna, 2007). Mariotti (2006) further recommended investigating cognitive and meta-cognitive perspectives of proof learning in the context of teachers' training.

THE FLOW OF A PROOF

The concept of flow is embedded in the way mathematicians think about proof and proving. Thurston (1994) writes that "when people are doing mathematics, the flow of ideas and the social standard of validity are much more reliable than formal documents" (p.169). The present research introduces the notion of flow of a proof, which attempts to encapsulate various aspects of proof presentation in such a way that the flow can be used as a pedagogical tool. The approach is holistic in nature; thus an initial characterization of flow relates to: (i) the way the logical structure of the proof is presented in class; (ii) the way informal features and considerations of the proof and proving process are incorporated in the proof presentation; (iii) contextual factors (mathematical and instructional). The flow of a proof is expected to affect students' cognitive and affective responses to the proof and the research aims to study those effects in situations where the flow is recognized by the students and when it is not. When students recognize the flow, they are expected to be able to notice and reflect on

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main formal and informal ideas of a proof and on the way those ideas relate to each other, so that insight, intuition and formalism are balanced.

The flow of a proof can be analysed from a global and from a local perspective. The global perspective is concerned with the way a proof is divided into modules and with the choices made by the lecturer as to how to present those modules in class so they are combined and communicated to the students as an intelligible, unified, coherent and hopefully engaging story. The local perspective involves the examination of single formal or informal arguments in the proof using argumentation theory, particularly the full scheme of Toulmin's model (Toulmin, 1958). In this paper the focus is on the global perspective of the flow of a proof. Contextual aspects are also briefly addressed. Specifically, a global flow analysis is applied to the proof of the Euclidean algorithm as presented to a class of prospective secondary level mathematics teachers during their first year of training. The effect this flow had on the way the prospective teachers understood the main ideas of the proof is discussed, and possible explanations and alternatives are presented.

METHOD

The research was carried out in a class of 38 prospective secondary level mathematics teachers taking a course in Number Theory during their first year of training. This paper refers to the lesson in which the proof of the Euclidean algorithm for finding the greatest common divisor of two natural numbers was taught. A summary of the proof as it was presented in the lesson is depicted in Figure 1.

$\underline{\textit{Lemma 1:}}$ *Let* $a,b,q,r \in N$. *If* $a = bq + r$ *then* $gcd(a,b) = gcd(b,r)$

$\underline{\textit{Lemma 2:}}$ *Let* $a,b \in N$. $b \mid a$ *if and only if* $gcd(a,b) = b$

$\underline{\textit{Proof of Euclidean algorithm:}}$

Given $a,b \in N$, $a > b > 0$

$\exists!\ q_1, r_1$ *such that* $a = q_1 b + r_1$; $0 \le r_1 < b$ [*division with remainder*]

$gcd(a,b) = gcd(b,r_1)$ [*lemma 1*]

if $r_1 = 0$ *then* $gcd(a,b) = b$ [*lemma 2*]

if $r_1 > 0$ *repeat the process*:

$\exists!\ q_2, r_2$ *such that* $b = q_2 r_1 + r_2$; $0 \le r_2 < r_1$ [*division with remainder*]

If $r_2 = 0$ *then* $gcd(a,b) = gcd(b,r_1) = r_1$ [*lemmas 1&2*]

if $r_2 > 0$ *repeat the process.*

After k steps: $r_{k-2} = q_k r_{k-1} + r_k$; $0 \le r_k < r_{k-1}$ *and* $gcd(r_{k-2}, r_{k-1}) = gcd(r_{k-1}, r_k)$

The process is finite [*well-ordering principle*] $\Rightarrow \exists n \in N : r_{n+1} = 0$

$gcd(a,b) = gcd(b,r_1) = gcd(r_1,r_2) = ... = gcd(r_{n-1}, r_n) = r_n$

$\underline{\textit{Numerical example}}$:

$a = 225,\ b = 120$

$225 = 1 \times 120 + 105$

$120 = 1 \times 105 + 15$

$105 = 7 \times 15 + 0$

$\Rightarrow gcd(225,120) = 15$

Figure 1: The proof of the Euclidean algorithm and the example given in the lesson

After the lesson the students were asked to answer a questionnaire relating to cognitive and affective aspects. The questions in the cognitive part concerned global and local aspects. The "global-oriented" questions asked the students to specify main ideas of the proof, to illustrate the proof with an example and to explain why the algorithm is finite. The affective part included questions relating to interest, confidence and significance. The lecturer of the course was informed in a very general manner that the research relates to teaching proof but was not instructed before the lesson how to present the proof and did not see the questionnaires. The lesson was observed and audio recorded. A researcher took notes on the lesson as well as remarks and responses of students and of the lecturer. In addition, two types of post-lesson interviews were conducted: (i) a reflective interview with the lecturer; (ii) two individual interviews with students, designed to probe their questionnaire answers, the way they perceived (if at all) the flow of the proof and mathematical or pedagogical factors that improved or impeded their understanding.

These data enabled examination and analysis of (i) the way the proof was presented by the lecturer; (ii) the way the proof and its presentation were received by the students; (iii) the correlation between the two.

FINDINGS AND DISCUSSION

Context

Thurston (1994) writes that "[for him] it became dramatically clear how much proofs depend on the audience" (p.175). A proof is not presented in a vacuum but in the context of a lecturer with firm beliefs and convictions, and to an audience with certain characteristics. Therefore, a proof might communicate some mathematical concepts and ideas in one context but others in another context.

The participating students can be roughly divided into two groups: students who are starting their academic studies, and students that finished their studies in another academic discipline but are currently qualifying as mathematics teachers. The latter are more experienced academically. Those two groups were sitting in separate areas.

The lecturer is experienced and knowledgeable. He regularly teaches the course in different academic institutions to different populations. He lectures frontally but also patiently answers students' questions. He is enthusiastic about the course content and this is reflected in his teaching. During the post lesson interview, he expressed awareness of the existence of two groups in the class as well as his belief that the proof presentation should be based on purely mathematical considerations rather than audience dependent, that the proof should be presented as accurately and formally as possible, and that it is especially important for prospective mathematics teachers to be exposed to formal proof even if they are not going to teach such proofs later.

Analysis of the proof presentation

The lecturer started the lesson referring to the topic of previous lessons, the properties of a number that divides a specific other number, and to today's topic, a number that is a common divisor of two numbers. The lesson ended with the lecturer stating the topic of the next lecture: analyzing the efficiency of the algorithm. These opening and closing statements situate the lesson within a larger frame of reference. They contribute to the sense of building a complete mathematical theory, and situating the theorem as part of a theory. Thus a mathematical context is provided.

Next, the lecturer defined the greatest common divisor (gcd) and introduced some of its basic properties without formally proving them. He showed a naïve way of finding gcd(220,125) and raised the question: What if the two numbers a,b for which we try to find the gcd are very large and have many divisors? He thus motivated the need for an efficient way of calculating gcd(a,b). Next he demonstrated the use of Euclid's algorithm for finding gcd(220,125), basing the algorithm on two lemmas, which he then proved, lemma 1 formally and lemma 2 verbally. Finally he described the algorithm using general notations involving indices as in Figure 1) and proved it formally. He used the well-ordering principle to show that the algorithm is finite. Hence the lesson can be divided into four modules (Figure 2).

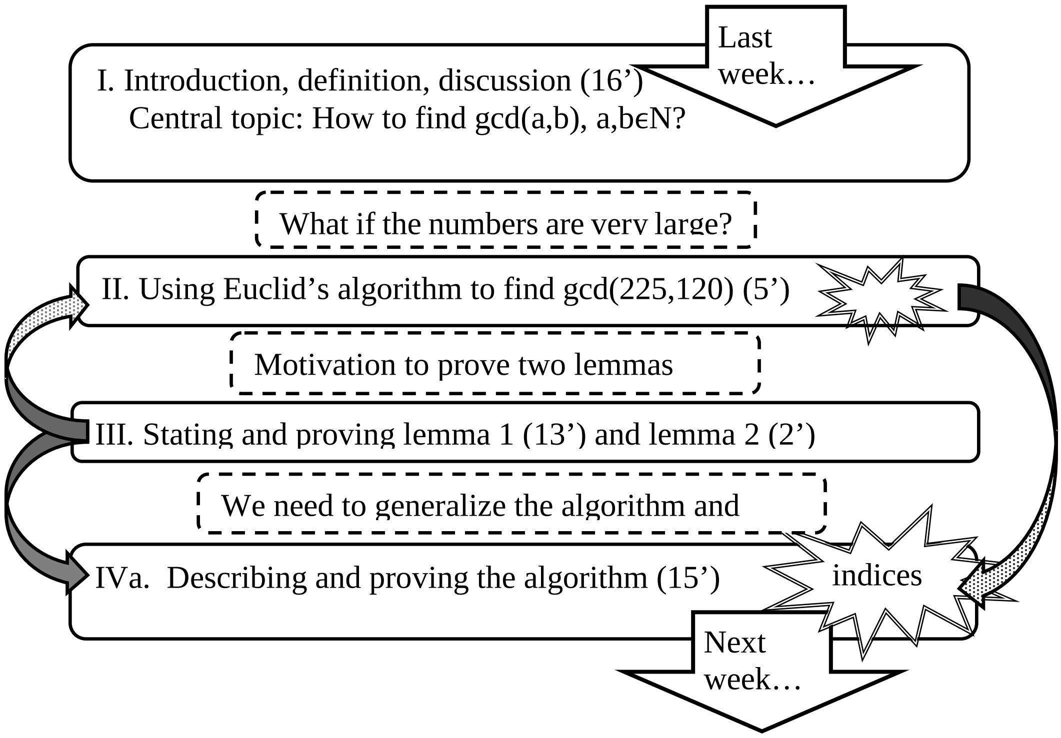

Figure 2: Schematic global analysis of the flow of the proof

The schematic presentation in Figure 2 is based on an analysis of the lesson and provides a compact visualization of the order and duration of the lesson's modules, the way they were tailored to each other and their interrelations. For instance the lemmas in module III were used during the numerical example of module II but were also motivated by it; similarly the lemmas are used in the general proof in module IV. The dashed boxes in Figure 2 represent the transitions between the modules. We interpret these transitions as the lecturer's attempt to transform a collection of modules into a unified, coherent and possibly engaging story. The stars represent places that might disturb the flow. The star in module II reflects the demonstration of the algorithm on a numerical example before it was formally introduced. The star in module IV represents the transition to index notation as in Figure 1.

The global proof presentation involved many choices made by the lecturer, choices which are directly connected to the flow of the proof. In the following subsections we will examine how these choices are reflected in the way the lecturer and students answered the question "What were the main ideas of the proof?"

Main ideas of the proof as stated by the lecturer and as stated by the students

During the post-lesson interview, the lecturer was asked what he sees as the main ideas of the proof (ideas 1-4 in Table 1). The students were asked the same question in the questionnaire. Fifteen students answered the questionnaire and thirteen of them answered this question. A summary of the students' answers is presented in Table 1.

The proof's main ideas	Nb of students
1. The division algorithm	5
2. A common divisor of a,b is a divisor of any linear combination of a,b	1
3. Examining smaller and smaller numbers until a trivial situation is reached	2
4. The algorithm is finite (well-ordering principle)	7
5. $a = bq + r \Rightarrow \gcd(a,b) = \gcd(b,r)$	8
6. $b \mid a$ if and only if $\gcd(a,b) = b$	6
7. Iterations	3
8. The remainder sequence is monotonically decreasing	1
9. Getting a deeper understanding of the divisors of a,b	1

Table 1: The main ideas of the proof as stated by students and lecturer

Table 1 reveals that none of the lecturer's main ideas was mentioned by a majority of the students. This demonstrates the variety that might exist in a group of students who participate in the same lesson, as well as the existence of gaps between the main ideas as perceived by the lecturer and the students.

Idea 1, the division algorithm, was referred to by only 5 students although Euclid's algorithm uses it repeatedly. During his interview the lecturer referred to idea 2 as one of the underlying ideas of the proof, casually mentioning that this is actually Lemma 1, but only one student stated this idea in his answer, and this student also stated idea 5 as if it was a separate idea. Interestingly, ideas 5 and 6, which are simply Lemmas 1 and 2, were mentioned by a relatively large number of students. Idea 6 was not mentioned by the lecturer at all; also during the lesson, he did not accord it much importance: he explained it briefly and wrote it on the board without proof. Idea 4 referring to the algorithm being finite was explicitly stated by only 7 students although it was very apparent throughout the lecture: it was discussed informally and even quite enthusiastically, and was also proven formally. Similarly, idea 3, which is at the heart of the algorithm was mentioned by only 2 students, or possibly also by the 3 students who stated idea 7: "iterations" which may be a less accurate way of expressing idea 3.

A possible explanation for these gaps may lie in the global flow of the proof (Figure 2). The lemmas were stated and proved right after the numerical example, using simple algebraic language, followed by a complicated general formulation involving indices. This might have caused the students to consider the lemmas as the main part of the proof, or perhaps they were the last part of the lesson they felt comfortable with. This explanation is supported by the following extract taken from the interview with Moran, one of the students. She stated ideas 5 and 6 as the main ideas of the proof and explained that they seem more applicable than others and that she feels more comfortable with algebraic expressions than with verbal explanations.

Interviewer: So is this because these expressions are written in algebraic form?

Moran: Yes, but these are expressions that even after a month has passed I can still understand what I wrote here, I don't have a problem with that… But I don't have any idea how to repeat the proof. To explain to you verbally what the lecturer wrote? I don't have a clue… But that's me.

Daniel, the other interviewed student, listed ideas 1, 3, 4, and 7. When asked to describe the proof in a general manner he referred to using the division algorithm repeatedly in order to reduce the numbers, and stated that the algorithm is finite. When asked about the connection between the gcd of the numbers in the beginning and in the end of the process he did not answer but said this was "a missing link".
The students were also asked to self-estimate their level of understanding of the Euclidean algorithm and of its proof. The results are summarized in Table 2.

Understanding level of the…	Very high	High	Reasonable	None
Euclidean algorithm	3	4	7	1
proof of the algorithm	1	1	10	3

Table 2: Students' self-estimations of their algorithm/proof understanding level

Out of 15 students who answered the questionnaire only two (seven) estimated themselves as having high or very high level of understanding of the proof (of the algorithm). The lecturer was asked to relate to possible difficulties encountered by students during the lesson. He stated that based on his past experience he demonstrates the algorithm on a numerical example before the general proof, and this raises students' involvement in the lesson and improves their understanding of the general proof; he perceived no particular difficulties. Taking into account his belief that considerations for proof presentation are purely mathematical, the data in Table 2 may be interpreted as showing a lack of adaptation of the flow of the proof to the audience.. This is also supported by the following excerpts from the interviews:

Moran: The board is filled with words and I am completely lost, I can't understand where the beginning is and where the end... I am constantly writing because I need to figure it out at home, but during the lesson it's like I am writing Chinese…

Daniel: I am taking two courses and this lecturer is by far better than the other … He's very organized, answers every question, I don't have much criticism, but … somehow, during the proofs I'm getting lost. It suddenly happens, I am telling myself: "Hey, wait a minute, where am I?"…

Possible alternative flows

Hence the question arises whether alternative flows based on the same modules might produce higher levels of student comprehension. Figure 3 presents two options.

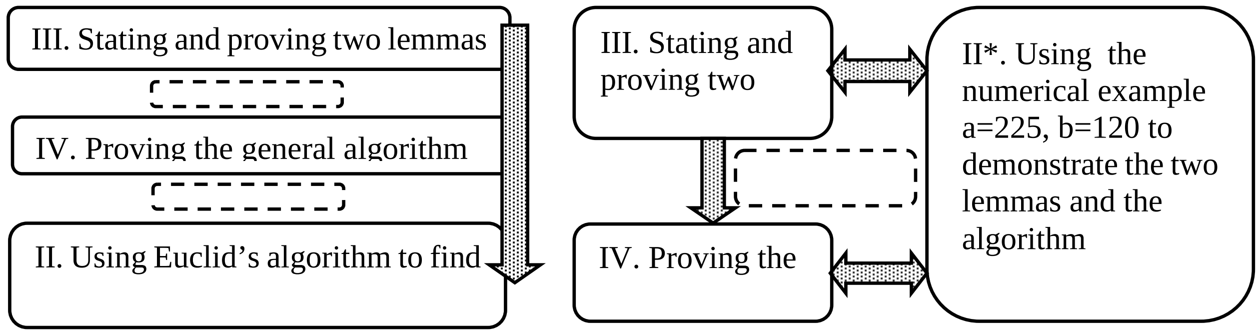

Figure 3a: (left) & 3b (right): Alternative global proof flows

Figure 3a represents a somewhat more conventional textbook type of flow, linearly proceeding from the lemmas to the algorithm's general proof and ending with an example. One characteristic of this flow is the temporal proximity between the example and the general algorithm. That proximity can also be achieved by the flow depicted in Figure 3b. Here the numerical example is used for framing the lemmas as well as the general proof, by going back and forth between formal deductions and their numerical illustration. This flow might also help to adjust the pace of the lesson to the students and to solve a problem indicated by Moran relating to the gap between the experienced mathematician and the students:

Moran: For him to say gcd – it's just gcd, it's clear. For me it's the first time I encounter this expression and it takes me ten minutes to understand what he means. Like, wait a minute, what's gcd? And then to see "a|b". Ten times during the lesson… whenever he wrote a|b I wrote a verbal note "a divides b" so when I read it at home I can remember that it means "a divides b" and not "b divides a". So I'm stuck on that - and he's running …

The decision to favour one type of flow over the other is of a didactic nature and the effect of using such alternative flows is a topic for further research.

CONCLUSIONS

In this paper, the notion of flow of proof was introduced, and global flow analysis was used to discuss the gap between the lecturer's and his students' answer to the question: What are the main ideas of the proof of the Euclidean algorithm? The analysis demonstrates the potential to produce a deep understanding of a multifaceted learning scenario, but many questions remain open. The analysis of local features of the proof presentation and the relation between the local and the global features is one further topic for research. Another one is the effect of the flow on affective factors and the intertwining between the cognitive and the affective factors.

The lecturer had a clear view of the main ideas he intended the students to have at the end of the lesson and he presented these ideas accordingly. Nevertheless, our data show that the outcome of the lesson was less than intended. Using a different flow, perhaps missing from the lecturer's "didactic arsenal", might help. In addition, the lecturer did not correctly assess the students' proof comprehension difficulties. The lack of suitable didactic strategies and adequate assessment methods is consistent with findings of Weber (2010) and Alcock (2010), and it impedes the creation of the type of communication that should exist in the mathematics classroom as Hanna (2007) recommends. This communication is perhaps particularly important in the training of future mathematics teachers, in accordance with Mariotti (2006) and Dreyfus (2000). Hopefully, studying the different aspects of the flow of a proof in a way that will shape flow as a didactic tool will help turning proof presentation in class into deep mathematical communication between teacher and students.

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THE EFFECTS OF MODEL-ELICITING ACTIVITIES ON STUDENT CREATIVITY

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This paper presents one part of a comprehensive study examining the implications and consequences of model-eliciting activities (MEAs) on mathematically talented and gifted students' development in several dimensions. This part focuses on the effect of a MEA intervention program on students' creative thinking. The method was based on pre-test and post-test forms of the Figural Torrance Tests of Creative Thinking (TTCT). The participants were 71 school students with diverse cultural backgrounds who are members of "Kidumatica" math club. Some of the students participated in a control group and the rest participated in the intervention program. The TTCT pre- and post-tests were administered to both groups. Findings indicated that MEAs have the potential to develop and improve students' creativity.

INTRODUCTION

In an era dominated by changes driven by cutting-edge discoveries, technological revolutions and innovative science, the development of students' abilities to create something novel and valuable has become crucial. Acknowledging the importance of mathematics and science education in this development, the OECD (2012a) set the following goal for its innovative education conference: "to present and discuss evidence and policies in mathematics and science education that can lead to better skills in thinking and creativity" (p.16).

Moreover, our ever-changing and highly competitive world requires an educational system which can provide students with authentic learning experiences that will encourage and cultivate their creative potential and give them opportunities to practice it, while exposing them to unfamiliar real-word problematic situations (OECD, 2012b; Lesh & Sriraman, 2005). The development and promotion of students' abilities to solve problems creatively have been explored by many educators and education researchers in all domains using different approaches (Guilford, 1950; Torrance, 1974; Amabile, 1996; Chamberlin & Moon, 2005).

Model eliciting activities (MEAs) give the student opportunities to deal with non-routine, open-ended "real-life" challenges. These authentic problems encourage the student to ask questions and be sensitive to the complexity of structured situations, as part of developing, creating and inventing significant mathematical ideas (Lesh & Thomas, 2010; Amit & Gilat, 2012; Gilat & Amit, 2012). However, the implications and consequences of such model problem-solving on the development of students' creative thinking have only been addressed in a few studies (Chamberlin & Moon, 2005). The current study was therefore aimed at exploring the effects of a MEA intervention program on the development of students' creative-thinking skills, using the widely used Torrance Test of Creative Thinking (TTCT) which has become highly recognized in the field of education and in the business world (Kim, 2011).

Creativity: definition and assessment

Guilford (1950) noted the importance of creativity as a research topic, emphasizing the implications of educational research and practices related to creativity and creative abilities, which should be encouraged and developed among students. Guilford was concerned with the creative behavior of the problem-solver, which he described as a sequence of thoughts and actions resulting in a novel production, and he defined creativity as the divergent production operation that identifies a number of different types of creative abilities, such as originality, fluency, flexibility, and elaboration. This definition, and the scoring of these four components, has served as the basis for many creativity tests (Guilford, 1967; Torrance, 1974; Sternberg, 2006). Guilford (1967) researched and developed a divergent thinking test known as Guilford's Alternative Uses Task, in which the test-takers are asked to list all of the uses they can think of for a common object, such as a cup, paperclip, or newspaper. Torrance (1974) developed the Torrance Test of Creative Thinking (TTCT) based on Guilford's definition of divergent thinking. In the present study, we used the TTCT-Figural to measure participants' creativity. According to Cramond (1993) and Torrance (1977), the figural test is presumed to be fair in terms of gender and race and for persons who have various language, socioeconomic, and cultural backgrounds (as cited in Kim, 2006).

MATHEMATICAL MODELING, PROBLEM SOLVING AND CREATIVITY

The parallelism between problem solving and creative thinking has been addressed in several studies (Guilford, 1967; Amabile, 1996; Sriraman, 2009). Guilford (1967) argued that problem solving and creative production "have so much in common that they are basically the same phenomenon" (p. 312). However there are a great variety of problem-solving tasks, applied in many subjects and in many different domains, that can stimulate, encourage and develop students' creativity. The literature reveals some features that characterize the problem or task involved in the creative problem-solving process: non-routine and heuristic, complexity of information, and multiple cycles (Mumford, Mobley, Uhlman, Reiter-Palmon, & Doares, 1991;Amabile, 1996; Sriraman, 2009)

Non-routine task - Amabile (1996) claimed that creative problem solving requires a heuristic task in which the path to a solution is unknown, rather than an algorithmic task which involves a known execution of a preexisting algorithm, without any exploration of other possible pathways. Sriraman (2009) argued that the creative mathematical process entails a non-routine problem, a problem with complexity and structure which requires not only motivation and persistence but also considerable reflection. Mathematical modeling activities based on "real-life" problem situations are none routine, authentic tasks with a high level of complexity, in which students are given the opportunity to construct powerful ideas relating to interdisciplinary data (Lesh & Doerr, 2003;Lesh & Sriraman, 2005).

<u>Complexity of information</u> - Mumford et al. (1991) underlined the importance of the information involved in the task, emphasizing the need for categorization and reconstruction of information rather than just recalling and understanding. MEAs differ from traditional “word problems” which define static assumptions involving givens and goals (Lesh & Doerr, 2003, Della & Cynthia, 2010). Mathematical modeling activities are based on “real-life” problem situations in which students are given the opportunity to construct powerful ideas relating to interdisciplinary data (Lesh & Sriraman, 2005). The ambiguity and the complexity of the data expose students to different approaches, multiple pathways and different or innovative mathematical solutions (Chamberlin & Moon, 2005; Amit & Gilat, 2012).

<u>Multiple cycles and solutions</u> - The literature reveals the importance of tasks that involve multiple cycles of exploration, in which the generation of new ideas, responses, new pathways, and alternative solutions are invented or discovered (Guilford, 1967; Munford et al., 1991); Amabile, 1996; Sriraman, 2009). The model-development processes involve a series of recursive cycles consisting of developing, testing, and revising phases in which a variety of different ways of thinking are repeatedly expressed, tested, and revised or rejected (Lesh & Doerr, 2003; Lesh & Thomas, 2010).

METHODOLOGY

In this paper, we describe part of a multidimensional study aimed at revealing the implications of a MEA intervention program on students' cognitive abilities, creativity and creative mathematical thinking. The intervention program focuses on exposing students to a variety of authentic mathematical modeling activities developed according to the six principles of MEAs: reality, model construction, self-evaluation, model documentation, model generalization, and the simple prototype principle and modeling process theory (Lesh, Amit, & Schorr, 1997). The study included two groups. The experimental group participated in the intervention program, while the control group did not. The experiment lasted 7 months; before the start of the intervention program, a pre-test was administered to students in both groups to assess their level of creativity. The intervention program included three different workshops that were administered by the researchers in four experience lessons lasting 75 minutes each. Each intervention workshop was made up of three parts: a warm-up activity, a MEA and a poster-presentation session (Figure 1). Upon completion of the program, both groups were evaluated with a post-test.

<u>The warm-up activity</u> consisted of a math-rich reading passage and readiness questions, serving to generate students' interest and motivation, and to introduce the context for the modeling activity. This activity ensured that students would have the initial knowledge to solve the modeling task, including factual knowledge and cognitive and technical skills, so that their solution would stem from their own experience (Lesh & Thomas, 2010).

<u>The modeling activity</u> consisted of a data section and a modeling task. The data section contained tables, images or drawings, with the third part of the readiness questions often referring to these structures. The modeling task consisted of a short question or statement, asking the student to solve a mathematically complex problem for a hypothetical client (Amit & Gilat, 2012). This part was performed in groups of 3 or 4 students, and was likely to involve several iterative cycles in which the students need to significantly test and revise their thinking about the situation.

<u>In the poster-presentation session</u>, each group prepared and presented a poster showing their results, in which they explained, justified and mathematically communicated their solution.

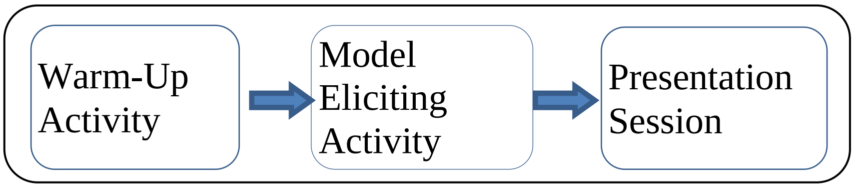

Figure 1 - Typical modeling workshop

Research questions

To what extent, if at all, does experience in eliciting mathematical models for "real-life" situations develop and improve creativity in talented and gifted students?

Are there any gender differences in the development of creativity among students that gain experience in eliciting mathematical models?

Participants

Participants in this study included 71 "high-ability" and mathematically gifted students in the 5th through 7th grades who are members of the "Kidumatica" math club. The “Kidumatica” program provides a framework for cultivation and promotion of exceptional mathematical abilities in youth from varied socioeconomic and ethnic backgrounds. The study consisted of a control group of 24 students (10 girls and 14 boys), and an experimental group of 47 students (14 girls and 33 boys) who participated in the intervention program. Both groups were exposed to the same amount of weekly mathematical enrichment activities at “Kidumatica” (such as logic, problem-solving, number theory, etc.) (Amit & Neria, 2008). While the experimental group participated in the intervention-program activities, the control group worked on other mathematical activities. Pre- and post-test comparisons were made in both the control group and the experimental group; the control group's results served as a baseline to which pre-test and post-test results from the experimental group were compared.

Instrument

This study utilized the standardized TTCT-Figural. The TTCT displays adequate reliability and validity (Kim, 2011) as a measure of creativity. The TTCT is one of the most commonly used measures of creativity in education and educational research, and has been translated into over 35 languages (Kim, 2006; Yuan & Sriraman, 2010). The test has two forms: A and B. The two forms of the figural test were used as pre and post-tests, respectively, and were scored according to the Streamline Scoring Procedure (Torrance, 2008). The test consists of three activities: picture construction, picture completion, and repeated figures of lines or circles. For each activity, students are asked to complete the drawing by turning it into something meaningful and imaginative. A translation (into Hebrew) and back-translation cycles were handled by the first author and two education researchers.

FINDING AND RESULTS

Research question 1

Significant differences were found between the TTCT pre- and post-test results among students participating in the intervention program and among control students. Both groups showed improvement from the pre- to post-test, although the experimental group showed relatively higher improvement (Figure 2).

To determine whether, and to what extent, experience in eliciting mathematical models for "real-life" situations develops and improves creativity in talented and gifted students, we examined the results of the TTCT-Figural pre- and post-tests from the control and experimental groups by performing repeated measure ANOVA with post-hoc analysis using Bonferroni correction. The results indicated significant differences between post- and pre-tests—$F(1, 69) = 30.84$, $p < 0.000$, $\eta^2 = 0.31$—indicating that both the control and experimental groups had improved over the course of the experiment. Moreover, results indicated a significant interaction between the groups and the time (pre-test to post-test)—$F(1, 69) = 9.15$, $p < 0.003$, $\eta^2 = 0.12$—indicating a significant difference between the experimental group's improvement and that of the control group. Post-hoc tests were conducted to evaluate pairwise differences among the means. In the pre-test, the mean score of the experimental group was 41.28 (SD = 22.9) and that of the control group, 42.5 (SD=24.5); in the post-test, the respective mean scores were 67.11 (SD = 24.59) and 50.08 (SD = 28.37) (Figure 2). This indicates that although both groups started with almost the same creative potential according to the TTCT-Figural, after the MEA intervention program, the experimental group exhibited higher improvement than the control group.

