What do mathematicians wish to teach teachers in secondary school about mathematics?

(article begins on next page)

PME 42

July 3-8, 2018 --- Umeå, Sweden

Proceedings

Of the 42nd Conference of the International Group for the Psychology of Mathematics Education

Editors: Ewa Bergqvist, Magnus Österholm, Carina Granberg, and Lovisa Sumpter

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July 3-8, 2018 --- Umeå, Sweden
www.pme42.se

Cite as:

Bergqvist, E., Österholm, M., Granberg, C., & Sumpter, L. (Eds.). (2018). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3). Umeå, Sweden: PME.

Website: http://www.pme42.se

The proceedings are also available via http://www.igpme.org

ISSN: 0771-100X
ISBN (volume 3): 978-91-7601-904-7

Logo Design: Catarina Rudälv and Amanda Rudälv
Printed by CityPrint i Norr AB, Umeå

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FRAMING THE SOCIAL DYNAMICS OF SMALL GROUP WORK IN ADOLESCENCE AS PEER CULTURES OF EFFORT AND ACHIEVEMENT

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This study applies the concept of peer cultures, which involve the values and concerns, habits and routines, and roles that students develop through sustained interaction with one another, to characterize the social dynamics of mathematics small group work. Each dimension was coded in time sample intervals in 30 small group audio- recordings from 27 American 6th-9th grade classrooms. The major dimensions of peer cultures could be reliably coded in mathematics small groups, and variations in frequency and quality of each dimension were evident. Coding of 23 more groups will occur; analyses will continue to document the frequency and quality of these dimensions, as well as co-occurrences of the dimensions within small groups. Results inform understanding of and supports for productive small groups for adolescents.

SMALL GROUPS AS PEER CULTURES

Small group work is intended to create peer-to-peer interactions in which students use one another as resources for learning. It is a popular and prevalent instructional format in middle and secondary mathematics classrooms in the U.S. & internationally (Fulkerson, 2013; U.S. Department of Education, National Center for Education Statistics, 2003). In small group work, students must negotiate social as well as structural and cognitive demands of a task (Barron, 2003), but for adolescent students, the social dynamics can fully undermine productive small group experiences (i.e., McFarland, 2001). The field lacks a unifying framework that captures common and influential social dynamics; such a framework would extend theorizing about small group learning as well as guide educators' support of productive small group work.

We apply Corsaro and Eder's (1990) concept of a *peer culture* to characterize key dimensions and processes of the social dynamics of small groups. Teachers set the group membership, task, and expectations for mathematical work and social interaction, but students appropriate the work through interpretive reproduction, taking what the teacher provides (i.e., task demands and expectations for working together) and aligning it with their own needs and interests (Corsaro & Eder, 1990). A peer culture, or the "stable set of activities or routines, artifacts, values, and concerns that children produce and share in interaction with peers" (p. 197) emerges, governed by *values and concerns, routines and habits*; and *roles* of its student members.

Studies of American, Australian, and Dutch youth show that in general, adolescents' *values and concerns* favor classroom disruption and disregard for effort over academic

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 3-9). Umeå, Sweden: PME.

engagement (Galvan, Juvonen & Spatzier, 2011; deBruyn & Cillessen, 2006), which may undermine group functioning (McFarland, 2001). However, small group experiences may lead students to renegotiate values to favor cooperation and respect of classmates (Gilles & Boyle, 2005). Work practices and helping behaviors are prominent *habits and routines* that occur within small groups. Small groupwork in Canada featured individualistic and collaborative work habits (Esmonde, Brodie, Dookie, & Takeuchi, 2009) as well as socializing and resistant work habits in American classrooms (McFarland, 2001). Peer helping can be adaptive (i.e., expansive, informative, and explanatory), expedient (involving the correct answer without explanation), or avoidant (ignored, neglected) (Ryan & Shim, 2012). *Roles* are the status positions assumed as students negotiate the academic demands and expectations of the task as presented by the teacher within the peer culture. Studies of Canadian and American teens suggest that roles include social loafers, turn-sharks, facilitators, experts, and socially dominant students (Barron, 2003; Esmonde et al., 2009; Linnenbrink-Garcia et al., 2011). When focused on effort and achievement, these dimensions of peer cultures bear significant influence over adolescents' academic outcomes (Hamm, Hoffman, & Farmer, 2012).

In the proposed study, we describe and report an iterative process and preliminary findings for formally capturing the key dimensions of small group peer cultures in middle and secondary math classes. Our research questions were: 1) To what extent and in what ways can the concept and dimensions of a peer culture be applied to the social dynamics of mathematics small groupwork?, and 2) Are there meaningful differences in frequencies and qualities of key dimensions of peer cultures across small groups?

METHOD AND ANALYTIC PLAN

Eleven middle and high school teachers from one rural and low-resourced, and one municipal and well-resourced school district in the American Southeast participated. Teachers identified specific class periods for observation, in which they used small groupwork of their own planning. In total, 3 6th, 6 7th, 12 8th, and 6 9th grade classrooms serving African American, Asian American, Latino, and White students, as well as a small number of students whose families had recently immigrated to the U.S., participated. Across classrooms, 161 small groups were observed and audio recorded by two researchers; group size ranged from 2 - 5 students (56.5% mixed gender).

All observed lessons followed the teacher's lesson plan without intervention by the research team. Student groups worked on a variety tasks appropriate to the grade level and content focus and sequence of their courses. Tasks included, for example: (1) finding areas and perimeters of circular and rectangular parts of a stained glass window, (2) finding the volume and surface area of a cylinder and a tube, (3) finding missing angle measures in various kinds of triangles, (4) analyzing central tendency and spread of data distributions, (5) analyzing quantities in two-way frequency tables and Venn diagrams, (6) modelling situations with linear relationships, and (7) comparing different representations of linear relationships.

We randomly assigned the 161 small group recordings to one of three distinct coding/analytic phases; this proposal involves analysis of the first phase of 54 groups (coding/analysis of these groups will completed prior to presentation). The complete enactment of small group work in each classroom was analyzed. The relative and absolute time allocated to small group work in the observed classrooms varied according to teachers' lesson plans, ranging from about a third of the class period to nearly the entire length of the class period. Since the length of class periods varied considerably (most either 47 or 85 minutes) and available time for group work also varied (13 to 82 minutes), we present results in terms of percentage of minutes of available group work time in the class period.

We used time sampling (1-min intervals within the identified groupwork time) procedures to capture the presence and prevalence of each peer culture dimension during small groupwork. Based on theory and empirical studies, the first 3 authors created an initial code list and working definitions: *Values and concerns* (i.e., statements about effort and achievement, as well as affect); *habits and routines* (i.e., adaptive, expedient, and avoidant helping; collaborative, individualistic, and socializing work practices); and *roles* (e.g., socially dominant, entertainer, social loafer, facilitator). The team independently coded three audio-recordings in Atlas-TI v8.1; calculated inter-rater agreement; and refined codes and code definitions. The team independently applied these codes to another audio-recording and assessed inter-rater reliability. Following strong inter-rater agreement and additional discussion, the team coded group recordings independently, calculating reliability after every 9 independent codings. This approach generated both frequency and narrative data for analysis.

PRELIMINARY RESULTS

To date, the team has coded 30 of the 54 audio recordings. For Research Question 1, the applicability of the concept and dimensions of peer cultures to small groups in mathematics, our preliminary results reveal a) refinement of the *a priori* peer culture codes; b) reliable coding of the three primary dimensions, as well as some sub-dimensions of peer cultures.

For finding a) we will briefly present an overview of the refined codes to capture the three dimensions and sub-dimensions of peer cultures, with illustrations from group recordings.

Dimension	Code Definition	Group Example
Values/Concerns		
• Peers' ability (not smart)	Statements positioning self, groupmate, or group as not as capable or as smart as others.	Two is easier than one, that's why we gave it [problem] to you! (group laughs)
Habits and Routines		
• Expedient Help-Seeking/Providing	Requests for and provisions of answers or procedures for getting the right answer.	S1: Wait, are the slopes the same? S2: Yes.
• Collaborative Work Habit	Multiple students contribute to joint problem solving. Group members interactively attend to the task, each other, and strategies, and solutions. Speakers and listeners have consequential roles.	S1: Wait, so if you find the perimeter of each rectangle, can you just times it that way, and then like, … S2: I feel like we should find the perimeter of just one full circle and multiply it by 10 … S3: Perimeter is 8.14, since each diameter is 1… S2: No, that's area, that's radius, remember? S3: S that's still … In unison: … 8.14….
Roles		
• Facilitator	One member who helps group make progress on the task, i.e., by getting the group started, back on task, or by seeking input from others. Affirming statements alone are insufficient; statements must move the group forward.	[following intervals of socializing among all group members] OK, we gotta write something down! [and group re-engages]

Table 1: Excerpt from coding dictionary and coding illustrations from groups.

For finding b), early in the coding process, we have established reliable coding of the primary dimensions and some sub-dimensions. For *values and concerns* and *roles,* this phase has involved identifying the presence or absence of this dimension, prior to formally coding for distinctions within the dimension. Specifically, for *values and concerns,* coding has focused on detecting indicators of values and concerns about effort and achievement through group members' statements and affect, with a code assigned if a statement reflective of values and concerns was present. Thus far, coders agreed 84% of the time that a statement reflecting values and concerns was present in an interval. For *roles,* coding has focused on identification that a student within the group has assumed a role. Coders agreed 76% of the time that a student within a group assumed a role. Quotations associated with the *values and concerns,* and *roles* codes will be generated for further differentiation of each dimension in the next phase of coding/analysis. Finally, we can reliably differentiate within two sub-dimensions of *habits and routines*: work habits (collaborative, individualistic, and socializing, with 78%, 87%, and 78% inter-observer agreement, respectively), and helping practices (providing and seeking, with 89% and 88% inter-observer agreement, respectively).

With respect to Research Question 2, variations in the frequencies and qualities of each dimension, our preliminary results suggest that all dimensions are present in groups but range significantly in both quantity and quality of occurrence. Descriptive statistics of frequency counts revealed that statements that referred to group members' *values and concerns* about effort and achievement occurred infrequently, on average in 19.08% of intervals (*SD* = 18.72). Preliminary review of the quotations associated with the *values and concerns* code suggest that these statements tended to focus on group members' ability; the need to be correct; the desire to finish quickly; and about the ease of the task. With respect to *roles,* an individual student assumed a role in roughly one-fourth of small group work time (*M* = 25.82, *SD* = 15.19). Preliminary review of the quotations associated with this code suggested that students acted as facilitators, entertainers, or were socially dominant.

With respect to *habits and routines*, helping behaviors of any form occurred relatively infrequently, present in only 9% of intervals on average (*SD* = 11.01), and included both adaptive and expedient helping. Work habits (i.e., collaborative, individualistic, and socializing) were coded at over 95% of all intervals. Early in this first phase of coding, we attained a high level of inter-observer agreement for specific types of work habits, and thus can report actual percentages of their occurrence. The work habits of groups tended to be collaborative, constituting 56.81% of small group work time (*SD* = 28.69*).* In contrast, group work habits were characterized as individualistic in 22.69% of intervals (*SD* = 20.59) and socializing characterized groups' work habits in 20.04% of intervals (*SD* = 21.04). Thus, groups were engaging in the task together for the majority of small group time, although significant amounts of time within small groups did not involve collaboration, and were actually off-task.

Preliminary examination of the quotations associated with collaborative work habits in particular suggests important nuances to consider in subsequent phases of coding/analysis. Specifically, collaboration may reflect a co-regulated process, in which multiple students participate, and group members maintain joint attention to the task, each other, strategies, and solutions. But students may also collaborate in a competitive fashion, characterized by domination of problem solving by multiple members, and resistance to taking up, extending, or encouraging other ideas reflecting an effort to own rather than share the problem solving process. Collaboration may also be assymetrical, with group members deferring to one or two members, in a hierarchical manner. Finally, uncoordinated or disjointed collaboration may occur, in which multiple group members are engaged with the task but in parallel, without tracking, listening to, taking up or extending peers' ideas. We have begun formal analyses of the quotations associated with the collaboration code that will enable us to determine formal codes to apply in the next phase of coding and analysis.

CONCLUSIONS

Systematic study and characterization of the social dynamics of small group work lays a critical foundation for understanding how students take up the cognitive and discourse demands of small group work, and provides a basis for developing tools to help teachers create and support productive social dynamics in small groups. The results of the proposed study reveal how an established framework for understanding peer social dynamics can be applied to understand the highly variable and influential nature of small group social dynamics in adolescence.

The literature on small group work provided a sufficient background to map dimensions of peer cultures on to small group social dynamics, but distinctions within the values and concerns of groups, and the roles enacted by students proved difficult to code reliably in the first round of coding. Efforts instead focused on reliably capturing these broad dimensions, and identifying a diverse corpus of quotations from which to establish more nuanced codes in the second phase of the study. Dimensions of habits and routines (i.e., helping behaviors and work habits) proved to be readily and reliably codeable, in part likely reflecting the well-established literature on helping, and on collaboration more generally. Differentiated coding of the broad dimension of habits and routines in the first phase will support more differentiated characterization of this aspect of peer cultures.

Very preliminary analyses of the descriptive statistics for each code revealed considerable variability in the prevalence of each dimension of peer culture. Qualitative analysis of the students' dialogue reveals additional variability and richness in how these dimensions are realized as students engage with a mathematics task. Future analyses will focus on the co-occurrences of these dimensions, in an effort to profile peer cultures that vary in their productive orientation toward successful small group work.

References

Barron, B. (2003). When smart groups fail. *The Journal of the Learning Sciences, 12,* 307–359.

Corsaro, W. A., & Eder, D. (1990). Children's peer cultures. *Annual Review of Sociology, 16,* 197–220.

deBruyn, E., & Cillessen, A. H. (2006). Popularity in early adolescence: Prosocial and anti-social subtypes. *Journal of Research on Adolescence, 21,* 1–21.

Esmonde, I., Brodie, K., Dookie, L., & Takeuchi, M. (2009). Social identities and opportunities to learn: Student perspectives on group work in an urban mathematics classroom. *Journal of Urban Mathematics Education, 2,* 18–45.

Fulkerson, W. O. (2013). 2012 National Survey of Science and Mathematics Education: Status of middle school mathematics. Chapel Hill, NC: Horizon Research, Inc.

Galvan, A., Spatzier, A., & Juvonen, J. (2011). Perceived norms and social values to capture school culture in elementary and middle school. *Journal of Applied Developmental Psychology, 32,* 436– 353.

Gilles, R.M., & Boyle, M. (2005). Teachers' scaffolding behaviors during cooperative learning. *Asia-Pacific Journal of Teacher Education, 33*, 243-259.

Hamm, J. V., Hoffman, A., & Farmer, T. W. (2012). Peer cultures of academic success in adolescence: Why they matter and what teachers can do to promote them. In A. Ryan & G. Ladd (Eds.), *Peer relationships and adjustment at school* (pp. 219–250). New York, NY: Information Age.

Linnenbrink-Garcia, L., Rogat, T.K., & Koskey, K.L.K. (2011). Affect and engagement during small group instruction. *Contemporary Educational Psychology, 36*, 13–24.

McFarland, D.A. (2001). Student resistance: How the formal and informal organizations of classrooms facilitate everyday forms of student defiance. *Sociology of Education, 107* (3), 612–678.

Ryan, A.M. & Shim, S.S. (2012). Changes in help seeking from peers during early adolescence: Associations with changes in achievement and perceptions of teachers. *Journal of Educational Psychology, 104,* 1124–1132.

U.S. Department of Education, National Center for Education Statistics (2003). *Teaching Mathematics in Seven Countries: Results From the TIMSS 1999 Video Study* (NCES 2003–013 Revised), by James Hiebert, Ronald Gallimore, Helen Garnier, Karen Bogard Givvin, Hilary Hollingsworth, Jennifer Jacobs, Angel Miu-Ying Chui, Diana Wearne, Margaret Smith, Nicole Kersting, Alfred Manaster, Ellen Tseng, Wallace Etterbeek, Carl Manaster, Patrick Gonzales, and James Stigler. Washington, DC.

WHICH IS SMALLER…? PARTIAL UNDERSTANDINGS AND MISCONCEPTIONS ABOUT MULTIPLICATION AND DIVISION BY FRACTIONS

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This study addresses difficulties students in grades 6 and 8 have in extending the meaning of multiplication and division from whole numbers to fractions. A research questionnaire and student interviews revealed various partial understandings of multiplication and division by fractions. Using matched pairs of modelling tasks, we compared how students interpret and apply different models of multiplication and division in tasks involving fractions. This enabled us to evaluate the sophistication of their conceptions and uncover their misconceptions. In particular, we uncovered a misconception that seems unique to rational numbers expressed as fractions: students conflated multiplication and division when modeling "part of".

INTRODUCTION

When Michael, in grade 6, wrote the symbol ">" to complete the mathematical expression (MEX) $72\times\frac{3}{4}\ \square\ 72:\frac{3}{4}$, we were convinced that he was holding the misconception *multiplication makes bigger and division makes smaller*, but our theory was not supported by Michael's explanation: "In both the multiplication and division exercises, the result will be less than 72, but the division will decrease [it] more". This explanation, which is partially correct, seems to involve conceptions of order. It led us to wonder how Michael's sense of how multiplication makes smaller differs from his sense of how division makes smaller.

BACKGROUND LITERATURE AND RESEARCH FRAMEWORK

Research on the extension of mathematical operations involves both semantic (Buzaglo, 2002) and psychological dimensions. In discussing students' partial understandings of multiplication and division, we begin with the known gap between understanding of multiplication and division of integers and understanding of these concepts in the context of rational numbers. This gap reflects the difficulty that students have in extending the meaning of these operations. Initial studies addressed extensions to decimals. In choice-of-operation tasks for multiplication word problems, students correctly solved problems when the multiplier was greater than one (in particular, an integer) more frequently than those with a multiplier less than one. The influence of the magnitude of the multiplier is referred to as the *multiplier effect*. Interviews with students revealed that they did not see a connection between problems

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 11-18). Umeå, Sweden: PME.

even when the interviewer tried to draw their attention to the similarity (Bell, Swan, & Taylor, 1981).

The phenomenon of choosing different operations for similar word problems was labeled by Greer (1987) as *nonconservation of operations* in the sense of Piaget. Division was the incorrect operation chosen most frequently to model multiplication word problems with decimal multiplier less than one. The reason given for this choice was: "the result must be smaller (than the multiplicand)". Similarly, Graeber & Tirosh (1989) found that preservice teachers incorrectly chose multiplication for division word problems with decimal divisor less than one "because the result must be greater (than the dividend)". In other words, the misconception that *multiplication makes bigger* and *division makes smaller* influences student choice of operation in word problems.

The theoretical framework of *conceptual change* has been used to explore students' difficulties in assimilating new scientific and mathematical concepts and to predict difficulties that might arise when new knowledge seems incompatible with what was learned previously. According to this approach,

> … understanding of scientific and mathematical notions that are not compatible with what the individual already knows is not an "all or nothing" situation; rather, there are intermediate states of understanding wherein elements of the prior knowledge are combined with elements of the incoming, incompatible, information to produce *synthetic conception* (Vamvakoussi, Vosniadou, & Van Dooren, 2013, p. 308).

Using this approach, Vamvakoussi and Vosniadou (2004) identified intermediate levels of understanding the concept "*density of rational numbers"*. For example, some students correctly stated that there are infinite numbers between two numbers when the numbers were represented as decimals, yet did not make this claim with two fractions.

Prediger (2008a, 2008b) examined knowledge of multiplication of fractions combining the framework of conceptual change with theories from mathematics education research. She suggested that formal, algorithmic and intuitive *components* of this knowledge express *levels* of understanding and emphasized the utility of this view for determining the depth and causes of obstacles to understanding. She also found that mental models of multiplication that seamlessly expand from the natural numbers to fractions present fewer obstacles. For example, in an *acting-across* model of multiplication, one quantity is a rate which acts on the second quantity (such as viewing the cost of a tank of gas as the price per liter acting across the volume). This model extends naturally to situations involving fractions. In contrast, the *part-of* model which has no parallel in the natural numbers is more difficult to assimilate.

Following Prediger, our analysis integrated data from multiple tasks; expanding her work, our data included both multiplication and division tasks. Tasks were designed to uncover students' models of multiplication and division of fractions. In addition, tasks were purposefully constructed in pairs so that correctly answering only one in the pair would reveal nuances in the sophistication of their mental models and shed light on the nature of their partial understandings.

METHOD

Research goal

Our primary goal was to determine how sixth-grade students construe multiplication and division with fractions, focusing on misunderstandings specific to rational numbers expressed as fractions. We hypothesized that some misconceptions of multiplication and division of rational numbers are unique to their expressions as fractions and the contexts in which they appear; as such, one might conjecture that these misconceptions would not occur with decimal representations. We also evaluated eighth-grade students' understandings of operations with fractions and compared these results with the results for sixth grade students.

Participants

The population in this study was Israeli students in grades six and eight: 213 sixth-grade students, evaluated after they had studied multiplication and division of fractions, and 267 eighth-grade students. For each grade level students came from ten different mathematics classes — two classes at each of five different schools.

Research instruments

The research instruments included a questionnaire administered to students from both grades and in-depth interviews with the sixth-grade students after they had completed the questionnaire. The questionnaire contained 31 items assessing conceptual understanding of multiplication and division, rather than procedural knowledge. Some of the word problems in the questionnaire were similar to those that students encounter in the initial stages of extending multiplication and division from whole numbers to fractions. Here we present four pairs of items that each includes an integer and a fraction less than 1.

Items 1-2: Determining a MEX for a multiplication word problem (multiplier < 1).

Write the appropriate MEX for the problem. The numbers in the problem must appear in your MEX.

(1) One meter of fabric costs 30 shekels. What is the price of $\frac{3}{5}$ of a meter of fabric?

(2) Rina has 30 shekels. She bought a pencil case with $\frac{3}{5}$ of her money. How much did the pencil case cost?

Items 3-4: Writing a word problem for a given MEX.

(3) Write a word problem where the answer will be the solution for the MEX $40 \div \frac{2}{5}$.

(4) Below is the beginning of a word problem. Complete the problem in any way you like so that the solution will be the answer for the MEX $\frac{2}{3} \times 60$.

Question: Ronen has 60 shekels ... ______________________________.

Items 5-6: Estimating results.

(5) Circle: The answer to $63:3/7$ is a) **greater than** 63, b) **less than** 63, c) **equal to** 63.

(6) Circle: The answer to $96x5/8$ is a) **greater than** 96, b) **less than** 96, c) **equal to** 96.

Items 7-8: Determining a MEX for a division word problem (divisor < 1). (Same instructions as for items 1-2.)

(7) It takes ¾ of an hour to bake a cake. How many cakes can you bake in 24 hours if you bake one immediately after the other?

(8) 24 whole pizzas were brought to a party. Each child ate ¾ of a pizza. How many children ate pizza if all the pizza was eaten?

RESULTS

The findings below illustrate students' partial understandings of the operations, evidenced by answering only one of the two items in the pair correctly.

Items 1-2: Determining a MEX for a multiplication word problem (multiplier < 1). Item 1 is an *acting-across* problem and item 2 is a *part-of* problem.

Our results were as follows: (a) 39% of sixth graders and 24% of eighth graders wrote different MEXs for the two problems; (b) 27% of sixth graders and 24% of eighth graders used multiplication for the first problem but not for the second, illustrating their difficulty seeing a *part-of* problem as a multiplication model. Of these students, 21% in grade 6 and 6% in grade 8 chose multiplication for problem 1 and division for problem 2; (c) Some used subtraction: with sixth graders 14% and 1% for problems 2 and 1, respectively; with eighth graders 23% and 5%, respectively.

The most common explanation for using division was "we need to find a part of 30". The interviews clarified their thinking: $30:3/5$ is intended to calculate "⅗ of 30"; they computed $30:3/5$ via a series of operations with whole numbers, namely $30:5x3$ (divide by 5, then multiply by 3). This method is commonly used in grades 4 and 5 to find part of a quantity. In contrast to arguments that deal with rules of size, e.g., "because division makes smaller", this argument gives meaning, albeit incorrect, to the entire expression.

In the following interview, Yael explains why she chose multiplication for problem 1 but division for problem 2:

> Because ⅗ is the length, and that times the price of the whole fabric, and that multiplies the other, and this gives a smaller number because this is a smaller fraction, and that simply shows you how much it costs. Here [item 1], 30 is how much a meter of fabric costs, it's not how much money she has and how much she bought……I think the only difference

[between items 1 and 2] is that here [item 2] it is ⅗ of her money, from her money – and here [item 1], we want to know how much the fabric costs [sic].

As with *acting-across* problems with integers, problem 1 gives the value per unit and asks to find the value of x units. Problem 2 is a *part-of* problem; it is distinct from multiplication word problem with integers. Yael ascribes to the number “30” different roles in the two problems: In problem 1, “30” is the **value** corresponding to 1 meter of fabric, while in problem 2, “30” is the **amount** that needs to be divided. Yael did not consider problem 1 to be a problem where one needs to calculate ⅗ **of** 30. Semantically, problem 1 is static, asking “how much it cost”, while problem 2 is dynamic, seeking “how much money she has” and then “how much she bought”. The process of finding a portion of a quantity, emphasized in problem 2, is active. Problem 1 encourages proportional reasoning through the use of two units--meters and shekels--while problem 2 encourages thinking about a direct operation between the two numbers given in the problem.

We will briefly discuss the choice of subtraction for problem 2. The most common reason for this error was that students saw this problem as similar to dynamic subtraction problems with integers and sought “how much money Rina had left". Although they wrote $30-3/5$, interviews revealed that they intended “30 minus ⅗ of 30”. Selecting subtraction to calculate "how much remains" was instinctive, and many changed their decisions during their interviews. Problem 1, as previously stated, has a static semantic structure.

Items 3-4: Writing a word problem for a given MEX. The difficulty in writing word problems is well known and includes basic writing skills, e.g., formulating an applicable question from information found within a story. In problems with fractions, it is important to properly ascertain the “whole” of which the fraction is a part.

We found that 15% of sixth graders and 24% of eighth graders wrote clear one- or two-stage **multiplication** word problems for both multiplication and division exercises with a fraction as an operator acting on an integer. Most of the word problems were *part-of* problems. For example: In a class of 40 students, how many went on a trip if ⅖ of the class participated?; Ronen had 60 shekels and bought a shirt with ⅔ of his money. How much did it cost? It is possible that the opening offered for the multiplication problem naturally led them to write a *part-of* problem. Nevertheless, writing such a word problem also for division by fraction indicates a lack of complete understanding of both operations.

Items 5-6: Estimating results. Twenty-two percent of sixth graders and 31% of eighth graders selected answer b (the result is “less than …”) for both items.

Items 7-8: Determining a MEX for a division word problem (divisor < 1). Thirty-three percent of sixth graders and 27% of eighth graders wrote different MEXs for the two problems. Of them, 26% of sixth graders and 22% of eighth graders correctly used division for problem 8 but multiplication for problem 7.

This error appears (e.g., Graeber & Tirosh, 1989) because the answer to the problem is expected to be greater than the dividend, and because *multiplication makes bigger*. Yet, even though in both problems the answer is expected to be greater than 24, almost one quarter of the students in each grade choses multiplication for problem 7 but not for problem 8. Furthermore, during the interviews, no one suggested "multiplication because the answer must be more than 24".

Two-thirds of the interviewees who used multiplication in problem 7 understood the context correctly, that is "one needs to find how many times ¾ goes into 24", yet insisted on multiplication because "you need ¾ plus ¾ plus ¾ plus .... i.e., you need to multiply". To them, multiplication is the repeated addition of ¾. They used the strategy of "building up" from the divisor until the dividend is reached, which is a strategy found in studies regarding division of integers (Mulligan, 1992). In contrast, in problem 8, "the pizzas are divided up", division is appropriate.

Another student's explanation for this error is the perceived difference between the two: "In 7 they are baking (making = multiplication) and in 8 they are eating (cutting = division)". The act of division in 8 is clear: begin with the dividend and *divide it* into parts; the fact that the divisor is a fraction played no role in their correct choice. Similarly, the fraction did not influence their erroneous choice of multiplication for problem 7.

Success in choosing a division MEX for item 8 did not necessarily indicate a complete grasp of division exercises. To wit: about a quarter of the students in both grades wrote the correct exercise ($24:3/4$) in problem 8, but incorrectly reasoned that the resulting number of $63:3/7$ would be less than 63 (item 5). In addition, 43% of the student in both grades wrote the correct division exercise for item 8, yet could not successfully reverse their thinking; these students were unable to write a word problem modeled by the division exercise $40:2/5$ (item 3).

DISCUSSION & CONCLUSIONS

We found different levels of understanding of the concepts of multiplication and division by fractions. There was considerable evidence of inconsistency in thinking (and performance), along with the students' inability to recognize these inconsistencies. Using the *conceptual change approach* allowed us to interpret these inconsistencies as intermediate stages in the change process: understanding multiplication and division by fractions is a gradual process where older elements of knowledge are improperly integrated with newer ones.

With respect to writing *different* MEXs for two word problems with identical mathematical structures and numbers, students who wrote the correct MEX for only one of the problems indicate a partial understanding of the operations. The explanations that students gave for their choice of operation also pointed to partial understanding: for example, a student might explain that multiplication is appropriate for one problem "because it gives a smaller result" (indicating a change in the perception that *multi-*

plication always makes bigger), yet not give this explanation for the second problem, even though the solution is also smaller.

For one of the quotative-division problems (item 7), they explained the (incorrect) choice of multiplication because "one must find how many times ¾ is contained in 24", yet they did not ask such a question with item 8. Students correctly answered only one of the division problems did not totally understand the quotative-division model. It appears that at intermediate levels, students are influenced by different characteristics of word problems and do not successfully identify the mathematical structure that is common to both. Students who successfully answered both multiplication problems (items 1 & 2), probably understood multiplication by fractions in the concept of meaning (part-of) and not only in the concept of order (multiplication makes smaller).

We found older calculation strategies mixed into choice of operation for the word problems. Students who incorrectly used division for a *part-of* problem were influenced by the familiar method from grade four to find a part of a whole: division by the denominator of the fraction and multiplication by its numerator. One can say that they incorporated into their knowledge expansion their previous knowledge about finding a portion of a quantity and not just awareness that division as something "that reduces". Students who used multiplication for division problem by explaining "we have to find how many times ¾ goes into 24" were influenced by the "building-up" strategy used in division problems with whole numbers. In both cases they had difficulty applying these strategies correctly using the numbers in the problem.

We found an intermediate level of understanding with respect to the misconceptions *multiplication makes bigger* and *division makes smaller*. Some students stated that multiplication by a fraction leads to "less than" but also claimed this in the case of *division* by a fraction. Students at this intermediate level appear to be locked into only one of the two misconceptions. Like Michael, whom we presented in the introduction, they also believe that both of the mathematical operations "make smaller". The varied results presented above suggest that some students hold the conception that multiplication by a fraction decreases and the misconception that division by a fraction finds a part of.

Other examples of an intermediate level of understanding were shown in the writing of word problems. Some students wrote *part-of* problems for both multiplication and division by a fraction. Conflating the meanings of the operations was not found in studies of decimals, indicating that this misconception could be attributed to the special way that fractions are written and to the unique *part-of* model associated with fractions.

As mentioned, when a student writes different MEXs for paired word problems, he has failed to attain full multiplicative reasoning. Harel (1995) claimed that being in this "naive-interpretist" stage for word problems is an inevitable interim stage in developing the concept of multiplicativity, yet once a child acquires multiplicative thinking, he recognizes identical mathematical structure in ostensibly different word problems.

We are convinced that teachers must discuss the meanings of the operations to help students through the "naive-interpretist" stage. Furthermore, it is essential to give students practice exercises with both operations, to eliminate confusion and sharpen their understanding of the meaning of each and the difference between them.

In summary, this study adds to the understanding of the conceptual change required to extend multiplication and division to fractions. It also provides questionnaire items and analysis methods for discerning obstacles to extending these operations that are unique to fractions. In addition, conflation of multiplication and division specific to fractions indicates that further study is needed.

References

Bell, A., Swan, M., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational Studies in Mathematics*, *12*(4), 399–420.

Buzaglo, M. (2002). *The logic of concept expansion.* Cambridge: Cambridge University Press.

Greer, B. (1987). Nonconservation of multiplication and division involving decimals. *Journal for Research in Mathematics Education, 18*(1), 37–45.

Harel, G. (1995). From naive-interpretist to operation-conserver. In J. Sowder & B. Schappelle (Eds.), *Providing a Foundation for Teaching Mathematics in the Middle grades* (pp. 143–165). New York: SUNY Press.

Mulligan, J. T. (1992). Children's solutions to multiplication and division word problems: A longitudinal study. *Mathematics Education Research Journal, 4*(1), 24–42.

Prediger, S. (2008a). The relevance of didactic categories for analysing obstacles in conceptual change: Revisiting the case of multiplication of fractions. *Learning and Instruction, 18*(1), 3–17.

Prediger, S. (2008b). Discontinuities for mental models: A source for difficulties with the multiplication of fractions. *Proceeding of ICME-11-Topic Study Group, 10*, 29–45.

Tirosh, D., & Graeber, A. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics, 20*(1), 79–96.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14*, 453–467.

Vamvakoussi, X., Vosniadou, S., & van Dooren, W. (2013). The framework theory approach Applied to mathematics learning. In S. Vosniadou (ed), *International Handbook of Research on Conceptual Change* (2nd ed, pp. 305-321). New York, USA: Routledge.

DOCTORAL PROGRAMS' CONTRIBUTION TO BECOMING A MATHEMATICS EDUCATION RESEARCHER

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The knowledge and skills that a mathematics education (MathEd) researcher should have and to what extent doctoral programs (DPs) contribute to this researcher were explored through the written responses of 37 doctoral students studying in the field of MathEd in Turkish, European and North American DPs to an open-ended survey. Findings addressed that doctoral students prioritized research and MathEd related knowledge and skills the most. Generic skills, career skills, critical research skills and habits of mind were stated the least. Participants evaluated their knowledge and skills and DPs' contribution to them as mostly sufficient. However, more courses and experiences were needed. Scholarly climate and human resources were the strongest aspects of DPs. Research opportunities for doctoral students needed improvement.

KNOWLEGDE AND SKILLS IN DOCTORAL PROGRAMS

The aim of doctoral education is to provide knowledge and skills to doctoral students in a specific field of study that they will become scholars and independent researchers (Mendoza & Gardner, 2010). Mathematics education (MathEd) researchers are expected to have several types of knowledge and skills to pursue a wide range of roles and duties in multiple contexts such as university, schools, classrooms, and national and international research communities (Hiebert, Lambdin, & Williams, 2008). On the other hand, disciplinary cultures of knowledge production with emphasis on apprenticeship in doctoral programs (DPs) have evolved to newer cultures and practices prioritizing capabilities, dispositions, and other ways of knowledge production equally important as expertise and knowledge (Lee & Boud, 2009). The recent global focus on the future of doctoral degree addresses that doctorates should also have generic and global capabilities (Cumming, 2010) including networking, collaboration, communication and problem solving (Hopwood, 2010).

The purpose of doctoral education in the field of education has been debated without a conclusion often through what one becomes at the end of the doctoral education (Gardner, Hayes, & Neider, 2007). There are multiple and often conflicting views of preparing doctoral students MathEd field which might be better explored by focusing on DP practices (Ferrini-Mundy, 2008). Such an investigation can take place at three levels from local (doctoral students) to intermediate (doctoral institutions), to more abstract level (Lee & Boud, 2009). DPs in the field of MathEd (including DPs such as, Teacher Education) have mostly been investigated in terms of number and content of available courses (Ferrini-Mundy, 2008). However, preparation of doctoral students

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 19-26). Umeå, Sweden: PME.

for being competent MathEd researchers requires scholarly contexts where students are encouraged to develop and integrate several skills and critical perspectives by being involved in scholarly communities (Haser, 2017; Middleton & Dougherty, 2008), which cannot be explored by focusing only on courses.

Studies have shown that doctorates are expected to have several competencies; yet, to what extent doctoral education can provide these competencies is questionable (Mowbray & Halse, 2010). Knowledge and skills associated with one's field of study are considered as one aspect of these competencies along with dispositions and behaviours (Durette, Fournier, & Lafon, 2016). Such competencies may also include a more comprehensive and integrated set of dispositions, habits of mind which are "intangible attitudes, values, and characteristics that cannot be seen or casually observed", and "more tangible and observable" skills and abilities for research practice and communication (Gardner, et al., 2007, p. 294). Sinclair, Barnacle and Cuthbert (2014) have explored studies about doctoral education to document the factors that may contribute to doctoral students' becoming active researchers. They found that active and productive supervisor, active research culture and department, sense of becoming a peer or independent, development of collaborative capacities, conceptualization of success in doctorate across contexts, socialization into research practice and emotional engagement with one's research studies were important in becoming an active researcher. Their findings suggested that when the conceptions of the purpose of the doctorate among the students, faculty, and institutions were similar, the students were likely to become active researchers.

The recent global focus on doctoral practices (Lee & Boud,2009) addresses that there is a need to explore how DPs prepare doctoral students for the knowledge and skills of conducting MathEd research (Ferrini-Mundy, 2008). Therefore, this study explored how doctoral students studying in the MathEd field identify knowledge and skills that a MathEd researcher should have, to what extent they have these knowledge and skills, and their views about how DPs contribute these knowledge and skills through an open-ended survey. The aim was to provide doctoral students' perspectives to the improvement of DPs. The findings can further be compared to the views of DPs' faculty and administrators and crosschecked with the content of the courses to evaluate the effectiveness of DPs and to increase DPs' contribution to becoming a researcher.

METHOD

The study was designed as a qualitative survey study in which doctoral students working in the field of MathEd were asked six questions about the knowledge and skills that a MathEd researcher should have, and to what extent they had and DPs contributed these knowledge and skills.

Participants

The participants of the study were a total of 37 doctoral students from Turkish DPs (21 participants) and international (non-Turkish) DPs (16 participants). Thirteen of the international students were in North American DPs and the rest were in European DPs. Turkish participants were reached through e-mails that were sent to graduate assistants and professors (to be forwarded to the doctoral students) working at DPs focusing on MathEd at Turkish universities. International participants were reached through the e-mails that were sent to their professors, who were conveniently known to the researcher, and through an online platform. The participants were at different stages of their doctoral studies (before dissertation or focus on dissertation). Table 1 summarizes participants' stages in their DPs for Turkish and international doctoral students.

	Turkish	International	Total
Stage in the Program	Focus on Disst.-17	Focus on Disst.-11	Focus on Disst.-28
	Before Disst.-4	Before Disst.-5	Before Disst.-9

Table 1: Participants' status in the program.

Comparisons between DPs in different countries or among doctoral students were not the purposes of the study. Therefore, findings were presented without reference to the countries or participants' stage in the DPs.

Data Collection and Analysis

Data of the study were collected by a qualitative open-ended survey. The first part of the survey had 10 demographic questions about participants' DPs, status in the DPs and scholarly activities. The second part had six questions asking their ideas about (1) the knowledge and skills a MathEd researcher should have; whether and how their DPs (2) were supporting these skills (3) or not (with references to the specific experiences); (4) self-evaluation of their knowledge and skills; (5) their supervisors' expectations; and (6) the strengths and needs-improvement aspects of DPs (three for each). The survey and the consent form was first sent to a small group of Turkish doctoral students as a text file via e-mail to respond and to reflect on the comprehensiveness and wording. The responses were received mostly via e-mail to the researcher as a digital text file and a small number of them were received as printed. These responses and reflections did not result in any change in the survey and it was sent to a larger group of doctoral students and professors in Turkey via e-mail. Professors were kindly asked to forward the survey to the MathEd doctoral students in their institutions. Filled-out surveys were received as digital text files via e-mail directly from the participants.

The English translation of the survey and the consent form was sent to international doctoral students via (i) e-mail asking the conveniently reached professors to forward and (ii) a digital survey tool which was announced to the students through the social media outlet of an association of European MathEd doctoral students and researchers. One of the professors revised the demographic questions to address the doctoral students' stage in the DPs in a specific European context and the survey was distributed to the students in that context with revised demographic questions.

The wording of the questions in the survey intentionally asked participants to identify knowledge and skills, not capabilities or competencies because knowledge and skills would not probably require further explanation. Identifiers such as, dispositions and values, would probably need further explanation as they have different meanings for individuals. However, asking knowledge and skills might have limited participants' responses and caused them not to consider certain competencies.

The thematic analysis (Braun & Clarke, 2006) was employed to analyse the data in the second part of the survey. First, responses to the first question were read several times and initial codes were generated. Then, data were coded by these codes and codes were grouped in eight potential themes (some from the literature) almost simultaneously, resulting in theme-code-coded data clusters. This helped to clarify the meaning of the themes and see if the themes reflected the set of codes addressing the same concept. Responses to questions (2) and (3) were analysed in terms of DP experiences that the participants expressed for the knowledge and skills they identified and responses to question (6) were analysed in terms of participants' evaluations of the strengths and needs-improvement aspects of the DPs by using analysis steps described above.

FINDINGS

Doctoral students identified eight major types of knowledge and skills that a MathEd researcher should have: *Knowledge of the MathEd field*, *research knowledge and skills*, *knowledge of teaching and learning process*, *communication skills*, *career skills*, *critical research skills*, *habits of mind*, and *generic skills*. DPs contributed to these knowledge and skills through *courses*, *implementations and projects*, and *program culture in terms of support*, *interactions and role models*. Yet, many participants expressed that more courses and/or project experiences were needed to improve MathEd researcher traits. Doctoral students evaluated the extent of their knowledge and skills as they *had* it, they had it to some degree and still working on *improving*, or they *did not have* it. Doctoral students' views about knowledge and skills that a MathEd researcher should have, whether DPs contributed to these knowledge and skills, whether or to what extent they had these knowledge and skills, and their views about their supervisors' expectations (*expecting* or *not expecting*) from them were reported together for each type of knowledge and skills. Then, their views about DPs' strengths and needs-improvement aspects were presented.

Knowledge, Skills and Doctoral Programs

Doctoral students most frequently expressed that a MathEd researcher should have *knowledge of the MathEd field (N=34, f=70)* which included mathematics knowledge especially in the field of expertise, knowledge of mathematics curriculum, theories and trends of MathEd research, pedagogical content knowledge, and technological pedagogical content knowledge. DPs contributed to this knowledge mostly through courses and projects, and supervisors expected them to have it. However, participants stated that they would like to have more courses in DPs addressing these components. Although many doctoral students expressed that they either had or were improving this knowledge, several of them stated that they did not have all components sufficiently.

Research knowledge and skills (N=32, f=59) was the second most frequent category which addressed knowledge and skills of research designs and data analysis methods in quantitative and qualitative trends, searching literature and identifying research problems. DPs contributed to this category mostly through courses, and less with projects. However, more courses and experiences were needed. Participants stated that they either had or were improving research knowledge; however, many also expressed that they did not have these knowledge and skills sufficiently. Supervisors were perceived as they expected these from doctoral students to a considerable degree.

Knowledge of teaching and learning process (N=23, f=36) was important for the participants. Knowledge of schools and classrooms, contexts, knowledge of teaching demands and students, and knowledge of theories were in this category. DP courses and projects helped participants to gain or improve this knowledge. They did not state much about what their programs lacked in this aspect. Supervisors were also not perceived that they would expect this knowledge from the students.

Communication skills (N=24, f=35) including communication of research outputs or process to all interested parties, contacting or finding scholarly communities, and attending conferences were important for participants. DPs did not contribute much to these skills. Some participants expressed that courses or DP culture should be more supportive to improve these skills. They mostly stated that they had or were improving communication skills and their supervisors expected them to have these skills.

Career skills (N=19, f=24) addressed skills such as writing for publications and grants, evaluating the quality of the journals and conferences, project management, and collaboration. However, DPs did not help students to improve these skills much and they expressed they needed more experiences or rather a DP culture to improve these skills. They mostly expressed that they had or were improving career skills. Supervisors were perceived that they would expect participants to have these skills.

Critical research skills (N=11, f=17) were not stated much frequently. Some participants expressed evaluating the significance of a study, developing perspectives for a research problem or implications of a study, and critical evaluation of literature as important skills of being a MathEd researcher. These participants stated that courses were useful for these skills and that they had or were improving these skills. These skills, however, were not perceived as expected by the supervisors much.

Habits of mind (N=10, f=18) such as, perseverance, patience, curiosity, and motivation, were expressed by some participants. This was rather seen as a personal trait that could be improved by personal efforts. Those who expressed these habits also indicated that they mostly had or were improving them. Supervisors were perceived as they expected the participants to have these habits.

Generic skills (N=5, f=9) included time management, balancing between professional and personal life, and motivating other people. Doctoral students' competence in these skills varied and they perceived their supervisors expected them to have these skills. They did not state DPs' contribution to these skills much.

Strengths and Needs-Improvement Aspects of Doctoral Programs

Doctoral students evaluated DPs' strengths and the aspects of the DPs that needed improvement in ten major categories: *Scholarly climate*, *human resources*, *research opportunities, program structure*, *courses*, *MathEd specific experiences*, *connections*, *collaboration*, *teaching opportunities* and *communication*. Scholarly climate and human resources were identified as strengths of the DPs. Courses in the DP, research opportunities, MathEd specific experiences and DP structure were identified as both strengths and needs-improvement aspects. Collaboration, communication and teaching opportunities in the programs were identified as aspects to be improved.

The strongest aspect of DPs was the *scholarly climate (N=21, f=30)* they provided. Doctoral students expressed that new approaches and ideas were welcomed in the DPs and they received constructive feedback and support for independent research. DPs had scholarly communities where scholarly activities were supported, and students gained perspectives and increased their motivation. Such strength was also perceived in *human resources (N=18, f=20)* where faculty members' experiences and support for students made DPs more valuable. Other students strengthened DPs by attending scholarly activities. DPs provided *connections (N=9, f=9)* for students to international collaborators, networks and other fields to improve their knowledge and skills.

Certain features of the *program structure (N=12, f=18)* such as, being able to select one's supervisor, courses, and research topic, were considered among the strengths of DPs. Doctoral comprehensive examination (in some of the DPs), was a strength when it prepared the students for their further studies. The variety and the content of the *courses (N=12, f=13)* in the DPs were satisfactory and beneficial for many participants. *Research opportunities (N=15, f=17)* such as, research knowledge and practices of different designs, data analyses, project writing, and being able to work in projects provided considerable learning opportunities. Part of the strength was about the *MathEd specific experiences (N=9, f=11)* such as, MathEd research projects, courses, and researchers which were supporting participants' studies.

However, *program structure (N=21, f=16)* needed improvement to provide more funding opportunities, specific supports for international students (in non-Turkish DPs), and manageable work load. *Course (N=11, f=14)* availability was an important drawback for some of the participants that there were times they could not find courses for their needs or *MathEd specific courses (N=7, f=7)* on important concepts. They specifically indicated that *research preparation (N=14, f=22)* should start earlier with introductory courses and there should be support for scholarly language improvement and publication. *Collaboration (N=9, f=11)* and *communication (N=7, f=8)* between the faculty members and doctoral students, and among doctoral students needed improvement in order to provide students better DP experiences. Insufficient *teaching opportunities (N=8, f=8)* in schools and at university level courses were addressed as important issues to be improved.

CONCLUSIONS AND DISCUSSIONS

Findings showed that doctoral students in the field of MathEd prioritized knowledge and skills of research methods, MathEd specific issues, teaching and learning process, and communication for a MathEd researcher. These categories mostly included components which could be learned through courses and resources, or observed in scholarly activities such as conferences. On the other hand, generic skills, career skills, critical research skills and habits of mind were not stated much by the doctoral students despite their importance for DPs (Cumming, 2010). These skills included skills that are not easily observed or qualifications that are developed over time and experience. Doctoral students' evaluation of DPs showed that they seemed to consider courses as the major DP experience and focused on their learning in the courses. The related questions in the survey asked to indicate courses and other experiences in the DPs in relation to the knowledge and skills they listed. Although most of the participants were focusing on their dissertation studies, they did not mention about the contribution of this experience, or of working with their supervisors during this process to their knowledge and skills.

Interestingly, participants criticized DPs for not providing better research opportunities, communication and career skills with emphasis on collaboration, which are important factors for becoming an active researcher (Sinclair, et al., 2014). These skills could have been perceived more positively within encouragingly perceived scholarly climate and human resources of DPs. Yet, they needed DP contexts providing more research opportunities with effective collaboration with professors and other doctoral students, and sharing of research processes and outcomes through more structured workshops and meetings. Indeed, being a part of a research study (other than the dissertation study) was found to be a key experience in becoming an independent researcher for MathEd doctoral students (Haser, 2017).

These findings showed that the focus on content and number of courses in MathEd DPs does not provide a better picture of the nature of DPs' practices and the types of competencies they address (Ferrini-Mundy, 2008; Middleton & Doughtery, 2008). Exploring these practices and competencies through the views of doctoral students, supervisors, and DP administrators might provide more grounded improvements in MathEd DPs to support becoming an active MathEd researcher.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77-101.

Cumming, J. (2010). Doctoral enterprise: a holistic conception of evolving practices and arrangements. *Studies in Higher Education*, *35*(1), 25-39.

Durette, B., Fournier, M., & Lafon, M. (2016). The core competencies of PhDs. *Studies in Higher Education*, *41*(8), 1355-1370.

Ferrini-Mundy, J. (2008). What core knowledge do doctoral students in mathematics education need to know? In R. E. Reys & J. A. Dossey (Eds.) *U.S. doctorates in mathematics education: Developing stewards of the discipline* (pp. 63-71). Providence, RI: AMS.

Gardner, S. K., Hayes, M. T., & Neider, X. N. (2007). The dispositions and skills of a Ph.D. in education: Perspectives of faculty and graduate students in one college of education. *Innovative Higher Education*, *31*, 287-299.

Haser, Ç. (2017). Key experiences in becoming an independent mathematics education researcher. *Canadian Journal of Science, Mathematics and Technology Education.* DOI: 10.1080/14926156.2017.1327090.

Hiebert, J., Lambdin, D., & Williams, S. (2008). Reflecting on the conference and looking toward the future. In R. E. Reys & J. A. Dossey (Eds.) *U.S. doctorates in mathematics education: Developing stewards of the discipline* (pp. 241-252). Providence, RI: AMS.

Hopwood, N. (2010). Doctoral experience and learning from a sociocultural perspective. *Studies in Higher Education*, *35*(7), 829-843.

Lee, A. & Boud, D. (2009). Framing doctoral education as practice. In D. Boud & A. Lee (Eds.) *Changing practices of doctoral education* (pp. 10-25). Oxford: Routledge.

Mendoza, P. & Gardner, S. K. (2010). The PhD in the United States. In S. K. Gardner & P. Mendoza (Eds.), *On becoming a scholar: Socialization and development in doctoral education* (pp. 11-26). Sterling, VA: Stylus Publishing.

Middleton, J.A. & Dougherty, B. (2008). Doctoral preparation of researchers. In R. E. Reys & J. A. Dossey (Eds.) *U.S. doctorates in mathematics education: Developing stewards of the discipline* (pp. 139-146). Providence, RI: AMS.

Mowbray, S. & Halse, C. (2010). The purpose of the PhD: Theorizing the skills acquired by students. *Higher Education Research & Development*, *29*(6), 653-664.

Sinclair, J., Barnacle, R., & Cuthbert, D. (2014). How the doctorate contributes to the formation of active researchers: What the research tells us. *Studies in Higher Education*, *39*(10), 1972-1986.

INSTRUMENT TO ANALYSE DYADS' COMMUNICATION AT TERTIARY LEVEL

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In this paper, we focus on the development of a theoretical instrument based on the interactive-cooperative-active-passive-framework to analyse dyads' communication processes in collaborative face-to-face learning scenarios. We can show that the adaption of this framework to the analysis of time-sampled video recordings is successful and that a dependency between dyads' communicational behaviours and their learning outcome may be present.

THE MAMDIM-PROJECT

The transition from secondary to tertiary education is known as a complex problem area, especially in mathematics (Gueudet, 2008). Within this context, the use of new instructional media like video tutorials, podcasts or commented presentations is expanding (Bausch et al., 2014), whereas at the same time a lack of research in this field is stated (Biehler, Fischer, Hochmuth, & Wassong, 2014). The mamdim–project (learning mathematics with digital media during the transition from secondary to tertiary education) explores the usage and benefit of different digital instructional media focusing on descriptive statistics in cooperation with four German universities (University of Applied Sciences Pforzheim, Offenburg University of Applied Sciences, Bielefeld University and Brandenburg University of Technology Cottbus-Senftenberg). We conducted a pilot study to improve the study design (pre-test | intervention | post-test) and the test items at two universities in 2015, for first results see Salle, Schumacher, & Hattermann (2016). In this paper, we take our data from the main study that took place in 2016 and 2017 with about 300 students.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

The role of language has been an important issue in mathematics education for a long time (Austin & Howson, 1979), a broad overview about contemporary research is given in Morgan, Craig, Schuette, & Wagner (2014). Students' learning of mathematics evolves in interaction and is enclosed in language and communication (Steinbring, 2015). Therefore, communication processes including sharing ideas verbally or in writing processes play a fundamental role in learning mathematics from a constructivist point of view. There is a significant number of studies showing that collaborative learning in small groups does not necessarily yield greater learning outcomes (Barron, 2003), whereas other researchers stress the advantages of working collaboratively to promote learning (Dillenbourg, Baker, Blaye, & O'Malley, 1996). At first sight, these results seem to be contradictory, but can be explained with the help of the

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 27-34). Umeå, Sweden: PME.

CAP-framework by Chi & Menekse (2015), see also Menekse, Stump, Krause, & Chi (2013) for the origins of this framework. We will use this framework in an adapted manner to analyse our data.

The Interactive-Constructive-Active-Passive (ICAP)-framework

The ICAP-framework explains what type of interaction is most effective in collaborative learning situations. To develop this framework, a first step consists of identifying observables that characterise individual's learning, called *engagement activities.* Regarding their benefit of learning, these activities can be classified and rank ordered in *passive engagement*, *active engagement* and *constructive engagement* (Chi & Wylie, 2014). To give an example, *passive* reading means reading silently without trying to integrate the text in present knowledge. *Active* reading is characterised by reading aloud in certain passages and by highlighting specific words or information. An activity such as self-explaining, taking notes or making drawings defines a *constructive engagement* while reading (Chi & Menekse, 2015). Different cognitive processes are triggered by different forms of engagement. The CAP-hypothesis claims that constructive engagement is superior to active engagement, in which active engagement dominates passive engagement with respect to greater learning outcomes. Many studies support the CAP-hypothesis, for an overview see Chi and Wylie (2014). With the help of the CAP-framework, it is possible to explain the mentioned contradictory research results. For example, a single active learner will only learn more in a dyad if his partner is an active learner or a constructive learner. A constructive learner will profit from group learning only, if his partner is at least an active learner and there will be no benefit if his partner is only a passive learner (Chi & Menekse, 2015). To understand better how dyads learn, it is possible to classify each individual's utterances as passive, active or constructive and to widen the CAP-framework to the ICAP-framework, in which the fourth engagement category *interactive* occurs, only making sense in collaborative learning scenarios. *Interactive* learning meets two criteria: a) both partners' utterances must be primarily constructive, and b) a sufficient degree of turn taking must occur (Chi and Wylie, 2014). In this respect, interactive learning is beneficial to both constructive learners. To assess each dyad's communicational behaviour, Chi and Menekse (2015) calculate a so-called *dialogue pattern score*, i.e. a rational number between 1 and 3. Dyads whose communication is mainly driven by utterances of a single learner, while the other partner remains passive, are classified with a score close to 1. At the other end of the spectrum, dyads whose communication is shaped by a high degree of constructive interaction between the learners, will be given a dialogue pattern score close to 3.

Based on the ICAP-framework, we will develop a theoretical instrument to analyse dyads' communication processes regarding their interactivity while working with digital instructional material.

The following research questions will be considered:

- In which way is the ICAP-framework applicable to analyse dyads' communication processes, which learn with a digital instructional medium?
- Are there dependencies between the interactivity of the face-to-face communication as measured by the dialogue pattern score and students' learning outcome in a pre-post-test-scenario?

STUDY DESIGN

11 pairs of electrical engineering or economics students at the University of Applied Sciences Pforzheim worked with a digital script (comparable to an interactive-pdf-file) as instructional medium in a moodle-environment. The material – dealing with measures of central tendency (e.g. arithmetic mean, median, harmonic mean) as the object of learning – encompassed 21 slides, containing definitions, formulas, explanations, examples and short multiple-choice questions.

As an example, the slide in figure 1 deals with the harmonic mean. The slide is followed by two examples illustrating the difference between the arithmetic and harmonic mean. In the first example, a car is driven the same amount of time on five distinct days with different average speeds and in the second example, a car is driven the same distance with different average speeds five times a week. The total average speed is calculated using the arithmetic mean in the first and the harmonic mean in the second example.

Measures of central tendency

The harmonic mean

The harmonic mean of n numbers $x_1, x_2, ..., x_n$ is defined by

$$\bar{x}_h = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

The question arises in which situation the harmonic mean has to be used to calculate the average. The answer is given in the following.

If the values in question x_i are given as ratios, e.g. speeds (= length/time), it is crucial whether the data in the data set is referring to the length (nominator) or time (denominator). If the data refer to the time, i.e. the **denominator**, the average speed is calculated using the **arithmetic mean**; but if the data refer to the length, i.e. the **nominator**, the **harmonic mean** has to be used.

This is illustrated in the following example:

Figure 1: Slide from the instruction material (translation by authors)

Subsequently the students were asked to answer the following multiple-choice question regarding the difference between the harmonic and arithmetic mean:

A garden centre creates a substrate by blending same masses of four different soils. These soils are known to have the following densities: Soil A: 710 kg/m³; Soil B: 920 kg/m³; Soil C: 830 kg/m³ and Soil D: 1000 kg/m³. Calculate the average density of the substrate.

Possible answers were 851 kg/m³, 865 kg/m³ and 857 kg/m³ with the first one being the correct answer (calculated using the harmonic mean). The second (wrong) answer is calculated using the arithmetic mean of the four values and acts as a distractor. After answering the question, students got a direct feedback whether their calculation was correct or not, which is the main difference to a paper-and-pencil environment. Since the ICAP-framework, on which our analysis is based, originated from the analysis of paper-and-pencil situations, this moodle-environment acts as a logical first choice to adapt the framework to the analysis of computer-environment situations before using it in situations like learning with video tutorials.

Before and after the digital media intervention students had to take a test consisting of both multiple-choice items and open questions regarding the overarching topic (descriptive statistics). For a detailed overview of the test items used, see Salle, Schumacher & Hattermann (in prep).

Methodology

The students' computer screens were captured and the utterances and the image of the two learners were videotaped. To analyse the recordings, the time-sampling method by Bakeman and Gottman (1997) was used, in which each video was organised into segments of 10 seconds. After that, all segments in which the students communicated (verbally) for at least five seconds about the mathematical aspects relevant to the material at hand were identified. We will call this type of communication *meaningful* in the following.

We decided to restrict our analysis of the data we collected to the students' communication processes taking place while they focus on the material that deals with the harmonic mean that we described above. Compared with other measures of central tendency, the relative difficulty of the topic provides a richer source for possible student interaction and thus a suitable area of focus.

Following Chi and Menekse (2015), we regard a student as "active" if he or she repeats or restates ideas from either his partner or the material at hand while a "constructive" learner elaborates on ideas, raises questions or explains something in response to a question. To adapt the ICAP-framework to our situation (time-sampled videos of dyads in a collaborative learning situation), we derived the following coding scheme: For each 10-second-segment that contains meaningful verbal communication we decided for each student individually whether this student took an active, a constructive or a passive part in the conversation.

The following quote where both partners act as constructive learners is taken from dyad 114 by working on the "car-example", described in the study design:

Student 1: So, why do I need to use the harmonic mean in one case and the arithmetic mean in the other case?

Student 2: Here *(while pointing at the screen)* he drives the car always the same amount of time and there *(points to the other example)* he drives the car always the same distance, but needs a different amount of time.

Here, the first student raises a question with regard to the material at hand while the second student directly picks up on this question and tries to explain the mathematical content with respect to the learning material.

Based on this analysis, every 10-second-segment containing meaningful mathematical conversation is now rated on an ordinal scale ranging from a score of 1 to 3. Table 1 gives a summary of the definitions of these scores which resemble the definitions by Chi and Menekse (2015).

score	Description Student communication in this 10 second segment…
1	… is dominated by one student. *(active-passive or constructive-passive)*
2	… is driven by both partners, but not interactively. *(active-active, constructive-active, constructive-constructive)*
3	… has two constructive partners contributing interactively to it. *(constructive-constructive and interactive)*

Table 1: Coding scheme of communication interactivity

We adapt the dialogue pattern score as described in the theoretical background by Chi and Menekse (2015) to analyse the 10-seconds-segments in the following way. The number of times each score occurred is counted for every dyad and the dialogue pattern score is calculated by taking the weighted average of the occurrences: For example, the communication of dyad 001 (see table 2) included in total 23 coded segments with meaningful communication. 5 of those segments were score 1 segments, 7 of them reached score 2 and the remaining 11 of them got a score of 3. From this, we can calculate their dialogue pattern score as

$$\frac{1 \cdot 5 + 2 \cdot 7 + 3 \cdot 11}{23} \approx 2.26.$$

Therefore, the communication of dyads with a dialogue pattern score close to 1 is affected predominantly by single student contributions without interaction between the partners while higher scores represent a higher level of verbal interaction.

RESULTS

Counting all segments with a given code and calculating the dialogue pattern scores for each individual dyad yields the following table 2:

	# of segments with score …				
Dyad	… 1	… 2	… 3	dialogue pattern score	normalised gain score
001	5	7	11	2.26	0.34
002	5	1	0	1.17	0.20
003	3	2	0	1.40	0.27
004	4	0	0	1.00	0.28
005	11	1	4	1.56	0.65
006	2	2	0	1.50	0.41
112	16	7	2	1.44	0.13
113	15	12	3	1.60	0.43
114	5	3	2	1.70	0.46
115	0	9	1	2.10	0.41
117	0	5	0	2.00	0.56

Table 2: Dyads' dialogue pattern and normalised gain scores

In total, three of the 11 pairs (001, 115, 117) achieved a dialogue pattern score above (or exactly) 2.0 while the score of two dyads (002, 004) is close to (or exactly) 1.0. The median of all scores is 1.56 with 5 dyads scoring below it.

To explore possible dependencies between the dialogue pattern score and the learning outcome of the dyads, we calculated the so-called *normalised gain score* for each dyad which relates pre- and post-test results using the following formula exactly as presented in Chi and Menekse (2015):

$$normalised\ gain\ score := \frac{posttest\ \%\text{-}pretest\ \%}{100\ \% - pretest\ \%}$$

Here post- and pre-test results are the averages of both partners' results as percentage figures. For an individual learner this number relates the percentage points he or she *actually gained* between pre- und post-test to the percentage points he or she *could have* gained. For example, a student scoring 25 % in the pre-test and 50 % in the post-test achieved a gain score of 0.33 (he gained 25 percentage points out of 75 percentage points he could have gained).

According to the ICAP hypothesis a higher dialogue pattern score should on average be an advantage for dyads with respect to their learning outcome. To test this hypothesis and answer our second research question, we took the average normalised gain score of the dyads with a dialogue pattern score of at most 1.5 and of those with a dialogue pattern score of above 1.5 respectively. We chose this threshold because at this score dialogues in which both partners are verbally active become dominant. The results are presented in the following figure:

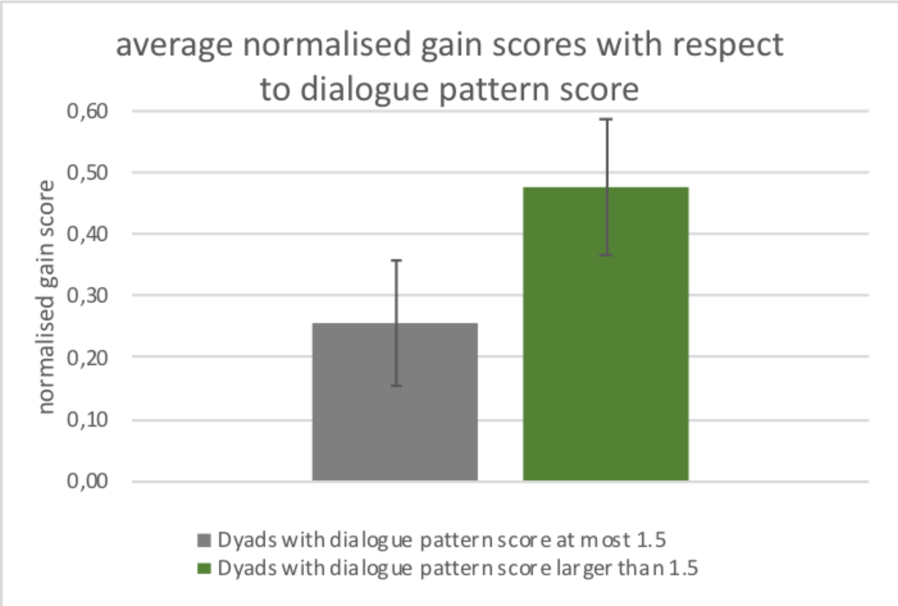

Figure 2: average normalised gain scores with standard deviation.

The normalised gain score of those dyads with a lower dialogue pattern score is M = 0.26 (SD = 0.10). This average rises to M = 0.48 (SD = 0.11) for those dyads with a higher dialogue pattern score. The difference in means is significant, t = -3.325, p = 0.009.

CONCLUSION AND PERSPECTIVES

The ICAP-framework has been adapted to analyse time-sampled recordings of dyads learning with a digital instructional medium. Using this method, the predicted results from the ICAP hypothesis – dyads communicating in a constructive manner benefit with respect to their learning outcome – could be replicated. In our ongoing research, we will validate our adaption of the ICAP-framework in the context of students working with different digital instructional media like video tutorials which comprise more of the unique features of a computer-environment. If this undertaking succeeds, our approach will be used to identify those digital instructional media that promote interactive constructive communication between students and that influence the learning outcome in a positive way. Additionally, this method can be used to investigate the benefit of collaborative learning in computer environments compared to single learners in more detail.

Acknowledgements: The project xxx is funded by the German Federal Ministry of Education and Research BMBF (grant 01PB14011).

References

Austin, J. L., & Howson A. G. (1979). Language and mathematical education. *Educational Studies in Mathematics, 10,* 161-197.

Bakeman, R. & Gottman, J. M. (1997). *Observing interaction. An introduction to sequential analysis*. 2. ed. Cambridge [u.a.]: Cambridge University Press.

Barron, B. (2003). When smart groups fail. *Journal of the Learning Sciences, 12,* 307-359.

Bausch, I., Biehler, R., Bruder, R., Fischer, P., Hochmuth, R., Koepf, W., & Wassong, T. (2014). VEMINT – Interaktives Lernmaterial für mathematische Vor- und Brückenkurse. In I. Bausch, R. Biehler, R. Bruder, P. Fischer, R. Hochmuth, W. Koepf, S. Schreiber, & T. Wassong (Eds.), *Mathematische Vor- und Brückenkurse: Konzepte, Probleme und Perspektiven* (pp. 261-276). Wiesbaden: Springer Spektrum.

Biehler, R., Fischer, P., Hochmuth, R., & Wassong, T. (2014). Eine Vergleichsstudie zum Einsatz von Math-Bridge und VEMINT an den Universitäten Kassel und Paderborn. In I. Bausch, R. Biehler, R. Bruder, P. Fischer, R. Hochmuth, W. Koepf, S. Schreiber, & T. Wassong (Eds.), *Mathematische Vor- und Brückenkurse: Konzepte, Probleme und Perspektiven* (pp. 103-122). Wiesbaden: Springer Spektrum.

Chi, M.T.H, & Wylie, R. (2014). The ICAP Framework: Linking Cognitive Engagement to Active Learning Outcomes. *Educational Psychologist, 49*(4), 219-243.

Chi, M.T.H, & Menekse, M. (2015). Dialogue patterns in peer collaboration that promote learning. In L. B. Resnick, C. Asterhan, & S, Clarke, (Eds.), *Socializing Intelligence Through Academic Talk and Dialogue* (pp. 263-274). American Educational Research Association.

Dillenbourg, P., Baker, M., Blaye, A., & O'Malley, C. (1996). The evolution of research on collaborative learning. In E. Spada, & P. Reiman (Eds.), *Learning in humans and machine: Towards an interdisciplinary learning science* (pp. 189-211). Oxford: Elsevier.

Gueudet, G. (2008). Investigating the secondary–tertiary transition. *Educational Studies in Mathematics, 67*(3), 237–254. doi:10.1007/s10649-007-9100-6.

Menekse, M., Stump, G., Krause, S., & Chi, M. T. H. (2013). Differentiated overt learning activities for effective instruction in engineering classrooms. *Journal of Engineering Education, 102*, 346-374.

Morgan, C., Craig, T., Schuette, M., & Wagner, D. (2014). Language and communication in mathematics education: an overview of research in the field. *ZDM Mathematics Education, 46*(6), 843-853. doi:10.1007/s11858-014-0624-9.

Salle, A., Schumacher, S., & Hattermann, M. (2016). The Ping-Pong-Pattern – Usage of notes by Dyads during Learning with Annotated Scripts. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proc. 40th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 147-154). Szeged, Hungary: PME.

Salle, A., Schumacher, S., & Hattermann, M. (Hrsg.) (in preparation). *Mathematiklernen mit digitalen Medien – Ergebnisse des mamdim-Projekts*. Berlin, Heidelberg: Springer.

Steinbring, H. (2015). Mathematical interaction shaped by communication, epistemological constraints and enactivism. *ZDM Mathematics Education, 47*(2), 281-293.

STRIVING FOR EQUITY: HOW POLICY SHAPES OUR UNDERSTANDING OF RACE IN MATH CLASS

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The purpose of this research is to present initial findings related to how federal education legislation in the United States frames racialized students in mathematics. By relying on Critical Race Theory and governmentality, I am able to highlight how race is considered in both extant mathematics education literature and current legislation. This allows for a discussion regarding how the use of race in policy actually impacts the types of research completed and how teachers perceive their students in class. To conclude, I join the calls of other mathematics educators who suggest that in order to attain equity, teachers and researchers must first actively work to counteract deficit narratives about racialized students.

STATEMENT OF PURPOSE

K-12 mathematics classes exist amongst a myriad of policy documents that influence the focus on particular topics within our classrooms. Many of these policy documents, including the *Principles and Standards for School Mathematics* (NCTM, 2000), the *Common Core State Standards for Mathematics* (The Common Core State Standards Initiative [CCSSI], 2015), and current federal education legislation in the United States (U.S.), all purport to be working towards equitable educational goals. In particular, there is a focus on the existence of achievement gaps in mathematics education between racialized students and their white peers. However, while these policies aim for equity, ideals of racial justice are often missing from the implementation of policies. To that end, this paper explores my initial findings from U.S. federal education legislation to show how racialized students are framed within the legislation. I conclude with a look at how this framing currently impacts our mathematics teaching, education, and research while also looking beyond current policy to call for changes in how we address race as a way to encourage a positive and lasting impression on racialized students in our mathematics classes.

This research stems from a desire to ensure that all students have equitable access to mathematics by articulating how the U.S. accountability system shapes a societal understanding of achievement in mathematics. Through knowing and understanding how federal legislation limits the practice of mathematics education, there is an increased ability for mathematics educators, researchers, and teachers to create space for meaningful and creative mathematics in classrooms. More specifically, my research aims to explore how achievement and accountability narratives prompt particular deficit narratives around the mathematical ability of racialized students, which adds to

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 35-42). Umeå, Sweden: PME.

the master-narrative that racialized students cannot succeed in mathematics (Nasir, Atukpawu, O'Conner, Davis, Wischnia, & Tsang, 2009). The importance of focusing on racialized students and mathematics is derived from the mandatory requirements for mathematics, the prominence of mathematics within the curriculum, as well as the impact of gate-keeping that mathematics can have on students' future life choices. My pointed focus on African American students in particular is based on the fact that in the most recent NAEP data available (U.S. Department of Education, 2015), Black students once again had the lowest percentage of students able to gain a proficient status on the assessment which is meant to gauge the overall national competency of twelfth grade students in mathematics.

THEORETICAL FRAMEWORK

To provide my research with a rationale as well as a focus for my literature review, the types of questions I have asked, the choice of methods, and analysis, I rely on Critical Race Theory (CRT) and governmentality. The goal of CRT is to eliminate racial oppression as part of the larger project of eradicating all subordination in society (Berry, 2008; Gutiérrez, 2013; Taylor, Gillborn, & Ladson-Billings, 2016). CRT is used in education research to recognize and illustrate how race, racism, and the process of racialization have played a substantial role in education research, teaching, policy, and legislation (Taylor, Gillborn, & Ladson-Billings, 2016). The application of CRT in mathematics education, more specifically, involves acknowledging "how tracking practices, teacher expectations, intelligence testing, and other curricular practices have subordinated people of color" (Berry, 2008). The second part of my theoretical framework is governmentality, which according to Foucault (1991) relies on history to show how the governmentalization of the state occurred. In particular, this requires the use of a selective or partial history chosen specifically to follow a path of ideas and how they are defined over time in particular ways (Foucault, 1991). Taken together CRT and governmentality allow for a way to look at how government structures have been able to define how race is related to the goal of equity in mathematics education. Together these two frameworks establish a way to center race as a main element shaping the experiences of students and teachers within the K-12 school system, while also providing the conditions to look beyond current policies to encourage an increased potential to achieve equity in mathematics education.

REVIEW OF THE LITERATURE

Currently, there is limited research in mathematics education that deals explicitly with how racialized students and their experiences are conceptualized in mathematics classes. More often, the reference to race in many studies is directly related to the reporting of data, which includes observations addressing disaggregated data (Carroll, 1997; Dorn, 2007; Wei, 2012), or determining if policy mandates can be feasibly met (Koyama, 2012; Stiefel, Schwartz, & Chellman, 2007). However, this process of centering the disaggregating of data in research limits the ways in which race and racialization can be considered in the analysis of data. The disaggregation process is limiting because it ignores structural impediments and sociopolitical context. These barriers and the sociopolitical environment are disregarded when the experiences of all Black students are summed into a single number, i.e. the average score.

Some notable exceptions to the trend of relying solely on disaggregated data to explain racialized experiences in mathematics are Gutstein (2007), Gutiérrez (2000, 2008), Berry (2004), and Martin (2009a). These authors often work within equity or social justice as a way of framing their research with an explicit focus on race in mathematics education. One way that researchers use this lens to center race is to highlight a particular teacher or mathematics department that is successfully working to help racialized students achieve in mathematics (Gutiérrez, 2000; Martin, 2009a). Additionally, there are researchers who look at how racialized students interact with mathematics (Davis et al., 2007; Gutstein, 2007; Moses & Cobb, 2001). A second way that researchers look at how students interact with mathematics is to look at the larger structure of schooling and how racialized students are placed within that structure. This research involves highlighting historical and cultural mechanisms that continue to impact the perception of racialized students within the school system. The goal of this research is to alter historical patterns of disenfranchisement and create spaces for racialized students to succeed in mathematics education, work that is exemplified by the Algebra Project (Davis, et al., 2007; Moses & Cobb, 2001; Solórzano & Ornelas, 2002).

Another area of mathematics education research that deals explicitly with race, works in relation to teacher bias. This research deals with the structure of schooling by highlighting elements of the hierarchy that exists in mathematics education and is established through the teacher nomination process. Researchers observing this phenomenon want to determine how teachers are directly or indirectly influencing student promotion and achievement through the mathematical hierarchy (Berry, 2004, 2008; Faulkner, Stiff, Marshall, Nietfield, & Crossland, 2014; Riegle-Crumb, 2006; Riegle-Crumb & Humphries, 2012). One of the ways that researchers have looked at the indirect contribution to racialized students placement in mathematics is through teacher perceptions of which students belong at a particular level of the mathematical hierarchy (Berry, 2004, 2008; Riegle-Crumb & Humphries, 2012). This indirect influence of teachers is important to acknowledge because if teachers work within the master-narrative that racialized students cannot achieve in mathematics the teachers will bring that implicit bias to student recommendations, further influencing the life choices of racialized students.

The importance of this research cannot be overstated, however, when taking CRT and governmentality into consideration, there is a missing discussion of how structural elements outside of K-12 also play a role in delineating how racialized students are seen in mathematics education. In particular, I believe that a closer look at how racialized students are framed within U.S. federal education legislation can provide insight into how equity is conceived of in mathematics education research, and how it can be altered going forward. To that end, this research explores data that I have collected while completing my PhD research to answer the questions: How are racialized students positioned in accountability policies? And what does this mean for mathematics educators and researchers?

METHOD AND DATA SOURCES

The method used is Critical Discourse Analysis (CDA) which allows for a way to search for and analyze the underlying ideology inherent in education discourse (Fairclough, 2010). More specifically for this research, CDA works within CRT and governmentality to recognize and elaborate on the existence of race and racial terminology used in governing documents which then influence action that occurs in K-12 classrooms. As a way to explore how racialized students are framed within federal education legislation in the U.S. I chose to look at the Elementary and Secondary Education Act of 1965 (ESEA) and its subsequent reauthorizations through to the present version, the Every Student Succeeds Act of 2015 (ESSA). My choice of federal legislation is based on the fact that it exists at the same policy level as both the NCTM (2000) *Standards* and the *Common Core Standards* (CCSSI, 2010) which both function at a national level to outline potential standards for teaching and learning in mathematics. Additionally, since its original inception, ESEA through reauthorizations such as the No Child Left Behind Act of 2002 has had an increasingly direct impact on mathematics classrooms and the potential perceptions of students therein. Using these documents allowed me to search through publicly available policy for words that were both explicit in their reference to race, such as race, racial, color, Negro, black, and African American, and those words that might be considered implicit or coded references to race, such as minority, diversity, segregation, desegregation, and integration. The engagement with both past and present federal education legislation allows me to extract an overall understanding of how racial references have shifted over time, while also elaborating on how current framing allows for mathematics educators to engage with race more explicitly.

OBSERVATIONS

After gathering the information from all of the reauthorizations of ESEA, there emerged four temporal shifts based in the amount of both explicit and inferred racial language used within the legislation, presented in Table 1 below.

	1965-1970	1972-1978	1981-1988	1994-present
Pieces of legislation	4	4	3	3
Average use of racial terminology	0.75	70.5	14	62.3

Table 1: Breakdown of racial terminology within U.S. federal education legislation

The shifts in time and vocabulary also outline the fluctuating importance of race and racial terminology since ESEA was initially passed in 1965. The next few paragraphs outline important events which have occurred in the political landscape that help to explain why particular language changed, ending with a short outline of the themes present within the current reauthorization of ESEA.

During the first time period, from 1965-1970, federal education legislation mentioned race exactly three times. The number of occurrences is so small largely because the legislation and the Johnson administration relied on the Civil Rights Act of 1964 to prevent federal funding from going to racially segregated schools. This tactic of using the Civil Rights Act was meant to counteract Jim Crow Laws in the South and allow for more funding to go to economically deprived school districts all while keeping explicit references to race out of the legislation (Jennings, 2015). The drastic increase in racial terminology that presented itself in the 1972-1978 time period was a direct result of introducing the Emergency School Aid Act of 1972 to the legislation. This money was meant as a way to eliminate minority group isolation through the funding of magnet school initiatives and was meant as the main way for the federal government to encourage desegregated schools. Relatedly, racial terminology was almost exclusively kept to sections that dealt with desegregation. And while there were initiatives mentioned to increase access to mathematics at this time, none of these sections referenced race in either explicit or inferred terminology.

The 1980s saw President Reagan change tactics and attempt to completely eliminate federal responsibility for education generally and desegregation more specifically (Jennings, 2015). Thus there is a drastic decrease in the use of racial terminology, as well as the elimination of the Emergency School Aid Act from the reauthorizations of ESEA during the Reagan years. Finally, the time period from 1994 to the present saw the reestablishment of the Emergency School Aid Act maintaining connections to magnet schools and desegregation, but also saw an increase in racial terminology beyond those sections that was not present in earlier reauthorizations. For example, with increased language around accountability and achievement there came specific requirements for districts and states to outline how programs would have an impact on racialized students in particular.

When looking at ESSA on its own, there are four themes that emerge from the use of racial terminology which are mentions related to mathematics, reporting and data, teachers, and desegregation language. The theme of mathematics is associated with the use of explicit racial terminology twice, where both sections acknowledge that racialized students are underrepresented in mathematics classes. Reporting and data on the other hand, which had seven occurrences of racial language over three and a half sections, specify reporting requirements and categories for data disaggregation that mention race as one of the categories needed to receive funding. The two sections that deal with teachers indicate that money is to be used to increase and address who is teaching racialized students. Finally, the 37 mentions in eight sections that relate to desegregation, outline the importance of magnet schools to desegregation efforts, as well as priority guidelines to give more money to schools that increase racial diversity.

DISCUSSION

The ways in which racial terminology is used throughout the reauthorizations of ESEA, but especially in present legislation, outlines some of the ways in which mathematics education research and teaching can begin to reconceptualize the relationship between racialized students and mathematics. First, is the importance of being able to meaningfully link mathematics teaching and learning with race, especially within policy. Both of the sections that mention mathematics, do so to outline ways that states and local educational agencies can apply for money to help underrepresented groups receive a well-rounded education, which specifically links racialized students and mathematics. For K-12 mathematics teachers this is particularly important since the stated purpose of ESSA is “to close educational achievement gaps” (2015, p. 8), and one of the largest achievement gaps exists between black and white students in mathematics (NAEP, 2015). That being said, in order to move beyond perpetuating deficit narratives around the achievement gap, and instead taking the sociopolitical turn that Gutiérrez (2013) suggests, research associated with linking racialized students and mathematics should take into consideration larger societal discussions of race. For mathematics this would include looking beyond test scores to links between housing, income or wealth patterns, teacher turnover, and implicit bias as ways of acknowledging how systemic and structural issues related to race play out in test scores.

Second, while maintaining statistical information about who is teaching racialized students and how racialized students are performing on assessments, these reporting mechanisms need to go farther. By only collecting particular types of data, this process limits the ways in which mathematics education teachers and researchers can then engage with their students, because they are hyper focused on test scores. This is not to say that all teachers do this, but that research and policy give this impression when it is so often repeated.

Finally, given that an overwhelming majority of racial terminology continues to target desegregation suggests that despite almost 50 years of explicit attempts to integrate educational facilities in the U.S. segregation is still a problem. Therefore, while a bit beyond what can be achieved in the realm of this study, this finding highlights the need for future research in educational policy around desegregation and its impact on mathematics education.

CONCLUSION

It is easy to call for the end of achievement gaps and to work towards equitable educational goals, however as Rochelle Gutiérrez (2013) suggests, when research and policy becomes detached from issues around power, it becomes much more difficult to actually make the changes being sought after. Therefore, as mathematics educators, practitioners, researchers, and policymakers striving for equity there needs to be more acknowledgement of how students are framed in legislation as a way to alter our preconceptions of racialized students in mathematics classrooms. This research joins the calls for a continued explicit discussion of race and its varied, but very real, impact on racialized students in our mathematics classrooms emphasized by some researchers (Gutiérrez, 2008, 2013; Martin, 2009b). Furthermore, the discussion highlights the need to discover alternative ways to discuss students' mathematical knowledge so that the master-narrative that racialized students, and black students in particular, are incapable of doing mathematics is not continually reinforced.

References

Berry, R. Q., III (2004). The equity principle through the voices of African American males. *Mathematics Teaching in the Middle School, 10*(2), 100-103.

Berry, R. Q., III (2008). Access to upper-level mathematics: The stories of successful African American middle school boys. *Journal for Research in Mathematics Education, 39*(5), 464-488.

Carroll, W. M. (1997). Results of third-grade students in a reform curriculum on the Illinois state mathematics test. *Journal for Research in Mathematics Education, 28*(2), 237-242.

Common Core State Standards Initiative (CCSSI). (2015). *Common core state standards for mathematics.* Retrieved from: http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Davis, F. E., West, M. M., Greeno, J. G., Gresalfi, M. S., & Martin, H. T. (with R. Moses & M. Currell). (2007). Transactions of mathematical knowledge in the Algebra Project. In N. S. Nasir & P. Cobb (Eds), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 69-88). New York: Teachers College Press.

Dorn, S. (2007). *Accountability Frankenstein: Understanding and taming the monster*. Charlotte, NC: Information Age Publishing.

Elementary and Secondary Education Act of 1965, Pub. L. 89-10, 79 Stat.

Every Student Succeeds Act of 2015, Pub. L. 114-95.

Fairclough, N. (2010). *Critical discourse analysis: The critical study of language* (2nd ed.). Routledge.

Faulkner, V. N., Stiff, L. V., Marshall, P. L., Nietfield, J., & Crossland, C. L. (2014). Race and teacher evaluations as predictors of algebra placement. *Journal for Research in Mathematics Education, 45*(3), 288-311.

Foucault, M. (1991). Politics and the study of discourse. In G. Burchell, C. Gordon, & P. Miller (Eds.), *The Foucault effect: studies in governmentality* (pp. 53-72). Chicago, IL: The University of Chicago Press.

Gutiérrez, R. (2000). Advancing African-American, urban youth in mathematics: Unpacking the success of one math department. *American Journal of Education, 109*(1), 63-111.

Gutiérrez, R. (2008). A "gap-gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education, 39*(4), 357-364.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *The Mathematics Teacher, Equity Special Issue, 44*(1), 37-68.

Gutstein, E. (2007). "And that's just how it starts": Teaching mathematics and developing student agency. *Teachers College Record, 109*(2), 420-448.

Jennings, J. (2015). *Presidents, Congress, and the public schools: The politics of education reform.* Cambridge, MA: Harvard Education Press.

Koyama, J. P. (2012). Making failure matter: Enacting No Child Left Behind's standards, accountabilities, and classifications. *Educational Policy, 26*(6), 870-891.

Martin, D. B. (2009a). *Mathematics teaching, learning, and liberation in the lives of black children*. New York: Routledge.

Martin, D. B. (2009b). Researching race in mathematics education. *Teachers College Record, 111*(2), 295-338.

Moses, R. P. & Cobb, C. E., Jr. (2001). *Radical equations: Math literacy and civil rights.* Boston, MA: Beacon Press.

Nasir, N. S., Atukpawu, G., O'Connor, K., Davis, M., Wischnia, S., & Tsang, J. (2009). Wrestling with the legacy of stereotypes: Being African American in math class. In D. B. Martin (Ed.), *Mathematics teaching, learning, and liberation in the lives of black children* (pp. 231—48). New York: Routledge.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for schools mathematics*. Reston, VA: Author.

No Child Left Behind (2002). No Child Left Behind Act of 2001, Pub. L. 107-110.

Riegle-Crumb, C. (2006). The path through math: Course sequences and academic performance at the intersection of race-ethnicity and gender. *American Journal of Education, 113*, 101-122. ISSN-01956744

Riegle-Crumb, C. & Humphries, M. (2012). Exploring bias in math teachers' perceptions of students' ability by gender and race/ethnicity. *Gender & Society, 26*(2), 290-322.

Solórzano, D. G., & Ornelas, A. (2002). A critical race analysis of advanced placement classes: A case of educational inequity. *Journal of Latinos and Education, 1*(4), 215-229.

Stiefel, L., Schwartz, A. E., & Chellman, C. C. (2007). So many children left behind: Segregation and the impact of subgroup reporting in No Child Left Behind on the racial test score gap. *Educational Policy, 21*(3), 527-550

Taylor, E., Gillborn, D., & Ladson-Billings, G. (Eds.) (2016). *Foundations of critical race theory in education* (2nd ed.). New York: Routledge.

U. S. Department of Education, National Assessment of Educational Progress (NAEP). (2015). National achievement level results: Mathematics grade 12. Retrieved from: https://www.nationsreportcard.gov/reading_math_g12_2015/#mathematics/acl

Wei, X. (2012). Are more stringent NCLB state accountability systems associated with better students outcomes? An analysis of NAEP results across states. *Educational Policy, 26*(2), 268-308. doi: 10.1177/0895904810386588

CAN STUDENTS CONSTRUCT NON-CONSTRUCTIVE REASONING? IDENTIFYING FUNDAMENTAL SITUATIONS FOR PROOF BY CONTRADICTION

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The purpose of this study is to identify and empirically corroborate a fundamental situation (Brousseau, 1997) for constructing "proof by contradiction." We identified the four elements of a fundamental situation: i) obtaining strong conviction; ii) negating the given proposition naturally without being aware of the assumption; iii) finding a contradiction easily; and iv) noticing the origin of the contradiction. Based on this study, a new research question arises: How can students construct "proof by contradiction" using teacher support?

INTRODUCTION

"Proof by contradiction" (PbC) is one of the most valuable types of reasoning in mathematics and mathematics education. However, students have specific cognitive and didactic difficulties in negating propositions and using laws such as the excluded middle (Antonini & Mariotti, 2008). Thus, although some authors have proposed didactic suggestions to help students overcome PbC difficulties (e.g., Wu Yu Lin & Lee, 2003; Antonini & Mariotti, 2008), in our opinion, many students are still unable to resolve these difficulties. One possible reason for this may be an overlooked component in the studies of students. In other words, almost all students who are analyzed in studies of PbC are either supplied PbC by their teachers before they engage in constructing PbC for the first time, or they have already been taught PbC before they engage in research.

In contrast, we believe that in order to understand a concept, students must construct knowledge by themselves (with their teacher's support). We assume that students cannot fully understand a concept if teachers or others tell them about it beforehand. Therefore, suggestions provided by the previous studies are inadequate as they are derived from observations of students whose understanding of PbC is not sufficient. In clarifying the conditions that enable students to construct PbC by themselves with their teacher's support, findings of previous studies become more meaningful, paving the way for elaboration and further research. Thus, our study aims to do the following:

P1: To identify a fundamental situation (Brousseau, 1997) for constructing PbC

P2: To corroborate the identified fundamental situation empirically

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 43-50). Umeå, Sweden: PME.

THEORETICAL BACKGROUND AND METHODOLOGY

The theoretical background for this study is based on the Theory of Didactical Situations (TDS; Brousseau, 1997), and the methodology adopted is didactical engineering, particularly *a priori* and *a posteriori* analysis (Artigue, 1992) within the framework of TDS. We used TDS because it is one of the most scientific theories in the discipline. Learning is defined in TDS as follows: "The student learns by adapting herself to a *milieu* which generates contradictions, difficulties and disequilibria, rather as human society does. This knowledge, the result of the student's adaptation, manifests itself by new responses which provide evidence of learning" (Brousseau, 1997, p. 30, italics in the original). This definition aligns with our assumption that students must construct knowledge by themselves.

TDS assumes that students construct mathematical knowledge in didactical or adidactical situations. Since any mathematical knowledge has been historically incubated in some situation, there always exist situations wherein it can be constructed. Because not all situations are replicable in educational settings, TDS assumes that *all* mathematical knowledge has at least one fundamental situation (FS) that can become a didactical situation (Brousseau, 1997, p. 30). However, FSs are not always easily identified by mathematics educators, and PbC does not typically employ constructive reasoning (in the sense of *intuitionism*). Thus, an FS for constructing PbC has not yet been identified. In TDS, on identifying an FS based on theory, we corroborate it through *a priori* and *a posteriori* analyses: first, by designing a didactical situation based on the FS (*a priori* analysis); second, by trying to realize this situation in an actual mathematics classroom; and third, by corroborating our hypothesis about the FS underlying the design.

FUNDAMENTAL SITUATION OF PROOF BY CONTRADICTION

Indirect argumentation seems to be a natural way of thinking (Freudenthal, 1973, p. 629). Thus, an FS for constructing PbC should enable students to employ indirect argumentation and develop this into a PbC. However, previous research suggests that ruptures between indirect argumentation and PbC may occur. Mathematicians and mathematics educators have pointed out the specific difficulties of PbC (e.g., Wu Yu Lin & Lee, 2003; Antonini & Mariotti, 2008); we distinguish between three types here in order to identify our FS.

D-I: Difficulties in considering PbC as an option and in carrying out the method of PbC

When students try to prove a proposition, they usually do not consider using indirect proof, including PbC, as an option. Although they may consider PbC suitable for proving a given proposition, they often give up constructing PbC mid-way. Several difficulties in the process have been reported: negating the proposition, formalizing and interpreting the negation (Wu Yu, Lin & Lee, 2003), finding a contradiction, and so on.

D-II: Difficulties in accepting the result of a PbC

Even if one is able to prove a proposition using PbC, the result may not seem acceptable: “I think this is one source of frustration, of the feeling that we have been cheated, that nothing has been really proved, that it is merely some sort of a trick—a sorcery—that has been played on us” (Leron 1985, p. 323).

D-III: Difficulties in grasping the very structure of PbC

PbC has a specific structure, that is, when one assumes the negation of a true proposition *P*, a contradiction comes into being implying that the negation is false and *P* is true. Thus, one needs to know the *theory* and the *meta-theory* (Antonini & Mariotti, 2008) of PbC.

In Japan, students engage in PbC in mathematics when they are in the 9^{th} grade and learn that the square root of 2 is irrational. However, since they have not been introduced to PbC until then, they face D-I, D-II, and D-III all at once. This confuses them. Additionally, knowing the structure of PbC is necessary for overcoming D-I and D-II, that is, students must have already overcome D-III to resolve D-I and D-II. Therefore, before students engage in PbC, they should engage in PbC in FSs in which they are required to face and overcome only D-III.

In this study, we focus on an insight from Dawkins & Karunakaran (2016), according to which, research on student learning of mathematical proofs should pay greater attention to the role of mathematical content. Thus, in order to avoid D-II, FSs for PbC should enable students to surmise that the proposition to be proved is true. For example, students who have already accepted that the square root of 2 is irrational have less trouble accepting the PbC in order to prove it (Antonini & Mariotti, 2008, p.407). In addition, in order to avoid the emergence of D-I, an FS should enable students to negate the sentence naturally and formalize the proposition to be proved. Such situations enable students to find a contradiction easily because they autonomously begin to enquire into what statements can hold in the false world. Items (i) – (iii) (Figure 1) are a summary of the above consideration.

A fundamental situation (FS) for constructing proof by contradiction is one in which students must do the following four things:

(i) Be strongly convinced that the proposition to be proved is true

(ii) In investigating the *milieu*, they must construct a false world by naturally assuming the negation of the proposition (without being aware of the assumption).

(iii) Easily find a contradiction in the false world

(iv) Notice that they make the assumption themselves and that this is the origin of the contradiction

Figure 1: A fundamental situation for constructing proof by contradiction.

However, even if a student is able to find the contradiction and conclude that a proposition is true, s/he may still reason this using indirect argumentation rather than indirect proof. Because the core of PbC lies explicitly in assuming the negation of a true proposition, students must make such assumptions after they negate and formalize propositions. In order to do this, students must identify the origins of a contradiction. Thus, we have added (iv) to Figure 1.

Figure 1 is our proposal for a possible fundamental situation for constructing PbC. In the next section, we corroborate this by *a priori* and *a posteriori* analysis.

DESIGN AND *A PRIORI* ANALYSIS

The subjects of our analysis are 9th grade students who come across PbC for the first time (as mentioned earlier). These students have already learned basic direct proofs in geometry and algebra, algebraic skills and concepts, and the notion of irrational numbers. They have also learned—but not proven—that the square root of 2 cannot be represented as p/q (where p and q are disjointed integers and q is not equal to 0). In their textbook, PbC is introduced in order to prove this. We thus designed a mathematics lesson as shown in Figure 3. The teaching protocol employed in this lesson followed the "problem-solving lesson" model presented in Figure 2.

Our experimental lesson was conducted in June 2016 in a junior high school attached to a national university. This experiment was conducted during one lesson (50 minutes) on 40 students (20 males/20 females). The teacher was the students' regular mathematics teacher, and is one of the authors of this study as well. We did not investigate students' pre-conceptions, because such an investigation may affect students' performance in the study. However, our reflection on the experiment revealed that none of the students seemed to know PbC well before the experiment; even after students found a contradiction, they did not to try to construct PbC by themselves. Instead, they all needed the teacher's support to shift from indirect argumentation to indirect proof.

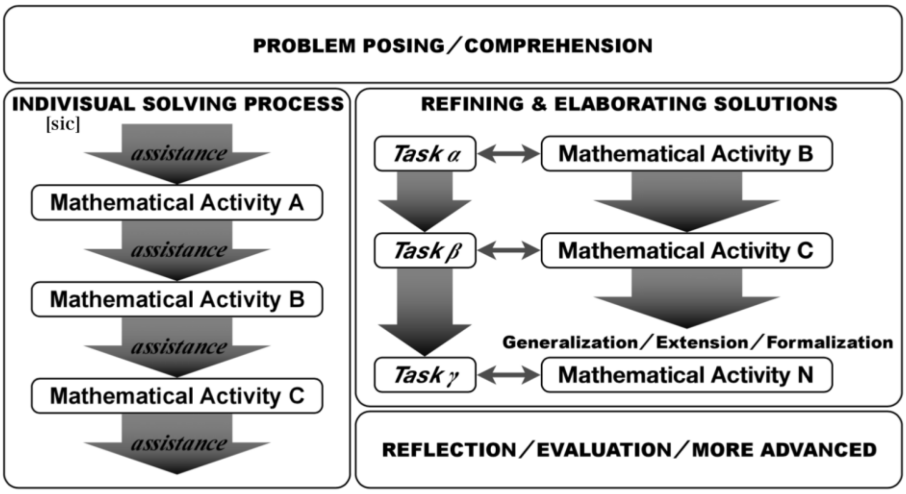

Figure 2: Lesson model (Mizoguchi, 2015, p. 627; reprinted with permission).

TS: Teacher's support

Problem

Let a,b be rational numbers. Do there exist a,b such that $a+b\sqrt{2}=0$? If these do exist, show all a,b and explain why there are no other. If these do not exist, explain the reason. (It is known that $\sqrt{2}$ is irrational number.)

Mathematical Activity C

Students infer the answer is only $a=b=0$ by inserting any value into a,b.

TS1: Is it just that you cannot find it?

TS2: Can you explain the reason ?

Mathematical Activity B–1

Students observe $a=-b\sqrt{2}$, and they become aware of the fact that the right side is a rational number, and the left side is an irrational number.

TS1: Can you show that $-b\sqrt{2}$ is irrational number ?

TS2: If $b=0$, so? Can you use known knowledge by using deformation of the formula?

Mathematical Activity B–2

Students observe $-\frac{a}{b}=\sqrt{2}$, and they become curious about the fact that the right side is a rational number, and the left side is an irrational number.

TS1: Can you explain your curiousness around the inference?

TS2: Can you find the root of the curiousness?

TS3: If $a=b=0$, can you obtain $-\frac{a}{b}=\sqrt{2}$?

Mathematical Activity A

Students become aware that, if one assumes $b\neq 0$, there appears the curiousness. For this reason, they conclude that the assumption is not correct thus the answer is only $a=b=0$.

TS1: Can you explain why the answer is only $a=b=0$?

TS2: What is the structure of your explanation?

Figure 3: Lesson designed to corroborate the FS identified in this study[1].

RESULTS AND *A POSTERIORI* ANALYSIS

In the lesson, the teacher posed the problem to the students and shared with them the property that the square root of 2 cannot be represented as a common fraction. We obtained data from video recordings and the students' worksheets. Only the problem and name fields are written in their worksheets. We banned eraser use so that we could examine all the ideas that students produced. During the "individual solving process" phase (Figure 2), students tried to solve the problem on their worksheets, and the teacher supported them verbally and individually, following the plan in Figure 3. The teacher was careful to align his support appropriately in keeping with the students' levels of progress. In the "refining and elaborating solutions" phase, the teacher picked students to present their own solutions (in the order of the mathematical activities C, B-1, B-2, and A) and all the students refined and elaborated their own solutions through discussion involving the entire class.

(a) When we solve $a+b\sqrt{2}$,

$b\sqrt{2}=-a$, then $\sqrt{2}=-\frac{a}{b}$

Both a and b are rational numbers…

(b) If there are any a and b that satisfy $a+b\sqrt{2}$

When we solve $a+b\sqrt{2}$,

$b\sqrt{2}=-a$

$\sqrt{2}=-\frac{a}{b}$

Both a and b are rational numbers. So $-\frac{a}{b}$ is a rational number too. Thus, $\sqrt{2}$ is a rational number too; however this contradicts the fact that $\sqrt{2}$ is an irrational number, so there are no a and b that satisfy $a+b\sqrt{2}$

$(a,b)=(0,0)$

(c) If there are any a and b that satisfy $a+b\sqrt{2}$ $(\underline{a\neq 0, b\neq 0})$

When we solve $a+b\sqrt{2}$,

$b\sqrt{2}=-a$

$\sqrt{2}=-\frac{a}{b}$ If $b=0$, I can not divide both sides, then $\sqrt{2}=-\frac{a}{b}$ by b, so I assume $\underline{b\neq 0}$.

Both a and b are rational numbers.

So $-\frac{a}{b}$ is a rational number too. Thus, $\sqrt{2}$ is a rational number too; however this contradicts the fact that $\sqrt{2}$ is an irrational number, so there are no a and b that satisfy $a+b\sqrt{2}$ when $a\neq 0, b\neq 0$.

Next, I insert $a=0$ into $a+b\sqrt{2}$, so $0+b\sqrt{2}$. Thus, $b\sqrt{2}=0$, so $\underline{b=0}$.

From this result, if we insert $b=0$ into $a+b\sqrt{2}$, it becomes $(a,b)=(0,0)$ too.

For above reasons, the answer is only $(a,b)=(0,0)$.

Figure 4: Male student Y's worksheet (translated into English by the authors, underlined by the student; (a), (b), and (c) added by the authors for convenience).

In the experimental lesson, all the students completed mathematical activity C successfully, and almost all the students completed B-1 or B-2 successfully in the first phase, that is, they found a contradiction (although some students described it as "strange"). Student Y (male) is one of the students who successfully constructed PbC. Figures 4 is an example of students' answers (translated here from their native language). In this example, the teacher supported him in constructing PbC (activity A), but PbC seemed difficult for him. In the "refining and elaborating solutions" phase, Student Y's presentation was mathematically sound and hence was accepted by the other students (See Figure 4 (c)). Next, the teacher presented: "When we need to prove a supposition, if we assume the opposite to be true and derive a contradiction, then, the initial supposition to be proved is considered true. We call this method 'proof by contradiction.'"

Here, let us focus on Student Y's problem-solving process. As soon as the "individual problem-solving process" phase began, Student Y thought the answer was only $(a, b) = (0,0)$ and that $\sqrt{2}=-\frac{a}{b}$ was contradictive. To indicate this, he wrote (a), as shown in Figure 4. However, he was puzzled by the contradiction and wrote, "Both a and b are rational numbers…" Thus, the teacher supported him by following TS-1 for B-2 in

Figure 3. Five minutes later, he finished writing indirect argumentation (b). Although it was a persuasive argument, he did not pay attention to his implicit assumption that ($a \neq 0, b \neq 0$). Hence, the teacher supported him by following TS-2 for B-2 in Figure 3. Ten minutes later, he finished writing a mathematically acceptable PbC (c). While in the "refining and elaborating solutions" phase, Student Y explained (c) to the other students after another student had explained B-2. However, some students could not find the essential difference between these two explanations. Thus, the teacher asked all the students, "The explanation by Y is very similar to another explanation (B-2). What is the important difference between them?" and asked Student Y to explain it. Student Y said, "Umm… $-\frac{a}{b}$, oh, sorry. Well… there is $b\sqrt{2} = -a$ in my explanation, well… we cannot divide $b\sqrt{2}$ by b" (the original was spoken in his native language), and Student Y pointed out that the assumption $b \neq 0$ is important. This showed that he noticed the importance of assuming negation of the proposition to be proved.

Student Y's problem-solving process (shown by (a), (b) and (c)) was in accordance with our design. Three observations support this claim: first, in (a), he surmised that the solution was only $(a, b) = (0,0)$ and found a contradiction in a false world, where the negation of the proposition to be proved was assumed; second, he made an indirect argument (b); and finally, he developed (b) into (c), that is, PbC, by detecting the origin of the contradiction and noticing that the negation of the true proposition was implicitly assumed. Thus, these empirical observations corroborate the fact that our designed lesson can produce a didactical situation and that our proposed situation in Figure 1 is an FS for constructing PbC.

IMPLICATION

The purpose of this study was not to design a "good" lesson, but to identify an FS for constructing PbC, and to corroborate it. Therefore, although not all the students were able to construct PbC by themselves in this lesson, the value of our findings cannot be undermined. Given the fact that Student Y (and some other students) constructed PbC by themselves (with the teacher's support), we may conclude that Figure 1 is valid as an FS. Designing a "good" lesson according to Figure 1 is thus a future task for mathematics teachers rather than for researchers. Our findings also imply a new research question: How can students construct PbC by themselves with their teacher's support? Future researchers investigating students' cognitive and didactical difficulties with PbC should expand their foci to the processes of construction of PbC by learners. Researchers should also investigate the differences between the processes underlying success and failure in constructing PbC.

We have three future tasks. First, we must investigate the processes of students who construct PbC by themselves, especially to examine whether or not they are able to use PbC by themselves, with their teacher's support (D-I), and whether or not they accept the results of PbC (D-II). Second, we must identify fundamental situations for overcoming D-I and D-II. In other words, we must design curriculum for understanding

PbC. Third, we must investigate the effects of applying previous studies' didactical suggestions to our teaching practices.

Notes

[1] They do not know that $-b\sqrt{2}$ is irrational. Thus, when students solved it in accordance with B-1, we supported their shift to B-2.

Acknowledgments

This work was partially supported by JSPS KAKENHI Grant Number 17H00147.

References

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: what is specific to this way of proving?. *ZDM*, *40*(3), 401-412.

Artigue, M. (1992). Didactical engineering. In R. Douady & A. Mercier (Eds.), *Research in Didactique of Mathematics: Selected papers* (pp. 41-65). Grenoble: La Pansée Sauvage.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics* (V. Warfield, N. Balacheff, M. Cooper & R. Sutherland, Trans.). Berlin, German: Kluwer.

Dawkins, P. C., & Karunakaran, S. S. (2016). Why research on proof-oriented mathematical behavior should attend to the role of particular mathematical content. *The Journal of Mathematical Behavior*, *44*, 65–75.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

Leron, U. (1985). A Direct approach to indirect proofs. *Educational Studies in Mathematics*, *16*(3), 321- 325.

Mizoguchi, T. (2015). Functions and equations: Developing an integrated curriculum with the required mathematical activities, *Proceedings of the 7th ICMI-East Asia Regional Conference on Mathematics Education*, 625-637.

Wu Yu, J.-Y., Lin, F.-L., & Lee, Y.-S. (2003). Students' understanding of proof by contradiction. In N.A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (Vol. 4, pp. 443–449).

MATHEMATICS DISCOURSE IN SMALL GROUPS

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This study of secondary classrooms examined students' mathematics discourse in small group learning environments. Audio-recorded conversations from naturalistic observations of classrooms provided data for investigating the learning environments students created and experienced in their small groups. The discourse framework and related coding scheme we utilized revealed key differences in the frequency and quality of students' explaining and questioning.

FOCUS OF THE STUDY

Small group learning environments promote opportunities for conceptual learning and powerful mathematical work (e.g., Mercer, 2005; Veenman, Denessen, vanden Akker, & van der Rijt, 2005). Facilitating students' group work presents challenges for teachers because their influence on what transpires is indirect. In this paper we report on foundational work of the Peers Engaged as Resources for Learning study of small group learning environments in secondary mathematics classrooms, addressing the research question: How can mathematics discourse among students working in small groups be characterized to reveal differences in the frequency and quality of their explaining and questioning about the mathematical work?

CONCEPTUAL FRAMING AND RELATED LITERATURE

We conceptualize the small group learning environment to comprise three major elements: the mathematics task (Stein, Grover, & Henningsen, 1996), the discourse related to the mathematics content (Sztajn, Heck, & Malzahn, 2013), and the social peer dynamics among the group members (Hamm & Hoffman, 2016). In this sense, we are broadly interested in what Ryve (2011) distinguishes as Discourse (the culture), in that the small group learning environment is a micro-culture in the classroom. It is shaped by and, in turn, shapes these three elements to constitute the opportunities students have to learn mathematics by engaging with content and with one another. The focus in this investigation, though, is the mathematics discourse (the conversation), so that our attention here, using Ryve's descriptors, is first on *discourse as Such*, because we seek to describe the frequency and quality of explanations and questions that occur in conversations among students working in small groups. We reiterate, however, that our attention to mathematics discourse always pertains its role in shaping students' engagement and opportunities; that is, we investigate discourse because it is one of the vital factors that forms the learning environment in each small group. In this sense, our study also examines *discourse as Constitutive* of small group learning environments.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 51-58). Umeå, Sweden: PME.

Our operational definition of discourse includes four dimensions: explaining, questioning, listening, and using multiple modes of communication (Sztajn, Heck, & Malzahn, 2013). We are interested in how working in small groups promotes, hinders, or otherwise influences the expression, exploration, interrogation, and representation of mathematics ideas among students through their communication. Here we focus on communication in speech, so we limit our attention to explaining and questioning.

Explaining is declarative speech, which may be tentative or definitive, through which students state an idea. We accept speech that is specifically intended to share the idea with other students or speech that may essentially be a student talking to her/himself, because in either case the act of speech allows other students access to the idea. Explaining can take many forms in terms of its mathematical content. Students may simply state a mathematical result or answer (e.g., The area is 10.). With or without providing a specific result, a student may name or describe a mathematical procedure or may voice the procedure as it is being applied in the course of working on the task (e.g., I found the average.). In explaining, a student may share a mathematical justify-cation for an answer or a procedure (e.g., The range will increase because we added an outlier.). Noting or describing mathematical comparisons or connections, among different answers/procedures or between an answer/procedure and the context of the task, or a context used as an analogy or example, (e.g., My multiplication and your addition account for the same parts.) are also forms of explaining.

Questioning is interrogative speech, which may be asked of another group member, or oneself, or may essentially be undirected. Questioning can represent uncertainty or doubt on the part of the speaker, who may be expressing uncertainty about her/his own thinking or about another student's idea. It may also represent a general or specific invitation for another student to respond. Questioning takes various forms, parallel to explaining, in terms of mathematical content. That is, a question may ask for an answer or procedure, which can be in closed form (e.g., What does x equal? Should we multiply?) or open form (e.g., How did you find x?). Questioning may also seek justification for an answer or procedure (e.g., Why did you multiply?), or may ask for a connection (e.g., How does dividing relate to the problem?).

Working on mathematics in small groups creates a unique learning environment for students. In it, they are able to share their own thinking and have access to the thinking of peers. By sharing their ideas, students can refine their thinking in terms of precision, justification, and meaning making (Barron, 2003; O'Donnell, 2006; Webb & Palinscar, 1996). The small group learning environment also shapes the experience of mathematics itself. In this environment, communication becomes an essential part of knowing and doing mathematics (Sfard & McClain, 2002). Attending to explaining and questioning in students' conversations provides a window on their engagement with the mathematics content and with mathematics communication among peers (Moschkovich, 2007), offering insights into the learning opportunities the small group learning environment affords (Zahner and Moschovich, 2010).

METHODS

Data for the study were taken from the naturalistic phase of a multi-year study, when 6th, 7th, 8th, and 9th grade mathematics teachers and their students were observed and audiorecorded as they engaged in small group work as part of each teacher's own lesson plans. Qualitative analysis of audiorecords utilized a codebook for capturing the presence and prevalence of various kinds of student talk, but 1-minute intervals, that occurred in the small group setting. Quantitative results consider the frequency and patterns of talk for various student groups.

Sample and context.

Study participants included eleven volunteer middle and high school teachers from one rural and low-resourced, and one municipal and well-resourced school district in the Southeastern US. Each teacher identified one to three classrooms in which they used small group work, resulting in three 6th, six 7th, twelve 8th, and six 9th grade classrooms engaged in the study. These classrooms served a mixture of African-, Asian-, Latino-, and White-American students, and a few students who recently immigrated to the US.

In one class period in each classroom, the entire class period was observed by two researchers and audiorecorded using one recorder for each small group of students and one recorder that the teacher wore. Across classrooms, 161 small groups were observed and recorded; group size ranged from 2 - 5 students. About half of the groups (56.5%) were mixed gender.

All observed lessons followed the teacher's lesson plan without intervention on the part of the research team. Accordingly, student groups worked on a variety tasks appropriate to the grade level and content focus and sequence of their courses. Tasks included, for example: (1) finding areas and perimeters of circular and rectangular parts of a stained glass window, (2) finding the volume and surface area of a cylinder and a tube, (3) finding missing angle measures in various kinds of triangles, (4) analyzing central tendency and spread of data distributions, (5) analyzing quantities in two-way frequency tables and Venn diagrams, (6) modelling situations with linear relationships, and (7) comparing different representations of linear relationships.

We assigned the 161 group recordings, stratified by classroom, to one of three coding and analytic phases. To address the research question for this study, we analyzed 26 of the 54 recordings assigned to the first phase (to be completed for presentation), which is designed to establish coding definitions for *a priori* codes suggested by theory/research and to identify and define potential emergent codes. We used time sampling (1-min intervals) to capture the frequency of occurrence of each code.

Analysis.

Analysis of small group episode recordings drew on a coding scheme adapted from the Mathematics Discourse Matrix (Sztajn, Heck, & Malzahn, 2013), which provides indicators of student talk that characterize types of explaining and questioning.

The explaining category includes six codes for capturing ways students might share their mathematical thinking or work in the small group context. “Explaining: Answers,” applies when students share or simply acknowledge/verify answers, correct others’ answers or express a value judgment about other students’ mathematical contributions without elaboration. “Explaining: Procedures” indicates when students name or describe their methods, procedures, or procedural ideas, including restating or building upon a solution method that was already shared. “Explaining: Justifications” is used for statements providing a reason or rationale for an answer, procedure, or broader idea, including restating justifications made by others. “Explaining: Connections to students’ work” applies when students make mathematical connections across their own and/or others’ explanations, state generalizations, or compare work to identify similarities or differences, including resolving differences. “Explaining: Connections to context” indicates when students make a connection between their work and the context of the problem or use an analogy or a context, or another mathematics idea, to make sense of a problem. When students share an explanation that is difficult to follow or has an unclear purpose, “Explaining: Ambiguous” is used.

The questioning category similarly provides six codes for questions that could be observed during small group work. “Questioning: Short-response,” describes questions designed to establish correctness of an answer, procedure, or idea, to lead to correct answers, or to verify steps in a procedure. This code also applies when students ask questions to clarify term(s) used in a solution method or idea that another student shared. “Questioning: Open-ended,” applies to questions that invite elaboration about answers, procedures, or other ideas. “Questioning: For justifications,” applies to questions designed to elicit reasons or rationale for answers, procedures, or other ideas. “Questioning: For connections to students’ work,” is applied to questions related to connecting/comparing across the group’s mathematical ideas, including identifying similarities and resolving differences. “Questioning: For connections to context,” applies to questions intended to relate ideas to the context of the problem, seek or consider an analogy, or connect to other mathematics ideas. Questions that do not have a clear purpose, or whose meaning is not clear, are coded “Questioning: Ambiguous.”

Students’ talk in small groups often includes not only explanations and questions, but also statements/questions relating to the requirements of the mathematical task as an assignment. Three codes were added to identify comments students make to manage the mathematics task in the small group: “Managing: Reading the task” (verbatim) “Managing: Restating the task” (in own terms) and “Managing: Reporting progress” (what is complete or still to do, what is understood or not understood).

Students in small groups also engage in non-mathematical, or “off –task”, talk. The occurrence of such talk is very frequent; the majority of one minute intervals included an instance of non-mathematical talk. Periods of silence or uninterpretable utterances are also frequent. The codes “Non-mathematical talk” and “No talk” were limited to only minutes in which no Explaining, Questioning, or Managing talk was evident.

The resulting Small Group Discourse Codebook was developed and revised by a team of three researchers to address the study's research question. The researchers collectively tested the codebook with four randomly selected recordings from another part of the study to develop common understandings of *a priori* codes, inform revisions and additions, and identify illustrative examples for each code.

After establishing codes for observable talk that were consistently interpreted across researchers, randomly selected recordings were coded by pairs of researchers. The team met to reconcile coding and further refine the codebook to strengthen consistency of interpretation. Average agreement was 75.3% in round 1 and 90.2% in round 3 of the training, which completed coding for 8 recordings (4 per district). Given the high level of agreement, further coding proceeded by randomly assigning recordings to individual researchers, with 20% being double coded to ensure continued reliability. Agreement on double-coded recordings ranged from 75% to 81%. The data reported here come from coding of 26 recordings, 13 from each district, with one excluded because the recorder had been turned off early in the class period.

RESULTS

The complete episode of small group work in each classroom was treated as the unit of investigation, divided into single minutes for coding. The relative time devoted to small group work varied according to teachers' lesson plans, ranging from about a third to nearly the entire length of the class period. Since the length of class periods varied considerably (28 to 85 minutes, most either 47 or 85 minutes) and available time for group work also varied (27 to 57 minutes), we present results as percentages of available group work time in the class period that received each code of interest.

Across all recordings, students were engaged in talk about their assigned task 87% of the available time, on average, ranging from 40% to 100%. Considering the broadest categories, an average of 65% (ranging from 14% to 100%) of the available time included talk coded as explaining. An average of 43% of the time included talk coded as questioning (from 12% to 89%). An average of 30% of the time included talk coded as managing the task (from 0% to 81%).

In terms of explaining, providing only answers (58%) or procedures (31%) were the most frequently occurring types. Offering justifications occurred, on average, 18% of the time, ranging from 0% to 59%. Identifying connections among students' work occurred an average of just 6% of the time, ranging from 0% to 43%; and connections to a broader context, on average only 2% of the time, ranging from 0% to 19%.

Considering students' questioning, the most frequently occurring types were short-answer (an average of 32% of the time, ranging from 12% to 73%) and open-ended questions (average 9%, from 0% to 25%) that called for only answers or procedures in response. Questions seeking justification occurred on average just 5%, with a range from 0% to 30%, of the available time. No questions in any group called for a response involving connections among students' ideas or to a broader context.

Examining the full episodes of group work to identify differences in the frequency and nature of talk among groups, several notable patterns emerged. These patterns are distinguished by the percent of available time in which explanations and questions involved either justification or connection (among student ideas or to a broader context of the problem), which we considered deeper instances of talk because they engage students in thinking beyond answers and procedures. Examining these deeper instances led to five patterns identified among the 25 group episodes (also see Table 1).

In 5 episodes both explaining and questioning occurred in 59% or more of the available time; deeper instances were found at least fairly often in both explaining and questioning, 12% to 59% of the available time. These episodes provided *Very Frequent Opportunities* for deeper learning through talk.

In 3 episodes both explaining and questioning occurred at least 40% of the time, and deeper instances, almost all involving justification rather than connection, were found fairly often, 17% to 38% of the available time, in either questioning or explaining, but occurred on only limited occasions in the other category. These episodes provided *Frequent Opportunities* for deeper learning through talk.

In 5 episodes explaining and questioning occurred at least 24% of the time, and deeper instances were found either fairly often in one category but not at all in the other, or were found in limited instances in both categories. These episodes provided *Occasional Opportunities* for deeper learning through talk.

In 7 episodes both explaining and questioning occurred at least 14% of the time; deeper instances were identified in limited cases in one or the other category, but not in both. These episodes provided *Limited Opportunities* for deeper learning through talk.

In the remaining 5 episodes both explaining and questioning occurred at least 12% of the time, but deeper instances were almost entirely absent from both categories. These episodes provided essentially No Opportunities for deeper learning through talk.

Opportunities	N	Explaining	Deeper Explaining	Questioning	Deeper Questioning
Very Frequent	5	82 to 100	23 to 59	59 to 89	12 to 30
Frequent	3	61 to 76	14 to 38	42 to 48	3 to 17
Occasional	5	55 to 81	11 to 39	24 to 59	0 to 4
Limited	7	28 to 96	2 to 21	14 to 67	0 to 7
No	5	14 to 67	3 to 4	13 to 46	0

Table 1: Percent of available time groups engaged in each type of talk to provide opportunities for deeper learning.

CONCLUSIONS AND IMPLICATIONS

Mathematics discourse plays a central role in the learning environment that students create and experience during small group instruction. Investigating the frequency and nature of students' talk for explaining and questioning about mathematics content during episodes of group instruction revealed differences in the opportunities these episodes offered for students to engage with the mathematics content of the tasks assigned for group work, and with one another's thinking about the mathematics of the tasks. The opportunities provided in these episodes were particularly distinguished by the frequency with which instances of explaining or questioning involved deeper mathematical purposes, either justification for an answer or procedure, or connection among students' contributions, to the context of the problem, or to other mathematical ideas. Accordingly, the analytic approach we employed addressed our research question, because we were able to detect important differences in mathematics discourse among episodes of small group work. We assert that these differences distinguish various small group learning environments in terms of the frequency of opportunities they provide students for deeper mathematics learning.

Our conclusion is that attending to specific aspects of discourse in terms of conversation, focusing on both frequency of occurrence and depth of purpose, aids in understanding how mathematics discourse among students constitutes a particular learning environment (Ryve, 2011) within an episode of small group instruction. The signifycance of this result, in our broader work and to the field, is that this approach to analyzing small group work provides a means to quantify and categorize the frequency and quality of mathematics discourse occurring during small group instruction. With this approach, investigations, including large scale studies, of at least four types of questions about small scale learning environments can be supported.

First, studies of supports for students that directly aim to improve mathematics discourse occurring in small group learning environments could use this approach to trace changes in the frequency and quality of resulting discourse.

Second, research into other factors that shape the small group learning environment can incorporate these measures of mathematics discourse. Studies of task design and implementation, including teaching actions, as well as peer-to-peer social dynamics, could be use this approach to examine the mathematics discourse that transpires under different conditions.

Third, studies that consider multiple factors, such as those above, alongside discourse, as constituting the small group learning environment, can adopt this approach to measure mathematics discourse. Such studies, then, can relate experiences in small group learning environments to various outcomes of interest—conceptual learning, orientations to mathematics, and views of peers as mathematical resources.

Fourth, studies of small group discourse can use the same approach to look beyond the frequency and depth of purpose occurring in the discourse to examine the flow of instances within episodes and specific student interactions that produce variations in the learning environment and the opportunities for engagement and learning it provides to each student.

References

Barron, B. (2003). When smart groups fail. *The Journal of the Learning Sciences, 12*(3), 307–359.

Esmonde, I., & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. *Journal for Research in Mathematics Education, 44*(1), 288-315.

Hamm, J.V. & Hoffman, A. (2016). Teachers' influence on students' peer relationships and peer ecologies. In K. Wentzel and G. Ramani (Eds)., *Handbook of Social Influences on Social-Emotional, Motivation, and Cognitive Outcomes in School Contexts*. New York: Taylor & Francis.

Mercer, N. (2005). Sociocultural discourse analysis: Analysing classroom talk as a social mode of thinking. *Journal of Applied Linguistics, 1*(2), 137–168.

Moschkovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics, 27*(1), 24-30.

O'Donnell, A. M. (2006). The role of peers and group learning. In P. Alexander & P. Winne (Eds.), *Handbook of Educational Psychology* (2nd ed.) (pp. 781-802). Mahwah, NJ: Lawrence Erlbaum.

Ryve, A. (2011). Discourse research in mathematics education: A critical evaluation of 108 journal articles. *Journal for Research in Mathematics Education, 42*(2), 167-199.

Sfard, A., & McClain, K. (2002). Guest editor's introduction: Analyzing tools: Perspectives on the role of designed artifacts in mathematics learning. *Journal of the Learning Sciences, 11*(2-3), 153-161.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. American Educational Research Journal, 33, 455-488.

Sztajn, P., Heck, D., & Malzahn, K. (2013) *Project AIM: Year three annual report.* Raleigh, NC: North Carolina State University, Chapel Hill, NC: Horizon Research, Inc.

Veenman, S., Denessen, E., van den Akker, A., & van der Rijt, J. (2005). Effects of a cooperative learning program on the elaborations of students during help seeking and help giving. *American Educational Research Journal, 42*(1), 115–151.

Webb, N. M., & Palincsar, A. S. (1996). Group processes in the classroom. In D. Berliner & R. Calfee (Eds.), *Handbook of Educational Psychology* (pp. 841–873). New York, NY: Macmillan.

Zahner, W., & Moschkovich, J. (2013). The social organization of a middle school mathematics group discussion. In *Modeling Students' Mathematical Modeling Competencies* (pp. 373-383). Springer, Dordrecht.

THE RELATION OF CHILDREN'S PERFORMANCES IN SPATIAL TASKS AT TWO DIFFERENT SCALES OF SPACE

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This study investigates the relation between performances of fourth-graders in spatial tasks with depictive material in the classroom and orientation tasks in real space. The children completed a paper and pencil test and a map-based orientation test on campus. A correlational analysis revealed that the children's performances in small-scale spatial tasks are related to their performances in large-scale spatial tasks. Moreover, classes of small-scale tasks that require mental transformations concerning the self and concerning objects are related to large-scale tasks that involve the update of the self-to-landmark relations in real space and the map-environment relation, respectively. Both classes contributed to the prediction of performances in map-based orientation tasks that require a constant update of map-self-landmark relations.

INTRODUCTION

Solving spatial tasks is recommended in geometry classes in primary school since doing so helps children to "grasping space", i.e. it contributes to a child's thoughtful interaction with the three-dimensional space in which they live, play and move (Freudenthal, 1973). The demands on spatial tasks in geometry education are therefore twofold: on the one hand, they should foster a child's ability to interact successfully with space. On the other hand, spatial tasks should allow a child to integrate and enrich individual spatial experiences while solving them. In order to accomplish both goals, spatial tasks should ideally be introduced into geometry classes in both abstract-depictive spatial settings in the classroom *and* in concrete-navigational spatial settings in real space (OECD, 2004, p.36).

Current studies in mathematics education emphasize the importance of spatial tasks in both contexts but typically investigate those in settings that include only written or small material (e.g. Logan et al., 2017). Researcher may do so because they assume that the performances in spatial tasks in depictive settings equal the performances in navigational settings in real space. However, empirical evidence on whether and to which extent performances in both contexts are related has never been provided. This study addresses that gap at a conceptual and empirical level.

THEORETICAL BACKGROUND

Cognitive psychologists conceptualize spatial tasks with depictive material, such as paper and pencil tests, as *small-scale spatial tasks*, since they rely on a stimulus that can be perceived from one single vantage point. They conceptually contrast them to

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 59-66). Umeå, Sweden: PME.

large-scale spatial tasks that require the locomotion of the subject towards multiple viewing points in order to be completed and may require the successful interpretation of a spatial representation such as a map (Montello, 1993, Hegarty et al., 2006).

In order to comply spatial tasks at both scales of space, children typically need to understand the interplay between different spatial positions in space and the visual appearance of object configurations. Hereby, the child needs to be able to encode and mentally manipulate three changing relations between the different objects, the self, and the environment: *object-to-environment relations*, *self-to-object relations* and *self-to-environment relations* (e.g. Hegarty et al., 2006).

Small-scale spatial tasks can be differentiated according to two classes of mental transformations demands that are necessary in order to solve them: (1) tasks that require *object-based transformations*, i.e. tasks that require the mental movement of a set of objects in the environment (OB*)*, and (2) tasks that require *egocentric perspective transformations*, i.e. tasks that require the mental movement of the own point of view in relation to a set of objects *(EGO)*. Both classes have been found to be distinct not only on the conceptual level, but also on an empirical level (e.g. Kozhevnikov et al., 2006).

Large-scale spatial tasks can also be conceptualized in a differentiated way according to different task demands. The memorizing of landmarks (important recognizable "objects") without providing maps has been studied under the perspective of individual differences in the performance to keep track of changing self-to-landmark relations in the environment that enable the formation of a *cognitive map* (e.g. Hegarty et al., 2006). *Static map use*, that focuses on aligning a map with the environment in order to draw directional inferences from it while not moving in space has been studied with respect to individual differences in the performance of recognizing and correcting misaligned relations between the map and the environment (e.g. Shepard & Hurwitz, 1984). Finally, *dynamic map use*, that requires the subject to keep track of the self-location and orientation on the map while moving in space or to navigate to landmarks, has been investigated with respect to individual differences in the performance to update self-to-map, self-to-environment, and map-to-environment relations (e.g. Liben et al., 2008). Although it has been highlighted that large-scale spatial tasks need to be conceptualized in a differentiated way (e.g. Kozhevnikov et al., 2006), the distinction of the classes outlined above is less studied from an empirical point of view.

Divergent results have been reported concerning the relation between performances of children in small- and large-scale spatial tasks. Those have been shown to be either totally dissociated (Quaiser-Pohl et al., 2004) or partially related (Liben et al., 2013). The latter study as well as similar studies with adults (Liben et al., 2008, Kozhevnikov et al., 2006) highlighted the potential role of single OB and EGO tasks as common and unique predictors of diverse large-scale spatial tasks.

PURPOSE OF THE STUDY

The goal of this study was to investigate the relation between the performances in small-scale spatial tasks and the performances in large-scale spatial tasks of primary school children. We aimed to examine whether classes of paper and pencil tasks were reliable and unique predictors of different of map-based orientation tasks. Moreover, we intended to assess whether patterns of unique prediction where generalizable for classes of map-based orientation tasks.

METHOD

Participants and stimuli

240 (111 m, 129 f) fourth graders from the north of Germany participated in the study on the campus of our university. The children were aged between 9 and 12 years (m=10.29, SD=.48). Each child completed a paper and pencil test in a group and a map-based orientation test in large-scale space individually at the same day with a break of at least 20min for cognitive recover.

Paper and Pencil Test

The Paper and Pencil Test consisted of eight small-scale spatial tasks, four of them measuring performances in tasks that require egocentric (EGO) transformations and four of them measuring performances in tasks that require object-based (OB) transformations. We developed EGO tasks mostly from the scratch and designed tasks that require the children to relate field views of various object configurations to the corresponding positions in plan views. One task was an adoption of the Guilford-Zimmermann-Boat test for children. The OB tasks consisted of an adoption of Ekstrom's Card Rotation Test, an adoption of the Vandenberg Mental Rotation Test, and adoption of the Paper Folding Test for children. We further designed a task that requires the children to imagine going along a path on a map and decide on each crossing whether they turned left or right.

We tested the quality of our tasks in a pilot study with N=222 children, making sure that our self-developed test has acceptable psychometrical characteristics and is construct-valid (EGO tasks are empirically separable from OB tasks).

Map-based orientation test

The map-based test consisted of eight tasks with three items each that were integrated in a treasure hunt on the campus (Table 1). One task was performed at the starting location in the beginning (*Rot*) and two at the end (*MDisk, MFlag)* of the treasure hunt. For all other tasks, we subsequently led the children to three flags (*Dots, Dir, HP, Read*) and finally encouraged them to place the disks on the campus (*Disks*). During the whole test, the children were not allowed to turn their map.

The test consisted of tasks that operationalized cognitive mapping processes (CM), the performance of mentally aligning a map in space in order to draw inferences from it while being static (MapUse) and the performance of keeping the orientation on where they are on a map while moving in space (MapOrtn). Those tasks represented the underlying construct in the large-scale test.

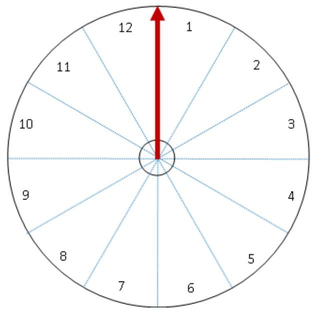
Task	Description	Measure
MFlags /MDisks (CM)	Requires the child to point to the locations of the flags/disks without using a map.	Correctness of the directions was measured with the help of an arrow and circle device that served as help for indicating directions.
Rot (MapUse)	Requires the child to indicate directions of landmarks on the map while taking different canonical viewing directions in the real space.	
Read (MapUse)	Requires the child to indicate directions of landmarks on the map while standing next to a flag.	
HP (MapUse)	Requires the child to point to the starting point.	
Dir (MapOrtn)	Requires the child to indicate the viewing direction once arrived at a flag.	
Dots (MapOrtn)	Requires the child to indicate the location of the current position with a coloured sticker on the map after walking from flag to flag.	Deviations of the stickers.
Disks (MapOrtn)	Requires the child to put a disk in the environment according to the location marked in the map.	Deviation of the disk.

Table 1: Large-scale spatial tasks in the map-based orientation test.

Data treatment

We encoded our data and analysed patterns of missing values in the map-based orientation test. We ensured that missing values are at least MAR and applied multiple imputations before further analysis. We computed 30 multiple imputations according to Si & Reiter's method (2013) using the R package *NPBayesImpute*, computed sum scores for all tasks and finally pooled the data sets using the R package *semtools*, which allowed us to extract one single empirical correlation matrix.

RESULTS

To investigate the relation of classes of small-scale tasks with the map-based orientation tasks, we computed the factor scores for EGO and OB. Both factor scores correlated with r=.64 ($p<0.001$). The result demonstrates that they indeed share a considerable amount of variance that will be considered further in our correlation analysis.

Relations between classes of small-scale tasks and single large-scale tasks

In a first step, we computed the pairwise correlations between the factor scores of the two classes of small-scale tasks and the set of large-scale tasks. As shown in Table 2, both classes correlate significantly with performances in the large-scale tasks. Only the performances in pointing towards the memorized locations of the flags did not correlate with either of two classes of small-scale tasks.

	MDisk	MFlag	Disks	Dir	Dots	Read	HP	Rot
EGO	.24**	.12	.41**	.28**	.43**	.38**	.20**	.20**
OB	.17**	.10	.42**	.29**	.45**	.40**	.23**	.26**
Residual EGO-OB	.18**	.07	.19**	.11	.19**	.16*	.06	.04
Residual OB-EGO	.01	.02	.21**	.15*	.22**	.21**	.14*	.18**

* two-tailed p< 0.05 ** two-tailed p<0.01

Table 2: Correlations and semipartial correlations for the task wise analysis.

To examine whether performances in EGO or in OB tasks predicted unique variance in the large-scale measures, we computed semipartial correlations (see also Table 2). After partialling out the shared variance between performances in EGO and OB tasks, for some large-scale tasks, only one of the two classes of small-scale task became significant, indicating that they predicted unique variance in the respective task. For instance, only the semipartial correlation between EGO and the performances in memorizing the locations of the disks became significant. Thus, cognitive resources that are unique to EGO tasks – performing egocentric transformations – appeared to have affected performances in the task *MDisk*, which requires updating of self-to-environment relations.

In two other cases, only the semipartial correlation with OB became significant. For those tasks, cognitive resources that are unique to OB tasks – performing object-based transformations while keeping the self-to-environment relation constant – affected the performances. OB tasks predicted therefore unique variance in two tasks that required the correct alignment of a map in space (*HP* and *Rot*). In the case of the task *Dir*, the semipartial correlation of EGO was also almost significant (p=0.07). For this reason, we did not interpret OB tasks to be unique sources of variance in this task. For the tasks *Disks, Dots,* and perhaps *Dir*, for both classes of small-scale tasks the semipartial correlations became significant. Thus, processing resources that are unique to OB tasks and unique to EGO tasks appeared to have affected the performance in those tasks that require keeping oriented after moving in space.

Relations between classes of small-scale tasks and classes of large-scale tasks

To further analyze the initial results at the broader level of classes of large-scale tasks, we performed a CFA using *R lavaan* in order to show that our tasks loaded on the factors that we derived from the literature. For the sake of shortness, we do not present the full analysis here. For each of the 30 data sets, the fit indices revealed a CFI>.99, a TLI>.98, RMSEA<0.05, and a non-significant chi-squared test showed that the model did not derive essentially from the data (see Hu & Bentler, 1999). We conjectured that the tasks in our map-based orientation test are clustered in accordance with the classes that we conceptualized from the literature. We then computed the corresponding factor scores and calculated correlations.

	Factor *CM:* Cognitive Mapping	Factor *MapUse:* Static Map Use	Factor 3 *MapOrtn:* Dynamic Map Use
EGO	.22**	.35**	.49**
OB	.15*	.41**	.50**
Residual EGO-OB	.15*	.11	.21**
Residual OB-EGO	.02	.25**	.25**

* two-tailed p< 0.05 ** two-tailed p<0.01

Table 4: Correlations and semipartial correlations between the performances in EGO and OB and the three classes of map-based orientation tasks.

As we show in Table 4, the correlation between EGO and OB with the class of cognitive mapping tasks was significant (p = 0.001 and p=0.02, respectively). In order to determine whether one of them predicted unique variance in tasks of cognitive mapping, we computed semipartial correlations. Once the shared variance of OB and EGO was partialled out, only the semipartial correlation between EGO and the first factor of large-scale tasks was significant (p=0.01), whereas the semipartial correlation between OB and the factor was not. Cognitive resources that are unique to EGO tasks, in particular egocentric mental transformations appear to have affected the performances in this self-to-environment representation factor. Similarly, correlations between EGO and OB with the class of static map use tasks, were highly significant (p<0.001). Once the shared variance between EGO and OB was partialled out, the semipartial correlation between OB and the second class was still significant (p<0.001), whereas the semipartial correlation to EGO was not (p=0.09). Thus, cognitive resources that are unique for object-based transformations, in particular the correct mental update of relations between objects and the environment, appear to have affected the performances in this map alignment factor. Finally, an analysis of the correlations between EGO and OB with the class of dynamic map use tasks revealed highly significant correlations (p<0.001). Even after partialling out the shared variance, both EGO and OB were still significant predictors when it came to dynamic mapping (p=0.001 and p<0.001, respectively). Thus, cognitive resources that are unique to EGO and to OB tasks, egocentric and object-based transformations, appear to have both affected the performances in this map-based orientation factor.

In summary, our empirical findings provide evidence that the children's performances in small-scale spatial tasks are related to the performances in large-scale spatial tasks.

The two classes of small-scale spatial tasks, EGO and OB both predicted the performance of large-scale spatial tasks at the level of single tasks and classes of them. After partialling out shared variance between EGO and OB tasks, however, we identified EGO tasks to be the only reliable predictor of cognitive mapping tasks and OB the only reliable predictor of static map use tasks. However, both classes are reliable predictors of dynamic map use tasks.

DISCUSSION

The results described above support the idea that spatial tasks should be used in a differentiated way in mathematics education. Our findings provide evidence that the performances of small-scale tasks are partially, but not fully related to performances in large-scale tasks. One possible explanation might be related to the underlying spatial abilities that enable solving those tasks with a certain performance. They probably rely on common cognitive processes that allow for the processing of small- and large-scale information such as the encoding of the spatial information and the representation in working memory (cf. Hegarty et al., 2006). Investigating these processes might be an important next step in mathematics education research. Our findings highlight, that large-scale tasks should be conceptualized in a differentiated way. Furthermore, the patterns of correlation reported within this study suggest a taxonomic classification of large-scale tasks that is analogous to one classification of small-scale spatial tasks. Indeed, tasks that demand egocentric mental transformations in small-scale space find their analogue on tasks that rely on a correct update of the self-to-landmark and self-to-environment relations, which can be interpreted as egocentric transformations in large-scale space (e.g. Kozhevnikov et al., 2006). Tasks that demand object-based transformations in small-scale space find their analogue in tasks that rely on updating processes between the map and the environment that can be interpreted as object-based transformations in large-scale space (e.g. Shepard & Hurwitz, 1984). Finally, dynamic map use tasks seem to be determined by a subsequent composition of egocentric transformations that allow to update self-to-map and self-to-landmark relations in the environment, and object-based transformations that allow to mentally updates the relation between the map and the environment while moving. This finding is in line with previous suggestions that dynamic map use requires two sets of mental transformations (Aretz & Wickens, 1992).

In future research, the relation between performances in small- and large-scale spatial tasks could be investigated not only at the level of classes of small-scale tasks, but also at the level of single tasks. This could point towards a set of good spatial tasks for practices in classroom and beyond. Furthermore, the relation could be studied at the latent level of the assumed underlying spatial abilities as well. Shifting the empirical investigations from the manifest to the latent level would result in an explicit modelling of measurement errors that probably allows for computing measurement error-free correlations.

Acknowledgements

We thank Dr. A. Robitzsch, IPN Kiel, Germany, for his support in the implementation of the multiple imputations in R.

References

Aretz, A. J., & Wickens, C. D. (1992). The mental rotation of map displays. *Human performance*, *5*(4), 303–328.

Freudenthal, H. (1973). The case of geometry. In *Mathematics as an educational task* (pp. 401–511). Springer, Dordrecht.

Hegarty, M., Montello, D. R., Richardson, A. E., Ishikawa, T., & Lovelace, K. (2006). Spatial abilities at different scales: Individual differences in aptitude-test performance and spatial-layout learning. *Intelligence*, *34*(2), 151–176.

Hu, L. T., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural equation modeling: a multidisciplinary journal*, *6*(1), 1–55.

Kozhevnikov, M., Motes, M. A., Rasch, B., & Blajenkova, O. (2006). Perspective-taking vs. mental rotation transformations and how they predict spatial navigation performance. *Applied Cognitive Psychology*, *20*(3), 397–417.

Liben, L.S., Myers, L.J., & Kastens, K.A. (2008). Locating oneself on a map in relation to person qualities and map characteristics. In C. Freska, N.S. Newcombe, P. Gärdenfors, & S. Wölfl (Eds.), *Spatial cognition VI*: Learning, reasoning, and talking about space (pp. 171–187). Heidelberg, Germany: Springer-Verlag.

Liben, L. S., Myers, L. J., Christensen, A. E., & Bower, C. A. (2013). Environmental-Scale Map Use in Middle Childhood: Links to Spatial Skills, Strategies, and Gender. *Child development*, *84*(6), 2047–2063.

Logan, T., Lowrie, T. & Ramful, A. (2017). Decoding map items through spatial orientation: performance differences across grade and gender. In B. Kaur, W.K. Ho, T.L. Toh & B.H. Choy (Eds.), *Proc. 41st Conf. of the Int. Group for the Psychology of Mathematics Education* (vol. 3, pp. 193-200). Singapore: PME.

Organisation for Economic Co-operation and Development. (2004). *The PISA 2003 assessment framework: mathematics, reading, science and problem solving knowledge and skills*. OECD Publishing.