Research question 2

The results indicated gender differences in the development of creativity in the experimental group, with the girls scoring higher than the boys on the TTCT tests (pre and post) and showing higher improvement. To further examine the question of gender differences, repeated measure ANOVA was conducted with post-hoc analysis. The results indicated significant differences between girls and boys—$F(1, 45) = 3.35$, $p = 0.073$, $\eta^2 = 0.513$. Post-hoc tests were conducted to evaluate pairwise differences among the means. The girls' mean pre-test score was 47.50 (SD = 23.15), and the boys' was 39.67 (SD = 23.14); in the post-test, the girls' mean score was 76.57 (SD = 15.47) and the boys' was 63.67 (SD = 26.7) (Figure 3). Based on this finding, there was a gender difference in the experimental group: the girls demonstrated higher creative improvement (ΔM29.07) and a higher final score (76.57) than the boys (ΔM24.0 and 63.67, respectively).

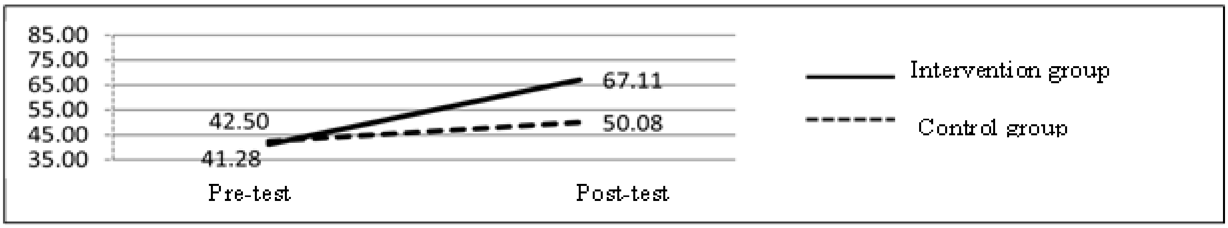

Figure 2 - Intervention and control groups' scores on TTCT-Figural pre- and post-tests

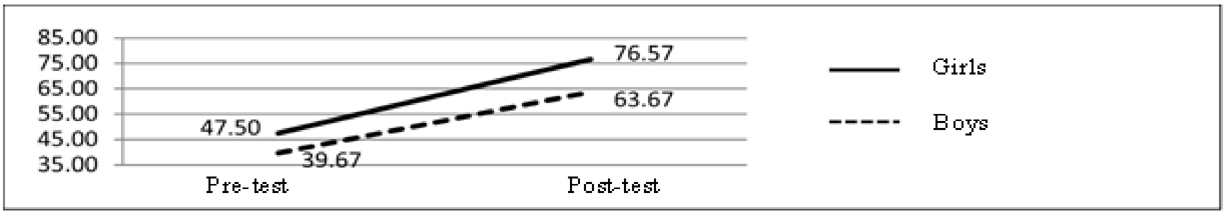

Figure 3 - Scores of girls and boys in the experimental group on the TTCT-Figural pre- and post-tests

DISCUSSION

In the first part of this study, the students participating in the experimental modeling program scored significantly higher on their creativity tests than the control group, although both groups showed improvement in their creativity scores. The development of students' creative thinking skills has been investigated in many studies on education, including mathematical education (Torrance, 1974; Kim, 2006; Baer & Kaufman, 2008; Yuan & Sriraman, 2010). Some of those studies used TTCT scores as an indicator of students' overall creative potential (Kim, 2006; Yuan & Sriraman, 2010). The MEA intervention program introduces students to challenging real-world problem-solving activities which was completely new to them (Lesh & Sriraman, 2005; Lesh & Thomas, 2010). Although this experimental program involved mathematically challenging problem solving, it required students to use their innovative and creative thinking skills (OECD, 2012a; Chamberlin & Moon, 2005; Lesh & Sriraman, 2005). Both control and experimental groups demonstrated significant development in their creativity, in accordance with Amit and Neria's (2008) finding of empowerment of students who are gifted or have high mathematical ability due to their participation in weekly high-enrichment workshops at "Kidumatica" math club.

Results from the second part of this study indicated that the girls participating in the intervention program were somewhat more creative than the boys. These findings coincide with Baer and Kaufman's (2008) conclusions, from their inclusive review of creativity studies, on gender differences in creativity, arguing that "the overall 'winner' in the numbers of studies in which one gender outperformed the other, it would be women and girls over men and boys" (p. 98). The participants in this research—both boys and girls—volunteered to participate in the mathematical enrichment program at "Kidumatica" due to their interest in mathematics and their desire to learn more (Amit

& Neria, 2008), which could indicate a great deal about their motivation (Amabile, 1996). According to Baer and Kaufman (2008) and Amabile (1996), motivation plays a large role in gender differences in creativity and has a much stronger effect on girls than on boys (Baer & Kaufman, 2008). In addition, the girls were young (5^{th} to 7^{th} grade) and according to some researchers “girls do not show less creative achievement until after high school” (Baer & Kaufman, 2008, p. 94). However, reports of studies on gender differences based on TTCT scoring are inconclusive (Torrance, 1977; Kim, 2006; Baer & Kaufman, 2008). This suggests a need for further research, with a higher number of participants and additional analyses of TTCT-Figural factors across control, experimental and gender groups. Although the presented pre- and post-test results are still preliminary and limited, overall this study clearly indicates a positive effect of the MEAs intervention program on the development of students’ general creativity. Furthermore it reveals young girls’ creative advantage over young boys.

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CONNECTING TEACHER LEARNING TO CURRICULUM

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Teacher effectiveness is influenced by individuals' knowledge. Traditionally, this knowledge is measured by the number of college mathematics courses taken. But a plateau effect indicates student learning is only slightly affected beyond a certain number of courses, suggesting that advanced classwork may encourage teachers' ***compression*** *and* ***abbreviation*** *of the mathematics. This report outlines a plan to increase teaching effectiveness using curricular materials designed for student* ***and*** *teacher learning. We posit that unpacking the mathematical knowledge inherent in tasks can provide entry points for student understanding and essential background knowledge* ***necessary*** *for teaching. Importantly, embedding professional learning in curriculum development may be critical to advancing mathematics learning as continued reductions in school district budgets restrict access to quality professional learning opportunities for in-service mathematics teachers.*

INTRODUCTION

It is widely established that teachers need to possess an understanding of mathematics that goes beyond the math they teach. Teachers of mathematics activate this additional mathematical knowledge when they differentiate problems to challenge students, listen to students' explanations of unconventional solution strategies to determine whether or not they are mathematically productive, or select assessment problems that are mathematically similar to the work done in class. However, for middle and high school teachers, the nature of that knowledge remains unclear. Research indicates that effective professional learning models incorporate aspects of content and pedagogy (Ball and Cohen, 1996; Borko, 2004; Putnam and Borko, 1997), however, little research has been done to "… understand how such useful and usable knowledge of mathematics develops in teachers" (Ball & Hill, 2004, p. 333).

> The research, then, is clear: teachers cannot teach what they do not know, and they cannot teach what they know if they do not have the skills to do so. Changing *teaching* is the single most powerful way to improve science and mathematics competency in the United States" (Sanders, 2004).

Research indicates that effective professional development relies on authentic classroom activities linked to the context of schools (Ball and Cohen, 1996; Rosebery and Puttick 1998; Garet 1999; Wilson and Berne 1999) and a rigorous examination of the concrete and site-specific challenges of teaching (Goldenberg and Gallimore, 1991).

We propose a pragmatic solution to advancing teachers' mathematical knowledge needed to teach effectively that is both timely and tied to classroom practice. We suggest that the widespread adoption of the *Common Core State Standards* (CCSS,

2013. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 337-344. Kiel, Germany: PME.

2010) in the U.S. and subsequent requirement that districts resequence, if not rewrite, their curriculum guides and unit assessments provides an opportunity to realign curriculum while improving teaching. Rather than assigning teachers the role of *treatment subjects* within a professional development intervention, teachers could be collaborators in designing and piloting curriculum revisions. Within this model and to better guide student thinking, teachers learn how children's ideas about a subject develop, as well as the connections between student ideas and the important concepts in mathematics (Schifter and Fosnot, 1993). This approach embeds professional learning in districts' systemic reform process and ensures the potential for continuous learning in an otherwise constrained educational and fiscal setting.

THEORETICAL FRAMEWORK

The use of curriculum materials in mathematics classrooms "is one of the oldest strategies for attempting to influence classroom instruction" (Ball & Cohen, 1996) and improve student learning. However, a significant gap remains between the *intended* vs. the *enacted* curriculum. Unfortunately, mathematics education researchers rarely make a *purposeful* distinction between the curriculum as outlined by curriculum designers and the curriculum as it is interpreted and enacted by classroom teachers (Stein, Smith, Silver, & Henningsen, 2000; Stein, Remillard, & Smith, 2006). This is a major factor contributing to the lack of fidelity in curriculum implementation (O'Donnell, 2008; Ball & Cohen, 1996). Further, curriculum developers have often failed to assess teachers' content and/or pedagogical knowledge (Sarason, 1982); and as a result fail to fully appreciate the necessity for teachers to deeply understand the content in order to implement the materials in a way that maximizes student learning (Dow, 1991). Clearly, curriculum materials should be created with closer attention paid to the process of curriculum enactment and teachers' learning of content (Ball & Cohen, 1996).

We suggest two models could be combined to increase the content knowledge of teachers and provide greater levels of curriculum coherence. First, teacher *Professional Learning Communities* (PLCs) should be formed in schools. PLCs have been shown to positively impact teacher learning and student achievement (Burdett, 2009; McLaughin & Talbert, 2006; Sparks, 2005) and are uniquely positioned to advance this work. Implemented properly, this approach could provide a low-cost model to increase teachers' content knowledge as well as encouraging collaboration and professionalism. And second, the PLCs should be tasked with creating *Educative Curriculum Materials* (ECMs), teacher materials designed to deepen teachers' knowledge of content, and increase teachers' ability to flexibly apply their mathematical knowledge across a variety of problem contexts (Davis & Krajcik, 2005). An ECM is not intended to script instruction, but rather is designed to "make teachers' learning central to efforts to improve education, without requiring heroic assumptions about each teacher's capacities as an original designer of curriculum" (Ball & Cohen, 1996, P. 7). A purpose of the ECM is to guide teachers away from developing or favouring a single approach to a problem–to recognize that this suggests to students that there is a *best* or *only* way to approach these types of patterns. We posit it will be necessary to go beyond traditional curriculum materials and teacher guides (including those found in reform curricula) to support teacher learning by helping teachers: (a) develop the specific mathematical content knowledge needed to teach students, (b) consider appropriate pedagogical content strategies to support student learning, (c) cultivate a classroom community focused on learning mathematics over time, and (d) reconsider the role students and the broader community play in mathematics learning (Schneider & Krajcik, 1999). Even though teachers routinely use textbooks as the primary classroom resource (Freeman & Porter, 1989; Sosniak & Perlman, 1990; Stodolsky, 1989; Woodward & Elliott, 1990), textbooks are not uniformly of high quality and can limit, rather than support, teachers' learning and developing professionalism (Ball & Feiman-Nemser, 1988; Woodward & Elliott, 1990). Davis and Krajcik (2005) note that to positively affect teacher learning, ECMs must reflect the complex classroom setting and incorporate all aspects of classroom instruction, including: "planning, lesson modification, assessment, collaboration with colleagues, and communication with parents" (p. 3). With this design, ECMs are thus "uniquely situated in the classroom, unlike other professional development opportunities" (Schneider, Krajcik, & Marx, 2000, p. 60) and subsequently, may prove to be more effective at bridging the gap between educational theory and classroom practice. Since ECMs are used by teachers as they plan lessons for their students, they necessarily access knowledge of content and pedagogy as they reflect on their students in a particular context. It is important to note that although ECMs have been presented as a promising option for advancing teacher learning (Ball et al., 1996; Schneider et al., 2000; Davis et al., 2005), little development work has been done in this area.

Use of PLCs mediates a criticism of the reform movement that argues educational reforms will not result in improved student learning unless the change process resides in schools, in individual teacher's classrooms (Elmore, 2007). Elmore (2007) suggests the current reform movement ignores "the weak incentives operating on teachers to change their practices in their daily work routines, and the extraordinary costs of making large-scale, long-standing changes of a fundamental kind in how knowledge is constructed in classrooms" (p. 24). PLCs can provide the type of school-embedded deep practice that presses teachers' understanding of content knowledge and the associated pedagogical strategies that will best support student learning (Lave & Wenger, 1991, Loucks-Horsley, 1998). Within this professional context, student and teacher interactions support continuous growth over time.

We strongly believe that to prepare students for success in mathematics, classroom activities must simultaneously develop *procedural* and *conceptual* strategies along with *problem solving skills*. We agree with the National Mathematics Advisory Panel that "Debates regarding the relative importance of these aspects of mathematical knowledge are misguided. These capabilities are mutually supportive, each facilitating learning of the others. Teachers should emphasize these interrelations; taken together, conceptual understanding of mathematical operations, fluent execution of procedures, and fast access to number combinations jointly support effective and efficient problem solving" (2008, p. xix).

Another key element in this approach is recognition of the need to emphasize the connected knowledge that is organized around the foundational ideas of a discipline. Learning is the result of an iterative development between conceptual and procedural understanding, where increases in one type of knowledge lead to increases in the other type of knowledge, which elicit new increases in the other (Rittle-Johnson, Siegler, & Alibali, 2001). It is not sufficient to simply provide students (or teachers) with expert models and expect them to learn (NRC, 2005). The connections between any particular procedure and the more ". . . fundamental principles and ideas appears to be the main road to adequate transfer" (Bruner, 1960).

> Teachers need to be able to apply their ideas to novel situations. . . With sufficient robust connections between specific, situated instances and more general principles, the connections should allow the teacher to identify new situations as occasions where the general principle might apply and to recognize ways of applying it as she adapts novel curriculum materials. (Davis, & Krajcik, 2005, p. 8)

PROPOSED ECM DESIGN

ECMs are teacher guides that include supports for teaching strategies *and* for teacher learning. They are designed to increase teachers' knowledge in specific instances of instructional decision making by helping them develop more general knowledge that can be applied flexibly in new teaching situations. This allows teachers in PLCs to explore the range of ideas that students teaching the lesson. The ECM prompts teachers to use classroom discussions to have students compare their individual solution bring to the problem prior to strategies to the strategies used by their classmates to solve the problem. In this manner, the teacher learns the mathematical knowledge needed to press student thinking and understanding, and to make the distinction between conceptual and procedural knowledge visible to students. In addition, the teacher is afforded opportunities to explore, discuss, and implement a variety of instructional "best practices" in a safe environment that support student engagement with the mathematical content.

Table 1 (below) provides an example page of a prototype ECM that would accompany a portion a lesson on proportional reasoning. The structure of the ECM is intended to provide a narrative context for the progression of the mathematics. Please note that the left-hand column describes the sequence of the actual task to be used during the student investigation, and provides possible discussion prompts. The middle column offers insights into the mathematics and prompts for teacher reflection. This column provides explanatory comments about the mathematics, brief questions that press the teacher's thinking about the underlying mathematics in the task, and explicit connections to important mathematical concepts that are possible extensions from the task. The right-hand column provides samples of student work that press teacher understanding of the mathematics students might uncover during the course of the lesson.

Class Activity (What you *do*)	ECM Notes (What you *think about*)	Student Samples (What you may *see*)
Study Buddies - Question 1 Lincoln Elementary pairs 2^{nd} and 6^{th} grade students as “Study Buddies.” Two-thirds of the 2^{nd} graders are paired with three-fourths of 6^{th} graders. Are there more 2^{nd} graders or more 6^{th} graders? How do you know? Think/pair/share: Each person spends 3-5 minutes working the task alone, and then works with a partner for 5-7 minutes to share and discuss their thinking. Ask students to use clear language when discussing their solutions and to define their variables if they use variables. When they say a quantity, they need to say what the quantity is representing (e.g. “two-thirds *of the 2^{nd} graders*”). Remind them that they are striving to understand each others’ solutions. Important Note: It is more important to spend plenty of time discussing details of solutions thoroughly than to end this activity after 10 minutes. The emphasis on discussion is true throughout the facilitation guide.	There are several ways to answer this question. If two-thirds of one number is the same as three-fourths of another number, the first number is larger. Look for examples that use: Tables, Picture, Diagrams, Numbers, Graphs, Proportional reasoning, Algebraic relationships. Take care to value all student representations. You can do this by (a) clearly explaining how the different approaches uncover mathematical information, and (b) unpacking the connections between the representations during the class discussion. Be *intentional* in deciding what order to have participants show and explain their solutions. Start the discussion by having the students that used tables, pictures, diagrams, numbers, or graphs share their work first. Have students present symbolic and proportional reasoning solutions last.	Following are several examples collected from high school students: Example 1: The student in the first example reasoned that since the ratio of paired 2^{nd} graders is smaller than the ratio of paired 6^{th} graders, the class of 2^{nd} graders must be larger. There are more 2^{nd} graders because a less percentage of 2^{nd} graders takes up a larger percentage of 6^{th} graders. Example 2: This student identified the ratios and then stated that “there are more 2^{nd} graders.” Since the student did not provide the rationale for this answer, some potential follow-up questions might include: How did you use the ratios to help you draw your conclusion? Or, Can you name a different pair of ratios that would also tell you that there were more 2^{nd} graders? $2^{nd} = 2/3$ $6^{th} = 3/4$ There are more 2nd graders. Example 3: Ask this student to describe the reasoning for using 1/3 and 1/4 , as well as how the use of decimals supports the conclusion Yes because 1/3 is bigger than $1/3 = .\overline{33}$; $\frac{1}{4} = .25$ Example 4: Ask this student to explain the relationship between 2/3 of one group and 3/4 of another group. Consider: Is one of these quantities larger than the other? What does that infer for the quantities to be compared? no, because $\frac{2}{3}$ is smaller than $\frac{3}{4}$.

Table 1: Sample ECM © Mathematics Content Collaboration Community (MC^3), 2008. Used by permission.

DISCUSSION

Teachers' mathematical knowledge has been measured largely by the number of college mathematics courses taken (Hill, Sleep, Lewis, and Ball, 2007).The mathematics taught in college courses, however, is often quite different than the mathematics used in teaching (Gilbert & Coomes, 2010). In fact, Monk (1994) found that beyond five courses, student learning was less affected by amount of mathematics teachers had taken. Research studies over the last two decades have suggested that while people with bachelor's degrees in mathematics may have a specific type of mathematical knowledge, they often lack what Ma (1999) described as a *profound understanding of fundamental mathematics* –a deep understanding of basic mathematical ideas. In fact, Adler and Davis (2006) suggest that advanced courses may encourage teachers' compression and abbreviation of mathematical knowledge. This situation is particularly problematic when considering teaching practice, since *unpacking* mathematical knowledge can provide critical entry points for students to understand, and therefore is *necessary* for teaching. This process has led to an increased focus on research into teachers' mathematical knowledge as it concerns the depth, connectedness, and explicit articulation of the specific mathematics of teaching (Ball, 2003; Ma, 1999); and further suggests that deepening the content knowledge that teachers need to teach effectively may require a vastly different approach to professional learning than has occurred in the past (Ball & Cohen, 1996; Schneider et al., 2000). It is important to note that "although many reform-based curricula are being developed, they have not been explicitly designed to support teachers' learning" (Schneider et al., 1999, p. 4), nor do they provide the necessary connections to essential mathematics knowledge for teaching.

ECMs go beyond the scope of traditional curricula and are created intentionally to support both student and teacher learning (Ball & Cohen, 1996; Schneider et al., 1999). The traditional approach to mathematics curriculum design has been to define and present mathematics as a hierarchy of procedural skills presented incrementally from basic arithmetic through more advanced subjects. This approach is in contrast to the way that students learn, developing mathematical understanding through the construction of increasingly detailed relationships between concepts (NRC, 2005). Working within PLCs, district specialists and teachers will create mathematics support materials in the form of ECMs and thus, bridge the gap between the *procedural* and *conceptual* strategies that are separate in most mathematics curriculum designs. As teachers use the materials to plan classroom lessons, they advance their own content knowledge. The materials help teachers to reflect deeply upon and unpack the content knowledge that they must know to teach concepts clearly and concisely, to interpret student responses, and address students' mathematical inaccuracies and misconceptions. ECMs, developed in teacher-led professional learning communities, may serve to address the profound lack of sustained professional learning that is the reality for public school teachers (Collopy, 2003). Rather than merely providing "guidelines" for teacher actions, educative curriculum materials provide teachers with insights about the ideas underlying the tasks and choices made for student activities

(Stein, Remillard, & Smith, 2000). Importantly, ECMs produced within PLCs will advance teacher learning while advancing teacher autonomy (Shkedi, 1998).

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CONSTRUCTING META-MATHEMATICAL KNOWLEDGE BY DEFINING POINT OF INFLECTION

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We present one case from a larger study designed to investigate ways to evoke a need for mathematical definitions among high school students and the process of constructing the definition following that need. The case describes a learning experience in which two students dealt with the notion of inflection point. We argue that the process of constructing a definition in this case like in other cases offers opportunities for constructing meta-mathematical ideas over and above the construction of the definition itself.

INTRODUCTION

Some mathematics educators argue that dealing with a definition is a necessary condition for understanding the concept that is defined. For instance Ouvrier-Buffet (2006) suggests that

> Apprehending a concept implies taking simultaneously all the following elements into consideration: examples and non-examples of the concept, allowing a first apprehension of the concept, the definition(s) of this concept and the proof of their equivalence, several representations of this concept and above all, the situations which allow the emergence of the concept and preserve its meaning. (p. 261)

Vinner (1991) stated that "the ability to construct a formal definition is a possible indication of deep understanding" (p. 97). But observing mathematical instruction reveals that this important issue is often neglected. Already 20 years ago, Borasi (1992) wrote:

> An analysis of the most popular syllabi and textbooks, as well as conversations with several mathematics teachers, soon made it clear that despite its importance, the notion of mathematical definition is rarely, if ever explicitly examined in precollege mathematics instruction. (p. 7)

Two decades later it seems as though nothing has changed. Students are rarely asked to deal with the roles and characteristics of mathematical definitions. If and when they encounter definitions, these usually come as polished statements formulated by the "authorities" – mathematics teachers, textbooks etc. In this paper, we suggest a way to involve students in mathematical activity concerning definitions, taking into account the elements suggested by Ouvrier-Buffet (2006). Our designs are intended to evoke a need for a definition, which leads the students to construct the definition.

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One of the characteristics of a mathematical definition is its arbitrariness. Definitions are man-made. They are not right or wrong, but accepted or rejected. The arbitrariness, says Borasi (1992), is not absolute. Definitions are mathematicians' attempts to create useful concepts compatible with our intuitive conceptions and they provide valuable tools to inquire into a mathematical situation.

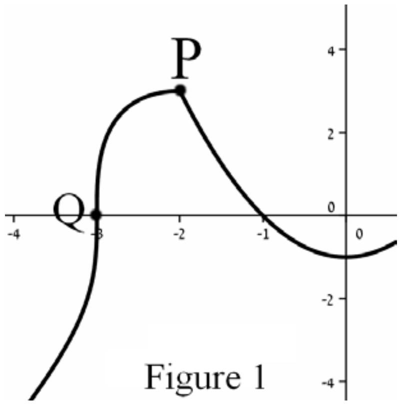


Figure 1

A concept that can nicely demonstrate the arbitrariness of a definition is *inflection point.* The arbitrariness can relate to questions like: does the function have to be continuous/differentiable at an inflection point? Does the 2nd derivative always equal zero at an inflection point? For example, are points P or Q in Figure 1 inflection points? It's a matter of conventional agreement which conditions we require and how we answer the above questions.

Tsamir and Ovodenko (2004) studied secondary school mathematics teachers' images and definitions of inflection points. They found that "commonly the concept image of inflection points included two types of points: those that fulfil the requisite $f'(x)=0$, and those that are (mis)placed in the spot where the curve bends" (p. 343). To the question "what is an inflection point?" most respondents provided concave-convex considerations, several suggested the insufficient condition $f''(x)=0$, but none of them referred to the continuity or differentiability of the function.

THE STUDY

The case presented in this paper is part of a larger study. The study investigates ways to evoke a need for a mathematical definition among high school students and examines different aspects of the process of constructing the definition following that need. Five activities were designed, dealing with different concepts: zero exponent, parabola as a locus, tangent to a graph at a given point, vertical asymptote and inflection point. Task-based interviews (Goldin, 2000) for each activity were carried out with four pairs of students. Every interview was videotaped and transcribed.

The process of investigating students' need for a definition of a concept and their construction of that definition might have different aims. One aim might be the construction of a concept that is accompanied with cognitive difficulties (e. g., the concept of derivative as a limit as described in Kidron, 2008). Another aim might be the construction of a concept (its conceptual understanding) and at the same time the ability to overcome the difficulties to coordinate between different registers (algebraic, graphic etc.), which offer different representations of the concept (e. g., the parabola example as described in Gilboa, Dreyfus & Kidron, 2011). The aim of the present study relates to the questions "what is a definition?", "what are the characteristics of a definition in mathematics?" and in particular to the arbitrariness characteristic of a mathematical definition. For this we designed a task-based interview dealing with the notion of inflection point. The activity includes examples that might conflict with the students' previous knowledge (for instance a point in which the 2nd derivative is not defined but the curve turns from concave to convex) and thus evoke the need for a definition.

The methodology for analysis (AiC)

We analyze the data using the RBC nested epistemic actions model for Abstraction in Context (Schwarz, Dreyfus & Hershkowitz, 2009). We chose this framework because it offers a systematic way to investigate processes of constructing knowledge and it takes into account the need for a new construct as part of the process of abstraction. According to AiC, a process of abstraction has three stages: the need for a new construct, the emergence of the new construct, and the consolidation of the new construct. Schwarz et al. (2009) claim that the need for a new construct is a prerequisite for the construction of the construct. They say:

> The need [for a new construct] may arise from of the design, from the student's interest in the topic or problem under consideration, or from combination of both; without such a need, no process of abstraction will be initiated. (p. 24)

The second stage is the central stage during which the new construct emerges. For analyzing the second stage, AiC suggests three observable epistemic actions: Recognizing (R) - the learner recognizes that a specific previous construct is relevant to the situation he or she is dealing with; Building with (B) - the learner acts on or with the recognized constructs in order to achieve a goal such as solving a problem; Constructing (C) - using B-actions to assemble and integrate previous constructs to produce a new construct. Hence R-actions are nested within B-actions, and B-actions are nested within C-actions. Constructing refers to the first time the learner uses or mentions a construct. Later uses may be part of the consolidation stage.

The story of one case

Two grade 11 students, Ruth and Adi, learning in a high level mathematics' track, were engaged in the task-based interview on points of inflection. In the first part of the interview, the students were asked to indicate inflection points of given graphs of functions; some of the graphs were accompanied by corresponding algebraic expressions. The students had learned about inflection points at school, and were influenced by that. Their concept image (in the sense of Vinner & Hershkowitz, 1980) of inflection point was identified on the basis of their answers as follows: An inflection point is a point on a graph of a function that fulfils the following requirements: (a) At this point the concavity of the graph turns from concave to convex or vice versa; the students called this the "pictorial explanation"; and (b) at this point the value of the 2nd derivative is zero and turns from positive to negative or vice versa; the students called this the "mathematical explanation". The case of the function $y = \sqrt[3]{x}$ (Figure 2a) caused a conflict between the two "explanations": The 2nd derivative of this function is not defined at the origin, but the transition from concave to convex at the origin is visually obvious. The following are some excerpts from the interview, in which Ruth and Adi express the conflict:

246 Ruth: It seems like there is [an inflection point], but maybe it is a hole on the graph. I mean, it just passes through this point but there is nothing there.

265 Adi: We actually say that there is no inflection point, but we see that it is inflecting.

308 Ruth: It’s like an asymptote of an inflection point… It’s supposed to be an inflection point but it was cancelled… It is an inflection without the point, but it is not an inflection point.

The students recognised (in terms of AiC) their previous construct of inflection point as reflected in their concept image as relevant to the task, but they realized that it is not sufficient for deciding whether or not there is an inflection point in this case. They understood that their “explanations” are not sufficient for determining in all cases whether a point is an inflection point or not. Hence, they felt a need for a definition of this concept expressed by Adi (387): "But if we don't have the graph, we don't see it, and she gave us only the equation, then we have to know if there is an inflection point, even if mathematically it isn't reasonable".

During the discussion of Figure 2a Adi (577) said: "Maybe about the 2nd derivative it’s not always true. Like, maybe it’s usually true but sometimes not". They also found out that the sign of the 2nd derivative turns from plus to minus, and they decided to "put the mathematical part in brackets". In terms of AiC, they constructed the knowledge element "It is not a necessary condition for an inflection point that the 2nd derivative equals zero".

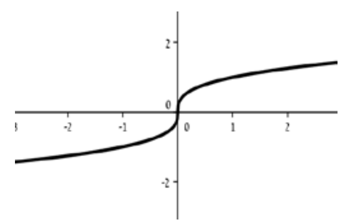
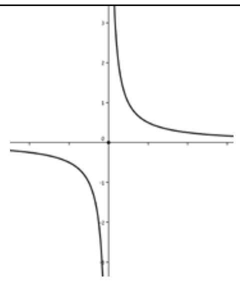
$y=\sqrt[3]{x}$	$g(x)=\begin{cases}\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$
	

Figure 2a Figure 2b

Toward the end of the interview, the students dealt with the function g(x) shown in Figure 2b. They discussed whether a point of discontinuity can be an inflection point. Here are some excerpts from this discussion:

842 Ruth: If it’s not continuous can it be an inflection point?

843 Adi: I don't know. If let’s say there wasn't a point at zero

844 Ruth: mhm

845 Adi: then we would say that

846 Ruth: It's like we said before

851 Adi: that we determined that there is, that's how we knew, and it didn't work out mathematically.

852 Ruth: Right. So here it is also doesn't work out mathematically.

855 Adi: You understand what I am saying? If I didn't have the zero then actually I wouldn't have an inflection point

856 Ruth: Yes, yes

857 Adi: and then it is like this… there is inflection.

860 Ruth: Right. Maybe it is. It is just for not having zero, zero.

861 Adi Maybe just because x equals zero it is telling us that there is an inflection point

862 Ruth Yes. It is not inflecting, that is to say it is inflecting but you can't actually see it

Building with (in terms of AiC) the previous construct "It is not a necessary condition that the 2^{nd} derivative equals zero", they agree that the origin in 2b is an inflection point of g(x). They also build with the idea that once you determine a definition, all specific cases that satisfy the definition are examples of the concept (846-852, 857), and they formulate their definition (867-876): "An inflection point is a point where the graph turns from concave to convex or vice versa, where the x and the y of the point belong to the domain of the function". They add that they have put the mathematical part in brackets because it is not always true.