Quaiser-Pohl, C., Lehmann, W., & Eid, M. (2004). The relationship between spatial abilities and representations of large-scale space in children—a structural equation modeling analysis. *Personality and Individual Differences*, *36*(1), 95–107.

Shepard, R. N., & Hurwitz, S. (1984). Upward direction, mental rotation, and discrimination of left and right turns in maps. *Cognition*, *18*(1), 161–193.

Si, Y., & Reiter, J. P. (2013). Nonparametric Bayesian multiple imputation for incomplete categorical variables in large-scale assessment surveys. *Journal of Educational and Behavioral Statistics*, *38*(5), 499–521.

WHICH ESTIMATION SITUATIONS ARE RELEVANT FOR A VALID ASSESSMENT OF MEASUREMENT ESTIMATION SKILLS?

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Measurement estimation skills are of significant importance for everyday life. In the last decades a lot of research results were generated describing students' estimation skills and strategies. Surprisingly, little attention has been paid to the basic question which types of situations are relevant for a valid conceptualization and operationalization of measurement estimation skills. Some studies refer to the basic structure of estimation conditions described by Bright (1976) whereas others ignore this question, though it is central to ensure validity of the empirical data. Following validity criteria and based on existing empirical findings on estimation strategies, we developed a comprehensive model of measurement estimation situations. This model provides a basis for the development of valid tests on measurement estimation skills as well as for the development of learning environments.

THEORETICAL BACKGROUND

Skills to estimate the attributes of objects (e.g., length, area) are of significant importance for everyday life as well as for professional expertise in various professions (Jones, Taylor, & Broadwell, 2009) and can be considered as a basis of measurement skills (Joram, Subrahmanyam, & Gelman, 1998). To date, research provides a lot of information on individuals' measurement estimation process, strategies and performance (e.g., Siegel, Goldsmith, & Madson, 1982; Joram et al., 2005). Moreover, empirical findings show that the teaching of estimation strategies is possible (e.g., Hildreth, 1983) and improves the accuracy in students' measurement estimation (e.g., Joram et al., 2005; Jones et al., 2009).

Most of the empirical studies used items representing specific estimation situations to collect data on estimation skills or strategies. Surprisingly, many studies did not address the question which types of estimation situations are relevant to elicit the skills or strategies aimed for from the considered students or adults. Ignoring the choice of estimation situations might result in a serious threat of validity of empirical data and its interpretations. We elaborate on this problem and suggest a comprehensive model of measurement estimation situations which satisfies validity criteria and integrates current research results on measurement estimation skills. For reasons of simplicity, we restrict our presentation on length estimation, though our model can probably be adopted for other attributes like area and volume as well.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 67-74). Umeå, Sweden: PME.

Measurement estimation and measurement estimation strategies

In the sense of Bright (1976) and other researchers, we consider measurement estimation as a mental process of determining a measurement for an attribute of an object without the aid of measurement tools. Central to this process is that the use of measure units happens mentally while other aids like benchmarks or body parts might be used as additional tools (e.g., estimate the length of a pencil in cm as a benchmark and then determine the width of the table by repeated use of the pencil as a tool).

Empirical research yields that children and adults mostly show a poor measurement estimation performance, that individual length estimation is in general more accurate than the estimation of area, volume, or weight and that estimation performance of students increase with grade (e.g., Siegel et al., 1982; Sowder, 1992; Joram et al., 1998). In order to understand estimation performance, Siegel et al. (1982) analyzed students' estimation processes and developed a process model describing the individual estimation process. This model particularly emphasizes the role of different estimation strategies in the estimation process and subsequent research provided evidence that the use of strategies predicts estimation performance (Joram et al., 2005; Jones et al., 2009; Huang, 2015). The most important estimation strategies (e.g., Siegel et al., 1982; Hildreth, 1983; Joram et al., 1998) are

- *unit iteration* as a mentally conducted measurement by a segmentation of the to-be-estimated object (TBEO) based on a given standard or non-standard unit and subsequent counting of the segments;
- *benchmark comparison* (or reference-point strategy) as a mental comparison of the TBEO with a distance represented by a benchmark or the sequence of the same benchmarks where the length of the benchmark is known or can be estimated;
- *decomposition/recomposition* as a process of mental decomposition of the TBEO into smaller parts, estimation of the length of each part by using one of the previously mentioned strategies and adding the estimates of all parts.

In addition to the estimation strategies, research points to further components of estimation skills. As the descriptions of the strategies make clear, domain-specific knowledge obviously plays an important role. This encompasses, for example, measurement knowledge related to the standards units (i.e., mm, cm, m, km and their interrelations in case of the metric system) as well as knowledge on benchmarks in a twofold way (e.g., knowledge on the approximate width of an A4 sheet which is given as a possible benchmark; knowledge of a suitable object that can be used as a benchmark for 10 cm). Besides these knowledge components, specific cognitive abilities contribute significantly to estimation performance. Models of estimation processes from cognitive psychology assume that (1) the TBEO is represented in the working memory, (2) this representation is estimated based on estimation strategies and information retrieved from the long-term memory and (3) the estimated length is finally checked in a monitoring process (e.g., D'Aniello, Castelnuovo, & Scarpina, 2015). From this follows that beyond knowledge on measurement, benchmarks and estimation strategies (stored as information in the long-term memory) individual working memory capacities play an important role.

As we already mentioned before, the relevant knowledge for the estimation process can be acquired in effective learning environments. This means in particular that students educated in different learning environments based on different curricula might possess different knowledge (e.g., different estimation strategies). Such differences become obvious when we consider students from two countries with different cultures, educational traditions and curricula. Differences in the learning context probably result in different benchmark knowledge since benchmarks are influenced by the cultural context. Differences might also occur in how students learn to implement estimation strategies. For example, some curricula emphasize the use of body parts as benchmarks for touchable TBEOs as suggested in Jones et al. (2009), other curricula may emphasize the strictly mental use of benchmarks to estimate imagined TBEOs.

Measurement estimation situations as a basis for research on estimation skills

In general, measurement estimation skills are inferred from the performance of an individual generating an accurate estimate for a required measurement of a given attribute of an object. To assess these estimation skills, individuals are asked to solve various estimation items. Such items represent estimation situations which can differ substantially and therefore might influence the estimation performance. For example, Pike and Forrester (1997) administered items representing estimation situations in a story context (ladybirds amidst a rainfall) and in a stereotypical mathematics textbook context. –It turned out that students' estimation performance was better in the mathematics textbook context – a phenomenon which is probably caused by the specific type of mathematical tasks and activities implemented in mathematics textbooks and the mathematics classroom. However, even when restricting items to one specific context (e.g., real life context), in each estimation situation there are characteristics which must be understood by students before performing an estimation process and which thus might influence the item difficulty. Accordingly, a thorough analysis of estimation situations implemented in test or interview items is necessary to ensure validity of the empirical data and inferred results. Surprisingly, in many published studies this challenge is addressed neither explicitly nor implicitly. In the following we shortly present three examples retrieved from the literature: first, the model of Bright (1976) who explicitly addressed this question and to which other articles refer (e.g., Sowder, 1992) and then the descriptions from the studies of Jones et al. (2009) and Siegel et al. (1982).

Model of Bright (1976)

In Bright (1976), eight types of estimation situations are described each as combination of three independent characteristics: (1) the object or the measurement is specified, (2) the TBEO is physically present/given or not, and (3) the unit of measurement is given or not. The eight situation types are divided into two parts, first the situation types where the object is specified (A) and second the types where the measurement is specified (B). The four situation types of class A are the usual estimation situations where students have to estimate a measurement for an attribute of an object, whereas the four situation types of class B are mainly interesting for instructional purposes (to support students in generating benchmark knowledge). It is mentioned (Bright, 1976, p. 90) that further subdivisions of the situation types could be made.

Structure in Siegel, Goldsmith, and Madson (1982)
In their article, Siegel and colleagues present an estimation process model based on findings of an interview study. The study relies on specific items suggesting the use of specific estimation strategies in order to elicit the cognitive processes of interest. Hence, there is an implicit model of different types of estimation situations structured by estimation strategies. Siegel et al. (1982) distinguish two problem types related to length measurement: benchmark problems and decomposition problems. However, in this case benchmark problems do not explicitly ask for the use of a benchmark or explicitly mention a benchmark. Instead, it is assumed that problems like “How long is a piece of manuscript paper?” are solved by the benchmark strategy. In contrast, the presented decomposition items explicitly describe decompositions ("If you took these cooking utensils and laid them end to end, how far would they reach?”). For both problem types the TBEOs were presented physically or by photographs. From the article it does not become clear whether the children and adults were allowed to touch the TBEOs during the estimation process.

Model of Jones, Taylor, and Broadwell (2009)
The article of Jones and colleagues on the use of body parts in the estimation process describes the Linear Measurement Assessment (LMA) which they used to test linear measurement estimation skills. The LMA is based on a model with five dimensions representing different types of length estimation situations (Jones et al., 2009, p. 1502): (1) estimating the length of an object while viewing the object; (2) naming an object from memory for different metric sizes; (3) estimating the lengths of large objects like a building; (4) metric estimation of objects that students can touch or distances they can pace; (5) using body parts as an aid to measure different objects. Analyzing the types of estimation situations, it turns out that different aspects like size or presence of the TBEO as well as the option to touch the TBEO are varied. Moreover, the situation type (2) is similar to one of those in Bright’s (1976) situation types of class B.

Summarizing the state of the art, it turns out that many studies in the field of measurement estimation skills do not explicitly elaborate on the choice of estimation situations for their data collections. Some studies refer to the model of Bright (1976), others provide own criteria for structuring the estimation situations. As Bright (1976) mentioned, his model can be refined and analyzing the other existing models, it turns out that specific types of estimation situations are not distinguished (e.g., situations in which a touchable benchmark is given or a representation of the TBEO’s length can be constructed by drawing a line). However, it is not clear what grain-size is relevant for research and educational practice in the field of length estimation skills.

RESEARCH GOAL AND RESEARCH APPROACH

Based on the previously presented theoretical background, we elaborate on the question what types of estimation situations are relevant for the assessment of estimation skills. As mentioned in the beginning, we restrict our presentation to length estimation. Our goal is to develop a comprehensive model on types of length estimation situations which ensures validity in case of assessments of length estimation skills. To establish validity, we follow the Standards for Educational and Psychological Testing (AERA, APA, & NCME, 2014). In chapter 1, the standards provide five sources of evidence which can contribute to validity: evidence based on (1) content, (2) response processes, (3) internal structure, (4) relations to other variables, and (5) consequences of the interpretation of results.

As presented in the previous section, empirical research yields that the following aspects are relevant factors for length estimation skills: knowledge in estimation strategies, measurement knowledge, benchmark knowledge, working memory capacity as well as context factors like culture (influences which estimation strategies are emphasized in the mathematics classroom) and characteristics of estimation situation in real life (in contrast to estimation situations in mathematics textbooks). These factors for length estimation skills contribute to the five sources of evidence for validity as follows (cf. AERA, APA, & NCME, 2014):

1. *Evidence based on content* asks for the relation between the construct (length estimation skill) and the requirements in the estimation situations. Hence, the situations must cover all relevant aspects of estimation situations in real life.
2. *Evidence based on response processes* asks for a fit between the construct (length estimation skill) and the observed performance in estimation situations. It must be ensured that the performance of mastering the estimation situations is based on the components (knowledge, working memory capacity) of estimation skills.
3. *Evidence based on the internal structure* asks for a fit between the structure of the construct (length estimation skill) and the performance in different types of estimation situations. Here, the influence of culture, educational traditions and curricula might play a role because the teaching of different estimation strategies can yield different performance profiles provided suitable estimation situations are represented by the items.
4. *Evidence based on relations to other variables* asks for correlations between the estimation performance and other variables. For example, performance should improve with the increase of benchmark knowledge or the grade of the students.
5. *Evidence based on consequences of the interpretation of results* means in particular to exclude unintended interpretations of the empirical results caused by construct irrelevant components or by construct underrepresentation. For example, culture might influence benchmark knowledge so that estimation situa-

tions with given benchmarks can be problematic in cross-cultural studies. Construct underrepresentation might occur if estimation situations are restricted to certain types (e.g. only touchable TBEOs) so that specific components of estimation skills are not required.

RESULTS

Based on the framework described by the criteria 1-5 in the previous section and existing models of estimation situations presented in the theoretical background, we identified six characteristics of estimation situations. The six characteristics are mostly pairwise independent; however, some combinations do not make sense in real life or even cannot occur so that these situations are excluded. Figure 1 gives an overview of the 72 estimation situations as compact tree diagram. It should be mentioned that there is a seventh characteristic which is not displayed in the tree diagram and which is related to the magnitude of the TBEO. Since length estimation of large objects is more challenging than length estimation of small objects (e.g., Jones et al., 2009), each estimation situation should be considered for small, medium, large, and huge objects.

There are manifold relations of the presented characteristics of estimation situations to the five validity criteria 1-5 developed in the previous section. Due to space limitations, we cannot explain and argue in detail how each characteristic of our model is related to the five criteria so that we restrict to some exemplary aspects.

The seven characteristics represent relevant aspects of estimation situations in real life and thus contribute to the first criterion (content validity). In real life, the TBEOs can be (i) physically present or not and (ii) touchable or not, (iii) there might be the opportunity to construct a representation of the same length (e.g., by drawing a line) and (iv) it can happen that no benchmark is given or that a specific benchmark (or more than one) is mentioned, is visible in real size, or is even visible in real size and touchable. Moreover, there are situations which (v) ask for estimates in standard units (e.g. metric units like cm) or in non-standard units (e.g., room width in number of floor tiles). If a benchmark is given, it can happen that (vi) its length is provided or not. Finally, (vii) in real life the TBEOs can vary in their length from small to huge.

In addition to the first criterion, the characteristics satisfy the other four criteria of validity. For example, it makes a strong difference for the cognitive load of the working memory if a TBEO is touchable or not or if a benchmark is visible in real size or not. Estimation processes are more challenging if individuals must imagine the TBEO and the benchmark because external representations are not available. Due to the high cognitive load, it can happen that only efficient estimation strategies can be processed and persons who only know complex strategies will show a lower performance in these situations. Hence, taking into account these characteristics of estimation situations contributes to validity corresponding to criteria 2-5 because varying performance in different types of estimation situations can be distinguished. As a second example, we want to mention that the seven characteristics allow the distinction of estimation situations which do or do not support specific estimation strategies. If a TBEO is touchable, the benchmark comparison strategy is much easier because a person can directly or indirectly use body parts as benchmarks (Jones et al., 2009). If representations can be constructed (e.g., by drawing a line of the length of the TBEO), this might support the decomposition strategy in case of medium sized TBEOs. The application of estimation strategies is associated with different knowledge components that depend on previous learning experiences. Hence, the seven characteristics allow distinguishing estimation situations which ensure on the one hand that persons with different knowledge on estimation strategies show different estimation performances (relevant for criteria 2-4) and avoid on the other hand that the construct length estimation skill is not underrepresented in the assessment (relevant for criterion 5).

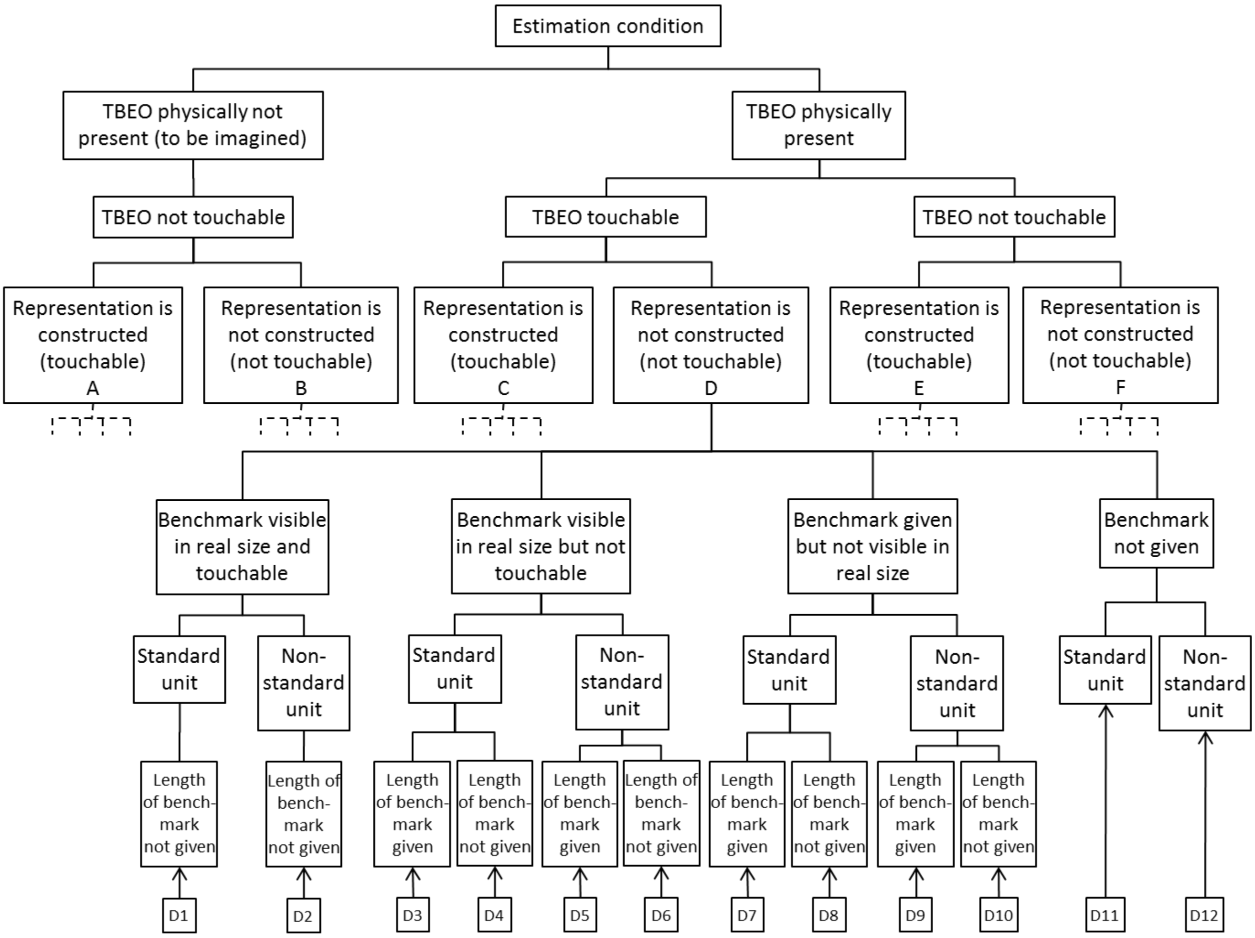

Figure 1: Model of 72 types of length estimation situations (12 types for cases A-F).

DISCUSSION

In this contribution we argued that in empirical research on measurement estimation skills, the choice of estimation situations is of great importance when collecting data. A low variation in the characteristics of estimation situations might result in a serious threat of validity of empirical data and the interpretations of the results. Surprisingly, many articles of empirical studies do not explicitly report how they have chosen the estimation situations for their assessment instruments and a reconstruction of this information is not possible. Only some studies refer to the existing basic model of Bright (1976) or give an explicit or implicit description of an own model.

For our analysis we adopted aspects of validity from AERA, APA, and NCME (2014) and combined these with the current state of research on relevant aspects of length estimation skills. The resulting five criteria for validity allowed developing a comprehensive model of types of estimation situations (Figure 1). In comparison to the models of Bright (1976), Siegel et al. (1982) and Jones et al. (2009), our model is much more detailed. It allows distinguishing more types of estimation situations and therefore gives opportunities for a more detailed analysis of estimation skills.

Currently, our arguments are purely theoretical. Hence, it is an empirical question how fine-grained the estimation situations must be mirrored by items in empirical studies. Depending on the respective research goal and the kind of data that should be collected, a coarser model with fewer estimation situations might be sufficient to assess length estimation skills. In such cases the model of types of estimation situations in Figure 1 can serve as an ideal to check for validity.

References

AERA, APA, & NCME (2014). *Standards for educational and psychological testing*. Washington, DC: American Educational Research Association.

Bright, G. W. (1976). Estimating as part of learning to measure. In D. Nelson & R. E. Reys (Eds.), *Measurement in school mathematics: 1976 yearbook* (pp. 87-104). Reston, VA: NCTM.

D'Aniello, G. E., Castelnuovo, G., & Scarpina, F. (2015). Could cognitive estimation ability be a measure of cognitive reserve? *Frontiers in Psychology, 6,* 1-4.

Hildreth, D. J. (1983). The use of strategies in estimating measurements. *Arithmetic Teachers, 30*(5), 50-54.

Huang, H.-M. E. (2015). Children's Performance in Estimating the Measurement of Daily Objects. In K. Beswick, T. Muir & J. Wells (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference,* (Vol. 3, pp. 73-80). Hobart, Australia: PME.

Jones, G., Taylor, A., & Broadwell, B. (2009). Estimating linear size and scale: Body rulers. *International Journal of Science Education, 31*(11), 1495-1509.

Joram, E., Gabriele, A. J., Bertheau, M., Gelman, R., & Subrahmanyam, K. (2005). Children's use of the reference point strategy for measurement estimation. *Journal for Research in Mathematics Education, 36(1),* 4-23.

Joram, E., Subrahmanyam, K., & Gelman, R. (1998). Measurement estimation: Learning to map the route from number to quantity and back. *Journal of Educational Review*, 6, 413-449.

Pike, C. D., & Forrester, M. A. (1997). The influence of number-sense on children's ability to estimate measures. *Educational Psychology, 17*(4), 483-500.

Siegel, A. W., Goldsmith, L. T., & Madson, C. R. (1982). Skill in Estimation Problems of Extent and Numerosity. *Journal for Research in Mathematics Education, 13*(3), 211-232.

RESPONDING TO TEACHERS: LEARNING HOW TO USE VERBAL METACOMMUNICATION AS A MATHEMATICS TEACHER EDUCATOR

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In this paper, I present the process of developing a framework for analysing verbal metacommunications, in the context of a new mathematics teacher educator working with in-service teachers of mathematics. The interest in analysing verbal Metacommunication arises from reflecting on the process of becoming a mathematics teacher educator, as I am learning how to respond in-the-moment to teachers of mathematics as they talk about teaching. Responding to teachers with verbal metacommunication appears to be significant in terms of supporting teachers in their own learning. There is currently no existing framework, within the mathematics education literature, for making systematic distinctions between types of verbal metacommunications in supporting group discussion.

BECOMING A MATHEMATICS TEACHER EDUCATOR

As a secondary school teacher of mathematics, I worked hard to set up a culture in my classroom where an overall aim of the year was linked to "being a mathematician". Over years of teaching the same tasks, I became attuned to hearing comments and observing actions linked to this aim. A powerful mechanism for building this culture was an ongoing commentary from me that went alongside the doing of the mathematics and in response to what the children were saying or doing. For example, a comment in response to a student who said, "I've noticed it's going up in twos" could imaginably have been "one thing mathematicians do is look for patterns" or "write that down as a conjecture to work on". As a teacher of mathematics, my teaching was "constantly organized by meta-comments" (Pimm, 1994, p.165) such that "the utterances made by students are seen as appropriate items for comment themselves" (p.165). Meta-commenting provided me with an alternative to evaluating student utterances, or responding directly to what was being uttered. Another purpose for commenting about the students' comments, was to create an image of a way of working that supported the students in their approach to working on mathematics, to establish a culture where students were motivated through asking their own questions and working on their own conjectures.

Almost two years ago I moved from secondary school mathematics teaching to a university, as a mathematics teacher educator working alongside a group of pre-service teachers of mathematics. In reflecting on sessions with the group of pre-service teachers, one issue that arose for me was around hearing and responding. Having been

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 75-82). Umeå, Sweden: PME.

attuned to hear and respond to comments in a mathematics classroom, I was able to respond as a mathematics teacher but was not yet able to respond as a mathematics teacher educator. From this awareness developed a motivation to research how I am becoming a mathematics teacher educator and a research project commenced.

Within the field of mathematics education there is a distinction made between what is termed the education of mathematics teacher educators where the focus is on teacher educators learning through formal courses and the mathematics teacher educator as learner where the emphasis is on “teacher educators’ autonomous efforts to learn, in particular, through reflection and research on their practice” (Krainer, Chapman & Zaslavsky, 2014, p.432). My study aligns with the second of these terms and concerns how I am learning to respond in becoming a mathematics teacher educator. Specifically, how to respond in-the-moment to pre-service teachers of mathematics and what, in addition to my classroom-attuned responses, I could be metacommenting upon.

VERBAL METACOMMUNICATION

The term metacommunication was introduced by Ruesch and Bateson (1951), where the concept was developed from detailed study of animal behaviour. Described as “an entirely new order of communication” (p.209) and defined as “communication about communication” (p.209), this new order of communication allowed Ruesch and Bateson (1951) to explain some complex and paradoxical attributes of social interaction. Any instance of interpersonal communication will consist of a “report” (p.179) aspect, synonymous with the content or data of the message, and a “command” (p.179) aspect, referring to the relationship between the communicants. According to Watzlawick et al. (1967), the report aspect of a message conveys information whereas the command aspect concerns how the communication is to be taken and therefore ultimately to the “relationship between the communicants” (p.33). It is the relationship aspect of communication, being a communication about a communication, that is, according to Watzlawick et al. (1967), “identical with the concept of metacommunication” (p.34).

Rossiter (1974) distinguished between two types of metacommunication: “that which is an ever-present aspect of all transactions and; that which constitutes additional commentary about communicative transactions” (p.36). The former type consists primarily of non-verbal cues, for example, tone of voice, body language or gesture, which can indicate whether the person communicating is, for example, serious or joking. These metacommunicational cues can provide information about how a message is to be interpreted “by indicating something about intentions and feelings of the message generator” (p.37). The latter type of metacommunication, which constitutes additional commentary, could be understood as simply ‘talking about talking’ and occurs whenever verbal and/or nonverbal communication becomes the topic of communication itself. The focus for this paper is on my verbal metacommunication in-the-moment of a discussion.

In terms of verbal communication, metacommunicational clues may be highly ambiguous and can be easily interpreted in entirely different ways. It follows that the ability to metacommunicate appropriately “is not only the condition sine qua non of successful communication, but is intimately linked with the enormous problem of awareness of self and others” (Watzlawick et al., 1967, p.34). The position, that it is the ability to metacommunicate appropriately that is essential for successful communication, provides a further rationale for my study. In particular, how do I use verbal metacommunication when responding to pre-service teachers talking about teaching? Furthermore, what is the process of learning to respond in-the-moment in a metacommunicative way?

I have also found myself reflecting on my responses when working with in-service teachers of mathematics. I am currently working alongside a group of ten secondary school mathematics teachers working and learning through collaboration to develop the mathematical reasoning of the children in their classrooms and in their wider departments. Between each meeting of the collaborative group, the mathematics teachers try out ways of working in their classrooms and work with other mathematics teachers in their departments to do the same. My role in the group is to support a discussion where the teachers share ideas and stories and learn from one another through reflecting on what they have been doing in school. It is in this setting where I began to develop a methodology for researching my learning as a mathematics teacher educator through paying attention to what I was noticing.

THE DISCIPLINE OF NOTICING AS A METHODOLOGY

In the context of my research, the connection between self-awareness; awareness of others and; my own ability to respond with metacomments, has become a meaningful one. Having audio-recorded the first of my discussions with the group of mathematics teachers, it was in the slow transcription of this discussion that I became aware of a shift in my attention at particular moments of a teacher speaking. In feeling this reaction in-the-moment of hearing the audio-recordings, I was “noticing” (Mason, 2002), making a distinction by distinguishing “some ‘thing’ from its surroundings” (p.33).

Mason’s (2002) description of the Discipline of Noticing as four “interconnected actions”, specifically: “Systematic Reflection”; “Recognising”; “Preparing and Noticing” and; “Validating with Others” (p.95), offers me a framework for my research methodology. In attending to what I notice in a systematic way as I transcribe the audio-recorded discussions, I am able to “mark” (Mason, 2002, p.33) so that I can “re-mark upon it later to others” (p.33). This marking seems to manifest itself as an uncomfortable feeling, or a sense of surprise or confusion and signifies when a moment has salience. In “recording” (p.33) these salient moments they have become available for further evaluation.

Based on the idea that something may be salient because of “some hidden assumption or bias” (Mason, 2002, p.248), I wanted to minimise this issue by utilising multiple perspectives and by practising “being in question” (p.248) through “seeking resonance with others in an ever-expanding community’” (p.248). In sharing these salient moments with others in the mathematics education community, I was “creating the conditions for the emergence of the as-yet unimagined rather than […] perpetuating entrenched habits of interpretation" (Davis, 2004, p.184). Through the process of self-reflecting and considering multiple perspectives, I began to understand learning to respond as a "recursively elaborative process of opening up new spaces of possibility by exploring current spaces" (p.184).

This process of sensitising myself to notice the types of comments that may prompt a metacommunicative response has been significant in terms of supporting me to consider possible ways of acting differently in the future, that is, becoming a mathematics teacher educator. Having worked for some time on developing these awarenesses through the slow transcription of the discussions with the group of teachers, and from the position that an ability to metacommunicate appropriately is essential for successful communication in supporting groups of teachers working collaboratively, my attention has now turned to analysing how I am responding at a metacommunicative level.

FRAMEWORKS FOR ANALYSIS OF VERBAL METACOMMUNICATION

Studies of the use of verbal metacommunication exist most predominantly within research on psychotherapy where the focus is on the relationship between the therapist and the client, and in research about the role of children's social pretend play. From literature related to more formal educational settings, I present two frameworks for analysis of verbal metacommunicative responses.

Firstly, Rossiter (1974) argues that to improve the ability to communicate at an interpersonal level, it is key to master the capacity to metacommunicate. In his paper (Rossiter, 1974), which concerns the instruction of "courses which focus on interpersonal communication" (p.36) based on the concept of metacommunication, Rossiter offers four functions (see Table 1) of "oral verbal communication about face-to-face interpersonal communication that is in process" (p.37).

More recently, Baltzersen (2013) contended that any metacommunicative utterance can be analysed in relation to all three of the following basic dimensions: What, how and when you metacommunicate. He originally investigated the impact of Metacommunication in the supervision process in higher education in Norway through linking survey questions to the "metacommunication concept" (p.128). Though initially methods appear limited in terms of the conceptualisation of this metacommunicational concept (specifically, indicators of metacommunication are linked to: discussing the supervision process and; clarification of tasks and roles in supervisions) his study does suggest that "metacommunication may have a substantial positive effect on the quality of communication in thesis supervision" (p.130). Based on these findings, Baltzersen goes on to ask the question, "What kind of metacommunication is important to create good supervision in higher education?" (p.130). Baltzersen's exclusive focus on verbal metacommunication enables him to develop a framework that, though not exhaustive, allows review of different definitions and examples of verbal metacommunication used in a one-one supervision context. Baltzersen (2013), as with Rossiter (1974), also offers four functions of verbal metacommunication (see Table 1).

The functions of metacommunication, described by both Rossiter (1974) and Baltzersen (2013), are presented in Table 1 in a way that demonstrates the parallels that I have drawn out from the two sets.

	Rossiter (1974, p.37)	Baltzersen (2013)
(1)	To focus conscious attention on the process of interaction	To create and establish a working alliance (p.133, p.135)
(2)	To clarify vague feelings about what is going on	To talk about intentions (p.133)
(3)	To determine if perceptions of what is happening coincide	To pose clarifying questions (p.135)
(4)	To provide direct feedback about speaker's communication behaviour	To evaluate some aspect of the relationship between the persons interacting (pp.133-134)

Table 1: Functions of verbal metacommunication presented in parallel (adapted from Rossiter, 1974, p.37; Baltzersen, 2013, pp.133-135).

To offer some further elaboration, I explore each pair of functions from Table 1 in turn. Firstly, Rossiter (1974) begins with what he describes as the "most important function of metacommunication […] that it focuses conscious attention on the process of interaction" (p.37). This attention to the process allows participants in the conversation to take a step back from the interaction itself and look at how the communication system is functioning. In the same sense, Baltzersen (2013) describes the need to create and establish a working alliance through agreeing on specific tasks; agreeing on goals; and identifying possible strains in the relationship between participants (p.133). Secondly, Baltzersen's suggestion that verbal metacommunication can function to communicate intentions through talking about what the speaker has said, or through disclosing or asking for opinions about the conversation, closely resembles Rossiter's clarifying "vague feelings about what is going on" (p.37) in that verbal metacommunication of this form can suggest how participants in the conversation arrived at their present state through paying attention to the process factors that influence emotional responses to the interaction itself. Thirdly, Rossiter's purpose of determining whether perceptions of what is happening coincide (p.37) concerns the need for perceptions to be made as explicit as possible so that other participants in the conversation know how to respond to them. In a similar vein, Baltzersen describes posing clarifying questions through clarifying the speaker's own prior opinion or another speaker's opinion; paraphrasing; repeating something said earlier; commenting on language use; and regulating others (p.135). Finally, Baltzersen suggests evaluating some aspect of the relationship between the persons interacting through explicating disagreement and highlighting one's own role or another person's role in the relationship (pp.133-134). Similarly, Rossiter recommends verbal metacommunication in order to draw attention to *how* a speaker is communicating through providing direct feedback about the speaker's communication behaviour. These pairs of functions form a framework with which some of my responses from discussions with the collaborative group of mathematics teachers are now analysed in the next section.

ANALYSING RESPONSES

Before using the framework (Table 1) for analysing my responses as verbal meta-communications, I needed to consider which responses could be fundamentally considered as verbal metacommunications (a communication *about* a communication), or alternatively, as a communication in direct response at the level of the discussion. In order to exemplify this distinction, consider the following two vignettes. Each vignette comprises a short extract of transcription taken from audio-recorded discussions with the group of mathematics teachers. Both vignettes provide a different paradigmatic example that are representative of a set of similar responses.

Vignette 1:

Teacher: I was just thinking of a time a couple of weeks ago when I was doing conversions and um, we were doing area and volume conversions, but part of the starter was just simple conversions and a kid from a top set was convinced that to get from millimetres to centimetres, you times by ten and even putting examples up he still was convinced no it was times by ten so even though he knows there are ten millimetres in one centimetre he still was convinced you times by ten so I don't really understand how to…

Tracy: Well it is, isn't it, you kind of are timsing by ten, it's ten times bigger, I guess maybe that's where that's coming from.

Vignette 2:

Teacher: I was just thinking back to a session I went to… and a lot of what we are discussing now here is very talk based, and is there almost a case with some of the things we are modelling to promote reasoning, we say a lot less, just show them, break it down into manageable steps, so I did this, linking area of rectangle to area of triangle, I taught that normally last term, it didn't go down very well.

Tracy: What do you mean by normally?

In vignette 1, the teacher is describing an issue with a student who was converting millimetres to centimetres. My response, "Well it is, isn't it, you kind of are timsing by ten, it's ten times bigger, I guess maybe that's where that's coming from", which I do not consider to be a verbal metacommunication, was a direct response at the level of the original communication. I was suggesting an explanation for the situation being described.

In vignette 2, the teacher is describing a lesson where he presented to the students, in silence, a series of images linking the area of a rectangle to the area of a triangle as an alternative to an approach he had used previously to teach the concept. He describes this previous approach as being taught "normally" to which I respond immediately with “What do you mean by normally?” In relation to the functions presented in Table 1, I would argue that the purpose of this response was “to determine if perceptions of what is happening coincide” through posing clarifying questions. Working on an account of the notion of “normally”, allows others to create an image of this teacher's classroom that might otherwise not be possible.

I now present one further vignette comprising of another short extract from a discussion with the group of mathematics teachers. I have chosen this final extract as a paradigmatic example of a response that I understand to be a verbal metacommunication but that becomes problematic when trying to describe it using the functions presented in Table 1. For context, the extract from vignette 3 follows on shortly from the extract from vignette 2 and is the same teacher speaking. Having described using the set of images for areas of rectangles and triangles, the teacher goes on to describe offering the students a problem, involving finding rectangles with equal area and perimeter. In the comment from vignette 3, the teacher is reflecting about having noticed a change in the energy of the students compared with previous lessons.

Vignette 3:

Teacher: Um, yeah, from what I thought would be kind of do and review of something at quite a low level and I’d have to really go over here’s how you do area, here’s how you do perimeter, actually it then turned into they did it all themselves, and you know in the class you get hands up all the time, it wasn’t sir help me, it was sir look at this, look at this, look at this I did it!

Tracy: Oh, that’s nice, so the difference was in hands.

In isolation, “Oh, that’s nice” is ambiguous. However, the second part of the response, “so the difference was in hands” offers an indication as to what I was valuing in that moment, using “so” as the link would suggest the “nice” was in recognition of the previous speaker’s acknowledgement of an observed difference, in this case, a different reason for hands going up. Is this communication about communication? Having made the comment myself, I do of course have an insider perspective. One awareness that I know I have is when a teacher talks about a change in their behaviour or that of their students. When this happens, I often find myself highlighting that a difference has been noticed and how this difference has been observed. One function of doing this is to direct the attention of others; to invite others to consider differences in their own classrooms and; to emphasise the importance of these types of observations as a classroom teacher working on their teaching. This function seems to me to be in a difference place to those in existing frameworks.

REFLECTING ON THE PROCESS OF LEARNING TO RESPOND

There is a motto of noticing which Mason (2002) alerts us to that is “I cannot change others, I can work at changing myself” (p.248). As a mathematics teacher, my conviction came from having an image of what teaching could look like and I worked hard to establish a verbal metacommentary that went alongside my students working on mathematics. In becoming a mathematics teacher educator through the process of sensitising myself to notice when a verbal metacommunication may be appropriate, and for what purpose, I am learning how to support and enable teachers working and learning through collaboration.

As I continue researching how I am learning to respond as a mathematics teacher educator, it is inevitable that further categorisations of verbal metacommunicative responses will emerge. One contribution to the field of mathematics education and, in particular, to mathematics teacher education and teacher educator learning might be a framework for systematically categorising verbal metacommunicative responses when working with teachers of mathematics. The classifications that emerge will principally be of value to me as a researcher of my own learning who is immersed in the process of developing this framework. By making these categorisations or distinctions, I am supporting further possibility of responding differently both now and in the future and I am reminded to return to an image of learning from Davis (2004) as a "recursively elaborative process of opening up new spaces of possibility by exploring current spaces" (p.184).

References

Baltzersen, R. K. (2013). The Importance of metacommunication in supervision processes in higher education. *International Journal of Higher Education, 2(2),* 128-140.

Davis, B. (2004). *Inventions of teaching: A genealogy*. New York: Routledge.

Krainer, K., Chapman, O., & Zaslavsky, O. (2014). Mathematics teacher educator as learner. In Lerman, S. (Ed.), *Encyclopedia of Mathematics Education*, 431-434. Netherlands: Springer.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge.

Pimm, D. (1994). Mathematics classroom language: Form, function and force. In Biehler, R., Scholz, R. W., Sträßer, R., & Winkelmann, B. (Eds.), *Didactics of mathematics as a scientific discipline*, 159-169. Dordrecht: Kluwer Academic Publishers.

Rossiter Jr, C. M. (1974). Instruction in metacommunication. *Communication Studies, 25(1),* 36-42.

Ruesch, J., & Bateson, G. (1951). *Communication: The social matrix of psychiatry*. New York: WW Norton & Company.

Watzlawick, P., Beavin, P., & Jackson, D.D. (1967). *Pragmatics of human communication. A study of interactional patterns, pathologies and paradoxes*. New York: Norton.

THE ROLE OF FINGER GNOSIS IN THE DEVELOPMENT OF EARLY NUMBER SKILLS

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The role of fingers in the development of early number skills has often been the focus of discussion in mathematics education, psychology and neuroscience. This study describes the findings of a longitudinal exploration of the mathematical development of children with Apert syndrome. Children with Apert syndrome are born with their fingers fused and even after surgery to separate them, do not often use their fingers spontaneously in activities involving number. Through observations over a 2 year period, the role of fingers in supporting learning and activities in numerical aspects of mathematics was seen to be complex and requiring good finger awareness and finger mobility. The findings suggest a possible explanation for the observation that some children who are low-attaining in mathematics are over-dependent on finger-use.

WHAT CAN CHILDREN WITH APERT SYNDROME TELL US ABOUT THE ROLE OF FINGERS IN THE DEVELOPMENT OF EARLY NUMBER SKILLS

The work discussed here describes the findings of a longitudinal 2-year study on the mathematical development of 10 children with Apert syndrome, between the ages of 4 and 9 years at the beginning of the study (Hilton, 2017). Apert syndrome is a rare syndrome which was first described by Wheaton in 1894, and investigated further by Apert in 1906 (Patton, Goodship, Hayward and Lansdown, 1988). There is an estimated a birth prevalence of Apert syndrome of approximately 1 in 65000, in North America and Europe (Cohen et al., 1992; Tolorova, Harris, Ordway and Vargervik, 1997). Children with Apert syndrome are born with complex fusions of their fingers and although they usually have surgery to release their fingers, they do not always gain five fingers (digits including thumbs) on both hands. In addition, children with Apert syndrome usually have limited mobility in their fingers, as the interphalangeal joints do not work properly. Although there is only limited literature on the mathematical development of children with Apert syndrome, the literature that does exist suggests that for these children, numerical activities are a particularly area of difficulty (Sarimski, 1997; Fearon and Podner, 2013). The present study shines a new light on the mathematical development of children with Apert syndrome and especially on the role of fingers in the development of early number concepts and early arithmetic. It also highlights the complex nature of the relationship between the use of fingers and problem solving in numerical calculations.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 83-90). Umeå, Sweden: PME.

The original research explored the strategies children with Apert syndrome use to help them solve numerical problems in mathematics and whether the children's hand anomalies impacted the range of strategies available to them.

THE THEORETICAL FRAMEWORK

The theoretical framework for the process of data collection was informed by constructivist grounded theory (Charmaz and Bryant, 2011) as it allowed for the possibility of unexpected and unanticipated findings. For the process of data analysis, the methods used were drawn from discursive analysis and thematic analysis. In order to collect data, a case study approach was adopted.

LITERATURE REVIEW

There have been a number of studies that have explored the link between finger gnosis and skills in arithmetic. It has been shown that touching objects when counting helps pre-school 4 year old children to count correctly (Alibali and DiRusso, 1999). This can help children to understand one-to-one correspondence and can relieve the pressure on working memory. Fingers can also help when trying to keep track of items and during calculations. With practice, children learn to map particular patterns on to particular numbers (Morrissey, Liu, Kang, Hallett and Wang, 2016). In other words, through repetition and practice, fingers can provide a sensorimotor embodied mapping of number patterns and their associated numerical relationships (Rinaldi, Di Luca, Henik and Girelli, 2016).

For these mappings to be effective requires an awareness of one's own fingers, or "finger sense", otherwise known as finger gnosis (Gerstmann, 1940) and finger mobility (Berteletti and Booth, 2015). Without this finger sense, it may be hard to identify one's own fingers in response to touch and request; make individual finger movements; and mirror the finger actions of others (Gerstmann, 1940).

In typically-developing children, finger gnosis develops quickly up to the age of 6 years and then continues to develop more slowly up to the age of 12 years (Strauss, Sherman and Spreen, 2006). Berteletti and Booth (2015) argue that the embodied actions of moving fingers as well as finger gnosis are significant in determining the role of fingers in early arithmetic. In addition, fingers are useful to keep track of items in a count (Andres, Seron and Olivier, 2007) or compare numbers presented symbolically (Sato, Cattaneo, Rizzolatti and Gallese, 2007). This evidence supports the findings from observational studies such as those by Hughes (1986) and Jordan, Huttenlocher and Levine (1992). However, this should be viewed within the context of finger-use in arithmetic being a learned, and not a spontaneous, activity (Crollen, Seron and Noel, 2011).

While there are cultural differences in the ways that children learn and are taught to use their fingers (Di Luca and Pesenti, 2011), it has also been suggested that "personal finger-counting habits influence the way numerical information is mentally represented and processed" (Berteletti and Booth, 2015, p.111) and stored in long-term memory (Di Luca and Pesenti, 2008). It seems likely that if fingers are used as a tool to support numerical calculations, the most significant factor is *whether* children learn to use their fingers rather than *how* they use them.

Jordan, Kaplan, Ramineni and Locuniak (2008) found that in kindergarten, children who used their fingers in calculations provided more accurate answers to questions. However, by the end of Year 3, those children who tended to be more accurate, used their fingers less frequently than those who made more calculation errors. As in the earlier study, Jordan et al. (2008) found that children from low-income families started kindergarten with less confident finger-use than their middle-income peers. Consequently, as the children from middle-income families were beginning to use their fingers less, children from low-income fingers continued to depend on their fingers for performing calculations. This suggests that it takes a considerable amount of time (in the region of 2 to 3 years) for children to transition from relying on fingers to help with arithmetic calculations to confidently using known facts and other strategies to support work with numbers.

Kaufmann et al. (2008), in a study involving 8 year old children and adults, used brain imaging techniques to explore the areas of the brain that are recruited when performing simple tasks involving number. In tasks involving non-symbolic representations of number, they found that although the children and the adults were able to complete the tasks successfully, children took longer. To explain this, the authors suggest that when making numerical comparisons using images of hands showing differing numbers of fingers, the children (but not the adults) recruited additional areas of the brain normally used for fingers. The authors suggest that fingers are an important stepping stone in the development of an abstract understanding of number.

Finger gnosis and fine motor skills have also been implicated in supporting the development of arithmetic and mathematical skills (Noel, 2005; Gracia-Bafalluy and Noel, 2008). Noel (2005) carried out assessments of finger gnosis with 41 children in Grade 1 and compared this with an assessment of their skills in mathematics one year later. A correlation was found between the children's level of finger gnosis in Grade 1 and their achievements in tasks involving number identification and simple arithmetic one year later. In fact, the relationship between finger gnosis and their achievement in mathematics was stronger than the relationship between tests of general cognitive ability and achievement in mathematics between Grades 1 and 2. This was followed up with an intervention study in which children were provided with a finger-differentiation intervention, twice a week for a period of 8 weeks. The children's finger gnosis and their numerical skills both improved, when compared to a control group (Gracia-Bafalluy and Noel, 2008).

Subitising, counting and the approximate number system

When children learn to make sense of numbers, there are many aspects of number that they need to come to understand. Subitising refers to the ability to enumerate small groups of objects without counting (Fuson, 1988). By the age of 3 years, children can usually subitise up to three objects. For adults, the maximum number is usually four (Hughes, 1986). Beyond subitising, it has been argued that there is a distinction between the ability to count and the ability to compare quantities (Dehaene, 2011).

Learning to count is no trivial task (Fuson, 1988) and all the principles of arithmetic that children learn at school are underpinned by an understanding of counting. The ability to count, though, is a human creation while the ability to compare quantities is a matter of survival (Dehaene, 2011). When we compare quantities we use our approximate number system (ANS) - a nonverbal mechanism for estimating the number of items in a set (Dehaene, 2011). This capacity is one that we also share with animals and must be distinguished from any symbolic or verbal representational system requiring accuracy. It has been suggested that there is a relationship between children's ANS and their attainment in mathematics (Halberda, Mazzocco and Feigenson, 2008) and that children who struggle with mathematics are more likely to have a poor ANS (Mazzocco, Feigenson, and Halberda, 2011).

RESEARCH METHODS

Semi-structured interviews and clinical interviews (Ginsburg, 1981) were used together with in-class observations of the children. The semi-structured interviews were designed to assess number knowledge, arithmetic skills and mathematical understanding.

Six or seven school visits were made to each of the children over the 2 year period of the study. When interviewing the children, the clinical interview approach made it possible to gain more in-depth understanding of the children's thinking. The interviews were audio recorded and later transcribed.

For the purpose of reliability, the mathematics-focused questions were based on existing assessments that had been reported in the literature. Due to the age range and developmental range within the children studied, a range of assessments was used. The assessments selected focused on number system knowledge, skills in arithmetic and strategies used for solving problems.

The children's Approximate Number System (ANS) was explored using "Panamath" (Halberda, Mazzocco and Feigenson, 2008), in order to establish whether there was a relationship between children's skills in this area and their knowledge and understanding in work on number and arithmetic.

The children's working memory was assessed, as this has been implicated as a potential reason for children's low attainment in mathematics (Raghubar, Barnes and Hecht, 2010). This was done with the "Working Memory Test Battery for Children (WMTB-C)" (Gathercole and Pickering, 2001)

Finally, the children's finger gnosis was assessed, as this was likely to be delayed in children with Apert syndrome and has been associated with knowledge and skills in number and arithmetic. For this an assessment of finger gnosis based on Gracia-Bafalluy and Noël (2008) was used.

FINDINGS

In the study group there was no relationship between ANS and attainment in mathematics. One of the lowest attaining children had the highest ANS scores. The children displayed a range of strengths and weaknesses in their working memory assessments, but an area of strength for all the children was the area of visuospatial skills. In terms of the mathematics assessments, there was enormous variation, but the focus for the purpose of this discussion will be on the use of fingers to support calculation.

Only one of the 10 children began to use his fingers without prompting and even he started very late (at 9 years of age). Initially school staff said that they did not encourage children to use their fingers because the children found it hard to move their fingers. The consequence of this was that when calculations took them beyond their working memory capacities, they were often unable to complete the activities. Joe, aged 7 years (who had four fingers on his left hand and five fingers on his right hand and good working memory skills) provides a good example of this:

Caroline: Right, which number is closer to seven, is it four or nine? [using visual array]

[9 second pause and then Joe points to the 9]

Caroline: Nine is closer. Why?

Joe: Because...ummm…nine minus two is seven

Caroline: Yep and what about the four?

Joe: Four…plus three

Caroline: So is that why nine is closer? [Joe nods]

Having seen this confidence the next example was a surprise:

Caroline: OK, how much is two plus four? [Joe is still for 5 secs] you can use your fingers, or I can give you some counters. How much is two plus four? [pause]

Caroline: Do you know what it would look like? Should I write it down for you?

Joe: Yeah [I write 2+4 on a piece of paper]

Caroline: Do you know how to do it?

Joe: No

Joe had a good working memory in most areas and he seemed to rely on this very heavily when doing numerical calculations. However when his working memory failed, he had no strategy to fall back on. He did eventually do this particular calculation with counters, but he needed prompting in order to see that this was a possible means of solving the problem.

Compare this with Emily, also aged 7 years (who had five fingers on her left hand and four fingers on her right hand) who had been doing finger gnosis training for at least 4 months and had then continued to use her fingers for mathematical calculations:

Caroline: Can you work out thirteen add 39? You can write it down if it helps… thirteen add thirty nine [spoken slowly as Emily writes 13+39] [pause]

Caroline: Do you know what it will be?

Emily: No I don't know what the answer is because…the trouble is the twelve….and I've got to add another ten on

Caroline: Yeah so what do you think this might be? [as I point to the calculation that Emily has written down] [pause] What's the strategy you could use to work it out?

Emily: Umm...nine and three...nine, ten, eleven, twelve [using fingers]. Now… fifty add two is fifty two [writes = 52]

For Emily fingers were a tool that she could use effectively to support with her calculation and to enable her to offload some of the work away from her working memory. This flexible use of fingers, as one tool among many, enabled Emily to complete the calculation quickly and efficiently.

CONCLUSIONS

Fingers seem to have a particular role to play in the development of children's early number skills. This study provides a new perspective because of the opportunity it provided to observe the implications on children's mathematical development when fingers were not used as a means of accessing and supporting numerical activities. When they were used, fingers provided a more reliable model than tools such as counters.

As the children in the study began to "know" their fingers, they could use them as tools to access the mathematical problems they sought to solve. This method was more reliable and easier than asking children to count out a given number of counters, especially as once children "know" their fingers, they do not need to count and so do not make the errors that often occur when counters, or similar tools, are used to help with solving numerical problems.

This study highlights in great detail, the special role that fingers can play in supporting children with arithmetic calculations. It identifies the need for practice in using fingers and specifically in developing finger gnosis at an early age in order to support sensorimotor development and to optimise the opportunities for children to develop mathematical confidence and competence.

The present study also suggests that if finger gnosis is not well-developed, children can experience a mismatch between their visual finger representations and the sensorimotor experience. If children's finger gnosis is poorly developed, it seem likely that their fine motor skills will also be affected, as they will find it hard to identify individual fingers. This is an area that deserves further exploration as a potential explanation for the observation that some children fail to use their fingers to help with mathematics, while others become over-dependent on the visual representation without a genuine "feel" for the numbers that their fingers represent.

References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive development*, *14*(1), 37-56.

Andres, M., Seron, X., & Olivier, E. (2007). Contribution of hand motor circuits to counting. *Journal of Cognitive Neuroscience*, *19*(4), 563-576.

Berteletti, I & Booth, J.R (2015). Finger Representation and Finger-Based Strategies in the Acquisition of Number Meaning and Arithmetic. In D. B. Berch, D. C. Geary & K. M. Koepke (Eds.). *Development of Mathematical Cognition: Neural Substrates and Genetic Influences* (Vol. 2) (pp.109-139). London: Academic Press.

Charmaz, K., & Bryant, A. (2011). Grounded theory and credibility. In Silverman, D. (Ed.) *Qualitative research* (3rd ed.). London: Sage.

Cohen, M.M., Kreiborg, S., Lammer, E.J., Cordero, J.F., Mastroiacovo, P., Erickson, J.D. … & Martínez-Frías, M.L. (1992). Birth prevalence study of the Apert syndrome. *American journal of medical genetics*, *42*(5), 655-659.

Crollen, V., Seron, X., & Noël, M. P. (2011). Is finger-counting necessary for the development of arithmetic abilities?. *Handy numbers: finger counting and numerical cognition*, 25.

Dehaene, S. (2011). *The Number Sense [How the Mind Creates Mathematics]*. 2nd Ed. Oxford: Oxford University Press.

Di Luca, S., & Pesenti, M. (2008). Masked priming effect with canonical finger numeral configurations. *Experimental Brain Research*, *185*(1), 27-39.

Di Luca, S., & Pesenti, M. (2011). Finger Numeral Representations: More than Just Another Symbolic Code. *Frontiers in Psychology*, *2*, 272.

Fearon, J.A., & Podner, C. (2013). Apert syndrome: evaluation of a treatment algorithm. *Plastic and reconstructive surgery*, 131(1), 132-142.

Fuson, K. C. (1988). *Children's Counting and Concepts of Number*. New York, NY: Springer-Verlag.

Gathercole, S. E., & Pickering, S. (2001). *Working memory test battery for children (WMTB-C)*. London: The Psychological Corporation.

Gerstmann, J. (1940). Syndrome of finger agnosia, disorientation for right and left, agraphia and acalculia: Local diagnostic value. *Archives of Neurology and Psychiatry*, *44*(2), 398.

Gracia-Bafalluy, M., & Noël, M. P. (2008). Does finger training increase young children's numerical performance? *Cortex*, *44*(4), 368-375.

Ginsburg, H. (1981). The Clinical Interview in Psychological Research on Mathematical Thinking: Aims, Rationales, Techniques. *For the Learning of Mathematics*, 1(3), 4-11.

Halberda, J., Mazzocco, M. & Feigenson, L. (2008). Individual differences in nonverbal number acuity predict maths achievement. *Nature*, 455, 665-668.

Hilton, C. (2017). *The mathematical development of children with Apert syndrome* (Doctoral dissertation, UCL (University College London)).

Hughes, M. (1986). *Children and number: Difficulties in learning mathematics*. Oxford: Wiley-Blackwell.

Jordan, N. C., Huttenlocher, J., & Levine, S. C. (1992). Differential calculation abilities in young children from middle-and low-income families. *Developmental Psychology*, *28*(4), 644.

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2008). Development of number combination skill in the early school years: when do fingers help?. *Developmental Science*, *11*(5), 662-668.

Kaufmann, L., Vogel, S. E., Wood, G., Kremser, C., Schocke, M., Zimmerhackl, L. B., & Koten, J. W. (2008). A developmental fMRI study of nonsymbolic numerical and spatial processing. *Cortex*, *44*(4), 376-385.

Kinsbourne, M., & Warrington, E. K. (1962). A study of finger agnosia. *Brain*, *85*(1), 47-66.

Mazzocco, M. M., Feigenson, L., & Halberda, J. (2011). Impaired acuity of the approximate number system underlies mathematical learning disability (dyscalculia). *Child development*, *82*(4), 1224-1237.

Morrissey, K. R., Liu, M., Kang, J., Hallett, D., & Wang, Q. (2016). Cross-Cultural and Intra-Cultural Differences in Finger-Counting Habits and Number Magnitude Processing: Embodied Numerosity in Canadian and Chinese University Students. *Journal of Numerical Cognition*, *2*(1), 1-19.

Noël, M.P. (2005). Finger gnosia: a predictor of numerical abilities in children? *Child Neuropsychology*, *11*(5), 413-430.

Patton, M.A., Goodship, J., Hayward, R. & Lansdown, R. (1988). Intellectual development in Apert's syndrome: a long term follow up of 29 patients. *Journal of medical genetics*, *25*(3), 164-167.

Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and Individual Differences*, *20*(2), 110-122.

Rinaldi, L., Di Luca, S., Henik, A., & Girelli, L. (2016). A helping hand putting in order: Visuomotor routines organize numerical and non-numerical sequences in space. *Cognition*, *152*, 40-52.

Sarimski K. (1997) Cognitive functioning of young children with Apert's syndrome. *Genet Counsel 8*, 317–322.

Sato, M., Cattaneo, L., Rizzolatti, G., & Gallese, V. (2007). Numbers within our hands: modulation of corticospinal excitability of hand muscles during numerical judgment. *Journal of Cognitive Neuroscience*, *19*(4), 684-693.

Strauss, E., Sherman, E. M., & Spreen, O. (2006). *A compendium of neuropsychological tests: Administration, norms, and commentary*. New York, NY: Oxford University Press.

Tolarova, M.M., Harris, J.A., Ordway, D.E., & Vargervik, K. (1997). Birth prevalence, mutation rate, sex ratio, parents' age, and ethnicity in Apert syndrome. *American journal of medical genetics*, *72*(4), 394-398.

PROFESSIONALISATION OF PROSPECTIVE TEACHERS THROUGH THE PROMOTION OF COGNITIVE DIAGNOSTIC COMPETENCE

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Teacher's knowledge about student's cognition is important in order to recognize the deficits of the students, to analyse them and to give appropriate support (Kunter et al. 2013). Thus, the presented DiMaS-net project focus on the professionalisation of prospective teachers regarding their diagnostic competence. A specific seminar for becoming secondary teachers was developed and with a pre-post Design the increase of teachers' diagnostic competence was investigated. In this paper we will describe the teacher training and present first results concerning the improvement of the perceived self-efficacy.

THEORETICAL BACKGROUND

Research projects on professional knowledge of teachers

The basis of quality teaching is the knowledge and skills acquired in the training in theoretical and practical phases of teacher training (Bromme, 2008), whereby Shulman´s taxonomy forms the basic framework for describing teachers' professional knowledge. He distinguishes between the four knowledge dimensions of general pedagogical knowledge (GPK), content knowledge (CK), curricular knowledge (CK) and pedagogical content knowledge (PCK) (Shulman, 1986; 1987). Shulman characterises the latter as a "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman 1987, p. 8).

This taxonomy is well known in teacher education research and is contained in many review articles (e.g. Baumert et al., 2010; Ball et al., 2008). It often forms the basis for various models, including those in research projects such as COACTIV, TEDS-M and the Michigan Group. Kunter et al. (2013) have developed a model for the professional competence of teachers that also includes the professional knowledge of teachers. It is subdivided into the competence areas of content knowledge, pedagogical content knowledge, pedagogical/psychological knowledge, organizational knowledge and counseling knowledge. Particularly relevant for the DiMaSnet project is the pedagogical content knowledge which in this model is subdivided into knowledge about the didactic and diagnostic potential of tasks as well as their cognitive requirements, knowledge about the mathematical thinking and the conceptions of pupils, and knowledge about various explanatory possibilities (Kunter et al., 2013). The study COACTIV thus placed a special focus on the knowledge of teachers (not students) about mathematical student cognitions.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 91-98). Umeå, Sweden: PME.

The research group around Ball and Hill also sees the "knowledge of content and students" as a part of mathematical knowledge for teaching and refers to the "knowledge of common student conceptions and misconceptions about particular mathematical content" (Ball et al., 2008, p. 401). The dissertation by Heinrichs (2015) lays a special focus on the process of error detection, identification of causes and subsequent handling of the error under the generic term diagnostic competence.

Diagnostic competence

Horstkemper (2006) describes diagnostic competence "as the basic qualification of all teachers" (p. 4, translation of the author), as it has, among other things, a great significance for dealing with heterogeneity, individual advancement and the support of learning processes (Bos & Hovenga, 2010). However, this is not a universal, but rather an area-specific ability, which according to Heinrichs' findings cannot be transferred from one mathematical content to another (Heinrichs, 2015; Spinath, 2005). Already Ginsburg (1977) recognized "the child´s failure is often the result of a procedure, which is organized and has sensible origins" (p. 49). If the teachers or trainee teachers receive knowledge about various misconceptions, it is easier for them to identify mistakes in a lesson (Reiss & Hammer, 2013). Misconceptions can be revealed by appropriate diagnostic tools and methods, and the prospective teacher can help the learner to correct the error (Lorenz, 1984).

Within our project we have set ourselves the aim to train prospective teachers in analysing and interpreting students' thinking processes and misconceptions that lead to mistakes in mathematics lessons. We aim at drawing their attention to the fact that faulty student cognitions are the cause of typical student errors and difficulties (Kunter et al., 2013).

The following definition clarifies the construct and the goal of the project DiMaS-net:

> Cognitive-diagnostic competence includes the teachers' conceptual mathematical content knowledge and knowledge of preferred ways of learners working and their thinking about mathematical topics that are explored using a variety of diagnostic methods.

If the prospective teacher is able to recognize, analyse and classify a student's misconception in a concept, then he or she has the possibility to design an insightful learning process as a learning opportunity (Kunter et al., 2013). In addition, practical experience (such as a diagnostic interview) can be a useful learning environment for developing diagnostic competence (Hascher, 2008).

Perceived self-efficacy

Another construct that we used within the framework of our study is the perceived self-efficacy regarding the recognition of thought and misconceptions. It refers to the social-cognitive theory of Bandura (1997) and for him "perceived self-efficacy is concerned not with the number of skills you have, but with what you believe you can do with what you have under a variety of circumstances" (p. 37). The perceived self-efficacy can be distinguished by its degree into generality, specificity and area specificity, whereby the teacher's perceived self-efficacy is a good example of the area specificity. Accordingly, individual statements include "the convictions of teachers to successfully master difficult demands of their professional life even under adverse conditions" (Schwarzer & Jerusalem, 2002, p.40, translation of the author). Teachers with a high perceived self-efficacy conceive a challenging teaching concept and show more patience in dealing with students having learning difficulties (Schwarzer & Jerusalem, 2002). In addition, literature shows a positive correlation between perceived self-efficacy and performance, which could possibly indicate a positive relationship between perceived self-efficacy and the recognition of thought and misconceptions (Schoreit, 2016).

RESEARCH QUESTIONS

The desiderata that are attempted to be clarified within our project arose from the theoretical background knowledge research.

1) How does cognitive diagnostic competence change in specific mathematical subject areas after prospective teachers of mathematics have participated in a diagnostic seminar?

2) What influence do diagnostic interviews have on the development of cognitive diagnostic competence?

3) What influence does dealing with known errors in literature have on the development of cognitive diagnostic competence?

4) How does the perceived self-efficacy of mathematics prospective teachers change regarding cognitive diagnostic competence by participation in the diagnostic seminar?

Only the last research question will be discussed in more detail within this paper.

DESIGN OF THE STUDY

The study described above was carried out as part of the DiMaS-net project (diagnosis and individual promotion of mathematics teaching in secondary schools through networking teacher education and training), which was financed by the “Quality Initiative for Teacher Education” programme of the Federal Ministry of Education and Research in Germany. Within the study, a four-hour block seminar of 180 minutes each, as intervention and data collection material were developed and piloted in winter term 2015/16 and summer term 2016. The seminar was held under the theme "diagnosis and support in teaching mathematics in secondary schools" and was aimed at prospective teachers for secondary schools. Subsequently, a revision took place, as well as the main study in winter term 2016/2017 and summer term 2017. Thematically, the arithmetic topics of "whole numbers and percentages" were in the focus of both diagnosis and support. 124 prospective secondary school teachers took part in the intervention and were divided into four experimental conditions.
The first experimental group (EG1) visited the complete seminar and also conducted a diagnostic interview between the third and fourth seminar. The second experimental

group (EG2) also participated in the complete seminar, but did not conduct a diagnostic interview. The third experimental group (EG3) received only a 90-minute input on errors and associated notions and misconceptions in the field of whole numbers, and the fourth experimental group (EG4) is a waiting group that instead attended a seminar about media in mathematic lessons.

CONTENT OF THE SEMINAR

In the first seminar session, the prospective teachers dealt with general topics related to diagnosis. Terms such as competence, professional competence and diagnostic competence were defined, and examples were given of how process- and product-oriented diagnosis can take place in mathematics lessons. As a process-oriented diagnostic option, the diagnostic interview was intensively examined.

The second seminar session covered typical mistakes and associated thinking processes in the subject areas of whole numbers and percentages. They were collected and recorded in a mind map by analysing tasks with the corresponding incorrect student solutions. The student solution considerations were based on a certain scheme. It is based on the general mathematical competences of the educational standards in Germany and was clarified in a process diagram within the study. Accordingly, each student solution is examined according to the following points:

- K6 - comprehending the given task
- K2 - devising solution strategies
- K3 - writing down necessary equations as mathematical models (if the task was contextual)
- K5 - working technically, calculating
- K3 - translating the solution back into the given context

The following task is typical of the tasks discussed.

At a construction site, a large hole is being dug, which will later become the basement of a detached house. Construction workers dig a 3-meter-deep hole. After consultation with the site supervisor, the hole must be dug out by another 2 meters in depth. How deep must the hole be dug in total?

Student solution:

$$-3\,m-(-2\,m) = -3\,m+2\,m = -5\,m$$

A total of 5 m must be dug deep.

First of all, the prospective teacher asks himself whether the student (whose solution is considered) was able to grasp the content of the task. For example, if he/she has not been able to extract all important information from the task text or if he/she has misunderstood the task text, then this mistake is based on a “K6-deficit”.

Afterwards it is checked whether the student is able to identify a helpful solution strategy. Through the scheme, the prospective teacher can examine the student solution step by step and thus give a differentiated statement about possible deficits and underlying thought processes.

The third seminar session allowed prospective teachers to work with the FIMS (Failure Diagnostic Interviews in Maths lessons of secondary schools) developed within the project. Video excerpts were analysed which show prospective teachers during the interview. Based on the interview situation, the prospective teacher receives detailed information about the mathematical competences of the student and an insight into the thought processes, procedures and solutions.

Depending on the examination conditions, some participants conducted a diagnostic interview with a pupil of their choice between the third and fourth seminar. During the fourth seminar session the prospective teachers discussed this interview situation and reported about their experiences. The rest of the fourth seminar session was devoted to the topic of necessary support for errors and faulty ideas in the fields of whole numbers and percentages.

The seminar design took into account the sources for the development of perceived self-efficacy in order to influence them. If a person achieves his or her own successes, this has the strongest effect on perceived self-efficacy due to his or her own efforts and performance. Since the prospective teachers themselves conducted a diagnostic interview, this first source of perceived self-efficacy was taken into account in the seminar. The second-largest impact has "vicarious experiences through observations of behavioural models" (Schwarzer & Jerusalem 2002, p. 42 translation of the author). In the third seminar, participants worked with video sequences, observed other students during the interview and learned from their mistakes if necessary. In addition, linguistic motivations such as "You can do this" and their own emotional arousal, such as fear, can also have an influence on perceived self-efficacy.

METHODOLOGY

The present study has a quasi-experimental design to check hypothesis. For quantitative data collection within a pre- and post-questionnaire, a performance test was used to determine the diagnostic competence and a questionnaire with 16 items was administered that recorded the constructs motivation and perceived self-efficacy.

The instrument for investigating perceived self-efficacy

At the beginning and the end of the seminar, the participants answered an identical questionnaire (quest.) on perceived self-efficacy averaged in 8 minutes, in which they assessed themselves about the recognition of students' thought processes and misconceptions in the subject areas whole numbers and percentages. The scales used in our study were adapted from existing scales. The perceived self-efficacy construct consisted of 7 items and had a reliability of $\alpha = .87$ within the piloting. Reliability in the main inspection is also satisfactory: $\alpha\_Pre = .851$ and $\alpha\_Post = .874$.

On a unipolar rating scale with the six verbal levels False (1), Mostly false (2), More false than true (3), More true than false (4), Mostly true (5) and True (6), the prospective teacher could give a positive or negative answer. The following example is representative of the raised scale of perceived self-efficacy:

I trust myself to diagnose the thoughts and misconceptions of my students.

The discriminatory power of the items is bigger than .500 in the pre-questionnaire and bigger than .552 in the post-questionnaire. To examine the change in perceived self-efficacy caused by the complete intervention (experimental condition 1 (EG1)), descriptive values are first presented and with the help of a t-test, it is examined whether the pre- and post-questionnaire differ significantly. The individual experimental conditions are then evaluated by an analysis of variance (ANOVA).

RESULTS

The first table present descriptive dates from the experimental condition 1 (EG 1).

	N	m	SD	emp. min.	emp. max.
pre-quest.	34	3,634	0,7763	2,3	5,6
post-quest.	33	4,377	0,6027	2,4	5,3

$t(30)=4.628$, $p< .01$, $d=0.748$

Table 1: descriptive dates from the EG1

Analysis by means of a t-test for dependent samples shows a significant difference with a medium effect (effect size Cohen´s d= 0.748) in the mean values between the pre- and post-questionnaire. Through the intervention, prospective teachers of EG1 have raised their perceived self-efficacy regarding the recognition of thinking and misconceptions. Now the different experimental conditions are compared with each other.

Perceived self-efficacy N	EG1	EG2	EG3	EG4
pre-quest.	34	23	40	23
post-quest.	33	23	36	20

Table 2: Number of participants in the experimental conditions

Perceived self-efficacy	EG1		EG2		EG3		EG4	
pre m/SD	3,634	0,7763	4,100	0,7556	3,651	0,9005	3,765	0,5157
post m/SD	4,377	0,6027	4,503	0,4871	3,889	0,8239	3,757	0,6938

Table 3: Mean value and standard deviation within the pre- and post- questionnaire

When comparing the individual experimental conditions, differences can already be found regarding the mean difference. For example, the experimental group 4 (EG4) shows no changes and the difference in group EG1, which has carried out the whole intervention including diagnostic interviews, is biggest compared to the other mean differences.

An analysis of variance (ANOVA) with measurement repetition taking into account the experimental condition as covariate shows a significant interaction effect between the independent variables time of the questionnaires and the experimental condition ($F(1{,}041) = 3{,}551$; $p = 0{,}017$; partial $\eta^2 = 0.095$). The experimental condition thus has a medium effect on the perceived self-efficacy. Furthermore, it will be investigated between which experimental conditions the changes between pre- and post-questionnaire are significantly different.

	EG1/ EG3	EG1/ EG4	EG1/ EG2	EG2/ EG3	EG2/ EG4	EG3/ EG4
F-value	5,710	7,947	2,465	0,247	1,828	1,294
p-value	0.020*	0.007**	0.123	0.621	0.184	0.261
partial η^2	0.083	0.142	0.046	0.005	0.045	0.025
selectivity	0.653	0.789	0.338	0.078	0.261	0.200

Table 4: Significant differences between the experimental conditions

Significances only occur in one pair of experimental conditions. The group (EG1) which took part in the total intervention differs significantly from groups EG3 (group with 90-minute whole numbers) and EG4 (waiting group). However, there is a big effect (effect size partial η^2= 0.142) regarding EG4, and only a medium effect (effect size partial η^2= 0.083) regarding EG3. Hence there are only significant differences between EG1 and EG3 and also EG1 and EG4. In particular, the significant difference between EG1 and EG4 shows that the complete intervention has an impact on perceived self-efficacy.

SUMMARY AND OUTLOOK

The evaluation of the data shows that an increase in perceived self-efficacy is possible through the intervention (EG1). The comparison of the different experimental conditions also shows an influence on the perceived self-efficacy. On the one hand, this can have a positive effect on teaching, as prospective teachers will try to understand the students' thinking and misconceptions more intensively. On the other hand, the increased perceived self-efficacy could indicate an increased diagnostic competence through intervention, since it is known from the literature that performance and perceived self-efficacy are related. This will be evaluated in the next phase of our project.

References

Ball, D. L.; Thames, M.; H. Thames, G. Phelps (2008): Content Knowledge for Teaching. What Makes It Special? In: *Journal of Teacher Education* 59 (5), 389–407.

Bandura, A. (1997): *Self-Efficacy*. New York: Freeman.

Baumert, J. et al. (2010). Teachers’ Mathematical Knowledge, Cognitive Activation in the Classroom, and Student Progress. *American Educational Research Journal, Vol. 47,* No. 1, 133-180.

Bos, W. & Hovenga, N. (2010): Diagnostische Kompetenz - besser individuell fördern. In: *Schule NRW* (8), 383–385.

Bromme, R. (2008): Lehrerexpertise, Teacher´s skill. In: W. Schneider und M. Hasselhorn (Eds.): *Handbuch der Pädagogischen Psychologie* (Bd. 10, pp. 159–167). Göttingen: Hogrefe.

Ginsburg, H. (1977): The psychology of arithmetic thinking. In: *The journal of children´s mathematical behavior,* 4(1), 1–89.

Hascher, T (2008): Diagnostische Kompetenz im Lehrerberuf. In C. Kraler & M. Schatz (Eds.): *Wissen erwerben, Kompetenzen entwickeln* (pp. 71-86). Münster: Waxmann.

Heinrichs, H. (2015): *Diagnostische Kompetenz von Mathematik-Lehramtsstudierenden*. Wiesbaden: Springer Spektrum

Horstkemper, M. (2006): Fördern heißt diagnostizieren. In: G. Becker, M. Horstkemper, E. Risse, L. Stäudel, R. Werning und F. Winter (Eds.): *Diagnostizieren und Fördern* (Bd. 24, P. 4-7). Seelze: Friedrich.

Kunter, M.; Baumert, J.; Blum, W.; Klusmann U.; Krauss, S. & Neubrand, M. (Eds, 2013). *Cognitive Activation in the Mathematics Classroom and Professional Competence of Teachers - Results from the COACTIV Project.* New York: Springer

Lorenz, J. H. (1984): Gibt es für Schüler einen Grund Fehler zu machen? In: *mathematik lehren* 5, 40–43.

Reiss, K.; Hammer, C. (2013): *Grundlagen der Mathematikdidaktik. Eine Einführung für den Unterricht in der Sekundarstufe.* Basel: Birkhäuser.

Schoreit, E. (2016): *Kompetent und trotzdem ängstlich?.* Kassel: University Press

Schwarzer, R. & Jerusalem, M. (2002). Das Konzept der Selbstwirksamkeit. In: *Zeitschrift für Pädagogik, 44. Beiheft,* 28–53.

Shulman, Lee S. (1986): Those who understand: knowledge growth in teaching. In: *Educational Researcher* 15 (2), 4–14.

Shulman, Lee S. (1987): Knowldege and teaching: foundations of the new reform. In: *Harward Educational Review* 57, 1–22.

Spinath, B. (2005): Akkuratheit der Einschätzung von Schülermerkmalen durch Lehrer und das Konstrukt der diagnostischen Kompetenz. In: *Zeitschrift für Pädagogische Psychologie* 19 (1/2), 85–95.

WHAT DO MATHEMATICIANS WISH TO TEACH TEACHERS IN SECONDARY SCHOOL ABOUT MATHEMATICS?

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This study investigates what mathematicians wish to teach teachers about what mathematics is. Data source included interviews with five research mathematicians who taught advanced mathematics courses to practicing secondary school teachers. Analysis revealed that expanding teachers' knowledge about what mathematics is was one of the main objectives of the interviewees. They referred to three aspects: (1) the essence of mathematics, (2) doing mathematics, and (3) the worth of mathematics. This paper characterizes and illustrates each aspect.

INTRODUCTION

In many countries, the education of secondary school mathematics teachers traditionally includes a strong emphasis on advanced mathematics courses at the college or university level, taught by mathematicians. This tradition has been reconsidered in recent years, and the relevance of academic studies of mathematics to secondary school mathematics teaching is being debated (e.g., Dreher, Lindmeier, & Heinze, 2016; Even, 2011; Murray et al., 2015; Wu, 2011). As part of a comprehensive study that examines what might be the relevance and contribution of academic mathematics courses, taught by research mathematicians, to teaching mathematics in secondary schools, the current study examines what mathematicians who teach such courses wish to teach teachers about what mathematics is.

THEORTICAL BACKGROUND

The empirical research literature on the relevance and contribution of academic studies of mathematics to teaching secondary school mathematics suggests a potential contribution at two levels of subject-matter knowledge. One level concerns knowledge of specific contents (e.g., Even, 2011; Zazkis & Leikin, 2010). For example, when interviewed about the contribution of their academic studies of mathematics to their teaching in secondary school, some mathematics teachers reported that they used the knowledge of specific topics they acquired to respond to students' questions or to enrich topics they taught. Yet, most studies reported on contribution at a more general epistemological level of knowledge about the nature of mathematics, about what mathematics is (e.g., Adler et al., 2014; Even, 2011; Zazkis & Leikin, 2010). For example, teachers reported that academic mathematical studies expanded their knowledge in aspects, such as, doing mathematics as problem solving, the role of intuition in doing mathematics, and the use of mathematics in other disciplines. These

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 99-106). Umeå, Sweden: PME.

new understandings enabled them to better represent the discipline of mathematics in their teaching.

As learning is shaped by teaching, a question arises: What do mathematicians, who teach academic mathematics courses to teachers, wish to teach teachers about the nature of mathematics? The existing literature concerning mathematicians' positions regarding academic mathematics studies of teachers is rather limited. It mainly comprises forewords appearing in mathematics textbooks intended for teachers (Klein, 1933/2016) and position papers written by a number of mathematicians who publish on educational topics (e.g., Wu, 2011; Ziegler & Loos, 2014). In most of these publications the mathematicians address the level of the nature of mathematics, emphasizing the importance of narrowing the gap between "what (research) mathematicians take for granted as mathematics and what teachers and educators perceive to be mathematics" (Wu, 2011, p. 382). Yet, different mathematicians suggest attending to different aspects. For example, Howe (Howe & Ma, 1999) stresses the characteristic of coherence and connectedness of mathematics:

> I would like to highlight the concern ... for the connectedness of mathematics, the desire to make sure that students see mathematics as a coherent whole. … A teacher who is blind to the coherence of mathematics cannot help students see it. (p. 885)

Wu (2011) emphasizes the importance of the fundamental principles of mathematics (e.g., definitions provide the basis for logical deductions), while Ziegler and Loos (2014) aim at broadening teachers' view of mathematics:

> …give them [teachers] a panoramic view on mathematics: …an overview of the subject, how mathematics is done, who has been and is doing it, including a sketch of main developments over the last few centuries up to the present (p. 9).

As seen, teachers report that academic mathematics studies contributed to their knowledge about the nature of mathematics, which mathematicians view as an important component of teachers' knowledge. Yet, our review of the literature reveals the deficiency of conceptual frameworks that could be used to examine what knowledge about the nature of mathematics mean. Moreover, empirical research that examines what mathematicians wish to teach teachers about the nature of mathematics is lacking. Our study addresses both these shortcomings of current research.

METHODS

Setting and Participants

The study was situated in a master's program, designed for practicing Israeli secondary school teachers of science and mathematics. A bachelor's degree in mathematics or in a mathematics-related field is required for admission to the mathematics strand of the program. A major component of this strand comprises eight academic mathematics courses, designed and taught by research mathematicians. Four of these courses deal with topics in the school curriculum at an advanced level: algebra, analysis, geometry, and probability and statistics. Three courses deal with the use and application of mathematics in other domains: computer science, natural sciences (applied mathematics), social sciences and everyday technologies. One course appraises the history and philosophy of mathematics. In addition, a final project that involves an independent study of an unfamiliar mathematical topic is carried out under the guidance of a mathematician.

Five of the seven mathematicians who teach in the mathematics strand of the program participated in the study. All are prominent research mathematicians, who usually teach only mathematics master and doctoral students. The five participating mathematicians teach all the mathematics courses in the program but two: algebra and the use of mathematics in computer science.

Data Collection and Analysis

The main data source included individual semi-structured in-depth interviews with the participating mathematicians. The aim was to learn what the mathematicians seek to teach teachers about mathematics and to reveal their views regarding the relevance and contribution of academic mathematics studies to secondary school mathematics teaching. The interviews included two main questions. The first focused on the general teaching goals of the mathematicians in the program. The second question focused explicitly on their teaching goals regarding what mathematics is. The interviews took about an hour and were recorded in audio. Additional data sources were participant observations in three courses: geometry, analysis, and the history and philosophy of mathematics, documented by field notes. The aim was to strengthen the internal validity of the study.

Data were analysed qualitatively. First, a full transcript of the interviews was made, followed by open coding and categorization in an iterative and comparative process. The aim was to identify what, if at all, the mathematicians wished to teach teachers about the nature of mathematics. In addition to the authors, three graduate students in the field of mathematics education participated in the coding process of about 20% of the data. All disagreements were resolved by discussion, so a consensus was reached.

FINDINGS

Analysis revealed that enriching, expanding and deepening teachers' knowledge about what mathematics is was a central goal of all the participating mathematicians. This became apparent at an early stage of all interviews, when the mathematicians were asked about their teaching goals in the program, before the topic of the nature of mathematics was explicitly raised by the interviewer. All the mathematicians stated at this point that knowledge about what mathematics is was as a major teaching objective. For instance (to ensure confidentiality, all the participating mathematicians are referred to as males, denoted as M1-M5),

> Interviewer: You teach the course... and advise students [in their final project]. What are your main objectives when you do these things?
>
> M2: The main objective is to explain what mathematics is...

All the interviewees mentioned also that they wished to enrich teachers' knowledge of specific mathematical contents. However, they attributed less importance to this goal. For example, M5 said, "This is actually what I think we need the entire program to concentrate on... to teach the mathematical method ... it is possible to teach it almost through any topic." Three mathematicians (M2, M3, M5) explicitly said that it was most likely that teachers would forget the specific contents they learned in the program. Thus, what they wished for was that teachers would remember different aspects of knowledge about mathematics:

> In my opinion, most of the material we teach will be lost, because they [the teachers] will forget it within half a year... But what we want to remain is the ability to understand, the ability to use the mathematical method... to have logic in what they do (M5).

Analysis of the interviews generated three aspects of the nature of mathematics that the participating mathematicians wanted to teach teachers: (1) the essence of mathematics, (2) doing mathematics, and (3) the worth of mathematics. Each of these aspects comprises two or three characteristics. In the following we describe and illustrate the characteristics of each aspect.

The essence of mathematics

The aspect *essence of mathematics* deals with the question: *What is this discipline called mathematics?* Data analysis revealed three main characteristics of the discipline of mathematics that the interviewees wished to teach teachers: (1) wide and varied (2) rich in connections (3) structured deductively.

Wide and varied

According to the participating mathematicians, teachers need to know that mathematics is a wide and varied discipline, which has many domains and many facets. For example, "I wish that what would happen to the teacher in the program… that the teacher would come and discover that there are many worlds in mathematics" (M4). The mathematicians stressed that teachers should be "introduced to different aspects of modern mathematics" (M1) and understand that mathematics continues to develop towards new and varied directions. M4 exemplified it: "… my last two lectures are always on chaos. The purpose of it, first, is to show them that mathematics is a science of the 21st century."

The mathematicians added that it is important for teachers to know that even areas that are familiar to them, such as, probability and algebra, are much broader than they commonly envision. That there exists mathematical knowledge in these areas –unfamiliar to most teachers – that helps to answer mathematical questions that the teachers' limited current knowledge cannot answer.

Rich in connections

All the mathematicians said that it was important for teachers to know that mathematics is rich in connections and that in order to properly understand mathematics one must be familiar with these connections. For example, M3 said that teachers need to be aware that there should not be disconnected fragments of knowledge in mathematics, and M1 emphasized "internal connections among the mathematical concepts and topics”. M1 demonstrated this type of connections, using the concepts of circle and ellipse. He argued that the connection between these two concepts is often misunderstood as if ellipse is a degenerate circle. However, the contrary is true because a circle is a degenerate ellipse.

Deductive structure

All the mathematicians said that teachers need to understand the deductive structure of mathematics. They referred to general characteristics and to elements of the deductive structure. General characteristics concerned with mathematics as a consistent science that is based on the laws of logic, its foundation lies on universal truths, and thus mathematics is not arbitrary: everything has a reason. For example, M2 said that one of his goals was:

> …to show that in mathematics, proofs and definitions, it is not that someone in the Ministry of Education determined those things, that it should be done this way and not another way. That it comes from natural and long-term attempts to understand things (M2).

With regard to the elements of the deductive structure, the mathematicians spoke of the need to understand in depth the roles of axioms, definitions, theorems and proofs. For example, three mathematicians mentioned the key role a successful definition could have and what a good definition is: "[The teachers should know] that in order to give definition one must be able to answer the question why this word stands here and what it signifies" (M5).

Doing mathematics

The aspect *doing mathematics* deals with the question: *How is mathematics done?* Data analysis revealed three main characteristics of mathematical activity that the mathematicians wished to convey to teachers: (1) asking questions, (2) thinking and understanding, and (3) using intuition and formalism.

Asking questions

The mathematicians argued that teachers need to know that a fundamental part of doing mathematics is asking questions. For example, "Questions are more important than answers. Once one is able to ask questions, a giant step forward has been taken" (M2). Two types of questions were mentioned. One type is questions arising from the mathematics itself. For example, "Also, one should inspect, for each mathematical theorem, what it offers, what the outcome would be if I change the conditions a little bit" (M1). The other type is questions arising from outside of mathematics. For example, "Why do planes flying to New York from Tel-Aviv fly over Canada?" (M1), and "Why does the time of sunset in spring and autumn change quickly but in winter and summer it hardly changes?" (M3).

Thinking and understanding

All the mathematicians emphasized the centrality of thinking and understanding in doing mathematics. For example, one said: "One of the primary goals was to show that there is mathematics beyond that [the technique] – also the thinking features…" (M1). And another, "I hope very much that they will not forget the main idea that in mathematics things need to be explained…" (M2). The mathematicians emphasized that understanding in mathematics involves understanding the purposes and meanings of what one does: "…when they do something, they have to be able to formulate exactly what they are doing, why they are doing that, what they could do differently" (M5). They added that thinking is difficult, and one must make an effort in order to do that, stressing that thinking should not necessarily be done quickly.

Using intuition and formalism

The mathematicians emphasized that teachers need to know that using intuition is an integral part of doing mathematics, especially at the initial stage of problem solving. Precision and formal representation come at a later step and they do not reflect the process in which mathematics done. For example:

> …we [mathematicians] often develop things intuitively. You think that one thing is true but once you try to prove it you realise that you need to slightly change the phrasing…
>
> Generalizations come at the end, not at the beginning. It's not that we understand the whole theorem at the beginning. We usually develop something small and gradually realize that there is a bigger picture, and at the end we give a big beautiful picture. (M4)

The worth of mathematics

The aspect *worth of mathematics* deals with the question: *What good is it to engage in mathematics?* Data analysis revealed two main characteristics: (1) the practical worth of mathematics, and (2) the worth of mathematics per se.

The practical worth of mathematics

All the mathematicians emphasized the need for teachers to know that solving practical problems is, and has always been, an important motivation to engage in mathematics; that mathematics is not just a theoretical science disconnected from the physical world, but rather a tool for solving real life problems. For example: "[Mathematics] is things related to life, to usages…" (M3). They explained that through its uniform language and its modeling possibilities, mathematics helps to solve problems that arise in different disciplines, contributing to fields, such as navigation, geography, physics, biology, economics, technology, astronomy, medicine, computers, and more. The mathematicians stressed that teachers need to understand "how [historically] people arrived at certain things" (M1). For example, "In my view, it is simply unacceptable that they will talk about derivatives without knowing why Newton developed this topic" (M4).

The worth of mathematics per se

The mathematicians spoke also about the need for teachers to be aware of the worth of engaging in mathematics per se. Yet, this characteristic was less stressed. In this regard, they spoke about the engagement in mathematics as a challenging and creative activity, which develops rational and logical thought. They added that the beauty and aesthetics of mathematics gives much joy.

CONCLUDING REMARKS

Our study provides important information regarding an issue that has been hardly studied, namely, what mathematicians wish to teach secondary school teachers about mathematics. As shown, the mathematicians who participated in our study aimed at enriching, expanding and deepening teachers' knowledge about what mathematics is. These findings are in line with teachers' reports about the contribution of academic studies of mathematics to teaching secondary school mathematics (e.g., Even, 2011; Zazkis & Leikin, 2010). However, in contrast to those studies, our study provides detailed information about what knowledge about the nature of mathematics might mean to mathematicians. Analysis of the responses of the mathematicians who participated in our study generated three aspects, each comprises two or three characteristics, that together could serve as a conceptual framework for analysing teacher knowledge and practice related to the general epistemological level of knowledge about the nature of mathematics. This framework is presented in Figure 1.

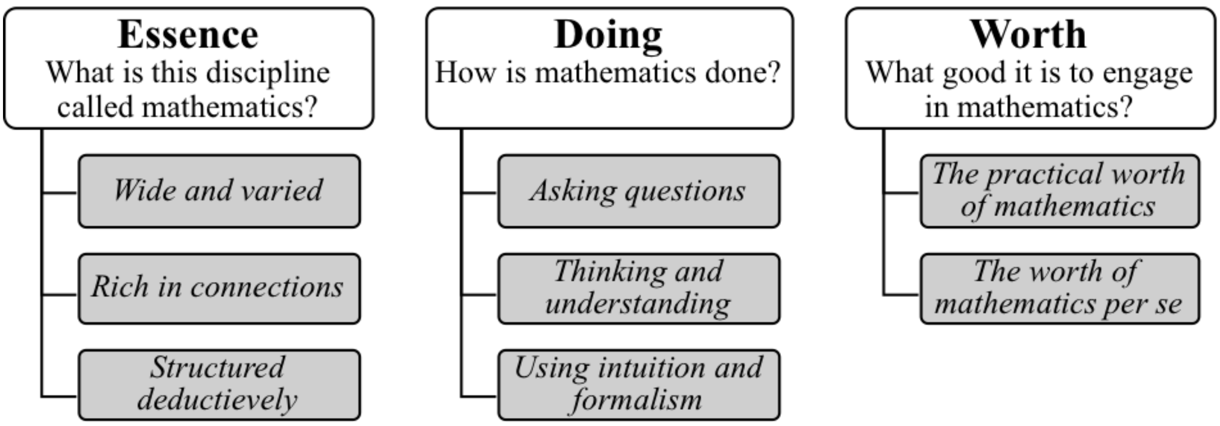

Figure 1: What mathematicians wish to teach teachers about the nature of mathematics.

Follow-up research is needed in order to examine the usefulness of this framework for the much-needed research on the contribution of academic mathematical studies to secondary school teachers' knowledge and practice. For example: How applicable is this framework for capturing what secondary school teachers learn about the nature of mathematics during their academic studies of mathematics? To what extent is this framework related to cultural and societal factors?

References

Adler, J., Hossain, S., Stevenson, M., Clarke, J., Archer, R., & Grantham, B. (2014). Mathematics for teaching and deep subject knowledge: Voices of Mathematics Enhancement Course students in England. *Journal of Mathematics Teacher Education*, *17*(2), 129–148.

Dreher, A., Lindmeier, A., & Heinze, A. (2016). Conceptualizing professional content knowledge of secondary teachers taking into account the gap between academic and school mathematics. Erscheint in. In *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*.

Even, R. (2011). The relevance of advanced mathematics studies to expertise in secondary school mathematics teaching: practitioners' views. *ZDM - International Journal on Mathematics Education*, *43*(6–7), 941–950.

Howe, R., & Ma, L. (1999). Knowing and teaching elementary mathematics. *Notices of the AMS*, *46*(8).

Klein, F. (2016). *Elementary Mathematics from a Higher Standpoint* (2nd ed., Vol. Volume I: Arithmetic, Algebra, Analysis). Berlin Heidelberg: Springer. (Original work published 1933)

Murray, E., Baldinger, E., Wasserman, N., Broderick, S., White, D., Cofer, T., & Stanish, K. (2015). Exploring Connections Between Advanced and Secondary Mathematics. In *Proc. 37th of the North American Chapter of the Int. Group for the Psychology of Mathematics Education* (pp. 1368–1376). Michigan State University.

Wu, H. H. (2011). The mis-education of mathematics teachers. *Notices of the AMS*, *58*(3), 372–384.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*(4), 263–281. https://doi.org/10.1080/10986061003786349

Ziegler, G. M., & Loos, A. (2014). Teaching and Learning "What is Mathematics." In *Proc. of the Int. Congress of Mathematicians* (Vol. 4, pp. 1203–1215). Seoul.

GESTURES AS EMBODIMENTS OF VARIABLES AND ALGEBRAIC EXPRESSIONS

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Researchers have investigated how students may represent indeterminate quantities (variables or unknowns) through expressions in natural language, non-numerical symbols, and external representations, implicitly treating indeterminate quantities much as if they were known quantities (Radford, 2011). Here I will focus on the following research question: How do sixth graders' gestures reflect their work with indeterminate quantities and the ways in which they operate on those quantities? Specifically, the present study provides evidence that: 1) sixth graders used gestures as visual representations of indeterminate quantities; and 2) students combined gestures into embodied forms of algebraic expressions.

INTRODUCTION

The present study aims to address the role of gestures in the development of students' algebraic thinking concerning the use of variables and algebraic expressions. This is an underexplored area, which in this study I will focus on by addressing the following research question: How do sixth graders' gestures reflect their work with indeterminate quantities and the ways in which they operate on these quantities? Gestures observed in the present study can be defined as spontaneous motion of hands and arms that co-occur with speech (McNeill, 1992).

Radford (2011) describes algebraic thinking as follows: "What characterizes thinking as algebraic is that it deals with indeterminate quantities conceived of in analytic ways. In other words, you consider the indeterminate quantities (e.g. unknowns or variables) as if they were known and carry out calculations with them as you do with known numbers" (p. 310). The author demonstrated this by using an example of a second grader working on extending a geometric pattern, in which an element at position n consisted of a row of n white squares plus one shaded square placed on top of another row of n white squares. The pattern corresponded to the function y = 2n+1, where n denoted the position in the ordered sequence and y denoted the number of squares in the pattern. The student extended the pattern to the 25^{th} position by saying, "What is 25 plus 25? After that you add 1!" In this example the second grader operated on an instance of an independent variable n=25 (as indeterminate quantity) by carrying out calculation (25+25)+1. Although she struggled to find the sum 25+25, she described the element at the 25^{th} position as a rule (in analytic ways) rather than as a value of 51. Whereas Radford (2011) finds indeterminate quantities in students' linguistic referents to "instances of the independent variable" (p. 310) and Brizuela

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 107-114). Umeå, Sweden: PME.

(2016) in a student's non-numerical inscription "?" to represent the unknown number of candy in a candy box, Cooper and Warren (2011) find them in external representations as points on a number line representing an unknown value. The present study makes a contribution to the body of literature on students work with indeterminate quantities in analytic ways (Cooper & Warren, 2011; Brizuela, 2016; Radford, 2011), by providing evidence that students do so through gestures. Specifically, in the present study I claim that 1) sixth graders used gestures as visual representations of indeterminate quantities; and 2) students combined gestures into embodied forms of algebraic expressions.

METHOD

Data

Data were selected from a collection of 378 classroom videos of 64 mathematics teachers in grades 5-9 from 9 districts, participating in a 3-semester long graduate-level professional development program aimed at improving teaching of mathematics, between the years 2011 and 2013. All participating teachers were asked, but not required, to allow researchers to videotape in their classrooms both at the beginning of their participation and at several points during the three semesters. Data used in this study are two 38-minute long video recordings of a single sixth grade mathematics lesson. At the time of this lesson, the teacher was nearing the end of her second semester of participation. The analysis presented here focuses on students' gestures at a single time point, and not on the teacher's change throughout the program. Each of the two videos was made by one of the two program researchers who recorded different aspects of the same lesson while also engaging with students and asking them to explain their thinking. Data were selected because of the prominent use of gestures among nearly half of the students.

Participants

Participants were thirteen sixth grade students arranged in four groups in a public school in New England, engaged in algebraic generalizations of a geometric pattern. Six of the students from three different groups used gestures to explain their thinking. Out of those I selected two students for analysis, Theo and Sophia, not from the same group, who used gestures to describe a geometric pattern in a general case, not specific to any particular position in the geometric pattern.

Methodology

I selected and transcribed the video episodes in which students' gestures co-occurred with their speech, here referred to as gesture-speech pairs. As the conceptual framework for this paper, I drew on literature that views gestures as semiotic resources (Arzarello, Paola, Robutti, & Sabena, 2009; Sabena, Radford, & Bardini, 2005; Radford, 2014) and, as such, convey information that closely relates to accompanying speech (Goldin-Meadow, 1999; McNeil, 1992). With this framework in mind, I first made interpretations about the information conveyed in speech and then analyzed students' gestures and made interpretations about the information they conveyed related to the context found in speech. Finally, I looked into students' gestures for evidence of use of indeterminate quantities in analytic ways.

Task and Lesson Flow

The teacher introduced the task as a real-life scenario asking students to make predictions for the number of tiles needed to enclose a garden of a varying length and constant width of one. During a whole-class discussion, for each of the first three elements of the geometric pattern (see Figure 1 left), the teacher, under a document camera, laid down the green tiles representing the garden spaces and then asked students to make predictions for the number of tiles needed to enclose it. Following students' predictions, she enclosed the garden with tiles and moved to the next element. Lastly, she asked students, "What patterns have you started to notice?" The first student to respond immediately took a covariation approach (Confrey & Smith, 1995) when he said, "You add a square foot to the garden, and it increases by two tiles on the outside." Another student noticed that three tiles were needed at each end to enclose any garden of the constant width of one, "one side is always three tiles." The teacher then announced that she would refer to the sides as "ends" (Figure 1 left depicts teacher placing two fingers on each "end"). Soon after the teacher physically separated the "ends" (Figure 1 right), Theo rephrased the first student's statement while using the terminology "top" for the top row of tiles, and "bottom" for the bottom row of tiles, "'Cause since you add more green tiles (garden spaces), you had one more on the top and one more on the bottom." This classroom discussion prompted students to visualize the four corner tiles as parts of the "ends" and not of the "top" or the "bottom". Such visualization corresponded to the function y=2n+6, where y represented the number of tiles needed to enclose a garden of length n.

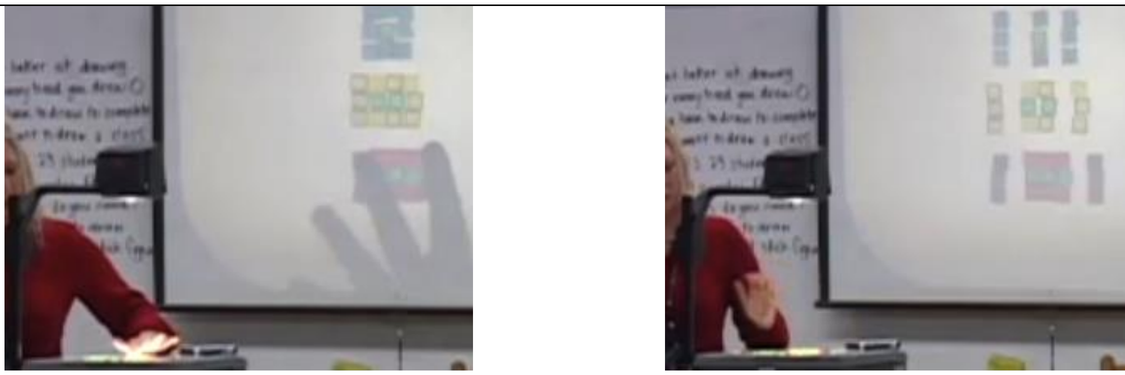


Figure 1: First three elements of the geometric pattern.

After the initial whole-class discussion, students were given blocks (instead of tiles) and a handout and were sent off to work in their small groups. While students worked in groups, the teacher and the two program researchers circulated around the classroom and worked with each group.

The handout consisted of a table with two columns: "Length of Garden" prefilled with values 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 25, 30, 100, 1000, n (left column), and "Number of

Tiles" (right column) left blank for students to fill in. To the right of the table was extra space designated by the teacher for students to record their observations and patterns they notice. The bottom of the page contained the following prompts: "How can you find the number of tiles for any garden length?" and "Write the rule".

RESULTS AND CLAIMS

Episodes 1 and 2 in Figure 2 summarize the gesture-speech pairs used by Theo and Sophia to describe the general term. Gesture-speech pairs are labeled alphabetically with letters "a" through "e", speech fragments co-occurring with gesture are underlined, and student speech is presented in bold.

Episode 1 – Theo describes the general case

a **b** **c**

Teacher (in response to another student who wrote down "tiles times two"): And why do you need to multiply this (pulls her thumb and index finger close together and briefly sets them on the table) times two? What is that going to give you?

Theo: **The top** [a] (gestures so that his arm is aligned with the length of the blocks) **and the bottom** [b] (same gesture on the other side of the blocks).

Teacher: The top and the bottom. So the two is coming from needing a top and a bottom.

Theo: (nods)

Teacher: And what's the other part?

Theo: **You have two sides** [c] (aligns the pencil with one end).

Teacher: The ends, good. And how many do you need for the ends?

Theo: **Three**

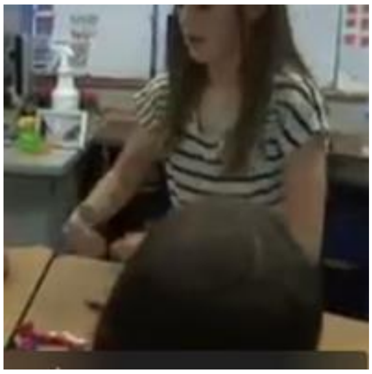
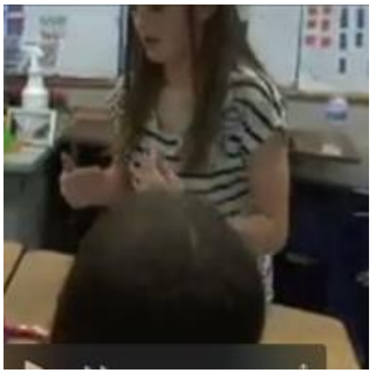
Teacher: Three on one side, and what else do you need?

Theo: **Three on the other side**

Teacher: A three on the other side. Perfect. And how many is that total?

Theo: **Six.**

Episode 2 – Sophia describes the general case

d  e  f 

Sophia: **I kind of noticed something - you always need like three on the side. You always need three on the side** (pauses then gestures two sides with each hand)[d]. **But - and you need - however many f(eet) - however long your um, plant , your, the length of your garden is that's how many tiles you need** (starts gesturing by forming two open palms pointing at each other, then stands up so that the camera can see her) **on <u>top</u>**[e] (moves her hand formation forward away from her) **and the <u>bottom</u>**[f] (moves her hand formation backwards towards her)**.**

Figure 2: Theo and Sophia's gesture-speech pairs.

Claim 1. Sixth graders used gestures as visual representations of indeterminate quantities

In what follows, I will argue that Theo's gestures (a and b) and Sophia's gestures (e and f) were visual representations of indeterminate quantities, the "top" and the "bottom".

Prior to Episode 1, Theo synchronized gesture and speech to explain that 46 tiles were needed to enclose a garden length 20, "Cause we did the twenty (places the pencil above the blocks at the left end) for the top (moves the pencil alongside the blocks to the right end) and twenty (places the pencil below the blocks at the left end) for the bottom (moves the pencil alongside to the right end) and the three (moves the pencil alongside the right end) on each side (moves the pencil alongside the left end) which equals forty-six". Evidently, Theo in an embodied way represented the equation 20+20+3+3=46. In Episode 1, the teacher posed a question ("And why do you need to multiply this times two?") to another student at Theo's table in response to his writing "tiles times two" on his paper. Theo offered an answer to her question by synchronizing utterances, "<u>The top</u> [a]", and "<u>the bottom</u> [b]", with the two identical open-palm gestures each on a different side of the row of blocks (garden spaces). Theo used these two identical open-palm gestures as visual representations of two identical entities, the two imaginary physical rows of tiles enclosing the garden from the top and the bottom. His gestures provided spatial orientation for each row of tiles as running parallel to the garden spaces from one end of the blocks to another. These gestures were also visual representations of two equal quantities, the number of tiles in each row, which added together demonstrated multiplication by two. Whereas in the specific case of garden length 20 Theo used the first two gestures as a visual representation of two rows of 20 tiles, the top and the bottom, the two gestures he used in Episode 1 (a and b in Figure 2) represented a general case as they had no reference to the number of tiles in the top and the bottom row. The quantities represented by these gestures (a and b) were thus unknown yet equal. This serves as evidence that Theo used gestures as visual representations of indeterminate quantities.

In Episode 2, without a prompt, Sophia started sharing her observations with students at her table "I kind of noticed". She described the general case in which she explicitly referred to the length of the garden as an unknown quantity, "however many f(eet) – however long your um, plant, your, the length of your garden is," and then used that to quantify how long the "top" and the "bottom" rows should be, "that's how many tiles you need <u>on top</u> [e] and the <u>bottom</u> [f]". She used two identical hand gestures, e and f, to visually represent two identical rows of tiles running parallel to the garden spaces. Besides spatial information, her gestures also conveyed quantitative information. Namely, for Sophia, the number of tiles was a property of the physical row of tiles as evident in her gesture-speech pairs, "that's how many tiles you need <u>on top</u> [e] and the <u>bottom</u> [f]". Although Sophia stated that the number of tiles in the top and the bottom row was unknown, "however many", at the same time she stated that they contained the same number of tiles "that's how many tiles you need <u>on top</u> [e] and the <u>bottom</u> [f]". The notion of equality is also supported by the two identical hand gestures. In summary, Sophia's gestures (e and f) were visual representations of two equal indeterminate quantities.

Claim 2. Students combined gestures into embodied forms of algebraic expressions

To provide evidence for Claim 2, I will now discuss the ways in which Theo and Sophia combined gestures as evidence that they were working with indeterminate quantities represented by those gestures as if they were known quantities, thus in analytic ways (Radford, 2011), and that these were embodied ways of representing algebraic expressions.

As argued in Claim 1, Theo's open-palm gestures (a and b) in Episode 1, for the "top" and "bottom" were visual representations of two equal indeterminate quantities. Theo used the conjunction "and" ("the top *and* the bottom") synchronized with repositioning of the hand, to combine the two indeterminate quantities represented by gestures, equal in size. This, I argue, is an embodied way of showing the algebraic expressions n+n. However, this was his response to teacher's question on why multiplication by two was needed, which means that Theo's two gestures were simultaneously embodiments of the equivalent expression 2n, showing multiplication by two as a repeated addition of two equal indeterminate quantities.

Theo went along with teacher's linguistic bid "and" to connect the "top" and "bottom" to the "ends" ("*And* what's the other part?") and described the "ends" with another gesture-speech pair (c), "You have two sides [c] (aligns the pencil with one end)". His pencil gesture c served as a visual representation of two equal fixed quantities which he immediately following the gesture described as “Three” on one side, “Three on the other side”, totaling “Six”. Theo, thus, combined the indeterminate quantities visually represented by the two gestures (a, b) in analytic ways as if they were known by adding 6 to their sum. This was his embodied way of showing the algebraic expression 2n + 6, which he eventually stated more explicitly when prompted to fill in the last row in the table for n number of tiles:

Teacher: So what are you going to do with the n now?

Theo: n times two plus six.

In Claim 1, I argued that Sophia’s gestures (e and f in Episode 2), just like Theo’s, were visual representations of two equal indeterminate quantities (“top” and “bottom”). Sophia used the word “and” in her speech, “on top [e] *and* on the bottom [f]”, synchronized with alternating her hand formation, forward (e) then backward (f) to show the top and the bottom, and therefore connected her two gestures into an embodied representation of the algebraic expression n+n. She represented the “ends” with another gesture-speech pair (d), “three on each side [d]”, a value that “always” stays the same. She used the word “but” synchronized with the repositioning of the hands, as a way to combine the “ends” (each of fixed length 3), with the “top” and “bottom” (each of an unknown length). The way she combined gestures in Episode 2 is an embodied way of showing the algebraic expression 3+3+n+n and evidence that Sophia worked with two equal indeterminate quantities in analytic ways as if they were known by adding their combined sum to the sum of the “ends”.

SUMMARY

In this paper I presented evidence and argued that sixth graders used gestures as visual representations of indeterminate quantities (Claim 1), and that students combined these gestures into embodied forms of algebraic expression (Claim 2).

To support Claim 1, I argued that Theo’s and Sophia’s gestures besides spatial also contained information which quantitatively characterized the top and the bottom row. In Theo’s case I contrasted the gestures he used in a specific case of garden length 20 to gestures he used in a general case described in Episode 1. Namely, Theo’s two gestures prior to Episode 1, as he faithfully retraced the 20 blocks from one end to another, at the top and at the bottom, were visual representations of the same fixed quantity of 20 tiles. In contrast to that, the two gestures (a and b) he used in Episode 1 were representations of two equal quantities without a regard to the number of tiles. In Sophia’s case I found evidence in her gesture-speech pairs (e and f) that the number of tiles at the top and at the bottom she thought of each as a property of a physical row of tiles, thus as evidence that each of her gestures (e and f) besides spatial orientation also reflected a notion of quantity, unknown yet the same.

To support Claim 2 I looked for transitions in speech and gestures as evidence that students combined gestures to represent addition, and multiplication as repeated addition, and by doing so operated in analytic yet embodied ways on the indeterminate quantities represented by those gestures. This in turn was an embodied way in which students represented algebraic expressions.

My findings complement those by Radford (2011), Brizuela (2016), and Cooper & Warren (2011) by providing evidence that students use indeterminate quantities and operate on them in analytic ways through gestures. This is a growing body of litera-ture on the development of algebraic thinking in ways other than the manipulation of symbols written in standard algebraic notation. Implications of the present study for research and instruction involve taking students' gestures into account when looking into students' algebraic thinking.

References

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97-109.

Brizuela, B. M. (2016). Variables in Elementary Mathematics Education. *The Elementary School Journal, 117*(1), 46-71.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education* 26, 66-86.

Cooper, T., & Warren, E. (2011). Years 2 to 6 students' ability to generalize: Models, representations and theory. In J. Cai & E. Knuth (Eds.), Early algebraization: A global dialogue from multiple perspectives. Advances in Mathematics Education Monograph Series (pp. 187–214). New York, NY: Springer

Goldin-Meadow, S. (1999). The role of gesture in communication and thinking. *Trends in cognitive sciences, 3*(11), 419-429.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago, IL: University of Chicago Press.

Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives. Advances in Mathematics Education Monograph Series* (pp. 303-322). New York, NY: Springer

Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal, 26*(2), 257-277.

Sabena, C., Radford, L., & Bardini, C. (2005). Synchronizing gestures, words and actions in pattern generalizations. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (pp. 129–136). University of Melbourne, Australia.

SOLVING COMBINATORIAL COUNTING PROBLEMS: PRIMARY CHILDREN'S RECURSIVE STRATEGIES

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The idea of recurrence is of fundamental importance in different areas of mathematics. One of these is the field of combinatorics, which provides many problems to introduce the idea of recurrence at an early stage of students' mathematical thinking. So far, there is still insufficient knowledge regarding the use of recursive strategies for combinatorial counting problems in primary schools. This paper therefore presents the results of a qualitative study with primary children of the third grade who solved analogous combinatorial problems by recursive strategies.

THEORETICAL BACKGROUND

The recurrence principle and its importance in solving combinatorial counting problems

The field of combinatorics is described as the art of enumerating and counting all the possible ways in which a given number of objects may be mixed and combined to make sure not missing any possible result (Bernoulli, 1713 German translation Haussner, 1899). From a mathematical perspective there are three approaches to solve combinatorial counting problems: systematic listing, counting principles and combinatorial operations (Schrage, 1996). Systematic listing and counting strategies can already be applied in primary school with so far developed knowledge and skills. The consideration of counting strategies in primary school is fundamental since these are forming the bridge between listing strategies and combinatorial operations (Höveler, 2018).

One of these central counting principles is the principle of recurrence. This principle is based on the mathematical idea of a recurrence relation which in general describes a way to "define a function by an expression involving the same function" (Schrage 1996, p. 194). More detailed it may be understood as the following rule which defines a(n) in terms of a(1), a(2), …a(n-1):

> "Let a(1), a(2), …be a finite or infinite sequence of numbers. If some initial values a(1), a(2), … a(k) are known and if for all n>k there is a rule which defines a in terms of a(1), a(2), … a(n-1), then every element of the sequence can be calculated according to this rule." (Schrage 1996, p. 194)

The recurrence principle is as well as the multiplication principle and further counting principles of particular importance as almost any counting problem can be solved by their skillful application. In addition, combinatorial operations can be derived from these principles (Schrage, 1996). Thus, based on a recursive solution of a combination

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 115-122). Umeå, Sweden: PME.

problem without repetition with four objects, then with five and with six objects, each time choosing two of them, a general recursive formula for n given objects for combinations without repetition can be obtained (for details see Höveler, 2014). Likewise, general formulas can also be developed for other combinatorial operations based on small problems by recursive considerations (Schrage, 1996).

Current state of research on children's recursive strategies in solving combinatorial counting problems

Previous studies, dealing with the application of counting strategies and possible occurring mistakes, focused primarily on the multiplication principle (e.g. Lockwood, 2010, Lockwood & Caughman, 2016). Concrete information about children's recursive strategies in the context of combinatorial problems are rare. Early investigations of Piaget and Inhelder (1975) give hints that children at elementary school age already use the idea of recurrence instead of counting all units particularly. Later studies also indicate the use of recursive strategies (e.g. Lack, 2009). But so far little is known about these counting strategies, as most studies with primary students (e.g. English, 1991, 1993; Maher & Martino, 1996; Maher, 2005) generally focused on solving combinatorial enumeration problems ("Which outcomes are possible?") and students listing strategies. There are also indications that children in secondary school solve combinatorial problems with recursive approaches: Shin and Steffe (2009) for example, investigated in a yearlong teaching experiment with two 7th grade students, how these students dealt with enumerative combinatorial problems. The results show that besides additive and multiplicative enumeration they also used recursive multiplicative enumeration. Further concretizations of these recursive strategies or the occurrence of systematic errors are missing.

These studies show that learners of different ages use recursive strategies to solve combinatorial counting problems. Although this is known and furthermore the considerable importance for the development of combinatorial thinking is obvious with regard to its subject matter, primary children's recursive strategies have not been studied so far.

THE STUDY

Aim of the study

Due to the afore mentioned importance of counting principles in general and the recurrence principle in particular, one main focus of a qualitative study on third graders strategies in solving combinatorial counting problems was, to answer the following research question: *Which counting strategies do third graders use to solve combinatorial counting problems and what is the relationship between primary children's strategies and the conventional mathematical approaches?* One aim was to find out if and, if so, which recursive strategies learners use and what difficulties they encounter.

Data collection and tasks

Information was gathered from individual, clinical interviews (Ginsburg, 1997) lasting 30 to 45 minutes. Overall 63 third graders from different schools were divided randomly in three groups. Every group of children got one set of combinatorial problems (set 1: combinations without repetition, set 2: combinations with repetition and set 3: arrangement without repetition). Each set of problems contained two isomorphic combinatorial problems in different contexts to find out, if primary children identify isomorphic structures and how they use these when solving the problems. To investigate children's recursive strategies each problem consisted of a basic and an extended task in which the number of elements of the basic task successively increased (see table 1, for further tasks see Höveler, 2018).

Basic task	Four teams want to play a soccer tournament. Each team plays once against each other team. How many games are there in total?
Extended task	How many games are there in total, when five (six, seven, ten) teams take part and each team place once against each other?

Table 1: Basic and extended task exemplified by the soccer problem

Unlike most of the previous studies the question "How many outcomes are possible?" was posed, instead of asking "Which outcomes are possible?" This question offered the opportunity to solve the stated problems by listing and counting strategies.

Data analysis

The interviews were video-recorded and transcribed, afterwards analyzed in two steps by central elements of the Grounded Theory (Glaser and Strauss 1967): First, classes of children's strategies were built. Afterwards relationships between their strategies, including the underlying concepts, and mathematical principles were identified by constant comparison. In this article, the identified strategies are described and it is named if and when these strategies lead to a correct result.

RESULTS

Children's recursive combinatorial counting strategies

In total the four recursive strategies "assumption of proportionality", "extension of groups", "forming new groups" and a combination of "extension of groups" and "forming new groups" were reconstructed. These strategies will be illustrated by an example below. Afterwards, it will be considered to what extent the desired number of solutions has been determined by means of these strategies.

"Assumption of proportionality"

Learners whose approach is based on the "assumption of proportionality" focus on how many outcomes with a fixed object appear in the set of outcomes. The solutions are created in many cases purely mentally in a few exceptional cases, the learners also create the number of solutions.

Situation: Leon has already determined the amount of soccer matches of four teams. Afterwards he is asked to determine the amount of matches with 5 teams and suggests that there are 9 matches in total.

1	I:	Aha, why nine? (...) How did you get that out?
2	L:	Since [points to the notes from the first task note] three are added to the six from the tournament with four teams.
3	I:	Mm and why three?
4	L:	Because each team plays three times, just like the teams before [tapping the blue pennant].

As the example shows, the value from the previous task which corresponds to the number of objects with a fixed element is added to the determined number of the previous task ("Three are added to the six from the tournament with four teams"). Children assume that the number of outcomes with this fixed element remains constant compared to the task already solved and transfer this to the new element ("Because each team plays three times, just like the teams before"). In previous investigations on combinatorial counting problems, there are no explicit findings that represent an existence of the "assumption of proportionality". But this strategy is named in the context of the generalization of patterns. For example, Akinwunmi (2012) describes that sequences of patterns, in addition to recursive and explicit structuring, are solved by assuming that a proportionate growth of the sequence.

"Extension of groups"

Within the "Extension of groups" in most cases the newly added objects are created. The strategy is exemplified by Lara's solution of the ice-cream problem ("Here are four different flavors of ice cream. How many different sundaes with two scoops are possible, if the order of scoops does not matter?"):

Situation: Lara has solved the basic ice cream problem and structured her solutions. She then finds out how many outcomes are possible with five different ice cream flavors under the same conditions (blueberry, which is colored in blue is added) and creates a total of 14 solutions.

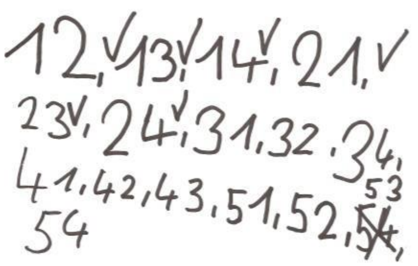
1	L:	Because we already had some [points to the solutions of the basic task] and the 10 were and then still 4 are added [taps on the four solutions with a blue tile], would be 14.
2	I:	Can you explain why there are 4 new solutions?
3	L:	Because then there are 4 blues again [again she points to the four solutions with blue tiles] because there are four different colors. Yellow and blue [taps on the corresponding ice cream cones], green and blue [taps on the corresponding ice cream cones], red and blue [taps the corresponding ice cream cones] and black and blue [taps on the corresponding ice cream cone].

As evidenced by Lara's actions and statements, children expand each of the already formed classes by an outcome which contains the new object ("Because then there are four more blue ones [again pointing to the four solutions with blue tiles] because there are four different colors").

"Forming new groups"

The strategy "Forming new groups" was used independently of the previous structuring strategies to determine the number of all figures with the new element. The newly added objects are also created in most cases.

> Situation: Jasmina has already solved the basic two-digit task (arrangement without repetition), then she is asked to find out how many two-digits numbers there are with 5 different digits under the same conditions.
>
> 1 J: Um, wait now, 12 plus the ones with fifty… 51, 52, 53, 54, … plus 4. Means 4 solutions with every digit, 16 in total.



The example shows that the task is again solved by adding the new objects with the new element to the already determined number of objects ("12 plus […] plus 4"). In this case a new group is formed for the newly added elements ("plus the ones with fifty").

The assumption about the completeness of the new objects in the new group differs among the learners:

- a) the number of created objects with the new element matches with the number of previously created figures in a fixed class (assumption of proportionality).
- b) the new element must be combined with all other elements in all possible ways.

In most cases, the desired number of outcomes was created with the underlying consideration in b) and only in a few cases on the basis of the assumption of proportionality (see a).

Combination "extension of groups" and "forming new groups"

This recursive strategy is a combination of the strategies "Extension of groups" and "Forming new groups".

> Situation: Sara has solved the basic ice cream problem and structured her solutions. She then finds out how many outcomes are possible with five different ice cream flavors under the same conditions (blueberry, which is colored in blue is added) and creates a total of 15 solutions.
>
> 1 S.: Then I do not need to write them down anymore.
>
> 2 I.: Aha?
>
> 3 S.: Then all I need is to add the ice-cream. with blueberry.
>
> 4 I.: Do you know how many of these there would be?

5 S .: Four I think. Oh no, its five. Every group has one more [points to the four classes], and blueberry-blueberry has to be added.

The example of Sara shows that children using this strategy add an object with the new element to each created class (“Every group has one more”) and create a class which contains all missing outcomes (“with the new element and blueberry-blueberry has to be added”).

Recursive strategies and number of outcomes

As stated in the previous section learners determine the number of outcomes for the extension of the problems by four different recursive strategies. For the further development of combinatorial thinking it is of interest to figure out if these strategies ensure that the required amount of outcomes is created and counted. Related results are presented in table 2. It has to be taken into account that the number of determined solutions to the basic task does not in every case match with the required number of the basic task. Therefore a distinction is made between the requested number ($a_{(n-1)}$) and the individually determined number ($a^*_{(n-1)}$) in the basic task.

		Combination without repetition	**Combination with repetition:**	**Arrangement without repetition:**
Requested number (n=5, k=2)		$\mathbf{a_{(n-1)} + 4}$	$\mathbf{a_{(n-1)} + 5}$	$\mathbf{a_{(n-1)} + 8}$
Determined number by	“Assumption of proportionality”	$\mathbf{a^*_{(n-1)} + 3}$	$\mathbf{a^*_{(n-1)} + 4}$	$\mathbf{a^*_{(n-1)} + 6}$
	“Extension of groups”	$\mathbf{a^*_{(n-1)} + 3}$	$\mathbf{a^*_{(n-1)} + 4}$	$\mathbf{a^*_{(n-1)} + 6}$
	“Forming new groups”	**a)** $\mathbf{a^*_{(n-1)} + 3}$ **b)** $\mathbf{a^*_{(n-1)} + 4}$	**a)** $\mathbf{a^*_{(n-1)} + 4}$ **b)** $\mathbf{a^*_{(n-1)} + 5}$	**a)** $\mathbf{a^*_{(n-1)} + 6}$ **b)** $\mathbf{a^*_{(n-1)} + 8}$
	Combination “extension of groups” and “forming new groups”	$\mathbf{a^*_{(n-1)} + 4}$	$\mathbf{a^*_{(n-1)} + 5}$	$\mathbf{a^*_{(n-1)} + 8}$

Table 2: Comparison of the required number of new objects and the number of new objects determined by the respective strategy

Results show that despite the consistent application of different systematic approaches, the learners do not determine the required amount of figures (table 2). The “assumption of proportionality” was, as well as the “extension of groups”, observed across all combinatorial problems. Regardless of whether the amount of outcomes was calculated or counted. This strategy was identified independently of the previous approach for the basic task. The "extension of groups" on the other hand, only occurred when the odometer strategy (see English, 1993 for details) was used to solve the basic task. Both strategies systematically produced a result that was in the case of combinations with and without repetition one less and in the case of arrangements without repetition exactly two less than the requested number (see table 2). "Forming new groups" was the most frequently used recursive strategy. It was also identified in the solution of all tasks and used regardless of the previous strategy. As stated before, in some cases children's assumptions about the completeness of the number of solutions in the new group were based on the assumption of proportionality (see previous section). In this case (see table 2, "forming new groups a)) the determined number was less than the requested. Otherwise, however, the correct number of solutions has been determined based on this strategy (see table 2, "forming new groups b)). The combination of "extension of groups" and "forming new groups" was used across all combinatorial figures, but only if the odometer strategy was used in advance for structuring. In all cases the requested number of outcomes was found.

DISCUSSION AND CONCLUSION

The results of this study indicate that third graders already use different recursive counting strategies to solve combinatorial problems. The four recursive strategies outlined in the previous section have been applied to all extended tasks regardless of the implicit combinatorial operation and context. At the same time, the results show that some systematic errors can be observed in third graders' recursive approaches. This applies in particular to the strategies "assumption of proportionality" and "extension of groups" which in no case led to a complete solution.

Which conclusions can be drawn from these results and which further investigations are necessary?

The results provide important information for diagnosis in the context of combinatorial problems and for individual support of learners: The results show that learners do not come to a wrong result by accidentally forgetting a solution. This result is based on one of three recursive strategies where the assumption of proportionality or the addition of groups leads to an incorrect number. Since many learners systematically determined an incorrect number of outcomes using a recursive procedure, an explicit discussion of recursive strategies and possible systematic mistakes should be made in the classroom. The study was conducted with third-graders solving three different combinatorial counting problems in two contexts. In this respect, additional studies are needed to identify further strategies and to make generalizations. It is to examine to what extent the identified strategies are general recursive strategies which are also used to solve other combinatorial problems and furthermore if these are typical for primary children without prior knowledge or used regardless of learner's age and prior knowledge.

References

Akinwunmi, K. (2012). *Zur Entwicklung von Variablenkonzepten beim Verallgemeinern mathematischer Muster*. Wiesbaden: Vieweg.

Bernoulli, J. (1713). *Wahrscheinlichkeitsrechnung (Ars Conjectandi) (Band 1).* Translated and published by R. Haussner (1899). Oswalds Klassiker der Exakten Wissenschaften.

English, L. D. (1991). Young children's combinatoric strategies. *Educational Studies in Mathematics, 22*(5), 451–474.

English, L. D. (1993). Children´s strategies for solving two - and three- dimensional combinatorial problems. *Journal of Research in Mathematics Education, 24*(3), 255–273.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge, UK: Cambridge University Press.

Glaser, B. & Strauss, A. (1967). *The Discovery of Grounded Theory: Strategies for Qualitative Research.* New York: Aldine De Gruyter.

Höveler, K. (2014). *Das Lösen kombinatorischer Anzahlbestimmungsprobleme: Eine Untersuchung zu den Strukturierungs- und Zählstrategien von Drittklässlern.* Doctoral dissertation. Dortmund: Technical University Dortmund. Ressource document: http://hdl.handle.net/2003/33604. Accessed: 9 January 2018.

Höveler K. (2018). Children's Combinatorial Counting Strategies and their Relationship to Conventional Mathematical Counting Principles. Hart E., Sandefur J. (Eds.), *Teaching and Learning Discrete Mathematics Worldwide: Curriculum and Research* (pp. 81–92). ICME-13 Monographs. Springer, Cham

Lack, C. (2009). *Aufdecken mathematischer Begabung bei Kindern im 1. und 2. Schuljahr.* Wiesbaden: Vieweg + Teubner

Lockwood, E. (2010). An investigation of post-secondary students' understanding of two fundamental counting principles. *The Electronic Proceedings for the Thirteenth Special Interest Group of the MAA on Research on Undergraduate Mathematics Education.* Raleigh, NC: North Carolina State University.

Lockwood, E. & Caughman, J. S. (2016). Set partitions and the multiplication principle. *Problems, Resources, and Issues in Mathematics Undergraduate Studies, 26*(2), 143-157.

Maher, C. A. (2005). How students structure their investigations and learn mathematics: insights from a long-term study. *Journal of Mathematical Behavior*, *24*, 1–14.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education*, 27(2), 194–214.

Piaget, J. & Inhelder, B. (1975). *The origin of the idea of change in children.* London: Routledge and Kegan Paul Ltd.

Schrage, G. (1996). Analyzing Subject Matter: Fundamental Ideas of Combinatorics. In T. Cooney, S. Brown, J. Dossey, G. Schrage & E. Ch. Wittmann (Eds.), *Mathematics, Pedagogy and Secondary Teacher Education* (pp. 167–220). Portsmouth: Heinemann.

Shin, J. & Steffe, L.P. (2009). Seventh Graders' use of additive and multiplicative reasoning for enumerative combinatorial problems. In S. L. Swars, D. W. Stinson & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 170-177). Atlanta, GA: Georgia State University.

PRE-SERVICE MATHEMATICS TEACHERS' WHOLE-CLASS DIALOGS DURING FIELD PRACTICE

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This paper is based upon an intervention study where pre-service teachers plan whole-class mathematical dialogs together with their mentor and lecturer. Learning to conduct dialogs is increasingly in focus in teacher education, and in this paper, we examine one whole-class dialog to learn more about its nature. We show that the pre-service teacher fails to involve several pupils in the dialog at the same time, leading to a series of shorter dialogs with one pupil at a time. In the dialog, the communication often ends up being teacher-dominated.

INTRODUCTION AND BACKGROUND

Mathematical reasoning is important for children's later achievement in mathematics (Nunes, Bryant, Sylva, & Barros, 2009). Differences in pupils' mathematical thinking and reasoning could be attributed to the type of questions teachers ask (Kazemi & Stipek, 2001). However, questions posed within mathematics classrooms across the world typically fail to provide pupils with opportunities to reason about mathematical concepts or to explore mathematical connections (Hiebert al., 2003). Asking questions that probe pupils' thinking is a complex skill that requires thoughtful planning (Manouchehri & Lapp, 2003). According to Henning and Lockhart (2003) prospective teachers pose questions quickly with few follow-ups, giving little time for the pupils to expand their answers. Leading whole-class conversations includes asking questions or posing problems to begin a discussion, monitoring pupil participation during discussion, and responding to pupil ideas. Grossman, Hammerness and McDonald (2009a) argue that "each of these is critical to the practice as a whole and represents practices that novice teachers can begin to develop in teacher education and the early years of teaching" (p. 281). Thus orchestrating whole-class conversations in mathematics are pointed to as an example of core practices in teacher education (Grossman et al., 2009a; Lampert et al., 2013).

Ghousseini and Herbst (2016) argue that different pedagogies need to be implemented in teacher education to prepare pre-service teachers for doing complex work of teaching like leading classroom mathematics (2016, p. 79). They show that by using representations of practice, decomposition of practice and approximation of practice (as introduced by Grossman et al. (2009b)), the pre-service teachers were given different opportunities to learn. The pre-service teachers in this study conducted a whole-class dialog in their field placement as an approximation to practice. We are interested in learning more about the nature of such whole-class dialogs conducted by

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 123-130). Umeå, Sweden: PME.

pre-service teachers. Learning more about what features are prominent in these dialogs will help us inform how we teach this complex skill in teacher education. In this paper, we set out to investigate one dialog conducted by a pre-service teacher by combining two analytical frameworks.

THEORETICAL FRAMEWORK

To analyze the whole-class dialogs we use two frameworks by Drageset to code the teacher actions (2014) and the pupil comments (2015). The first framework provides 13 types of teacher actions, falling into three superordinate categories (Drageset, 2014). The first of the superordinate categories is *redirecting actions;* actions where the teacher redirects the pupils' attention by either asking a *correcting question*, *advising a new strategy* or *putting aside* a pupil's comment. The second category is *progressing actions*, in which teachers' different ways of moving the lesson forward is included. The actions *simplification* and *closed progress detail* are used to simplify the problem or to ask a specific question (typically with only one correct answer) to move the pupils one-step ahead in the solution. *Open progress initiatives* are, on the other hand, questions that does not limit the possible responses, and a *demonstration* is when the teacher takes over and solves the problem by himself. The third category is *focusing actions,* actions the teacher's uses to put emphasis on certain things. This category is itself divided in two; *request for pupil input* and *pointing out.* The teacher can request pupil input by asking them to *enlighten detail*s, asking for *justification* or to *apply to a similar problem*. He can also *request assessment from other pupils*. The teacher points out either by *recap*ping at the end of the dialog or by making the pupils *notice* something during the dialog (Drageset, 2014, p. 297-298).

The second framework has 21 initial categories of pupil comments grouped into five superordinate categories (Drageset, 2015). The superordinate categories are *explanations, pupil initiatives, partial answers, teacher-led responses* and *unexplained answers*. Responses from *explanations* and *teacher-led responses* were most prominent in the data material analyzed in this paper. Therefore, we present these two superordinate categories with subcategories in detail. For the remaining categories we refer to Drageset (2015, p. 38). The superordinate category explanations distinguish between three different kinds of explanations the pupils make: Explaining what they are doing and how (*explain action*), explaining why something is true (*explain reason*) or explaining a concept (*explain concept*). There are a number of different teacher-led responses, and typically, these comments were correct responses to basic tasks and arose as a result of the teacher reducing complexity (Drageset, 2015, p. 37). The five different kinds of teacher-led responses were: *Correct as a response to closed progress details, correct as a response to simplification, confirm or reject teacher suggestion, quote teacher* and *off track*. Combined, the frameworks give detailed information about the communication in the whole-class dialog, and allow us to look for emerging patterns.

METHODS

Context

The project is situated within Norwegian initial teacher education for primary school, which at the time of the study was a four-year long integrated program. Each year the pre-service teachers have 5-7 weeks of field practice as well as parallel studies in education and different subject matters. The pre-service teachers have their field practice in groups of four.

Official documents (KD, 2009) states that field practice and theoretical studies are equal arenas for learning and professional development, and mentors in contracted schools are regarded as teacher educators. The mentors are paid, and allotted time, for mentoring the pre-service teachers two hours daily. The mentor in this study holds a master's degree in mathematics education. Due to the project, the lecturer in mathematics is part of the planning and post-lesson mentoring.

Participants and data collection

A group of four pre-service teachers were voluntarily recruited from a program with special emphasis on science and mathematics. They have their field practice with third-graders. At the time of the study the pre-service teachers are in their third year, taking courses in mathematics and education. Mathematical dialogs with pupils is a substantial part of the mathematics courses, focused both in the course literature and in lectures where video of professional teaching is watched and analyzed. As an intervention, the pre-service teachers were asked to conduct whole-class dialogs in field practice, and videos of their teaching was used as a tool in the mentoring. The pre-service teachers, together with the mentor and lecturer, planned for productive mathematical whole-class dialogs understood as dialogs where pupils can reason in mathematics and develop a deep understanding for mathematical concepts. Our understanding builds on Sfard & Kieran's (2001) definition of productivity:

> The term productivity (...) refers to discourse which can be proved to have some concrete lasting effect: the discourse has led to the solution of a problem, it influenced participants' thinking and ways of communication, it changed their mutual positioning, it became richer in rules and concepts (p. 50).

The whole-class dialog analyzed in this paper was "Hannah's" dialog on a string of addition problems (36+40, 36+43, 36+46, 63+20 and 63+29). This is one of six instructional activities also used by Lampert et al. (2013) when they work with novice teachers. They claim that such activities enable the mentors to better predict the challenges the pre-service teachers will encounter in the classroom, making the pre-service teachers well prepared before conducting the dialog. The whole class dialog lasted for 28 minutes, and was videotaped. During the dialog the third-grade pupils were all seated at the front of the classroom and had no access to individual writing materials. We therefore found it sufficient to capture the dialog using only one video camera pointed at the pre-service teacher and the white board. This captured well the utterances of both the pupils and the pre-service teacher.

Data analysis

To answer the research question, we analyzed the video using the data analysis software Studiocode. We first coded all utterances in the dialogs, directly on the video, using predefined codes from the two frameworks from Drageset (2014, 2015). This generates a timeline where each code is given a row as seen in Figure 1. The rectangles we see correspond to utterances coded with the different codes. E.g. the rectangle labeled 18 on the line "explain action" means that from 00:21:57 to 00:22:15 a pupil was explaining what or how to do something, and this was the 18^{th} time so far in the dialog that a pupil had explained an action.

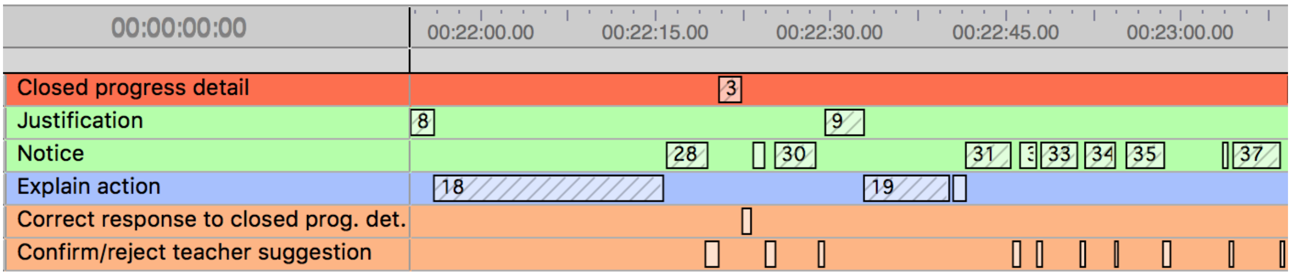

Figure 1: Part of the timeline from Hannah's dialog with Knuth about his solution to 63+29. The first three rows are teacher actions, the last three are pupil comments.

In this process, we also kept track of, and coded, which pupil made each utterance and the time provided to think after a question was posed. To manage the large number of utterances, we then partitioned the dialog into five segments, according to the five different addition problems. By visually examining the timelines, we found that the dialog segments showed a recurring pattern of teacher dominated communication. In the timeline in Figure 1 we see this pattern, particularly in the last part of the segment.

To better understand the dynamics of the communication, we chose to transcribe the dialog and at the same time coding with the same codes as the video. The analysis developed going back and forth between the coded video and the coded transcripts, asking questions to the data. The question "who got to speak?" revealed that many pupils were given time to speak, but the dialog mainly consisted of shorter dialogs between the teacher and one pupil at a time. This led to a refined partition of the whole-class dialog into 11 segments, where each segment contains the dialog about one pupil's solution to an addition problem. The next step was to analyze these 11 segments to understand what characterizes these shorter dialogs. This involved looking closer at the mathematical content of the dialogs, and to each of the segments we asked: "What is the strategy used by the pupil?", "How is this strategy articulated by the pupil?", "How does the teacher handle the utterance? E.g. are other pupils engaged in interpreting the utterance and how is the strategy represented on the board?", "What was the communication pattern?" We used tables to identify patterns or similarities/differences in the segments.

Concerning ethics, we adhered to guidance from the Norwegian Research Committee (NESH). All names are pseudonyms.

RESULTS

As already alluded to in the methods, the analysis of the whole-class dialog shows that the dialog consists of a series of shorter dialogs between the pre-service teacher and one pupil, sometimes involving a few other pupils. The analysis also shows that the pupils are given a lot of time to think after questions have been posed, wait time is used 13 times, often lasting more than 30 seconds.

Further, the shorter dialogs often follow a similar pattern. The pre-service teacher asks a pupil to share how they found an answer to an addition problem. When the pupil has shared his strategy, the teacher is interested in learning more about the solution, but this part of the dialog becomes teacher dominated. The teacher often repeats every step of the pupil's explanation, occasionally stopping to ask for a *closed progress detail* or to ask the pupil to confirm that the teachers' interpretation was correct. This is often done by repeating something the pupil says. Such *pointing out* actions done during a dialog is coded as *notice*, rather than *recap* which according to Drageset is when the teachers sum up the dialog and moves on to something new. We illustrate this finding with the following excerpt from Hannah's dialog with Knuth about 63+29. We go into the dialog after Knuth has established that 60+20 is 80. How each utterance is coded is written in bold italics at the end of every utterance.

Hannah:	Can you tell me again? ***Justification*** (Justification 9 in Figure 1.)
Knuth:	I added. Behind 8 I added 9, and then I added 3 more afterwards. (…) So it was 92. ***Explain action***
Hannah:	I think I understand. You started with this number [83]? ***Notice***
Knuth:	Yes. ***Confirm or reject teacher suggestion***
Hannah:	But you took away the 3 (covers the 3 on the board)? ***Notice***
Knuth:	Yes. ***Confirm or reject teacher suggestion***
Hannah:	Because you thought it was a bit easier to just work with 80 first (writes 80 on the board)? ***Notice***
Knuth:	Mm. ***Confirm or reject teacher suggestion***
Hannah:	And then you added that 9 (points to 9 in 29, writes +9 after 80 on the board)? ***Notice***
Knuth:	Yes. ***Confirm or reject teacher suggestion***
Hannah:	And then you had to remember to add that 3 you had (points to 3 in 83)? ***Notice***
Knuth:	Yes. ***Confirm or reject teacher suggestion***
Hannah:	Was it like that? ***Closed progress detail***
Knuth:	Mm. ***Confirm or reject teacher suggestion***
Hannah:	And then you got that 80+9 is 89 (points to the board)? ***Notice***

Knuth: Yes. ***Confirm or reject teacher suggestion***

Hannah: Plus 3 is? ***Closed progress detail***

Knuth: 92. ***Correct as a response to closed progress detail***

Hannah: 92 (writes =92). That was a clever way. ***Notice***

The communication pattern that arises here is a dialog where the teacher either uses a *pointing out* action like *notice* or a *closed progress detail* question. Knuth responds with a *teacher-led response*. These pupil responses are typically very short, often just one word, allowing the teacher to take over the responsibility to articulate the strategy. We see in this dialog that Knuth is often just saying “yes” to confirm what the teacher said. This is typical for all the dialog segments. There is a total of 142 pupil utterances in this dialog, 42 of them fall into the category *confirm or reject teacher suggestion*, meaning the pupil is saying “yes”, “no” or “mm” (a confirming sound). The reason this teacher-led pattern occurs might be that the pre-service teacher wants to make sure that the strategy was articulated in such a way that all the pupils were able to follow. However, this over-use of *pointing out*-actions may make these actions lose their focusing ability, since everything is highlighted.