At this point, we argue that the students have constructed a definition for inflection point, namely the one they formulated. Next, the following dialogue took place (I denotes the interviewer):

879 I: O.K. this is your definition. Does it work for all the examples we had?

883 Adi: Ah, yes. If we leave it like that. But if we want to include the mathematical part, because that is what we were taught

884 Ruth: it changes the answers.

903 Adi: I have a question. Is our goal to determine if there is an inflection point from a graph without a formula or from a formula without a graph and get the same answers? Let's say if there is an inflection point or there isn't an inflection point in both cases?

906 I: More or less. Your answer to the question 'what is an inflection point?' should work for all the examples.

910 Adi: So it depends on our decision. Because if we had determined that it is not [an inflection point, referring to the origin in Figure 2a] then we could keep the mathematical part.

911 Ruth: Right. The question is if you need the mathematical part or not.

912 Adi: You can understand directly from the word inflection that there is a change in the concavity or in the convexity that it is inflecting.

913 Ruth: But mathematically it doesn't work.

914 Adi: But they cannot take a concept and say that it is like this mathematically. Maybe it was translated incorrectly.

In terms of AiC, we say that Ruth and Adi constructed a new knowledge element – the arbitrariness of a definition – in the context of the definition of inflection point. They again question the necessity of $y'' = 0$ and raise the question whether an inflection point is a mathematical concept, characterized by certain algebraic conditions but not necessarily accompanied by a graphic expression, or whether it is a visual idea. Adi (914) even suggests that the term inflection may have been translated incorrectly into Hebrew (which, in fact, is not the case; the Hebrew term is best translated back into English as “twisting point”, a term that expresses the notion at least as well as “inflection point”). Using the arbitrariness (building with it, in terms of AiC) they then construct a new definition that includes $y'' = 0$ as a necessary condition for inflection points. They determine that according to this new definition, g(x) (Figure 2b) has no inflection point not because of the lack of continuity but because the 2nd derivative is not defined at the origin, and that the same holds for $y = \sqrt[3]{x}$.

DISCUSSION

Is there a happy end to the story? One might say that the last definition the students constructed is the same as their concept image at the beginning of the learning experience, but we argue that, in spite of the similarity, they constructed new mathematical knowledge and learned an important lesson. Prior to the learning experience, the “pictorial explanation” and the “mathematical explanation” existed as parallel (and not necessarily compatible) registers in the students’ minds. In the two examples, the “pictorial explanation” had the upper hand and the “mathematical explanation” was simply not always germane for the students. Adi's saying in line 903 and the dialogue in lines 910-914 demonstrates that as far as the definition is concerned, the students have difficulties coordinating between the two registers. Constructing the arbitrariness character of definition enabled the students to reinforce the status of the “mathematical explanation” and to allow it to have the upper hand. Determining that the origin in Figure 2a is not an inflection point, despite of what is visually obvious, confirms that they understood the difference between a definition and a description; they understood that a definition is not a description with some pictorial explanation.

In the process of constructing “their” definition, Ruth and Adi achieved some aims of the process of constructing a definition:

- They understood the difference between a definition and a description.
- They understood that a definition must not be only pictorial but must coordinate registers, the graphical representation and what the students called “the mathematical explanation”.
- They realized the arbitrariness character of definition, at least in the current context.

In addition, two more ideas can be inferred from our case study:

(1) The case demonstrates a situation that was identified in previous studies (e. g., Alcock & Simpson, 2009; Edwards & Ward, 2004; Vinner, 1991; Vinner & Dreyfus, 1989): Even when students had been exposed to the mathematical definition of a concept they frequently didn't use it when solving problems that required the use of the definition. The concept image dominated the concept definition. In our story, the students constructed the knowledge element "It is not a necessary condition for

inflection point that the 2nd derivative equals zero", which could and did, at first, lead them to construct the conventional definition. But eventually they abandoned it for the benefit of their concept image: an inflection point is a point where the second derivative exists and $y'' = 0$.

(2) The students did not construct the conventional definition of inflection point, which requires continuity. They exploited the fact that, as we noted in the introduction, the definition of an inflection point is to some extent a matter of conventional agreement and the arbitrariness character of the definition permits different definitions. AiC researchers are not interested primarily to analyse if the students constructed the conventional constructs. The AiC analysis focuses on what the students actually did construct and how they progressed. The RBC epistemic actions model, a systematic tool for analysis offered by AiC, enabled us to look beyond the "last line" of the present story, namely that supposedly nothing new was learned. Using this tool for analysis of constructing processes revealed to us the important constructing process of arbitrariness of a definition, which in turn served as a building block for the construction of the new definition that includes $y'' = 0$ - a construct which, superficially, looks like the students' old concept image, but through the lens of AiC turns out to be a bona fide mathematical definition based on mathematical grounds.

The case we presented in this paper demonstrates the importance of letting students construct mathematical definitions, following the emergence of a need for the definition. The larger study, which is still in progress, includes other examples where the need for a definition is evoked and the definition is subsequently constructed. These examples demonstrate further opportunities to achieve important aims during the process of constructing a definition (see also Gilboa et al., 2011), and to construct other meta-mathematical ideas than the arbitrariness of a definition, for instance *essence* (what constitutes a mathematical definition) or *consistency* (a definition must not cause a contradiction among the mathematical system in which it defined), constructs that contribute to the mathematical knowledge of the students over and above the knowledge of specific definitions.

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DIFFICULTIES IN THE CONSTRUCTION OF EQUATIONS WHEN SOLVING WORD PROBLEMS USING AN INTELLIGENT TUTORING SYSTEM

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We present results of a research conducted with secondary students (15-16 years old), which addressed the effects of the teaching of algebraic solving of arithmetic-algebraic word problems by using the software Hypergraph Based Problem Solver (HBPS). We show students' performances in which they construct equations by making one of the unknowns appears isolated in one side of the equation. Students' actions and comments suggest that this tendency stems from the difficulty in interpreting the equal sign as a comparison between quantities.

Keywords: word problem solving, intelligent tutoring systems, algebra, equal sign

INTRODUCTION AND AIMS

There are studies that relate the students' previous arithmetic experience with the displayed difficulties when they solve word problems in an algebraic way. Behr, Erlwanger, & Nichols (1976) and Kieran (1981) agreed that students use to give a procedural character to the equal sign, that is, they conceive this sign as a signal to do something. In the same way, Stacey & MacGregor (2000) exposed that students often have difficulties to think in the equation, given their propensity to calculate. They identified also different perceptions of equations: a formula for working out the answer, a narrative describing operations yielding a result and a description of essential relationships.

In addition, the use of technology in teaching and learning the algebraic way of solving word problems has occupied an important role in the developed educational research along last decades. Intelligent tutoring systems (ITS) have been designed and evaluated with the aspiration to exploit the possibilities of one-to-one tutoring. Some examples are *AnimalWatch* (Beal, Arroyo, Cohen, Woolf, & Beal, 2010) or *MathCAL* (Chang, Sung, & Lin, 2006).

In this paper we provide results of a case study in which it is observed how secondary students solved word problems in an algebraic way using an ITS called Hypergraph Based Problem Solver (HBPS) (Arnau, Arevalillo-Herráez, Puig, & González-Calero, 2013). In particular, we will focus on analyzing the students' difficulties when constructing the equation (or equations) that solve a word problem.

THEORICAL FRAMEWORK

In this study, we adopt the theoretical and methodological perspective of Local Theoretical Models (LTM), that was presented in Filloy (1990) and was fully developed in Filloy, Rojano, & Puig (2008). Next, we present the formal competence model and the fact that HBPS was intentionally built under the constraints of our competence model has led us to consider it an ideal element in our teaching model.

Competence in solving algebraically word problems requires, among other factors, competence in the process of translating the text of the statement problem in natural language to algebraic language (Filloy, Puig, & Rojano, 2008). Overall, we can describe this requirement through the first four steps of the ideal sequence of steps in which we can break down the Cartesian Method (CM): 1) the analytic reading of the statements of the problem to transform it to a list of quantities and relations among quantities; 2) choosing a quantity (or several quantities) which one designates with a letter (or several different letters); 3) writing algebraic expressions to designate the other quantities, using the letter (or letters) introduced in the second step and the relations found in the analytic reading made in the first step; 4) writing an equation (or as many independent equations as the number of letters introduced in the second step) based on the observation that two (non-equivalent) algebraic expressions written in the third step designate the same quantity; 5) transforming the equation into a canonical form; 6) the application of the formula or the algorithm of solution to the equation in canonical form; and 7) the interpretation of the result in terms of the statement of the problem (Puig, 2010). The CM is the method with which is usually introduced the teaching of algebraic solving of word problems, although CM is not usually presented explicitly. The operation of HBPS forces the student to complete one step before moving to the next one. This could promote the reflection on the ideal step sequence of the CM and improve the competence in translating the text from natural to algebraic language. In fact, and in line with this thesis, Arnau et al. (2013) put forward results of an empirical study that shows how the use of HBPS produces an increase of the competence level in algebraic solving of word problems.

THE INTELLIGENT TUTORIAL SYSTEM HBPS

The tutor HBPS stands out for its ability to offer flexibility to the user during the resolution maintaining appropriate tutoring on users' actions. So, HBPS is able to: 1) support the resolution following different paths of resolution; 2) allow algebraic resolutions with one or more letters; 3) support arithmetic resolutions when the characteristics of the problem permits it; and 4) provide immediate feedback to the solver.

We will illustrate briefly how the tutor HBPS works using the problem *The tea*, but first we will make an analysis of the problem. The problem reads as follows:

> We have two types of tea: one from Thailand at 5.2 €/Kg. and other one from India at 6.2 €/Kg. How many kilograms of tea from India we have to add to 45 kilos of tea from Thailand to obtain a mixture at 5.75 € / Kg?

The analytical reading constitutes the first step of the CM and transforms the problem into a list of quantities and relations. A possible reading for *The tea* could be:

> Quantities: price of a kilo of tea from Thailand (*Put*), price of a kilo of tea from India (*Pui*), price of a kilo of tea mixture (*Pum*), kilos of tea from Thailand (*Ct*), kilos of tea from India (*Ci*), kilos of tea mixture (*Cm*), total price of tea from Thailand (*Pt*), total price of tea from India (*Pi*) and total price of tea mixture (*Pm*).
>
> Relations: $Put \cdot Ct = Pt$, $Pui \cdot Ci = Pi$, $Pum \cdot Cm = Pm$, $Ct + Ci = Cm$; $Pt + Pi = Pm$

Second and third steps in HBPS are expressed in an explicit definition of all the involved quantities in the analytical reading. The definition of a known quantity demands the assignment of a numerical value, while the definition of an unknown quantity involves the assignment of a letter or an expression. This (algebraic or arithmetic) expression reflects the relation between this quantity with other quantities. In HBPS, after selecting the problem, initially it is only shown the statement and the *Quantity definition panel* (see Fig. 1)

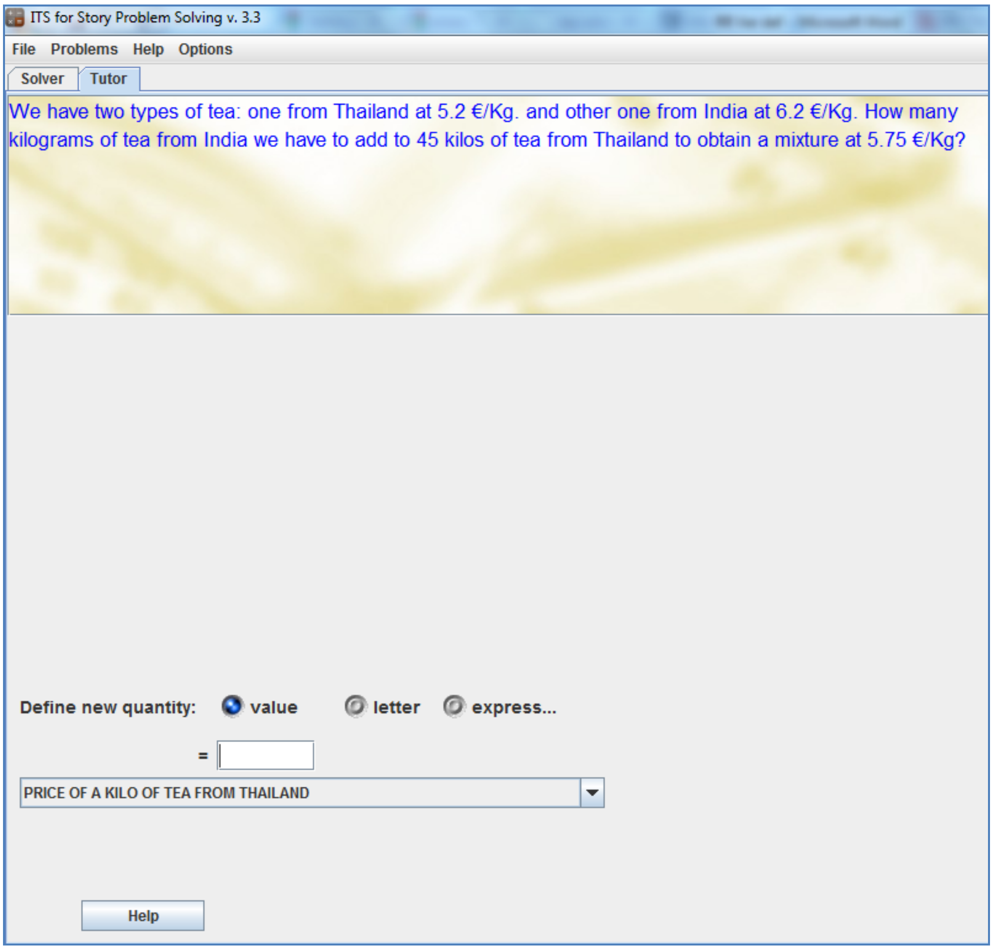

Fig. 1. After selecting the problem *The tea*.

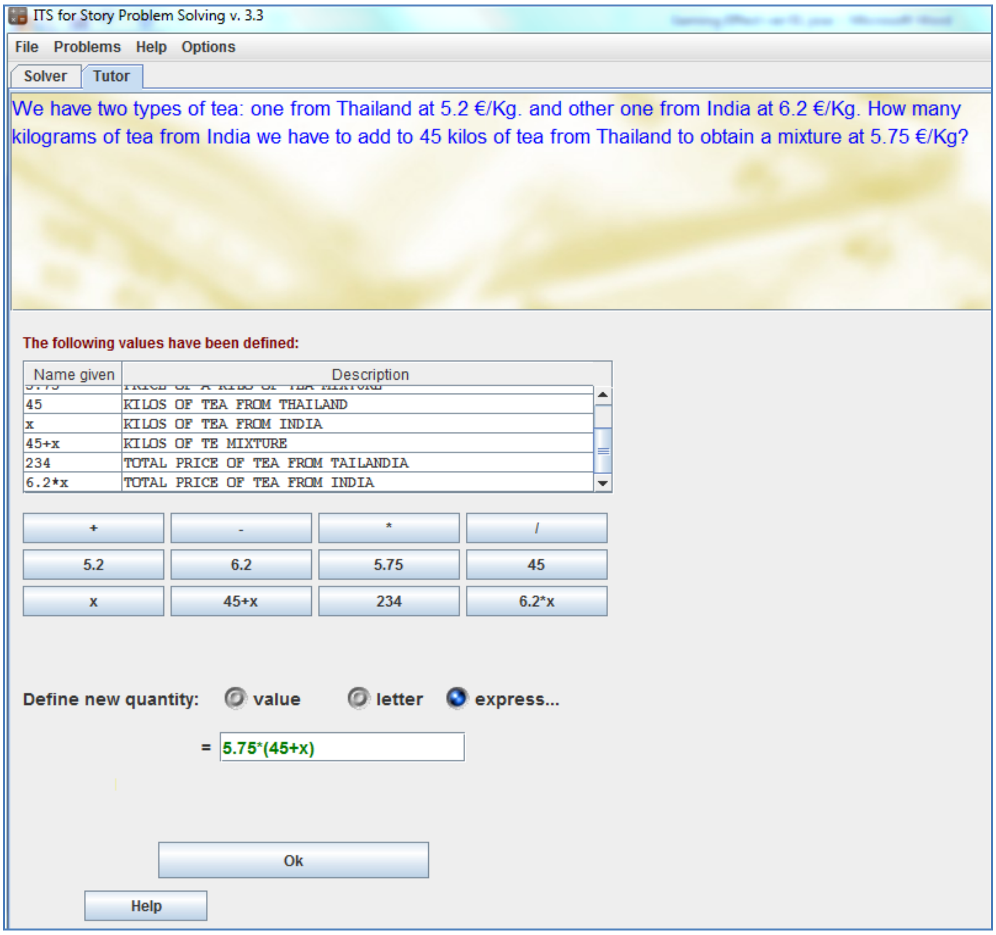

Fig. 2. Finishing the third step of the CM

All the quantities defined by the solver are stored and displayed in a table and can be consulted at any time during the resolution. Fig. 2 is a screenshot of the resolution at the time of declaring the unknown quantity, *total price of tea mixture* (*Pm*). As the figure shows, the solver has opted to use only one letter (x), assigning it to *Ci*, and expressing the other unknown quantities in terms of this letter. The only exception is *total price of tea from Thailand* (*Pt*), which has been represented by the arithmetic expression $5.2 \cdot 45$ ($Put \cdot Ct$), and that the tutor calculates directly (234). In summary, *Cm* is represented by $45 + x$ ($Ct + Ci$), *Pi* by $6.2x$ ($Pui \cdot Ci$), and *Pm* by $5.75\ (45 + \mathrm{x})$ ($Pum \cdot Cm$).

At the time the solver has assigned expressions to all quantities (which means to complete the third step of the CM), HBPS automatically activates the *Equation construction panel* (fourth step of the CM). In the present example, the program will allow only the construction of one equation, because only one letter has been used previously. Fig. 3 shows the *Equation construction panel* at the time in which the solver introduces the correct equation. The solver is building an equation on the dual representation of the quantity *total price of tea mixture* (*Pm*): $5.75(45 + x)$ (using the relation $Pm = Pmu \cdot Cu$) and $6.2x + 234$ (using the relation $Pm = Pt + Pi$).

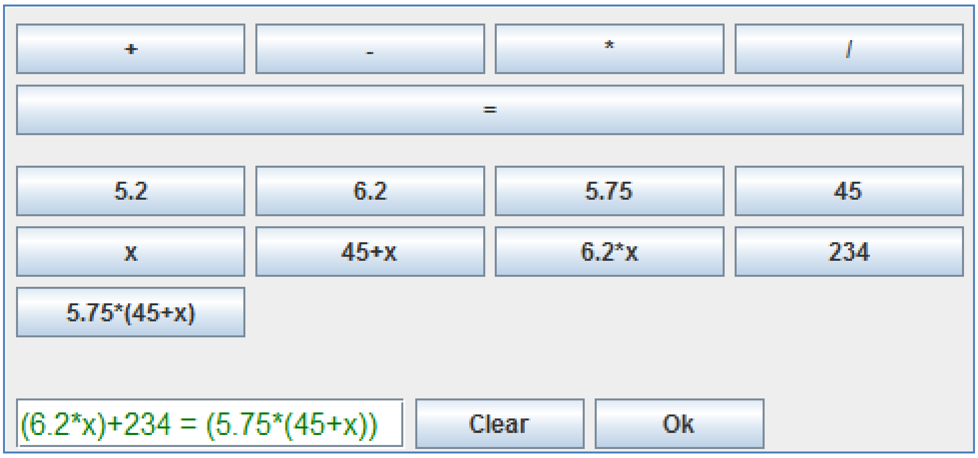

Fig. 3. Finishing the fourth step of MC

THE EXPERIMENTAL DESIGN

We analyze excerpts of protocols obtained from recordings of two pairs of students while they solve the problem *The Tea* in an algebraic way using HBPS. The students were part of a natural group of 36 students in the fourth year of secondary school (15-16 years old) in a Spanish public secondary school. They had received prior instruction in the algebraic way of solving word problems.

The pair P & M in the problem *The tea*

We begin the analysis when students started the fourth step of CM. Until then, P & M had used two letters and had assigned expression to all unknown quantities in terms of these two letters. In particular, the letter x represents the quantity Ci and the letter y the quantity Pm. The remaining unknown quantities were expressed as follows: Cm as $45+x$ (using the relation $Ct + Ci = Cm$); Pi as $6.2x$ (using the relation $Pui \cdot Ci = Pi$); and Pt as 234 (result of using $Put \cdot Ct = Pt$, where Put and Ct are quantities informed in the statement of the problem). At that moment, the relations $Pum \cdot Cm = Pm$ and $Pt + Pi = Pm$ remained unused. They should construct the equations on them.

In the dialog that follows, M rereads the statement and proposes to start writing an equation in explicit form (y = …) (item 1). However, when M asks what the letter y represents, P shows doubts (item 4). In fact, it is M who finally looks for it in the defined quantities table to make sense to the letter. Furthermore, it is M who enters correctly the first of the two equations using the relation $Pt + Pi = Pm$ (item 6).

1. M: So ... how many kilos of tea from India must be added to 45 of tea from Thailand to get the mixture at 5.75? (She rereads the statement).

   […]

2. P: y is equal to… I think <u>first we have to put down y is equal to</u>…
3. M: Let´s see … So what was it?
4. P: Mmm… I think it´s the cost of…
5. M: The price of tea (after consulting in the defined quantities panel by dragging the scroll). I mean this, five point seventy-five, which costs a kilo by the kilos you have. Right?

6 M: (M writes $y = 234 + 6.2*x$).

M seems to appear surprised by the fact that they have not finished yet (item 8). She might not be aware that they must construct as many equations as used letters. Besides, P says that "they already have" *Pm* and now they must calculate *Ci* (item 9). In principle, this could be interpreted as if each equation allows them to calculate an unknown or that the second equation must be built on a dual expression of the quantity *Ci* (which is represented by the letter *x*). But later, while discussing how to construct the second equation, P affirms "Listen to me, M, in this (equation) forget the prices, because the prices are supposed to be in the above one (referring to the first equation)". From this, it could be deduced that P intends to build the second equation using only quantities belonging to the magnitude weight, but not prices. However, the relation between these quantities from magnitude weight has been already used to express the quantity *Cm* (it was assigned $45 + x$ using the relation $Ct + Ci = Cm$).

7. P: OK. (M presses the OK button and the program validates the equation).
8. M: <u>We have to do two!</u> (She laughs surprised).
9. P: Ok. Now…<u>We have already the total price of the mixture</u> (point at the first equation). Now <u>we have to make the kilos</u>.

The verbalization of M appears to verify that she considers all that is left to do is to calculate the quantity *Ci* (*x*) (item 10). This enhances the interpretation that they consider that the other unknown, *y* (*Pm*), has been calculated. Again it becomes evident how the idea of calculating a quantity leads them to write equations with an isolated letter on one side. Finally, they failed to build the equation correctly. This is not surprising since the quantity *Ci* (*x*), on which they intended to build the last equation, does not belong to the only unused relation ($Pum \cdot Cm = Pm$) and, accordingly, a second expression for it could not be obtained.

10. M: We are supposed to need to know the kilos of tea from India. Right?
11. P: *x*… no, wait…
12. P: *x*. I think now we have to write *x* is equal to… the tea from India is equal to…
13. M: The total minus forty-five… all the mixture of tea minus forty-five … (She starts to write *x*=…).

The pair D & E in the problem *The tea*

Before the fourth step, D & E represented with the letters *x* and *y* the quantities *Ci* and *Cm*, respectively. The remaining unknown quantities were expressed as follows: *Pm* as $5.75y$ (using the relation $Pum \cdot Cm = Pm$); *Pi* as $6.2x$ (using the relation $Pui \cdot Ci = Pi$); and *Pt as* 234 (result of using $Put \cdot Ct = Pt$). At that moment the unused relations were: $Ct + Ci = Cm$ and $Pt + Pi = Pm$.

D asks what they can isolate to write the equation (item 1). E's reply clarifies that they have to isolate the quantity which is asked for in the statement (item 2). Finally, D changes his mind and decides to isolate the letter *y* (item 5). In this case, the tendency to construct equations in explicit form does not constitute an obstacle, and they construct correctly an equation using the relation $Ct + Ci = Cm$.

1. D: What can we isolate?
2. E: The *x* that is what is asked for, right?
3. D: Yes, it's asked for the *x*, that is the kilos of tea from India. (He writes *x* =…)
4. E: What must be added to forty-five kilos of tea if every kilo of tea from Thailand costs five point two...?
5. D: Maybe… *y* equals forty-five plus *x*. (He writes $y = 45 + x$. The program validates the equation).

To construct the second equation, D says that "it has to be the *x*" (item 8), which is much more significant if we consider that his last verbalization reflects they initiated the construction of the equation without knowing about which quantity operated (item 10).

6. D: Now…
7. E: *x*…
8. D: Now, maybe, it is not the *x*, is the price… or what? Yes, it has to be the *x*… (Write *x* = …).
9. E: I mean, the *x* is if every… if it´s forty-five times…
10. D: What is the *x*? The kilos of tea from India… the price of … no, no… and the price of tea from India…? (He writes $x = (6.2*x)/x$. The program reports the error).

D & E modify their strategy (item 14), but again D seems to only consider the option of writing equations in explicit form. This time he tries with the unknown *y*. Finally, they failed to build the equation correctly. As the relation that remained to use ($Pt + Pi = Pm$) does not include the quantities Ci (x) and Cm (y), it was impossible for D & E to construct an equation following the criterion of maintaining one of the letters isolated on one side.

11. E: No. Maybe it´s divided by *y*…
12. D: By *y*? No, because the *y* is… ok, let´s try. (He writes $x = (6.2*x)/y$. The program reports the error).
13. D: No.(…) What can we isolate?
14. D: We need another equation. (He writes *y* = …). The *y* is the kilos of tea… of tea mixture…

CONCLUSIONS

We have described the emergence of a tendency to construct equations in which one of the unknown appears isolated on one side when solving problems using HBPS. The format of the equation and students' comments seem to indicate that, by expressing the equation in this way, they intended, although it was not possible, to calculate the value of the quantity linked to the letter. In the analyzed cases, the students gave priority to this mechanical construction against the need to address the remaining unused relations, which would allow them to build the equation. Sometimes this led them to failure to finish the problem because the letter that they sought to place isolated on one of the sides did not appear in the only remaining unused relation, and therefore, it was not possible to construct a second expression for that quantity. We can conclude that this tendency is a reflection in the algebraic solving of word problems of the difficulty, already reported, to interpret the equal sign as a comparison between quantities rather than a signal to do something.

On the other hand, during the teaching of functions in secondary school it is very usual to encourage students to represent functions in explicit form. It could be reflected in the way students build equations. In order to limit the possible influence of this fact, we are currently designing an experiment with students in second year of secondary school (11-12 years old) (not yet exposed to the representation of functions in explicit form) to study if this tendency is also present.

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STUDENTS' PERSONAL RELATIONSHIP WITH THE CONVERGENCE OF SERIES OF REAL NUMBERS AS A CONSEQUENCE OF TEACHING PRACTICES

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Our research focuses on the teaching of series and on the consequences of institutional choices on students' learning. Our analyses of textbooks and teaching practices led us to conjecture that the existence of some implicit contract rules in the existing praxeologies to teach series may have an impact on students' learning. The analyses of the students' responses to a questionnaire suggests that, although organised around the application of convergence criteria, existing praxeologies do not seem to prevent the development of ideas of convergence linked to the use of the potential infinity.

INTRODUCTION AND BACKGROUND

Infinite series of real numbers (*series* in what follows) have played a key role in the historical development of Calculus and have many applications within mathematics (such as the writing of numbers with infinite decimals, or the calculation of areas by means of rectangles), and also outside of the field of mathematics (as the modelling of situations such as the distribution of pollutants in the atmosphere, or of medication in the blood system). These elements may justify their position in the introductory Calculus courses in many countries.

In Canada, the organisation of education and official curricula is under the jurisdiction of each province. In the province of Québec, compulsory education finishes at the age of 16 and students who wish to pursue university studies need to follow two years of pre-university studies (called *collégial*) before they can enter university. Students who want to pursue scientific or technical careers will have an introduction to Calculus during the *collégial* studies.

Research literature about series is scarce and it has mostly focused on their learning, but not on their teaching. Regarding their teaching, Robert (1982), stated in a pioneer work that inadequate conceptions of convergence of sequences and series found in university students in France could be, in part, due to the exercises used in teaching.

Regarding their learning, Kidron (2002, pp. 209-211) summarised some main difficulties specifically linked to series: 1) the infinite sum as a *process* or as an *object*; 2) the intuition of the infinite process as a *potentially infinite* process or as an *actual infinite sum*; 3) the reading of the equality $S = a_0 + a_1 + \dots + a_n + \dots$ from left to right or from right to left, which is cognitively different; 4) the *concept definition* of *infinite sum* is not necessarily linked to the *concept image*; 5) symbolic notation. In particular, based on the work of Fischbein *et al* (1979), Kidron stated that "the 'finitist' character of our intellectual schemes might cause difficulties when we deal with the notion of Infinite Sum" (p. 210) and added that "the natural concept of Infinity is in fact the

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concept of "Potential Infinity", which is simply a process that goes on without end, like counting without stopping" (p. 210). Other difficulties in the learning of series can be found in González-Martín, Nardi & Biza (2011).

Our literature review led us to reflect upon whether or not the teaching of series takes into account these learning difficulties. For this reason, we decided to analyse how series are presented in *collégial* textbooks and the possible consequences linked to their teaching. For the first stage, we analysed a sample of 17 textbooks used in *collegial* studies in Québec from 1993 to 2008 (González-Martín, Nardi & Biza, 2011), paying special attention to the praxeologies (see next section) privileged by textbooks. Our main results can be summarised as follows:

- Series are usually introduced through praxeologies which do not lead to a questioning about their applications or their importance (*raison d'être*).
- Praxeologies tend to introduce series as a tool in order to later introduce functional series, but the importance of series *per se* is usually absent.
- Praxeological organisations tend to ignore some of the main difficulties in learning series identified by research.
- The vast majority of tasks concerning series are related to the application of convergence criteria, or to the application of algorithmic procedures.