In Table 1 we give an overview of how frequently the different categories of teacher actions and pupil comments were used. We see here that the teachers *pointing out* actions account for the majority of the *focusing actions*.

Teacher actions	Number of instances	Pupil comments	Number of instances
Redirecting actions	9	Explanations	33
Progressing actions	62	Pupil initiatives	2
Focusing actions (total)	63	Partial answers	9
– request for pupil input	18	Teacher-led responses	68
– pointing out	45	Unexplained answers	11

Table 1: Total number of teacher actions and pupil comments in the different categories in the dialog.

In Table 1 we also see that the dialog contains almost no utterances that Drageset defines as *redirecting actions*, only nine times during the dialog. The redirecting actions are the teachers’ tools to control the direction the dialog takes, and are used to advise strategies, dismiss pupil solutions or correct pupil responses. This lack could indicate that it is more important for pre-service teachers to allow a majority of the pupils to share their strategy than to keep a short and focused dialog.

The analysis of the mathematical content in the dialog segments shows that many strategies shared in the different dialogs are similar. In fact, only three different strategies are shared in the 11 segments; adding tens and ones separately, using previously known answers (e.g. 46 is 3 more than 43 so 36+46 must be 3 more than 36+43) and making equivalent addition problems (e.g. 63+29=60+32). Even so, there is little difference in the attention given to each pupil regardless of how many times that strategy has been discussed beforehand. This emphasis on strategies that have previously been presented might also confuse pupils who identify that the strategies are the same. As mentioned above, the pre-service teacher hardly ever used *redirecting actions* in the dialog, losing the ability to move quickly past strategies previously shared. One reason to discuss the same strategies several times could be to make the pupils compare and discuss why the strategy always works. However, the pre-service teacher writes little or nothing on the board that represents the pupils' strategies, and accordingly this would have been difficult to carry out.

DISCUSSION

Our study is carried out in an authentic context in an ordinary elementary class. We show that, unlike in the study by Henning and Lockhart (2003), the pre-service teacher took time posing questions and asked follow-up questions to give pupils time to expand their answers. Despite this, we reveal that the dialog is not without problems. Pupils are mainly talking to the teacher, and the teacher is taking too much of the responsibility to articulate the pupils' explanations.

We argue that the overall nature of the dialog is that the pre-service teacher attempts to balance the challenge of hearing strategies from many pupils, and at the same time make all these strategies understood by the rest of the class. The lack of redirecting actions is compatible with the desire to let many pupils explain their strategy, normally the redirecting actions are used to put aside suggestions that the teacher does not want to pursue in the dialog. This repeating of strategies results in little progress in the dialog as a whole and the pupils are not engaged in each-others thinking. Hence, the dialog is not a productive whole-class dialog as defined by Sfard and Kieran (2001).

Our study sustains what previous research has shown, that conducting whole-class dialogs is a challenging task. The pre-service teachers had read a lot about conducting such dialogs, and had attempted it in previous years of their teacher education program. They were also well prepared for their discussions, after having planned together with peers and two mentors. Ball and Forzani (2009) argue that if there is an unknown to questions posed in classrooms, it is what pupils' responses will be. The mentors can anticipate many challenges that the pupils will encounter, but they cannot prepare them for every possible challenge. This shows the importance of incorporating practice on core practices in real teaching situation in teacher education programs. Our study also successfully shows that the two frameworks developed by Drageset (2014, 2015) can be combined to give an overall picture of the classroom interaction. It would be interesting to continue analyzing several pre-service teachers' dialogs using the same approach to see if they follow the same pattern, or if there are a number of different patterns that appear in pre-service teachers' dialogs. Another possibility would be to examine differences and similarities between novices and experienced teachers.

References

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education, 60*(5), 497-511.

Drageset, O. G. (2014). Redirecting, progressing, and focusing actions – a framework for describing how teachers use students' comments to work with mathematics. *Educational Studies in Mathematics, 85*, 281-304.

Drageset, O. G. (2015). Different types of student comments in the mathematics classroom. *Journal of Mathematical Behavior, 38*, 29-40.

Ghousseini, H., & Herbst, P. (2016). Pedagogies of practice and opportunities to learn about classroom mathematics discussions. Journal of Mathematics Teacher Education, 19, 79-103.

Grossman, P., Hammerness, K., & McDonald, M. (2009a). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: Theory and practice, 15*(2), 273-289.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009b). Teaching practice: A cross-professional perspective. *Teachers College Record*, 111(9), 2055–2100.

Henning, J. E., & Lockhart, A. (2003). Acquiring the art of classroom discourse: A comparison of teacher and prospective teacher talk in a fifth grade classroom. *Research for Educational reform,* 8(3), 46-57.

Hiebert, J., Gallimore, R., Garnier, H., Giving, K. B., Hollingsworth, H., Jacobs, J., ... Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 Video Study*, NCES (2003-013), U.S. Department of Education. Washington DC: National Center for Education Statistics.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal*, *102*, 59-80.

KD (Ministry of Education and Research) (2009). Report No. 11 to the Storting (2008-2009). *The teacher. The Role and the Education. Oslo: Ministry of Education and Research.*

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., ... Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education, 64*(3), 226-243.

Manouchehri, A., & Lapp, D. A. (2003). Unveiling student understanding: The role of questioning in instruction. *Mathematics Teacher, 96*(8), 562-566.

Nunes, T., Bryant, R., Sylva, K., & Barros, R. (2009). *Development of maths capabilities and confidence in primary school*. London: Department for Education. http://dera.ioe.ac.uk/11154/1/DCSF-RR118.pdf

Sfard, A., & Kieran, C. (2001). Cognition as communication: rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42-76.

INVESTIGATING SECONDARY SCHOOL STUDENTS' EPISTEMOLOGIES THROUGH A CLASS ACTIVITY CONCERNING INFINITY

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In this paper, we report findings from a pilot study investigating school students' epistemologies of mathematics by using novel mathematics definitions. Students aged 17 and 18-year-old in Italy and the UK were asked to complete a worksheet that used a numerical approach to determine the sizes of infinite sets and were, then, invited to attend focus group interviews about their experience with the material. Thematic analysis of the interviews reveals that this approach is useful to distinguish between naïve and advanced epistemologies and using unseen mathematical definitions can help enrich our understanding of epistemologies held by students of school age.

BACKGROUND TO THE STUDY

Students' beliefs about mathematics have often been connected to their engagement with the subject (Muis, 2004), their behaviour as problem solvers (Schoenfeld, 1989; Muis et al. 2015) and their self-regulation strategies (Muis, 2007). However, understanding what these beliefs are and how to best measure them has generated a lively methodological debate in the epistemological beliefs literature (see for example Limon, 2006). Many (e.g. Muis et al. 2014) find the most common questionnaires used so far, and in general quantitative methods alone, to be unsuitable for such investigations. Criticisms to the use of large scale surveys include the inability to ascertain that there is a shared meaning of key words between the researchers designing the surveys and the students filling them in (Muis et al. 2014), and doubts have been recently raised that large scale questionnaires cannot be used across diverse cultural contexts (Mogashana et al. 2012). In this pilot study, we tested a qualitative methodology for the investigation of school students' epistemological beliefs. We hypothesised that, by documenting the reactions of secondary school students when asked to work with a definition of infinity (a concept that they would have encountered at this point in their education) very different from the one they have been used to, we may gain insight into their epistemology of mathematics. We report preliminary findings from this pilot study and we suggest some directions for future research.

STUDENTS' EPISTEMOLOGIES OF MATHEMATICS

Francisco (2013) makes a strong argument for the need of more studies investigating secondary school students' epistemological beliefs about mathematics and observes that many findings regarding school students are assumed to be true only because

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 131-138). Umeå, Sweden: PME.

they are found to be true for university students and not because they originate from empirical research involving school-age students. For example, Perry (1970) found that college students are likely to hold naïve epistemologies when they start their university studies and many researchers have therefore assumed this would be the case for school students too. Francisco (2013) also notices the disagreement on what are considered to be epistemological beliefs and how these can be studied. For the scope of our study, we adopt the definition of epistemological beliefs found in Hofer (2001): these are beliefs about knowledge and knowing, including:

> . . . the definition of knowledge, how knowledge is constructed, how knowledge is evaluated, where knowledge resides, and how knowing occurs. (Hofer 2001, p. 355)

This definition is only deceivingly simple, but it is one that has drawn widespread agreement amongst researchers in this field (Limon, 2006). A comprehensive review of the literature regarding epistemological beliefs about mathematics by Muis (2004) finds, among its main results, that epistemological beliefs about mathematics hinder rather than help students learning and that these beliefs have a clear impact on the students' academic progress. The author also reviews the evidence of the impact of such beliefs on problem solving activities and mathematics learning more in general and finds that, amongst the most non-availing beliefs school student hold, are that in mathematics there always exist one right answer and that every problem has one right answer only. A subsequent review of the literature by Depaepe et al. (2016) found similar results but noticed that in the years since Muis's (2004) review there has been much variety of methodologies employed to study students' epistemological beliefs well beyond the use of large scale quantitative surveys. This finding reflects the methodological issues raised at the start of this paper. Given that school students' epistemological beliefs about mathematics have been linked to many aspects of their engagement with the subject e.g. to problem solving habits (Schoenfeld, 1989), mathematical achievement and conceptual change (Mason, 2003), it seems important to have solid methodologies to investigate such beliefs. Hence, we ask the following research question:

RQ: What can the students' reactions to the introduction of an alternative approach to a familiar but difficult mathematical concept tell us about their epistemological beliefs about mathematics?

As familiar concept we selected infinity and we suggested an alternative definition of the measure of an infinite set as this definition is in stark contrast to what students would have encountered during their studies. Similar methods could however have been employed by choosing to use definitions from non-standard analysis, or by using the superreal number system proposed by Tall (1980). In the following paragraph, we summarise some research on students' understanding of infinity as some of these findings will also be reflected in our data.

STUDENTS AND INFINITY

Mathematics education has been preoccupied with the way in which students make sense of infinity because this concept is crucial both for the way in which it underpins several ideas from analysis and calculus; and for the understanding of set theory and the concept of cardinality. Many approaches have been used to make sense of students' understanding of infinity and, while it is beyond the scope of this paper to offer a comprehensive review of this literature, we will just mention a few ideas which will be useful for our analysis later on. Monaghan (2001) observes that students often perceive infinity as a process (the process of counting without ending, or a process that goes on and on – also defined as potential infinity, see also Kidron and Tall, 2015) while an object view of infinity would require students to regard infinite sets as completed totalities. Monaghan (2001) also points out that a process view of infinity is at odds with the classical concept of cardinality (actual infinity) and creates conflict when students encounter Cantorian set theory. In this setting students prove that a proper subset of a set and the set itself have the same cardinality if the two sets are countable and infinite. This creates conflict as it is obviously not the case for finite sets. Paradoxes are also used to elicit students' understanding of infinity. For example, Mamolo and Zazkis (2008) report that most difficulties with paradoxes concerning infinity are caused by the conflict of a potentialist (infinity perceived as a process that may go on forever such as counting) and an actualist (an object perceived in its entirety which has infinite size, such as the natural numbers) interpretation of infinity. They also notice that the experience that the students have of reality often gets in the way of the understanding of paradoxes.

MATERIALS

To construct materials for our investigation we introduced students to a numerical treatment of infinity due to Yaroslav Sergeyev (see Sergeyev (2003)). The basics of this treatment can be developed within a conservative extension of Peano Arithmetic, as shown in Lolli (2015). The intuitive idea behind Lolli's theory is that, within a model of arithmetic that contains infinitely large numbers, one may identify a cut-off point for N, the set of natural numbers. A new arithmetical term ① (read: gross-one) is used to denote this cut-off point. Suitable axioms then enable the construction of a theory of numerical measures of infinite parts of N. For instance, in view of these axioms, the initial segment of a model that is bounded by ① is such that any two subsets in bijective correspondence are assigned the same measure, which is smaller than ①. In particular, even and odd numbers are assigned the same measure, smaller than ① and denoted by ①/2. Thus, the whole part relation typical of finite collections is preserved for infinitely large ones.

METHODS

The study was carried out at two sites, in Italy and in the UK. At the first site participants were Year 11 to 13 students (aged between 16 and 18) in a private school in the

South of England. We first held a 90-minute session where they were asked to work in groups of 4 or 5 on the worksheet we designed. The worksheet guided them through five exercises involving grossone including: doing field arithmetic with ①; computing the sum of geometric progressions with an infinitely large number of terms as a strategy to study geometric series and investigating the Thomson lamp paradox (Berresford, 1981) without appealing to ① or by appealing to ①. After the session, we held 2 focus group interviews with nine participants. At the second site participants were students in 6 classes of fourth and fifth year of high school (aged between 17 and 19) attending 2 secondary schools in the south of Italy. There were 77 and 12 students who took part in sessions designed as the previous ones using the same worksheet, which had been translated by the second author of this paper. After the sessions, we held 6 focus group interviews (structured this time as class discussions and thus involving all students who had taken part in the activities). Altogether we collected 8 focus group interviews and observed 6 sessions. The focus group interviews were audio recorded. Thematic analysis (supported by analysis of the field notes taken during the observations) was carried out on the interviews transcripts with focus on the evidence of students' difficulties with the concept of infinity and hints of their epistemological beliefs concerning mathematics. The project was approved by the Research Ethics Committee of the institution where the second and third authors work.

THE DATA

The data were analysed both to investigate misconceptions that students hold about infinity (mainly through discussion of the Thompson Lamp task) and to look for indications of their epistemological beliefs about mathematics. During the analysis of the interview data we found agreement with many previous studies regarding students' understanding of infinity. For example, concerning the discussion on the Thompson lamp paradox, we observed how students' concrete intuitions interfered with the formulation and handling of the paradox, just as Mamolo and Zazkis (2008) found in their study. When a UK student was asked about her thoughts on the solution of the Thompson Lamp paradox she replied:

> Student (UK): The person would die before the end of the process!

We also observed evidence regarding students' tendency to reason in terms of potential (infinite counting) rather than actual infinity, in accordance to what Monaghan, (2001) found.

> Student (UK): Since infinity, there is no actual number for infinity, if you think there will always be 1 more…

Some of the students stated that using the new definition could remove some of what they perceived to be incongruences in the Cantorian approach, such as for example that in the case of infinite countable sets, a set and one of its proper subsets can have the same size.

Student (IT): It is a strange idea [*having various sizes of infinity*] but very intuitive. It allows us to understand a new concept of infinity. Before this we thought that infinity minus a quantity was infinity. Now we can see this better – that an infinity can be smaller than another infinity.

Therefore, students seem to engage in a meaningful way with this concept. Regarding students' epistemologies about mathematics we observed two distinct approaches amongst the students we interviewed: that of *rejection* of the new formulation of infinity or *acceptance* of this formulation. We argue here that these two stances are linked to students' views of knowledge and knowing in mathematics.

Rejection: I think this is a contradiction…

During the observations of the sessions with the students we noticed how all students engaged with the material and worked together through the exercises. However, the follow up interviews revealed that some of the students could not accept that there would be a different definition of a concept they had already encountered. The extract below is from one of the focus group interviews with the Italian students:

Student 1: I think this [*the definition of ①*] is a contradiction - it is a concept which I cannot make mine because it is in contradiction to what I know…

Interviewer: … contradictory because it has both characteristics of infinity and characteristics of finite numbers?

Student 1: Yes ...

Student 2: If you consider it as an infinite big number it is not contradictory because in the end this is not [*the*] infinity

Student 3: It is one of the characteristics of grossone… continuously increasing…

(IT focus group interview)

From this extract emerges a distinct sense of unease on the part of the students and especially of Student 1. They seem to be torn between being able to use formally a definition that they have been given (analysis of the written work produced during the group work sessions revealed that many students managed to find a solution for the Thompson Lamp using grossone) but being unable to accommodate this definition in their beliefs about mathematics. The quote below (collected in a separate focus group) can also be interpreted as manifestation of this unease.

Student (IT): I can't think of subtracting an infinitely large number from an infinitely large number - where do I get to? I don't get to zero for sure . . .

In this case we may argue that, for these students, mathematics is either right or wrong and that an alternative definition of a familiar concept cannot be accommodated because it appears to be in contradiction to what they have studied and taken to be right.

Acceptance: It does kind of work as *i*…

Unlike the previous group, other students not only appear able to accommodate this new concept in their knowledge about mathematics but could work with it without perceiving it as incompatible with what they already knew:

> Student (UK): It does kind of work as *i*, that you have your real part and your imaginary part and ...like i^2 would be minus one . . .

Or:

> Student (IT): It is like *i* - you don't know what is *i* but you know that i^2 is -1.

Indeed, the parallel that these students draw with the imaginary unit *i* is revealing. We know from the history of mathematics and Cauchy's famous remark that 'We completely repudiate the symbol $\sqrt{-1}$, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it' (Nahin, 2010) that the mathematics community took much time to accept this new mathematical object especially because it contradicted (or seemed to contradict) much of the mathematics known before. We interpret this ability to see the similarities between these objects, *i* and grossone, as evidence of an advanced view of what mathematics is. Moreover, another student remarked:

> Student (UK): Because […] I mean they say infinity isn't a number but then […] there is an argument for and against that.

In this extract, we can infer that this student is considering that perhaps there may be different ways of defining mathematical concepts and perhaps more than one interpretation is possible. This may be an indication of a more advanced mathematics epistemology, one where not every statement is true or false and that recognises mathematics as the product of a social construction.

DISCUSSION

The aim of this study was to test a novel qualitative methodology to investigate students' epistemological beliefs about mathematics. We tested whether asking secondary school students to work through a worksheet introducing a new conceptualisation of infinity, unseen and somewhat incompatible with some of their existing knowledge, could provide a strategy suitable to expose secondary school students' epistemological beliefs. We chose an alternative view of infinity and how to measure the size of infinite sets as this approach is in contrast to the way in which students have been exposed to the concept of infinity in their studies. How to understand and measure school students' epistemological beliefs about mathematics is an important topic as these beliefs impact on most aspects of their learning and engagement with the subject (Muis, 2004). Indeed, both Muis (2004) and Depaepe et al. (2016) found in their reviews that the beliefs held by students regarding mathematics were hindering rather than facilitating their learning, making the issue of measuring these beliefs (and eventually influencing them) all the more important. Through thematic analysis of the focus group interviews held after the class activities we found that we could distinguish at least two separate understandings of how mathematics is structured and operates, i.e. two different mathematics epistemologies held by the students participating in the study. Some students held a naïve view close to an absolutist position, according to which mathematics is perceived as a fixed body of knowledge that cannot change (Depaepe et a. 2016). This view manifested itself in the unease felt by the students who were able to work formally through the definitions and concepts given but could not accommodate those in their understanding of infinity because they perceived them to be in stark contrast with what they already knew. Other students held a more advanced view in line with a fallibilist view of mathematics, which perceives this subject as socially constructed hence open to revisions and changes. This view manifested itself in the parallel that some students drew between the introduction of grossone and the introduction of the imaginary unit *i*. These students were able to accommodate the idea that some mathematical definitions may change and that different (even contrasting) definitions of the same concept may exist in mathematics. Therefore the call for caution voiced by Francisco (2013) that not all school students hold naïve epistemologies of mathematics seems to be justified. This finding partially answers our research question by showing that such methods can potentially elicit students' epistemological beliefs and can help understanding their structure. Moreover, following the idea that epistemological beliefs impact on conceptual change and that more sophisticated epistemologies such as those related to fallibilist views of mathematics promote conceptual change (Pintrich, 1999), we would argue that our methodology can not only elicit such epistemologies but also stimulate re-thinking of previously held beliefs by kindling cognitive conflict in the students. More extensive data collection and testing the use of other concepts (such as the superreals, Tall, 1980) could refine this methodology and contribute to our understanding of students' epistemologies but also could potentially help students refine their own epistemologies of mathematics.

References

Berresford, G. C. (1981). A Note on Thomson's Lamp Paradox. *Analysis*, *41*(1), 1-3.

Depaepe, F., De Corte, E., & Verschaffel, L. (2016). Mathematical epistemological beliefs. In J. A. Greene, W. A. Sandoval, & I. Braten (Eds.), *Handbook of epistemic cognition* (pp. 147–164). Routledge.

Francisco, J. M. (2013). The mathematical beliefs and behavior of high school students: Insights from a longitudinal study. *The Journal of Mathematical Behavior*, 32(3), 481-493.

Hofer, B. K. (2001). Personal epistemology research: Implications for learning and teaching. *Educational Psychology Review*, *13*(4), 353-383.

Kidron, I., & Tall, D. (2015). The roles of visualization and symbolism in the potential and actual infinity of the limit process. *Educational Studies in Mathematics*, 88(2), 183-199.

Limon, M. (2006). The domain generality–specificity of epistemological beliefs: A theoretical problem, a methodological problem or both?. *International Journal of Educational Research, 45*(1), 7-27.

Lolli, G. (2015). Metamathematical investigations on the theory of Grossone, Applied Mathematics and Computation, 255(1), 3-14.

Mamolo, A., & Zazkis, R. (2008). Paradoxes as a window to infinity. *Research in Mathematics Education*, 10(2), 167-182.

Mason, L. (2003). High school students' beliefs about maths, mathematical problem solving, and their achievement in maths: A cross-sectional study. *Educational Psychology*, *23*(1), 73-85.

Mogashana, D., Case, J. M., & Marshall, D. (2012). What do student learning inventories really measure? A critical analysis of students' responses to the Approaches to Learning and Studying Inventory. *Studies in Higher Education*, *37*(7), 783-792.

Monaghan, J. (2001). Young peoples' ideas of infinity. *Educational studies in Mathematics*, 48(2), 239-257.

Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of educational research*, *74*(3), 317-377.

Muis, K. R. (2007). The role of epistemic beliefs in self-regulated learning. *Educational Psychologist*, *42*(3), 173-190.

Muis, K. R., Duffy, M. C., Trevors, G., Ranellucci, J., & Foy, M. (2014). What were they thinking? Using cognitive interviewing to examine the validity of self-reported epistemic beliefs. *International Education Research*, *2*(1), 17-32.

Muis, K. R., Psaradellis, C., Lajoie, S. P., Di Leo, I., & Chevrier, M. (2015). The role of epistemic emotions in mathematics problem solving. *Contemporary Educational Psychology*, *42*, 172-185.

Nahin, P. J. (2010). *An imaginary tale: The story of* $\sqrt{-1}$. Princeton University Press.

Perry, W. G. (1970). *Forms of intellectual and ethical development in the college years: A scheme*. New York: Holt, Rinehart and Winston.

Pintrich, P. R. (1999). Motivational beliefs as resources for and constraints on conceptual change. In W. Schnotz, S. Vosniadou, & M. Carretero (Eds.), *New perspectives on conceptual change* (pp. 33–50). Oxford, UK: Elsevier Science.

Sergeyev, Ya.D. (2003). *The Arithmetic of Infinity*. Rende: Orizzonti Meridionali (Kindle Edition 2013).

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for research in mathematics education*, 338-355.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. Issues in education, 4(1), 1-94.

Tall, D. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity. *Educational Studies in Mathematics*, *11*(3), 271-284.

WATCHING MATHEMATICIANS READ MATHEMATICS

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This report contributes to the debate about whether expert mathematicians skim-read mathematical proofs before engaging in detailed line-by-line reading. It reviews the conflicting introspective and behavioural evidence, then reports a new study of expert mathematicians' eye movements as they read both entire research-level mathematics papers and individual proofs within those papers. Our analysis reveals no evidence of skimming, and we discuss the implications of this for research and pedagogy.

INTRODUCTION

Proof is central to mathematical practice, so understanding proof and proving is an important goal of most mathematical curricula (Hanna, 2007). Furthermore, at least in advanced mathematics courses, students spend considerable time learning mathematics by studying proofs (Selden & Selden, 2003). Consequently, several research groups have investigated the processes by which students engage with written proofs (Inglis & Alcock, 2012; Ko & Knuth, 2013; Mejía-Ramos & Weber, 2014).

A complementary approach is to examine expert mathematical practice, with researchers arguing that if we want students to develop expert-like behaviours, we require accurate understanding of those behaviours (RAND, 2003; Weber, 2008; Wilkerson-Jerde & Wilensky, 2011). In this report, we address an unresolved issue from studies on expert reading (Inglis & Alcock, 2012, 2013; Mejía-Ramos & Weber, 2014; Weber, 2008; Weber & Mejía-Ramos, 2011, 2013): that of whether mathematicians skim-read mathematical texts before carefully reading line by line.

The skimming hypothesis was generated when Weber (2008) interviewed eight mathematicians about their behaviour while validating research-level proofs. Many explained that they would often skim-read before reading line by line. For example, one described "first try[ing] to understand the structure of the proof, to get an overview of the argument that's being used" (p.441); another described first reading through the proof "to get the flow of it" and then going back to "get the details" (p.441).

Inglis and Alcock (2012) investigated this hypothesis by asking mathematicians and undergraduates to validate purported proofs and recording their eye movements as they did so. They found no evidence of initial skimming—participants typically did not fixate on the last lines of purported proofs until approximately half way through their reading attempts. Citing earlier methodological work (e.g., Nisbett & Wilson, 1977), Inglis and Alcock therefore suggested that introspective evidence about mathematical practice should be regarded with caution. Weber and Mejía-Ramos (2013), however, criticised this argument, in part because the proofs Inglis and Alcock used were too

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 139-146). Umeå, Sweden: PME.

short to give meaningful results about expert practice. Inglis and Alcock (2013) concurred that their purported proofs were considerably shorter than those encountered in mathematical research (largely because their expert/novice research design required proofs that were accessible to first-year undergraduates).

Certainly mathematicians believe that they skim-read: Mejía-Ramos and Weber (2014) reported that 92% of mathematicians responding to a large-scale survey agreed with the statement “When I read a proof in a respected journal, it is not uncommon that I skim the proof first to comprehend the main ideas of the proof, prior to reading the proof line-by-line”. They also asked participants about their reading behaviour when refereeing; again, large majorities of participants claimed to skim-read and check for validity in this context. They therefore suggested that it would be strange if Alcock and Inglis’s (2012) failure to find such behaviour reflected actual mathematical practice. But whether mathematicians actually skim-read remains an open question and, in this report, we investigate whether skimming is evident in mathematicians’ eye movements when they read research-level mathematics.

METHODS

Participants, apparatus and procedure.

To determine whether mathematicians skim-read before reading line by line, we recorded mathematicians’ eye movements while they read research papers drawn from their own fields. Participants were ten permanent members of staff (assistant professor level or above) from a UK University. All had doctorates and numerous published academic papers. Five were applied mathematicians, four were pure mathematicians, and one was a statistician. Eight different nationalities were represented.

Each participant was asked to select a research paper that they planned to read but had not yet begun; these papers were forwarded to the researchers prior to the experimental session. To protect the anonymity of participants, we do not report which papers were chosen. However, they included published journal articles, pre-prints from the arXiv, and a short monograph. Topics included Bessel functions, algebraic geometry, group theory, and the modelling of physical and biological phenomena. The papers varied in length: the shortest was 4 pages and the longest 53.

Each participant took part individually in a quiet room. Eye movements were recorded with a Tobii T120 Eye-Tracker, set to sample at 60Hz. The T120 is a remote eye-tracker with two binocular infrared cameras under a 17” TFT monitor; it typically achieves eye-position tracking accuracy of 0.5°. Stimuli were displayed on a screen that participants viewed (without head restriction) from a distance of approximately 60cm. For each participant, the eye-tracker was calibrated with a 9-point display.

Participants were told that they would be shown their paper and that they should read it as if intending to write a review for MathSciNet, an online database of short reviews of published mathematical papers. All participants were familiar with the guidelines for MathSciNet, which state:

> In most cases the review should state the main results, together with enough notation to make the statements comprehensible to someone already familiar with the field. The main ideas of the proof should be sketched when this is feasible.

This instruction was designed to ensure that all participants would read for comprehension rather than some other purpose (such as checking validity). We believed that if skim reading were a common feature of mathematicians' reading behaviour, then these instructions would be likely to reveal it.

After the instructions were displayed and explained verbally by the experimenter, the first page of the participant's research paper was displayed and the experimenter left the room. Participants could move sequentially through the pages of their papers using cursor keys, and were provided with pen and paper to make notes if they wished. On completing the task, they stopped the recording and called the experimenter. There was no time restriction, and participants' reading times varied between 17 and 65 minutes.

Data analysis.

Our analysis uses the fact that, when viewing a static image, eye movements consist of fixations (short stationary periods, usually lasting 150-500ms) and saccades (rapid movements between fixations). During saccades, no information can be processed (e.g., Matin, 1974), so fixation locations suffice to determine the path of a participant's attention (for a substantial review of eye-movement research see Rayner, 2009). Our strategy was to create, for each participant, a scatter plot with time on the *x*-axis and paragraph in the paper on the *y*-axis. Because eye-movement data are noisy (blinks or random head movements can cause single fixations away from the location of attention (Inglis & Alcock, 2013), we then fitted curves to these plots using LOESS regression (also known as "locally weighted scatterplot smoothing"). This technique fits connected quadratics to local sections of a scatterplot (e.g., Cleveland, 1979), and permits fitting a curve to data without making *a priori* assumptions about the shape of the curve. If participants adopted initial skim strategies, we would expect their fixation plots to look like that shown in Figure 1.

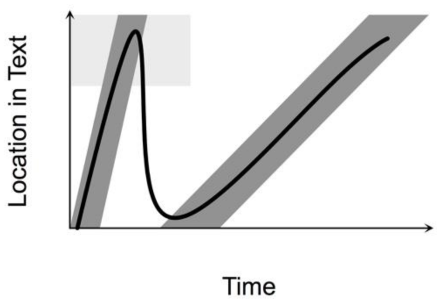

Figure 1: The type of fixation plot and LOESS curve we would expect if a participant had adopted an initial skimming strategy.

We operationalised this by evaluating whether each participant's LOESS curve entered the light grey box in the top left of Figure 1: if the focus of attention entered the last third of the reading material within the first third of their reading attempt, we coded this as a skim (cf. Weber & Mejía-Ramos, 2013).

RESULTS

We first examine global reading behaviour, reporting on each participant's reading of their entire paper. We take this approach because, in research-level mathematics, proofs cannot normally be read in isolation: papers typically introduce novel definitions, ideas and techniques before presenting a proof. We then examine local reading behaviour, illustrating participants' reading of their papers' first self-contained arguments. This allows us to compare more directly with earlier discussions of skim reading (Inglis & Alcock, 2012; Inglis & Alcock, 2013; Mejía-Ramos & Weber, 2013; Weber & Mejía-Ramos, 2013), which have typically involved single proofs.

Global reading behaviour.

Figure 2 shows individual paragraph-by-time fixation plots for all ten participants. There appeared to be three broad categories of attention movement. Some participants (1, 2, 4, 5, and 7) read in an approximately linear order, beginning at the start of the paper and progressing to the end with few moves to non-adjacent paragraphs. Others (8, 9, and 10) moved their attention in a piecewise linear fashion: they started with a linear approach, then re-read certain sections in detail, again linearly. Finally, two participants (3 and 6) appeared to adopt different approaches. In the post-experiment debrief, Mathematician 3 reported that he had not understood the introduction to his paper and had therefore failed to make substantial progress beyond the first few pages. This is consistent with his eye movements, which include a series of linear attention moves within the first 30 paragraphs. Mathematician 6 had relatively few fixations (in any location) in the latter half of his reading attempt. He made a large number of notes, so we attributed this to his eyes being largely off screen during this time.

Despite this variety in reading behaviour, no mathematician used a skimming strategy: in no case did the LOESS curve enter the last third of the paper in the first third of the reading time. Some graphs (1, 4, 6, 7 and 8) *did* show a small number of single fixations in the key area, but these were so few that we attributed them to participants scrolling forward to the reference sections of their papers (they had to view each page in turn, explaining the "trails" of fixations leading up to the reference sections in plots 4, 6 and 8). Even for participants who read in a piecewise linear fashion, reading behaviour can be distinguished from the skimming strategy detailed by Weber (2007), because the second and third reading attempts did not involve the whole text and/or took place at a substantially faster rate than the initial reading attempt.

If initial skimming were a common feature of mathematicians' reading behaviour, it is extremely unlikely that we would have found no skims in our data. A skimming rate of zero out of ten is significantly lower than 50%, sign test p = .002, and significantly lower than the 92% figure found by Mejía-Ramos & Weber (2013), binomial test $p = 1.03\times10^{-11}$. Of course, it is possible that our operationalisation of skimming was faulty, and we consider this possibility in the next section and the discussion.

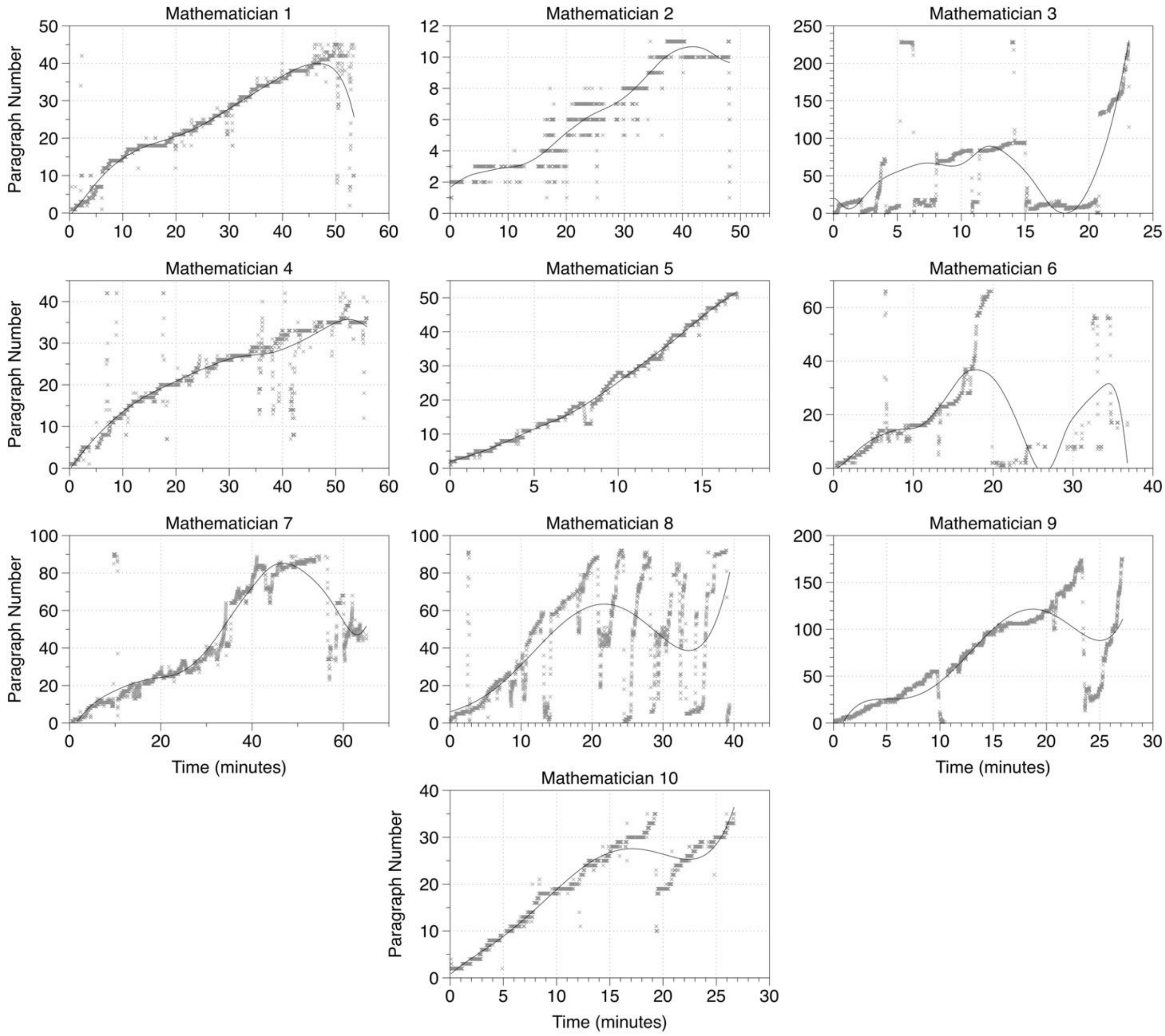

Figure 2: Paragraph Number by Time fixation plots for each participant, together with associated LOESS curves (second order, smoothing parameter 0.3).

Local reading behaviour.

We found no evidence of skimming in participants' attention while they read entire papers. But each of their papers included multiple shorter arguments, some of which formed self-contained paragraphs. Because our global analysis focused on between-paragraph eye movements, it is therefore possible that we missed the skimming behaviour hypothesised by Mejía-Ramos and Weber (2013) because this takes place within paragraphs. To investigate this possibility, we identified the first self-contained argument in each paper (typically a proof of a lemma or proposition, or the derivation of a model of a physical/biological process), and conducted a line-by-line analysis of the corresponding participant's attention for this argument.

Two illustrative fixation plots are shown in Figure 3. The wide graph shows every fixation on the relevant areas of each paper, although it is clear that many of these fixations did not contribute to genuine reading attempts (single fixations were probably due to random eye-movements or to flicking through the pages). Because of this we have magnified the sections of the plots that we judged to be the first attempt to read through the self-contained arguments, and plotted the associated LOESS curves. In our judgement, neither these participants nor any others could be said to have used a skimming approach — the full set of these plots (one for each participant) can be inspected at: https://doi.org/10.6084/m9.figshare.5733510.v1.

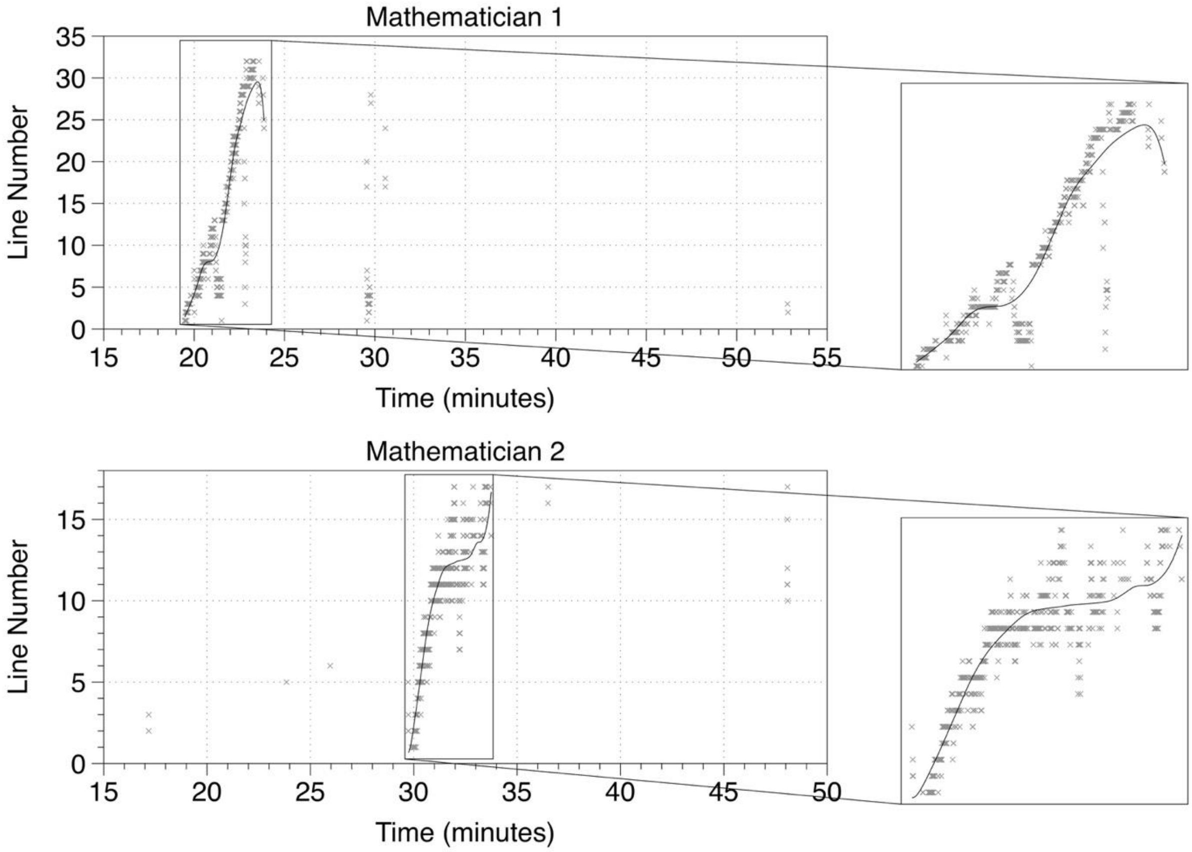

Figure 3: Line Number by Time fixation plots for the first argument in the paper for Mathematicians 1 and 2. The first clear-cut reading attempt is been magnified, together with its associated LOESS curve (second order, smoothing parameter 0.3).

DISCUSSION

Mejía-Ramos and Weber (2013) found that 92% of mathematicians claimed to understand the structures of proofs by skimming them before reading in detail. We have no reason to believe that our sample was unrepresentative of expert mathematicians—our participants worked in various areas of pure and applied mathematics and statistics, and were from eight different countries—yet we found no evidence of skimming in our data. The probability of this occurring if the introspective accounts are correct is vanishingly small, so we think it unlikely that skimming as operationalised in our study is fundamental to mathematicians' behaviour.

We briefly discuss two possible accounts for this finding, drawing out the implications of each. One account is that mathematicians simply do not skim. This would raise methodological concerns: where introspective claims are inconsistent with behavioural evidence, we must decide how to interpret the results of methodologically distinct studies. In such a situation, one might argue that introspective evidence should simply be ignored (e.g., Lyons, 1986; Nisbett & Wilson, 1977). Alternatively, however, it could be that we incorrectly operationalised what it means to skim when reading mathematics. When 92% of participants agreed that they would often “skim [a] proof to comprehend the main ideas…prior to reading [it] line-by-line"”, perhaps they were referring to a much longer process than either we or Weber and Mejía-Ramos (2013) believed. Perhaps, for instance, the entire reading attempts we recorded in this experiment (which lasted up to an hour) should be classified as skim-reads. Perhaps it is only after a relatively long “skim” that mathematicians go back and re-read mathematical arguments line by line, or perhaps in normal circumstances mathematicians only skim and line-by-line reading is relatively rare. We suggest that disentangling these possibilities requires ethnographic studies of mathematical practice (cf. Greiffenhagen & Sharoock, 2011). Such studies would form a worthwhile contribution to the literature on mathematicians’ reading behaviour.

In the meantime, we can comment on a broader issue. Our data revealed considerable variety in mathematicians’ reading behaviours, as is apparent in Figure 2. It thus contributes to a growing body of evidence on diversity in expert mathematical behaviour (e.g., Inglis, Mejía-Ramos, Weber & Alcock, 2013; Weber, Inglis & Mejía-Ramos, 2014). We do not yet know what causes these differences. Is behaviour driven by individual differences among mathematicians? Or perhaps by the mathematical content or structures of papers or proofs? What prompts a decision to re-read a section, or to skip ahead? However, we can observe that such findings complicate arguments that we should teach students expert-like behaviours (e.g., RAND, 2003; Wilkerson- Jerde & Wilensky, 2011). If expert behaviour is heterogeneous, as suggested by this study and others, then basing instruction upon it is a non-trivial task.

References

Cleveland, W. S. (1979). Robust locally weighted regression and smoothing scatterplots. *Journal of the American Statistical Association*, *74,* 829–836.

Greiffenhagen, C. & Sharrock, W. (2011). Does mathematics look certain in the front, but fallible in the back? *Social Studies of Science*, *41,* 839–866.

Hanna, G. (2007). The ongoing value of proof. In P. Boero (Ed.), *Theorems in school: From history, epistemology and cognition to classroom practice*. Rotterdam: Sense, pp. 3–18.

Inglis, M. & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*, 358–390.

Inglis, M. & Alcock, L. (2013). Skimming: A response to Weber and Mejía-Ramos. *Journal for Research in Mathematics Education*, *44,* 471–474.

Inglis, M., Mejía-Ramos, J. P., Weber, K. & Alcock, L. (2013). On mathematicians’ different standards when evaluating elementary proofs. *Topics in Cognitive Science*, *5*, 270-282.

Ko, Y.-Y. & Knuth, E. J. (2013). Validating proofs and counterexamples across content domains: Practices of importance for mathematics majors. *Journal of Mathematical Behavior*, 32, 20–35.

Lyons, W. E. (1986). *The Disappearance of Introspection*. Cambridge, MA: MIT Press.

Matin, E. (1974). Saccadic suppression: A review and an analysis. *Psychological Bulletin, 81*, 899–917.

Mejía-Ramos, J. P. & Weber, K. (2014). Why and how mathematicians read proofs: Further evidence from a survey study. *Educational Studies in Mathematics, 85,* 161-173.

Nisbett, R. E. & Wilson, T. D. (1977). Telling more than we can know: Verbal reports on mental processes. *Psychological Review*, *84*, 231–295.

RAND (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND Corporation Mathematics Study Panel.

Rayner, K. (2009). The 35th Sir Frederick Bartlett Lecture: Eye movements and attention in reading, scene perception and visual search. *Quarterly Journal of Experimental Psychology*, *62*, 1457–1506.

Selden, A. & Selden, J. (2003). Validations of Proofs considered as texts: can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*, 4–36.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education*, *39*, 431–459.

Weber, K., Inglis, M., & Mejia-Ramos, J. P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist*, *49*(1), 36-58.

Weber, K. & Mejía-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics*, *76*, 329–344.

Weber, K. & Mejía-Ramos, J. P. (2013). On mathematicians' proof skimming: A reply to Inglis and Alcock. *Journal for Research in Mathematics Education*, *44*, 464-471.

Wilkerson-Jerde, M. & Wilensky, U. (2011). How do mathematicians learn math? Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics*, *78*, 21–43.

ENHANCING TEACHER NOTICING USING A HYPOTHETICAL LEARNING TRAJECTORY

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Since noticing has been identified as a critical component of teaching expertise, researchers have tried to identify contexts to its development. These studies assume that growth in teachers' noticing expertise can be inferred from their professional discourse. Prior research has also shown that teachers' noticing development in teacher education programs is challenging if no framework or guide to support pre-service teachers in their noticing is provided. In our study, 29 pre-service teachers used a hypothetical learning trajectory as a guide to interpret students' fractional thinking. Results show that the use of a hypothetical learning trajectory improves pre-service teachers' professional discourse on students' mathematical thinking and then, enhances noticing.

THEORETICAL BACKGROUND

Teacher noticing, a critical component of teaching expertise, can be seen as a "movement or shift of attention" (Mason, 2011, p. 45), or a set of three inter-related skills: attending, interpreting, and deciding to respond (Jacobs, Lamb, & Philipp, 2010). Although many researchers have understood noticing in terms of these three skills, it is useful to think about noticing at a fine-grained level. Mason (2011, p. 47) highlights that people can notice different things at different times in different ways, and sees these fine-grained processes as *holding wholes, discerning details, recognising relationships, perceiving properties, and reasoning on the basis of agreed properties*. These micro-level processes can be seen as the mechanisms behind the three inter-related skills of noticing. For example, when teachers attend to students' strategies, they are discerning the details of students' thinking. Similarly, teachers often interpret students' mathematical thinking by taking into account the details discerned, which requires them to recognise relationships between the identified mathematical details and the characteristics of students' mathematical thinking.

Examining teacher noticing hence hinges on how researchers investigate the micro-structure of attention. With the aim of developing and honing teachers' noticing of students' mathematical thinking, mathematics educators have studied noticing in many different contexts—video clubs (van Es, & Sherin, 2008), lesson study (Lee, & Choy, 2017; Amador, & Carter, 2018), written students' answers (Sánchez- Matamoros, Fernández, & Llinares, 2015), mentoring conversations (Seto & Loh, 2015), or narratives (Ivars, & Fernández, 2018). A common important assumption underlies them all:

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 147-154). Umeå, Sweden: PME.

growth in teachers' noticing expertise is inferred from their professional discourse. In other words, the development of noticing, perceived as a shift from general strategy descriptions to descriptions that included teachers' reasoning based on mathematically relevant details of students' mathematical thinking, can be seen from how teachers discuss mathematical thinking. Consequently, changes in pre-service primary teachers' discourse on students' mathematical thinking indicate changes in their noticing expertise.

According to recent research, developing teachers' noticing of students' mathematical thinking without a guide or focus that supports pre-service teachers learning (Wilson, Mojica, & Confrey, 2013) is challenging. This begs the question of the type of foci needed to support teachers when developing their noticing expertise. Here, we see student's hypothetical learning trajectory as a potential support for teachers when they try to identify learning goals, interpret students' mathematical thinking, and respond with appropriate instruction. More importantly, hypothetical learning trajectories provide pre-service teachers with a specific language to describe students' thinking (Edgington, Wilson, Sztajn, & Webb, 2016). In this context, we hypothesise that providing pre-service teachers with a student's hypothetical learning trajectory (HLT) will help them elaborate a more detailed discourse on students' mathematical thinking, and therefore, enhance their skill of noticing. Our research question is: Does the use of a HLT help pre-service teachers elaborate a detailed discourse on students' mathematical thinking?

METHOD

Participants and context

Twenty-nine pre-service teachers (PTs) participated in this research. They were attending a course on the teaching and learning of elementary mathematics as part of their degree to become a primary school teacher. As part of this course, these PTs participated in a learning environment aimed at developing their noticing of students' fractional thinking. The emphasis on fractional thinking is partly motivated by the part-whole meaning of fractions, one of the most problematic concepts in elementary school maths. Seeing a HLT as a useful means "for teaching concepts whose learning is problematic generally" (Simon & Tzur, 2004, p.101), we designed a HLT of the part-whole meaning of fraction to guide PTs in noticing students' fractional thinking.

Simon's (1995) conceptualisation of a hypothetical learning trajectory includes three components: (i) a learning goal, (ii) a hypothetical learning process (hypothetical learning trajectory proficiency levels of thinking) and (iii) a set of learning activities. The learning goal is to understand the part-whole meaning of the fraction concept. Following a literature review on how students' thinking about the part-whole concept of fraction develops over time (Battista, 2012; Steffe, & Olive, 2009), we considered three different levels of students' mathematical thinking (hypothetical learning trajectory proficiency levels; Figure 1). We also included a set of learning activities to help students move through different levels of thinking (proficiency levels): activities of identifying and representing a fraction given a whole, activities of identifying and representing a whole given a part and, activities of comparing fractions (using continuous and discrete contexts, and proper and improper fractions).

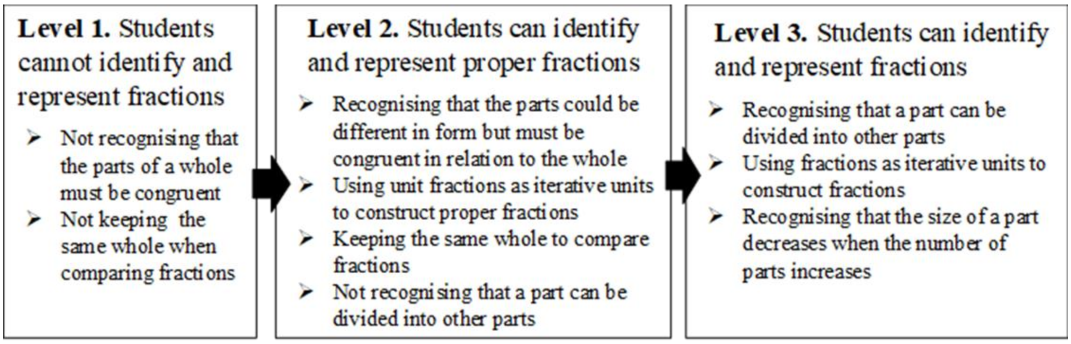

Figure 1: Proficiency levels of the HLT.

PTs participated in a learning environment that was organised around six sessions lasting two hours each. The first two sessions focused on mathematical elements related to the part-whole concept of fraction, and provided PTs opportunities to work on fraction activities and analyse video-clips of primary school students solving the same fraction activities. In the last four sessions, the HLT was introduced and, PTs had to complete three tasks (Task A, Task B, and Task C), in which they have to use the HLT to notice students' mathematical thinking.

The tasks

The three tasks have the same structure: one or two primary school activities, the answers of three primary school students (or pair of students) to these activities with different proficiency levels, and the following four questions: *Q1)* Describe the primary school activity taking into account the learning objective: what mathematical elements does the student need to know to solve it? *Q2)* Describe how each pair of students has solved the activity identifying how they have used the mathematical elements involved and the difficulties they have had with them. *Q3)* What are the characteristics of students' thinking (related to the proficiency levels of the HLT) that can be inferred from their responses? Explain your answer. *Q4)* How could you respond to these students? Propose a learning objective and a new activity to help students progress in their thinking.

In Task A, three pairs of primary school students' answers to an activity of identifying a proper fraction were presented (Figure 2, adapted from Battista, 2012).

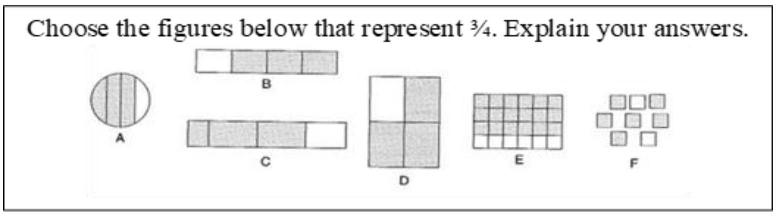

Figure 2: Activity of identifying a proper fraction (Task A).

The three pairs of students' answers show different characteristics of the HLT proficiency levels (Table1).

Mathematical Elements \ Primary School Students	Víctor & Xavi (Level 1)	Joan & Tere (Level 2)	Félix & Álvaro (Level 3)
The parts of a whole must be congruent	No*	Yes	Yes
A part can be divided into other parts	No	No	Yes

* *No* means that the pair of students have difficulties with the mathematical element and *Yes* means that they used properly the mathematical element

Table 1: Characteristics of the three pairs of primary school students answers in Task A

The primary school activity in Task B consists in comparing proper fractions (*Activity: Which is greater 4/5 or 3/4? Explain it with a picture or words*). Three pair of students' answers were presented showing different characteristics of the HLT proficiency levels (Table 2).

Mathematical Elements \ Primary School Students	Marta & Vicent (Level 1)	Ana & Iván (Level 2)	Louis & Núria (Level 3)
The wholes must be the same to compare	No	Yes	Yes
Inverse relationship between the number of the parts and the size of each part	No	No	Yes

Table 2: Characteristics of the three pairs of primary school students answers in Task B

Task C includes the answers of three students to two activities (Figure 3): in activity 1, a proper fraction has to be identified and in activity 2, the whole has to be reconstructed when a fractional part is given, in this case, an improper fraction.

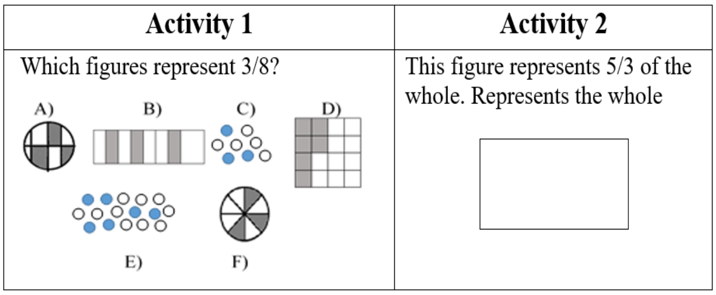

Figure 3: Activities of Task C.

Table 3 shows the characteristics of the three students' answer to these activities.

	Student 1 (Level 1)		Student 2 (Level 2)		Student 3 (Level 3)	
Mathematical Elements \ Activity	1	2	1	2	1	2
The parts of a whole must be congruent	No	No	Yes	Yes	Yes	Yes
A part can be divided into other parts	No		No		Yes	
Use a part (unit fraction) as an iterative unit to reconstruct the whole		No		No		Yes

Table 3: Characteristics of the three primary school students answers in Task C.

Analysis

Data of this research are PTs' answers to questions Q2 and Q3 of the three tasks. We carried out an inductive analysis considering if PTs had (i) identified the mathematical elements in the students' answers (discerning details); and (ii) interpreted the students' thinking relating the mathematical elements identified in the students' answers to the different proficiency levels (recognising relationships between the mathematical elements identified and the HLT proficiency levels). Initially, a subset of PTs' answers was independently analysed by three researchers regarding the above two foci. Next, we compared our results discussing the discrepancies until we reached an agreement. Afterwards, we carried out the complete analysis constantly revising our categories. At the end of this analytical process, two main categories emerged: (i) *Interpreting through the three tasks*: PTs who interpreted students' mathematical thinking relating the mathematical elements with the proficiency levels in the three tasks and (ii) *Difficulties in at least one task*: PTs who had difficulties using the mathematical elements to interpret students' mathematical thinking at least in one of the tasks.

For each of the latter categories, three subcategories emerged regarding the pre-service teachers' professional discourse. These subcategories differ on PTs' capacity to focus their attention on the relevant mathematical details of students' answers: i) *Evidencers*: PTs who interpreted students' thinking providing details from students' answers ii) *Non-evidencers:* PTs who interpreted students' thinking but did not provide details from students' answers iii) *Adders*: PTs who interpreted students' thinking providing details from students' answers but adding unnecessary information. Through the three tasks, PTs were classified as consistently Evidencer (PTs classified as Evidencer in the three tasks), consistently Non-evidencer (PTs classified as Non-evidencer in the three tasks) and progress from Non-evidencer or Adder to Evidencer (PTs who shifted from of the Non-evidencer or Adder group to the Evidencer group).

RESULTS

Five different pre-service teachers' profiles emerged from the analysis of the data regarding how they interpreted students' mathematical thinking and the discourse provided (Table 4).

Ways of interpreting students' mathematical thinking	Discourse provided			
	From Non-evidencer or Adder to Evidencer	Consistently Evidencer	Consistently Non-evidencer	TOTAL
Interpreting through the three tasks	7	8		15
Difficulties in at least one task	4	9	1	14
TOTAL	11	17	1	29

Table 4: Profiles of PTs regarding how they interpreted and the discourse provided.

Twenty-eight out of the 29 PTs interpreted students' mathematical thinking providing details from students' answers in the last task. Seventeen out of them provided details from students' answers in the three tasks (consistently Evidencers). Eleven PTs interpreted students' answers providing a less detailed discourse in task A and Task B (without providing details from students answers or adding unnecessary information-groups of Non-evidencers and Adders, respectively) but in task C, they used a more detailed discourse providing details from students' answers to support their inferences.

Next, we show through excerpts of answers given by the PT25, how these 11 PTs improved their discourse (in the sense of giving a more detailed discourse) from task A to task C. This PT25, in Task A, interpreted students' mathematical thinking relating the mathematical elements identified in students' answers with the HLT proficiency levels. However, she did not provide details from students' answers to support her inferences (Non-evidencer). For instance, she wrote:

> Joan and Tere recognise that the parts of the whole must be congruent. They identify fractions in continuous contexts but they have difficulties in discrete contexts. They do not recognise that a part can be divided into other parts. This pair of students is at Level 2.

This PT25, in Task B, started to provide details from students' answers to support her interpretation of students' mathematical thinking (Evidencer; emphasis is added in the details):

> Louis and Núria (Pair 3) have acquired the mathematical element inverse relationship since they notice that 4/5 needs 1/5 to build the unit and 3/4 needs 1/4 to build the unit. Then, as 1/5 is shorter than ¼, they know that 4/5 is greater. They are at level 3.

The excerpt above shows how PT25, in Task B, used some details from students' answers to support her claims. For instance, when she said "4/5 needs 1/5 to build the unit and 3/4 needs 1/4 to build the unit. Then, as 1/5 is shorter than ¼, they know that 4/5 is greater" and related it with the mathematical element "inverse relationship" showing an improvement in the professional discourse used to interpret students' thinking. In Task C, she interpreted students' mathematical thinking providing also details from students' answers (Evidencer; emphasis is added in the details):

> Student 2→In the activity 1, he has acquired the mathematical element the parts of a whole must be congruent since he answers that figures A and B do not represent 3/8 and figure F is 3/8. Furthermore, he claims that Figure D is 6/16 and does not recognise Figures C and E as fractions. Therefore, he does not have acquired the element a part can be divided into other parts. In activity 2, he does not know how to solve the activity since he thinks that the figure given is the unit (3/3). Although he splits the figure in congruent parts using 1/3 as an iterative unit, he does not know how to work with improper fractions. This student is at level 2 since he understands that the parts of a whole must be congruent, he uses the unit fraction as iterative unit but he has difficulties with the mathematical element a part can be divided into other parts and has difficulties with improper fractions in discrete contexts.

DISCUSSION AND CONCLUSIONS

The fact that 28 out of the 29 PTs provided, at least in the last task, a more detailed discourse including details from students' answers to support their claims suggests that the HLT helps PTs improve their professional discourse and, this can be seen evidence of noticing enhancement. In fact, 11 out of the 29 PTs improved their discourse through the three tasks. This improvement let them to progress from elaborating a less detailed discourse (adding unnecessary information or not providing details from students' answers), to entering a more detailed discourse providing details from students' answers. Progress in their discourse was evidenced by the amount of details provided. Therefore, progress in their discourse is a sign of improving the way they noticed students' mathematical thinking since they were able to focus their attention on the mathematical details of students' answers. At the same time, they also provided evidence from students' answers, which could be understood as an increase in sensitivity to the details of the learning situations (Mason, 2011).

In this sense, enhancing noticing can be understood as a virtuous circle in teacher education programs in which the HLT is a critical element. Whether noticing is displayed by discourse, introducing a HLT as a guide can improve PTs' professional discourse since it helped them focus on details, and enhance their noticing skill. In other words, HLT "may assist teachers in leveraging students' existing understandings" (Wilson, Sztajn, Edgington, Webb, & Myers, 2017; p 571), providing them with a structure that facilitates the generation of a professional discourse, which includes interpretations based on evidence (on details of students' answers).

Our study provides teacher educators with types of tasks that they can use to help pre-service teachers enter in a more detailed professional discourse to attend to the details of students' answers and their different mathematical levels of thinking. Nevertheless, more research is needed to examine whether improvements in professional discourse can help pre-service teachers make instructional decisions based on students' mathematical thinking.

Acknowledgements

This research was supported by the projects EDU2014-54526-R and EDU2017-87411-R from MINECO, and by a FPU grant FPU14/07107 from the Ministry of Education, Culture and Sports (Spain).

References

Amador, J., & Carter, I. (2018), Audible conversational affordances and constraints of verbalizing professional noticing during prospective teacher lesson study. *Journal of Mathematics Teacher Education, 21*(1), DOI 10.1007/s10857-016-9347-x

Battista, M.T. (2012). *Cognition-Based Assessment and teaching of fractions: Building on students' reasoning*. Portsmouth, N.H. Heinemann.

Edgington, C., Wilson, P. H., Sztajn, P., & Webb, J. (2016). Translating learning trajectories into useable tools for teachers. *Mathematics Teacher Educator*, *5*(1), 65-80.

Ivars, P., & Fernández, C. (2018). The role of writing narratives in developing pre-service primary teachers noticing. In G. Stylianides & K. Hino (Eds.), *Research Advances in the Mathematical Education of Pre-service Elementary Teachers. ICME-13 Monographs*. Springer: Cham.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Lee M.Y., & Choy B.H. (2017). Mathematical teacher noticing: The key to learning from lesson study. In E. Schack, M. Fisher, & J. Wilhelm (Eds.), *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks. Research in Mathematics Education* (pp. 121–140). Springer International Publishing.

Mason, J. (2011). Noticing: Roots and branches. In M. G. Sherin, V. R. Jacobs, & R. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 35–50). New York: Routledge.

Sánchez-Matamoros, G., Fernández, C., & Llinares, S. (2015). Developing pre-service teachers' noticing of students' understanding of the derivative concept. *International Journal of Science and Mathematics Education*, *13*(6), 1305–1329.

Seto, C., & Loh, M. Y. (2015). Promoting mathematics teacher noticing during mentoring conversations. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proc. 39th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 153–160). Hobart, Australia: PME.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 114–145.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the Hypothetical Learning Trajectory. *Mathematical Thinking and Learning*, *6*(2), 91–104.

Steffe, L., & Olive, J. (2009). *Children's fractional knowledge*. Springer Science & Business Media.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education*, *24*(2), 244–276.

Wilson, P. H., Mojica, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior*, *32*(2), 103–121.

Wilson, P. H., Sztajn, P., Edgington, C., Webb, J., & Myers, M. (2017). Changes in teachers' discourse about students in a professional development on learning trajectories. *American Educational Research Journal*, *54*(3), 568–604.

USING EQUATIONS TO DEVELOP A COHERENT APPROACH TO MULTIPLICATION AND MEASUREMENT

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We explicate connections between multiplication and measurement that hold promise for developing a more coherent approach to core topics in the K-12 mathematics curriculum. Within research on multiplication, there has been an ongoing conversation about the extent to which topics in the multiplicative conceptual field (Vergnaud, 1983, 1988) should or should not be unified under a single meaning for multiplication. Within research on measurement, specific types of quantities (e.g., length, area, volume, and angle measure) have often been treated as separate topics (e.g., Smith & Barrett, 2017). We start with notion of equal-sized units that are the basis for both multiplication and measurement and develop an approach for integrating these two core strands of school mathematics into a more coherent whole.

INTRODUCTION

Both the operation of multiplication as a model of problem situations and the measurement of quantities rely on equal-sized units. As a consequence, multiplication and measurement are intertwined throughout a central swathe of school mathematics from whole-number multiplication to linear equations and beyond. Although learners may experience the wide range of topics related to multiplication and measurement as initially disjoint, developing a coherent view of such topics is a desirable educational goal both because it can support and reinforce an interconnected knowledge base and because seeking and identifying common structure across diverse situations reflects a core value of the mathematics community.

We are by no means the first to point out connections between multiplication and measurement of quantities but, we will argue, the extant theoretical research on both topics falls short of the coherent perspective for which mathematics education should strive. In response, we examine how an equation of the form $N \cdot M = P$ can be interpreted in a consistent way across diverse situations that contain measured quantities. After identifying some key limits of current perspectives on connections between multiplication and measurement, we will explicate how we coordinate perspectives on measurement and equations to achieve greater consistency. Although our presentation is theoretical, we emphasize that it is informed by our experience helping future teachers achieve a coherent view of topics that span elementary, middle, and secondary grades.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 155-162). Umeå, Sweden: PME.

BACKGROUND

We begin by positioning our contribution with respect to strands of research on multiplication, measurement, and equations.

Meanings for Multiplication

One strand of research on multiplication has sought to identify psychological primitives that provide the conceptual basis for the arithmetic operation. As examples, researchers have proposed repeated addition (e.g., Fischbein Deri, Nello, & Marino, 1985), splitting (e.g., Confrey, 1994; Confrey & Smith, 1995), and units coordination (e.g., Steffe, 1988, 1994). These perspectives have not emphasized equations or addressed directly interpretations of the equal sign.

Other strand of research has employed at least two perspectives on multiplication to analyze reasoning with measured quantities (e.g., Schwartz, 1988; Thompson & Saldanha, 2003). In the first perspective, a multiplicative comparison is established between a measurement unit and some attribute that is segmented or partitioned by that unit. The magnitude of the attribute is so many times that of measurement unit (e.g., the length of a wooden plank is 4 times the length of a 1-meter plank). In the second perspective, a new quantity is formed through a multiplicative composition of two already established quantities. Schwartz characterized multiplication in such cases as a referent transforming operation, while Thompson and Saldanha discussed a new measurement unit formed by the product of the two initial units—for instance, a unit of area is formed by the product of two units of length. In our reading, these researchers do not attempt to reconcile the two perspectives on multiplication, one in which units are preserved and one in which units are transformed. From our point of view, this is an important limitation if coherence across contexts is a primary goal.

Still other perspectives on multiplication (e.g., Boulet, 1998; Davydov, 1992) have characterize connections between multiplication and measured quantities not in terms of transforming units but rather in terms of coordinating measurement with two different units. In particular, Davydov argued that multiplication situations are characterized by combining smaller into larger units and then coordinating measurement with the smaller and larger units. Thus, one could have the ultimate goal of measuring a given volume of liquid in terms of cups but first use gallons to obtain an intermediate measure which could then be converted into cups. The perspective we present builds most directly on that of Davydov's.

Diverse or Unified Interpretations of Multiplication?

Vergnaud's (1983, 1988) construct of the *multiplicative conceptual field* (MCF) highlighted the diverse range of topics and problem situations that are related to multiplication. Recognizing that a wide range of topics and problem situations in school mathematics are related to multiplication raises the following fundamental question: Can such a range be brought together under a single, consistent meaning for multiplication?

Researchers have taken different perspectives on the extent to which multiplication can (or should) be viewed in a unified or consistent way across situations (e.g., Anghileri, 1989; Boulet, 1998; Greer, 1992). Anghileri and Greer have emphasized distinctions among situations. To illustrate, Greer argued that equal-group and rate situations can be conceived in similar terms with clear asymmetry between the multiplier and multiplicand, but that Cartesian products and rectangular areas lack such asymmetry: One can interchange the role of the length and the width of a rectangle when computing its area. In response, Boulet extended Davydov's (1992) notion of combining smaller into larger units; argued for consistent application of the distinction between the multiplier and multiplicand across positive integers, negative integers, and positive rational numbers; and demonstrated how this distinction could be applied across the range of situations discussed by Greer. In our reading, Boulet identifies multiplication with iterating and division with partitioning. We agree with Boulet's goal of achieving a consistent interpretation for multiplication, but (as explained below) rely on identifying multiplication and division with questions asked about situations that contain equal-sized groups of units.

Diverse or Unified Approaches to Measurement?

Past research and curricular approaches to measurement have often treated different types of quantities (e.g., length, area, volume, and angle measure) as separate topics (e.g., Smith & Barrett, 2017). Smith and Barrett argued for a coherent approach to measurement of all quantities based on seven principles that include using a smaller unit to partition a larger quantity into equal-sized parts, using smaller units to exhaust (tile) a larger quantity, and subdividing or grouping units to create hierarchical structures. These researchers also echoed the perspective discussed above that new quantities can be created through multiplicative composition of already established quantities, but did not articulate a specific meaning for multiplication or consider the role that meanings for equations play in interpreting multiplication.

Meanings for the Equal Sign

Research on algebra (e.g., Stephens, Ellis, Blanton, & Brizuela, 2017) contains numerous reports that students often struggle with meanings for the equal sign. When using the operational meaning, students interpret the equal sign as an indication to compute an answer. When using the relational meaning, students interpret the equal sign as a statement of equivalence between two quantities or expressions (p. 389). From our perspective, the distinction between operational and relational interpretations of the equal sign does not address issues of measurement squarely.

MULTIPLICATION AS COORDINATED MEASUREMENT

Figure 1 shows our quantitative definition of multiplication based in measurement. It applies to situations in which there is a quantity (the product amount) that is simultaneously measured with two different measurement units. For this reason, we characterize our perspective as one of coordinated measurement. Here is the connection to Davydov's (1992) work discussed above. We make five initial points.

N	$\bullet$	M	$=$	P
How many base units make one group exactly?		How many groups make the product amount exactly?		How many base units make the product amount exactly?

Figure 1: A quantitative meaning for multiplication based in measurement

First, we use the terms "base units" and "groups" as generic terms, and each new problem situation requires identifying what is 1 base unit and what is 1 group. N, M, and P are numbers, each of which is the answer to a measurement question. P describes the measure of the product amount in terms of base units, and M describes the measure of the product amount in terms of groups. N describes how base units and groups, the two approaches to measuring the product amount, are related.

Second, whereas Davydov (1992) concentrated on whole numbers, we include fractions using a definition based in measurement. In that definition, the unit fraction $1/b$ is defined by partitioning a quantity formed by 1 part into b equal-sized parts. The measurement perspective comes in by asking how many of the original 1 part makes any one of the b equal-sized parts exactly. The fraction a/b is the quantity formed by any a copies of $1/b$. (see also National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Thompson & Saldanha, 2003). This meaning for fractions can be applied either at the level of base units when applied to N or to P or at the level of groups when applied to M.

Third, division is characterized as multiplication with an unknown factor (Beckmann & Izsák, 2015). What makes a situation a division situation, given the use of base units and groups to measure a product amount, is either a question about the measure of 1 group in terms of base units, a question about the value of N, or a question about the measure of the product amount in terms of groups, a question about the value of M. The former is often referred to as partitive or sharing division, and the latter is often referred to as quotitive or measurement division (e.g., Greer, 1992). Notice that, whereas sharing is often associated with partitive division and measuring is often associated with quotitive division, in our discussion measuring is associated with both types of division. The distinction between partitive and quotitive division lies in what is being measured—1 group or the product amount—and what is being used as a unit of measure—1 base unit or 1 group.

Fourth, the definition is stated as an equation, which requires interpreting the equal sign. The equal sign is not interpreted using the operational meaning. The discussion of division above underscores this point as what is unknown could be N or M. At the same time, the equal sign does not state a relationship among quantities directly. Rather, the equation states a relationship among numbers derived from measurement.

Fifth, in contrast to Greer's (1992) discussion of contexts in which multiplication is asymmetric and contexts in which the operation is symmetric, *N* and *M* cannot be interchanged. Each has a distinct role in measurement. Put another way, multiplication is commutative when calculating but not when modelling problem situations (e.g., 3 cookies on each of 7 plates is not the same as 7 cookies on each of 3 plates).

EXAMPLES

We present three examples to demonstrate how the perspective on multiplication summarized in Figure 1 addresses limits of past research identified above.

Example 1: Measuring length

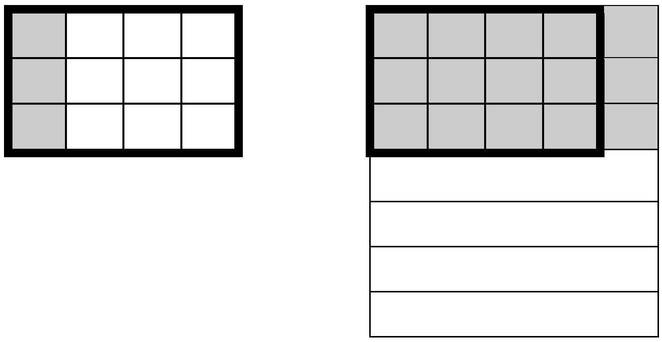
Consider once more the 4-meter plank of wood. One can view the plank as segmented into four separate lengths each of which is 1 meter, and one can make a multiplicative comparison in which the plank is 4 times as long as the 1-meter unit. Following the meaning of multiplication in Figure 1, this situation can be captured by the following equation: $1 \cdot 4 = 4$. Here, because the value of *N* is 1, there is 1 base unit (1 meter in this case) in 1 group. The answer to the measurement question how many base units make the product amount exactly is 4 because four 1-meter units make the length of the plank. The answer to the measurement question how many groups make the product amount exactly is also 4 because 4 groups make the length of the plank. There is nothing special about using a whole number in this example. If the plank were 4/5 meters, the equation $1 \cdot 4/5 = 4/5$ could be interpreted similarly using the measurement sense of fractions discussed above. Furthermore, there is nothing special about lengths, our meaning of multiplication applies to any multiplicative comparison between the magnitude of an attribute and a specified unit.

Example 2: Measuring rectangular area

Now consider the area of a 3 meter by 4 meter rectangle. One can interpret the equation $3 \cdot 4 = 12$ through the meaning of multiplication shown in Figure 1 as follows. In Figure 2a, 1 base unit is one mini-square, 1 group is one column (shaded), and the product amount is outlined in bold. Then, the value of *N* is 3, because 3 base units make 1 group exactly, and the value of *M* is 4, because 4 groups make the product amount exactly.

Notice that *N* and *M* do *not* refer to units of length and that, in contrast to prior accounts discussed above, there is no transformation of referents and no multiplicative composition of two units of length to create a new unit of area. The reason that the familiar *length* • *width* = *length* formula gives correct answers is that measures of lengths and widths, in this case in terms of 1-meter units, have the same values as numbers of base units in 1 group and numbers of groups in the product amount. Once again, there is nothing special about using whole numbers. If the dimensions of the rectangle were 3/7 meters and 4/5 meters, the equation $3/7 \cdot 4/5 = 12/35$ could be interpreted similarly using the measurement sense of fractions discussed above. In Figure 2b, 1 base unit is the large square, one group is three rows (shaded) where each row is 1/7 of the base unit, and the product amount is outlined in bold where each column is 1/5 of the group. Furthermore, there is nothing special about squares as base units for area: One could take a base unit to be a rectangle with the dimensions 1 centimeter by 1 meter.

Notice also that we have used the same meaning of multiplication to measure both lengths in example 1 and rectangular areas in example 2. This is in contrast to past research on reasoning with measured quantities in which, as discussed above, at least two distinct perspectives on the meaning of multiplication have been used. Thus, taken together, these two examples are one demonstration of how our perspective affords a more coherent approach to multiplication and measurement of quantities.



(a) (b)

Figure 2: (a) Measuring area with whole numbers. (b) Measuring area with fractions.

Example 3: Measuring angles

We have chosen angle measure as our final example to demonstrate how our perspective on multiplication can handle topics typical of secondary school, at least in the United States. In this case, we want to see how the formula for arclength, $r \bullet \theta = s$, is a particular case of our $N \bullet M = P$ equation. Figure 3 shows two concentric circles. Circle 1 has radius 1 centimeter, and Circle 2 has radius r centimeters. In this case, 1 base unit is 1 centimeter, one group is the radius of Circle 2, r plays the role of N because it gives how many centimeters make the radius of Circle 2 exactly, and θ plays the role of M because it gives how many radii (i.e., groups) make the arclength (i.e., the product amount) exactly. If one imagines that Circle 1 remains fixed and that Circle 2 dilates, either increasing or decreasing in size, the value of r will vary but the number of radii that make the product amount exactly, θ, will remain fixed. In this case, our $N \bullet M = P$ equation expresses a constraint on the measures of the radius and arclength on Circle 2 as they covary.

CONCLUSION

Our conjecture, to be pursued in future research, is that the coordination of multiplication and measurement we describe can be extended across most if not all of the diverse topics and situations that Vergnaud (1983, 1988) included in the MCF. If our conjecture is borne out, then the field will have a theoretical lens that unifies a large swathe of important mathematics to a greater extent than has been accomplished currently. This would then make possible approaches to mathematics education in which an explicit, quantitative meaning of multiplication is applied consistently across topics and across grades. A main implication for instruction is designing experiences in which students (and teachers) are supported as they seek affordances in problem situations for measuring in base units and in groups and for interpreting equations in terms of values derived from such measurement. Exactly how such experiences might be effectively sequenced is an open question and one which are currently investigating with future mathematics teachers.

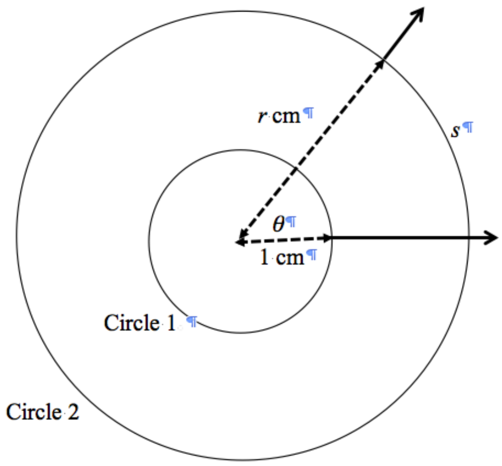


Figure 3: Measuring angles in radians.

Acknowledgements

This research was supported by the National Science Foundation under Grant No. DRL-1420307. The opinions expressed are those of the authors and do not necessarily reflect the views of the NSF.

References

Anghileri, J. 1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics, 20*(4), 367-385. doi: 10.1007/BF00315607.

Beckmann, S., & Izsák, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education, 46*(1), 17–38.

Boulet, G. (1998). On the essence of multiplication. *For the Learning of Mathematics, 18*(3), 12–19.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291-330). Albany: State University of New York Press.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Davydov, V. V. (1992). The psychological analysis of multiplication procedures. *Focus on Learning Problems in Mathematics, 14*(1), 3–67.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3-17.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning.* (pp. 276-295). New York: MacMillan.

National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *The common core state standards for mathematics*. Washington, D.C.: Author.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence Erlbaum.

Smith, J. & Barrett, J. (2017). Learning and teaching measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 355–385). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 119-140). Reston, VA: Lawrence Erlbaum Associates & National Council of Teachers of Mathematics.

Steffe, L. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany: State University of New York Press.

Stephens, A, Ellis, A., Blanton, M., & Brizuela, B. (2017). Algebraic thinking in the middle grades. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 386–420). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Shifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 127-174). New York: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (ST ed., pp. 141-161). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Erlbaum.

A WRITTEN, LARGE-SCALE ASSESSMENT MEASURING GRADATIONS IN STUDENTS' MULTIPLICATIVE REASONING

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We examine a written, large-scale assessment that assessors can use to infer and measure gradations in students' scheme for whole number multiplicative reasoning. To design such an instrument we drew on Tzur's notion of fine grain assessment, which is used to distinguish two stages in the construction of a scheme: participatory and anticipatory. We briefly present the assessment items, the validation process, and reliability statistics–Cronbach's alpha, Rasch modeling, and student response patterns from students (N=492) in grades 3 and 4 (~ages 8-10), including distinctions in item difficulty levels. We discuss implications for large-scale assessment design and implementation.

In this study, we extend two recent studies from our large research project investigating elementary students' development of multiplicative reasoning (Hodkowski, Hornbein, Gardner, Jorgensen, Johnson, & Tzur, 2016)[1]. We report on a written assessment designed and implemented to infer into students' multiplicative reasoning. Such an assessment faces the challenge of finding an adequate alternative to the labor-intensive method of interviewing students. Whereas task-based, cognitive interviews afford inferring students' reasoning from their interactions with assessors, a large-scale assessment must allow valid and reliable inferences based solely on student responses. To face this challenge, we built on Norton and Wilkins' (2009) use of quantitative methods to measure students' reasoning based on models researchers obtained through interviews with small numbers of students. We expand the work of Norton and Wilkins by focusing on conceptual gradations that Tzur and colleagues (Tzur & Simon, 2004; Tzur, 2007) have postulated within students' reasoning—the *participatory and anticipatory* stages.

Researchers across the world have been studying students' challenges with whole numbers multiplicative reasoning (Lamon, 2007). In our work, we stress that such reasoning involves more than knowing multiplication facts and/or developing procedural skills. It includes students' meanings for multiplication (Steffe & Cobb, 1998), their insights into multiplicative relationships between numbers (Bakker, van den Heuvel-Panhuizen, & Robitzch, 2015), and their coordination of different kinds of units to form new units (Tzur, Johnson, McClintock, Xin, Si, Woodward, Hord, & Jin, 2013). In this study, we address the following problem: How can a written, large-scale assessment be used, in place of interviews, to infer gradations in students' scheme for whole number multiplicative reasoning?

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 163-170). Umeå, Sweden: PME.

THEORETICAL AND CONCEPTUAL FRAMEWORK

Assessing assimilation (schemes)

As humans, we cannot directly observe the reasoning of others. Through interviewing or written methods, researchers can make inferences about others' reasoning. We draw on Piaget's (1985) core notion of *assimilation*—a cognitive intermediary between observable "stimuli" and "responses"—as a lens to make such inferences. Von Glasersfeld (1995) explained that assimilation, and reasoning, are made possible by a three-part cognitive building-block—a *scheme*. Schemes comprise: (1) a recognition template (situation) that triggers one's goal; (2) an activity to accomplish that goal; and (3) an effect that one mentally anticipates, or notices retroactively, to ensue from that goal-directed activity. We designed our assessment to measure gradations in students' mental use of schemes for multiplicative reasoning.

Schemes for whole number multiplicative reasoning

Researchers explicitly distinguished multiplicative reasoning from successfully determining answers to multiplication problems (Bakker et al., 2015; Tzur et al., 2013). We infer that students engaging in multiplicative reasoning can use schemes to keep track of and coordinate different kinds of units. For example, consider this task: "Julia has 6 towers, each made from 3 stacking cubes. How many cubes did Julia use to make the towers?" A student may draw all cubes and correctly count them one-by-one. In contrast, a student engaging in multiplicative reasoning with an assimilatory scheme would coordinate three kinds of units: composite units (e.g., towers), the magnitude of each unit (e.g., cubes per tower), and units of 1 (e.g., total of individual cubes).

Tzur et al. (2013) identified six schemes for multiplicative reasoning. Our study focuses on assessing the first one, termed *multiplicative double counting* (mDC), which marks the shift from additive to multiplicative reasoning. A student having the mDC scheme could recognize a situation as consisting of two different kinds of units, set the goal to find the total of 1s in them, trigger the activity of simultaneously distributing and counting (keeping track of) accrual of 1s and of composite units (e.g., 1 tower is 3, 2 towers are 6, 3 towers are 9, … 6 towers are 18), and anticipate a new kind of unit as a result of her activity.

Gradations in students' schemes

When assessing students' reasoning, we do not mean that having a scheme is like flipping an "on-off" light switch. Rather, we distinguish gradations in schemes through Simon and Tzur's (2004) constructs of participatory and anticipatory stages, which differentiate a student's ability to bring forth a scheme. In the anticipatory stage, a student can independently, and spontaneously, do so. In the participatory stage, a student needs prompting to bring forth a goal-directed activity and its effect of an emerging scheme. We acknowledge that prompting can take different forms. In this paper, we focus on prompting involving additional supports, provided to a student through her sensory perception. For example, in the task involving Julia and the 6 towers, a hint could be a picture showing one completed tower and just a single cube for each of the remaining towers. If a student is at a participatory stage, such a hint may bring forth her activity of counting 1s and composite units, and thus may enable her to engage in multiplicative reasoning and to provide a correct response.

Fine grain assessment

To measure participatory-anticipatory gradations in students' mDC scheme, we adapted Tzur's (2007) fine grain assessment. In fine grain assessment, assessors begin with tasks that include no hints, then move to subsequent tasks including increasing levels of hints. We stress that the purpose of hints is *not* to funnel students to a certain solution method and/or correct answer. Rather, the purpose of hints is to provide students with opportunities to bring forth existing schemes. Including a hint-free task prior to other tasks allows assessors to infer the stage of a student's assimilatory scheme based on her or his solutions to tasks—first without and then with hints.

METHODS

We report on our methods to develop and implement a written assessment that targets gradations in a foundational scheme that indicates students' emerging multiplicative reasoning: multiplicative double counting (mDC). To design the assessment, we drew on the expertise of our large, diverse project team, which includes mathematics educators, a mathematician, research methodologists, and language experts.

The mDC assessment: Problems, items, and sub-items

The mDC assessment we developed contains five word problems. The first problem served as a screener problem (1-digit addition, 8+7), to foster initial success. The next four problems, together, allow inferring the stage of a student's mDC scheme (see multiplicative reasoning problems #2-5 below). To assess participatory-anticipatory gradations, each problem comprises at least three items. We designed the first item to be "hint-free." The subsequent items included increasing levels of hints. With each increasing level of hint, we intended to provide students opportunities to bring forth their mDC scheme. Therefore, hints provided increasingly specific information about the different kinds of units in the situation. Furthermore, to assess students' text comprehension, in each problem we included sub-items for which students filled in blanks with information given in a problem statement. For example, in Problem #3, students filled in this blank: "Alex put __ towers in the box." (see Figure 1).

In Problem #2, we focused on students' iteration of a composite unit (e.g., a tower of 3 cubes) to determine if it could constitute a larger composite unit (e.g., a tower of 24 cubes). In Problem #3, we intended for students to distribute items of one composite unit (3 cubes per tower) over another unit (6 towers) to find the total number of 1s in the compilation of composite units (total of 18 cubes). In Problem #4, we intended students to keep track of composite units (4 teams of 5 players each). We asked them to determine the correctness of a hypothetical student's (Joy) statement that, through "skip-counting: by 5, she found there were 35 players in all. In Problem #5, given a total number of items (28 cookies), we intended for students to segment this total by iterating a given composite unit (4 cookies per bag) to determine the total number of composite units (bags) that constitute the total.

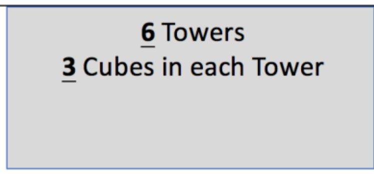
In this paper, we focus on Problem #3. Figure 1 shows the first, "hint-free" item.

The picture to the right shows a box.

Alex put ***6*** towers in the box.

Alex made each tower with ***3*** cubes.

The numbers on the picture show this.



A: Alex put _____ **towers** in the box.

B: Alex made **each tower** with _____ **cubes**.

C: How many **cubes in all** did Alex use to make ***6*** towers? (fill in blank): _______

Figure 1: mDC Assessment Problem #3, Item 1; Hint-free.

The second and third items of Problem #3 followed the same format as the first item. These items included increasing levels of hints. Figure 3 shows level 1 and level 2 hints for items 2 and 3, respectively. The level 1 hint included additional diagrammatic information about the activity of iterating a unit of "tower." The level 2 hint included additional diagrammatic information about the units composing the towers to be iterated, the "cubes per tower."

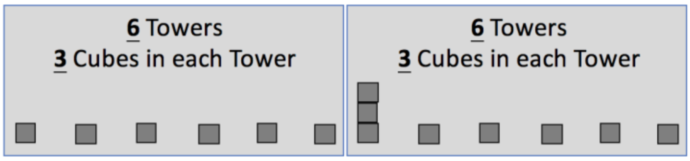


Figure 2: Problem #3, Items 2 and 3: Hint level 1 at left; Hint level 2 at right.

Validity and Reliability

We addressed construct validity through a five-phase process. Initially, Tzur created an expert draft for each problem. Second, he shared the draft with a content expert who gave feedback, with changes. Third, the project team worked on that version, leading to more revisions of language and diagrams. Fourth, this version went through an expert panel review. We gave this version to three experts in the field, who evaluated the problems and items, responding by: "keep," "change as follows," or "omit." The experts suggested a few revisions, but not any omissions. Fifth, Tzur conducted individual, cognitive interviews with five children to check the entire assessment. Issues arising from those interviews led our team to make further, finer revisions.

We addressed construct validity along three lenses: language, potential gender or cultural biases, and mathematical consistency with the multiplicative reasoning we intended to measure (Hodkowski et al., 2016). The mathematics educators and mathematician worked closely to address mathematical consistency. We drew on the language and literacy experts on our project team to develop word problem statements appropriate for students learning English as an additional language. In addition, we included situations familiar to the student population with whom we worked.

Tzur conducted cognitive interviews with 26 fourth-graders to determine the extent to which their responses to items on the mDC assessment and additional tasks he posed correlate with his inference into their mDC scheme. To determine if the mDC assessment could actually serve as a proxy for students' reasoning, as opposed to just the child's ability to correctly solve each item, we used the Kendall's Tau-b statistics to calculate agreement between Tzur's inferences and the score obtained from the written assessment items (Ktb=0.883, 2-tailed $p<.0005$). Thus, we claim the data of students' performance on items on the mDC items indicate students' engagement in multiplicative reasoning (mDC scheme).

Student population, student numbers, and assessment administration dates

Students participating in our study were from three different elementary schools in one small and one large public school district. Both districts were in the metropolitan area of a large US city. About 85% of the students in our study identified as students of color, and 70% were learning English as an additional language.

We report results from three administrations of the mDC assessment to students in 3rd and 4th grades: Spring 2016, Fall 2016, and Spring 2017. We report results from a total of 492 student assessments, produced by 404 unique students (some assessed twice or three times). Table 1 disaggregates the assessment totals by student grade and administration date. We analyzed data from all 492 available assessments, because they reflect students' assimilation (or lack thereof) of the problems into their mDC scheme in far-apart administrations. This larger number allowed us to further investigate gradations in students' reasoning.

Grade	Spring 16	Fall 16	Spring 17	Total
3	81	146	117	344
4	26	83	39	148
Total	107	229	156	492

Table 1: Numbers of students taking the mDC assessment by grade and date.

Like researchers across the world (e.g., Bakker et al., 2015), we experienced challenges implementing this large-scale study with students in schools. One challenge included obtaining student and parent consent, which impacted our data analyses on sets of disaggregated data. To address challenges, we worked with teachers and school personnel to determine protocols and times beneficial to both parties.

Data entry

Six graduate research assistants (GRA) were trained to enter the student responses to the mDC assessments. To increase reliability, one GRA read the student's responses out loud and the other entered those into a spreadsheet as is. The first GRA could see and was asked to verify that responses entered correctly for every student. We coded no response as "9999" and an unreadable or incoherent answer as "5555."

Analysis: Cronbach's alpha, Rasch modeling, Student response patterns

To calculate overall item consistency of all four multiplicative reasoning problems (#2-5), we used the Cronbach's Alpha statistics for all items and sub-items. Moreover, we employed the principal component method of exploratory factor analysis to confirm that all items loaded onto a single principal component (here, the construct of mDC scheme), and that no additional factors could be extracted.

We conducted Rasch analysis to determine item difficulties and person measures. We tested if students who did not bring forth the mDC scheme on a hint-free item could do so on items containing hints. We used Rasch modeling with hint-free items and with items containing any form of hint (both level-1 and level-2 hints). Students bringing forth the mDC scheme on hint-free items would have an anticipatory stage of the scheme. Students having the participatory stage of the mDC scheme would bring forth the scheme after receiving the diagrammatic hints. We hypothesized that Rasch analysis would indicate items containing hints to be consistently less difficult than hint-free items. Next, we examined a Wright map, which organizes both persons and items by logits ranging from -3 to +3. Ideally, in a Wright map, the distribution of items should show a wide range of item difficulties, with more items in the middle than at the extremes, and each item on a unique difficulty level.

Besides Rasch modeling, we also examined students' response patterns for hint-free items and items containing hints. We grouped the data into four response patterns: (1) Hint-free Correct, Hint Correct; (2) Hint-free Correct, Hint Incorrect; (3) Hint-free Incorrect, Hint Correct; (4) Hint-free Incorrect, Hint Incorrect. We coded "correct" for items containing hints if students provided a correct response for any level of hint.

RESULTS

mDC assessment consistency and factor analysis

Chronbach's alpha for the 8 items in the mDC assessment (263 cases = 53.5% of all 492), 4 hint-free items and 4 items with hint, is 0.907. This value reflects excellent inter-item consistency. Rasch item analysis indicated a very high consistency (0.98).

Rasch modeling and Wright map

Our Wright map showed item difficulties ranging from -2.3 to 1.3 logit scores. In a few cases, two or three items had the same logit score. For three of the four problems, analysis of our Wright map showed that the hint-free items were more difficult than the items containing hints. The most difficult item was the hint-free sub-item C on Problem #3 (logit score = 1.3); second to it was the sub-item C on Problem #3 that contained a hint (logit score = 1.15). Although these logit scores were fairly close, our analysis confirms the hint-free item to be more difficult. Furthermore, the Rasch item reliability to be 0.98 indicates that items have a hierarchy of difficulty. We found more distinct results for Problems 3, 4, and 5.

Students' response patterns

For each of Problems 2-5, we compared the four groups of student response patterns in respect to students' overall performance on the remaining assessment. Table 2 shows students' response patterns to sub-item C of Problem #3 (See Figure 1). As expected, for each item, students' average scores in the Correct-Correct group (N=169) were highest, and students' average scores in the Incorrect-Incorrect group (N=264) were lowest. For the other two groups, Correct-Incorrect (N=20) and Incorrect-Correct (N=39), students' average scores were between the two extremes.

	Hint-Free Correct	Hint-Free Incorrect	Total
Hint Correct	169	39	208
Hint Incorrect	20	264	284
Total	189	303	492

Table 2: Students' response pattern to Situation 2, sub-item C.

These results suggest we can measure gradations in students' multiplicative reasoning. Yet, gradations were not entirely clear-cut. Some students responded correctly to a hint-free item, then incorrectly to an item containing a hint (Correct-Incorrect, N=20). This seems to run counter to our conjecture that hints could provide students opportunities to bring forth their schemes. Despite this seeming discrepancy, the Correct-Incorrect group (N=20) scored lower on the overall mDC assessment than the Incorrect-Correct group (N=39), who responded incorrectly to the hint-free item. We infer that other factors, such as guessing, accounted for this result.

DISCUSSION

Based on our results, assessors can use the mDC assessment to measure gradations in students' mDC scheme for whole number multiplicative reasoning. Gradations include two stages, anticipatory and participatory, indicated by whether students demonstrated evidence of bringing forth a scheme before or after being given a hint.

To date, researchers have used small scale, labor intensive interview methods to distinguish students' anticipatory and participatory stages of conceptual development (e.g., Simon et al., 2016; Simon & Tzur, 2004). Our mDC assessment is a step toward measuring gradations in students' reasoning on a large scale. Although researchers have identified finer grained distinctions at the participatory and anticipatory stages (e.g., Simon et al., 2016), currently our assessment is only sensitive enough for researchers to use to measure distinctions between anticipatory and participatory stages. We would need further refinement to make more nuanced distinctions.

Distinguishing between anticipatory and participatory stages is useful for explaining a challenge common to teachers, termed "the next day phenomenon." For example, a student at a participatory stage may bring forth her mDC scheme to determine a total number of cubes, given 6 towers with 3 cubes in each. Yet, the same student may not bring forth her mDC scheme on a seemingly similar task. Students' participatory stage is a crucial and vulnerable stage in learning, and can explain, in part, why students may not yet be able to engage in multiplicative reasoning without additional support.

[1]Acknowledgment

This study was supported by the US National Science Foundation (grant 1503206). The opinions expressed do not necessarily reflect the views of the Foundation.

References

Bakker, M., van den Heuvel-Panhuizen, M., & Robitzsch, A. (2015). Effects of playing mathematics computer games on primary school students' multiplicative reasoning ability. *Contemporary Educational Psychology*, *40*, 55–71.

Hodkowski , N. M., Hornbein, P., Gardner, A., Johnson, H. L., Jorgensen, C., & Tzur, R. 2016, November). Designing a stage-sensitive written assessment of elementary students' scheme for multiplicative reasoning. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1581-1587). Tucson, AZ: The University of Arizona.

Lamon, S. J. (2007). Rational numbers and proportional reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 629-667). Charlotte, NC: Information Age Publishing.

Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *The Journal of Mathematical Behavior*, *28*(2), 150–161.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development*. Chicago: University of Chicago Press.

Simon, M. A., Placa, N., & Avitzur, A. (2016). Participatory and anticipatory stages of mathematical concept learning: Further empirical and theoretical development. *Journal for Research in Mathematics Education*, *47*(1), 63–93.

Steffe, L. P., & Cobb, P. (1998). Multiplicative and Divisional Schemes. *Focus on Learning Problems in Mathematics*, *20*(1), 45–61.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics*, *66*(3), 273–291.

Tzur, R., Johnson, H. L., McClintock, E., Xin, Y. P., Si, L., Woodward, J., Hord, C., & Jin, X. (2013). Distinguishing schemes and tasks in children's development of multiplicative reasoning. *PNA*, *7*(3), 85-101.

Tzur, R., & Simon, M. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Mathematical Educ. in Science and Technology*, *2*(2), 287–304.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C.: Falmer.

THEORIES ABOUT MATHEMATICAL CREATITIVY IN CONTEMPORARY RESEARCH: A LITERATURE REVIEW

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The paper at hand presents a systematic analysis of theoretical backgrounds in articles about mathematical creativity over the period of 2007 – 20016. Due to the multifaceted concept of creativity, various keywords were used for the literature study. Those keywords were identified in a search in relevant literature and ten years of PME proceedings. The coding of the articles as well as the inductively created category system is presented. As a result, we see that most authors refer to a multitude of descriptions to examine creativity. With this approach we were able to shed light to the characteristics of conceptualizations of creativity which are discussed.

INTRODUCTION

Creativity – also among students – is increasingly being explored – since it is also seen as a central component of modern technology in society (Leikin & Pitta-Pantazi, 2013). There are very different views about what creativity is and, accordingly, different approaches how creativity is examined – and different trends become apparent. We therefore see the need to get a clearer and systematic picture of the theoretical basis on which current research in this field is based. For this purpose, we conducted a configurative literature study (similar to Nilsson, Schindler, & Bakker, 2018) (see Gough, Oliver, & Thomas, 2013), where – in a first step – we systematically searched for adequate keywords in the proceedings of ten years of PME and – in a second step – developed a categorization system for the analysis of the articles. One result is that seven other words for the term creativity are used synonymously. In addition, the analysis shows that only one third of the considered articles define creativity, and the majority of articles indicate many different descriptions of creativity.

BACKGROUND

Mathematical creativity as a research topic is gaining increasing interest in recent years (cf. Singer, Sheffield, & Leikin, 2017). For example, a PsychINFO® keyword search for "math" combined with

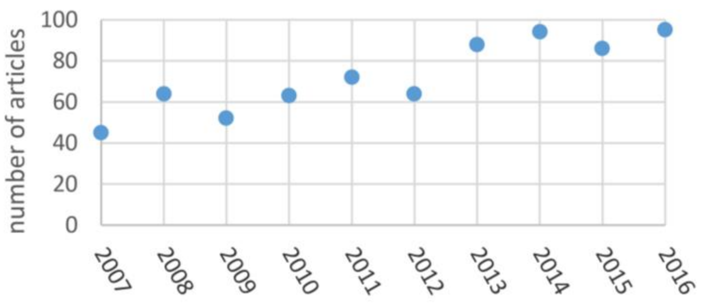

Figure 1: Number of articles found in the database PsycINFO® with the keywords *math** and creativity-related keywords. In total, 723 articles were found.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 171-178). Umeå, Sweden: PME.

“creativ*”, “innovat*” and similar expressions (see below for more details) reveals a doubling of the number of articles related to the topic in the last 10 years (see Figure 1).

For this interest in creativity and its importance, there are many reasons. To name a few, creativity is considered to be important in problem solving, making innovations, and being a responsible citizen (Kim, Roh, & Cho, 2016). Barak summarizes:

> “It is evident that creative thinking skills, openness to change, flexibility, and the ability to cope with challenging tasks are essential for integration in today’s society and workplace, whereas specific skills and knowledge are rapidly becoming obsolete and new fields are emerging every few years.” (Barak, 2009, p. 345)

However, there is no single, universally accepted definition for creativity (Treffinger et al., 2002). The definitions that are used are often vague like “[creativity can be defined] as the process of producing something that is both original and worthwhile” (Sternberg & Sternberg, 2011, p. 479). For research, however, it is important to have well-defined terms and concepts (Rhodes, 1961). Therefore, vague concepts need to be discussed and sharpened. This is especially true for research on subject-specific creativity, and it is appropriate to conduct a thorough review to record which definitions and theories are used in research on mathematical creativity.

In this article, the preparation and implementation of a configurative literature study is presented. The first part deals with the selection of keywords to conduct the review. The second part presents the results of a descriptive analysis of our review.

RESEARCH QUESTIONS

It is of interest to find out how elements such as theories, models or – more generally– conceptualizations on the subject of mathematical creativity are used in contemporary research. This raises the following research questions:

1. Which words are used synonymously for creativity and thus result in a keyword for a configurative literature search?
2. How is creativity in contemporary research conceptualized?

METHODOLOGY

In order to analyze the theories, models, and other elements which are mentioned in the theoretical parts of papers about mathematical creativity, we conducted a configurative literature study, similar to Nilsson et al. (2018) who based their research on Gough et al. (2013). For our purpose, we adopted this approach because we not only wanted to extract theories but also include smaller remarks, which do not have the claim of a theory.

Searching for keywords and articles

There is no accepted definition of creativity and there are many different conceptualizations of the term (Treffinger, Young, Selby, & Shepardson, 2002). Therefore, it is important to be aware that there might be other signal words than “*creativ**”, which describe/conceptualize creativity. To create a list of appropriate search terms aiming at finding articles for review, a systematic screening of conference and handbook articles was conducted. For this screening, all PME proceedings from 2007 to 2016 were included as well as several handbooks like E*ncyclopedia of creativity, invention, innovation and entrepreneurship* edited by Carayannis (2013), Sternberg's (1999) *Handbook of creativity*, or handbooks that are specific for mathematics education, e.g. Gutiérrez, Leder, and Boero's (2016) *The second handbook of research on the psychology of mathematics education*. In these proceedings and handbooks, all articles with “creativ*” in their title were browsed for their keywords to find expressions that are used synonymously for creativity. For our list (see results Section), we omitted terms that describe components of creativity; for example. Lerman (2014) described mathematical creativity as a combination of fluency, flexibility, originality, and elaboration. These words were not included in the list of keywords, because these words are too everyday-linguistic and not do not have sufficient specificity.

To search for articles, PsycINFO® has been chosen as a database which is one of the most frequently used databases for behavioral and science research (American Psychological Association, 2017). As this database covers many different journals from different fields, we combined our keywords with *math**. As another restriction, only those articles are included which are published in a journal listed in the Web of Science (WoS) (with a focus on “education & educational research”, “education, scientific disciplines”, and “education, special”). This decision was made as an objective criterion to ensure a certain quality of the articles. However, this prerequisite to use only journals listed in the WoS leads to the exclusion of some journals that are relevant to mathematics education, like *ZDM – Mathematics Education* (see Discussion).

We have decided to search within the last ten years for appropriate articles. To make sure not to miss any articles because of infrequently updated databases, we chose the ten-year period from 2007 to 2016.

Screening the articles: focus on titles, abstracts, and keywords (criteria for inclusion of articles)

Only articles with creativity as a central topic (compared to, e.g., “creative methods to draw graphs”) should be included in the review. Therefore, titles, abstracts, and keywords were scanned. Articles that could not be clearly included or excluded were discussed in an expert discussion and included in the case of doubt.

Coding the Articles: Analysis of the theoretical parts

For the remaining articles, we focused on theoretical parts similar to Nilsson et al. (2018). If no theoretical part was labelled in an article, everything up to the research question(s) and methods was analyzed. This step is the last to exclude articles, which did not focus on creativity.

To compare the conceptualizations of creativity and the ways in which theories or other concepts on creativity are used in the selected articles, inductive categories were developed through a qualitative content analysis (see Mayring, 2015, for details). For better readability, we call a phrase with a reference to literature a "*statement*" which built the analysis unit for our research. In consequence, if there was a phrase which did not have any reference to other literature, it was not referred to be a statement. We clustered the statements from each article and then built categories according to the emerging clusters.

RESULTS

Extraction of keywords and articles

The extraction of keywords that are used synonymously to creativity was done for two different data types: more than 10 different handbooks from the fields of creativity and mathematics education and the PME proceedings from 2007 to 2016.

For the handbooks, we were able to extract a list of four central keywords (Table 1; first column). When searching for keywords in the PME proceedings, we found eight keywords (Table 1; second column) which are representative in several contributions.

We see that there is an overlap in both of the data types and that the list extracted from the handbooks is completely included in the list emerged from the PME proceedings. The latter seem to give a more varied picture of possible synonyms of or concepts related to creativity. For the search in PsycINFO®, all eight keywords are used.

Figure 2 summarizes the steps within the search procedure and the number of articles that were found and that remained after each step.

Keywords from handbooks	Keywords from PME
Creativ*	Creativ*
Innovat*	Innovat*
Invent*	Invent*
Divergent think*	Divergent think*
	Illuminat*
	Bisociat*
	Overcom*fixation
	Aha*

Table 1: Keywords extracted from handbooks and from pro PME proceedings 2007 – 2016.

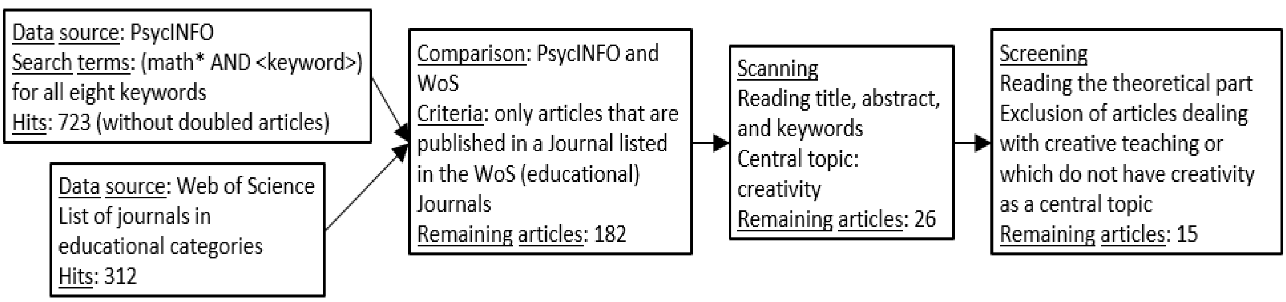

Figure 1: Schematic search procedure and inclusion criteria.

Initially, 723 articles that fit the search terms were found in the database. After the alignment with the list of journals in the WoS, 182 articles were selected. The titles, abstracts, and keywords of these articles were scanned, resulting in a selection of 26 articles of which the theoretical part was read (see methodology section). Of those 26 articles, however, only 15 articles were on topic and reviewed. Thus, in total, 8 % (15 of 182) of the articles using keywords from our list and being published in journals that are listed in the WoS actually dealt with research on creativity and were included in the further analysis. Table 2 shows the numbers of articles sorted by the journals in which they have been published. *Thinking Skills and Creativity* is most often represented – half of the articles were published here.

Journal	# articles
Thinking Skills and Creativity	7
Asia Pacific Education Review	2
Gifted Child Quarterly	2
Educational Psychology	1
High Ability Studies	1
Innovations in Education and Teaching International	1
Technology, Pedagogy and Education	1
Total	15

Table 2: The number of articles that were finally included into the analysis and the journals in which they have been published.

Coding the articles: Analysis of the theoretical parts

Seven different categories of references arose inductively from the statements in the articles: (1) *Definition*: Statements, which are clearly labeled with an expression like "defined as". (2) *Components*: Statements, which provide a closed list of properties to describe creativity. (3) *Description*: Statements, which describe creativity but do not refer to a closed list (as components). (4) *Development*: Statements, which hint either at special programs or trainings to foster creativity; or statements, which describe developments of creativity in e.g. students. (5) *Integration*: Statements, which show that the mentioned aspect is seen as an aspect of a bigger construct (e.g., giftedness). (6) *Relation*: Statements, which show a link to another construct and are not an integration (e.g., achievement). (7) *Assessment*: Statements, which deal with the assessment of creativity. Table 3 shows the numbers of articles, which include at least one statement indicating each category.

Categories	# articles
Definition	4
Components	4
Description	14
Development	9
Integration	6
Relation	14
Assessment	7
Total	58

Table 3: Number of articles, with statements in the listed categories.

Definition. It is striking that in only four articles, statements referring to *definition* were found. For example, Daugherty and White (2008) refer to Torrance and write: "Torrance (1965, 1988) defined creativity as sensing gaps in information, formulating solutions that complete the information, testing these solutions, and communicating the results" (p. 31). Ayas and Sak (2014) commit the statement: "creativity usually is defined as the ability to generate ideas or products that are novel and useful (Boden, 2004; Cropley, 1999; Mayer, 1999; Piffer, 2012; Plucker, Beghetto, & Dow, 2004; Sak, 2004; Sternberg & Lubart,1995)" (p. 195) and refer thereby to more authors. We see – also with the inclusion of the other two statements which refer to the category *definition* – that the authors show different emphasis: The spectrum ranges from a feeling (Daugherty & White, 2008) to specific abilities (Ayas & Sak, 2014) to the properties of products (Ayas & Sak, 2014; Barak, 2009; Kim et al., 2016).

When analyzing categories *description* and *components,* we see (Table 4) that the majority of the articles presents at least four statements assigning to these categories.

To get a more detailed insight, we will now focus on one particular article which presents a broad variety of statements: Ayas and Sak's (2014) "*Objective measure of scientific creativity*". In addition to the above stated *definition*, the authors also provide statements assigned to the categories *components* or *description*. With the following quote Ayas and Sak *compose* scientific creativity as a process of three stages "These three processes [referring to Scientific Discovery as Dual Search; SDDS] guide the entire process of scientific creativity from formulation of hypotheses, through experimental evaluations to decisions to accept or reject hypotheses" (ibid., p. 197). Additionally, the authors *describe* a variety of aspects of creativity and cover different scopes of application. They refer, for example, to the domain-specificity of creativity: "The evidence for domain specificity of creativity is found both in broadly defined cognitive domains (e.g., mathematical, linguistic, and musical) and in narrowly defined tasks or content domains (e.g., poetry writing, story writing, and collage making) (Baer, 1998)" (ibid., p. 196). In other parts, further statements are presented, partly with contrary conclusions. The focus in the authors' study is a computer based Assessment of Creativity, which is why many *assessment* statements are made. In total, Ayas and Sak (2014) cover all categories.

Overall, this shows which categories are covered in a theoretical part of an article and whether assumptions are based on definitions or rather on descriptions. It is also possible to reconstruct the extent to which research is conducted either within creativity or whether the focus is on linking to other constructs, such as SDDS and computer-based assessment and how these elements are characterized.

DISCUSSION AND OUTLOOK

The aim of this article was to analyze how creativity is described and conceptualized in contemporary research. A two-step procedure was conducted: In the first step, the review was preceded by systematically searching for synonyms of creativity. This step was necessary because there is no uniform definition for the subject area and, therefore, a large number of views and descriptions exist in parallel. We found eight keywords which were used for the research: *creativ**, *divergent think**, *innovat**, *illuminat**, *invent**, *aha**, *bisociat**, and *overcom** *fixation*. These words are seen as central to creativity in the considered sources and are often used synonymously. The lack of a clear definition (Singer et al., 2017) and a high number of definitions (Treffinger et al., 2002) is espoused by other researchers as well. After the hit list was filtered, only about 8% (26 out of 182) of the total articles remained for analysis. This was due to the fact that some keywords appear in other contexts (e.g., "the paper *illuminates* the research question”). Further, creativity also appears to be used as an umbrella term for many different approaches to research.

In the second step, the selected articles were carefully read. It was possible to create inductive categories that were used for sorting statements from the articles. These categories are *definition*, *components*, *description*, *development*, *integration*, *relation*, and *assessment* of creativity. With these categories, it is possible to analyze the contents of theoretical parts of the articles included and to obtain essential information: It is noticeable that although there are so many different existing definitions, only in a few articles, statements could be assigned to the category *definition*. Rather, a large number of *descriptions* and *components* were used to contour creativity. As with the search for keywords, the analysis of the theoretical parts also shows that many different concepts are described and composed within individual articles. However, this multiplicity makes it difficult to grasp the authors’ emphasis. In order to meet this challenge, the following refinements are suggested:

First, the list of journals included in the review should be extended by adding journals that are renowned in mathematics education research but are not listed in the WoS. Second, the categories presented here could possibly be further specified. Third, the assignment of statements to categories should be used to present the considered articles in networks with the aim of exploring the inherent meta structure of articles on mathematical creativity. Fourth, with the networks constructed in this way, it might be possible to recognize larger connections and to grasp different theories on the subject of mathematical creativity. This approach could already be pursued in the field of statistics education research (Nilsson et al., 2018). Fifth, current research could be examined even more closely: It would be very interesting to see to what extent the assumptions used in theoretical parts are empirically implemented in methodological parts in recent articles and to analyze which limitations or extensions exist.

With the aim of categorizing and systematizing research in a very broad field such as creativity, another step has been taken to focus better on different approaches. Since precise theoretical basics are central to research (Bikner-Ahsbahs, Knipping, & Presmeg, 2015), we suggest concentrating on clearly outlined assumptions, even though there is no uniform definition of creativity, in order to be able to further decode mathematical creativity and ensure a joint discussion.

References

American Psychological Association. (2017). *PsycINFO®: A World-Class Resource for Behavioral and Social Science Research*. Retrieved from www.apa.org/pubs/databases/psycinfo/psycinfo-printable-fact-sheet.pdf

Ayas, M. B., & Sak, U. (2014). Objective measure of scientific creativity: Psychometric validity of the Creative Scientific Ability Test. *Thinking Skills & Creativity, 13,* 195–205.

Barak, M. (2009). Idea focusing versus idea generating: A course for teachers on inventive problem solving. *Innovations in Education and Teaching International, 46*(4), 345–356.

Bikner-Ahsbahs, A., Knipping, C., & Presmeg, N. C. (Eds.). (2015). *Approaches to qualitative research in mathematics education: Examples of methodology and methods*. Advances in mathematics education. Dordrecht, New York: Springer.

Carayannis, E. G. (Ed.). (2013). *Encyclopedia of creativity, invention, innovation and entrepreneurship.* Dordrecht: Springer Dordrecht.

Daugherty, M., & White, C. S. (2008). Relationships among private speech and creativity in Head Start and low-socioeconomic status preschool children. *Gifted Child Quarterly, 52*(1), 30–39.

Gough, D., Oliver, S., & Thomas, J. (2013). Learning from Research: Systematic Reviews for Informing Policy Decisions: A Quick Guide. *A paper for the Alliance for Useful Evidence*. London: Nesta.

Gutiérrez, Á., Leder, G. C., & Boero, P. (Eds.). (2016). *The second handbook of research on the psychology of mathematics education: The journey continues*. Rotterdam: Sense Publishers.

Kim, M. K., Roh, S., II, & Cho, M. K. (2016). Creativity of gifted students in an integrated math-science instruction. *Thinking Skills and Creativity*, *19*, 38–48.

Leikin, R., & Pitta-Pantazi, D. (2013). Creativity and mathematics education: The state of the art. *ZDM – Mathematics Education*, *45*(2), 159–166.

Lerman, S. (Ed.). (2014). *Encyclopedia of Mathematics Education*. Dordrecht: Springer.

Mayring, P. (2015). Qualitative Content Analysis: Theoretical Background and Procedures. In A. Bikner-Ahsbahs, C. Knipping, & N. C. Presmeg (Eds.), Advances in mathematics education. *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (pp. 365–380). Dordrecht, New York: Springer.

Nilsson, P., Schindler, M., & Bakker, A. (2018). The Nature and Use of Theories in Statistics Education. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International Handbook of Research in Statistics Education* (pp. 359–386). Cham: Springer International Publishing.

Rhodes, M. (1961). An Analysis of Creativity. *The Phi Delta Kappan*, *42*(7), 305–310.

Singer, F. M., Sheffield, L. J., & Leikin, R. (2017). Advancements in research on creativity and giftedness in mathematics education: Introduction to the special issue. *ZDM*, *49*(1), 5–12.

Sternberg, R. J. (Ed.). (1999). *Handbook of creativity*. Cambridge: Cambridge Univ. Press.

Sternberg, R. J., & Sternberg, K. (2011). *Cognitive psychology* (6. ed.). Belmont, Calif.: Wadsworth Cengage Learning.

Treffinger, D. J., Young, G. C., Selby, E. C., & Shepardson, C. (2002). *Assessing Creativity: A Guide for Educators.* Storrs: University of Connecticut, The National Research Center on the Gifted and Talanted.

20 YEARS OF MATHEMATICS MOTIVATION MIRRORED THROUGH TIMSS: EXAMPLE OF NORWAY

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Student motivation is important for recruitment to further STEM education and carrier. Over the last decades, Norway has allocated many resources for recruitment to STEM, especially for girls, making it important to explore how students' motivation has changed across time. Using data from TIMSS, this paper explores the changes in Norwegian students' motivation in mathematics across time (N= 43 366), including differences across grades (4 and 8) and gender over the past 20 years. Measurement invariance analysis and multi-group CFA was conducted in Mplus. Findings indicate an increase in motivation (self-concept, intrinsic and extrinsic) across time for both grades, and higher motivation in favour of boys. These findings have implications for policy making and teaching practices in mathematics classrooms in Norway.

INTRODUCTION

In the past 20 years the International large-scale study, Trends in International Mathematics and Science Study (TIMSS), has gathered information about students' motivation in mathematics (and science). Since its inception in 1995, Norway has participated in the Survey, thus allowing for a unique opportunity to follow whether and how students' motivation for mathematics has changed over the years. While the data allow us to examine students' motivation across gender and grades (i.e., grades 4 and 8), so far it has not been investigated whether students' perception of motivation has been constant across different TIMSS cycles. Using data from 1995 to date, the paper investigates this issue, focusing on the mathematics domain.

MOTIVATION

Over the past decades students' motivational beliefs have been regarded as the driving force behind their learning and academic success (Wigfield, Tonks & Klauda, 2009), even in those situations when students are challenged by difficult and demanding tasks (Skaalvik & Skaalvik, 2009). At the same time the driving force we conceptualise as motivation can be observed through an array of both internal and external dimensions.

Basic concepts

When observing students' motivation to learn, much of the educational research draws on the distinction introduced by Deci and Ryan (1985), within the scope of their Self-Determination Theory (SDT). The basic concepts described are those of intrinsic and extrinsic motivation (Deci & Ryan, 1985; Ryan & Deci, 2000; Deci & Ryan 2008). The former is defined as the inherent tendency to seek out novelty and challenges, to

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 179-186). Umeå, Sweden: PME.

explore, and to learn. The construct itself describes person's natural inclination toward spontaneous interest, and exploration that is so vital to cognitive and social development of every human being (Csikszentmihalyi & Rathunde, 1993). In the context of the school and learning of mathematics, students who are intrinsically motivated to learn mathematics find the subject to be interesting and enjoyable (Deci & Ryan, 1985).

The latter construct, extrinsic motivation, refers to the drive that comes from external rewards (i.e., praise, career success) and the performance of an activity so as to achieve some separable outcome (Ryan & Deci, 2000). SDT suggests that extrinsic motivation can largely fluctuate and vary it its nature (Ryan & Connell, 1989). It can be personally endorsed (e.g., a student practicing mathematics due to the fact s(he) recognizes value of the activity for later career choice) or by compliance with an external regulator (e.g., a student is practicing mathematics as to adhere to parents' control).

In their later work Deci and Ryan (2008) focused their attention on the separation between autonomous motivation and controlled motivation. Autonomous motivation encompasses intrinsic motivation and the types of extrinsic motivation in which people have identified with an activity's value, making their actions very much self-endorsed. Controlled motivation, on the other hand, entails external and introjected regulation. Since in the TIMSS 2015 framework clear reference is given to the concepts of intrinsic and extrinsic motivation (Hooper, Mullis, & Martin, 2013), in the remaining part of the text we use this terminology.

Motivation and achievement across age and gender

A number of studies have found an association between motivation and achievement (Bøe & Henriksen, 2013; Froiland & Davison, 2016). Furthermore, research steadily shows that intrinsic motivation is more closely related to achievement (Becker, McElvany, & Kortenbruck, 2010) and external rewards to be diminishing students' intrinsic motivation (Deci, Koestner, & Ryan, 1999). At the same time we may easily argue that students are not intrinsically inclined to all subjects, so fostering motivation through extrinsic rewards may be essential for some teachers and parents. Nonetheless, studies also show that successful students may easily internalize their extrinsic motivation to increase performance, especially in an environment that promotes value of competence and autonomy (Ryan & Deci, 2000).

Much of the research also focuses on the relationship between motivation and age and motivation and gender. What studies show is that students' motivation to learn mathematics decline with age, especially in the context of their intrinsic interest to be involved with mathematics (Gottfried, Fleming, & Gottfried, 2001; Wendelborg, Røe, Federici, & Caspersen, 2015), while associations with achievement grow stronger (Kaarstein, 2017). Gender differences relative to both motivational patterns and achievement in mathematics are widely reported (Lazarides, Rubach, & Ittel, 2017). In addition, studies show that students' high motivation for a particular domain like mathematics is also accompanied with their high self-concept in the domain (i.e., perceived competence in mathematics; Seaton, Parker, Marsh, Craven, &Yeung, 2014)

Research focus

Departing from the TIMSS framework depicting students' motivation to learn mathematics and students' subject specific self-concept (Hooper, Mullis, & Martin, 2013), we examine whether Norwegian students' perception of motivation in mathematics has been constant across different TIMSS cycles including differences across grades (4 and 8) and gender over the past 20 years.

METHOD

Sample

In this paper we use data from grades 4 and 8, including all TIMSS surveys Norway has participated in. These include; TIMSS 1995 (N = 10 212 students), TIMSS 2003 (N = 8 475 students), TIMSS 2007 (N = 8 735 students), TIMSS 2011 (N = 6 983 students), and TIMSS 2015 (N = 8 959 students). Norway did not participate in the TIMSS 1999 round.

Constructs

In the development of the items that are included in the student questionnaires on motivation, TIMSS has grounded its constructs into the work of Deci and Ryan's (1985) concepts of intrinsic and extrinsic motivation, accompanied with the students' self-concept (Hooper et al., 2013). All analyses in this paper use that same division. Notably TIMSS framework constantly keeps up with the developments that take place in society, school and research as to ensure consistent measurement quality of the concepts included in the student background questionnaire, while at the same time ensuring that these changes are minimal for the comparison purposes across the cycles. Students' motivation is measured by asking the students how much they agree with the statements, as shown for example in Table 1. For each statement, students may choose between four options: agree a lot, agree a little, disagree a little and disagree a lot. As we are examining students motivation for mathematics across the span of 20 years, items that were chosen to be included in the analyses of both intrinsic and extrinsic motivation and self-concept are grounded on four criteria: 1) at least three items must be included in each construct, 2) the statements selected should have been included in at least four cycles of TIMSS, 3) the items included have been the same in all the cycles, and 4) the items must be present in contextual questionnaire for both grades.

Example of items pertaining to intrinsic motivation is given in Table 1. Extrinsic motivation, which TIMSS only included in grade 8, is put together by statements like: I need to do well in mathematics to get the job I want, I need to do well in mathematics to get into the of my choice, I think learning mathematics will help me in my daily life and I need mathematics to learn other school subjects.

Self-concept items investigate whether child perceives that (s)he is usually doing well in mathematics, is good in mathematics, observes mathematics as more difficult for him(her) than other peers in the class and learns mathematics quickly.

Items	TIMSS cycles				
	1995	2003	2007	2011	2015
I enjoy learning mathematics					
I wish I did not have to study mathematics					
Mathematics is boring					
I learn many interesting things in mathematics					
I like mathematics					
I like any schoolwork that involves numbers					
I like to solve mathematics problems					
I look forward to mathematics lessons					
Mathematics is one of my favourite subjects					

Table 1: Example items in grade 4 on intrinsic motivation. A comparison is made relative to the 2015 cycle. An area is shaded if an item has been used in the cycle.

The way we have defined the constructs in the study allows us to examine students' intrinsic motivation in all cycles except the one in 2003. Self-concept and external motivation can be investigated from TIMSS 2003 and for all subsequent cycles.

Analyses

Preparation of the data included reversing the negative items and transforming the scale by assigning 0 to the “disagree a lot” choice and a 3 to the “agree a lot” choice. Missing data were treated with Robust maximum likelihood (MLR), an option within the Mplus.

All the analyses were done in Mplus 8 (Muthen & Muthen, 1998-2017) using a multi-group approach (i.e., several groups can be analyzed in the same model). The hierarchical structure of the TIMSS data (i.e., students belong to classes belonging to schools) was taken into account using the option type = Complex in Mplus. In this study students are grouped according to 1) cycles, 2) grades, and 3) gender. The purpose of running multi-group analyzes is to check whether the differences between the groups (i.e., cycle, grade and gender) are significant. Confirmatory factor analysis (CFA) for each construct was calculated for all groups and latent means for each construct were estimated. In order to investigate whether different groups have the same perception of the construct, Measurement Invariance analyses (MI) were used. Where results indicate the same understanding of the concept by different groups, a comparison can be made.

RESULTS

Measurement invariance

The results from the MI analyses show that all constructs were scalar invariant across the examined cycles. Scalar invariance implies that the means of a construct are comparable.

Constructs	ΔCFI	ΔTLI	ΔRMSEA	ΔSRMR
Intrinsic motivation (4)	-0.007	-0.005	0.019	0.008
Intrinsic motivation (8)	-0.007	0.002	-0.002	0.003
Extrinsic motivation (8)	-0.027	-0.030	0.049	0.020
Self-Concept (4)	-0.020	-0.019	0.029	0.016
Self-Concept (8)	-0.003	-0.004	0.021	0.008

Table 2: Results of the measurement invariance models across the cycle (cut off points are: ΔCFI < 0.01, ΔTLI < 0.01, ΔRMSEA < 0.02, ΔSRMR < 0.015).

All constructs were scalar invariant across gender in grade 4. In grade 8, scalar invariance was found for intrinsic motivation (except in 1995) and extrinsic motivation (except 2011). No scalar invariance was found for self-concept in grade 8.

We could not establish scalar invariance for any of the constructs across grades.

Changes over time

Figure 1 shows the trend in changes across the cycles for each motivational aspects and each grade respectively. A horizontal axis represents different TIMSS cycles (e.g., T95 refers to TIMSS 1995 cycle all the way to T15 representing TIMSS 2015). The vertical axis shows the latent means (standardized).

After a significant decrease in students' intrinsic motivation from 1995 to 2007 in grades 4 and 8, a steady increase is visible for the period 2007 to 2015. From 2003 to 2011 in grade 8 we can also observe a significant increase in students' extrinsic motivation, keeping this high plateau in 2015 as well. No significant differences are visible between values for 2011 and 2015 cycles for extrinsic motivation.

Finally, when it comes to self-concept in mathematics students in grade 4, there is a steady increase during the period from 2003 to 2011. Although somewhat lower values are visible in 2015 there are no significant differences between this and the previous cycle. A steady and significant incline is also visible for the period 2003 to 2015 when observing students' self-concept in grade 8, but the changes are not as steep and are insignificant when comparing adjunct cycles (e.g., 2003 with 2007 and 2011 with 2015).

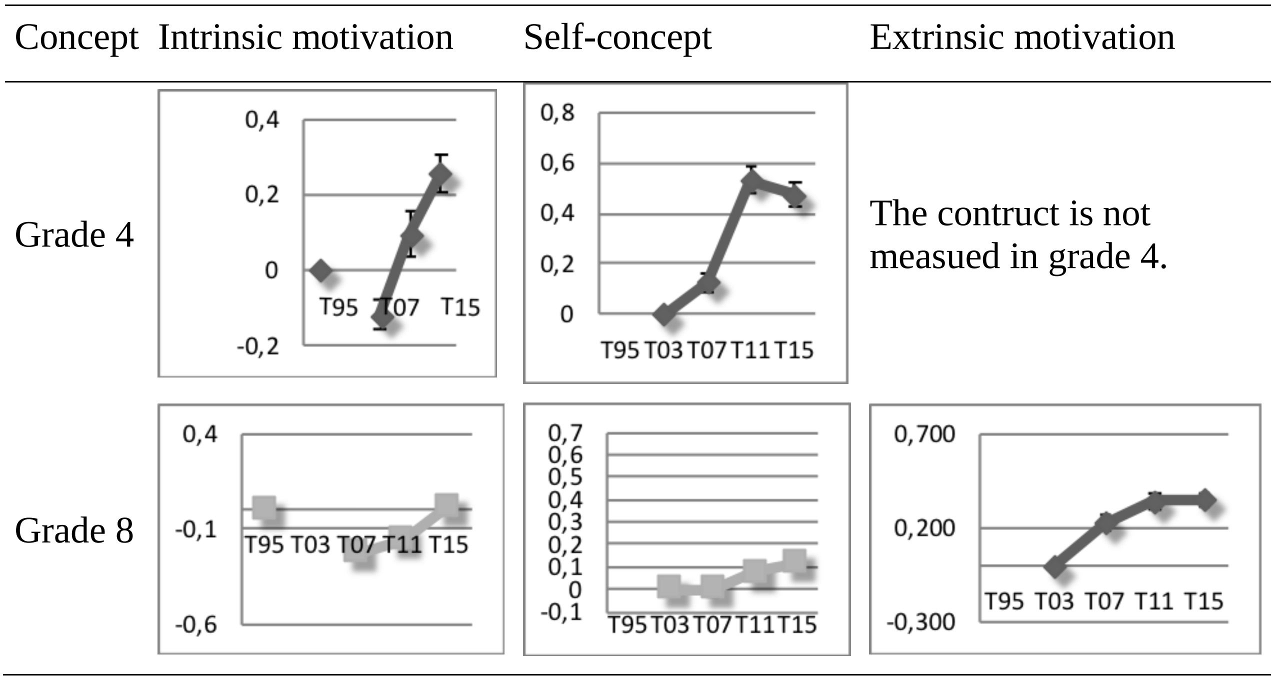

Figure 1: Changes in the observed constructs across the TIMSS cycles (1995-2015).

Differences between boys and girls

When observing the gender differences, across the cycles, boys seem to be more motivated to do mathematics. For extrinsic motivation (grade 8), this gap seems to diminish and disappear. In grade 4 on the other hand the gap seems to increase for self-concept.

Constructs	1995	2003	2007	2011	2015
Intrinsic motivation (4)	-0.086	-	-0.100*	0.049	-0.066
Intrinsic motivation (8)	noMI	-	-0.031	0.013	**0.138***
Extrinsic motivation (8)	-	**0.272***	**0.123***	noMI	0.030
Self-Concept (4)	-	0.082	0.023	**0.163***	**0.139***
Self-Concept (8)	-	noMI	noMI	noMI	noMI

Table 3: Gender differences across cycles (*denotes significance, if **bold** in favour of boys; noMI denotes scalar invariance was not established).

CONCLUSION AND DICUSSION

The results clearly indicate scalar invariance for all constructs across the TIMSS cycles from 1995 to 2015. This means that the students' perception of motivation has not changed over time and that for students who participated in TIMSS in 1995 the meaning of intrinsic motivation, for example, was the same as for the students participating in TIMSS 2015.

At the same time we have not established scalar invariance across grades, indicating that we cannot draw the conclusion that students' motivation is lower in grade 8 than in grade 4 as indicated in multiple international reports for most countries (e.g. Mullis, Martin, & Loveless, 2016). However, our results point to caution when interpreting this result, as these constructs may not be invariant across grades in other countries as well. Finally, with respect to the construct related to self-concept in grade 8, its meaning is problematic across all examined cycles when comparing boys and girls. The lack of construct invariance gives little grounds for any meaningful comparison.

When observing changes over time an important finding for the Norwegian school system and mathematics teaching is an increase in students' motivation. Although this increase is steeper in grade 4 than in grade 8, it is consistent across the cycles and is not in line with the major results reported for other European countries; that students' recruitment and hence motivation for mathematics is in decline (OECD, 2016).

Given the amount of resources invested in the Norwegian education system in promoting STEM choices for girls it is interesting to observe that boys still seem to be more motivated for mathematics than the girls. Although we do not claim causal links between these two processes, it is notable that in a society with such a clear focus on equity between the sexes, such clear differences still exist and that the gap between boys and girls in perceiving own competence related to mathematics even exist in grade 4. At this point it remains to be seen if the trend will be kept in 2019, with the new TIMSS round, or whether a shift in how boys and girls go about mathematics will occur. In any case, these will set the course on how teachers, teacher' educators and policy makers plan their programs and actions in promoting student agency and career choice in STEM.

References

Becker, M., McElvany, N., & Kortenbruck, M. (2010). Intrinsic and extrinsic motivation as predictors of reading literacy: *A longitudinal study. Journal of Educational Psychology*, *102*(4), 773-785.

Bøe, M. V., & Henriksen, E. K. (2013). Love it or leave it: Norwegian students' motivations and expectations for postcompulsory physics. *Science Education, 97*(4), 550–573.

Csikszentmihalyi, M., & Rathunde, K. (1993). The measurement of flow in everyday life: Toward a theory of emergent motivation. In J. E. Jacobs (Ed.), *Developmental perspectives on motivation* (pp. 57-97). Lincoln: University of Nebraska Press.

Deci, E. L., & Ryan, R. M. (1985). *Intrinsic motivation and self-determination in human behavior*. New York: Plenum.

Deci, E. L., & Ryan, R. M. (2008). Self-Determination Theory: A Macrotheory of Human Motivation, Development, and Health. *Canadian Psychology*, *49*(3), 182-185.

Deci, E. L., Koestner, R., & Ryan, R. M. (1999). A meta-analytic review of experiments examining the effects of extrinsic rewards on intrinsic motivation. *Psychological Bulletin, 125,* 627-668.

Froiland, J. M. & Davison, M. L. (2016). The longitudinal influences of peers, parents, motivation, and mathematics course-taking on high school math achievement. *Learning and Individual Differences*, *50*, 252-259.

Gottfried, A. E., Fleming, J. S., & Gottfried, A. W. (2001). Continuity of Academic Intrinsic Motivation From Childhood Through Late Adolescence: A Longitudinal Study. *Journal of Educational Psychology, 93*(1), 3-13.

Hooper, M., Mullis, I. V. S., & Martin, M. O. (2013). TIMSS 2015 Context Questionnaire Framework. In I. V. S. Mullis & M. O. Martin (Eds.), *TIMSS 2015 Assessment Frameworks* (pp. 61-83). Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Kaarstein, H. (2017). Elevers motivasjon - TIMSS 2015. *Tangenten*, *28*(2), 17-22.

Lazarides, R., Rubach, C., & Ittel, A. (2017). Adolescents' Perceptions of Socializers' Beliefs, Career-Related Conversations, and Motivation in Mathematics. *Developmental Psychology, 53*(3), 525-539.

Mullis, I. V. S., Martin, M. O., & Loveless, T. (2016). *20 Years of TIMSS: International Trends in Mathematics and Science Achievement, Curriculum, and Instruction*. Boston: Boston College and International Association for the Evaluation of Educational Achievement (IEA).

Muthén, L.K., & Muthén, B. (1998–2017). *Mplus user's guide* (7th ed.). Los Angeles, CA: Muthén & Muthén.

OECD (2016). *Equations and Inequalities: Making Mathematics Accessible to All, PISA*, Paris: OECD Publishing.

Ryan, R. *M., &* Connell, J. P. (1989). Perceived locus of causality and internalization. *Journal of Personality and Social Psychology*, *57,* 749-761.

Ryan, R.M., & Deci, E.L. (2000). Self-Determination Theory and the Facilitation of Intrinsic Motivation, Social Development, and Well-Being. *American Psychologist*, *55*(1), 68-78.

Seaton, M., Parker, P., Marsh, H.W., Craven, R.G., & Yeung, A.S. (2014) The reciprocal relations between self-concept, motivation and achievement: juxtaposing academic self-concept and achievement goal orientations for mathematics success. *Educational Psychology*, *34*(1), 49-72.

Skaalvik, E. M., & Skaalvik, S. (2009). Elevenes opplevelse av skolen: sentrale sammenhenger og utvikling med alder. *Spesialpedagogikk, 74*(8), 36-47.

Wendelborg, C., Røe, M., Federici, R. A., & Caspersen, J. (2015). *Elevundersøkelsen 2014. Analyse av elevundersøkelsen 2014.* Trondheim: NTNU Samfunnsforskning

Wigfield, A., Tonks, S., & Klauda, S. L. (2009). Expectancy-value theory. In K. R. Wentzel & A. Wigfield (Eds.), *Handbook on motivation in school* (pp. 55–76). New York: Taylor Francis.

CRITERIA FOR KNOWING A GEOMETRICAL OBJECT: THE ENACTIVIST PERSPECTIVE

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This article proposes several criteria for students to know a geometrical object and identifies the trigger for evolving interactions between the teacher, the students, and the learning environment from an enactivist perspective. This article focuses on the identification of the key concept of bringing forth a world to interpret students' geometrical behaviors. Based on a qualitative research methodology, a theory for generating geometrical objects was suggested and exemplified through the analysis of third-grade mathematics lesson, from which three criteria; (I) theoretical, (II) possible, and (III) actual; were identified. It was observed that an important trigger for the evolving interactions was that the students were able to engage physically and theoretically in an open situation.

INTRODUCTION

Mathematics education research based on enactivism has taken various directions since it was first examined (Varela, Thompson, & Rosch, 1991; Marurana & Varela, 1992). Enactivists believe that *knowing* is a dynamic action and have sought to understand the nature of this action in learners. As the Enactivist theory deals with the various rich and complex interactions involved in knowing as action, it could be used as a conceptual tool to explain mathematical, cognitive phenomena.

Mathematics education research could be advanced by comparing key concepts such that the *viability* interpretation in constructivism leads to *bringing forth the distinct worlds of significance* inherent in enactivism (Proulx & Simmit, 2013, 2016). The latter is a key concept as it suggests that a mathematical object could be generated by associating it with the appropriate world (Kieren & Simmt, 2009), thereby making the students' actions more meaningful. To examine this more closely, this article focuses on the criteria needed by students to come to know a geometrical object through their actions and to be able to analyze and explain the phenomena in class, which also allows for an analysis of the students' mathematical behaviors.

This article seeks answers to the following research questions:

RO1: What criteria do students adopt to generate an object and determine whether it is geometrical or not?

RO2: What factors are important to initiate and evolve the interactions between the teacher, the students, and the learning environment?

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 187-194). Umeå, Sweden: PME.

THEORETICAL CONSIDERATIONS

Key ideas behind enactivism: Focusing on knowing a geometrical object

Enactivism's theoretical foundation is that *knowing* is a biological phenomenon (Maturana & Varela, 1992; Varela, 1992); that is, knowing has biologically evolved as effective action through environmental effects and the human actions in that environment (Maturana & Varela, 1992, pp.173-181). Therefore, as a series of effective actions develops the significant world in which an individual exists and lives, understanding environmental factors is critical to mathematical knowing research as it is necessary to understand the world the individual is within because this determines which objects can emerge in the individual's known environment and which objects the individual can generate in this world.

Maturana and Varela (1992) claimed that knowing an object involved being able to distinguish it:

> The act of indicating any being, object, thing, or unity involves making *an act of distinction* which distinguishes what has been indicated as separate from its background. Each time we refer to anything explicitly or implicitly, we are specifying a *criterion of distinction,* which indicates what we are talking about and specifies its properties as being, unity, or object. (p. 40)

This statement claims that to know a concrete object in an environment, it needs to be distinguished from all other objects so that others know what is being talked about. Therefore, to know a geometrical object that has both figural and conceptual characteristics (Fischbein, 1993), it is necessary to clarify its *criterion of distinction* visually and in language (Simmt & Kieren, 2015, p. 311). As a geometrical object has an inter-objective nature, which could be seen as a social dimension that is consensual with others, these aspects need to be considered when discussing geometrical objects (Simmt & Kieren, 2015, pp. 310-313). However, these comprehensive approaches do not always provide answers to the process associated with the emergence of appropriate geometrical inter-objects or what methods are used to bring this forward (Simmt & Kieren, 2015, p. 313) if the geometrical nature and the critical actions required to form them are unknown.