The second stage of the research consisted in analysing *collégial* teachers' use of textbooks, and whether, through their practices, they attempt to do something different from what is usually presented in the textbooks (González-Martín, 2010). Our interviews with five teachers revealed that they considered the textbook they use as adequate for the teaching of series, and that their practices tended to mostly reproduce what was presented in their textbooks.

As a consequence of the results of these two stages, we conjectured the existence of some implicit *contract rules* in the teaching of series in the *collégial* institutions in Québec. In González-Martín (2013) we discussed two implicit rules implying that students do not need the definition of what a series is to solve the tasks given to them, and also that applications of series are not important. For the purposes of this paper, as we are interested in the notion of convergence, we only discuss the following one:

Rule 1: "The notion of convergence can be reduced to the application of convergence criteria".

This rule has been chosen for this paper because the convergence of a series is a key element, but research has identified many difficulties with the notion of convergence. In addition to this, our analysis of textbooks showed that an average of 77% of the tasks given to the students are related to the study of the convergence or the sum of given series applying criteria (González-Martín, Nardi & Biza, 2011, p. 578), seeming to confirm *Rule* 1. We believe that this rule is a direct consequence of *R4*. It could also be a consequence of *R3*: instead of directly addressing the main difficulties concerning series, its teaching focuses on the application of rules and criteria which reduce series and their complexity to some algebraic manipulations. One consequence of this rule is that students might become able to apply criteria to decide whether a series is convergent or not; however, what *being convergent* means, or whether the limit is actually reached or not, are key questions which are not addressed. In particular, in González-Martín (2013) it was apparent that when students define what a series is: 1) a very small number of students (5/32) was able to provide a definition with no erroneous elements; 2) among them, mentioning that the sum could converge or diverge seemed to be an important thing to mention for only one student (1/5).

To verify whether *Rule 1* has an impact on *collégial* students' learning of series, we decided to create a sample of students and to apply a questionnaire. Let us define first the main elements of our theoretical framework, before clearly stating our objectives.

THEORETICAL FRAMEWORK

As we recognise the important role of institutional choices in the learning of mathematics, and the repercussions of these choices, our research follows an anthropological approach (Chevallard, 1999).

Chevallard's (1999) anthropological theory attempts to achieve a better understanding of the choices made by an institution in order to organise the teaching of mathematical notions. This theory recognises that mathematical objects are not absolute objects, but entities which arise from the practices of given institutions. These practices can be described in terms of tasks, techniques used to complete the tasks, technologies which both justify and explain the techniques, and theories which include the given discourses. According to this theory, every human activity generates an organisation of tasks, techniques, technologies and theories which Chevallard designates as *praxeology*, or *praxeologic organisation*. A praxeological analysis allows us to characterise the institutional relation to mathematical notions within given institutions. This institutional relation is mainly forged through the exercises (or tasks), and not only through the theoretical explanations (Kouidri, 2009). Praxeological analyses are useful to describe praxeological organisations, but also to identify the existence of (sometimes implicit) *contract rules*, which are rules that the institution fosters through its practices around a mathematical notion and which contribute to determine the institutional relation to a mathematical notion. This institutional relation and its *contract rules* play an important role in the development of the learners' *personal relationship* with the mathematical notions s/he learns within the institution. Chevallard states that "from this personal relationship, the learner will constitute what one could designate as being 'knowledge', 'know-how', 'conceptions', 'competencies', 'mastery' and 'mental images'" (1988/89).

In our case, our objective is to have elements to characterise *collégial* students' *personal relationship* with the convergence of series and to see if this *personal relationship* seems to have a strong relation with the implicit *contract Rule 1* we have identified in the teaching processes.

METHODOLOGY

To verify the impact of the *contract Rule 1*, among others, on the students' *personal relationship* with series, we created a sample of 32 students in their first year of *collégial* studies (where series are introduced) after the teaching of series had occurred. These 32 students had had three different mathematics teachers, who we name teachers *A*, *B* and *C*. Our sample consists of 4 students from teacher *A* (referred to as students A1 to A4), 14 students from teacher *B* (referred to as students B1 to B14), and 14 students from teacher *C* (referred to as students C1 to C14).

We constructed a questionnaire with 10 questions, aiming to asses the students' learning about series, as well as to verify our conjectures about the impact of different *contract rules* on their learning. The questionnaire was administrated in May 2011 during one of their courses (approximately 55 minutes in duration), and the students participated voluntarily.

In this paper, we discuss the students' responses to the two following questions:

Question 7:

Consider the infinite sum $\sum_{n=1}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$ where we add infinitely +1 and -1. This series was studied in 1703 by the mathematician Guido Grandi. According to you, what would be the sum of this series?

a) The result is 0. b) The result is 1. c) The result can be 0 or 1.

d) The result is ½. e) The result is infinite. f) The result does not exist.

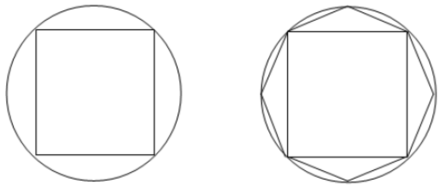
g) Other (explain).

Explain your answer.

Question 9:

Let's consider a circle with a given area *A*. Let us now suppose that:

- x_1 is the area of the square inscribed in the circle.
- x_2 is the area of the 4 isosceles triangles which, together with the square, form a regular octagon inscribed in the circle.



- x_3 is the area of the 8 triangles isosceles which, together with the octagon, form a regular 16-sided polygon inscribed in the circle.
- Etc.

Would you agree with the statement that $\sum_{n=1}^{\infty} x_n = A$? Justify your answer.

Figure 1. Questions 7 and 9.

In the next section, we present and comment on the results obtained from these questions.

DATA ANALYSIS

Question 7 (Q7)

The distribution of responses to this question is the following:

<table>
<tr><td>The result is 0</td><td>Idea that each time that you add 1, you subtract it</td><td>A3, B11</td></tr>
<tr><td>The result is 1</td><td></td><td>A4, B7</td></tr>
<tr><td rowspan="2">The result can be 0 or 1</td><td>“At infinity, you don’t know if it’s even or odd, so both are possible” or “For n even it’s 1, for n odd, it’s 0”</td><td>B2, B4, B9, B13</td></tr>
<tr><td>Other</td><td>B10, C12</td></tr>
<tr><td>The result is ½</td><td></td><td>None</td></tr>
<tr><td>The result is infinite</td><td></td><td>C1, C4</td></tr>
<tr><td rowspan="6">The result does not exist</td><td>No explanation</td><td>C9</td></tr>
<tr><td>It’s divergent</td><td>A2, B13, C5, C8</td></tr>
<tr><td>There’s no definite sum for this series</td><td>C2, C7</td></tr>
<tr><td>It’s always 0 and 1 in alternation – It’s like sine and cosine – The numbers cancel each other.</td><td>B6
C3, C6</td></tr>
<tr><td>At infinity, we wouldn’t know if it’s 1 or 0</td><td>B5, B8, B12
C10, C14</td></tr>
<tr><td>Unclear explanation</td><td>B1</td></tr>
<tr><td rowspan="2">Other</td><td>Wrong application of a criterion</td><td>A1</td></tr>
<tr><td>Explanations involving “the series diverges”, “undefined” or “there won’t be a result”</td><td>B3
C11, C13</td></tr>
<tr><td>I don’t know</td><td></td><td>B14</td></tr>
</table>

Table 1: Responses to Question 7.

This question was used by Bagni (2005) with students who hadn’t previously studied series. This might explain why the distribution of answers in our study is different, and half of our students answer that the result does not exist. However, only ten students (10/32) give a detailed explanation for this fact. We believe that the distribution of our results is quite surprising, since this series is usually presented in the introductory examples; however, it seems that the fact that the students have not solved any task involving it leads to the fact that they do not remember how to interpret it. We can also see some justifications which seem to call for the use of the potential infinity (A3, B2, B4, B9, B11, B13, C12), especially in the students of teacher *B*. We also note that some students use the term “diverge” meaning “tends to infinity”, whereas others use it meaning “does not converge” (mostly those of teacher *C*).

Interestingly, these students have spent a great amount of time studying the convergence of far more complex series through the application of convergence criteria, in the algebraic setting. Indeed, some students tried to answer the question (wrongly) by applying algebraic techniques (A1, A4, B7, B10). However, the students seem to be blocked when the study of the convergence does not require the application of any criterion. The fact that praxeologies seem to focus so often on when a series converges or tends to infinity, seems to have an effect on the students’ reaction when a series does not converge.

Question 9 (Q9)

The distribution of responses to this question is the following:

<table>
<tr><td rowspan="7">Yes</td><td>At some point, x_∞ will be infinitely small, but will form the whole area with the previous ones</td><td>B8
C8</td></tr>
<tr><td>The more we add figures, the more the whole figure will look like a circle</td><td>B2, B9</td></tr>
<tr><td>It’s a very precise approximation – The whole area will be almost completely filled – We approach the area of the circle – Tends to the area of the circle</td><td>B1
C3, C5, C10, C11</td></tr>
<tr><td>The error tends to zero</td><td>C2</td></tr>
<tr><td>We can see it</td><td>C13</td></tr>
<tr><td>Unclear justifications</td><td>A2, A3, B5, C14</td></tr>
<tr><td>No explanation</td><td>A1, C12</td></tr>
<tr><td colspan="2">No</td><td>B3, B4, B6, B11
C1, C9</td></tr>
<tr><td colspan="2">Other</td><td>A4, B7, B10, B12, B13, B14, C4, C6, C7</td></tr>
</table>

Table 2: Responses to Question 9.

In this case, the use of a question in the geometric setting seems to have a stronger impact in the students’ interpretation of the situation. Only 10 students (10/32) (none of them from teacher *A*) seem to be able to interpret the phenomenon, although 7 of them (B1, B2, B9, C3, C5, C10, C11) seem to use arguments implying that we *approach* the circle as much as we want, suggesting the use of the potential infinity.

This makes us wonder whether or not when students conclude the convergence of a series applying a criterion, they are convinced that the series reaches the result, or they believe that the series *gets very near* the result? We must say that in the case of teacher *A*, all the students were able to study the convergence of the series $\sum_{n=1}^{\infty}(\frac{1}{3})^{2n}$ (Q5 in the questionnaire) applying criteria, but none replied correctly to Q7 and Q9. From teacher *B*, among the 6 students who correctly studied the convergence of the series in Q5, 3 of them were unable to provide a correct response to both Q7 and Q9; however, among the 8 students who did not correctly study the convergence of the given series (Q5), 2 replied correctly to Q7, one replied correctly to both Q7 and Q9, and 5 did not reply correctly to either Q7 or Q9. From teacher *C*, among the 7 students who correctly studied the convergence of the given series (Q5), only one replied correctly to both Q7 and Q9, 4 replied correctly only to Q7, and one only to Q9. These results seem to suggest that the capacity to correctly apply criteria does not guarantee the development of tools useful for interpret situations where convergence may or may not be involved.

FINAL REMARKS

Our results, both from the textbook analysis and the teaching practices, seem to confirm the presence of *contract Rule 1*: the study of convergence is mostly reduced to the application of criteria in the algebraic setting. And as we conjectured, some students seem able to apply criteria to decide whether a series is convergent or not; however, what *being convergent* means, whether the limit is actually reached or not, and the study of cases which cannot be tackled algebraically seem to be questions out of reach for many students.

To sum up, the main characteristics of the existing praxeologies for the teaching of series (*R1* to *R4*) in *collegial* studies suggest a set of implicit *contact rules*. In González-Martín (2013), we discussed the rules implying that students do not need the definition of what a series is to solve the tasks given to them, and also that applications of series are not important. Our current findings seem to support the existence of a rule implying that the study of convergence can be reduced to the application of convergence criteria. The existence of these implicit rules seems to clearly have an impact on students' *personal relationship* with series and their convergence. In the interviews with the teachers, it seemed apparent that they reduced series to the study of the convergence of series, which is reduced to the application of algebraic techniques (González-Martín, 2010). But what a series is, what being convergent (or not) means, and how to reason about these questions seem to be issues which are not present in the praxeologies, and they are generally absent from the students' *personal relationship* with series. Some difficulties with series (as the use of the potential infinity) are not taken into account by existing praxeologies, and we can see them appear spontaneously in our students.

Our next steps consist in concluding our analyses of the questionnaires to have a whole portrait of the student's *personal relationship* with series, and to make connections with teaching practices. We hope our results can contribute to the debate about whether the current teaching of series could be improved, and how this improvement can be made.

ACKNOWLEDGEMENTS

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IMAGINATION AND TEACHING DEVELOPMENT

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This paper sets out to provide a reason for the limited effectiveness of mathematics teaching development activity with a project set up to establish communities of inquiry. Teaching development is set out briefly, using constructs from community of practice theory and activity theory. Development and research methodologies arising from these theoretical perspectives are summarized and illustrated. Data comes from an episode that embraces two consecutive workshops that took place within the project. It is concluded that many forces combine to align teachers to their school practice, development is a slow extrapolation of practice based on shared ideas that, when supported by experience, are meaningful in the imagination of teachers.

INTRODUCTION

This paper reports from a mathematics teaching developmental research project introduced to the PME community by Jaworski (2004). The project set out to establish communities of inquiry of teachers from grades one to thirteen and university teacher educator/researchers (didacticians). It was theorised initially from a communities of practice (CPT) perspective. Later activity theory (AT) became a useful framework within which to theorise development. Teaching development is usually a slow process; Liljedahl (2010) reports on teachers' "rapid and profound change", but acknowledges that instances of such change are rare. Design research has produced encouraging results from the collaboration between teachers and mathematics educators (Cobb & Gravemeijer, 2008). However, problems arise as carefully thought out designs become adapted to the practice of teachers who have not been part of the design process (Artigue, 2009). Developing teaching through establishing communities of inquiry is an approach that is intended to draw teachers into the developmental process so that they become co-learners in partnership with didacticians. The approach is thus sensitive to the knowledge resources of teachers and didacticians, and is intended to ensure sustained teaching development. However, Jaworski (1998) has reported that teachers' adoption of a research stance within their practice is jerky and 'evolutionary'. Two workshops, about 18 months into the project, lead to critical reflection on founding principles of the project. This report contributes to understanding approaches to mathematics teaching development.

TEACHING DEVELOPMENT FROM CPT AND AT PERSPECTIVES

A principled account of development can be worked out within CPT from the "modes of belonging" described by Wenger.

> To make sense of these processes of identity formation and learning, it is useful to consider three distinct modes of belonging: (1) *engagement* – active involvement in mutual processes of negotiation of meaning (2) *imagination* – creating images of the world and

2013. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 369-376. Kiel, Germany: PME.

> seeing connections through time and space by extrapolating from our own experience (3) *alignment* – coordinating our energy and activities in order to fit within broader structures and contribute to broader enterprises. (Wenger, 1998, pp. 173-174)

From this perspective teaching development rests on teachers being actively *engaged* and involved in making sense of their practice: its constitution, goals, conditions, constraints, opportunities, etc. Teachers may move beyond their own immediate experience by *imagining* how practice is in other teachers' classes, and what this might mean for their own practice. Teachers *align* themselves to their practice by accepting, understanding and working within the culture, customs, rules, regulations, norms and requirements of their practice.

The introduction of inquiry into teachers' practice entails a significant development of CPT. Jaworski (2006) argues that inquiry transforms alignment into critical alignment, through which teachers are empowered to examine critically the tensions, contradictions and constraints of their practice, and actively seek changes to overcome them. I have argued elsewhere (Goodchild, 2011) that the introduction of inquiry as a developmental tool constitutes a paradigm shift and that the resulting developmental research founded on communities of inquiry belongs to a critical research paradigm. This shift is accompanied by the need for an alternative theory to conceptualise development because imagination and extrapolating from experience do not account for goal directed actions motivated by a desire to improve practice. Engeström's (2001) explanation of expansive learning within an AT frame provides an alternative conceptualisation. Briefly, expansive learning runs in cycles of successive phases of internalisation and externalisation. Participation begins with internalisation of the norms, routines, actions and operations of culturally established activity. This is followed by a period when the tensions and contradictions inherent in activity become apparent and give rise to individual innovations; this is a phase of externalisation. Within a community of inquiry, inquiry is the catalyst, tool and mechanism of the externalisation phase. Innovations become accepted by the community and a new phase of internalisation begins as they are accommodated within the activity.

Concepts in the analysis of events arise from these two theoretical perspectives, from CPT: *engagement*, *imagination*, *extrapolating*, and *alignment*; from AT: *tensions*, *contradictions*, and *expansion*; in addition, *inquiry* and *critical alignment* are used.

DEVELOPMENT AND RESEARCH METHODOLOGIES

Briefly, teachers were invited to join the project with the intention of establishing school teams as inquiring communities embraced within the whole project community. Project activities included school team meetings, and six workshops each year, which included all project participants. Didacticians visited schools and observed lessons when invited by teachers. In workshops, teachers reported activity with their classes, engaged in discussions about mathematics and didactical issues, and planned for classroom activity. Didacticians planned workshop programmes responding to teachers' wishes and feedback. Didacticians also presented 'inquiry' as a cycle of planning and implementing, observing and reflecting, and feeding back into fresh planning. The developmental activity was based upon a number of fundamental principles, for example: teachers have substantial knowledge and experience that must be recognised and shared; teachers have the authority for what happens in their classrooms; the inquiry cycle is an effective approach to developing teaching and learning mathematics; and teachers take the lead in preparing for their own classes, with support and fresh ideas offered by didacticians. Thus, didacticians would suggest starting points for planning, but designs for classroom activity would be the result of teachers' collaborative planning activity.

Research focused on development in learning mathematics, teaching mathematics, and the developmental process. Data were generated by video- or audio- recording every project event; including school-based activities when a didactician was present. Also, documentary evidence and the textual productions from group sessions in workshops were collected. The data are naturalistic in that they were gathered from development-inspired events rather than actions motivated by the research. Data are analysed from a symbolic interactionist perspective in which teachers' meanings emerge in their interaction with each other, their students and didacticians. An abductive approach (Meyer, 2010) to interpret the evidence is taken.

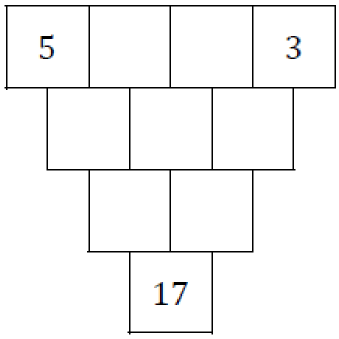
AN EPISODE EXPLORED

The episode used in this report was originally chosen to provide an example of (effective) activity within the project. However, the initial stages of analysis revealed the episode to be disappointing because it appeared to contradict claims for effectiveness. Reflection on the episode led to the realisation that it held useful lessons about effective engagement of didacticians with teachers in developmental research. Thus the episode was analysed to address the question: What interferes with creative innovation in teaching mathematics?

The data includes about three hours of recordings of discussions that took place between teachers from two upper secondary schools and didacticians in 'group' and 'plenary' sessions that occurred within two consecutive workshops. Two extracts taken from within the first five minutes of the first session in the first workshop are reproduced below. These introduce the discussions that took place over the two workshops, and illustrate the analytic process. Extracts are translated into English and paraphrased to make better sense for the purpose of reporting. Analysis takes place in the original language, transcripts support work with the raw data, in this case video recordings and some documents prepared for the workshops.

At the end of the first year of the project, at the teachers' request, it was agreed that workshops would be based on explicit curriculum themes. Moreover, it was agreed that a substantial amount of time in workshops would be spent within groups working at similar grades preparing for classroom activity. During the autumn preceding the episode examined here, workshops had focused on probability and geometry. In the spring semester the three workshops focused on teaching and learning algebra, the episode here is based on the first two 'algebra' workshops.

Some days before the first ‘algebra’ workshop, tasks had been sent out to schools with the request that school teams discuss the task amongst themselves in preparation for the workshop. The tasks were based on the theme of addition triangles (see figure 1) that were presented at a variety of levels of demand that might be adapted to meet the learning needs of students at most grades. When the teachers from the upper secondary classes (grades 11-13) met they had discussed the tasks in school teams as requested, but rather than engaging with the tasks they came with their own, alternative, proposal for their group discussion.



In this ‘addition triangle’, the values in the cells (below the first row) are the sum of the values in the cells immediately above.

Work out the values that must be written in the empty cells.

Fig. 1: Example of ‘addition triangle’ task.

Extracts from the first group session

Olav: But I just thought about typical problems, or errors, that I have observed lately in tests and such. For example, if it stands two x divided by x, they have, two x over x, so they say we have two x on top, we take away one x, and so we have x left. Or if there is x plus four over x, but here it is OK to shorten, we cancel that (x) and that (x), isn’t it? For example, Paul (a colleague) and I had a task, they (students) should make a quadratic equation, we were focusing on quadratic equations. We have one side (draws a rectangle and writes x) which is x, and that (side, points and writes) is x minus two, isn’t it, it is two less than the other side. They (students), the area is given, should make a quadratic equation. So we set it up like this, x times x minus two is equal to fifteen, for example. And so we get, obviously, x squared minus two. This (points to x-2) we set in parentheses. It is such things that repeat. And it repeats in first class (grade 11), and again in the second class (grade 12), and it persists, we can see it still in the third class (grade 13). No matter what we do so it appears that we cannot get rid of it.

Summary of errors described:

$\frac{2x}{x} \sim \frac{xx}{x} \sim x$; $\frac{x+4}{x} \sim 4$; $x.x - 2 = 15 \sim x^2 - 2 = 15$

Osvald: It was a thought, as we were all together, that then, possibly could arise some good ideas, THAT (emphasised) I have good experience with. So another could say oh no, THAT I have good experience with. And I believe we feel that we fight (struggle) and many of us will come a little bit further. It is the same things that we have problems with every year, and then we have not been good enough.

Major themes embedded in these extracts

The teachers had spent time in their school teams discussing the tasks sent out in advance but the tasks did not meet the teachers' needs or expectations and did not motivate their engagement. The teachers perceived the workshop as providing an opportunity for them to work together with teachers from another school on issues that they experienced as 'meaningful' in their practice. Their shared concerns about teaching and learning algebra are the errors made by their students. These errors have been observed for many years, they are recognised by all the teachers present, they appear to be resistant to correction as the students repeat them in every grade at upper secondary level. The teachers accept that they are challenged to find new ideas and improve the effectiveness of their teaching and they look to each other to share and generate fresh approaches.

Teachers were **engaged** with the project's activities, both within their school team and participation in the workshop. Their **alignment** to the project's goals is ambiguous, they were ready to discuss, but on themes of their own choosing. The belief that they could learn by sharing and developing experiences with each other reveals the potential of **imagination** that might enable them to develop (**extrapolate**) their practice. The acknowledgement of the recurrent resistant errors, which they felt unable to address effectively, indicates **tension** within their practice, between their repertoire of teaching approaches and desired outcomes.

About three hours of recordings is analysed, the outcome is summarised in table 1.

Alignment
Teachers' are subject to many interacting 'forces' including: the curriculum, time, textbook, examinations, order of content within and through grades, and student expectations.
Responsibility for mathematics education is shared across institutions – primary school, lower secondary, upper secondary and university-each dependent on the former.
Students' conceptual understanding of algebra is the responsibility of lower sec. school.
Imagination
Experience leads to deeply rooted belief: algebra requires "drill" and must be memorized.
Ideas are distributed by sharing with other teachers who have experiences 'similar to ours'.
Didacticians' ideas not relevant to 'our' practice, tasks not usable, suited for other grades.
Developing conceptual understanding takes time, which is not available (cf. alignment).
Learning algebra can be supported through applications, such as in geometrical tasks.
Students can be supported in working on algebraic tasks by reducing ambiguity in the presentation of expressions (such as enclosing 'x-2' within parentheses: '(x-2)').

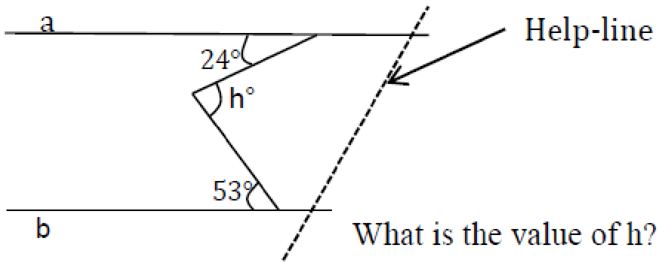
Replacing letters with numbers can help students to spot their errors.
Engagement Teachers want to work on tasks that they can see are usable in their own classes. Teaching mathematics follows well-established routines and is informed by beliefs about mathematics and mathematical competencies.
Tensions Curriculum goals do not fit with students' competencies. Students enter the grade level without the understanding that should be developed earlier. Students lack conceptual understanding, there is no time to develop it (cf. alignment). Algebra requires mastery of techniques incompatible with an inquiry stance. Quick repetition does not have a lasting impact but shortage of time prevents longer engagement with under-developed concepts. Students seek 'rules', use the textbook to find 'rules', this undermines inquiry approaches. New content is added to the curriculum such as 'experimental geometry' which is interesting and well-adapted to inquiry – but the examination questions are unpredictable and the topic is placed at the end of the textbook, too late to influence students' attitudes.

Table 1. Summary of main points from the analysis

Didacticians responded by sending out four tasks as preparation for the second algebra workshop, these are shown in figure 2. When these fresh tasks were considered, the teachers were unenthusiastic. The first task they felt was appropriate for lower secondary classes. The second task came in for the strongest reaction:

> Stefan: … I will advise most strongly against using that task with my students … I will never have used that in a lesson …

The third task was dismissed; possibly because, as it became evident later, at these upper secondary schools mixed numbers (whole part + fraction) are excluded from students' experience - from classes and their textbooks - because it is believed mixed numbers are confused as algebraic expressions (whole part X fraction). The fourth was a development of a geometry task that had been discussed in a previous workshop, and this was the only task that the teachers would admit to being usable in their classes, and agreed to discuss further in the group.

1. I will make a frame around a rectangular picture. If the material for the frame is 1 cm broad, what is the length of frame material I need?
2. Is the following sometimes, always or never true? $\frac{x+4}{x} = 4$ Explain why.
3. $2 + 2 = 2 \cdot 2$; $3 + 1\,{}^{1}/_{2} = 3 \cdot 1\,{}^{1}/_{2}$; $4 + 1\,{}^{1}/_{3} = 4 \cdot 1\,{}^{1}/_{3}$ ⋯ *true or false*?
   Can you continue this: 5 +, 6 +, 7 +; … 10 +; 59 +; -7 +; ${}^{2}/_{3}$ +
4. 

What is the value of h?

A conversation in the staffroom: "A pupil came to me today and claimed that he had solved the 'M' problem with the help-line shown in this figure. Is it possible?

Fig. 2 Tasks sent out before the second algebra workshop (*format changed*)

DISCUSSION

It is questionable how realistic the notion of critical alignment is for the teachers described in this paper. Their alignment to practice is maintained by many forces that appear to resist innovation. On the surface there appears to be a significant contradiction in teachers' behaviour. They have observed the same errors in students' algebraic reasoning over decades, and their attempts to address the errors appear ineffective, they seek new ideas. Nevertheless, they are dismissive of novel approaches suggested by didacticians and remain convinced that the approaches that have consistently failed (drill and memorising) are essential. The teachers claim that they want new ideas but appear to reject them when they are offered. However, these two positions may not be contradictory if the new ideas they seek are about 'better ways' to do drill and technical skill development, which appear to be the only options given the conditions of the practice in which they are engaged.

Didacticians approached the teachers' challenge with tasks that were intended to stimulate discussion and initiate teachers' design activity, and enter into an inquiry cycle to explore fresh approaches to teaching algebra. However, the teachers did not see the suggestions as relevant or applicable in their classrooms. One explanation for the teachers' response can be inferred from the very opening statements reproduced above; the teachers were seeking fresh ideas grounded in experience to which they could relate. The exercise of imagination entails, in Wenger's words 'extrapolating from (their) own experience' (1998, p. 173). This explanation leads to the conclusion that teaching development within communities of inquiry (for these teachers) must attend to more than developing a principled understanding of inquiry. New ideas need to be more than starting points for discussion and design, they have to be embedded in classroom practice that is meaningful within the teachers' experience.

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GOING BEYOND TEACHING MATHEMATICS TO IMMIGRANT STUDENTS: TEACHERS BECOMING SOCIAL RESOURCES IN THEIR TRANSITION PROCESS[1]

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Starting from the constructs 'transitions' and 'mathematical identities', and using data from our empirical research to support our arguments, we present a theoretical discussion on how the concept of 'social resource' may explain the teachers' role in immigrant students' transition processes and construction of identities. The narratives of immigrant students about their own experiences and expectations in relation to their learning of mathematics help us to illustrate how students ascribe a role to their teachers that goes beyond merely teaching mathematics. Teachers become social resources, playing a significant role both in meaning-making and in the reconstruction of identity processes.

TRANSITIONS

In the field of mathematics education, the concept of transition has often been linked to the learning processes taking place when students move from one academic level to the next, having to face more complex mathematics concepts. However, this is only one of the many particular meanings attributed to a general construct.

In our research, and from a socio-cultural perspective, transitions result from the fact that people live in multiple contexts of practice and experience multiple ruptures, since what may be taken for granted in one context may no longer be valid in another. As in Abreu, Bishop and Presmeg (2002), we regard the transition process as part of a dynamic relationship between the learner and the contexts between which s/he transitions; a process, multiple in itself, that involves inputs from both the individuals and the socio-cultural structures in which they act and interact, where individual agency goes along with the social dynamics, and where the individual is constantly negotiating with the context.

Gorgorió, together with her colleagues, participated in the project lead by Abreu, Bishop and Presmeg that resulted in the above-mentioned publication. In Gorgorió, Planas and Vilella (2002), we reported on the socio-cultural conflicts that immigrant students experience in their mathematics lessons as well as the teachers' understanding of those conflicts. Our analysis was of transitions being negotiated at a face-to-face interaction level and our data came, essentially, from classroom observation. Our focus was on conflict as the externalization of the discrepancies between the different meanings attached, by teachers and students, to the various learning situations which take place in the classroom.

Our interpretation of transition was:

> We understand the construct 'transition' not as a moment of change but as the experience of changing, of living the discontinuities between the different contexts, and in particular

2013. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 377-384. Kiel, Germany: PME.