When knowing a geometrical object, a possible distinction criterion is recognizing the isomorphism (Greer & Harel, 1998) inherent in the distinct worlds that emerge through the evolutionary process. For example, a triangle can be represented through drawing, constructing, or describing; however, even though its nature depends on the method used to make it, it is possible to recognize the uniformity; that is, the isomorphism or invariance between all triangles as they are structurally the same regardless of position, magnitude, or material attributes. When seeking to bring forth a geometric world through classroom communication, even though multimodal factors affect students' geometrical conceptions, learners need to be able to observe the significant properties rather than the attributes, and see them as the main theme.

What worlds are brought forth to know a geometrical object?

Classrooms are complex environments made up of artifacts with educational intention, concrete models that represent concepts or matters, noticeboards for learning history, relationships between teachers and students, and feedback from learning with others. However, the significance of these aspects depends on each learner. When manipulating materials such as concrete objects or diagrams, we draw on knowledge from not only the physical world in which such concrete objects could exist, but also the geometrical world, which is mediated by theoretically thinking about the creation of the object as a model (cf. de Freitas & Sinclair, 2014, pp. 200-224). Therefore, bringing the world forth is complex in the classroom environment.

Proulx and Simmt (2013) claimed that thinking differences between the pairs are often related to either the physical or geometric world; therefore, the method for knowing and acting is highly dependent on the world that is brought forth. Further, when creating visual images of parts that cannot be directly perceived in the physical world such as a concrete object from various viewpoints, and then experimentally transforming, disposing, and reconstructing them, these visual images always have a possibility to transcend reality; therefore there is a co-defining relationship between cognition as action and bringing a world forth.

This co-defining that results in updating or revising has been referred to as a learner's engagement in a problematic situation; the dynamics of understanding mathematics as a change in the method for knowing objects. Pirie and Kieren (1994) claimed that a complementarity in the actions and expressions leads to subsequent actions, with the meaning of the actions differing from the meaning of the expressions dependent on the level of understanding; therefore, problems as enacted objects could vary because of the differences in meaning. Ideally, experience needs to be applied through abstraction/concretization and generalization/specialization rather than from within the same world where the knowledge becomes stable and transmittable.

Although Pirie and Kieren (1994) described bringing worlds forth as a recursive model process, this article examines the dynamics of the students' actions. To do this, a method of thinking in each world was employed based on virtuality or possibility.

METHODOLOGY

This article focused on a third-grade mathematics lesson that was collaboratively designed by the author, student teachers, and another practicing teacher. As the objective of this research was to examine the processes behind the bringing forth of distinct worlds, a qualitative research methodology was adopted (Flick, 2009).

The main focus of the lesson was to develop triangles using several methods, with the objective being to motivate the students to geometrically and logically perceive the figures through a range of learning activities. The main activities were focused on (1) developing the figures and (2) geometrically and logically demonstrating the figures. First, the teacher gave the definition for a point as the intersection of a line and a curve, and then demonstrated that any line could be constructed by connecting two points. Students were then asked to position a line, a circle, an isosceles triangle, and a regular triangle within a given figure (Fig. 2).

The lesson activities, which involved the entire class, were recorded on cameras at the back and front of the classroom by two prior research collaborators. While one video recording focused on student interactions, the other filmed the entire class, both of which were supported by the researchers' individual field notes. By referring to the student worksheets and the researcher field notes, after the lesson, there were discussions on the students' thinking processes and action intentions. These descriptions and transcripts as well as the teacher and student actions, tool usage, and the various expressions on the whiteboard were analyzed. By focusing on the various teacher, students, and learning environment interactions such as the concrete model and diagram on the whiteboard, the criteria for the knowing of geometrical objects were identified.

RESULTS

First, the teacher asked the students to watch the process of connecting the same-length sides to construct a concrete equilateral triangle (Fig. 1); while expressing the definition, the student attempted to touch hidden parts of the object to confirm that it was a regular triangle. The students used geometrical terms and expressions that differed from everyday language, such as "regardless of congruence of all edges, I am not sure that this angle is same as this one because this is hidden by the teachers' hands", which indicated that they were bringing forth their own worlds based on the lesson content; the perceptual figures drawn on the whiteboard, the manipulatives, a task to find any regular or isosceles triangles, and the use of tools such as rulers and a compass.

Fig. 1: Touch gesture by a student.

Following four steps; (a) drawing a certain line, (b) drawing a large circle by placing the compass needle on any position on a given line, (c) confirming that two points were generated as intersections of a line and a circle, and (d) drawing a small circle by placing a compass needle on the right-hand point; the teacher drew a figure on the whiteboard using a ruler and compass, and established six points actually by placing a red mark on each point (Fig. 2).

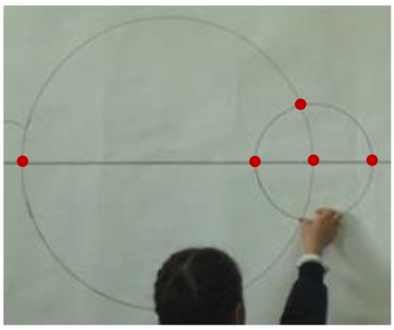

Fig. 2: Embodied points on a figure.

Because the point was a small, filled circle drawn in the third-grade mathematics textbook, no points initially existed on Fig. 2, as it was supposed to exist theoretically in a world in which the rule for making figures differed. The students' statements of "now I see" and "I can see some points" implied that they did not appear to have difficulty understanding that the methods for a figure's existence can change depending on its construction rule. However, when the red marks with the position and magnitude were marked on the whiteboard, it became a certain, possibly existing object for the students.

This lesson's main objective was to find regular or isosceles triangles within Fig. 2. As described above, although there are various criteria for a triangle, no triangle existed actually or theoretically in Fig. 2; therefore, as they needed to generate triangles so as to be able to discuss any triangle in the same way as the point (Fig. 3), the final activity was to generate and demonstrate a triangle.

Fig. 3 altered the structure previously created by the students, as it shifted from a closed structure in which two circles with their centers on a given line intersected with each other to an open structure which allowed the students to generate other figures by connecting the points with defined positions. The discussion gradually shifted to the generation of possible triangles as students connected the points and decided whether the triangles were or were not isosceles. For example, the following transcript was student S's explanation that the drawn, special triangle was an isosceles triangle.

Teacher: Why, in the middle of our discussion, could S see that it was an isosceles triangle?

Some Students: Because he could see!

Student S: All radii have the same length. Well, two parts are connected [pointing to radius of a small circle on the whiteboard], and the exterior connected parts are the same, so this is an isosceles triangle.

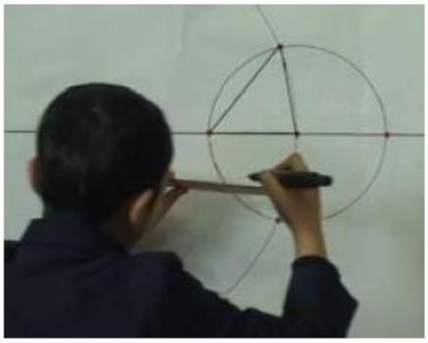

Fig. 3: Generating a possible triangle.

Although this demonstration was theoretical as the students use some figural properties to assess whether the triangle was isosceles, it was inadequate as established mathematical proof. Some students identified the two sides student S mentioned as the radii of a small circle; however, few were able to accept the statement that "all radii were equal" as the rationale for the demonstration.

Following this communication, another student demonstrated that the triangle was not regular by measuring each side (Fig. 4); they measured the length of side CA by placing the compass needle at one end and rotating it to CB to check whether the two sides were equal (Fig. 4; left). Then, they compared the length of CA (CB) to the third side, BA, by placing the compass needle on B. The transcript of this student's statement was as follows:

Student Y: The length of this reaches here [pointing to point on BA in Fig.4; right], and the black one is left, so it is an isosceles triangle.

Student Y: I don't think that it is a regular triangle.

Teacher: What do you think?

Teacher: Raise your hands if you think it is an isosceles triangle.

(All students raised their hands.)

Strictly speaking, rather than actually measuring with a ruler, student Y only compared one side with another using a compass. After the teacher's questioning, all students accepted the method and the results that this was an isosceles triangle and not a regular triangle.

When drawing the figure on the whiteboard, the teacher deliberately omitted any specification of the length, which meant that only the relative position of each figure

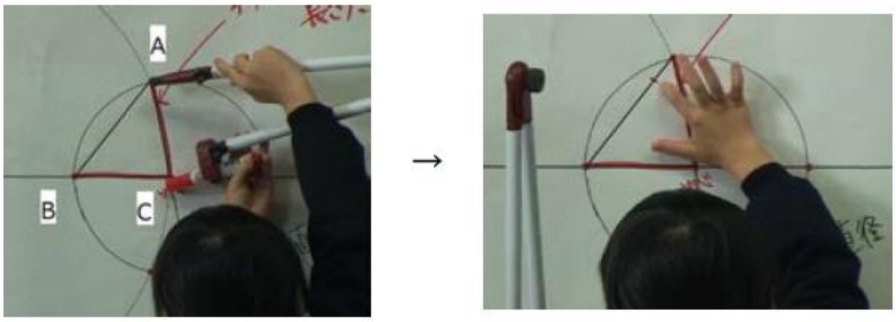

Fig. 4: Demonstration by measuring and comparing.

was able to be determined. Therefore, the positions for points A, B, and C (Fig. 4) could change depending on the circle drawn by the teacher; the shape of the triangle and the lengths of its sides could also then change. While measuring or comparing actions according to criteria (III) is effective and accepted in the real world; however, in the mathematical and geometrical world it is not. After this demonstration, students generated some triangles by combining and connecting six points, and concluded that while some isosceles triangles could theoretically exist, regular triangles could not exist on this special, given figure.

DISCUSSION AND CONCLUSION

Three criteria for knowing geometrical objects

The students' real knowing actions were not necessarily simple during the lesson. During a post-discussion about the students' actions, at least three criteria were identified to determine whether a triangle was regular: (I) if the triangle fulfills the definition; (II) if students were able to construct the same kind of triangles based on the given construction procedure and definitions; and (III) if the students could perceive and manipulate the constructed figure as a concrete object.

Criterion (I) is focused on **theoretical knowing** as it depends on the technical language (see student S's statement), and criterion (II) is focused on **possible knowing,** as the definition implies a method for generating possible, special objects (see student actions in Figs. 2 and 3); therefore, both are acceptable *in the mathematical world*. Criterion (III), however is related to **actual knowing** because body movements and physical sensations are preferred (see the touch gesture in Fig.1 and student Y's demonstration) and accepted in the *physical world*. Each criterion's effectiveness was determined by the students as participants in the classroom conversations.

In the sequence of activities to identify the figure as a geometrical object, criteria (I), (II), and (III) assisted the students to determine whether the triangle was isosceles, regular, or neither. As was seen, the theoretical demonstration based on geometrical properties by student S in Fig. 3 and the measuring and comparing by student Y in Fig. 4 were acceptable effective methods. Rather than going forward or folding back beyond these distinct worlds, as implied in Pirie and Kieren (1994), the students appeared to inhabit both distinct worlds to know the objects and change their problem generation and solution priorities. When the third graders came to know the geometrical object, they were able to apply several effective criteria from these distinct worlds; therefore, any object could be seen as mathematical if the criteria applied.

In particular, criterion (III) demands that students apprehend the figures through bodily actions and sensations such as touch and gesture, (II) demands that they purposely generate the objects according to the endorsed procedure and in reference to the geometrical definition, and (I) demands that all generated objects be accepted if they fulfill the definition and the students can generate other objects based on these definitions. An endorsed procedure could be equated to an algorithm, such as the four steps followed to create the figure in Fig. 2, which is different from constructing an object to fulfill a certain definition. For third graders, these objects were dealt by applying the various criteria to identify the isomorphic relationships.

An open situation as the trigger for evolving interactions

The endorsement of the object's existence based on the several identified criteria was an effective and collective action for the third graders, enabled them to distinguish objects from the background, and to convince themselves that the geometrical must exist. This endorsement of existence made the students pay attention to the background of the object; that is, in the lesson, the individual triangle was known from its relationships with the given line and circles, which was realized based on the specific geometrical definitions and properties in the geometrical world, and was actualized by drawing and constructing it in the physical world, all of which evolved through the interactions between the object and its background.

In the lesson, a trigger for the modification of the criteria priorities and the evolving interactions between the students and the figure, as exemplified in Figs. 2 and 3, were conducted in an open situation that allowed the students to act both physically and theoretically as they adopted another rule to generate the figure. All students physically generated the geometrical object, and then attempted to demonstrate its existence theoretically, and vice versa. The tangled methods concerning the existence of an object initially confused the students; however, they gradually determined their own actions and defined rules within their own created worlds.

Acknowledgement

This work was supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI (23730829).

References

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body*. New York, NY: Cambridge University Press.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139-162.

Flick, U. (2009). *An introduction to qualitative research* (4th ed.). London: Sage Publications Ltd.

Greer, B., & Harel, G. (1998). The role of isomorphisms in mathematics cognition. *Journal of Mathematical Behaviour*, *17*(1), 5-24.

Kieren, T., & Simmt, E. (2009). Brought forth in bringing forth: the inter-actions and products of a collective learning system. *Complicity: An International Journal of Complexity and Education*, *6*(2), 20-27.

Maturana, H., & Varela, F. (1992). *The tree of knowledge*, Boston, MA: Shambhala.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding. *Educational Studies in Mathematics*, *26*(2-3), 165-190.

Proulx, J., & Simmt, E. (2013). Enactivism in mathematics education: moving toward a re-conceptualization of learning and knowledge. *Education Sciences and Society*, *4*(1), 59-79.

Proulx, J., & Simmt, E. (2016). Distinguishing enactivism from constructivism: engaging with new possibilities. In Csikos, C. et al. (Eds), *Proc. 40th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 99-107). Hungary.

Simmt, E., & Kieren, T. (2015). Three "moves" in enactivist research: a reflection. *ZDM*, *47*(2), 307-317.

Varela, F. (1992). Autopoiesis and a biology of intentionality. In McMullin, B. (Ed.). *Proceedings of the workshop "Autopiesis and Perception"*(pp. 4-14).

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge: MIT Press.

MICRO-EVOLUTION OF DOCUMENTATIONAL WORK IN THE TEACHING OF THE VOLUME OF REVOLUTION

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In this paper, we draw on the documentational approach to analyse the evolution of one experienced secondary teacher's work towards the teaching of the topic of "volume of revolution". He used a range of paper and computer based resources including the software Autograph. Data were collected in observations of three lessons on this topic taught to two different groups of 16-18 years old students and a follow up interview with the teacher where he was asked to reflect on his choices in these lessons. The findings illustrate teacher's documentational work with the used resources, and his schemes of use – aims, rules of actions, operational invariants and inferences – and identify the micro-evolution, namely the small changes and the rationale behind these changes, of these schemes across the lessons.

INTRODUCTION

Teaching is a complex profession that requires teachers to interact with, and promptly respond to, a range of factors in their teaching environment. As a result, teachers' practices are not merely a reflection of their plans and beliefs. Other factors also come into play, such as teachers' and students' personalities and epistemologies, institutional constraints, unexpected circumstances, time issues and available materials (Nardi, Biza, & Zachariades, 2012). These factors should be taken into account when studying teachers' practices (Herbst & Chazan, 2003). Indeed, Lerman (2013) suggested that research should avoid "implied telos about 'good teaching' [… and] study what happens in practice and offer multiple stories of that practice" (p. 623). In this paper, we report findings from Kayali's PhD study that investigates mathematics teachers' ways of tuning the different elements in their working environment, especially when using mathematics-education software (i.e. software designed for mathematics teaching and learning purposes). Specifically, we look at teachers' 'live' practices within specific contexts and examine consistencies and potential gaps between intended and actual practices (Kayali & Biza, 2017). Here, we draw on the documentational approach (Gueudet & Trouche, 2009) to analyse data of three video-recorded lesson observations of one teacher's work on the topic of "the volume of revolution", and a follow up audio-recorded interview with him where incidents from the observation were discussed in order to respond the research question: "How does teacher's documentational work change across lessons, if it changes, and why?".

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 195-202). Umeå, Sweden: PME.

THE DOCUMENTATIONAL APPROACH

The *documentational approach* (Gueudet & Trouche, 2009) explores teachers' work with *resources*. The term resource here has a wider definition; it can be an artefact, a teaching material, a social interaction or anything that influences a teacher's activity (ibid.). This approach, also, refers to Adler's definition of resource "as the verb re-source, to source again or differently" (2000, p. 207). According to the documentational approach, teachers while interacting with resources develop *schemes of use*. A scheme of use adopts a set of resources to be used across different situations according to specific procedures (Gueudet, 2017). It consists of the *aim of the teaching activity* (e.g. to teach about the volume of revolution); *rules of action*, which represent teacher actions (e.g. solving past-exam questions on the volume of revolution); *operational invariants*, which are the reasons adopted by a teacher to justify her stable actions in a range of similar situations (e.g. it is useful to use Autograph and the textbook to introduce the formula); and, *inferences* (e.g. it would work better if I present the image from the textbook first). A teacher develops a *document* when she associates a set of resources with the scheme of use of these resources (ibid.). Document can be "thought of as the verb *document*: to support something (here the teacher's professional activity) with documents" (Gueudet & Trouche, 2009, p.205, italics in original). A teacher's documentational work includes the set of resources encountered, collected, amended or developed by that teacher for a specific goal (ibid.). The documentational approach offers lenses for exploring the evolution of a teacher's documents, which in turn "contributes to the study of her professional evolution. Naturally, such a study must not be limited to the material aspect of documents, but must also investigate the evolution of usages […] and operational invariants" (ibid., p. 211). In this study, we aim to explore the characteristics of one teacher's document by investigating his set of resources and schemes of use during the teaching of three lessons on the volume of revolution, taking into account the justifications he made during the lessons and in the follow-up interview.

METHODOLOGY

This paper reports outcomes from a PhD project conducted in the UK by the first author. The study looks at upper secondary mathematics teachers' documentational work, specifically schemes of use that also concern mathematics-education software. It employs qualitative analysis based on an interpretative research methodology (Stake, 2010). In this paper, we discuss three video-recorded lesson observations and the follow-up interview of one participant, George. At the time of the data collection, George had 15 years of teaching experience mostly in upper secondary education (ages 16-19). The follow up interview was conducted after the initial analysis of the three video-recorded observations. The interview questions focused on the teacher's main steps and choices that were identified during this initial analysis. In the interview, George was invited to reflect and comment on these specific choices (e.g. the use of Autograph). The follow up analysis of George's responses in the interview and actions during the lessons was performed by using the documentational approach. Specifically, the analysis identified the used resources as well as the schemes of use: aims of the teaching activity, rules of actions, operational invariants and inferences in the context of the observed lesson and summarised them in a documentational work table, similar to the one used by Gueudet (2017) in her analysis of university teachers' work. Here, a simplified version of this table was produced (Table 1), summarizing the rules of actions and operational invariants related to the aims of teaching about volume of revolution and preparing students for exams, in the first lesson. Any changes to the rules of actions or operational invariants in the second or third lessons are discussed in the data analysis section, where we also address the used resources and inferences.

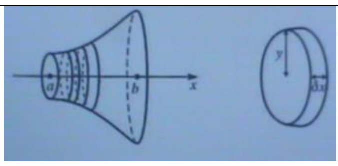
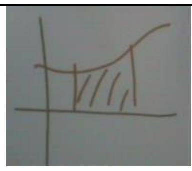
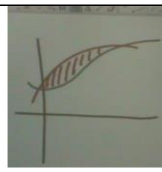
THE VOLUME OF REVOLUTION: THE THREE LESSONS

The data presented here are from three lesson observations. Each lesson was 50-minute long and taught to two mixed gender groups (here G1 and G2) of Year 13 students (17-18 years old). George was teaching the same topic, "volume of revolution", to G1 (first and third lessons) and G2 (second lesson). In the first lesson to G1, George started by asking the students about the formula for the area of a circle. He, then, used Autograph (a dynamic environment with visualising graphs affordances, see http://www.autograph-maths.com) to show the graph of $y=x(x\text{-}3)$, which he had pre-prepared. George rotated the graph to show the students that it was done in 3D mode. After that, he applied trapezium rule (which the students had seen before) on the area between the graph and the x-axis. He used a small number of divisions to show how the trapezium rule gives an underestimate of the area. Then, he used Simpson's rule (not known to the students at that time) to shade the area between the graph and the x-axis. He commented that this rule was more accurate and that the students were going to learn more about it in the next lessons. Afterwards, he rotated the shaded area around the x-axis, and he got a shape which he described as a "pointy sphere", a "Pacman", or a "smarty". Then, George opened another graph, also pre-prepared, this time of $y=\sqrt{x}$, and showed the students the rotation of the area between this graph and the x-axis, around the x-axis. After that, he tried to use Autograph to show the students slices of the solid on the screen and to lead them to the formula of the volume of revolution. After trying a few commands in the software, the demonstration was not clear and George did not seem satisfied, but he still kept trying to illustrate how the formula can be explained by using the graph on Autograph. Then, he moved to talking about who came up with the integration notations and explained that integration is like "sum", that was why the symbol for integration ($\int$) is like an (s) shape. His next step was to invite students to practice on questions from the textbook (Wiseman & Searle, 2005) when he spotted an illustration of the formula (Figure 1, ibid, p. 108). He asked his students to look at this illustration and he explained the formula again by using the image. Having done that, he started solving textbook questions on the board, explaining that he was starting with an "easy example" (Figure 2). Then, he displayed the formula sheet on the interactive whiteboard, which seemed to have reminded him that he had not explained the formula of the volume of revolution for rotations around the y-axis. So, he went quickly through this formula by advising the students to replace y by x and the dx by dy

in the initial formula. Next, he showed the students some past-exam questions and started solving one on the board. He also answered students' questions on the topic. At the end of the lesson, George gave the students paper copies of the formula sheet.

During the second lesson, to G2 this time, George followed similar steps to introduce the volume of revolution. However, in this lesson, instead of using Autograph to explain the formula for the volume of revolution, he did so by displaying on the board a pre- scanned copy of the textbook illustration (Figure 1). Another difference is that, in this lesson, he solved two examples from the textbook, the one he solved in the first lesson and another one. As a result, he did not have the time to solve past exam questions within the lesson, although he mentioned that students should solve some of these. A third difference was the additional example of $y = \sin(ax+b)+c$ he presented on Autograph. With this example, he used a, b and c to transform the graph; the rotation of which gave different shapes that seemed very impressive to the students. During this lesson, George recalled two questions asked by students in the first lesson and answered them. Towards the end of the lesson, he pointed out some questions in the textbook which were too difficult and exceeded exam requirement.

The third lesson was again for G1 where George devoted some time to quickly review the idea and formula of volume of revolution. He showed the same example used in the previous two lessons $y=x(x\text{-}3)$, and, then, used the textbook illustration to explain how the formula was deduced. He also used cards to remind the students of the formulae. He mentioned that there were two types of questions: "easy ones" (Figure 2) and "more difficult" ones (Figure 3). Then, he proceeded with a past-exam question solution and a presentation of its mark scheme. After that, he gave the students some time to solve questions independently until the end of the lesson.

 $\int_a^b \pi y^2 dx$		
Figure 1: Textbook illustration (Wiseman & Searle, 2005, p.108) and the formula for volume of revolution.	Figure 2: An "easy" question.	Figure 3: A "more difficult" question.

In an interview conducted four months later, George was invited to reflect on his way of teaching the volume of revolution. He said that he found the textbook diagram better than anything he could do on Autograph. He added that he liked using both the software and the textbook. He said that the software enabled him to show different shapes and added "fun" to the lessons. When using Autograph, he mentioned that he used familiar functions to reinforce students' previous knowledge. Specifically, he used $y = \sin(ax+b)+c$ to reinforce students' previous knowledge about transformations. He added that the use of past exam questions came in response to students' requests and needs to practice for the exam.

DATA ANALYSIS

The resources George used in these three lessons were: interactive white board; board, curriculum of year 13; textbooks; past exam questions and mark schemes; past teaching-experience; students' previous knowledge; calculators; notebooks; Autograph and pre-prepared graphs; formulae sheet; personal website; school website; and, formulae cards. Although the formula cards were on display next to the board all the time and shown to the students only in the second and third lessons while the formulae sheet was used in the first and second lessons, we would say that the resources stayed almost the same across the three lessons.

The schemes of use George followed during all three lessons had the same aims: a specific aim "teach students about the volume of revolution", and a more general one "to prepare students for the exams". In Table 1, we have summarised two elements of his scheme of use during the first lesson: rules of actions (numbered R1, R2… in the first column following the order of events during the first lesson), and operational invariant (numbered O1, O2… in the second column not in chronological order). In the second and third lessons, during which George introduced the volume of revolution to G2 and continued working on the topic with G1, the operational invariants stayed the same. Although, most of the rules of actions remained the same, we observed some differences in their appearance in George's teaching and in their order. In the rest of our analysis we focus on these differences in the rules of actions by making references to the R1-R20 in Table 1. We also discuss the inferences in George's scheme of use as those were identified in the three observations and the follow-up interview.

In the second lesson, George started by R15 (Table 1): "Use the formula sheet to show the formula", and then showed the textbook diagram (Figure 1) on the interactive whiteboard. After that, he followed R2-R8 by showing graphs of function and rotations of areas on Autograph, in a way similar to the way he followed in the first lesson. Then, he proceeded with R12: "Use the textbook diagram to explain the formula" without attempting to do R9 "Introduce the formula for volume of revolution using Autograph". Next, he followed R13-R19. In the third lesson, with G1, George continued working on the volume of revolution topic by quickly going though R2-R7, then moved to R12 followed by R19. In the last two lessons, R9 "Introduce the formula for volume of revolution using Autograph" was not a rule of action. In the interview, George commented on this by saying that the textbook diagram "show[ed] it cut up a little bit easier […] and [was] better than anything [he] could do on Autograph" (O5). As a result, George's first inference from the three lessons is that he found *the textbook diagram more helpful in explaining the formulae*. He added that he found it useful to have "both Autograph and the textbook". This leads us to his other inference: *it is useful to use both Autograph and the textbook as resources.* Another inference is related to *the functions entered on Autograph and how these were chosen to expand and build on students' previous knowledge* (O3).

Rules of action	Operational invariants
R1. Remind students of the formula of circle's area	O1. Autograph helps students visualise the volume of revolution in 3D
R2. Use Autograph to show the students the graph of $y=x(x\text{-}3)$ drawn in 3D	
R3. Connect new ideas with students' previous knowledge	
R4. Use trapezium rule and Simpson's rule on Autograph	O2. With Autograph we "can make weird shapes and have fun"
R5. Explain that Simpson's rule is more accurate than trapezium rule in this case	
R6. Rotate the shaded area around the *x*-axis	O3. Using a familiar graph helps "reinforce previous knowledge"
R7. Show the students different positions and rotations of the shaded area	
R8. Show another example prepared previously for the graph of $y=\sqrt{x}$	O4. Using pre-prepared graphs helps in focusing on the new topic and saves time
R9. Introduce the formula for volume of revolution using Autograph	
R10. Give an idea about the history of integration notation	
R11. Explain why integration is used to find the volume of revolution	O5. The textbook diagram works better than Autograph in terms of explaining the formula
R12. Use the textbook diagram to explain the formula	
R13. Solve an example from the textbook	
R14. Start with an "easy" question	
R15. Use the formula sheet to show the formula	O6. The use of exam-style questions is in response to students' needs, and to give them some practice for the exam
R16. Explain the formula for rotations around the *y*-axis	
R17. Give tips to the students	
R18. Answer students' questions	
R19. Use past exam questions to give students idea about how they are tested on the formula of revolution	
R20. Give hard copies of the formula sheet	

Table 1: Rules of actions and operational invariants for the first lesson's scheme

George commented during the interview on his choice of functions to graph on Autograph, and specifically his use of the sine function during the second lesson:

> Partly from using that in previous lessons. So, knowing that that is going to give an interesting shape, and from playing around with sine graphs and things like that in previous lessons. So, using functions that they were aware of […] So, a transformation of the sine curve I think, we were doing that with it. What I'm also doing there I am also reinforcing or going back over making sure that they know about their transformations. So, I'm kind of

> teaching two topics at once. So, although we are doing this volume of revolution, I am also reminding them of what they do when they do their transformations because I know they are going to get asked about that one

George also thought that using the sine function is "more interesting than using polynomials". However, he only showed the sine function in the second lesson and not in the first or third. It was not clear from the data collected whether he showed the sine function to the G1 in a different lesson, or whether he chose not to show it to them for any reason. However, in the interview he admitted explicitly that what he did with the sine function on Autograph was a good choice for the volume of revolution topic. In terms of the use of past-exam questions, George used these after solving one example from the textbook in the first lesson (R19). In the second lesson, he mentioned he was going to solve past-exam questions, but the lesson finished before he did. In the third lesson, he solved a past-exam question and explained its mark-scheme on the board. When asked about these choices, George mentioned that it was in response to students' needs that he now used past-exam questions frequently (O6). He added that students felt that not all textbook questions were exam-style questions, and some were even "more difficult" than exam questions (which is something he pointed to in the second lesson). It was also because he wanted to give his students some practice for the exam. As a result, he chose to use past-exam questions for every topic he taught. Finally, we noted that George did not have the time to solve past exam questions in the second lesson, maybe because he chose to solve two textbook examples although this was not evident in the data, which do not indicate the warrant of this choice.

DISCUSSION

The resources George used stayed the same throughout the three lessons. However, we noticed differences in the way these resources were used. George's experience with Autograph in the first lesson, led him to amend the way he used it in the next lessons by deciding to use it for visualization of the concept of volume of revolution, but not to explain the formula. Hence, based on the experience of the first lesson which became a resource for George in the following lessons, we noticed his inference in relation to the textbook diagram (Figure 1) being preferred for the purpose of explaining the formula for the volume of revolution. Also, we observed some variation in the order of the rules of actions between the three lessons, reflecting the interplay between Autograph and the textbook. In terms of the use of past-exam questions, George considered these an important resource for every topic. During the interview, it was not possible to focus on every change or difference from one lesson to another (e.g. not using the sine function in the first and third lessons) because the interview was done a few months after the observations and this is one of the limitations of this study. In general, from the data collected and by using the documentational approach we explored how George's practices evolved, how he reflected on that, how he re-sourced his experiences, and what inferences he adopted during and after teaching these three lessons. Findings from our analysis demonstrate the potencies of the documentational approach in our insight into teachers' 'live' work by capturing also the dynamic nature of this work. Observations of lessons supported by further evidence from interviews and reflections from the teachers can explore the micro-evolution, namely the small changes and the rationale behind these changes, of teachers' documents from one lesson to another. We consider this micro-evolution in this instance as re-scheming from one lesson to another, implying that the teacher was scheming "again or differently" (Adler, 2000, p. 207) or recycling his scheme from one lesson to another.

Acknowledgements

This research is funded by a studentship from the Faculty of Social Sciences, University of East Anglia.

References

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education*, *3*(3), 205-224.

Gueudet, G. (2017). University teachers' resources systems and documents. *International Journal of Research in Undergraduate Mathematics Education*, *3,* 198-224.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71*(3),199-218.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the Learning of Mathematics, 23,* 2-14.

Kayali, L., & Biza, I. (2017). "One of the beauties of Autograph is … that you don't really have to think": Integration of resources in mathematics teaching. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Conference of European Research in Mathematics Education* (pp. tbc). Dublin, Ireland.

Lerman, S. (2013). Theories in practice: Mathematics teaching and mathematics teacher education. *ZDM, 45*, 623-631.

Nardi, E., Biza, I., & Zachariades, T. (2012). 'Warrant' revisited: Integrating mathematics teachers' pedagogical and epistemological considerations into Toulmin's model for argumentation. *Educational Studies in Mathematics, 79*(2), 157-173.

Wiseman, G. & Searle, J (2005). *Advanced Maths for AQA: Core Maths C3 and C4*. Oxford: Oxford University Press.

Stake, R. E. (2010). *Qualitative research: Studying how things work*. New York: Guilford Publications, Inc.

SUPPORTING PRESERVICE TEACHERS' IN-THE-MOMENT NOTICING

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The purpose of this paper is to discuss the factors influencing preservice teachers' in-the-moment noticing that emerged from a research on noticing skills. We set up an environment where preservice teachers worked with a group of students on mathematical tasks for 24 weeks. We focused on the mathematical opportunities occurred during task implementations and how preservice teachers respond to those opportunities as an indicator of their noticing skills. We analysed their interactions with students and coded their responding actions according to our coding scheme. We recognized that not only preservice teachers' knowledge or noticing skills but other factors such as research setting, students' prior knowledge and tasks were likely to influence how they responded to students' mathematics.

INTRODUCTION

As parallel to studies on improving quality of education, training programs for both in-service and preservice teachers (PSTs) are reformed to justify such needs of current system (Cochran-Smith & Villegas, 2015). Because teachers' knowledge and skills have an impact on students' achievement, which in turn influence the quality (Campbell et al., 2014), scholars investigate for effective ways of improving PSTs' professional knowledge and skills. Recently, many studies are conducted on teachers' noticing skills not only because there is a mutual relationship between noticing and teachers' pedagogical content knowledge (PCK) (van Zoest et al., 2017) but also it refers to the "guidance" or "facilitator" role of a teacher in a student-centred classroom environment such that she tries to elicit students' thinking and support their understanding (van Zoest et al., 2017).

The studies on noticing are mostly based on analysis of own or others' videos of teaching in terms of oral or written reflections (e.g., Barnhart & van Es, 2015). As different from such studies, in this study we arranged an environment that enabled us to collect data about PSTs' in-the-moment noticing while they were working with a group of four students on the tasks prepared by the research team. We videotaped each PST's interactions with students and analysed PSTs' actions according to our coding scheme. We also asked PSTs to reflect on each task implementation both orally and written. We attempted to triangulate all data to understand the nature of PSTs' noticing skills. In this paper, we will discuss some issues which were likely to influence PSTs' in-the-moment noticing.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 203-210). Umeå, Sweden: PME.

THEORETICAL FRAMEWORK

In this study we used Jacobs and his colleagues' (Jacobs, Lamp, & Philipp, 2010) definition of *professional noticing of students' thinking* to describe PSTs' noticing skills. They defined noticing as having three interrelated components as attending to students' strategies, interpreting their understanding and deciding how to respond to students. Although these components can be easily differentiated in oral or written reflections of teachers, it is hard to make such a distinction while analysing in-the-moment noticing. Because interpreting students' understanding is a mental process, it may not be identified explicitly unless the teacher interprets students' strategies aloud during the instruction. Therefore, we assessed PSTs' interpretation of students' understanding within their responding actions.

Although Jacobs and his colleagues' (Jacobs, et al., 2010) definition provides an idea of what noticing involves in, it is difficult to decide what is noteworthy to attend in students' strategies or thinking. Leatham and his colleagues (Leatham, Peterson, Stockero, & van Zoest, 2015) defined *Mathematically Significant Pedagogical Opportunity to Build on Student Thinking* (MOST) to address such difficulty in noticing studies. They described a MOST instance as a composition of three sequential components such that it should be emerged from student's mathematical thinking, be mathematically significant and be a pedagogical opportunity. They noted that students' misconceptions or incomplete reasoning might be a MOST instance as well as their correct answers based on use of different strategies or approaches. However, such correct or incorrect answer of a student should be in the context of that particular lesson and have a potential to support other students' understanding of the current subject.

Because MOST provides more tangible impetus for the initial step of noticing, in this study, we analysed PSTs' in-the-moment noticing in terms of how they noticed the MOST instances occurred during task implementation. Therefore, we first identified MOST instances occurred in each video and then we analysed whether the PST attended to that instance or not and if she did, how she responded to that instance.

METHODOLOGY

Participants

A total of 10 preservice mathematics teachers participated in this study in 2016-2017 academic year. Seven of them attended to study for two semesters while others attended only one semester. They were all undergraduate students such that four of them were sophomore (Asya, Aydan, Ayla, Aysun) four were junior (Bahar, Berna, Beste, Burak) and others were senior (Ceren, Ceyda) students. Although the number of courses they took was varying, they had already taken some core mathematics and pedagogy courses such as calculus and educational psychology.

Research Setting

The study was conducted under a university-school collaboration program between a large university in Turkey and a local middle school in the neighbourhood of the university. In the line of the collaboration, as the research team, we took the responsibility of administration of an elective mathematics course offered for the seventh grade students in the school. We also offered an elective course for PSTs in the university such that PSTs were expected to implement mathematical tasks in that elective course.

At the beginning of the semester, we talked about tasks, task design and implementation, students' common misconceptions, and effective ways of understanding and supporting students' mathematical thinking. We discussed these issues via some sample videos and student work that we had in our repertoire from our earlier studies. After one-month preparation period, we assigned a group of four students for each PST that they would work with in the school for a year. We attempted to make heterogeneous groups based on the test results we administered to students.

In the school, we followed a 3-step process for task implementation. At first, PSTs introduced the task to their groups and let them work on the task individually approximately for 20 minutes. The PSTs were not allowed to intervene but take some notes about students' work at this step. After individual work, students were asked to discuss their answers as a group. They were allowed to change their answers during the group discussion. As a final step, after group discussion, PSTs began to interact with students. They were supposed to address issues that they noted during the individual work or group discussion. They were told not to explain the correct solution immediately but allow students to figure out or convince each other about the correct answer. As students were discussing their answers with PSTs, they were not allowed to change their answers but write their new solution to a separate piece of paper. Each task implementation process was videotaped and students' worksheets and extra sheets were collected at the end.

After each task implementation, we immediately met with PSTs for oral reflection and talked about how the implementation went, how students performed on the task, whether they had been some unexpected events, how they addressed to students' mistakes or misconceptions. Then we asked them to write a reflection based on their videos and students' worksheets. Two days after the implementation we met with PSTs to discuss the following week's tasks.

As the research team, we prepared 20 tasks such that 5 of them were about numbers, 7 of them were algebra, 5 of them were geometry and 3 of them were data and statistics. We also asked PSTs to prepare at least one task for each content area for their own groups and implement them. We used 2 lesson hours (approx. 80 min.) for each task implementation and we spent a total of 24 weeks in the school.

Data Collection and Analysis

We collected data through videos, written documents, achievement tests, and belief scales. We videotaped all PST-student interactions and reflection sessions. We applied a PCK test to PSTs and a mathematics achievement test to the students. We attempted to learn about students' beliefs about mathematics and also PSTs' beliefs about teaching mathematics through Likert-type scales. We also collected PSTs' written reflections, their assignments and students' written work on tasks.

We used videos of task implementations to describe PSTs' in-the-moment noticing. However for a comprehensive evaluation of PSTs' noticing skills, we also used PSTs' oral and written reflections, their assignments and their PCK-test results. We developed a coding scheme to analyse PSTs' noticing. As similar to coding frameworks suggested in the literature (e.g., van Es & Sherin, 2002; Barnhart & van Es, 2015) we paid attention to the level of sophistication in PSTs' reflections and actions. However, we classified PSTs' in-the-moment responding actions under two categories as *answer-focused* and *understanding-focused* such that PSTs' attempts to *explain* the solution or *orient* students towards to correct answer through questioning are classified under answer-focused while their attempts to *explore* or *elaborate* students' thinking and understanding are classified under understanding-focused. We shared the responsibility of analysing each PST's data and then we met in weekly basis and discussed our coding. Thus, we justified agreement on coding of each PST's data.

As a part of our analysis we also paid attention to some factors which were likely to influence PSTs' in-the-moment noticing. Because there are not much studies on in-the-moment noticing, we decided to discuss these factors to shed light on further studies as well as interpret our findings in the line of such constraints.

FINDINGS

The overall aim of our research was to investigate the nature of PSTs' noticing skills. As we were trying to understand PSTs' noticing skills we also looked for what were likely to influence their noticing as well as how it would affect students' learning and understanding. We have already analysed the tests and the scales given to PSTs and students however we completed the analysis of videos and written reflections of 10 tasks implemented during the first semester. We recognized that some factors related to PSTs, students, tasks, and implementation process were likely to influence PSTs' noticing and responding actions.

Preservice Teachers' Knowledge and Skills

The participant PSTs were varying in terms of number of mathematical and pedagogical courses they took so far as well as their teaching experiences in the form of tutoring. Indeed, PSTs' ability to recognize MOST instances, decide how to act and manage a group work are related to their PCK (van Zoest et al., 2017). Because they had not taken much courses that would feed their PCK it was not surprising that they differed in terms of MOST instances they attended and their responding actions as well as test scores. In Table 1, PSTs' pre and post test scores and the frequencies of how they attended to MOST instances are given.

As a quantitative measure, we developed and administered a 21-item PCK test to PSTs at the beginning and at the end of the study. At the beginning of the study, out of 47 points, and amongst whom attended two semesters, Aydan got the lowest score as 20 and Bahar got the highest score as 28. Aydan was a sophomore where Bahar was a junior student. Bahar had one-year tutoring experience whereas Aydan did not have any. At the end of the study, we obtained an improvement in PSTs' PCK scores such that Wilcoxon signed rank test revealed that the difference between tests was significant at level .005 (p=.018).

			MOST		
PST	**Pre**	**Post**	**Missed**	**Attended Answer-focused**	**Attended Understanding-focused**
Asya	24.5	26.5	3	21	1
Aydan	20	34.5	5	14	
Ayla	27.5	33.5	4	19	
Aysun	22.5	35.5	3	16	1
Bahar	28	34.5		25	2
Berna	24.5	34.5	3	19	
Beste	25.5	30		22	
Ceren	26	30	5	22	1
Ceyda*	34.5	35.5	--	--	--
Burak*	26.5	29.5	--	--	--

Table 1: PSTs' test scores and distribution of MOST instances. *They participated in the study during the second semester.

As a qualitative measure we analysed PSTs' implementations and reflections. We identified a total of 186 MOST instances during the first semester. PSTs missed 23 of them such that Aydan missed 5 of her MOST instances while Bahar did not miss any of her MOSTs. We recognized that sophomore and junior PSTs paid more attention to make students learn from each other instead of intervening their discussions. Out of 163 MOST instances attended, in 158 of them PST used answered-focused responding actions rather than using probing questions to elicit students' thinking. There were cases where one of the students gave the correct answer and others unquestionably accepted it. For instance, in one of the tasks about integers some students did the following -180-90=-90 while others calculated the correct answer as -270. The students convinced their peers by saying that "when there are two minus signs then they would add the numbers." The PSTs did not force those students to explain their reasoning for -90 neither during group discussions nor during interactions. When they began to interact with students they mostly attempted to make students to figure out correct answer by giving reference to that group discussion.

Another issue related with PSTs' PCK as well as their noticing was having tutoring experience or not. Except Aydan, Ayla and Ceren, all PSTs had some sort of tutoring experiences with a single student or a large group of students. Such an experience contributed to their repertoire of students' difficulties and misconceptions in mathematics which was likely to influence their ability of attending to the MOSTs. For instance, as noted above, Bahar had one-year tutoring experience and she did not miss any of her MOSTs. However, while tutoring, the PSTs' major concern was to explain the content and solve the problems rather than support students' conceptual understanding. Because they were used to interact with students in that way, their reactions to MOST instance were likely to be in answer-focused fashion. For instance, out of her 27 MOST instances in 25 of them, Bahar attempted to either guide students to correct answers through questioning or she explained the solution of the task.

Students' Prior Knowledge

Students' prior knowledge was one of the determinants of both occurrence of MOST instances and also style of PST-student interaction. Students' misconceptions and incorrect procedures are source of MOST instances as well as their alternative solutions for given problems (van Zoest et al, 2017). The achievement level of the students that participated in this study was not high as inferred from the test results and students' work. Out of 50 points, the mean of pre-test was 9.28 and of post-test was 23.98. Therefore, almost all MOST instances (183 instances out of 186) we identified emerged from student's misconceptions or incomplete knowledge and thinking. Students' such lack of knowledge shaped PSTs' responding actions for MOST instances. Because students did not know the basis of the content that had been discussing nor its connections with other concepts, they failed to find the correct answer or explain their reasoning for their answers. Thus, PSTs' attempts for exploring and elaborating students' mathematical understanding did not work for many students when the MOST instance was based on students' misconceptions. Then, PSTs gave up asking for probing questions but make students to figure out or understand the correct answer or solution. That is, although PSTs wanted to address students' conceptual understanding as well as their procedural knowledge, because of students' lack of knowledge, they displayed one of the answer-focused responding actions instead of understanding-focused actions, as seen in Table 1.

Nature of Tasks

In this study, we used tasks as a medium for PST-student interactions such that we analysed PSTs' noticing in terms of the MOST instances occurred during the implementation of these tasks. We prepared the tasks in aligned with mathematics curriculum to provide students an opportunity to apply and explore the concepts that they had learned in their regular mathematics lessons. Therefore, the source and the number of MOST instances that we observed varied. For instance, in one of the integer tasks we made students to step up and down on stairs and then write the operations mathematically because we aimed to make students to discover operations with integers by relating them with real life situations. We observed that some of the students wrote -1-(-2)=-3 to represent "As getting 2 steps down from step #-1 you will reach step #-3." Although they recognized that the answer would be -3, they placed "-" sign for 2 because problem says "steps down". Indeed, it was one of the MOST instances that we expected to be occurred in this task. Thus, the PSTs attempted to respond such MOST instances during their interactions with students. In some of the tasks, we wanted students to make practice of some mathematical principles and procedures. For instance, in geometry tasks we asked students to calculate the area of given geometric figures. Because they needed to apply correct formula, we did not observe so many MOST instances in such tasks. In parallel to such differences in the nature of the tasks, PSTs' responding actions for those MOST instances also varied. For instance, for the integer task they tried to support students' conceptual understanding of integers but for geometry task their focus was to make students to apply correct procedure and find the answer.

Task Implementation Procedure

As mentioned above we asked PSTs to follow a 3-step implementation process: individual work, group discussion and PST-students interaction. We also asked them not to intervene during the individual work or group discussion but monitor students and encourage them to join in group discussion. The purpose of group discussion was to support students' collaborative work and peer learning. However, in some cases, students addressed to MOST instances during group discussions such that one of the students explained the procedures or told about the correct answers. Then, PSTs either preferred to ask students to repeat the correct answer or make a summary of what was discussed in the group or announce that it was discussed in the group so skipped to another instance. There were a few cases where PSTs asked for students' thinking about the MOST instance occurred before the group discussion but was corrected by the student then. Even though it was the case, the student did not want to tell much about their earlier thoughts but explained the correct answer.

Furthermore, there were some cases where PSTs could not manage the group such that they devoted too much time for group discussion. Then, either there was no room for PST-students interactions or PSTs could not discuss all MOST instances. We recognized that they were aware of those instances, since in their reflections they noted that they could not respond to some instances because time was up. Out of 23 missed MOSTs, they mentioned 9 of them in their written reports.

CONCLUSION

To increase the quality of education we need to equip teachers with various knowledge and skills that enable them to create rich learning environment for their students and also support their understanding and thinking (Campbell et al., 2014). In this study, we focused on PSTs' in-the-moment noticing skills such that we tried to understand the nature of their noticing in terms of whether or not they attended to the MOST instances and if so, how they responded to those instances. Although we tried to provide similar conditions for PSTs, we recognized that their academic background and experiences, the tasks used during the implementations, students' performances on tasks and the implementation procedure were likely to influence their noticing and responding actions. A variety in PSTs' background is acceptable and might be required to understand the nature of PSTs' noticing skills. However, students' lack of knowledge was an inhibitor for rich PST-student interactions such that PSTs were unable to encourage students to explain their thinking and reasoning. Thus, they mostly responded to students in terms of having them to figure out the correct procedures and solution. Furthermore, some tasks were procedural such that there was a little room for potential MOST instances and also PST-student interactions. Therefore, rich tasks should be used to investigate the nature of PSTs' noticing. Finally, to evaluate how a PST responded to a student when a MOST instance was observed in her work, other students should not be let to intervene. Otherwise, the student gave up explaining her reasoning or solution. Briefly, while setting up an environment for investigating noticing skills, the students, the tasks and the procedure should be determined cautiously.

Acknowledgment

This research was supported by The Scientific and Technological Research Council of Turkey (TUBITAK, Grant no: 215K049).

References

Barnhart, T. & van Es, E. (2015). Studying teacher noticing: Examining the relationship among pre-service science teachers' ability to attend, analyze and respond to student thinking. *Teaching and Teacher Education, 45,* 83-93.

Campbell, P. F., Nishio, M., Smith, T. M., Clark, L. M., Conant, D. L., Rust, A. H., Choi, Y. (2014). The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement. *Journal for Research in Mathematics Education, 45(4),* 419-459.

Cochran-Smith, M., & Villegas, A. M. (2015). Framing teacher preparation research: An overview of the field, part I. *Journal of Teacher Education, 66(1),* 7-20.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41(2),* 169-202.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46(1),* 88-124.

Van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education, 10(4),* 571–596.

Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student mathematical thinking that are worth building on in whole-class discussion. *Mathematical Thinking and Learning, 9(1),* 33-54.

CORE MATHEMATICAL TEACHING PRACTICES IN ALGEBRAIC AND FUNCTIONAL RELATIONS

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Teaching and learning algebra in school mathematics is challenging. The purpose of this study is to explore the core classroom practices that support students' development of algebraic thinking. By analysing 5 video-taped lessons of two middle school mathematics teachers who were in different career stages in Korea, we began identifying the essential aspects of classroom practices that create rich learning opportunities for algebraic thinking and that support students' access to those learning opportunities. Our findings also discuss what to consider for Korean mathematics teachers to support their students to develop agency, authority and identity in learning middle school levels of algebra.

INTRODUCTION

What classroom practices are more effective to promote students' conceptual development in algebra and algebraic thinking? Algebra has been characterized as a gateway to higher mathematical learning and success in the 21st century (NCTM, 2000). However, teaching and learning algebra in school mathematics is challenging. Although curriculum developers and educational researchers are beginning to explore the kinds of mathematical experiences that elementary and middle school students need in order to prepare them for the formal study of algebra in later grades (Carpenter, Franke, & Levi, 2003), there remains a noticeable disconnect between the research on the learning of school algebra and the research on the teaching of algebra; researchers still know relatively little about algebra teaching (Kieran, 2007). It's not yet clear how teachers can promote students' conceptual development, especially on the topic of algebra. The purpose of this study is to explore the core classroom practices that support students' development of algebraic thinking by looking at two expert teachers' lessons. Knowledge of instructional practices in one country may help teachers in another country ability to address the issues and challenges that hinder their students' learning of mathematics. Our study addresses the following research questions: (1) what are the characteristics of classroom practices for promoting students' development of algebraic concepts and practices in a Korean context?; (2) what kinds of learning opportunities are provided to support the students' development of algebraic concepts and practices?; and (3) how are students supported to access to those learning opportunities?

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 211-218). Umeå, Sweden: PME.

THEORETICAL FRAMEWORK

Teaching and Learning Algebraic Thinking

While algebra has been considered as a gateway to higher mathematical learning and success in the advanced learning, national assessments have discovered that many students struggle with higher level algebraic problems. They often have difficulties with translating from verbal to symbolic, communicating their reasoning, and justifying their methods. The importance of algebraic thinking is related to mathematical practices. Two types of teaching algebraic thinking that need to develop include: (1) engaging in algebraic representations with concreteness fading strategy, and (2) modelling from contextual algebraic problems.

First, the present study is driven by a concreteness fading method by focusing on algebraic representations. Concreteness fading is built based on Piaget's (1952) and Bruner's (1966) learning theories that emphasize the notion of transition from concrete to abstract representations. Both Piaget and Bruner suggest that students need to begin learning with concrete materials or visual representations and then proceed with abstract representations. Goldston and Son (2005) defined concreteness fading as the process of successively decreasing the concreteness of a simulation with the intent of eventually attaining a relatively idealized and decontextualized representation that is still clearly connected to the physical situation that it models. The previous research has proved this concreteness fading to be an effective method for learning scientific principles and mathematical rules (McNeil & Fyfe, 2012).

We also look at contextual algebraic problems building on Schoenfeld and the Teaching for Robust Understanding Project (2016). Schoenfeld and his colleagues articulate "robustness criteria" for contextual algebraic tasks that include: (1) reading and interpreting text, and understanding the contexts describe in problem statements; (2) identifying salient quantities in a problem and articulating relationships between them (3) Using algebraic representations of relationships; (4) Executing algebraic procedures and checking solutions; and (5) Explaining and justifying reasoning. While algebra is commonly defined as simply a form of "doing" instead of a "way of thinking", it is important to examine the opportunities to engage in these algebraic practices with the aforementioned criteria.

The Essential Aspects of Mathematics Teaching: Focusing on Student Learning Opportunities

There has been significant research and discussions on what makes good mathematics teaching and how we can characterize it internationally. There are various ways and factors that teachers make good mathematics *teaching*, as seen in many studies with various foci, such as teacher knowledge (e.g., Ball, Thames, & Phelps, 2008) or teaching practice (e.g., Mathematical Quality of Instruction, MQI: University of Michigan, 2006). These kinds of research have contributed to our understanding on the essential aspects of teacher knowledge and the act of teaching. However, teaching is a cultural activity (Stigler & Hiebert, 1999) so that the act of teaching may not be considered as absolute. There are many different moves and ways to make good teaching—which is, guiding students to learn. Thus we attempt to explore the opportunities to learn for students to develop algebraic concepts and to engage in mathematical practices particularly of algebraic thinking. To do so, we use Teaching for Robust Understanding of Mathematics (TRU Math) framework (Schoenfeld & the Teaching for Robust Understanding Project, 2016) as the TRU Math framework provides essential perspectives on students' learning opportunities (Schoenfeld, 2018) and how students access those opportunities. Also, as our data in this study is from Korean classrooms, TRU Math was successfully applied to see the essential features of learning opportunities in Korean classroom culture in the previous study (Kim, 2017).

TRU Math framework provides five dimensions: (1) the mathematics—the extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers; (2) cognitive demand—the extent to which students have opportunities to grapple with and make sense of important disciplinary ideas and their use; (3) equitable access to content—the extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core disciplinary content being addressed by the class; (4) agency, ownership, and identify—the extent to which students are provided opportunities to "walk the walk and talk the talk" to contribute to conversations about disciplinary ideas, to build on others' ideas and have others build on theirs, in ways that contribute to their development of agency, ownership, and the positive identities as thinkers and learners; and (5) formative assessment—the extent to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. We basically follow these five dimensions as our perspectives on seeing learning opportunities in our classroom observation data, but we particularly focus on algebraic thinking in more details when we see the first dimension, the mathematics.

METHODS

Research Contexts, Participants and Data Collection

This study is drawn upon a larger funded cross-cultural project. The purpose of the project is to identify the core mathematical knowledge and practice for teaching algebraic and functional relations. In this article, we focus on two teachers' classrooms. The first participant (Ms. L) was a 7th grade math teacher (15 years of middle school teaching experience) in the first author's University affiliated middle school and was recommended to participate in this study by her school principal. The second participant (Mr. K), a 7th grade teacher (31 years of middle school teaching), was a master teacher in Seoul, Korea and he was well-known in his school district. He used an alternative textbook that his textbook development team created. According to his team, the goals of the alternative textbook aimed at exploring which textbook supports student mathematical thinking and mathematical practices rather than pieces of knowledge, and helps students' self-discovery or self-invention of mathematical concepts. The teachers were interviewed and their everyday teaching and researched lesson teaching were observed and videotaped.

Main data sources for the study were 5 video-taped lessons: two lessons from Ms. L (the very first algebra introductory unit—introducing letters to represent variables—, and the very last unit of linear equations) and three lessons of linear functions unit from Mr. K. Teacher interviews were used as supplemental in this study.

Analysis

All the video-taped lessons were transcribed and chunked as episodes to be analysed. We firstly identified each teacher's focused algebra specific topic(s) and algebra specific learning opportunities. Here, we provide algebra specific learning opportunities from Mr. K's lessons as he used contextually rich algebra textbooks. We, then, characterized their classroom practices to describe how the learning opportunities were accessed with respect to TRU Math framework (see Schoenfeld et al., 2016 for more detailed information).

FINDINGS

We found that two teachers provided cognitively demanding learning opportunities by using contextually rich algebraic tasks. Most of classroom activities supported meaningful connections between procedures and concepts by allowing students to explain their ideas and reasoning. We also observed that two teachers structured classroom activities by emphasizing concreteness fading. In this section, we firstly describe the learning opportunities from contextually rich algebraic textbooks from Mr. K's cases. Then, we characterize both teachers' classroom practices with foci of similarities and differences in terms of the five essential aspects of mathematics teaching using TRU Math framework to describe how students in each classroom were supported to access to those learning opportunities. Due to the page limits, we provide the descriptions of two teachers' classrooms, but we will provide actual video data as the evidence for our analytical descriptions at the PME 42 session. The following subheadings are represented the core classroom practices of two Korean teachers.

Algebra Specific Learning Opportunities

The topic of Mr. K's observed lessons focused on a linear function, and he used the alternative textbook consisting of various contexts for development concepts and mathematical thinking process. Mr. K created and provided learning opportunities using the contextual problems in the textbook as follows.

First, Mr. K facilitated students to *interpret and understand contexts by analysing the contextual situations and helping them link the contextual scenario to linear relationships*. At the beginning unit of a linear function, Mr. K provided a contextually rich task which was a part of news article describing the relationships between heat waves and the number of deaths. Mr. K analysed the article to help students understand what the article said and helped students link the contextual scenario to linear relationships.

For example, he elicited students to seek how they could anticipate the number of deaths several years later when we assumed that the heat waves increased every year in a constant rate. Students started to discuss what they noticed and how they interpreted with regards to the rate of change. Second, Mr. K provided opportunities students to *identify quantities from the contextual scenario* and *to articulate relationships between them by making students discuss the quantities and their relationships*. He firstly asked students to write down what they found from the analysis of contextual problems, then asked them to talk about the relationships between the quantities and variables. For example, students were asked to articulate the rate of change of one variable when another variable changes constantly. Third, Mr. K helped students generate algebraic representations of the relationships between quantities using variable notations of x and y. Students had opportunities to discuss the algebraic notations of linear functions using x and y based on what they learned from the lesson unit of linear equations. Mr. K also provided another algebraic notion of a function, $y=f(x)$. Last, Mr. K supported students to *make connections between representations and concepts*. Students had opportunities to link concepts between prior concept of direct proportion and the current concept of linear function. They also had opportunities to talk about how the different algebraic notions and concepts of functions such *as* $y=ax+b$, $y=f(x)$, and functions as correspondence and as relationships. He used concreteness fading strategy to support students' conceptual development. He started with careful explanations of contextual situations and helped students to analyse the situations with their own words. He helped students create number line and a coordinate plane, which are visual representations, then students had opportunities to make connections among those representations and to introduce and create abstract algebraic notions.

Meaningful Mathematics Learning Opportunity

Both teachers' classroom activities were very focused on the targeted mathematical ideas and concepts. Korean mathematics lessons usually began with the review of previous lesson concepts, and this activity provided students opportunities to build on their prior knowledge and to make connection between procedures, concepts and contexts. Mr. K and Ms. L also provided this opportunity at the beginning of the lessons. During the main body of lesson, Mr. K's lesson structure had mostly whole class discussion but he also used small group activities when he thought students need to discuss around the tasks and questions he asked, particularly when students' thinking needed more elicited and their talks needed to be elaborated. Ms. L planed small group activity structure and she followed her lesson plan as she usually used group work tasks that she created and reorganized textbooks and lesson materials. In any structure, the main mathematical activities of both teachers' lessons and the tasks that they used were very focused and students had opportunities to think conceptually and to discuss the focused concepts in the tasks and in teacher questionings. Students also engaged in procedural process types of problems and this activity was also very focused around the concepts.

High Levels of Cognitive Demands

For this dimension, Mr. K's lessons mostly maintained high levels of cognitive demands while Ms. L's lessons showed different degrees of richness across episodes. Mr. K continuously supported students to engage in productive struggles with his team's invented textbook materials that contained a lot of contextual concepts and problems. His questioning pushed students to think further around the concepts and to reason abstractly based on the concrete contexts. On the other hand, some of Ms. L's lessons maintained high cognitive demands levels, but others showed middle or even low levels. For example, her group work tasks required students to create a poster using 16 cards (8 problem cards, and 8 solution cards) as a review of linear equation unit and they provided opportunities students to engage in productive struggles. Students were required to solve the equation problems and to match the cards with the solution cards, and to present and explain what they did. However, the teacher's guidance somewhat removed students' productive struggles by asking procedural process types of questions or just explaining what she thought rather than asking thinkable questions. However, we found that most tasks that both teachers provided students contained high level of cognitive demands for conceptual development, but the questioning strategies were shown differently across two teachers.

Structured Access to Mathematical Contents

Both teachers supported all students to engage in meaningful mathematical activities and to participate in learning opportunities. Both teachers' classroom norms were established as all students to engage in the mathematical discursive activity. Although there were uneven participations during whole class structure, Mr. K gestured or called unengaged students in a comfortable atmosphere to participate in classroom activities and discussions. Ms. L's classroom norms were more established as all students who were in different understanding levels shared their ideas. In particular, students were in low level of understandings asked their peers what they didn't understand during small group works. We thus coded most of the two teachers' lessons as level 3.

Limited Degrees of Students' Agency, Authority and Identity

Both teachers seemed to encourage students to discuss the targeted topics. However, Mr. K's lesson episodes were coded mostly as medium levels and sometimes high levels, while Ms. L's lesson episodes were coded mostly as low levels and sometimes medium levels. Both teachers' classrooms were not yet a very productive discourse community as the discursive interactions were more likely teacher initiated and students answered and they did not have opportunities to build on each other's ideas. However, Mr. K provided students more chances to elaborate that they thought so that they had opportunities to explain their ideas and reasoning. It helped students have sense of mathematical doers and thinkers, and form positive mathematical identities.

Limited Use of Formative Assessment

Mr. K's lessons were coded mostly as medium levels and sometimes high levels while Ms. L's lessons were coded mostly as low levels and sometimes medium levels. Ms. L quickly fixed the students' errors and misunderstandings when they were raised and her feedbacks were very directly rather than having a room for students to think further. On the other hand, Mr. K asked students to notice their errors and partial understandings with persistent questions when they were raised.

CONCLUSION AND DISCUSSION

As the beginning of investigating the big research agenda—what are the core classroom practices supporting students' algebraic conceptual development—, we characterized the two Korean middle school teachers' algebra classrooms with a focus on the quality of learning opportunities and how the opportunities were accessed. We also began identifying the learning opportunities to develop algebraic concepts and algebraic thinking using Mr. K's algebraically context rich classrooms as a case study. He provided rich opportunities students to investigate the contextualized scenarios and to understand the basic concepts of functions and linear relationships at the beginning of the lesson units. Mr. K had students fully engaged in understanding of the basic concepts and objects by providing opportunities students to analyse the situations, to identify the quantities to use, to articulate the relationships between quantities, notions and concepts, and to link the concepts and representations. These opportunities for algebraic concepts and practices were analysed as high levels of contents and cognitive demands. Most students were also encouraged to access these rich opportunities in both teachers, but students in Mr. K's classroom had more opportunities to develop their agency, authority and identity. Our findings imply that Korean teachers had already strong knowledge of mathematics and competency of reorganizing contents to teach and guide students appropriately. Now, it is time for them to have more opportunities to consider and discuss their students to develop their positive identity as mathematical thinkers and doers so that they can see themselves as having ideas worth to listen by others and being able to contribute to the mathematical learning in a classroom.

Acknowledgement

This works is supported by Dr. Ji-Won Son's Spencer Small Grant Number: Award 76669.

References

Ball, D. L., Thames, M. H., & Phelps G. (2008). Content knowledge for teaching: What makes it special*? Journal of Teacher Education, 59,* 389-407.

Bruner, J. S. (1966). *Toward a theory of instruction*. Cambridge, MA: Belknap Press of Harvard University Press.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *Journal of the Learning Sciences, 14*(1), 69–110.

Kim, H. (2017). Teacher Learning Opportunities Provided by Implementing Formative Assessment Lessons: Becoming Responsive to Student Mathematical Thinking.

McNeil, N. M., & Fyfe, E. R. (2012). “Concreteness fading” promotes transfer of mathematical knowledge. *Learning and Instruction, 22*,440e448.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Piaget, J. (1952). *The child’s conception of number.* London, Great Britain: Routledge & Kegan Paul

Schoenfeld, A. H. & The Teaching for Robust Understanding Project. (2016). *An introduction to the teaching for robust understanding (TRU) framework*. Berkeley, CA: Graduate School of Education. http://map.mathshell.org/trumath.php

Schoenfeld, A. H. (2018). Video analyses for research and professional development: the teaching for robust understanding (TRU) framework. *ZDM*, 1-16.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world’s teachers for improving education in the classroom.* New York: The Free Press.

University of Michigan (2006). *Learning mathematics for teaching. A coding rubric for measuring the mathematical quality of instruction* (technical report LMT1.06). Ann Arbor, MI: University of Michigan, School of Education.

TEACHER CAPACITY FOR PRODUCTIVE RESOURCES USE

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By compiling the analyses of the data from elementary teachers using a range of mathematics curriculum programs in the United States, this paper elaborates teacher capacity needed for productive resource use. The capacity elaborated in this paper includes (1) articulating mathematical points of the lesson and steering instruction toward the mathematical points, (2) recognizing affordances and constraints of the resource, (3) using the affordances of the resources, and (4) filling in the holes and gaps in the resources. Each of these aspects is explained along with examples from the data and related literature. This paper also discusses the need of nurturing proper operational invariants in teacher education (teacher preparation and professional development) and the role of resources in increasing teacher capacity.

INTRODUCTION

In this paper, I describe teacher capacity needed for using existing resources productively, based on a set of analyses of the data gathered in the *Curriculum Use for Better Teaching* (ICUBiT) project. In fact, the goals of the project were to identify components of the capacity that Brown (2009) calls *Pedagogical Design Capacity* (PDC, i.e., "a teacher's ability to perceive and mobilize existing curricular resources" in order to design instruction) and to develop tools to measure PDC. The data were drawn from 25 elementary teachers in grades 3–5 in the United States. These teachers were using five different curriculum programs (each program includes resources for students and teachers for daily lessons), ranging from commercially-developed to reform-oriented. Analyzing the content and pedagogical support of the five curriculum programs and analyzing how each teacher using their curriculum program to teach everyday lessons from various perspectives shed light on teacher capacity needed for effective use of existing resources. Specific aspects of the teacher capacity are described along with examples for the ICUBiT project and related literature in this paper. Eventually, this paper attempts to answer to the following question: What are the components of teacher capacity for productive resource use?

THEORETICAL BACKGROUND

I set teachers' work of using existing resources in a broad research context, although I use Brown's (2009) notion of PDC to conceptualize the capacity for using existing resources productively. This capacity is critical in teachers' documentation work and documentation system (Gueudet & Trouche, 2009). According to Gueudet and Trouche, teachers are engaged in documentation work, such as looking for resources and selecting tasks, and in this process they build documentation systems. They distinguish

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 219-226). Umeå, Sweden: PME.

between resources and documents. Resources are a range of artifacts for teaching, such as textbooks, software, and discussions with a peer teacher, whereas documents are evolving products of teachers' documentation work, which include resources, *usage* (action rules), and *operational invariants* (cognitive structure guiding resource use). How teachers use the resources is observable; in contrast, operational invariants are often invisible but can be interpreted from how they use the resources. In my analyses to explore teacher capacity for productive resource use, first I focus on teachers' usage, i.e., how teachers read, adapt, and use existing resources to teach mathematics lessons. Then, I infer teachers' operational invariants to make sense of the ways in which they used the resources. Examining teachers' use of resources along with their operational invariants supports the inquiry of teacher capacity needed for resource use.

I consider teacher decision making around using existing resources as *pedagogical reasoning and action* by Shulman (1987) and aspects of using knowledge in teaching practice as elaborated in Rowland, Huckstep, and Thwaites' (2005) notion of *knowledge quartet*. Both Shulman's notion of pedagogical reasoning and action and Rowland et al.'s notion of knowledge quartet include teaching practice that Remillard (1999) calls improvisation, or "on-the-spot curriculum construction" (p. 331), which indicates teacher moves that are not specified in the written lessons. Examining teachers' decisions on how to use resources to design instruction and their improvisations is eventually digging deeper into teachers' reasoning and knowledge in use, which helps explore teacher capacity for productive resource use.

The productiveness of using existing resources depends on the opportunity for students to learn during instruction. When the resources are used productively, enacted lessons must create opportunities for students to learn the mathematical points of the lessons with sufficient cognitive demand on the students (Kim, 2018). Students need to explore, reason about, and understand the target mathematics of the lessons. Therefore, teacher capacity for productive resource use should be examined in terms of whether the resource use supports students' learning of the mathematical points of the lessons, and what aspects of resource use support or do not support student learning.

Finally, I insist that exploring teacher capacity of productive resource use is based on the participatory relationship between teachers and resources (Remillard, 2005). Using notions of instrumentation and instrumentalization, Gueudet and Trouche (2009) also illustrate the mutual interaction between a teacher and resources in documentation work and documentation system. Teacher capacity needed for using resources productively is grounded in such bilateral influences that shape both parties. This relationship generates the research context that examines not only the components of the teacher capacity needed for using resources productively, but also the role of the resources in supporting teachers to develop such a capacity.

DATA SOURCES

In order to explore the capacity needed for productive use of existing resources, I drew on data gathered from 25 teachers in grades 3–5 in the *Improving Curriculum Use for*

Better Teaching (ICUBiT) project. These teachers were using five different curriculum programs (5 teachers per curriculum program), ranging from reform-oriented to commercially developed. The teachers were (1) asked to keep a Curriculum Reading Log (i.e., indicating parts they read, parts they planned for instruction, and parts influenced their planning on a copy of written lessons), (2) observed in three consecutive lessons in each of two rounds, and (3) interviewed after each round of observations. All observations were video-taped and all interviews were audio-taped. Then, both video- and audio-taped data were transcribed for analysis. Scrutinizing teacher capacity for productive resource use, this paper drew on a range of analyses on various aspects of resource use by the teachers, such as sequencing lessons, using intervention resources, and deciding whether to follow guidance in the written lessons (e.g., Kim, 2015, 2018, under review). In doing so, I documented patterns of the teachers' resource use, their effectiveness in terms of the mathematical points of the lessons, and teachers' rationale for their decisions. Searching for patterns in these analyses revealed critical components of teacher capacity for productive resource use. I also drew on literature related to teacher capacity and resource use.

TEACHER CAPACITY FOR PRODUCTIVE RESOURCE USE

Teachers made various decisions regarding how to use their curriculum program. Some decisions impacted enacted lessons positively toward students' learning of the mathematics they were supposed to; others did not. Although a lot of support features are provided in the written lessons, it is evident that teacher improvisations occurred quite often regardless of programs used (Kim, under review). Various teacher decisions on resource use, kinds of improvisations, and teachers' reasoning behind their decisions revealed different aspects of resource use and teacher capacity needed. Below, four specific aspects of teacher capacity for productive resource use are described along with brief examples from the data in the ICUBiT project. Although I describe them individually, they are interrelated components of teacher capacity, rather than mutually exclusive.