> between different school cultures and different mathematics classroom cultures (…) Transitions arise from the individual's need to live, cope and participate in different contexts, to face different challenges, to take profit from the advantages of the new situation arising from the change. Transitions include the process of adapting to new social and cultural experiences, and students need to be helped to understand the meanings of the new experiences and to reinterpret them and construct new ones based on their own individual meanings and values. (Gorgorió, Planas and Vilella, op.cit., p. 24)

Since our first approach to transitions, the group EMiCS[2] has been doing research on the processes of teaching and learning mathematics of immigrant students. One decade later, we revisit the construct transitions with a different perspective –that of the students– and with a slightly different focus –meanings as related to identities. Individual meanings arising from, and together with, socio-cultural meanings during transition processes influence the construction of the identities of those involved, by shaping what counts as knowledge and making available certain types of knowledge.

Transitions originate as a consequence of ruptures resulting from changes of contexts of social practice, changes within the person itself, or changes in the relations between persons and objects (Zittoun, 2007). Transitions require processes of adjustment to new life circumstances and involve multiple changes in reference frames and meanings and in relationships with people. These changes require people to modify routines and interpretations, explore new possibilities, and develop new ways of acting and interacting. According to Zittoun (2008), transitions include learning, identity change, and meaning-making processes; they also imply the reconstruction of identities and require new forms of knowledge and skills as they bring about the need to engage in meaning-making to confer sense to what is happening to the person.

We approach the processes of mathematics learning by immigrant students as transition processes, since they involve new contexts of mathematical practice, different relationships with both people and knowledge, and different ways of understanding actions and interactions in the classroom. In school transitions, students' difficulties are often related to processes of attribution of meaning to the learning situation, and to processes of identity reconstruction. As Meaney and Lange (2012), we consider the impact of transitioning between contexts, in which mathematical knowledge and ways of interacting around it are perceived differently, to be an issue of social justice.

CONSTRUCTING MATHEMATICAL IDENTITIES

According to Nasir (2002), the development of identities is tied to human activity, affiliation and meaning systems. We consider identity as located in cultural and social practices, which can vary between different institutions and different societies. However, we do not preclude identity as having an internal component, despite being crucially linked to the social and cultural practices. Joining Holland, Lachicotte, Skinner & Cain (1998), we claim that identity construction is a cultural and social process, where nonetheless the individual has agency. Identities are not only constituted by labels that people place on themselves. Identity is about how people become who they are, and how they come to figure out who they are, through the worlds they participate in and through their relationship to others within and outside these worlds. Martin (2007) establishes that mathematics identity (as he calls it) encompasses the dispositions and deeply held beliefs that individuals develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change their living conditions. A mathematics identity involves, in a constant negotiation, a person's self-understanding as well as how s/he is constructed by others in the context of doing mathematics.

By the students' "mathematical identity" we mean the academic identity that they develop as mathematics learners and users. This way, mathematical identities include how students view their own aptitude for mathematics and how they see themselves as users of mathematical knowledge, both in school and beyond. Students develop their mathematical identity through their participation in mathematical activities, their interpretation of their own experience as mathematics learners, their expectations about future (mathematics) education and about their uses of mathematics both in school and outside. Students' mathematical identities also include their sense of affiliation with the mathematical practices in their particular lessons and their identification with the norms and values regulating it. The development of mathematical identities is shaped by students' social positions in the mathematics classroom, their construction of mathematical knowledge and how students understand their experiences as mathematics learners.

Students' mathematical identities are dynamic rather than static, and are bound up in other social or cultural identities they may develop. Students' mathematical identities are constantly being reconstructed around students' perceptions of themselves as mathematics learners and how they are seen by significant others involved in mathematical activities. How students see themselves as mathematics learners, however, is as important as how they are defined by others, especially their mathematics teachers. Immigrant students' transitions have the potential to change the ways in which they interpret themselves and their roles both in school and outside. The ways in which they experience and have experienced their learning of mathematics in different contexts and how they interpret these experiences impact on their practices and on the identities they develop. Identities are constantly reconstructed by engaging in a practice and belonging to a group, but also by wanting to engage in real or imagined practices and belong to real or imagined worlds.

TEACHERS' ROLES IN TRANSITION PROCESSES

Transition processes need to be understood not only in terms of the experience of the person in transition, but also considering other people related to him/her that may share the experience in more or less direct ways. In Gorgorió et al. (op.cit), we claim that transitions do not depend solely on the individual experiencing them, since they are shaped by social representations, valorisations and expectations about the role, success and skills of each of the persons involved. In the context of school mathematics practice, transition processes are co-constructed. Together, the various classroom participants reconstruct the meanings attributed to different persons, situations and mathematical objects. Besides the students themselves, other significant persons are involved, among them teachers and parents. Teachers, both as individuals or representing the educational institution, are frames which are likely to facilitate or constrain transition processes. Mulat and Arcavi (2009), in their study of success in mathematics of students of Ethiopian origin in Israel, identify three key elements perceived by the students as a contribution to their achievement in mathematics, by shaping their identities. Among the key elements is the immediate environment consisting of students' mathematics learning experience and the role played by their teachers and parents.

The purpose of this theoretical essay is to shed light on what has been until now an unexplored area: we are in search of a construct that will help to describe teachers' roles in immigrant students' transition processes as perceived by the students themselves. The data we use here to exemplify our ideas come from a broader ongoing study aimed at understanding the transition processes of immigrant students learning mathematics in Catalan schools (for more details, see Costanzi, Gorgorió, and Prat, 2012). The examples come from the narratives obtained in semi-structured interviews where we explicitly ask immigrant students about their past and present experiences and their expectations about the future, with relation to mathematics learning.

When analyzing the interviews, in each student's narrative, we identify elements that may be considered as an evidence of a rupture in their lives as students –e.g. moving to Catalonia, changing teachers or schools, moving from primary to secondary schools, etc.– regardless of whether they explicitly talk about it as a changing reference frame or not. We also identify the ruptures that they explicitly refer to but that we couldn't have anticipated –e.g. a parent losing a job, the death of someone in the family. Once the ruptures are identified, we look for descriptions within the narratives which the students use to explain their trajectories as mathematics learners and the situations where teachers may have played a significant role.

Students' explanations are very illuminating in the way that they highlight how teachers may play a significant role. Nevertheless, we don't ignore the fact that students' cultural backgrounds contribute to highlighting particular aspects of their experiences. Neither do we ignore that each individual story is a co-construction of the interviewer, the researchers as analysts, and the student as narrator. Therefore, in terms of understanding the dynamics of the transition process, we also have to take into account the teachers' perspective and the actual practices in the classroom (e.g. Gorgorió and Prat, 2009 and 2011). This is the reason why we interview their teachers and observe some of their mathematics lessons in order to obtain complementary data.

When analysing the narratives, we identify which are the ruptures and key moments to change trajectories in ways that are significant for the students, and understand how students' identities grow. Trying to make sense of how students explain their teachers' role in their development as mathematics students, we see how teachers are regarded as important characters in their story, as important persons in the transitions out of ruptures. Thus, teachers appear in students' narratives as potential *social resources* (Zittoun, 2007), as people in the students' social networks that can be asked for, or who may offer, expert or relational support.

Thus Hina, a Pakistani girl, has told us: “they [the teachers] helped us to connect with the way that we had learned in Pakistan as these teachers had taught Pakistani students before”. When asked for a further explanation, she added: “they were aware of what kinds of the difficulties would Pakistani children face when they were here at the school”. Later on, when talking about the mathematics teacher she insisted on the fact that “teacher was aware of the way we learned mathematics in Pakistan, because before us the school had Pakistani children”. These short quotes illustrate how Hina refers to her teachers as social resources, both in singular and plural, as a result of their expert knowledge.

Hina also refers to her teachers as providers of relational support. She tells us: “Since I know the teachers well, I explained to the teachers about the reason of the low achievement of my sister”. She has also negotiated with them: “and I shared with them that the level of the other school had not been good. I told the teachers that we [Hina and her family members] would work closely with her so that she could improve her section”. In general, the teacher is the one offering help, support or advice, thus taking the role of meaning dealer, in the sense of someone that initiates the negotiation in order to include the student in the process. This meaning concerns social norms, socio-mathematical norms, norms of the mathematical practice and actual mathematical content.

Most often, it is the teacher who initiates the process and this has to do with personal traits of both actors and the status of “being an authority” that teachers have in many cultural groups, often reinforced because the student belongs to a minority and/or to a socially deprived group. In all cases, the interactions between teachers and students are mediated by representations that students and teachers have of teaching and learning of mathematics, representations that differ according to their cultural backgrounds. Less often, students seek help from teachers, but only when they feel familiar with them, as in Hina’s case. Felipe, a boy from Chile, provides us with another example. He tells us: “*if you met him* [the teacher] *two days ago, it's difficult to get close to him in a way where he'd help you with something you don't understand or don't know”*. In Felipe’s case the teacher is only a *potential resource*, since Felipe never asked for help.

We use the term *potential* when characterizing teachers as social resources not only because some students never ask for support, but also because in some narratives teachers are referred to as constraining the student’s opportunities. The examples provided until now only show teachers described as social resources who play a positive role. However, there also are narratives where the students consider that teachers have had a negative role in their transitions; in all of them the term “teachers” does not refer to individual teachers, but to the institution they represent. For example, when Paola, a Colombian girl, explains her experience when she arrived in Barcelona, she says: *“when I arrived they held me back one year because of Catalan”*. “They” refers to the teachers of her school, who cannot decide by themselves, and are only agents of the educational institution that regulates the adscription of students to groups.

In transition processes, current experiences as mathematics learners are interpreted on the basis of meanings and representations acquired in previous situations. Often students refer to teachers they have had in their previous schooling, before coming to

Barcelona. For instance, Hina tells us that she never leaves things unexplained because this impedes her further learning: *"I always seek clarity from my teachers. They like me asking.* (...) *I have developed* [this strategy] *with thanks of my mathematics teacher Mr. Zulfiquar from Pakistan"*. Zittoun (op.cit.) establishes "that people can also draw upon social knowledge to determine how to act with people in certain situations" (p. 199) and considers that kind of mobilizations to also constitute social resources. Therefore, we see that teachers may be referred to as social resources also at an intra-personal level.

We have already mentioned that teachers may be described as having a negative role in students' transitions. However, sometimes, even when students account for their teachers to have been of great help to them, when considering the student's trajectory as a whole, what we see is that the teacher limits the student's real opportunities to learn. Ronnie, a boy from Ecuador, tells us: *"since I'm not good enough at math, I won't be able to pass the entrance exam to go to university"*. However, he describes his mathematics teachers as having been of great help to him since he arrived. Ronnie is one of the cases that illustrate how identities are made available to students through classroom practices, by leading them to construct a limited mathematical knowledge, constraining their future possibilities. Her teacher tells us that, with an honest intention to make them feel that they succeed, she only proposes routine exercises, because of her perception of immigrant students to be low achievers.

Besides being social resources, teachers also play an important role in their attribution of different mathematical identities and in providing different opportunities for participation and access to knowledge to immigrant students. Therefore, teachers' role in students' transitions, even if perceived as positive by the student, may result in a limited construction of his/her mathematical identity. Teachers, both as individuals and as members of institutions, through the different models of school mathematics that they provide to their students, and through the different positioning of these students in their practices, open different spaces for students' construction of identities in their transition processes.

As an end note, we stress that although the mathematics teacher's main purpose is to teach mathematics, his/her role as a social resource is by no means a negligible one. The word *mathematics* could be replaced in the entirety of this article by the word *history* and the paper would still make sense. However, the fact that the above statement would make sense for a history teacher does not imply it would not apply to a mathematics teacher as well. Moreover, in mathematics and history this role should be interpreted in different ways, since mathematics education has a specific social purpose. Often immigrant students are unsuccessful in mathematics when the teacher fails to go beyond exclusively teaching the content of the subject or when they don't realize that their teacher may become a social resource for them.

FINAL REMARKS

In Catalonia, immigrant students are allocated to a group level on arrival usually according to their age. Some schools are positioned differently from others in terms of the students they provide for. There are teachers working at institutions with a high number of immigrant students per group, while there are others working with only a very limited number. There are several constraints in the practice of teachers having students from many nationalities, but at the same time, there is the risk that they become invisible when there are only a few immigrant students in a class. Therefore, the individual and institutional role of the teacher cannot be separated. Moreover, both teachers and institutions are impregnated with social representations and valorisations that shape their practices and their pedagogical models.

In conclusion we claim that mathematics teachers when acting as social resources not only may help the student to bridge the gap between the students' different reference frames, and therefore have a role in their reconstruction of meanings, but may also play a role in their reconstruction of identities. In their narratives, immigrant students account for their teachers as being a significant other in processes of meaning-making and reconstruction of identities, both crucial for their transition processes.

Notes

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DEVELOPING ONE-TO-ONE TEACHER-STUDENT INTERACTION IN POST-16 MATHEMATICS INSTRUCTION

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Recent developments in mathematics education emphasise the role of teacher-student interaction in developing students' conceptual understanding and knowledge construction, with a corresponding de-emphasis on the use of 'telling': the stating of facts and demonstration of procedures. This action research study investigated teacher-student dialogue during one-to-one interactions in one teacher's post-16 mathematics classroom. Four students volunteered to participate. Data sources included clinical interviews, student feedback interviews and an analytical log. The findings indicate that, although the teacher utilised more 'telling' than 'questioning' interventions, often these 'telling' actions served useful and necessary functions. Findings also indicate that the teacher's scaffolding skills developed during the project.

INTRODUCTION

Current reforms in mathematics education, influenced by a social constructivist view of learning, place dialogue at the heart of the development of conceptual understanding and mathematical thinking skills. Teachers are now seen as 'facilitators of learning' (Smith, 1996; Lobato, Clarke and Ellis, 2005) who manage discussion within a student's Zone of Proximal Development by employing suitable scaffolding and fading techniques (Wood, Bruner and Ross, 1976; Vygotsky, 1978). Underlying these ideas is a strong criticism of transmissive teaching styles, often referred to as 'teaching by telling'. However, there is very little in terms of specific guidance for teachers about how best to achieve these reform aims (Chazan and Ball, 1995; Smith, 1996; Baxter and Williams, 2010). This has led to what Baxter and Williams describe as the "dilemma of telling: how to facilitate students coming to certain understandings without directly telling them what they need to know or do" (p. 8).

This paper reports an exploratory action research study into this issue in the context of a 13-18 state-funded comprehensive (all-ability) school in England.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

In their systematic review, Kyriacou and Issitt (2008) note that research on teacher-student dialogue in the UK is scant, especially at the level of one-to-one interaction. What research there is into whole-class teaching generally reveals a prevalence of transmissive 'teaching by telling', and little evidence of effective scaffolding that might effect a shift towards student independence (Myhill and Warren, 2005; Kyriacou and Issitt, 2008). Reasons proposed for the prevalence of the transmission model include acknowledgement that scaffolding can be a difficult and uncomfortable task, carried out in a pressured environment; and that teachers' beliefs about the nature of mathematics, as well as their own schooling, can affect their

2013. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 385-392. Kiel, Germany: PME.

competence at scaffolding (Schoenfeld, 1992; Myhill and Warren, 2005). When effective scaffolding was observed, teachers were seen to hold back from telling, instead eliciting student thinking through the use of probing questions, along with carefully tailored questions and prompts that provided just enough guidance for breakthrough (Tanner and Jones, 2000; Goos, 2004; Cheeseman, 2009; Ferguson and McDonough, 2010). With reference to data on instruction in elementary arithmetic and geometry, Anghileri (2002) has identified several scaffolding strategies with particular reference to mathematics learning.

But it is simplistic to suppose that achieving effective teacher-student dialogue in mathematics teaching, and enhanced scaffolding behaviours, can be achieved by the effort to eliminate an ingrained habit of telling. Chazan and Ball (1995) propose that a blanket exhortation to avoid telling is inadequate because it ignores the importance of context. Lobato, Clarke and Ellis (2005) point out that many kinds of telling perform useful functions in the development of conceptual understanding, and can be reconciled with a constructivist viewpoint. These two sets of researchers, along with Baxter and Williams (2010), suggest that it is important to gain further understanding of the function of teacher actions through analysis of the intentions behind their scaffolding decisions.

METHODS

The motivation for the study arose from one teacher's sense of a mismatch between her pedagogical ideals and teaching practices. This teacher-researcher, the first author, had been teaching mathematics for two years at the beginning of the project. Her graduate pre-service teacher education programme had promoted the notion of teacher-as-facilitator, which she then aspired to realise in the classroom. However, once in post, she was troubled by an awareness that she was inclined to be too 'helpful', resolving her students' difficulties too directly (in the light of her own ideals) rather than scaffolding their own problem resolutions. She felt especially dissatisfied with her management of one-to-one interactions with students taking 'A-level', a post-compulsory advanced course normally required for university entrance. She therefore set out to improve the quality of the teacher-student dialogue with these students. With reference to her professional goals, and to the literature reviewed above, she framed the following research questions (stated in the first person). **RQ1**: What would a critical analysis of the form and function of my utterances reveal about the nature of my scaffolding strategies? **RQ2**: Can the form and function of my scaffolding interventions be changed as a consequence of investigation on my part? **RQ3**: What aspects of the scaffolding strategies employed are valued by the students?

These formalised RQs, with their emphasis on critical analysis and change, aligned with an interpretive stance, suggested an action research methodology. The time-frame available limited the number of action research cycles to two. After outlining her research aims to the 12 students in her Year 13 (student age 17-18) mathematics class, six male students volunteered to participate, from whom four were selected, partly in order to reflect different attainments within the class.

Clinical Interviews

In order to address RQ1 and RQ2, the teacher was audio-recorded interacting with single students in a one-to-one situation slightly removed from the bustle of the classroom – 'in vitro' rather than 'in vivo'. Each of these dyadic interchanges took the form of a clinical, task-based interview, in which the interviewer's responses are contingent on the subject's reactions to the task (Ginsburg, 1997). This means of eliciting student thinking by contingent prompting and probing is a similar discourse model to that involved in the type of local level on-the-fly scaffolding (van Lier, 1996; Brush and Saye, 2002) that the teacher aimed to develop in her own practice, and therefore provided a rich means of analysing her performance. These interviews took place during those lessons when a space adjacent to the classroom was available for the one-to-one interaction. In order to maintain further links with a familiar setting, each clinical interview was initiated by a task/question from the class 'A-level' textbook. Two questions judged to be sufficiently demanding for the participants to require assistance were selected for each cycle of intervention. Each interview was transcribed verbatim, along with paralinguistic aspects (pauses, interruptions and heavily stressed words), soon afterwards.

Student Feedback Interviews

In order to address RQ3, the four students took part in a semi-structured interview immediately after their clinical interview, using the same recording method. The following open questions were devised to enable the participant to reply without constraint, and to allow the teacher-researcher to probe for meaning as she judged appropriate:

Q1 Did you find any aspect of the teacher input helpful?
Q2 Was there anything that wasn't helpful?
Q3 Is there anything that might have been more helpful for me to do?
Q4 Is there anything you would like to add?

Analytical Log

In order to carry out the process of critical reflection inherent in the two action research cycles, the teacher used an analytical log in which to record her own evaluation of the clinical interviews. She also recorded thoughts, feelings and insights that arose during the process of analysing the feedback interview transcripts. As a result, the log had a narrative quality more characteristic of a journal of reflection. In this way the teacher's own reflexivity contributed to the analytical process, with a view to accessing the intentionality behind her utterances.

Data Analysis

Clinical interviews: First, the transcripts were coded to indicate the *form* of the teacher's dialogic interventions, distinguishing between questioning and telling. Next, following Lobato et al (2005), codes were added to identify the *function* of her utterances. Initially, six scaffolding categories were used, taken from Anghileri (2002): Checking, Convention, Demonstrating, Explaining, Focusing, Probing. These were eventually supplemented with six further emergent categories: Confirming, Directing, Funnelling, Parallel modelling, Prompting and Rephrasing. A brief description of these codes is given in Table 1.

Student feedback: Likewise, transcripts from the student feedback interviews were coded in the first instance according to the participant's perception of the 'helpfulness' or otherwise of a particular scaffolding intervention. Secondly, the function codes above were then assigned.

The analytical log was coded according to whether, in her reflective evaluation of the clinical interviews, the teacher had approved or been critical of each scaffolding intervention, and both the form and function codes were applied accordingly.

Function	Description
Checking	Checking for understanding
Confirming	Indicating the correctness of a student answer
Convention	Discussing a conventional norm (arbitrary knowledge)
Demonstrating	Showing, outlining a procedure
Directing	Providing instructions, advice or suggestions
Explaining	Conceptual content - saying 'why'
Focussing	Highlighting an important concept/idea
Funnelling	Leading student to a correct answer through a constraining series of questions and prompts
Parallel modelling	Modelling the solution to a similar, related problem
Probing	Drawing out student thinking
Prompting	Providing a hint to direct student's attention
Rephrasing	Rephrasing or summarising a student's utterance

Table 1: The 12 codes used in the analysis of function of teacher interventions

FINDINGS

The findings are now discussed with reference to each of the research questions.

RQ1: What did critical analysis of the form and function of my utterances reveal about the nature of my scaffolding strategies?

Analysis of the form of the teacher's scaffolding interactions in the first cycle suggested that she relied overwhelmingly on telling (113 out of 170, with the remaining 57 coded as questions). However, analysis of their function revealed that a large proportion of the telling actions were simple *confirmations* of the rightness or wrongness of student ideas, e.g.

Jack: Do you want it in the exact form?

Teacher: Yes, always leave it in the exact form unless you're asked not to.

Other than *confirming*, the most common telling categories were *explaining* conceptual content; *demonstrating* a procedure; *directing* by providing instructions, advice or suggestions; and outlining a mathematical or cultural *convention*, e.g.

Teacher: Yes, but you have to set it out right. You have to start by saying that you're finding the integral between two *x*-values.

Analysis of the self-critical content of the teacher's analytical log revealed that she was dissatisfied with instances where she employed telling to *demonstrate*, *direct*, *explain* or *funnel*; and also where she used questioning in order to *funnel*, e.g.

Teacher: What have you just found?

Jack: The *x* value.

Teacher: And what were you asked for?

In cases where a student was unable to recall a procedure, she reflected that *parallel modelling* (where a solution to a similar, often simpler, problem is modelled) would have been a more useful strategy than demonstrating using the question itself. In the cases where she was critical of her explaining interventions, she believed (on reflection) that it would have been more beneficial to have assisted the student with guidance involving probing (drawing out thinking) and prompting (with hints to allow the student to make a conceptual link). She also noted that there was a controlling element to her directing, sometimes due to lack of confidence (for example, when exploring a novel method). With regard to the *funnelling* instances, she reflected that she seemed to be hurrying the student towards the answer instead of allowing him more time to respond to her questioning.

Analysis of the 'approving' content of her analytical log revealed that she was more satisfied with instances when she had employed telling to *confirm,* discuss *convention*, and *parallel model*; and when she used questioning to *probe*, e.g.

Teacher: Could you have double-checked it in a different way?

Connor: My domain and range?

Teacher: Yes. Because your graph was wrong, how else could you have checked your range?

Connor: By confirming with the domain of the other one?

and to *prompt*, e.g.

Jack: You know when minus *e* to the *u* is differentiated? Does that become minus *ue* to the…

Teacher: Ah! What's the differential of *e* to the *x*?

Jack: Oh, does it stay the same?

Teacher: Yes, it does.

The teacher felt that *confirming* was a necessary part of her scaffolding strategy. She also judged that 'telling to share a *convention*' was the only way to impart arbitrary mathematical knowledge, or socio-cultural norms, and hence was a necessary intervention. Thus, she approved of an instance in which she directed the student on how to 'set out' his work, as this involved the sharing of a conventional norm. She noted that *probing* questions revealed student thinking, and, in the case of one individual, elicited his longest responses. Finally, she reflected that *prompting* questions enabled the student to work through problems more independently, whilst also allowing for the possibility of internalisation and transfer for future independent use.

RQ2: Can the form and function of my scaffolding interventions be changed as a consequence of investigation on my part?

Analysis of the form of the teacher's scaffolding utterances in the second cycle of clinical interviews revealed that she used a greater proportion of questioning interventions than pre-investigation (telling accounted for 79 out of 134 coded utterances, with the remaining 55 coded as questions). There were some notable changes in the function of her scaffolding interventions that may have resulted from the action-research investigation. In the second cycle she *demonstrated* and *explained* considerably less, having been critical of her use of those interventions previously. She *parallel modelled* more often, and also *probed* more often and more directly. The final observed change was that she was now utilising *indirect prompts* – a form of fading – which she had not done in the first cycle of clinical interviews.

Analysis of the 'self-critical' content of the second cycle analytical log revealed that she was dissatisfied with instances when she used questioning to *focus*, *funnel*, *probe* and *prompt*. A common theme can be detected in these criticisms: the observation that she was not giving the students sufficient time to think. The teacher finally noted that her lack of confidence with using an untried method had caused her to intervene and change the way one student was approaching a particular question.

Analysis of the 'approving' content of her analytical log indicated that the teacher was satisfied with many more of her scaffolding interventions than previously: specifically, instances where she employed telling to discuss *convention*, *direct* (when procedural content was involved), *focus*, *parallel model* and *probe*; and where she used questioning to *focus, parallel model*, *probe* and *prompt*. She also approved her use of indirect fading prompts, as the following log extract shows:

> He was still unsure of what to do, so I prompted him indirectly on what it might be a good idea to do now. It worked: he realised it would help to draw a diagram (a prompt I had been using previously during lessons). I think this is an example of 'fading' – replacing direct prompts with increasingly indirect ones in order that the student internalises the original prompt. He was then able to proceed without further intervention from me. I am pleased with this: it felt right.

RQ3: What aspects of the scaffolding strategies employed are valued by the students?

Analysis of the feedback interview responses from the first cycle of clinical interviews revealed that *discussing* a conventional norm, *explaining* and *prompting* were valued strategies. One student made the suggestion that *parallel modelling* would have helped him more – consistent with the teacher's own reflective evaluation.

Analysis of student responses from the second cycle of clinical interviews revealed that *prompting*, *parallel modelling* and *confirming* were valued scaffolding strategies. One student also suggested that more use of *demonstrating* would have helped him, specifically the use of diagrams to enable him to visualise the situation more easily.

CONCLUSION

From the analysis of this teacher's utterances it becomes apparent that this one-to-one teacher-pupil pedagogical interaction is far more complex than the commonly-held view, cited in Baxter and Williams (2010, p. 8), that "teachers should not lecture, demonstrate or 'tell'". These findings are consistent with Chazan and Ball's (1995) argument that context is all-important, and is a crucial consideration in the management of the dilemma of telling. This finding, coupled with the teacher-researcher's realisation that she had, indeed, been able to modify and develop her scaffolding strategies – to tell more selectively and question more skilfully – added to her confidence as an instructor. Moreover, she continues to use her coding framework as a reflective tool.

Such is the paucity of research into mathematics teacher-student dialogue (Kyriacou and Issitt, 2008), particularly at secondary level, that there is abundant scope for teacher-researchers to undertake related studies into 'contingent' (Rowland, Thwaites and Jared, 2011) interactions in their classrooms. With this purpose in mind, the coding framework devised and applied in this study would be a useful tool for other teachers wishing to examine and develop their scaffolding strategies. Furthermore, the impact of classroom pressures (such as classroom management, curriculum and testing) on teachers' scaffolding strategies – something that policy makers often seem to overlook – also merits further investigation.

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THE EFFECTS OF TWO INSTRUCTIONAL APPROACHES ON 3RD-GRADERS' ADAPTIVE STRATEGY USE FOR MULTI-DIGIT ADDITION AND SUBTRACTION

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In arithmetics education, two instructional approaches are suggested to teach children adaptive strategy use. The explicit approach encompasses the explicit teaching and practicing of selected strategies, whereas the problem-solving approach emphasizes the analysis of task characteristics and the individual generation of strategies. In a one-week intervention study with 79 3rd-graders from 17 school classes, we compared these instructional approaches. Results from post- and follow-up tests did not yield significant differences between the two approaches in adaptive strategy use or in accuracy. In comparison to a control group (consisting of the 162 classmates), the combined experimental groups showed a sustainable better achievement for adaptive strategy use whereas there was no significant difference concerning accuracy.

1 INTRODUCTION

In the last decades, mathematics educators called for a change in elementary arithmetic education by questioning the exclusive role of the standard (written) algorithms for the basic arithmetic operations. In particular, the acquisition of adaptive expertise, i.e. the ability of individuals to solve arithmetic computation tasks efficiently by flexibly choosing an appropriate strategy, is attached a more prominent role. Although adaptive strategy use for multi-digit computations is considered as an important skill, empirical studies revealed unsatisfactory results for primary school students (e.g. Selter, 2001; Torbeyns, Verschaffel, & Ghesquière, 2006). Accordingly, it seems that specific instructional approaches are necessary to organize effective learning opportunities for students. Unfortunately, hardly any empirical studies examining the effectiveness of instructional approaches to foster students' adaptive strategy use exist so far.

2 ARITHMETIC COMPUTATION STRATEGIES

There are various ways of categorizing computation strategies for multi-digit addition and subtraction (e.g. Threlfall, 2002, pp. 33ff.). We use a categorization with five idealized strategies for addition and subtraction problems which is described in prominent mathematics education books in Germany (cf. Table 1). We want to emphasize that these strategies are idealized strategies. Children obviously use more strategies, especially by combining two or sometimes three of these idealized strategies. The jump and the split strategy are universal strategies which can be applied

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for all addition and subtraction problems[1]. The other three strategies are advantageous only for specific problems and cannot be applied efficiently in general.

Jump strategy	Split strategy	Compensation strategy	Simplifying strategy	Indirect addition[2]
123 + 456 = *579*	123 + 456 = *579*	527 + 398 = *925*	527 + 398 = *925*	701 - 698 = *3*
123 + 400 = 523	100 + 400 = 500	527 + 400 = 927	525 + 400 = 925	698 + 3 = 701
523 + 50 = 573	20 + 50 = 70	927 – 2 = 925		
573 + 6 = 579	3 + 6 = 9			

Table 1: Idealized computation strategies.

2.1 Adaptive strategy use of primary school children

For examining children's skills of adaptive strategy use - which means children's skills of solving arithmetic computation tasks efficiently - we firstly have to describe what we understand by an "efficient" solution. We consider two dimensions: (1) For a given arithmetic task, one can check from a mathematical perspective which strategy (or strategies) need(s) the smallest number of solution steps. (2) There is a student performing these solution steps. From a psychological perspective, one can ask how much cognitive effort different solution steps require, which obviously depends on the knowledge and skills the individual has acquired so far (probably biased by affective variables like self-efficacy). Moreover, it is the question in which context an individual solves a task. In particular in school, it may happen that students follow a reference framework which is different from the mathematical perspective mentioned above (cf. socio-mathematical norms, Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009).

In empirical studies using assessment tests, it has to be defined normatively which strategies are considered as efficient for a given arithmetic task and which are not. The simplest way to do this is to restrict the criteria to properties of the given task. However, as done in many studies (e.g., Klein, Beishuizen, & Treffers, 1998), individual characteristics like knowledge and skills also have to be taken into account. In our research with 3rd-graders, we first identify the range of strategies which can be expected by the group of students under investigation (i.e., strategy repertoire in the sense of declarative knowledge as well as the fluent and accurate application of these strategies with low cognitive effort in the sense of procedural knowledge). Then for these strategies we analyze how they fit to the characteristics of a given task and, thus, provide a short solution (normative mathematical perspective).

As already mentioned in the introduction, empirical findings reveal an unsatisfactory proficiency of primary school students concerning the adaptive strategy use. Before they learn the standard algorithms, many students have a favourite strategy which they

[1] It is an open discussion how to deal with the split strategy in case of subtraction problems with regrouping. Some of the German textbooks introduce this strategy but avoid the notation of intermediate (negative) results.
[2] The indirect addition strategy is for subtraction problems only.

use as a standard procedure ignoring number characteristics of the given tasks (German students frequently prefer the jump strategy for subtraction tasks and the split strategy for addition tasks, Heinze, Marschick, & Lipowsky, 2009). Moreover, most students solely use the standard algorithms after they have been introduced (e.g., Selter, 2001).

2.2 Influence of the instructional approach

The unsatisfactory results concerning students' adaptive strategy use points to the role of mathematics instruction. Based on a literature review, we distinguish three idealized instructional approaches denoted as routine approach, explicit approach and problem-solving approach (see Heinze et al., 2009 for details). The *routine approach* is a traditional approach which follows the idea of "one strategy first". This means, that firstly only one strategy – in general, the jump strategy – is learned and practiced by the students so that it can be applied accurately as a routine procedure. After that other strategies and their adaptive use are presented in a sense that there exist so-called „computation tricks" or „advantageous computations" which are sometimes helpful.

The *explicit approach* is a reform-oriented approach which emphasizes the adaptive strategy use from the beginning. In an introductory phase, students invent their own strategies. After that it is up to the teacher to reduce the diversity of invented strategies to a set of main strategies (cf. Table 1) which are successively introduced and practiced by the students. By solving tasks and discussing different solutions, the latter follows two goals: the acquisition of routine expertise in strategy execution and of experience in adaptive strategy use. An example for an implementation of this approach is the realistic program design as implemented in the study by Klein et al. (1998).

The *problem-solving approach* has a stronger constructivist character than the explicit approach. It rejects the assumption that individuals "select" a strategy from an strategy repertoire (cf. Threlfall, 2002). Instead, each arithmetic task is considered as a new problem and, accordingly, students generate a specific solution strategy for this problem (based on their knowledge and experience and on the task characteristics). Hence, the teacher does not introduce any "official" strategy but continuously gives opportunities to analyze task characteristics, to solve problems and to discuss the efficiency of the students' solution strategies. Based on their experiences, students will accumulate knowledge on task characteristics and on skills in applying and judging individual strategies so that they will optimize their adaptive strategy use step by step.

There exist only a few empirical studies that investigated the influence of instructional approaches. For example, Klein et al. (1998) in a one-year quasi-experimental study with 2^{nd}-graders showed that students' adaptive strategy is improved more by an explicit approach than by a routine approach emphasizing one main strategy. Heinze et al. (2009) compared 3^{rd}-graders who were taught by textbooks following the routine, the explicit and the problem-solving approach respectively. They found an advantage for students taught by textbooks with the explicit and the problem-solving approach. However, for high achieving students there was no significant difference between the three groups. The latter result is in line with results from Torbeyns, De Smedt, Ghesquière, and Verschaffel (2009) showing that high achieving students taught by the routine approach can reach a high level of adaptive expertise.

3 RESEARCH QUESTION AND METHODOLOGY

Only a small number of empirical studies have investigated the influence of instructional approaches on the acquisition of adaptive expertise. Moreover, it is an open question whether the reform-oriented approaches (explicit and problem-solving approach) differ in their effectiveness. Accordingly, we conducted a strictly controlled experimental study addressing the overall research question on the effectiveness of the explicit approach and the problem-solving approach on students' adaptive strategy use and students' accuracy when solving arithmetic computation tasks. Implementing a one week intervention as a holiday course, we investigated specifically:

- Are there sustainable effects of a one-week teaching intervention on adaptive strategy use (in comparison to a control group)?
- Are there differences in the short-term and long-term effects of the instruction based on the explicit and the problem-solving approach respectively?

3.1 Sample, design and instruments

We focus on 3rd-graders (9-10 years old) because in Germany in the first half of the 3rd grade, the number domain is extended up to 1000. Hence, students learn addition and subtraction strategies for three-digit numbers. In the second half of grade 3, the standard algorithms are introduced. The sample for the intervention comprises 79 randomly chosen 3rd-graders from 17 classes of German primary schools. In principle, the children were randomly allocated to one of the two instructional approaches (randomization was controlled for general cognitive abilities, general mathematics achievement and socio-economic status). The control group encompasses the 162 classmates of the children who participated in the intervention. The intervention was organized as a one-week course at our research institute during fall holidays in October 2011. The overall intervention time was equivalent to 16 schools lessons (45 min) and accompanied by breaks for playing games and lunch. The lessons were taught by two trained research assistants following ideal-typical teaching scripts of the explicit and the problem-solving approach (a short overview is given in Table 2, scripts and material were approved by expert ratings). To limit the group size, we had two student groups for each approach (one group was taught in the first and one in the second holiday week). To control for teacher effects, both teachers taught each approach once.

Day	Explicit approach	Problem-solving approach
1	Repetition of numbers up to 1000 and small group discussions	
2	Discovery & practice of jump and split strategy, small group discussions of individual solutions	Distance of given numbers, decomposing numbers, categorizing tasks in easy, smart[3] and other tasks
3	Discovery & practice of indirect addition, compensation & simplifying	Categorizing tasks, generation of easy and smart tasks
	Solving tasks and comparing solutions in small group discussions	
4	Repetition of all strategies	Categorizing tasks and discussing individual criteria for categorization
	Solving tasks and comparing solutions in small group discussions	
5	Post-tests and interviews[4], closing session	

Table 2: Content of the one-week holiday course for both approaches

Data for adaptive and accurate strategy use was collected by trained university assistants with a pre-test 2 weeks before the intervention (T1), an immediate post-test (T2) and two follow-up tests after 3 months (T3, January 2012) and after 8 months (T4, June 2012). The tests at T4 were administered after the students learned the standard algorithms for addition and subtraction. The control group participated only in the testing at T1, T3 and T4 because the post-test at T2 was during holidays. Each test consisted of 8 multi-digit addition and subtraction tasks suggesting specific strategies as efficient solutions (e.g. compensation, simplifying or indirect addition, see Table 1). The four tests were linked by anchor items: consecutive tests had 6 common items and a core of 4 anchor items was part of all tests (403-396, 1000-991, 398+441, 502+399).

The item solutions were rated two times: firstly as correct or incorrect and secondly by the efficiency of the strategy for the given task. For the latter, we used a bottom-up procedure to develop a category system. The strategy efficiency was judged independently by two persons with an acceptable inter-rater reliability ($\kappa > .70$). For the results presented in this paper, we assigned 0, 1 or 2 points to each category depending on a normative rating whether the used strategy was efficient (2 points), not efficient (0 points) or "partly efficient" (1 point). The one-point category encompasses, for example, mixtures from efficient and inefficient strategies or inefficient strategies where a single simple step is processed mentally. These strategies are not inefficient but also not really efficient. For the statistical analysis, we scaled the raw data independently for accuracy and adaptivity using the IRT-based Rasch model (software ConQuest). This procedure allowed a mapping of the data of different tests on two

[3] "Easy tasks" can be solved immediately (e.g., 150 + 230), "smart tasks" easily by a specific strategy (e.g., 329 + 141). Obviously, the allocation of tasks depends on the individual.
[4] We also conducted interviews which are not discussed in this paper.

uni-dimensional scales (for adaptivity and for accuracy). We conducted the scaling twice: once for T1, T3 and T4 comparing the effects of the intervention with control group conditions (the control group did not participate in T2) and once for T1-T4 to compare the two intervention groups. For all scales, we obtained good item fits and the EA/PV reliability at each of the measurement points was satisfying (adaptivity: .74-.90 and accuracy: .68-.88). Due to the IRT modeling, the measurement units of the scales are logits and not absolute values. To give an idea about students' absolute accuracy performance, we can report a rate of correct solutions between 58% at T1 and 70% at T4 (although these values are not directly comparable between the different tests).

4 RESULTS

4.1 Effects of the teaching intervention in comparison to a control group

To analyze the effects of the one-week intervention, we compare the results of the children who participated in the intervention with those of their classmates for the pre-test (T1) and the follow-up tests (T3, T4). The comparison indicates whether the one-week intervention has specific sustainable effects in addition to the regular mathematics class. Since the results for all tests are allocated on the same scale, we use an ANOVA with repeated measurement (see Figure 1).

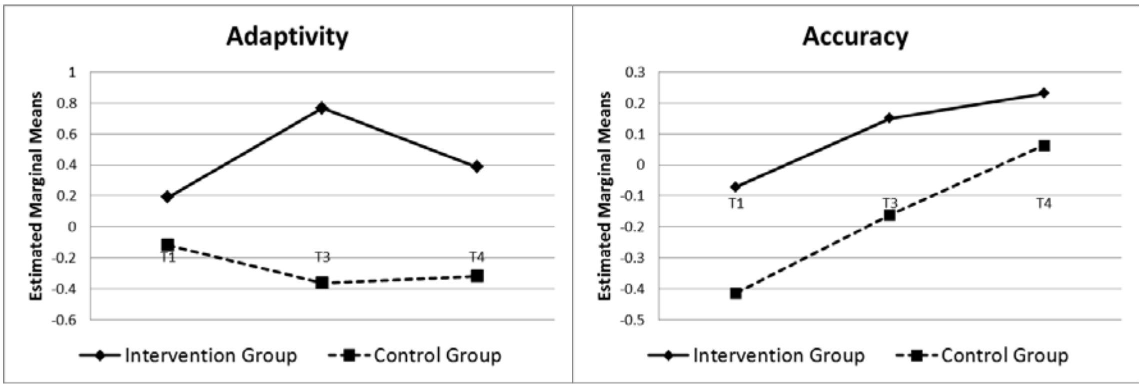

Figure 1: Development of adaptive and accurate strategy use of the children from the intervention (both groups combined) and their classmates (control group). For both scales, units of measurement are logits and not absolute values.

Regarding students' adaptive strategy use, the main effect "time" is not significant ($F(2,478) = 1.28$, $p = .28$) but we get a significant interaction effect "time*group" ($F(2,478) = 5.69$, $p < .01$, partial $\eta^2 = .03$). The latter can be traced back to the difference at T3 between the children who participated in the intervention and their classmates in the control group ($F(1,239) = 11.36$, $p < .001$, partial $\eta^2 = .04$). In case of accuracy performance, we get a main effect "time" ($F(2,478) = 15.12$, $p < .001$, partial $\eta^2 = .06$) but no interaction effect "time*group" ($F(2,478) = 1.10$, $p = .33$). As displayed in Figure 1, the accuracy performance of both groups increases similarly.

4.2 Effects of the instructional approach

For the comparison of the two reform-oriented instructional approaches, we use an ANCOVA with repeated measurement for T1-T4. Since we have data on general cognitive abilities for the children who participated in the intervention, we include this as a covariate. For students' adaptive strategy use, it turns out that we get significant main effects "time" for adaptivity ($F(3,228) = 4.32$, $p < .01$, partial $\eta^2 = .05$) and accuracy ($F(3,228) = 3.35$, $p < .05$, partial $\eta^2 = .04$). The interaction effects "time*group" for adaptivity ($F(3,228) = .79$, $p = .50$) and for accuracy ($F(3,228) = .60$, $p = .62$) are not significant (see Figure 2 for an illustration of the adjusted mean values; the values of Figures 1 and 2 are not comparable since we scaled two times, see 3.1).

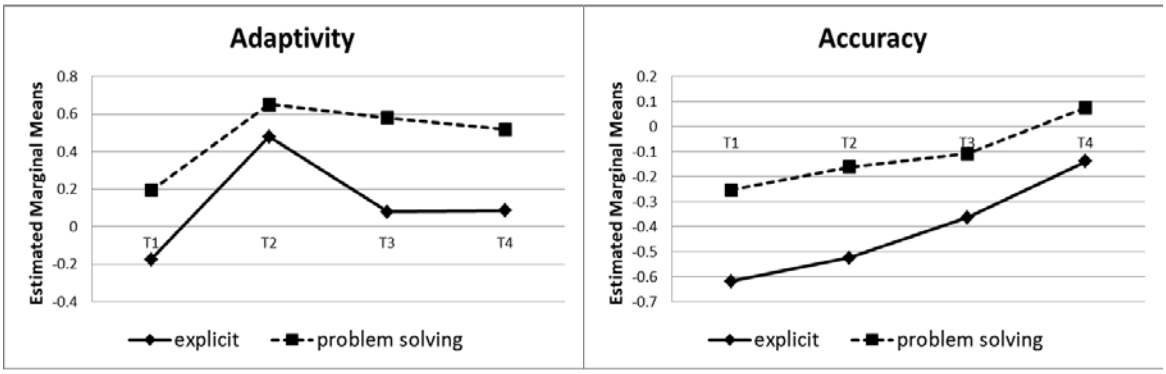

Figure 2: Development of adaptive and accurate strategy use of students taught by the explicit and the problem-solving approach (covariate: general cognitive abilities). For both scales, units of measurement are logits and not absolute values.

5 DISCUSSION

The findings presented in 4.1 indicate that the one-week intervention was successful in the sense that the participating students specifically improved their adaptive strategy use compared to their classmates. Even after 3 months, students who attended the intervention still show a better adaptivity in their strategy choice. After 8 months, when the students learned the dominant standard algorithms in their mathematics class, the difference regarding the adaptive strategy use becomes smaller (and is not significant anymore) which is in line with findings from other studies (see 2.1). However, it seems that for a certain time the comparatively short intervention "protects" the students against an unreflected application of routine procedures. Furthermore, the results show that the intervention, which aimed at a flexible and task specific application of different strategies, had no negative effects on students' accuracy. This result is interesting because it is a plausible assumption that students make more mistakes when they have to learn and to apply more than one strategy. Here, in contrast, it turns out that an increase in adaptivity is not to the disadvantage of accuracy. Summarizing the results of 4.1, we can state that the one-week intervention yielded a specific and sustainable intervention effect on students' ability for adaptive strategy use.

Comparing the explicit and the problem-solving approach, we cannot report significant differences in adaptivity or in accuracy (see 4.2). Although Figure 2 suggests a difference between the groups, we cannot report an adequate level of significance. At this stage, we are cautious with far reaching interpretations because different explanations are possible. For example, it is indeed a possibility that both approaches have more or less the same effect on a group level. Then we would conclude that it does not matter which approach a teacher follows. Since we only consider aggregated values on a group level in our statistical analysis, another explanation could be a compensation of positive and negative effects of each approach. So, it is possible that high achieving and low achieving students benefit in a different way from the two approaches. We collected a lot of additional data during the intervention by interviews on numerical knowledge, perceived socio-mathematical norms etc. In the following months, we will conduct further in-depth analyses to examine whether there are specific effects of the different approaches.

Acknowledgements

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PROBING STUDENT EXPLANATION

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Previous studies have produced several typologies of teacher questions. Often probing questions that require students to explain are included into the types of questions. However, studies that have created subtypes for probing questions are rare. The aim of this study is to elaborate on different ways of asking students to explain in mathematics. Altogether, 29 pre-service teachers' lessons were videotaped. The videotapes were coded for teachers' probing questions. After this, I used the grounded theory approach to create categories for types of probing questions: probing solution method, probing reasoning, probing argument, probing reason for something, probing meaning, probing extension, and unfocused probing questions. The types of probing questions are discussed in the paper.

INTRODUCTION

Several studies highlight the importance of asking students to explain. There is evidence that student explanation has a positive effect on learning and strengthens students' understanding (Rittle-Johnson, 2006; Wong et al., 2002). Explanations also make students' thinking visible and allow the teacher to proceed accordingly (Ruiz-Primo, 2011). Furthermore, asking students to explain is important in supporting dialogic interaction (McNeill & Pimentel, 2009) and in creating learning possibilities for other students too. According to Kazemi and Stipek (2001), teachers' ways of asking questions affect students' explanation behaviour. Hunter (2008) reported that students also learned to ask explanations from each others.

Traditionally good teachers are considered as good explainers. However, nowadays more emphasis is placed on getting the students to explain. In particular, inquiry-based mathematics teaching highlights the importance of student explanation (e.g., Kazemi & Stipek, 2001). In this study, inquiry-based mathematics teaching means that students work alone or in small groups to solve non-standard mathematical problems designed to potentially bring forth mathematical ideas related to the topic at hand while the teacher supports students' reasoning and orchestrates classroom discussion. In line with Stein, Engle, Smith, and Hughes (2008), inquiry-based mathematics teaching consists of launch, explore, and discuss and summarize phases. The teacher introduces the problems in the launch phase. Then, students work in small groups during the explore phase. Finally, students' solutions are discussed in the discus and summarize phase. The main idea of inquiry-based mathematics teaching is that a teacher has a crucial role in activating students to reason more and more mathematically and to build mathematical explanations for their findings (Hähkiöniemi & Leppäaho, 2012).

Questions that ask students to explain are usually considered higher order questions. For example, Kawanaka and Stigler (1999) consider higher order questions requesting

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an explanation or description of a mathematical object, solution method, or a reason why something is true or not true. They found that 9.6 % of German, 22 % of Japanese, and 1 % of U.S. eight-grade teachers' questions were higher order questions. The rest of the questions asked for a yes or no answer or to state a fact. Also other studies have found that the proportion of questions that ask for explanation is relatively small (e.g., Myhill & Dunkin, 2005; Sahin & Kulm, 2008).

Sahin and Kulm (2008) consider three types of mathematics teachers' questions: factual, guiding, and probing questions. Factual questions request a known fact, guiding questions give hints or scaffold solution, and probing questions ask for elaboration, explanation or justification (Sahin & Kulm, 2008). Sahin and Kulm used the following three criteria for identifying probing questions: (1) ask students to explain or elaborate their thinking, (2) ask students to use prior knowledge and apply it to a current problem or idea, (3) ask students to justify or prove their ideas. They found that the two sixth-grade teachers' use of probing questions varied from 17 % to 42 %.

Despite several classification schemes of teacher questions, studies that suggest different subcategories for questions that ask for explanation are rare. One such study is Kawanaka and Stigler's (1999) study. According to them, higher order questions may request (1) analysis, synthesis, conjecture or evaluation, (2) how to proceed in solving a problem, (3) methods that were used to solve the problem, (4) reasons why something is true, why something works or why something is done, or (5) other information. Kawanaka and Stigler (1999) found that the teachers in the three countries asked different kinds of higher order questions.

The aim of this study is to further elaborate on different ways of asking students to explain in mathematics. Particularly, I construct subcategories for different types of probing questions that request explanation. The subcategories help us to understand the complexity of explanation asking. The following research question guided the data analysis: In what ways do teachers ask probing questions that invite students to explain?

METHODS

Data collection

The data of this study is a part of a larger study on pre-service teachers' implementation of inquiry-based mathematics teaching. 29 pre-service teachers participated to an inquiry-based mathematics teaching unit. The unit included nine 90 minutes group work sessions about the ideas of inquiry-based mathematics teaching. For example, the pre-service teachers practiced how to guide students in hypothetical teaching situations (see, Hähkiöniemi & Leppäaho, 2012). Then, each pre-service teacher implemented one inquiry mathematics lesson in grades 7–12. All the lessons were structured in the launch, explore, and discus and summarize phases. Students used GeoGebra to solve problems in 17 lessons.

The lessons were videotaped and audio recorded with a wireless microphone attached to the teacher. The video camera followed the teacher as he or she moved around the classroom. When the teacher discussed with a student pair, the camera was positioned so that students' notebooks or computer screens could be seen. Students written notes were collected after each lesson. Additional data, which is not used in this paper, includes video recorded debriefing sessions, audio recorded stimulated recall interviews of the teachers, and video recorded work of focus students with additional camera.

Data analysis

Data was analysed using Atlas.ti video analysis software. All the teachers' subject related questions were coded to probing, guiding, and factual questions. The definitions for these codes were constructed on the basis of Sahin and Kulm's (2008) definitions. All teacher utterances which requested students to explain or examine their thinking, solution method or a mathematical idea were coded as probing questions. A teacher utterance was considered as a question if it invited the students to give an oral response. For example, utterances such as "explain" were considered as questions even though grammatically they are not questions. On the other hand, grammatical questions were not coded as questions if the teacher did not give the students a possibility to answer the question.

After this, all the probing questions were further analysed. The grounded theory (Glaser & Strauss, 1967) approach was applied. First I viewed to the probing questions several times to become familiar with them. Then, I clustered the probing questions into categories. I constructed the categories by interpreting what the teacher asks students to explain. I used the method of constant comparison (Glaser & Strauss, 1967) as I compared each coded question to the other questions coded to the same category. In addition, I compared how each question would fit to the other categories. After creating the categories, I examined the properties of the categories by viewing repeatedly the questions of a certain category. I also compared the categories to each other and explored relations between them. Through this process I organised the categories into main categories (see Table 1). Due to space limitations, only the main categories are discussed in this paper.

RESULTS

Altogether, the pre-service teachers asked 345 probing questions that is 25 % of all the subject related questions. The categories of probing questions are presented in Table 1. Below, I elaborate on the different types of probing questions that the pre-service teachers asked.

Type of a probing question	Frequency	Percentages
Probing solution method	96	28 %
Probing reasoning	70	20 %
Probing argument	37	11 %
Probing reason for something	61	18 %
Probing meaning	46	13 %
Probing extension	14	4 %
Unfocused probing questions	21	6 %

Table 1: The types of probing questions asked by the pre-service teachers' (n = 29).

Probing solution method

In these kinds of probing questions, a teacher asked students to explain how they solved a problem or what they did. For example, in an 8th grade lesson about percentages, a pair of students was solving how much juice can be made of 1.5 litres of concentrate when 30 % of the juice has to be concentrate. The students had solved the problem as shown in Figure 1, when the teacher came to talk with the students:

Teacher: Explain a little what you have done here [invites oral response].

Student: We took first 10 % which is this 0.5. Then we multiplied it by 7 to get 70 %. Then we added the 30 % to 70 %.

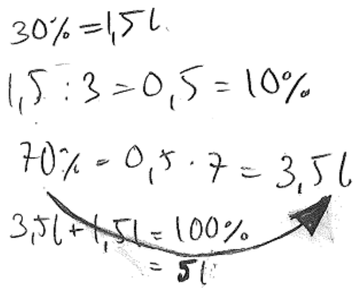

Fig. 1. Students' solution of how much juice can be made of 1.5 litres of concentrate when 30 % of the juice has to be concentrate.

The teacher's utterance was a question in a sense that it invited an oral response from the students. The question explicitly asked the students to explain what they did encouraging the students to explain how they solved the problem.

This category includes also questions that ask how students reached a solution without clearly expressing whether students should explain what they did or how they reasoned. For example, the teacher discussed with another student pair about the same task as above:

Teacher: Where did you get that kind of an equation [$x \cdot 0.30 = 1.5$]?

Student: Well, you need 30 % concentrate. So. This is 30 %. So, when x is multiplied by it we get 30 % of x which is 1.5.

In this case, the student actually responded by explaining the reasoning behind the equation.

Probing reasoning

This category includes questions in which a teacher asked students to explain what they are thinking, how they reasoned something, how something could be reasoned, how they invented something, or what kind of problem they have in their thinking. The difference to the previous category is that a teacher explicitly asked to explain reasoning or thinking. For example, in a 10th grade lesson about the contingence angle of two tangents to a circle, a student claimed that the sum of the central angle and the angle of contingence is 180° (see Fig. 2). Then, the teacher asked her to explain how she reasoned it:

Teacher: From which did you conclude it?

Student: Because the two other angles are 90, it becomes 180 [sum of the angles C and D], and because this is quadrangle, it is 360 [sum of the angles A, B, C, and D].

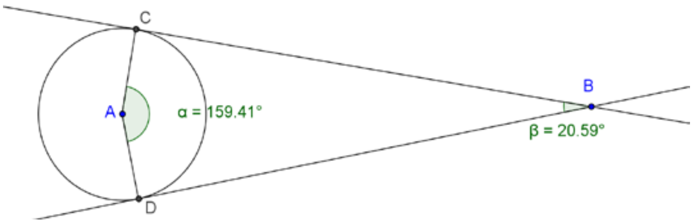

Fig. 2. GeoGebra applet for investigating the sum of the contingence angle and central angle.

In this case, the student responded by explaining her reasoning. However, sometimes students explained what they did even though the teacher asked about reasoning.

Probing argument

In these kinds of probing questions, a teacher asked students to give arguments by requesting justification, how students know something, or whether something really is as students claim. For example, in an 11th grade lesson about logarithm included the following whole class discussion about $\log_2 16$:

Student: We got 4.

Teacher: Yeah. What would be the argument?

Student: Because 2 to 4 equals 16. Isn't it? 4 to 2. I don't know.

In this case, the teacher asked the student to justify his answer but the student was not sure about the justification.

Probing reason for something

When probing reason for something the teachers asked students to explain reason why something is as it is or why the students did something. For example, in a 9th grade lesson about divisibility rules, a student claimed that a number is divisible by two if the last digit is even. Then, the following discussion occurred:

Teacher: What is the reason, could you..?

Student: Because they are divisible by two. […]

Teacher: Why is it enough to look at the last digit?

Student: Because if the last one would by odd, then the number could not be divisible by two. […]

Teacher: What is the reason that you can divide the whole number by two? I can see that you can divide four [by two].

Student: They are round thousands, round hundreds, round tens, to which only the digit in the end is added to. So it is the one digit which matters instead of the whole number. […]They are complete thousands, hundreds, and tens, which all are divisible by two, and therefore, the whole number is divisible by two if the last one is not odd.

In this episode, the teacher repeatedly asked the student to explain the reason for the divisibility rule noticed by the student. At first, the student seemed not to understand what kind of reason is asked for, but finally, when the teacher kept on asking, the student formulated mathematical explanation for the divisibility rule.

Probing meaning

This category includes questions in which a teacher asked students to explain the meaning of something. For example, in a 7th grade lesson about the concept of variable, the teacher asked about the formula that a student pair had constructed to describe a certain phenomenon:

Teacher: Tell about this. What does this mean [the students' formula $h \cdot 5 + 2$]?

Student: Every hour costs 5 Euros plus the 2 Euros entrance fee.

Often, the pre-service teachers did not explicitly ask for meaning as above but, for example, "what happens here?" In the latter case, students are still expected to explain what they mean by something. This category includes also questions that encourage students to explain more, and thus, clarify what they mean. For example, a teacher encouraged a student to explain the meaning of a figure by asking "What do you have in the figure?"

Probing extension

In these kinds of probing questions, a teacher asked students to explain how their solution method would work in a slightly different situation or how the problem could be solved differently. These questions invite to explain how a solution could be extended to a new direction. For example, a teacher asked this kind of question in an 11th grade lesson about continuity when a group of students said that a certain piecewise function is continuous because the graphs given by the calculator overlap:

Teacher: If you calculated it, what would happen? […] How could you calculate whether the graphs overlap without drawing the graphs?

Student: Is it possible to calculate the intersection points? If you substitute $x = 1$, it will not be possible.

In the above episode, the teacher's questions steered the students to consider using the equation of the function in addition to the graph of the function. The question also invited students to explain how they could do this. Thus, the question was a probing question which asked students to extend their solution to a new direction. The difference to guiding question is that students are invited to examine their solution in relation to the potential extension suggested by the teacher. In contrast, guiding questions help students to solve the problem in first place.

Unfocused probing questions

Unfocused probing questions invite students to explain but it is not expressed what should be explained. For example, this category includes the following questions: "Would you like to say something [about the solution of a problem]?" and "Do you have an idea?"

DISCUSSION

The results of this study show that there are several different types of probing questions. Although all probing questions request explanation, different things are asked to be explained. Previous studies have proposed several questioning typologies which often include questions that ask for explanation (e.g., Kawanaka & Stigler, 1999; Myhill & Dunkin, 2005; Sahin & Kulm, 2008). This study created subtypes for probing questions: probing solution method, probing reasoning, probing argument, probing reason for something, probing meaning, probing new idea, and unfocused probing questions.

Some of the categories of probing questions resemble those of previous studies. The category of probing solution method is similar to Kawanaka and Stigler's (1999) question types asking for how to proceed in solving a problem and methods that were used to solve a problem. The other question types of Kawanaka and Stigler (1999) do not have such a clear correspondence. For example, reasons are asked in the categories of probing reasoning, probing argument, and probing reason for something. When compared to Sahin and Kulm's (2008) three criteria of probing questions, the justification criteria is similar to probing argument and the criteria of applying previous knowledge resembles slightly the category of probing extension.

A relatively big proportion of the pre-service teachers' questions were probing questions when compared to previous studies (Kawanaka & Stigler, 1999; Myhill & Dunkin, 2005; Sahin & Kulm, 2008). Thus, it seems that pre-service teachers are prepared to ask probing questions. However, a large proportion (28 %) of the probing questions requested students to explain how they solved a problem. Kawanaka and Stigler (1999) reported even larger proportion of this kind of questions. Thus, teachers need to be aware of what they ask students to explain and ensure that students engage also in explaining their reasoning (cf. Kazemi & Stipek, 2001). However, students do not always explain their reasoning even though asked for. Thus, teacher needs to keep on asking reasoning with slightly different words as illustrated in the results (probing reason for something). In future research, it would be interesting to study how the types of probing questions are related to types of students' explanations.

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WHEN VISUAL AND VERBAL REPRESENTATIONS MEET THE CASE OF GEOMETRICAL FIGURES

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We present the first stage of a "study in progress" whose aim is to link visualization, students' construction of geometrical concepts and their definition, and students' ability to prove. We exemplify this stage in our research (visualization and concept formation), by means of paradigmatic examples, which reveal visual and verbal processes related to construction processes of geometric figures and inclusion relationships between figures and their attributes. Our results confirm known findings (e. g., the position of a shape affects its identification and the related inclusion relationships) and point to findings in a new direction, like the effect of the question's representation on students' responses concerning the inclusion relationships.

INTRODUCTION AND BACKGROUND

Quite many studies investigated the effect of visualization on the construction of basic geometrical concepts (e.g. Fischbein, 1993; Fujita & Jones, 2007; Hershkowitz et al., 1990; Vinner & Hershkowitz, 1980). Many other studies focused on proofs in geometry and reported on many difficulties students face in the process of proving and in the understanding of its meaning (e.g. Martin et al., 2005). There is definitely less research attempting to link these two trends of research with the aim of considering findings and insights from one trend as a vehicle for understanding and interpreting difficulties which were found in the other. This is quite surprising, considering the fact that the accepted way to teach geometry in school in most countries is a hierarchical division of contents and teaching approaches from intuitive to formal along the school years as if it is clear theoretically and practically that figural, intuitive and visual geometrical knowledge is a necessary condition for construction deductive geometrical knowledge. For example, the US NCTM curriculum standards (1989) claim that "the study of geometry in grades 5-8 links the informal explorations, begun in K-4, to the more formalized processes studied in grades 9-12" (p.112). This accepted way of teaching is justified by the hierarchical level structure of the van-Hiele theory (1958), which is the most comprehensive theory concerning teaching and learning geometry. This research report is part of "a study in progress" aimed at linking visualization, students' construction of geometrical concepts and their definitions, and students' ability to prove. Figure 1 is a schematic presentation of the overall goal of the research. In this RR, we exemplify the first stage of the study, visualization and concept formation, by means of a few paradigmatic examples, which reveal visual processes related to construction processes of geometric concepts.

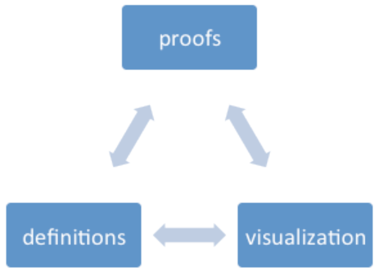


Figure 1: The aim of the research as a whole

Vinner & Hershkowitz (1980) and Tall & Vinner (1981), focused on the cognitive construction of mathematical concepts, and proposed a model of two components: the *concept definition* - the verbal description of the mathematical concept, which characterizes the concept mathematically, and the *concept image* - the cognitive structure that includes all the examples and the processes related to the concept in the learner's mind. Geometric concepts have a special status: Fischbein (1993) coined the term "figurative concepts" and explained that in our thinking the geometric shape is not only related to the formal definition, but are also linked to images.

Vinner & Hershkowitz (1983) and Hershkowitz et al. (1990), found that for each geometric concept there is at least one prototypical example. For example the square is a prototypical example of the quadrilaterals group. The prototypical examples are acquired first, and are therefore found in the concept image of most learners. Prototypical examples are usually a subset of the concept's examples with the longest "list" of attributes, the critical attributes of the concept and some attributes that are specific to that subset but are non-critical. These non-critical attributes have dominant visual properties, which have an effect on the construction of geometric concepts, and affect the classification and identification abilities, construction, and judgment concerning basic geometrical concepts. This phenomenon is in agreement with construction of concepts in everyday life (Rosch & Mervis, 1975). Hershkowitz et al. (1990) mentioned several ways to judge geometric shapes as examples of a concept: (i) Visual judgment: The student relies on the prototype as a visual frame of reference. For example, the prototype of altitude of a triangle is an altitude inside the triangle. This classification and identification level fits the first van Hiele level (visualization). (ii) Judgment by prototype attributes: Classification into examples and non-examples by checking the existence and non-existence of the special attribute of the prototype. This wrong judgement fits the second van Hiele level. (iii) Analytical classification: Students rely on the critical attributes of the concept as they appear in its mathematical definition. This correct judgment fits the second/third van Hiele levels.

The definition implies inclusion relationships between groups of the concepts' examples on the one hand, and groups of attributes of the same concepts on the other. These inclusion relations have opposite directions (Hershkowitz et al., 1990). For example, the squares are included in the parallelograms, but the group of critical attributes of the squares contains the group of critical attributes of the parallelograms.

THE STUDY (FIRST STAGE)

The goals of this stage: As mentioned above the overall goal is to investigate relationships between visualization and concept formation in relation to definitions and proofs. Here we will focus only on the role of visualization and definitions in processes of geometrical concept formation. The findings from this stage will be used as a basis for (i) defining the knowledge level of the research population in relation to populations in other similar research work, and for (ii) the next and more advanced stages in our study.

Population: The participants in the study are 112 tenth grade students from one regional high school in the Arab sector in the centre of Israel. The students learn with different teachers in three parallel classes, which are considered to be at the highest mathematical level among the seven parallel classes in this school. The teachers have a first degree in mathematics from the universities in the country and have more than ten years of experience in mathematics teaching.

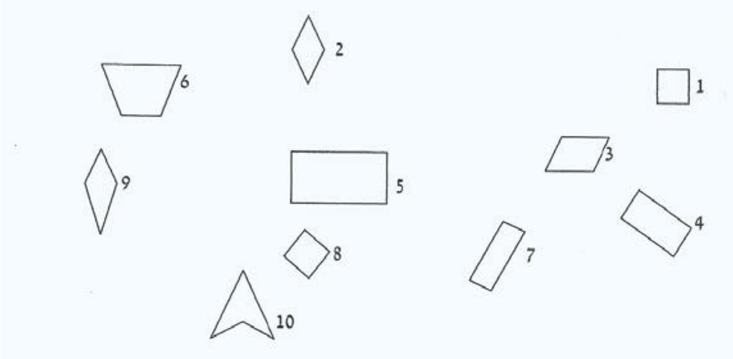
Methodology: The tools of the three-stage research include three questionnaires, distributed at time intervals sufficient for analysing the results of each questionnaire and use its findings in the design of the next questionnaire. In the present RR only questionnaire 1 is relevant. It deals with visualization (related to quadrilaterals), identification and construction of definitions, and the inclusion relationships between groups of quadrilaterals. After administering the questionnaire and analysing its results, 10 students were interviewed.

DATA COLLECTION AND FINDINGS

The data *of the first stage* were collected in 2012. Questionnaire 1 was administered to participants at the end of the first semester. We focus here on data that emerged from two tasks in this questionnaire, regarding the inclusion relationships between various groups of quadrilaterals, which was investigated in two different ways: visually (Figure 2) in the first task and verbally (Figure 3) in the second.

Data analysis from Task 1 (Figure 2): In Tables 1 & 2, we present only data concerning the squares (Figure 2, shapes 1 & 8), and their analysis. We relate quantitatively to students' knowledge about the inclusion relationships of squares in various other groups of quadrilaterals. This knowledge is investigated within a visual presentation and in two different positions of the squares. Table 1 shows that when the square is presented in an upright position, almost all students (98%) recognize the shape as a square, but 43% recognize it only as square without any inclusion relationships, whereas 18% consider the square as rectangle, rhombus and parallelogram, and another 7% also see it as a kite. In addition we can see that it is easier for students to see the square as a parallelogram (45%) than as rhombus (38%) or rectangle (32%) or kite (7%).

Task 1: Here we have the following shapes!



Write the numbers of all the shapes that are parallelograms___________
Write the numbers of all the shapes that are rectangles ______________
Write the numbers of all the shapes that are rhombuses_____________
Write the numbers of all the shapes that are squares_______________
Write the numbers of all the shapes that are kites

Figure 2: Task 1 (following NCTM, 1989)

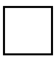
Signed as:	Square	Rectangle	Rhombus	Parallelogram	Kite	Total
	✓					48 (43%)
			✓			1 (1%)
				✓		1 (1%)
	✓	✓				2 (2%)
	✓		✓			8 (7%)
	✓			✓		12 (11%)
	✓	✓	✓			1 (1%)
	✓	✓		✓		6 (5%)
	✓		✓	✓		5 (4%)
	✓		✓		✓	1 (1%)
	✓	✓	✓	✓		20 (18%)
	✓	✓	✓	✓	✓	7 (6%)
Total	98%	32%	38%	45%	7%	100%

Table 1: Inclusion of an upright square in other groups of quadrilaterals

When we move from shape 1 to shape 8, in which the square is standing on its vertex, some results remain similar but others change dramatically (Table 2). Still, most students (90%) identify the shape as a square, but 21% consider it ONLY as a square, without any inclusion relationships. Another dramatic change is that many more students accept that this tilted square (shape 8) is rhombus (66% in comparison to 38% concerning the up-right square). All other data do not change appreciably from one position of the square to the other. These findings strengthen findings from other studies on the effect of the figure's position on its identification. It is worth to mention that these difficulties remain and can be found even within groups of teachers and teachers-students (Hershkowitz et al., 1990).

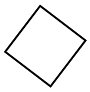
Signed as	Square	Rectangle	Rhombus	Parallelogram	Kite	Total
	✓					24 (21%)
			✓			7 (6%)
				✓		2 (2%)
	✓	✓				1 (1%)
	✓		✓			28 (25%)
	✓			✓		8 (7%)
			✓	✓		1 (1%)
	✓	✓		✓		3 (3%)
	✓		✓	✓		10 (9%)
	✓		✓	✓	✓	1 (1%)
	✓	✓	✓	✓		20 (18%)
		✓	✓	✓	✓	1 (1%)
	✓	✓	✓	✓	✓	6 (5%)
Total	90%	29%	66%	46%	7%	100%

Table 2: Inclusion of a tilted square in other groups of quadrilaterals

Data analysis from Task 2 (Figure 3): Like in Task 1, the kite is the most "problematic" figure - only 38% of the students identified the rhombus as a kite, and only about 20% identified the square as a kite. In addition, only 59% identified the square as a rectangle, 68% identified the square and the rhombus as parallelograms and 70% identified the square as a rhombus.

> Task 2. Answer the following questions (Circle the answer which seems as correct to you), and briefly explain your answers.
> Is a square a rectangle? Yes/no Explanation__________
> Is a square a parallelogram? Yes/no Explanation__________
> Is a square a rhombus? Yes/no Explanation__________
> Is a kite a parallelogram? Yes/no Explanation__________
> Is a square a kite? Yes/no Explanation_________
> Is a rhombus a parallelogram? Yes/no Explanation________
> Is a rhombus a kite? Yes/no Explanation_________

Figure 3: Task 2 in questionnaire 1 (following Fujita & Jones, 2007).

As Task 2 investigates verbally what was investigated visually in Task 1, it is interesting to compare the results. In Table 3 we present a comparison between the findings from Tasks 1 and 2 concerning the parallel items. It is interesting to realize in Table 3 that students better identified inclusion relationship verbally, when no drawings of the figures were given (Task 2), than when using the drawings (Task 1). We can interpret this result as follows: When asked about an inclusion relationship based on drawings, students tend to judge visually (van Hiele level 1), without worrying if the critical attributes of one figure contain the critical attributes of the other. On the other hand, when are asked verbally about the same inclusion relationship, they are pushed to rely on the critical attributes (van Hiele levels 2/3).

Concerning the reasons that students give for justifying their claims in Task 2, 25% to 32% give no reasons or reasons that are not geometrical, while 33% to 56% used the critical attributes of the included group (e. g., squares), rather than the critical attributes of the containing group (e. g., parallelograms). We consider this as prototypical reasoning. As the percentage of the prototypical reasoning increases, the number of correct answers concerning the inclusion relationship decreases. Only 13% to 42% of the participants use the critical attribute of the containing group, for justifying the inclusion relationships (e. g., the squares are parallelograms because they have one pair of opposite sides which are equal and parallel).

Task 2: Positive answer		Task 1: A correct identification
Is a rhombus a parallelogram?	69%	58% marked rhombus (shape 2) as parallelogram
Is a rhombus a kite?	39%	17% marked rhombus (shape 2) as a kite.
Is a rectangle a parallelogram?	77%	53% marked rectangle (shape 5) as parallelogram
Is a square a rectangle?	59%	33% -30% marked square (shapes 1 & 8) as rectangle.
Is a square a parallelogram?	68%	47% marked square as parallelogram.
Is a square a rhombus?	70%	38% marked a square (shape 1) as rhombus and 66% marked it for the tilted square (shape 8)
Is a square a kite ?	17%	7%-8% among the students that identify the shapes 1 and 8 as square identified it as a kite.

Table 3: Comparison between the results of parallel items in Tasks 1 and 2

Examples from interviews

The purpose of the interviews was to further explore issues that were revealed by the analysis of the questionnaire. We present two episodes concerning the issue investigated in Tasks 1 and 2:

Episode 1 (I – interviewer; A – Aseel, a student: discussing Task 1)

1	I:	Determine which shapes are kites.
2	A:	2, 9.
3	I:	2, 9 that's all? But, before you said that the square is a kite, you remember?
4	A:	Yes.
5	I:	You said that this shape is square, why you don’t say that it is a kite?
6	A:	Because it doesn't have the kite attributes.

7 I: A few minutes ago you said that a square is a kite.
8 A: (silent)
9 I: What's the problem with the square.
10 A: That the sides are equal.
11 I: And the kite?
12 A: Only the upper sides are equal and the bottom sides are equal.

Aseel presents a clearly prototypical judgment concerning the kite: Only shapes 2 and 9 are kites, because they are prototypical kites – they "stand" on a vertex and have two different pairs of adjacent equal sides. For Aseel, the square is not a kite, because it has this special attribute of the equality of all four sides, which prevents it from being an example of a kite, and this in spite of the interviewer’s clear hint (line 7).

Episode 2 (I – interviewer; R – Raya, a student: discussing Task 1)

1 I: A few minutes ago you said the square is rectangle.
2 R: Yes.
3 I: But here you mark shape no 1 as square but you don’t mark it as rectangle?
4 R: (silent).
5 I: Why you don’t mark it, what's its problem?
6 R: The way it looks!
7 I: So?
8 R: I want to change my answer.
9 I: What would you like to change?
10 R: "Is the square a rectangle?" before, I said yes.
11 I: And now?
12 R: No.

Raya uses a *visual judgment*: the square does not look like a rectangle. This fits the first van Hiele level. Raya uses a prototypical rectangle, which is in her concept image as a visual pattern (how it looks), she doesn't use any attributes.

CONCLUDING REMARKS

Our results confirm the findings of earlier studies concerning the effect of visual elements on the students’ geometrical concept formation. The source of these effects is the existence of prototype examples. It is surprising that when statements were given to the students verbally without visual dimension more students identified the inclusion relationships accurately and even knew how to explain them. In many cases the strong visual properties affect the students’ judgment and the classifications they make. When the task is represented verbally, these properties are hidden, but when the task is represented visually these properties become active. In summary, we have found that many students know the formal definition but do not make use of it when faced with a visual task representation.

Acknowledgment

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STUDYING MATH AT THE UNIVERSITY: IS DROPOUT PREDICTABLE?

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At German Universities, math course dropouts present a serious problem and have not been reduced significantly over the last several years. This makes a diagnostic test for high school students desirable, which helps advise students on their needs at the beginning of their studies. In the framework of a prep course for beginners in studies of math and physics, a test was developed to find out whether the beginners have a big gap of knowledge in relevant topics concerning high school mathematics. After the first six months, the individual performances in the test were compared to the students' exam results in mathematics. The predictions of the new test turn out to be significantly better than the predictions of the school examination marks.

INTRODUCTION

Dropout

Recent research on the dropout quota for Germany (Dieter, 2012) shows an approximate quota of 60% dropout for the studies in mathematics and mathematics teacher studies. We will refer to "dropout" in the sense that a student does not continue her or his studies in mathematics, but perhaps in a different subject. This dropout proved significantly higher than that in other subjects, like computer science and economics. Over the last decade, the quota has changed very little and continues to maintain this level.

This project was started under the hypothesis that the first year of high school studies needs a special focus. In the first two semesters, there is a dropout rate of about 40%. Afterwards, the portion of new cases of dropout decreases strongly from year to year. A significant gender difference is being observed in the dropout quota. In the first year, the dropout rate for men is about 10% smaller than for women, at 45%. This difference decreases over the years of study and loses significance after the third year.

German math courses

Since the transition from high school to universities concerns many details that are nationally specific, parts of this report may appear quite German. We hope to provide better insight into the situation, which may be interesting for the situations in other countries, too. The dropout rate is especially high at the beginning of the studies. Various reasons for the problems at the beginning are named, based on personal characteristics (e.g., Reason, 2004), as well as on institutional characteristics (e.g., Chen, 2012): The density of information is much higher at universities than it is at high school. Furthermore, the requirement for procedural competencies increases strongly; in particular, there is a high degree of deductive reasoning that was not learned in school. Math courses at German Universities are traditionally theoretical and

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proof-oriented and start with calculus and linear algebra in the first year. The situation concerning linear algebra is similar to that in France, see Dorier (2009).

THEORETICAL BACKGROUND

What is dropout?

The Organisation for Economic Co-operation and Development defines dropout as leaving the system of higher education without achieving a degree. These dropout quotas are not specific for a subject of study. Rather, it measures the overall effectiveness of the higher education system. To evaluate the success of a specific course of study, changes of subjects have to be taken into account. Therefore, a dropout quota of a specific course of studies can be defined as the percentage of students earning a degree in this course from all students having begun this course. This quota considers those students who start their studies in mathematics but earn a degree in another subject. The quota gives information about one specific subject rather than the higher education system.

Test Concept

The students were tested one month before their actual studies started in the framework of a prep course. Students in mathematics, physics, and pre-service teachers who want to teach mathematics at high school take these courses.

The aim of the test and the course was to decrease the difficulties of the beginning of university courses. Therefore, important fields of knowledge were identified, which are normally assumed to be known after school and necessary for the algebra and calculus courses. To find these fields, the school curricula were compared to the content of the university courses. As a result, the following themes were examined in the test: differential and integral calculus, trigonometrics, exponential and polynomial functions as well as systems of equations and inequalities, vector analysis, and foundations of algebra.

The formulated items asked for declarative knowledge, so procedural knowledge was not in the focus. The reason for this decision was the use of computers for an automatic evaluation. This made it possible to use the results for determining homogeneous seminar groups. The response formats included single and multiple-choice questions, as well as numeric answers. The test was scheduled for 90 minutes and was held at the beginning of the prep course. The one parameter Rasch Model was used to analyse the test, so to each person one person parameter was assigned to describe the test performance.

The test is divided into eight parts, described below, and for each part, one example is given.

Foundations of algebra

The first section deals with the different types of numbers, the domain of roots, fractions, and linear equations in one variable.

Example item: Choose the equivalent equations to $y = x + 5$ with x and y being real numbers.

$y^2 = x^2 + 10x + 25$ $\quad$ $y + 2 = x + 7$

$\sqrt{y} = \sqrt{x + 5}$ $\quad$ $13x = 13y - 65$

Equations and inequalities

Systems of linear equations with up to three variables and quadratic equations, as well as their graphical representation, are treated.

Example item: Find the solution set of $(x - 2)^2 < 4$.

$\mathbb{L} = \{x \in \mathbb{R} | x < 0\}$ $\quad$ $\mathbb{L} = \{x \in \mathbb{R} | x < 0 \; or \; x > 4\}$

$\mathbb{L} = \{x \in \mathbb{R} | x < 4\}$ $\quad$ $\mathbb{L} = \{x \in \mathbb{R} | x > 0 \; and \; x < 4\}$

Polynomials

The items on polynomials include computations of zeros, symmetry, images, and graphs. Rational functions and their domain, symmetry, and limits are also part of this section.

Example item: How many roots do the following functions have?

$f_1 = x^4 + x^2$ has ____ roots.

$f_2 = x^4 - x^2$ has ____ roots.

Exponential and logarithmic functions

This section deals with the calculation rules of exponentials and logarithms. Characteristics of the graphs, like monotony domain, image, and limits are also addressed.

Example item: Which of the following statements are true?

- The domain of definition of the function $f(x) = e^x$ is the set of the real numbers.
- The domain of definition of the function $f(x) = \log(x)$ is the set of the real number.
- The image of the function $f(x) = e^x$ is the set of real numbers.
- The image of the function $f(x) = \log(x)$ is the set of real numbers.

Trigonometric functions

Definitions on triangles and the unit circle, the periodicity zeros, maxima and minima, intersects, and the inverses of the trigonometric functions are treated.

Example item: Which of the following statements are true?

- The graph of the sine function and the graph of the cosine function intersect at $x = \frac{\pi}{3}$.
- The sine function has a local minimum at $x = \pi$.
- The set of all roots of the cosine function is $R = \left\{ l * \pi + \frac{\pi}{2} \middle| l \epsilon \mathbb{Z} \right\}$.

Vector geometry

The items on vector geometry deal with lines, planes, their intersects, and distances from points. Computation rules for matrixes are also addressed.

Example item: Do the points P(1/1/1), Q(2/3/4) and R(3/5/7) lie on one line?

Differential calculus

This section deals with derivatives of elementary function, differentiation rules, and the computations of maxima and minima, inflection, and saddle points.

Example item: Choose the derivative function of $f = x^2 * e^x$.

$f = x^2 * e^x$	$f = 2x + e^x$
$f = (x^2 + 2) * e^x$	$f = 2x * e^x$
$f = (x^2 + 2x) * e^x$	$f = \frac{1}{3} x^3 * e^x$

Integral calculus

Antiderivatives of elementary functions, integration rules, and computation of areas are treated in this section.

Example item: Choose an antiderivative of the function $f(x) = \sin(x)$.

$F(x) = -\sin(x)$	$F(x) = -\cos(x)$
$F(x) = \sin(x)$	$F(x) = \frac{1}{\tan(x)}$
$F(x) = tan(x)$	$F(x) = \cos(x)$

RESEARCH QUESTIONS

The official school leaving examination mark is the most frequently used criterion for awarding university places in Germany. Even though the quality of this mark has some deficiencies, it has proven to be a good predictor for marks at the university (Burton & Ramist, 2012) especially in Germany (Trapmann et al., 2007). Therefore, the implemented test only makes sense if its prediction of marks is higher than the prediction of the school leaving examination.

Concerning the gender differences in the dropout quota discussed above, it would be interesting whether this difference remains when controlled for in the test performance at the beginning of studies.

Last year, the school system in the Western federal states of Germany completed its transition from 13 years of school attendance to 12 years. In the last two years, both generations of students (13 and 12 years of attendance) entered the universities in the federal state of the university where this study was carried out. The different performances are another main interest of our study.

The following research questions are addressed:

- Is the newly developed test a better predictor for students' performances than the school leaving examination?

- Do female students perform in a different way from male students when they show the same test results at the beginning of their studies?
- Is there a difference in university exam grades depending on the time of school attendance?

METHODOLOGY

The regression analysis is based on a total of N=149 students for the calculus course and N=145 students for the algebra course. The students' exam results in a course as the dependent variable was predicted with six different independent variables. Exam results are graded from 1 (very good) to 5 (failed); the mark 4 (sufficient) is still a pass. The independent variables include the following:

- School leaving examination marks (SchEx)
- School leaving examination marks in mathematics (SchExMa)
- Person parameters of the diagnostic test (PerPar)
- Gender (Gen)
- Duration of school attendance (SchDur)
- Distance between leaving school and starting studies (SchDis)

School leaving examination marks are graded from 1 to 6, like the exam marks. The last three variables are dichotomous: in (Gen), (SchDur) and (SchDis) the value 0 stands for female, 12 years of attendance, and direct start of the studies after school resp., and the value 1 for, male, 13 years and a break of at least a year resp.

With the six independent variables, a multiple linear analysis of regression has been performed to predict separately the results of the exams. Therefore, the independent variables have been selected and included step-by-step into the model. In each step, the independent variable with the most significant correlation to the dependent variables is chosen and added to the regression model, as long as such a variable exists. It is also controlled for whether an already included variable loses significance because of newly included variables. When all variables adding prediction to the dependent variables are included and all other variables are excluded from the model, the procedure stops.

RESULTS

Results are presented according **to** the two main tests at the end of the first semester in the areas of calculus and linear algebra.

Calculus

First, the correlation among the calculus exam results and the independent variables has to be investigated. Since the highest correlation is between the person parameter of the test and the exam results, it is included in the model. Afterwards, the distance between leaving school and starting studies, as well as the duration of school attendance, was included in the model in the next two steps. None of the last three independent variables has a significant influence on the model and therefore, none of these variables has been included. In addition, none of the included variables has to be excluded due to becoming insignificant. The final model can be seen in Table 1.

Independent variable	R^2	b	β	T	Sig T
PerPar	0.437	-1.088	-0.695	-10.453	0.000
SchDis	0.461	-0.650	-0.213	-2.988	0.003
SchDur	0.480	0.198	0.147	2.071	0.040

Table 1: Stepwise multiple linear analysis of regression for the calculus exam

The first column in Table 1 shows the included variables of the model. Here, R^2 denotes the proportion of variance in the exam results, which can be explained by the independent variables included in this step. Therefore, it grows with each included variable. The variable “b” denotes the regression coefficients; the regression equation model for the prediction of the calculus exam results, including the constant, is thus:

$Calculus_{predict.}$=5.042-1.088*PerPar-0.650*SchDis+0.198*SchDur.

The variable “β” denotes the standardized regression coefficients. The standardized coefficients are those that are obtained when all variables are standardized before performing the regression analyses. With the standardized coefficients, the effects of the different variables can be compared on one scale. A higher standardized coefficient means a higher influence on the dependent variable. The last two columns show a T-test on the significance of the parameters being different from 0. With a p-value lower than 0.05, a variable is included into the model.

First, the analyses of regressions show that the diagnostic test possesses a higher prediction on the calculus exam results than the school leaving examination marks or the mathematics marks on the final school exam. This is why these examination marks are not included in the model. This is also reflected in the correlation indexes. The index for the diagnostic test is -0.591, higher than the overall school mark of 0.38 or the mathematics mark of 0.246. They do not provide any more information after including the diagnostic test results. However, this is not surprising, since the three variables all measure performances before the beginning of the studies.

The last variable not included is gender. Gender does not have an extra impact on the calculus exam after knowing the three included parameters. However, that does not mean there is no gender influence, because the influence might be hidden in the included parameters.

The standardized regression coefficients can also be compared. The quotient of two of these coefficients reflects the relative importance of the variable. The test parameter is by far the most important variable since it is more than three times greater than the parameter of the distance between leaving school and starting studies, and nearly five times greater than the parameters of the variable duration of school attendance. This is also reflected in the R^2, which is increased only a little bit by including these variables.

Linear algebra

Similar to the calculus exam results, the algebra results are predicted mostly by the diagnostic test results. However, in the algebra exam there are no further variables included in the model. The final model is a simple linear regression:

$Algebra_{predict.}$=5.235-0.836*PerPar.

While the algebra exam results correlate higher with the school marks (overall: 0.454 and math: 0.310) the correlation with the diagnostic test (-0.603) is still higher and therefore used in the regression model.

To compare the two exams, a second regression analysis was executed. This time the three variables included in the calculus exam were implemented “by force” into the model. This model can be seen in Table 2.

Independent variable	R^2	b	β	T	Sig T
PerPar	0.411	-0.860	-0.660	-10.056	0.000
SchDis	0.419	-0.303	-0.118	-1.686	0.094
SchDur	0.425	0.194	0.083	1.191	0.236

Table 2: Stepwise multiple linear analysis of regression for the algebra exam

As expected, the parameters for the two excluded variables are not significantly different from 0. Note that the b-coefficient of the diagnostic test results is smaller than for the calculus exam. Therefore, the influence is not as strong as for the calculus exam. This is also true for the distance between school leaving and starting the studies but not for the school attendance duration. However, they are also not significant.

DISCUSSION

It is well known that results involving regression models have to be interpreted very carefully. An example of this is the comparison of the group of students before the reform with 13 years of regular school attendance with the group of students with 12 years of regular school attendance. For this, quotients of beta values are considered. Assuming the same level of proficiency in the class test, the students with 12 years of school achieved better results in differential calculus than those with 13 years. The difference is significant, but it is not a strong effect, as the b-coefficient is at 0.2. This might be an indication that a difference is measurable, but usually is made up during the first months.

Those students who do not enter their studies in the same year when they left school clearly perform better (up to b=-0.65). However, there is a clear difference in the test performances of these two groups, even though not significant (p=0.065). This indicates that students who enter their studies not in the year they left school can make up the difference in the first six months of studies.

Interestingly, these results only concern differential calculus. In the area of linear algebra, the differences are not significant. This could underline that the character of linear algebra builds less intensively on knowledge required in school. Processes of abstraction and proof are more important than techniques needed for analysis, which are to some extent prepared at school.

Even though the dropout rate of female students is higher than that of their male peers, no gender effects could be deduced from the data. It must be concluded that there was no further gender effect when controlled for the other variables.

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CONFLICTING GOALS AND DECISION MAKING: THE DELIBERATIONS OF A NEW LECTURER

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Teaching, at any level, is a very complex activity and at university new lecturers are often left to unravel it themselves. Pedagogical goals are an intrinsic part of the process and the decisions a teacher makes are crucial in determining whether they are met. In this paper we use Schoenfeld's "Resources, Orientations and Goals" framework to shed light on a new university lecturer's resolution of conflicts between competing goals in two areas. We find that the framework has efficacy in analysing this resolution and the results have implications for the professional development of new lecturers.

BACKGROUND

In this paper we use Schoenfeld's framework (2010), that presents goal-oriented teaching as a decision making process, to examine the influences on the pedagogical practice of a lecturer in her first university position. The primary thrust of Schoenfeld's ideas is that decisions in teaching are made in order to meet goals (G) that the teacher has conceived. In turn, these goals are established and prioritised on the basis of the teacher's beliefs, values, dispositions, etc, called their orientations (O). Finally, during practice, resources (R) such as knowledge and physical entities are marshalled to help attain the goals. A teacher's resources, orientations and goals for a particular teaching scenario will be designated their ROG. Originally conceived as applying to school teaching, where it has been employed in research (Aguirre & Speer, 2000; Thomas & Yoon, 2011; Törner, Rolke, Rösken, & Sririman, 2010), it is becoming clear that it also has applicability to university teaching (Hannah, Stewart & Thomas, 2011; Paterson, Thomas & Taylor, 2011).

One area that this theoretical perspective can illuminate is the resolution of conflict between competing orientations and goals. Examples include the teaching of Ben, described by Jaworski (2006), whose conflict of orientations was between his belief in giving students freedom of choice and his control over their actions. In a similar vein, Thomas and Yoon (2011) analysed the resolution of conflicting goals by Adam. They showed how his goals to respect cultural influences on his students' learning and to meet the requirements of curriculum, time, and assessment led him to keep his primary goal to prepare students for future success at the expense of his goal to emphasise student-centred learning. At the university level, Paterson, Thomas and Taylor's (2011) study found that two experienced mathematics lecturers faced a conflict between their beliefs and goals as mathematicians and as teachers. In one case the teacher side won out and in the other the mathematician.

In order to investigate a teacher's ROG and its influence on practice it is important to build a supportive group around the practice. A key principle underpinning this is that of co-learning (Jaworski, 2001), where both the researchers (mathematics educators) and practitioner (university lecturer) view themselves as learners engaged in action and reflection. By studying the teaching practice, and reflecting on it critically, these co-learners become a community of inquiry (Wells, 1999; Jaworski, 2003). In this study all the authors are both mathematics education researchers and practitioners in university mathematics departments, with previous experience together of cooperative engagement in research based on a community of inquiry (Hannah, Stewart & Thomas, 2011). One aspect of this that we all subscribe to is the critical alignment of Jaworski (2006) where all the participants align with aspects of practice but can play the role of critical questioner to move practice forward. In this context we examined the role of the second author's ROG as she sought to resolve conflicts between competing goals during the first semester of her first university position.

METHOD

The case study described here involves Sepideh, a mathematics lecturer, starting a new position at a mathematics department in a research university in August 2012. Her teaching duties included two sections of a standard first course in linear algebra. Although she already had experience in teaching linear algebra, this was her first full time academic appointment, and the first time she had been given full responsibility for a course. The 45 students enrolled in each class were mainly engineering students together with some mathematics majors. The participants in the study were the new mathematics lecturer and two researchers who also had experience of teaching linear algebra. The data for this research comes from the new lecturer's diaries, recording major events that happened in her classes, her thoughts prior to teaching a topic, her reflections after the end of the course, and also from email correspondence, in the form of questions and answers exploring the lecturer's goals and orientations.

Data analysis was carried out using directed-content coding and analysis of the data, based on Schoenfeld's framework of Resources, Orientations and Goals (ROGs).

RESULTS

In this section we present two areas where the lecturer experienced conflicting goals. We analyse these conflicts in terms of her ROGs (see Figure 1 for the aspects of the ROG relevant to our discussion) and describe how these were resolved.

Conflict 1: Which order to teach?

Three of Sepideh's goals relate the issue of order of presentation of material.

Goal 1: To cover the syllabus within the allocated time.

This is, of course, an implicit goal of all teaching (R1, R2). Sepideh says, "In the beginning I basically taught very closely to the book. [R4]" This had the advantage of preserving the formal structure of the syllabus: "I know at least two other instructors that didn't cover transition matrices. How did they cope teaching this section then, if students didn't know what a transition matrix was?" On the other hand, she had the freedom (R3) to try different ways of presenting the material: "One of the mathematicians said: you need to feel your audience, if they are not really up to much formal ideas (namely the engineers) you can make it less proof orientated. [O7]"

Goal 2: To introduce every topic with a picture.

Goal 3: To show the importance of being able to understand the formal world.

These two goals both stem from Sepideh's beliefs (O1, O2 and O3) about the usefulness of Tall's theory of Three Worlds (Tall, 2004, 2008, 2010) as a way of understanding students' learning of mathematics.

Resources (knowledge, experience)

1. Knowledge of syllabus to be covered.
2. Knowledge of time constraints, both in the classroom (for covering the syllabus) and outside the classroom (for preparing teaching resources).
3. Freedom to adopt her own approach to the subject, bringing her research to the lectures.
4. Text-book set down for the course.
5. Other texts covering the same material in different ways.
6. Blackboard (especially for pictures).
7. PowerPoint slides
8. Data projector and document camera.
9. D2L online system.
10. Experience that students struggle with definitions and proofs in Tall's formal world.
11. Experience that students feel comfortable carrying out routine calculations in Tall's symbolic world.
12. Experience that students sometimes struggle to interpret diagrams or to connect them to the symbolic world.

Orientations (beliefs)

1. Belief that students learn by sampling Tall's three worlds and building their concept images as they go.
2. Belief that students should progress from the embodied world on to the symbolic world and finally to the formal world. (Modified during our discussions.)
3. Belief that, for full understanding, students need to grasp a concept in all three worlds.
4. Belief that although the text-book has some nice examples, it adheres too closely to a definition-theorem-proof style of exposition.
5. Belief that her exposition should proceed from easy material to harder material.
6. Belief that students should be interested in and engaged with a topic as soon as possible in order to promote understanding.
7. Belief that students in service courses will not be interested in definitions and proofs.
8. Belief that traditionally mathematicians use a written form of presentation, with students copying.
9. Belief that as many resources as possible should be made available online.
10. Belief that providing good, clear notes on PowerPoint slides has value for student engagement and understanding.
11. Belief that students should see the usefulness of mathematics.

Figure 1: The lecturer's resources and orientations related to the areas of conflict.

This theory is based on three mental worlds of mathematics: embodied, symbolic, and formal. The embodied world is where we think about the physical world, using "…not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuo-spatial imagery" (Tall, 2004, p. 30). The symbolic world is where actions, processes and their corresponding objects are realized and symbolized. The formal world comprises defined objects, presented in terms of their properties, with new properties deduced by formal proof. These worlds describe qualitatively different ways of thinking that individuals develop as new conceptions are compressed into more thinkable concepts (Tall, 2008). All three worlds are available to, and used by, individuals as they engage with mathematical thinking, and they interact so that "three interrelated sequences of development blend together to build a full range of thinking" (Tall, 2008, p. 3). Tall observes that "Although embodiment starts earlier than operational symbolism, and formalism occurs much later still, when all three possibilities are available at university level, the framework says nothing about the sequence in which teaching should occur" (Tall, 2010, p. 22). For example, Tall claims that many students learning mathematical analysis would be happy to think and operate entirely in the formal world, whereas others may prefer to think in terms of thought experiments and concept imagery. Hence, no single approach is privileged over another and decisions can be based on the objective of each course so we do not "inflict formal subtleties on students who are better served by a meaningful blend of embodiment and symbolism" (Tall, 2010, p. 25].

Despite this last remark, Sepideh prefers to follow the 'natural' sequence of development: first embodied, then symbolic, and finally formal. Hence her desire to start every topic with a picture: "I wanted to start from the embodied world (whenever possible)," even if experience is forcing her to modify her views:

> The more I teach and get experienced … the more I think about the same question of the progression of the worlds and the sequence in which they should be taught. I know I wouldn't start from the formal world (especially with abstract notations) as I don't achieve anything there [R10, O6, O7]. This leaves me with the embodied and symbolic worlds. I believe in the flow of easy to hard [O5], but I don't say the embodied is necessarily easier for students. I don't see that many overjoyed faces when I draw pictures for linear combinations of two vectors [R6], but I do see more pleasing responses when they see two vectors are scalar multiplied and added (using numbers and vectors) [R11]. I know some of my [pictures] are a bit confusing.

Hence also her desire to aim for the formal world: "I wanted them not to be afraid of [or] avoid the theorems and definitions that were in the book and see their usefulness," even though she knows (R10) that some students struggle in the formal world: "One student] said he failed linear algebra last semester because the instructor only did proofs and [theorems] with hardly any examples that students could do."

Trying to meet these goals simultaneously was fraught with difficulty:

> I went to the library and found the book by Lay and have been using it. The textbook for this course by Kalman and Hill is very proof orientated – I am not stressing the proofs as such. I also got the book by Poole which I am going to use teaching vectors (it has many [pictures] and I am planning to use them in my teaching).

Thus, initially, driven by the need to cover the syllabus (G1), she followed the text-book (R4) but found (O4) the book too close to the formal world as a starting point

(R10, O2). This drove her to seek other texts (R5). But deviating from the text-book comes at a cost:

> I spent a lot of time [R2] thinking about how to introduce my material and make everything interesting to capture my audience (the math majors and engineers) [G5, see below]. … I was constantly trying to make decisions about how to introduce the lesson [O6], which [pictures] to use [G2], which definitions would be useful [G3] and at the same time containing all the major points, which examples to use to back up my arguments, which theories and proofs to include.

Conflict 2: Presentation style

The second area of conflict that we present concerns decisions around the presentation style that Sepideh would use in lectures. It arose from two pairs of goals, goals 1 (see above) and 4, and goals 5 and 6.

Goal 4: To be the best possible teacher of mathematics.

Goal 1 was introduced in Conflict 1, and Goal 2 is a natural consequence of being a new lecturer in a mathematics department. She sees herself as a mathematician, stating:

> I am a mathematician—I don't need real life examples to make me interested in mathematics or make me interested in teaching mathematics. I love maths for the sake of maths—but my students are not like that, so I am prepared to change to meet their needs.

These two goals came into pedagogical conflict with the following two goals.

Goal 5: To make the presentation interesting and engaging for students to capture her audience.

Goal 6: To provide on-line for students as many of the notes and other resources as possible.

The conflict is confirmed by her response to a question we put to her "Going against my own beliefs [O6, O9] was also a conflict for me. I know writing on the board [R6] is not going to help but I still did it [O8]. By that I mean writing too many proofs and theorems and definitions [R10, O7] to the point of drowning everyone". We see through a series of four decision points in the lectures how the conflict is managed and the presentation style is gradually refined in an attempt to resolve the conflict. The first decision lasted through the opening lectures of the course. After a few weeks she notes the presentation conflict, or struggle, in these words:

> I feel like I answer all their questions quickly and write a lot on the board [R6], like a good mathematician does [O8, G4], and of course they copy. I tried making slides [R7], but it takes so long—I tried to summarise things and then put it on slides that seems to be more effective [G5]. Still struggle to present in the best way possible [Conflict 2].

We see the tension between the time it takes (G1) to do the examples live (Sepideh copied the examples from the textbook and wrote them on the board live. Later she typed them into her lecture slides and thus made them available online), like a mathematician (O8, G4), and using pre-prepared PowerPoint slides (R7). The initial compromise, likely fuelled by Goal 6, is to make summary slides, but to continue with hand written examples on the board (R6) as a prominent part of the lectures. Within another week or so the strong Goals 5 and 6 seem to be having more of an influence on the presentation style. She comments in her diary on her presentation style and its effect on students:

> I did some geometrical and algebraic examples on the board [G2]. Then showed more geometrical representations of subspaces on slides [O2, R7]. This time I made slides and put all the definitions, theories and some [pictures] on the PowerPoint slides [O11]. I think students got something out of my efforts as they started asking questions [O6]. I think that is a good sign, meaning that they can understand enough to see what is going on and ask questions [G5].

Hence, at this point, although there are still worked examples done by hand on the board (G4), there has been a move towards putting examples on the slides, along with 'definitions, theories and pictures'. Finding the balance is difficult, since as she notes in response to a question "I know having too many slides is not going to be effective [O6] but I still fell into the trap and did it." The following comment on student understanding indicates the growing presence of the goal to build understanding through engaging students in lectures (G5). By week 8 of the course we begin to see recognition of the influence of resource provision (G6) on her practice.

> I did two examples [R6, G4] one to find a basis for the row space where I reduced the matrix and read off a basis from the reduced matrix. I did another example and reduced the matrix but read a basis corresponding to the leading 1's columns from the original matrix. This worked much better…I found a nice summary from Poole in finding null space, row space and column space which I wrote on the board [R6, G4] and later posted them with my slides up on the server [R7, O9, O10, G6].

Some 9 weeks into the course we see a small move away from writing on the board in favour of a different resource. The diary entry makes it clear that the motivation here is to engage the students mentally (O6, G5) rather than have them copying all the time. In addition it facilitates the goal of putting the material online (O9, G6).

> I felt that the book didn't have many good examples about orthonormal vectors, so at the beginning of the class I put 3-4 examples on orthogonality, orthonormal vectors and showing inner products (using the 4 criteria) on the document camera [R8] and went through them. I told them not to copy and just listen [O6, G5] because I was going to scan those and make them available online [O9, G6]. We have a system called D2L (Desire to Learn) [R9] to post all of our resources there)…I try to make my classes as interactive as possible–I don't like it when I am writing and they just copy [O6, G5].

Overall the goal to write examples remains throughout but it has been moderated somewhat by the very strong goal to interest and engage the students [O6, G5]; to have them listen and think and not just be copying. Since she wants to provide copies of the examples to students, but doesn't want them to copy and it is not possible to provide them with electronic versions of notes written on the board (R6), Sepideh has found a way around this by using the document camera resource (R8).

DISCUSSION

Teaching at any level is a very complex practice and this study provides some further support for the view that Schoenfeld's (2010) framework is a useful vehicle for describing and analysing issues, such as conflicting goals, related to the complexity of university teaching. In this case study we have highlighted the trials of a young researcher, eager to bring her research into the classroom and as we described, at times this was not a straight forward task and was causing conflicts. We also have examined the difficulties of decisions in balancing competing goals. Furthermore, the lecturer facing this complexity was provided with no resources other than a textbook. As she says "That's ok if you only want to use the chalk and talk—but it is no way going to work if you have so many ideas for making your lectures better." Hence her efforts in producing new resources that were fresh and interesting, based on a new book, a new audience and her beliefs in what teaching should be, were very time consuming. She valued the freedom to choose that she was given and did not necessarily want a collection of prepared slides that might not have suited her beliefs and teaching style.

So what would have helped her to deal with difficult decisions about the order of presentation of the material and to encourage her natural teaching style? Our study suggests two ways forward. Firstly, it is clear that an increase in preparation time for new lecturers would have assisted. Allied to this would be systems by which resource ideas could be shared with new colleagues, reducing the preparation time pressures. Secondly, it appears that as part of her professional development the new lecturer should be provided with the opportunity to discuss issues with colleagues. One way to provide this would be in the form of a department mentor, an experienced colleague who provides the opportunity for a reflective discussion of teaching issues. Mathematicians have a reputation for not being good communicators. As Byers (2007, p. 7) declares: "Many mathematicians usually don't talk about mathematics because talking is not their thing-their thing is "doing" of mathematics". However, as Sepideh comments, while "Talking and confronting things is not easy, you discover things about yourself that you didn't know before" and so she described this project, and its formation of community of co-learners, as a great way of getting someone to talk about your teaching.

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MATHEMATICS TEACHERS' BELIEFS AND SCHOOLS' MICRO-CULTURE AS PREDICTORS OF CONSTRUCTIVIST PRACTICES IN ESTONIA, LATVIA AND FINLAND

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The article presents results from a cross-cultural comparison of mathematics teachers' beliefs. We report the differences in Estonian, Latvian and Finnish mathematics teachers' self-reported constructivist teaching practices. Additionally, we explore how much of these differences can be explained by teachers' beliefs and their schools' micro-cultures. Results indicate that teachers' beliefs and cultural context explain 15 % of the variation in constructivist teaching practice of mathematics teachers while the school micro-culture does not provide a significant additional effect.

TEACHERS' BELIEFS AND PRACTICES IN CONTEXT

Extensive research on teachers' beliefs has been inspired by assumption about the significant influence of teachers' beliefs on classroom practice. Lately such simplified view has been challenged by studies indicating complex and dialectical nature of relationships between teacher's beliefs and practice (Handal & Herrington, 2003) instigating the studies that put more emphasis on contextual factors. The presented study is based on the theoretical framework reflecting the role of culture, school micro-culture, and teacher beliefs in the formation of actual classroom practices. Our results of a cross-cultural survey of mathematics teachers' beliefs in Estonia, Latvia, and Finland show how beliefs and context are influencing teaching practices in these countries.

The latest research shows that only a contexture of manifold individual and contextual factors can help to elucidate the observed differences in teaching practices. Quite often, though, the educational research focuses on relationships between single internal or external factor and effective teaching practice, more seldom the larger number of variables are explored. This study is based on the theoretical framework (see Figure 1) including both individual factors of teachers, i.e. their teaching beliefs, and contextual factors on two different levels considering the interactive nature of internal and external circumstances.

Individual factor: teachers' beliefs

Based on the studies showing that beliefs shape teachers' decisions (Schoenfeld, 1998), it is evident that contextual factors impacting learning achievements, are mediated by the teachers. Teachers' beliefs about mathematics and its teaching and learning play a significant role in shaping teachers' instructional behaviour (Thompson, 1992). In this study beliefs are understood broadly as conceptions, views and personal ideologies that shape teaching practice. Currently educational community widely assumes that teachers' beliefs about the nature of teaching and learning include

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both "direct transmission beliefs about learning and instruction" or, so called, "traditional beliefs" and "constructivist beliefs about learning and instruction" (OECD, 2009).

Belief research in mathematics education has focussed on how teachers view the nature of mathematics, its learning and teaching, and teaching in general. What the teacher considers essential in mathematics as a discipline, would also influence the teaching methods they apply (Dionne, 1984; Ernest, 1991).

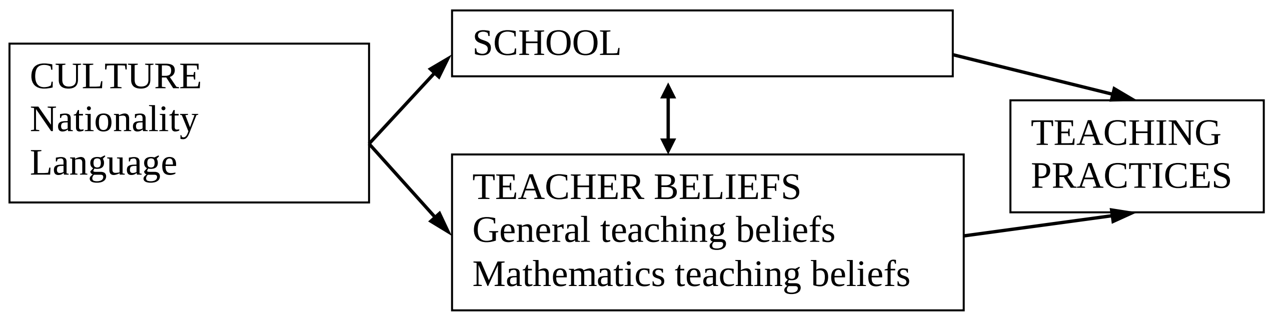

Figure 1. The theoretical framework for the study.

The results of our previous study (Hannula, Lepik, Pipere, & Tuohilampi, 2013) show that the mathematics teachers' general teaching beliefs were related to their beliefs on mathematics teaching. Teachers who believed more strongly in constructivist ideas also supported the process aspect of mathematics more, while those who held a more traditional view of teaching emphasized the toolbox aspect of mathematics. Yet, it is important to notice that there was no negative correlation either between constructivism and toolbox aspect or between traditional view and process aspect.

Contextual factors: culture and school micro-culture

Teaching does not happen in a social vacuum – at least two principal levels of contextual factors should be acknowledged in educational research by analogy with the interaction of microsystem, exosystem, macrosystem, and chronosystem described in Ecological Systems Theory (Bronfenbrenner, 1994). More comprehensive level of contextual factors is the overall cultural milieu manifesting in both official and unofficial aspects. Well-discerned official aspects include, for instance, the economical and social situation in the country, and national educational policy. The unofficial aspects of the culture do not always follow the national borders. They impact schooling through the values of education and the teacher-student relationships.

The teachers' actions are also constrained at the more specific level of contextual factors. The local micro-culture, i.e. shared vision, values, goals, beliefs, and faith in school organization (Deal & Peterson, 2009; Fullan, 2005) is reflected in the school rules and norms and in the way teachers collaborate. On the other hand, the teacher is an important actor of this micro-culture and may influence its development over time. The importance of school micro-culture has been found repeatedly in intervention studies. For example, in an evaluation of one large professional development program within mathematics education (Bobis, Clarke, Clarke et al., 2005), the aspects that were considered most effective were the practical resources and activities, the assessment process, the influence of significant people, classroom support, and the opportunity to share ideas.

Associations between the individual background, the school context, teachers' beliefs and practices and the learning environment are consistently found in a large number of countries (OECD, 2009). However, in different countries the content, scope and interaction of these variables considerably differ. So far, a few studies have compared teachers' beliefs across countries (e.g., Andrews, 2007; Andrews & Hatch, 2000; Felbrich, Kaiser, & Schmotz, 2012). Cultural differences determined, for instance, by religion or language within country may also relate to teachers' beliefs and practices. As the review on variables of mathematics education in high-performing countries has shown (Askew, Hodgen, Hossain, & Bretscher, 2010), high achievement of students could not be so much connected with specific mathematic teaching practice as to the cultural values. When cultural minorities are educated in their own schools, the different cultural values might also contribute to differences in approaches to teaching mathematics.

Focus on Finland, Estonia and Latvia: similarities and differences

The countries participating in presented study are geographical neighbours, though with different historical, economical, and social background. Finland, Estonia, and Latvia have similar school systems in several aspects. Pupils start school at the age of six or seven years, and compulsory school lasts nine years in each country. In compulsory school, pupils most often study in mixed-ability groups as there is no tracking. In Estonia and Latvia mathematical rigor, exact use of language, deductive approaches and reasoning were stressed under the Soviet system until the 1991. Later, the constructivist teaching approaches have been actively promoted, although not always embodied similarly strongly on all school stages (Pipere, 2005) or not always fully dismissing the transmission way of teaching mathematics (Lepik, 2005). In Finland student-centred approaches have been dominating and the national policy have emphasised mathematics and sciences since the 1990th. In all three countries teachers are trained at the university level. Currently about 25% of students in Latvia and 19% of students in Estonia attend schools with Russian language of instruction.

Our previous study (Hannula, Lepik, Pipere & Tuohilampi, 2013) indicates that Latvian mathematics teachers emphasize the constructivist teaching beliefs most, while Estonians are the strongest supporters for the traditional beliefs. In general, Finish teachers agree the least with both of these approaches. As to the differences within Estonia and Latvia according to the language of teaching, Russian speaking teachers put more emphasis on proofs. The school micro-culture – as reflected in teachers' perception of collaboration and recognition at their school – seemed to have a clear connection to constructivist practices in both Latvian subsamples and in Estonian speaking sample from Estonia but not in the other groups.

RESEARCH QUESTIONS

Based on the review of the literature, we consider our model (Figure 1) to capture the main factors explaining the differences in approaches to teaching practices. Drawing from the previous analysis, the cultural factors can be viewed as bearing influence not only on the local school micro-culture and teacher beliefs apart but also on how school micro-culture and teacher beliefs relate to each other (Hannula et al., 2013).

Therefore, in this paper we will explore the following research questions: 1) What kind of differences in mathematics teachers' support for constructivist approach in their teaching practice can be identified between the teachers from following cultural groups – Finnish, Estonian speaking Estonian, Russian speaking Estonian, Latvian, and Russian speaking Latvian, and 2) how much of the cross-cultural variation in teaching practice can be attributed to local micro-cultures and individual teachers' beliefs?

METHODS

Participants

In total the data was collected from 815 7-9th grade mathematics teachers in Estonia (n=333), Latvia (n=390), and Finland (n=92). A subsample of teachers from Russian speaking schools was collected in Estonia (n=99) and Latvia (n=96). The Estonian teachers' length of service of ranged from 1 to 59 years (M=22), the Latvian teachers' length of service of ranged from 1 to 44 years (M=23), and the Finnish teachers' length of service of ranged from 1 to 35 years (M=14). The data collection has been completed between 2010 and 2012.

Instrument

A seven-module questionnaire was devised to explore aspects of mathematics teachers' beliefs on mathematics teaching and their classroom practice. In this paper, we focus on four modules about teachers' (1) overall job satisfaction; (2) general beliefs on teaching and learning; (3) beliefs on mathematics teaching and learning and (4) self-reported teaching practices. Teachers responded to items in modules (1) to (3) using a 5-point Likert-scale from strongly agree to strongly disagree and to items in module (5) using a 4-point Likert-scale (from never to (almost) every lesson).

Based on the factor analyses of three modules we computed following sum variables: School micro-culture: Collaboration and recognition ($\alpha = .696$; 5 items); General teaching beliefs: Constructivist approach ($\alpha = .726$; 12 items), Traditional approach ($\alpha = .575$; 4 items); Mathematics teaching beliefs: Process ($\alpha = .731$; 9 items), Toolbox ($\alpha = .676$; 6 items), Proofs ($\alpha = .592$; 4 items). The sum variable for teaching practice was represented by Constructivist practices ($\alpha = .623$; 5 items).

Theoretical background, development and structure of the questionnaire as well as the sample items for first three modules are described more thoroughly in the previous papers (Lepik & Pipere, 2011; Hannula et al., 2013).

The last module measuring self-reported teaching practices consisted of eight items on how often teachers ask their students to engage in certain classroom practices, e.g.

"Decide on their own procedures for solving complex problems". Based on a factor analysis we identified three dimensions: Facts and routines, Constructivist practices and Use of computers. Because only the Constructivist practices formed a reasonably reliable scale, we will focus our analysis on this dimension.

Analysis

A one-way between subjects ANOVA was conducted to compare the effect of culture on teachers' self-reported use of constructivist teaching methods. The five different cultural contexts of our study were determined by the country, but in Estonia and Latvia also by the language of instruction at school.

We compared four different models for analysing the effect of different variables on constructivist teaching practices. The GLM univariate analysis used in this study allowed for analysis of the effects of both continuous and categorical variables. The first model included only the teacher beliefs (Constructivist approach, Traditional approach, Process, Toolbox, and Proofs) as independent variables. The following models included stepwise also the school micro-culture (Collaboration and recognition), the country (Finland, Estonia, Latvia), and the language (i.e. the Russian-speaking minorities). Through this analysis we can assess how much additional information we gain including the cultural context in the model.

RESULTS

The analysis of variance indicated a statistically significant effect of context on teachers' use of constructivist teaching for the five cultural contexts [$F(4, 801) = 6.211$, $p = 0.000$]. More specifically, Estonian speaking Estonian teachers and Finnish teachers indicate less frequent use of these methods than the other three groups (Table 1). The post hoc analysis (Tamhane) indicated that Estonian speaking Estonian group differed statistically significantly from Russian speaking Estonian teachers ($p=.009$) and both Latvian groups (Russian-speaking: $p = .025$; Latvian-speaking: $p = .004$).

Cultural context	N	Mean	Std. Deviation
Latvian	294	2.40	.369
Latvian Russian	95	2.43	.391
Estonian	232	2.28	.387
Estonian Russian	91	2.46	.440
Finnish	94	2.27	.495
Total	806	2.36	.406

Table 1. The mean and standard deviation for teachers' reported use of constructivist methods in the five cultural contexts studied.

The next level of analysis was to test the hypothesis that the cultural variation is mediated through the local micro-culture of the school and teachers' beliefs. The first model included only teachers general teaching beliefs (constructivist or traditional) and their view of mathematics (Process, Toolbox, Proofs) as independent variables. The model gives a statistically very significant prediction explaining over ten percent of the variation (partial eta squared = 0.129). The more detailed models increased the effect size only a little (Table 2).

	Model I (Teacher beliefs)		Model II (Model I + micro-culture)		Model III (Model II + country)		Model IV (Model III + language)	
Source	Sig.	Partial η^2	Sig.	Partial η^2	Sig.	Partial η^2	Sig.	Partial η^2
Corrected Model	.000	.129	.000	.133	.000	.143	.000	.150
Intercept	.000	.047	.000	.037	.000	.035	.000	.039
General teaching beliefs:								
Constructivist	.000	.022	.016	.018	.001	.016	.001	.017
Traditional	.049	.005	.063	.005	.072	.005	.082	.004
Mathematics teaching beliefs:								
Process	.001	.016	.000	.018	.000	.019	.000	.018
Toolbox	.655	.000	.579	.000	.394	.001	.377	.001
Proof	.024	.007	.037	.006	.024	.007	.079	.004
Collaboration and recognition			.162	.003	.216	.002	.229	.002
Country					.014	.012	.173	.005
Language							.088	.004
Country x Language							.072	.005

Table 2. Statistical significances and effect sizes of dependent variables on teachers' preference for constructivist teaching practices. Results of four alternative GLM Univariate models.

The two most important variables to predict use of constructivist methods were the teachers overall constructivist beliefs and their perception of mathematics as a process, which together predicted 3.5 percent of the variation in the final model. Independent from those also emphasis on proofs and overall traditional teaching beliefs had an influence in the teaching practices. However, these effects lost their statistical significances when country and language were added into the model.

Our preliminary analysis indicated a statistically significant correlation between the local micro-culture and teaching practice. The GLM univariate analysis reveals that the effect was fully mediated by teachers' beliefs.

Country had an effect on teaching practices ($p<.05$; partial eta squared = .012) that was independent from the teachers' beliefs. When the language was added into the model, the statistical significance was lost. These results indicate that the observed effect of the country is intertwined with the language of education.

DISCUSSION AND CONCLUSIONS

Our data indicated that the cultural context (country and language) of education influences mathematics teachers' preference for constructivist teaching methods. We found that teachers' beliefs were an important predictor for teaching preferences, while the local micro-culture had no effect independent from teachers' beliefs. Previous research has provided strong evidence for the importance of local micro-culture in teacher professional development. Our results are not in conflict with those results, but rather indicating that the influence of local micro-culture is realized through teachers' beliefs.

We also found that the country had an influence that could not be attributed to teacher beliefs only. This suggests that some contextual influences may influence practice while not influencing teachers' beliefs. An interesting observation was that the influence of some teacher beliefs was partially overlapping with cultural factors. Before including language in the model, teachers' emphasis on proof was statistically significant factor influencing teaching practices. When the language was included into the model, this effect was decreased. Such finding indicates that some observed influences of teacher beliefs on teaching practices may, in fact, be cultural influences.

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