Articulating mathematical points and steering lessons toward mathematical points

Using existing resources to teach mathematics, teachers first read and make sense of the written lessons. In doing so, they need to identify the mathematical points of the lessons and evaluate how well the lesson activities, tasks, and problems support students' learning of the mathematical points (Remillard & Kim, 2017; Sleep, 2009). Then, they need to organize lesson activities toward the mathematical points in instruction (Brown, Pitvorec, Ditto, & Kelso, 2009). Failing to articulate the mathematical points of the lessons, teachers orchestrate lessons activities away from the mathematical points (Kim, 2015, 2018, under review). In other cases, teachers identified the mathematical points properly and yet had hard time steering instruction toward the mathematical points, when challenged by students' difficulty understanding the mathematical idea (Kim, 2018).

One third-grade teacher in the ICUBiT project considered identifying and using keywords as the goal of the lessons on creating and solving multiplication and division story problems, and emphasized keywords instead of the meaning of multiplication and division in instruction (Kim, under review). Moreover, the teacher altered or omitted important lesson components that had a great potential to support students' understanding of multiplication and division. For example, she omitted a lesson component that asked students in pairs to come up with story problems for two related expressions (i.e., 6×3 and $18 \div 3$) so that students could see the differences between multiplication and division contexts. Instead of this task, the teacher asked students to generate a list of keywords for each of the two operations. The teacher made comments as students offered some expressions as keywords, whether each suggested word would be acceptable for each operation. In doing so, she lost an opportunity to highlight characteristics of multiplication and division in relation to each other. The loss of meaning continued in the following lesson when students were creating multiplication and division story problems. While focusing on keywords, such as *in all*, and *share equally*, the teacher did not use the important terms, such as *number of groups, number in each group*, and *equal groups*, to explain the characteristics of and differences between multiplication and division. As a result, after spending two days of generating multiplication and division story problems, still more than half of her students were not able to complete the task. On the third day of classroom observation, there was a range of student-generated story problems. Some students had stories but no questions; some students did not have multiplication or division contexts (addition or subtraction instead); some students had numbers that do not work well (34 things divided equally into 3 or 4 groups); some students had only one type of story problems (all multiplication or all division)

Articulating mathematical points and steering lessons toward the mathematical points are not limited to within individual lessons. Teachers need to articulate mathematical points of a series of lessons (an entire unit or a set of consecutive lessons) and teach students through a proper mathematical pathway so that the students can understand the connections and relationships in the mathematical points and develop a coherent mathematical storyline, or "a deliberate progression of mathematical ideas" (Sleep 2012, p. 954) across lessons. Teachers need to see how mathematical ideas are developed over a series of lessons, and sequence tasks and lessons according to this progression. Otherwise, students may have difficulty develop a proper understanding of the complete ideas across lessons. For example, sequencing tasks and lessons in a way that eased up on the first two days and then enacted a series of important explorations on one single day, a fifth-grade teacher forced students to make sense of common fractions (1/4, ¾, 1/8, 3/8, 1/3, 2/3, 1/6, etc.) and their percent equivalences in one day.

Recognizing affordances and constraints of the resources in use

As teachers read and make sense of the resources and identify the mathematical points of the lessons/activities/tasks, they can also recognize aspects/components of the lessons/activities/tasks that support or do not support students' learning of the mathematical points. In order to use existing resources productively, teachers need to recognize affordances and constraints of the resources they use, with respect to their students' learning of the mathematical points (Atanga, 2014; Choppin, 2011; Kim, 2015, 2018, under review; Kim & Son, 2017). Teachers who were not able to recognize the affordances may not use them in instruction. Also, teachers who do not recognize the constraints hardly try to make up the limitations. Depending on their evaluation of the affordances and constraints along with their students' need, teachers can decide whether they use, change, or omit components of lessons/activities/tasks, or add new elements to enact lessons (Kim, under review). Therefore, recognizing affordances and constraints is critical to use the existing resources productively.

For example, not seeing the usefulness of representations provided in the resources for subtracting a fraction from a whole number, one third-grade teacher totally dismissed the need for the representations (fraction circles or pictures, bars, and number line) in supporting students' conceptual understanding of the procedure for subtracting a fraction from a whole number (Kim, 2018). Even when students suggesting to use a representation, the teacher refused to use any. Mentioning that the representations were too simplified and tended to confuse students, the teacher did not recognize the affordances of the representations that support students' conceptual understanding of the procedure. As a result, for three days of listening to the teachers' explanations and using the procedure, the students in this class still had difficulty understanding why they did the way they did.

Using affordances

Recognizing the affordances of existing resources is important; so is using those affordances in instruction. Brown's (2009) definition of PDC includes both "perceive and mobilize" the existing resources. In particular, using those resources together as a coherent set seems critical in using the existing resources well (Atanga, 2014). Various components of the resources are designed to support students' learning of the mathematical points. Resources as a set rather than separate elements indicate the synergy that they can generate in supporting teachers to steer instruction toward the mathematical points. In the ICUBiT project, when using resources productively to teach lessons, teachers were using a range of elements provided in the resources toward the mathematical points of the lessons. Otherwise, as seen in the earlier example of the teacher focusing on keywords, teachers altered or omitted useful, important resources (e.g., representations and tasks), sometimes in place of additional elements they chose to do instead. In other cases, teachers used the affordances unproductively.

The fifth-grade teacher mentioned above recognized the usefulness of 10×10 grids to relate fractions and their percent equivalences (e.g., ¾ = 75%). But, the teacher used the grids not very effectively in the second observed lesson, by asking students to shade their own grids and write the fraction and the percent that each of their grids represented. Students shaded their grids randomly and wrote a fraction and percent pair only by counting the number of squares shaded (e.g., 79 squared shaded, so the grid represents 79/100 = 79%) without much focus on the relationship between fractions and percents. This was problematic because the mathematical point of the lessons was not about determining fraction-percent pairs of 10×10 grids shaded randomly. The written lessons were deliberately focusing on using the grids to relate common fractions and their percent equivalences, moving from easy fractions (e.g., 1/2 = 50%) to harder fractions (1/4 = 25%) and finally to more complex fractions (e.g., 1/3 = $0.33\frac{1}{3}$%).

Filling in holes and gaps properly

Recognizing constraints of the existing resources does not necessarily lead to productive ways of overcoming them, which is another important aspect of the capacity needed for effective use of existing resources. In the ICUBiT project, teachers tended to add new elements to the written lessons to enact them (Kim, under review). Some were intentionally added as planned; some were improvised in response to students. Whether these new elements are planned in advance or improvised during instruction, they have to support students' learning of the mathematical points of the lesson. Especially, those intended to overcome the constraints of the written lessons or improve the written lessons must be prepared carefully to increase the opportunity for students to learn the mathematical points of the lessons.

One teacher using a curriculum program whose individual lessons were designed for multiple class periods so that students could explore related mathematical ideas in depth over 2–3 days (Kim & Atanga, 2013, Kim, under review). In a lesson for 3 estimated days, students were asked to use base-ten pieces to measure the area of a coat, and compare and order large numbers. This lesson was designed for geometrical and numerical explorations combined. The students were using the concept of symmetry to effectively measure the area of a coat (only measuring a half of the area and doubling the number found) and making sense of the large numbers as the areas would be in thousands of one's pieces. As the lesson was complex in nature, detailed guidance was provided for teachers. However, there were still room for improvisations as the teacher enacted the lesson. Noticing that her students needed a review on symmetry before starting a task of finding the area of a coat, one teacher asked students questions about symmetry, which effectively supported students' work on the task.

One fourth-grade teacher, using a written lesson on mean that was focusing on only the procedure to find the mean of a set of numbers, asked students to use cubes to determine the mean of four different numbers in the introduction of the lesson. This, however, was not productive because using cubes were not supporting students to understand what mean really means. Basically representing the procedure of "add/combine them all and divide by four" by using the cubes, the teacher did not highlight the conceptual nature of mean, i.e., what the mean of the four numbers really represents.

There are no perfect curriculum resources that fit in any classroom situation; proper change, omission, or addition is needed as teachers are engaged in documentation work. Yet, the way teachers fill in the holes and gaps in the existing resources should be determined toward students' engagement with mathematical points.

SUPPORTING TEACHER CAPACITY DEVELOPMENT

Various approaches can be taken to support teachers to develop their capacity for productive use of existing resources. Two particular approaches are highlighted in this paper.

The data used for this paper revealed that the teachers in the ICUBiT project had certain operational invariants (cognitive structure guiding resource use) that Gueudet and Trouche (2009) explained as part of teacher documentation system. Unproductive use of existing resources is often rooted in operational invariants that are not appropriate (Kim, under review). For example, the teacher emphasizing keywords in multiplication and division story problems believed keywords helped students' learning of operations and solving story problems. Also, the teacher, not using representations in the lessons on operations with fractions, believed that representations were not helpful, but confusing students' thinking. Therefore, in order to support teachers to develop the capacity needed for productive use of existing resources, teacher education (i.e., teacher preparation and professional development) need to support teachers to examine their own operational invariants and generate such opportunities in individual teachers' documentation work and documentation system.

In addition, the role of curriculum resources in increasing teacher capacity is fundamental. Educative features in the resources support teacher learning (Davis & Krajcik, 2005). Above all, it is recommended that curriculum resources be designed to support teachers to clearly see the mathematical points of lessons, activities, and tasks. As evident in the teachers of the ICUBiT project, teachers may or may not recognize the mathematical points of the lesson as they read or glance at the guidance in the curriculum resources. Also, it is recommended that curriculum resources provide guides to make proper decisions, especially on various options teachers can choose from and in the case for improvisations in response to students. As in the ICUBiT project, teachers often make changes, omit lessons and lesson components, and add new elements. Proper guidance for these adaptations is critical for steering instruction toward the mathematical points of the lesson.

Acknowledgment

This paper is based on work supported by the National Science Foundation under grants No. 0918141 and No. 0918126.

References

Atanga, N. A. (2014). *Elementary school teachers' use of curricular resources for lesson design and enactment*. Unpublished dissertation in Western Michigan University.

Brown, M. W. (2009). The teacher–tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd, (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). New York: Routledge.

Brown, S. A., Pitvorec, K., Ditto, C., & Kelso, C. R. (2009). Reconceiving fidelity of implementation: An investigation of elementary whole-number lessons. *Journal for Research in Mathematics Education, 40*(4), 363–395.

Davis, E. A., & Krajcik, J. S. (2005). Designing educative curriculum materials to promote Teacher learning. *Educational Researcher*, *34*(3), 3–14.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71*, 199–218.

Kim, O. K. (2015). The nature of interventions in written and enacted lessons. In Beswick, J., Muir, T., & Wells, J. (Eds.), *Proceedings of 39th Psychology of Mathematics Education Conference,* Vol. 3. (pp. 153–160). Hobart, Australia: PME.

Kim, O. K. (2018). Teacher decisions on lesson sequence and their impact on opportunities for students to learn. In L. Fan, L. Trouche, C. Qi, S. Rezat, & J. Visnovska (Eds.), *Research on mathematics textbooks and teachers' resources*. Springer.

Kim, O. K. (under review). Teacher fidelity decisions and the quality of enacted lessons. *ZDM Mathematics Education.*

Kim, O. K., & Atanga, N. A. (2013). Teachers' decisions on task enactment and opportunities for students to learn. *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 66-73). Chicago, IL: University of Illinois at Chicago.

Kim, O.K., & Son, J. (2017). Preservice teachers' recognition of affordances and limitations of curriculum resources. In *Proceedings of 41st Psychology of Mathematics Education Conference,* Vol. 3. (pp. 57–64). Singapore: PME.

Remillard, J. T. (1999). Curriculum materials in mathematics education reform: a framework for examining teachers' curriculum development. *Curriculum Inquiry, 29,* 315–342.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211–246.

Remillard, J., & Kim, O. K. (2017). Knowledge of curriculum embedded mathematics: Exploring a critical domain of teaching. *Educational Studies in Mathematics, 96*(1), 65–81.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8,* 255–281.

Sleep, L. (2012). The work of steering instruction toward the mathematical point: A decomposition of teaching practice. *American Education Research Journal, 49*(5), 935–970.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1–21.

SECONDARY SCHOOL STUDENTS' APPRAISAL OF MATHEMATICAL PROOFS

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Research on the reading of proofs is an important area of proof research in mathematics education. As one aspect of the reading of proofs, we focus on 'proof appraisal' by students (that is, students' judgements about given proofs) and explore how students appraise different proofs of an identical statement. Using a simple proof and a generalisable proof of a statement, we analysed the results of a questionnaire completed by 39 Grade 8 secondary school students (13–14 years old). We show aspects of each proof that were appraised by the students, such as simplicity, and the relativeness of their proof appraisals. An implication is a possible 'gap' between the 'mathematical value' appreciated by students and that by researchers and teachers.

INTRODUCTION

Research on the reading of proofs has recently gained more attention in the mathematics education community, and there have been several types of recent studies in this research area (Komatsu et al., 2017). One type relates to students' comprehension of given correct proofs, such as whether students can understand key terms and statements in the proofs and illustrate a sequence of inferences with a specific example (e.g. Mejia-Ramos et al., 2012). Another type of study is of proof validation with students asked to determine the validity/invalidity of the purported deductive proofs (e.g. Inglis & Alcock, 2012). In this paper, we focus on another type of research in the reading of proofs, namely *proof appraisal*. We use this term to refer to students explaining their reasons for preferring a particular given proof.

Some existing studies have examined whether students appreciate certain aspects of proofs, such as verifying that statements are true and explaining why statements are true (e.g. Healy & Hoyles, 2000; Segal, 1999). However, research on the functions of proofs shows that the power of proofs is not only in the verification and explanation of statements (de Villiers, 1990; Hanna & Barbeau, 2008). Recently, Inglis and Aberdein (2014, 2016) have classified mathematicians' appraisals of proofs into four dimensions: aesthetics, intricacy, precision, and utility. Although mathematicians and students are different in various ways, such as mathematical maturity and interest in mathematics, it is anticipated that students' proof appraisals are also likely to be diverse.

To this end, we consider a specific setting where students are given multiple valid proofs of an identical statement and are asked to judge these proofs. This setting is different from those of existing studies that have contrasted valid proofs with insuffi-

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 227-234). Umeå, Sweden: PME.

cient arguments (e.g. empirical arguments) and with purported deductive proofs that actually include errors (e.g. Healy & Hoyles, 2000; Inglis & Alcock, 2012). We use multiple valid proofs of an identical statement because the given multiple proofs have in common the capability of verifying that the statement is true, and thus we expect it to be possible to elicit students' appraisals of various aspects of proofs other than the verification of the statement. Hence, in this paper, we address the following research question: How do students appraise different proofs of an identical statement?

FRAMEWORK: COHERENCE AND GENERALISABILITY OF PROOF

As a framework for classifying proofs that are different in terms of how they draw the conclusion of the statement, we employ the notions of *direct fit* and *familial fit* suggested by Raman-Sundström and Öhman (in press). *Direct fit* here refers to the relationship between a statement and a proof, while *familial fit* refers to the relationship between a proof and a family of proofs. In this paper, we focus on the notion of *coherence*, one aspect of direct fit, and that of *generalisability*, one aspect of familial fit. A proof is regarded as *coherent* if the proof is stated in the same terms as the statement that the proof addresses. A proof is regarded as *generalisable* if the idea of the proof can be used for a larger class of statements.

We explicate these notions by taking a statement as an example: the sum of the interior angles of a star octagon is 720° (this statement was also used in our questionnaire whose data are examined in this paper). Here, a star polygon is defined as a polygon constructed by connecting vertices while skipping the adjacent vertex. Figure 1 shows a star heptagon. Star polygons where the numbers of the vertices are odd (hereafter, star-odd polygons) can be drawn in one stroke, whereas star polygons where the numbers of the vertices are even (star-even polygons) cannot, but can be drawn by a combination of two polygons. Figure 2 shows two different proofs of the statement that the sum of the interior angles of a star octagon is 720°.

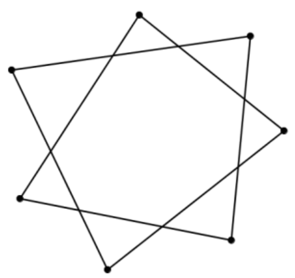
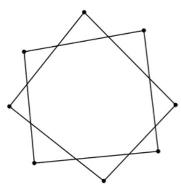


Figure 1: Star heptagon

- Proof A:
The interior angle sum of a quadrilateral is 360°. We multiply it by two because a star octagon consists of two quadrilaterals. Thus, the interior angle sum of a star octagon is 720°.



- Proof B:
There are eight triangles on the outside, and the interior angle sum of these triangles is 1440° (= 180° × 8). The white- and black-marked angles are not the interior angles of the star octagon, and these angles constitute two sets of the exterior angle sum of the 'inside octagon', namely 720° (= 360° × 2). Hence, the interior angle sum of a star octagon is 720° (= 1440 – 720).

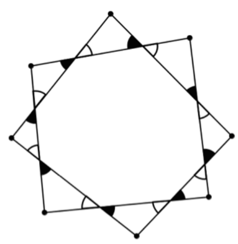


Figure 2: Two proofs of the identical statement

Proof A can be regarded as *coherent* because it uses only interior angles that are stated in the statement, whereas Proof B lacks *coherence* because it introduces the concept of exterior angles, which is not mentioned in the statement. With respect to *generalisability*, Proof B is *generalisable* because the idea of this proof (based on the constancy of the sum of exterior angles in any polygon) can be applied to all star polygons. For example, it is possible to prove that the sum of the interior angles of a star heptagon is 540° by calculating '$180 \times 7 - 360 \times 2$'. On the other hand, Proof A relatively lacks *generalisability* because the idea of this proof is not applicable to the star-odd polygon case. Note that the discussion here is just an example; some proofs may have both coherence and generalisability, and other proofs may have neither.

METHODS

Background

The study reported in this paper consisted of two parts; one part involved designing and implementing a series of proof tasks related to star polygons over four 50-minute lessons in a secondary school classroom (Komatsu et al., in press), and the other part involved conducting a questionnaire after the lessons to investigate how the students appraised two proofs of the aforementioned statement about the star octagon case. This paper presents the results of the second part. In a later section, we briefly describe the implemented lessons because the results of the questionnaire would be influenced by the features of those lessons.

Questionnaire and participants

We produced a questionnaire presenting two proofs almost identical to Proof A and Proof B shown in Figure 2 and then asking, "Which method do you use for finding the sum of the interior angles of a star octagon, Proof A or Proof B? Describe the reason for your choice". We decided to ask such a question because we expected that if students were requested to select either of the two proofs with explaining the reasons for their choices, they would express their appraisal of the advantages of the two proofs more explicitly.

This questionnaire was implemented with 39 eighth-grade students (13–14 years old) in a Japanese lower secondary school affiliated with a national university. The mathematical capabilities of the students were above average for Japan according to their teacher (the fifth author of this paper). The students had covered in class the knowledge necessary to understand the two proofs (e.g. the interior/exterior angle sum theorems of polygons).

Procedure of data analysis

Although our questionnaire was based on a single task, we obtained rich data where the students fully explained the reasons why they preferred one proof to the other proof. Hence, we analysed the students' responses in a qualitative way, coding their responses and then counting the number of students referring to each code in order to investigate what aspects of each proof tended to be appraised by the students. Our coding procedure was as follows. The first author split each of the students' descriptions of their proof appraisals into several segments; there were 97 segments in total for the 39 students (which shows the richness of our data). Temporal codes were then devised to denote these segments. After that, if certain codes were found to be similar, they were unified into a single code. The second author then checked the appropriateness of the coding, and any discrepancies were discussed until the authors reached a consensus (see Tables 2 and 3 for identified codes and their distributions).

Lessons implemented before the questionnaire

As mentioned earlier, our questionnaire was conducted after the implementation of a series of tasks about star polygons over four lessons. Because space here is limited, we give only a brief summary of the implemented lessons, in Table 1 (for more details, see Komatsu et al., in press). The students explored the sums of the interior angles of various star polygons in the lessons. As shown in Table 1, they had had the experience of constructing both Proof A and Proof B before the questionnaire. In particular, in the fourth lesson, they had recognised the generalisability of Proof B, where they found that this proof idea could be applied to all the star polygons. They also invented an algebraic expression for the interior angle sum of a star polygon: $180n - 720 = 180(n - 4)$ (where n is the number of the vertices of the star polygon).

Lesson	Star polygons investigated in each lesson
1st lesson	Star pentagon
2nd lesson	Star-even polygons (e.g. constructing Proof A for a star octagon)
3rd lesson	Star-odd polygons (considering 'outside triangles' and 'inside polygons' like Proof B)
4th lesson	Star-even polygons revisited (e.g. constructing Proof B for a star octagon) and then star polygons in general

Table 1: Summary of the implemented lessons

RESULTS

In the questionnaire, 27 students selected Proof A and 12 students selected Proof B. As a result of coding their proof appraisals, we found that most of the codes could be divided into two types. The first type showed what aspects of each proof the students appraised, while the second type represented the relativeness of the students' proof appraisals (e.g. whether they thought that their proof choices depended on situations). Below, we show the obtained results by type. English translations of the students' responses are rendered from the original Japanese by the authors.

Students' reasons for their proof appraisals

Table 2 shows our classification for the first type of codes, showing what aspects of each proof the students appraised and how many students referred to those aspects. For each proof, the sum of the numbers for all codes is larger than the number of students who selected the proof because there were cases where the proof appraisal by a given student was related to multiple codes.

	Selecting Proof A (n = 27)		Selecting Proof B (n = 12)	
Code	#	% in Proof A	#	% in Proof B
Simple	20	74%	2	17%
Brief	6	22%	1	8%
Understandable	4	15%	2	17%
Free from error	4	15%	0	0%
Immediate	3	11%	0	0%
Generalisable	0	0%	8	67%
Advantage of formula	0	0%	4	33%

Table 2: Students' reasons for their proof appraisals

For the selection of Proof A, the most frequent code is *simple* (indicating that the description of the proof is mathematically simple). This code is also related to other codes, such as *brief* (which means that the proof requires only a single calculation). Below are examples of students' responses for each of these (we use parentheses to show codes assigned to each response):

> S1: Because there are two polygons, if the sums of the interior angles of these polygons can be found, it can be easily solved. (*simple*)

> S2: Proof A does not require complicated calculations, and the answer can be found with a single calculation. (*simple* and *brief*)

As can be seen above, students selecting Proof A considered this proof to be simple and brief because it required only a property well-known by the students (the interior angle sum of a quadrilateral) and a single calculation. These students focused on a specific case mentioned in the questionnaire (the star octagon).

Students choosing Proof B had a different viewpoint, in which they took other star polygons into consideration; the most common reason for the selection of this proof is thus represented by the code *generalisable*. This code is also related to the code *advantage of formula*:

> S3: If we know that the sum of the interior angles of a triangle is 180°, calculate 180 × 8 since 180 × star octagon. Because the sum of exterior angles is always 360°, this method can be used for all cases. (*generalisable*)

> S4: $180(n – 4)$ -> 180 × 4 = 720. The formula is easy. It has applicability, it can be used for odd cases, and if we use the formula, other problems can be solved as well. Dividing into odd and even cases is bothering. (*advantage of formula*, *generalisable*, and *simple*)

As shown earlier, the idea represented in Proof B can be generalised to all star polygons, and students choosing this proof appreciated this generalisability. In the lessons implemented before the questionnaire, the students found that the sum of the interior angles of a star polygon can be expressed as 180(n – 4) (see the methods section). Some students, such as S4, mentioned the advantage of using this algebraic expression.

Relativeness of students' proof appraisal

The second type of codes, which represent the relativeness of the students' appraisals of the two proofs, is summarised in Table 3.

	Selecting Proof A (n = 27)		Selecting Proof B (n = 12)	
Code	#	% in Proof A	#	% in Proof B
Depending on situation	9	33%	1	8%
Appreciation of the other proof	3	11%	1	8%
Limitation of the selected proof	2	7%	0	0%
Criticism of the other proof	2	7%	6	50%

Table 3: Relativeness of students' proof appraisals

One code in this type is *depending on situation*: nine students selecting Proof A stated that they would choose Proof B if the number of the vertices of the star polygon had been different. Other relevant codes are *appreciation of the other proof* and *limitation of the selected proof*: some students choosing Proof A mentioned the value of Proof B as well as the limitations of Proof A:

> S5: When finding the sum of the interior angles of a star polygon in future, I will use Proof A in the case where the number of vertices is even, and Proof B in the odd case. (*depending on situation*)

> S6: When finding the sum of the interior angles of a star polygon, I feel that Proof B is good as it can be used for odd and even cases. For the even case where it is obvious that polygons overlap, I want to use Proof A. (*appreciation of the other proof*)

> S7: Although Proof A can be used only for the case where the number of vertices is even, it is simple, and thinking and calculation are easy. (*limitation of the selected proof* and *simple*)

Students can be regarded as relatively appraising Proof A if they referred to the code *depending on situation*, *appreciation of the other proof*, or *limitation of the selected proof*. This is because these students not only appraised Proof A, but also recognised the limitations of Proof A and the advantage of Proof B. In the questionnaire, 13 students selecting Proof A (48% of that group) appraised it relatively.

On the other hand, half of the students selecting Proof B explicitly criticised Proof A, making *criticism of the other proof* a common code for Proof B but not Proof A:

> S8: The star octagon case can be solved with the method of Proof A, but the star heptagon case etc. cannot be solved with the method of Proof A. The method of Proof B can be commonly used for all of star polygons, so I will use the method of Proof B. (*criticism of the other proof* and *generalisable*)

In relation to this, only one student referred to the code *depending on situation* for Proof B, and the same is the case for *appreciation of the other proof*. Thus, students selecting Proof B tended to appraise this proof absolutely rather than relatively.

DISCUSSION

In this paper, we have examined how secondary school students appraised two different proofs of the same statement. To this end, we employed Raman-Sundström and Öhman's (in press) notions of direct fit and familial fit to prepare two contrasting proofs. In the implemented questionnaire, the most common reasons for selecting Proof A and Proof B were respectively simplicity and generalisablity. Although more students preferred Proof A to Proof B, proof appraisals by almost half of the students selecting Proof A were relative, indicating that they recognised the limitations of Proof A and the value of Proof B in terms of generalisability.

Our findings may raise an issue for mathematics teachers and mathematics education researchers. Proof B can be generalised to all star polygons, and generalisation is much appreciated in the mathematics education community (e.g. Mason, 2002). Generalisable proofs, or proofs that can be used for different purposes, are highly evaluated in mathematicians' practice as well (Hanna & Barbeau, 2008; Weber & Mejia-Ramos, 2011). However, in our study, when asked to select either the simple proof or the generalisable proof, the students tended to prefer the former. This may relate to students' emerging mathematical values (Seah, 2016). It may be that there is a 'gap' between the type of mathematical value appreciated by students aged 13–14 years old and that by teachers and researchers (mathematics education researchers and mathematicians).

That said, a note of caution is that this 'gap' may have arisen from the specificity of our questionnaire where the students were shown a single case (the star octagon) and were asked to select a proof only for this case. In fact, as mentioned above, there were students who preferred the simple proof (Proof A) and, at the same time, appreciated the generalisability of Proof B. Hence, it would be necessary to explore further the gap found in this study by asking different types of questions and adopting different methodologies.

While our questionnaire was based on a single task and implemented with only a relatively small number of students, and, as such, we do not intend to assert the generalisability of all our results, several of the codes that we devised for representing students' proof appraisals may be useful beyond our study. Inglis and Aberdein (2014, 2016) classified mathematicians' proof appraisals, and intricacy (its opposite) and utility in their classification are respectively related to simplicity and generalisability among our codes. Given that simplicity and utility are observed in studies involving different groups (students and mathematicians), these codes, on the one hand, may likely be employed to represent proof appraisals in general. On the other hand, other codes used here (e.g. depending on situation and appreciation of the other proof) are probably best considered as being specific to our study, derived from the specific question in our questionnaire where the students were shown multiple valid proofs of a statement and were asked to show their preferences. Thus, these codes may be useful for capturing students' proof appraisals in similar contexts.

References

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17–24.

Hanna, G., & Barbeau, E. (2008). Proofs as bearers of mathematical knowledge. *ZDM – The International Journal on Mathematics Education, 40*(3), 345–353.

Healy, L., & Hoyles, C. (2000). A study of proof conception in algebra. *Journal for Research in Mathematics Education, 31*(4), 396–428.

Inglis, M., & Aberdein, A. (2014). Beauty is not simplicity: An analysis of mathematicians' proof appraisals. *Philosophia Mathematica, 23*(1), 87–109.

Inglis, M., & Aberdein, A. (2016). Diversity in proof appraisal. In B. Larvor (Ed.), *Mathematical cultures: The London Meetings 2012–2014* (pp. 163–179). Basel, Switzerland: Birkhäuser.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*(4), 358–390.

Komatsu, K., Fujita, T., Jones, K., & Sue, N. (in press). Explanatory unification by proofs in school mathematics. *For the Learning of Mathematics*.

Komatsu, K., Jones, K., Ikeda, T., & Narazaki, A. (2017). Proof validation and modification in secondary school geometry. *Journal of Mathematical Behavior, 47*, 1–15.

Mason, J. (2002). Generalisation and algebra: Exploiting children's powers. In L. Haggarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 105–120). London, England: Routledge Falmer.

Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics, 79*(1), 3–18.

Raman-Sundström, M., & Öhman, L. D. (in press). Mathematical fit: A case study. *Philosophia Mathematica*.

Seah, W. T. (2016). Values in the mathematics classroom: Supporting cognitive and affective pedagogical ideas. *Pedagogical Research, 1*(2), Article No: 53.

Segal, J. (1999). Learning about mathematical proof: Conviction and validity. *Journal of Mathematical Behavior, 18*(2), 191–210.

Weber, K., & Mejia-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics, 76*(3), 329–344.

LEARNING MATHEMATICS THROUGH ONLINE FORUMS: A CASE OF LINEAR ALGEBRA

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The aim of the study reported in this paper was to explore online interactions of twenty five high-school students in an asynchronous forum that accompanied a face-to-face course in linear algebra. The forum generated a considerable number of mathematical post-exchanges, the vast majority of which came from a small group of six students. The data analysis revealed a positive correlation between thread-initiation and achievements of students in the course. Students' activity in their self-initiated threads correlated with their activity in the threads initiated by their peers, which attests to the collaborative nature of the forum. In about half of the threads students sought verifications for their solutions to the assigned problems. The paper ends with a discussion on what one's online activity might indicate in terms of her course learning.

INTRODUCTION

In recent years, online asynchronous forums have become a common occurrence in university education. Explanations for this trend can be found in the evolvement of educational theories and practices. Indeed, social constructivism, that seems to dominate in our field, calls for a radical shift in our approach to teaching and learning, when technology is positioned as a powerful tool for facilitating this shift. From a practical perspective, online forums allow for more sharing, reflecting and retaining of ideas produced by learners than in a typical face-to-face instruction.

*O*nline *a*synchronous *f*orums (OAFs) seem particularly advantageous in undergraduate courses, where syllabi are dense, learning is mostly lecturer-centered and hundreds of students with different mathematical backgrounds are enrolled. Moreover, freshmen frequently lack the necessary skills for learning mathematics in a new academic setting. Accordingly, OAFs become a possible venue for mathematical interactions, in which these skills can be shaped (e.g., Jacob & Sam, 2008; Perkins & Murphy, 2006).

In many universities online forums accompany face-to-face courses in undergraduate mathematics. Some of these *course accompanying online asynchronous forums* (CAOAFs) mainly serve the course staff in making organizational announcements, while others comprise multiple threads with rich mathematical discussions where students seek and provide help with course materials to one another. The study reported in this paper is a part of a larger ongoing project on the mathematics learning that occurs in CAOAFs (see Kontorovich, 2018 for a first report). The data for this study comes from a particularly replete forum that accompanied a face-to-face course in linear algebra for high-school students.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 235-242). Umeå, Sweden: PME.

BACKGROUND AND CONCEPTUAL FRAMEWORK

Mathematical OAFs

Research in mathematics education has been mainly concerned with *rewarding* and *open* OAFs. By rewarding, I refer to the learning settings where a percentage of a final grade is assigned to students' participation in online discussions (e.g., Jacob & Sam, 2008). Accordingly, a rewarding interaction is structured by the assignment that was given to the learners and their participation is monitored and assessed by course instructors. While rewarding interactions occur in specially developed platforms, such as Canvas, HighLearn, Piazza, Moodle etc., open interactions take place in the World Wide Web. In open forums the participation is voluntary, anonymous, and not restricted to any particular course and theme. Van de Sande (2011) described open interactions as a mathematical help exchange between seekers and providers. Table 1 shows that some characteristics of CAOAFs are similar to rewarding forums while others are in common with open forums.

Types of participation in OAFs

Educational research has been concerned with various aspects of learners' participation in OAFs. Perkins and Murphy (2006) explored individual engagement in a rewarding OAF in the context of education. The categories that the researchers used for classifying participants' posts consisted of: *clarification*, which refers to all aspects of stating, clarifying and defining the discussed issues; *assessment*, which is concerned with evaluation of the argumentation in the discussion; *inference*, which is concerned with making generalizations and drawing connections; and *strategies*, which account for discussing possible actions and predictions of their outcomes. Perkins and Murphy (2006) found that the majority of students' posts were concerned with clarification and assessment.

In the context of a mathematical open OAF, van de Sande (2011) suggested that the posted queries are mostly concerned with textbook problems, and that the participants seek help with *construction of a solution*, *verification of a solution* constructed by them, and *construction of an explanation for a solution* taken from another source. In her study, van de Sande examined the mathematical activity of a help-seeker and associated it with a publication of full or partial solutions. As a result of the analysis, the researcher found out that help-seekers were active in nearly 60% of their posts.

Other studies identified patterns in learners' posts and used them for constructing participants' online profiles. For instance, in a course on programming languages, Shaw (2012) distinguished among four profiles: *askers* who mostly asked questions about the course contents and problems but avoided participating in discussions initiated by other learners; *repliers* who tended to enrich the discussions with solutions; *watchers* who browsed questions and solutions of other participants but the number of their posts was small; and *no action* who did not follow forum discussions. A statistically significant correlation was found between learner's participation type and their achievement. Specifically, repliers scored higher than askers, who scored higher than watchers, who scored higher than no action type.

Characteristics	Open OAFs	CAOAFs	Rewarding OAFs
Participants	registered members	learners who study the course	
Participants' identity	anonymous	usually a full name is displayed	
Participation	driven by participants interests and needs		driven by course assignments
Relevance of online discussions	relevant to participants with similar interests and needs	usually relevant to all students in the course	depends on the assignment

Table 1: Similarities and differences of OAFs

The intensity of participation in OAFs has been also addressed (Jacob & Sam, 2008). For example, van de Sande (2011) distinguished between *core* participants of the forum who post frequently and *peripheral* participants who post occasionally.

RESEARCH GOAL

The goal of the study reported in this paper was to explore students' interactions in an online asynchronous forum and their relations to achievements in the face-to-face course in linear algebra. The three questions that instigated the investigation were: (1) How does students' posting intensity correlate with their achievements in the course? (2) How do posting intensity in self-initiated and peer-initiated threads correlate? (3) What kind of help did the students seek and provide in their post-exchanges?

METHOD

Setting and data collection

As a part of a larger project, twenty-five ninth-graders were selected to participate in a prestigious three-year program at the renowned technological university in Israel. The selection was made based on students' interest in undergraduate education, recommendations of their teachers, and school achievements. The program was aimed at preparing for and engaging high-school students in undergraduate education in parallel with their regular school studies. The data for the present study was collected during the first program year in a preparatory course for linear algebra.

The course consisted of twenty-one weekly lessons, each of which lasted for two and a half hours. The course syllabus encompassed three central topics: (i) polynomials, (ii) matrices and equation systems, and (iii) vector spaces. After each lesson the course teacher provided students with a list of problems to solve at home. The solutions were not intended for submission, but variations of some of the problems appeared in a quiz in the following lesson. At the end of the course the final exam was assigned. Accordingly, the data on students' achievements came from the weekly quizzes and the final exam.

A CAOAF was opened in a popular social networking site in order to provide students with a platform for collaboration on homework assignments. The students came from different schools situated in different parts of Israel, and it was expected that a considerable portion of students' discussions would take place in the forum. The students were encouraged to use the forum for sharing and addressing each other questions and difficulties regarding the learned topics. The published posts comprised the data on students' online interactions.

Data analysis

The data was approached with a computer-mediated data analysis (CMDA) method to researching online behaviors. The method operationalizes the phenomena under consideration (in this case, students' online interactions) by creating coding catego-ries and exploring the relations between them with statistical means. Opposed to other studies where the explored number of posts was rather small (Jacob & Sam, 2008; Van de Sande, 2011), the forum under discussion contained thousands of posts-exchanges, a comprehensive analysis of which is still ongoing. In this study, Questions (1) and (3) were associated with thread-initiation, question (2) was explored in the Polynomials Unit. Overall, the analysis was conducted with posts in which the students explicitly addressed the mathematical content of the course.

In Questions (1) and (2) the correlations were examined in the IBM SPSS STATISTICS 23 software with Spearman's rho coefficient. The test is useful for analyzing a relation between ordinal variables, such as the number of initiated threads and achievements. Content analysis was used in Question (3) for revealing the type of help that students sought and provided in their thread-initiating posts. The analysis was approached with partially predefined categories that were presented in the second section of the paper (Krippendorff, 1980).

FINDINGS

Thread-initiation and course achievements

The explored CAOAF consisted of 334 mathematical threads. The average number of thread-initiating posts was 14.13 per course student (SD=25.09). A large SD indicates considerable differences in students' posting behaviors. Six students who posted the most intensively can be addressed as core participants of the forum, and together they initiated about 85% of all threads (see Table 2). Table 2 also shows that the thread-initiation intensity of almost all the course students was relatively stable during the year. For example, S2 and S3 were the most active in all the three units of the course; the participants that were peripheral in the first unit of the course remained in this status until the end of the course. Overall, the core group outperformed the peripheral group in the quizzes and in the final exam.

	S1	S2	S3	S4	S5	S6	Total
Polynomials Unit	13.6%	18.6%	22%	17%	6.8%	3.4%	81.4%
Matrices and Equation Systems Unit	9.8%	24.8%	25.6%	15%	5.3%	7.5%	88%
Vector Spaces Unit	9%	37.6%	24.8%	0.8%	6.8%	5.3%	84.3%
Course total	10.2%	28.9%	24.6%	9.5%	6.2%	5.8%	85.2%

Table 2: Proportions of threads initiated by core participants in the course

A strong and positive correlation was found between the total number of threads initiated by a student in the course and her achievements in the final exam (see Table 3). In the Vector Space Unit the correlation between the initiated threads and weekly quizzes was positive and moderate, and in the Matrices Unit it was positive and strong. In the Polynomial Unit the correlation was insignificant. The number of threads generated by the students in each unit moderately correlated with their achievements in the final exam.

Thread-initiating posts	Quizzes	Final Exam (range of scores: 15%-100%)
Polynomials Unit	$r_s = 0.472, p > .05$ (range of scores: 43%-90%)	$r_s = 0.448^{*}, p = .032$
Matrices Unit	$r_s = 0.733^{**}, p = .00$ (range of scores: 25%-98%)	$r_s = 0.515^{*}, p = .012$
Vector Spaces Unit	$r_s = 0.549^{**}, p = .007$ (range of scores: 33%-100%)	$r_s = 0.645^{**}, p = .001$
Total course account		$r_s = 0.658^{**}, p = .001$

Table 3: Correlations between students' thread-initiation and course achievements

Thread-initiation and participation in peer-initiated threads

The relations between students' thread-initiation and participation in each other's threads were explored in the Polynomials Unit. The findings are summarized in Table 4 and they show a moderately strong and positive correlation between the number of threads initiated by a course student (w) and the number of her peers' threads in which she participated (x). Also a moderately strong positive correlation was indicated between the number of students' posts in their self-initiated threads (y) and in the threads initiated by other students (z). In the other words, the more active a student was in her own threads the more actively she participated in the threads initiated by her peers. This finding indicates the collaborative nature of students' interactions in the CAOAF under scrutiny.

	w	x	y
x	$r_s = .666^{**}, p = .001$	-	-
y	$r_s = .981^{**}, p = .00$	$r_s = .728^{**}, p = .00$	-
z	$r_s = .618^{**}, p = .002$	$r_s = .969^{**}, p = .00$	$r_s = .69^{**}, p = .001$

Table 4: Correlations between students' posting behaviors

Help sought and provided in the threads

The content analysis of thread-initiating posts resulted in seven categories:

(1) Problem formulations – where thread-initiators sought for clarifications on formulations of homework problems (e.g., "*In question 1 it says that* $q(x) | p(x)$. *What does this | mean?*")

(2) Full solutions – where thread-initiators sought for help with problems without active contribution to their solution (e.g., "*Did anyone solve Question 2? I can't think of anything.*")

(3) Partial solutions – where thread-initiators provided some ideas for problem solutions and sought for help with their development (e.g., "*I found that* $x = -1$ *is a root of* $p(x)$ *with the multiplicity of 6. How do I find the remaining two roots?*").

(4) Verification of full solutions – where complete solutions were posted by thread-initiators (e.g., "*Hey, guys! Here are my answers. If you find any mistakes, please let me know*").

(5) Clarification on classroom material – where thread-initiators sought for explanations of the material discussed in the class (e.g., "*Can you explain the Rank-nullity theorem to me again? I didn't get it in the lesson*").

(6) Problem-solving strategies – where thread-initiators sought for strategies that are applicable to sets of problems (e.g., "*How should I approach problems asking to construct a mapping when the image of the base is given?*").

(7) Inference – where thread-initiators shared their conjectures that were not discussed in the classroom (e.g., "*I think that complex numbers are neither positive or negative, I think that they are sign-less.*"

In the case of core participants, the proportions of the threads in each category are shown in Table 5. Two remarks can be made on the table: First, nearly half of their thread-initiating posts were concerned with verification of complete solutions. Considering this category together with (3) and (7), it can be suggested that core participants of the forum were mathematically active (in the sense of van de Sande, 2011) in at least two thirds of their threads. Second, the posting profiles of students S4 and S6 are similar and they can be addressed as *solution verifiers*, as about 90% of their thread-initiation posts belonged to this category. The profiles of S1 and S3 are also similar, but their posting behaviors were spread among various categories.

	S1	S2	S3	S4	S5	S6	Total
(1) Problem formulations	6%	9.6%	12.5%	3.2%	20%	5.3%	9.5%
(2) Full solutions	6%	6.4%	10%	9.7%	20%	0%	8.6%
(3) Partial solutions	15.2%	10.6%	10%	0%	15%	5.3%	9.2%
(4) Verification of full solutions	33.3%	68.1%	36.3%	87%	25%	89.4%	52%
(5) Clarification of classroom material	12.1%	1%	13.8%	0%	5%	0%	7.7%
(6) Problem-solving strategies	18.2%	1%	5%	0%	15%	0%	6.5%
(7) Inference	9%	3.2%	12.5%	0%	0%	0%	6.5%

Table 5: Categorization of thread-initiating posts

SUMMARY AND DISCUSSION

A hand full of studies have been concerned with online mathematical interactions (e.g., van de Sande, 2011). This study contributes to the emerging body of knowledge by focusing on a forum that accompanied a face-to-face course in linear algebra for high-school students. The data showed that the forum generated a considerable amount of mathematical post-exchanges between the students, when the vast majority of posts came from a small group of six students. Notably, in the study of Jacob and Sam (2008), who explored rewarding OAFs with hundreds of participants, the number of core participants was also around ten. Hence, comes the question "what does an intensive participation in an online discussion indicate in terms of one's learning?

While the reported findings are limited by the specificity of the explored setting, they allow making conjectures for investigations in further research. The technological platform that contained the explored forum, indicated that the posted threads were attended by all the course students. In informal conversations, peripheral participants indicated that they regularly read online discussions and the discussions were rich enough to fulfil their academic needs; consequently, they did not initiate threads of their own. This argumentation is typical to the 'lurking' participants of online communities (Preece, Nonnecke, & Andrews, 2004). On the one hand, the idea that an online interaction that occurs between a small group of core participants suffices for the whole course seems reasonable and entails practical recommendations. On the other hand, a positive correlation was found between the number of students' posts and their course achievements. Also, core participants outperformed their peripheral peers in quizzes and the final exam. This finding is also in line with Shaw (2012), who revealed in the context of language programming that the achievements of core participants were higher that the achievements of peripheral ones.

A possible interpretation for the achievement gap can be based on Moore and Kearsley's (1996) notion of transaction distance that was introduced for capturing students' involvement in distance courses. The researchers argued that more dialogue between a student and instructor indicated a smaller transaction distance, which is a signal of students' greater involvement. In our case, the dialogue occurred between the students, but it still seems to indicate an engagement with the course contents.

Another interpretation for the achievement gap can be based on the considerable body of knowledge on students' interactions in collaborative learning settings. The connections of online interactions to this body of knowledge can be illustrated with Leikin and Zaslavsky (1997) who found that when divided into small groups, students helped one another with mathematical explanations and error detection. The students in this study also sought and provided help of a similar type. The positive relations between providing face-to-face help and achievements has been well-documented. The researchers explain that help-providing necessitates students to recall, reorganize and articulate the learned material in the new ways, which contribute not only to the help-seekers but also to help-providers. Possibly, help-providers go through similar processes when the call for help comes from online. The question that remains, is what makes some students answer such a call.

References

Jacob, S. M., & Sam, H. K. (2008). Measuring critical thinking in problem solving through online discussion forums in first year university mathematics. *Proceedings of the IMECS*.

Kontorovich, I. (2018). Why Johnny struggles when familiar concepts are taken to a new mathematical domain: Towards a polysemous approach. *Educational Studies in Mathematics*, *97*(1), 5–20.

Krippendorff, K. (1980). *Content analysis: An introduction to its methodology*. London: Sage publications.

Leikin, R., & Zaslavsky, O. (1997). Facilitating student interactions in mathematics in a cooperative setting. *Journal for Research in Mathematics Education*, *28*(3), 331–354.

Moore, M. G., & Kearsley, G. (1996). *Distance education: A systems view*. Belmont, CA: Wadsworth.

Perkins, C., & Murphy, E. (2006). Identifying and measuring individual engagement in critical thinking in online discussions: An exploratory case study. *Educational Technology and Society*, *9*(1), 298–307.

Preece, J., Nonnecke, B., & Andrews, D. (2004). The top five reasons for lurking: improving community experiences for everyone. *Computers in Human Behavior*, *20*, 201–223.

Shaw, R. S. (2012). A study of the relationship among learning styles, participation types, and performance in programming language learning supported by online forums. *Computers & Education*, *58*, 111–120.

Van de Sande, C. (2011). A description and characterization of student activity in an open, online, mathematics help forum. *Educational Studies in Mathematics*, *77*(1), 53–78.

ACTIVATION AND MONITORING OF PRIOR MATHEMATICAL KNOWLEDGE IN MODELLING PROCESSES

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In a qualitative study with eighth to tenth graders (N=18), we investigated whether the activation of prior mathematical knowledge would promote or interfere with solution processes as students solved modelling problems. In addition, we analyzed the role of metacognitive monitoring of knowledge activation. Participants with different prior mathematical knowledge solved modelling problems in which multiple solution approaches were possible. We found that the activation of inappropriate prior mathematical knowledge negatively impacted modelling. Negative effects of prior knowledge also occurred if a second solution for a problem was required because learners stuck to the prior knowledge of their first approach. Monitoring of knowledge activation was rarely found, even when it would have been helpful.

INTRODUCTION

Building a mental model of a real-world situation is particularly important for solving modelling problems (Leiss, Schukajlow, Blum, Messner, & Pekrun, 2010). To build a mental model, students have to structure and simplify the information presented in the problem statement. To decide what information is important, they need to have at least a rough idea of a corresponding mathematical model in mind. Thus, students have to activate prior mathematical knowledge at the very beginning of the solution process. However, an initial strong focus on mathematical issues might occur at the expense of the development of a situational understanding and could lead to solutions that are not adequate from a realistic perspective. Metacognitive monitoring of the activated prior knowledge is considered to play an important role in the decision to either use or ignore the activated prior knowledge. The present article investigates the interplay between prior mathematical knowledge, modelling activities, and monitoring of knowledge activation, with the aim to better understand under what circumstances the activation of mathematical knowledge promotes or interferes with modelling processes.

THEORETICAL BACKROUND AND RESEARCH QUESTIONS

Effects of Prior Knowledge and Monitoring of Prior Knowledge on Performance

Prior knowledge is considered to be an important predictor of performance (Dochy, Segers, & Buehl, 1999). But under certain circumstances, the activation of prior knowledge can have negative effects, as the activation of inappropriate knowledge while solving mathematical problems can lead to a search in the wrong part of the problem space (Kaplan & Simon, 1990). Certain mathematical contents seem to trigger

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 243-250). Umeå, Sweden: PME.

inappropriate activation of prior mathematical knowledge. Students were previously found to activate knowledge of proportional relations even when this knowledge was not suitable for the problem at hand. Reasons are seen in the dominant role linearity plays in classrooms and everyday contexts (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). Further, it can be hypothesized that knowledge about the topic that was taught most recently is often activated regardless of its appropriateness because, in most classroom situations, this knowledge is typically needed to solve exercises and to succeed on tests. Metacognitive monitoring of knowledge activation was found to be helpful to avoid negative effects of prior knowledge on performance (Stillman, 2011; Stillman & Galbraith, 1998; Van Dooren & Matthew, 2015).

Role of Prior Knowledge for Mathematical Modelling

The translation of a real-world situation into a mathematical model is at the core of mathematical modelling. The translation process requires initial modelling activities such as understanding, structuring, and simplifying the real-world situation in order to transfer it into an adequate mental model of the situation that can be further mathematized (Blum, 2015). Modelling problems often contain superfluous information, and identifying the important information becomes part of the activities of structuring and simplifying. Prior mathematical knowledge can be considered necessary to identify the information that is required to develop a mathematical model. Hence, anticipations of mathematical knowledge might be needed to successfully carry out initial modelling activities. On the other hand, impulsively activated mathematical knowledge has been suggested to promote superficial solutions in which situational constraints are neglected, especially if no metacognitive activities to monitor the activation of knowledge are conducted (Stillman & Galbraith, 1998). Cue salience and its interaction with prior knowledge is thereby seen as particularly important because it can trigger the activation of inappropriate knowledge. Activation of inappropriate prior mathematical knowledge and a lack of metacognitive activities devoted to monitoring knowledge activation might account for why students have trouble solving modelling problems, but little is known about the interplay between these factors and students' solution processes.

Research Questions

These considerations led us to pose the following research questions:

1. To what extent does the activation of prior mathematical knowledge promote or interfere with modelling processes?
2. Is metacognitive monitoring used to determine the appropriateness of the activated mathematical knowledge?

METHOD

Participants and Data Collection

The sample involved 18 eighth to tenth graders (9 girls and 9 boys between the ages of 14 and 16) from four middle-track classes (German Realschule) from two different schools. We selected participants by following the principle of maximum variation sampling (Patton, 2015, p. 283). As selection criteria, we focused on the background variables mathematical ability, reading comprehension, and prior mathematical knowledge. Mathematical ability was estimated with math grades and reading comprehension via a general standardized test (Leiss et al., 2010). Mathematical knowledge about circles could help or inhibit problem solving. Thus, we chose eight students who had not yet covered this topic in their mathematics classes and ten students who had studied this topic before participating in the investigation. The interviews were conducted individually. First, each participant worked on the problems "Wind turbine" and "Ferris wheel" using the think-aloud method to verbalize his or her approach (Figures 1 and 2). Second, a stimulated recall interview was conducted in which the participant watched the problem solving videos along with the interviewer and commented on his or her own (i.e. the student's) actions spontaneously or when requested to do so by the interviewer. At the end of the stimulated recall interview, students were asked to find a second solution for the "Wind turbine" problem.

"Wind turbine" problem

Wind energy is the fourth largest type of energy in Germany and is therefore an important part of energy production. Because wind turbines are very large, they are also called wind giants. Overall, a wind turbine is about 150 meters high. The radius of the windmill is 45 meters. This is exactly the length of one of the blades. The three blades are mounted at a height of about 95 meters on a so-called nacelle. The nacelle is rotatable so that the blades of the wind turbine can align themselves with the wind direction. The speed at which the blade tip rotates is about 40 meters per second at an average wind speed. If the wind blows too hard, the system switches off. At a medium wind speed, a blade will return to its initial position after 6 seconds.

How many meters will the blade tip cover in one turn of the wind turbine?

Figure 1: "Wind turbine" problem

"Ferris wheel" problem

The London Eye is the third largest Ferris wheel in the world. It stands directly on the banks of the Thames. Overall, the Ferris wheel is 140 meters high and has a huge diameter of 125 meters. From the highest point of the Ferris wheel, you can see for 40 km. For passengers to board and exit, the wheel does not have to stop because it turns very slowly. The speed is only 10 meters per minute. A ride on the Ferris wheel is expensive. It costs 25 euros but also takes 40 minutes.

At what altitude above the water level will a person be 10 minutes after boarding?

Figure 2: "Ferris wheel" problem

To stimulate the activation of different prior knowledge, we decided to use problems to which different solution approaches could be applied. The first problem "Wind turbine" can be solved by either calculating the circumference of the circle ($C=2\cdot\pi\cdot 45m\approx 283m$) or using the proportional relation of time and travel distance ($d=40m*6=240m$). For the "Ferris wheel" problem, constructing an adequate mental model of the situation and recognizing that the position of the gondola after 10 minutes is a quarter of one rotation are crucial to applying an appropriate solution method. The result can be calculated by adding up the length of the radius and the height of the base (125:2+140-125=77.5 m). The problem can also be solved with other approaches (e.g. trigonometric functions), but other approaches did not come up in the interviews.

Data Analysis

The problem solving and stimulated recall interviews were transcribed and sequenced. Sequences of the stimulated recall interviews were assigned to the related problem solving sequences in order to collect more indications of whether prior knowledge was activated. The transcripts were analyzed using qualitative content analysis (Mayring, 2014). A category scheme was used to code the sequences with regard to modelling activities, prior mathematical knowledge, metacognitive monitoring of knowledge activation, and the appropriateness of the solution. More specifically, the modelling activities were divided into initial modelling activities (understanding/structuring the problem) and later modelling activities. Prior mathematical knowledge was categorized into subcategories referring to different mathematical contents (e.g. circle calculation or proportional relations). The occurrence of metacognitive monitoring of knowledge activation was recorded. Different solution qualities (correct, partial, incorrect and processing canceled) and different qualities of the mental model of the situation (adequate, not adequate) were distinguished. Content-analytical quality criteria such as the stability and reproducibility of the analysis were tested by calculating intra- and inter-coder reliability for more than a quarter of the material with satisfactory agreement (Cohen's kappa calculated for each dimension ranged between $.691 \leq \kappa \leq .878$). Disagreements about the coding were discussed and validated consensually.

RESULTS

Because of space limitations, we present only the most important results and exemplarily sketch two examples of solutions to the "Ferris wheel" problem in which aspects of prior mathematical knowledge were found to promote or interfere with problem solving.

For the first research question, we analyzed what kind of prior mathematical knowledge was activated and how this knowledge interacted with the modelling processes. Learners who had prior knowledge of circle calculation often activated this knowledge ("Wind turbine" problem: 6 of 10 students; "Ferris wheel": 6 of 10 students). For the "Wind turbine" problem, they activated this knowledge even more often than knowledge that referred to proportional relations, although the approach of calculating the circumference of the circle is more difficult and prone to errors (circle calculation: 6 of 10 students; proportional relations: 2 of 10 students). This was found despite the fact that these learners also had prior knowledge of proportional relations, which was verified in the interviews. Learners without prior knowledge of circle calculation usually used prior knowledge of proportional relations ("Wind turbine" problem: 6 of 8 students; "Ferris wheel": 5 of 8 students). Regarding the supporting or interfering effect of the activated knowledge, we found a big difference between the problems. For the "Wind turbine" problem, in one third of the solution processes, knowledge of circle calculation or proportional relations was already activated in the initial modelling activities of understanding and structuring. In the largest number of cases (12 of 18 students), the activation of knowledge of circle calculation or proportional relations led to appropriate approaches and correct solutions. But after applying one approach, most learners had trouble applying a second approach. They tried to apply their prior knowledge of their first approach again, but they did not step back and activate their prior knowledge of other mathematical contents. The transcript below illustrates this difficulty as described by one of the learners.

> 29:25 158 Ella: So this problem, the first one ["Wind turbine"], I thought was relatively easy because, as I said before, you only had to calculate the circumference here. But the first solutions are always easy, but then to come up with the second … because then you are so fixated on one calculation and then you also think that it is now the only one. It's just difficult then to still be open to another way.

In the "Ferris wheel" problem, activation of knowledge about circle calculation or proportional relations in initial modelling activities was often found to be accompanied by inadequate mental models of the situation (15 of 18). For example, students who activated prior knowledge of circle calculations in initial modelling activities (5 of 10) figured out that this was not fruitful and either applied a second approach (3 of 10) or canceled their processing (2 of 10). On the other hand, the activation of prior knowledge of proportional relations (10 of 18) typically led to a single attempt in which the learners used this knowledge to calculate the distance traveled instead of the height above the water level as requested and reported the distance traveled as a result (10 min·10 m/min=100 m). Hence, in almost all cases, the activation of prior knowledge of proportional relations resulted in incorrect solutions.

The second research question was about the use of monitoring activities to control the activation of prior mathematical knowledge. Monitoring of knowledge activation was found only very rarely ("Wind turbine" problem: 2 of 18; "Ferris wheel" problem: 1 of 18). In particular, for initial modelling activities, no metacognitive monitoring was found at all. Moreover, there were no differences between the "Wind turbine" and "Ferris wheel" problems, even though for the "Ferris wheel" problem, it was essential to monitor one's knowledge activation in order to recognize the inappropriateness of certain prior knowledge. Moreover, we found indications that even if students identified contradictions in the solution, they did not change their solution. The case of Pia presented below exemplifies this issue.

In the following, two solutions to the “Ferris wheel” problem are sketched. In the first case, Tabea is a learner with high reading comprehension skills and high mathematical performance. Her solution process is characterized by a long period in which she engages in the initial modelling activities of understanding and structuring. Although Tabea has no prior knowledge of circle calculation, she activates such knowledge and mentions that “hopefully this has nothing to do with π.” She later explains that she knows about π because of a poster in her classroom. Her first idea is to calculate the circumference of the circle and divide the result, but she does not know how to do it. She mentions that “there must be something else that I have overlooked” and starts to read the problem statement again and transfers important information into a sketch (Figure 3). The sketch and her prior knowledge of fractions help her to recognize that 10 minutes corresponds to a quarter rotation. She calculates the length of the radius and interprets it as equal to the height she was searching for. However, her solution fails to take into account the base of the Ferris wheel.

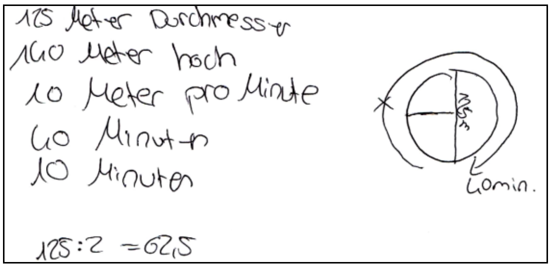


Figure 3: Tabea’s solution to the “Ferris wheel” problem

The second case is Pia, a student with rather weak reading comprehension skills and weak mathematical performance. Like Tabea, her process of solving the “Ferris wheel” problem begins with a long period in which she engages in initial modelling activities. She reads the problem statement several times and also sketches the situation. Pia uses prior knowledge of proportional relations to interpret the speed of 10 m per minute as “in one minute, I am ten meters high” and to create a table to calculate the distance traveled after ten minutes (Figure 4, left). In the sequences presented below, she writes down and comments on her solution.

18:06	37	[*pause*] So, I am not one hundred percent sure, but um.
18:10	38	[*writes*] After ten minutes, it is located at a height of 100 meters. Okay, I’m done.

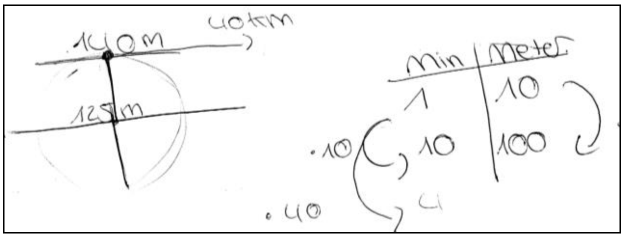
I: Are you at least satisfied with your solution?

P: No, not really, actually this is not right.

I: Okay, what is wrong?

P: That, if you are 100 meters high, you have actually only gone this far [draws a sketch (Figure 4, right) to explain the difference between the distance traveled and the altitude].

In the stimulated recall interview, Pia is able to explain that she is aware of the discrepancy between her solution, which presents the distance traveled, and the height she was searching for (Figure 4, right).



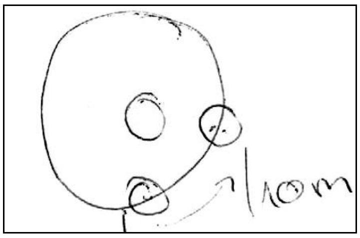


Figure 4: Pia's solution and the sketch in which Pia explains the discrepancy between her solution and the question

In summary, Tabea is one of the rare examples where the activation of inappropriate prior knowledge did not lead to an incorrect solution to the "Ferris wheel" problem (Tabea's solution was categorized as partially correct). On the other hand, in Pia's case, her prior knowledge of proportional relations was used to come up with a superficial solution, and even her recognition of discrepancies did not lead her to search for appropriate prior knowledge.

SUMMARY AND DISCUSSION

In the present study, we investigated whether the activation of prior mathematical knowledge would promote or interfere with solution processes in solving modelling problems. The positive or negative impact of the activated prior mathematical knowledge depended on the appropriateness of the knowledge. Students tended to activate inappropriate knowledge if some information in the problem statement looked promising at first glance but did not match the problem's demands. In these cases, especially the activation of prior mathematical knowledge in initial modelling activities was accompanied by inadequate mental models of the situation and incorrect solutions. This can be considered an indication that supports the hypothesis that impulsively activated mathematical knowledge can promote superficial solutions (Stillman & Galbraith, 1998). Prior knowledge of proportional relations and circle calculation were both activated frequently, even if these types of knowledge were not appropriate for solving the problem at hand. The inappropriate activation of knowledge of proportional relations is in line with previous research that demonstrated that students tend to overgeneralize proportional relations (Van Dooren et al., 2005). Students' frequent activation of prior knowledge of circle calculation indicates that the most recently learned subject is an important although unexplored factor that should be addressed in future studies. Further, it was found that learners had trouble finding a second solution because they stuck to the prior knowledge they had activated for the first solution. This indicates that a first solution impedes the search for a second solution, and this should be considered an aggravating factor when multiple solutions are required. A low occurrence of metacognitive monitoring was found, although in some of the solution processes, metacognitive activities could have helped students recognize the inappropriateness of the activated knowledge and might have stimulated a search for prior knowledge that was more appropriate. Therefore, a lack of metacognitive monitoring can also be considered as one reason for students' low success in solving the modelling problems (Stillman & Galbraith, 1998). Teaching methods that were found to stimulate monitoring activities such as prompting each student from the very beginning to find two solutions (Schukajlow & Krug, 2013) might help students recognize the inappropriateness of prior knowledge.

Despite methodological limitations such as the limited number of participants, our findings can contribute to a better understanding of the role that prior mathematical knowledge plays in modelling processes and might inspire further studies.

References

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do? In J. S. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 73-96). New York: Springer.

Dochy, F., Segers, M., & Buehl, M. (1999). The relation between assessment practices and outcomes of studies: The case of research on prior knowledge. *Review of Educational Research*, 69(2), 145-186.

Kaplan, C. A., & Simon, H. A. (1990). In search of insight. *Cognitive Psychology*, 22, 374-419.

Leiss, D., Schukajlow, S., Blum, W., Messner, R., & Pekrun, R. (2010). The role of the situation model in mathematical modelling-Task analyses, student competencies, and teacher interventions. *Journal für Mathematikdidaktik, 31*(1), 119-141.

Mayring, P. (2014). *Qualitative content analysis: Theoretical foundation, basic procedures and software solution*. Klagenfurt: Beltz.

Patton, M. Q. (2015). *Qualitative research and evaluation methods: Integrating theory and practice* (4th ed.). Los Angeles: Sage.

Schukajlow, S., & Krug, A. (2013). Planning, monitoring and multiple solutions while solving modelling problems. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 177-184). Kiel, Germany: PME.

Stillman, G. (2011). Applying metacognitive knowledge and strategies in applications and modelling tasks at secondary school. In G. Kaiser, W. Blum, R. B. Ferri, & G. Stillman (Eds.), *Trends in Teaching and Learning of Mathematical Modelling ICTMA14* (pp. 165–180). Berlin: Springer.

Stillman, G., & Galbraith, P. L. (1998). Applying mathematics with real world connections: Metacognitive characteristics of secondary students. *Educational Studies in Mathematics*, 36(2), 157-194.

Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2005). Not everything is proportional: Effects of age and problem type on propensities for overgeneralization. *Cognition and Instruction*, 23, 57–86.

Van Dooren, W., & Matthew, I. (2015). Inhibitory control in mathematical thinking, learning and problem solving: A survey. *ZDM*, 47(5), 713-721.

PRIMARY STUDENT’S DATA-BASED ARGUMENTATION – AN EMPIRICAL REANALYSIS

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Despite its importance for informed citizenship, empirical research into student's abilities in developing data-based argumentations is relatively scarce and needs to be broadened, in particular as far as primary students are concerned. In a reanalysis of data from more than 380 primary students, this research need is addressed. The study describes key elements of data-based argumentation in the intersection domain of statistical thinking and critical thinking, drawing on a framework focused on scientific reasoning. A corresponding coding affords insight into primary student's approach to data-based argumentation, both into their strengths and difficulties.

INTRODUCTION

Dealing with data has developed to be a standard curricular element in the primary mathematics classroom in many countries and learning goals aiming at student’s statistical literacy receive growing emphasis. Among other, statistical literacy should enable learners to critically evaluate whether and how claims which express specific interpretations of data are supported by the data. Evaluating such claims often requires data-based argumentation. However, the base of empirical evidence about student’s abilities related to developing and evaluating data-based arguments is still scarce and needs to be broadened, including a need for conceptualisations of how data-based argumentation interdepends with statistical thinking and critical thinking. In particular, relatively little is known whether already primary students are able to generate data-based arguments and what difficulties they may encounter.

This paper consequently addresses this research need. Based on a theoretical perspective which links theories related to statistical thinking and approaches to critical thinking, key requirements of data-based argumentation are described. The connecting framework affords the development of a coding which identifies key aspects of successful data-based argumentation. The results of the coding suggest that it is generally possible for primary students to successfully generate data-based arguments. Further, the analysis yields insight into potential difficulties of the learners.

The following first section introduces the theoretical framework for the analysis and leads to the research questions. We then report on sample and methods, present results, and discuss them in a concluding section.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 251-258). Umeå, Sweden: PME.

THEORETICAL BACKGROUND

Can primary students develop data-based argumentations? What obstacles may they encounter? – Figure 1 gives an overview of relevant aspects that may play a role for the development of data-based arguments: Firstly, primary students might struggle with requirements of *statistical thinking* (Kuntze, Aizikovitsh-Udi & Clarke, 2017, cf. Shaughnessy, 2007; Wild & Pfannkuch, 1999). For instance, students might not be able to read data from a diagram (Reading, 2002), or to deal appropriately with statistical variation (Watson & Callingham, 2003) – a requirement which appears often when working with statistical data. Wild and Pfannkuch (1999, cf. Gal, 2002) furthermore mention critical thinking skills or a critical stance as a necessary component of statistical thinking. However, they hardly explain in detail how statistical thinking elements may interact with critical thinking strategies. In particular, a link with the large body of existing theories about critical thinking is hardly made in Wild and Pfannkuch's model. This is why Figure 1 shows *critical thinking* (CT) as a second relevant area: When having to generate data-based argumentations, students may lack of CT strategies, e.g. they might not question given interpretations of data, be open towards alternative interpretations of data, search for potential contradictions with available data, etc. Approaches to CT have developed catalogues of strategies and dispositions related to CT on a non-content-specific level. For example, Ennis (1987) defined CT as "reasonable, reflective thinking that is focused on deciding what to believe and do" (p. 10). He distinguished dispositions, such as a *critical spirit* and being open-minded, from *skills*, which include questioning interpretations (own as well as those from other sources) and tacit assumptions (Ennis, 1989). However, beyond Ennis' work, there is a variety of different approaches to CT and a variety of CT definitions. Content-domain-related considerations related to statistical thinking have hardly been elaborated in these approaches.

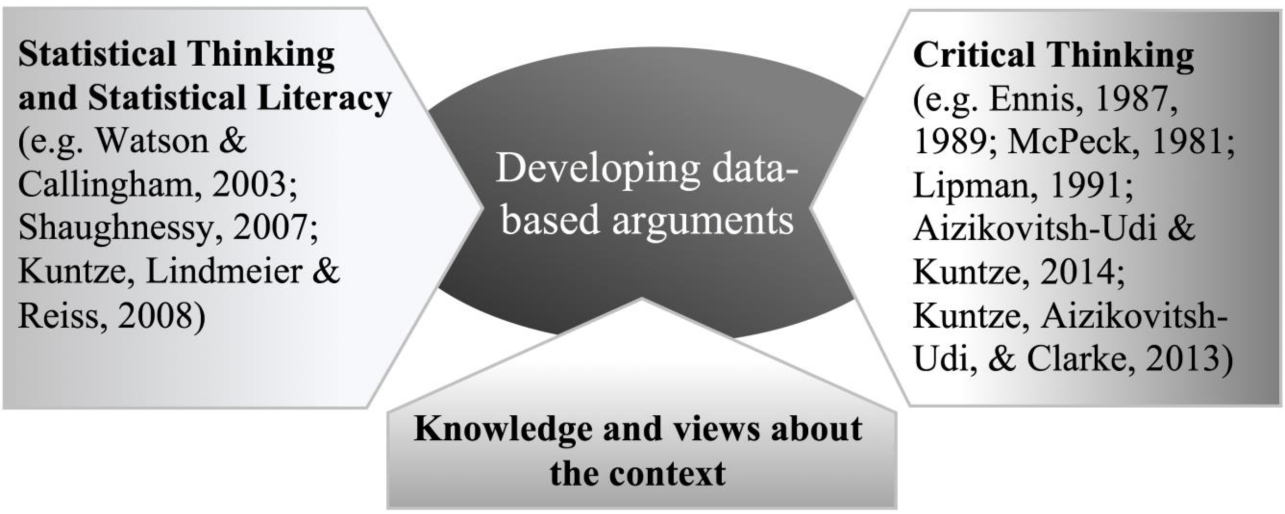


Figure 1: Interplay of statistical thinking/statistical literacy, critical thinking as well as knowledge and views about the context (cf. Kuntze, 2016).

We would like to add that the arrows shown in Figure 1 are meant to have intersections and to interact: For example, students' views about a situation context may hinder or support them in questioning interpretations of data about this situation context, i.e. to apply CT strategies. CT may support or even be part of statistical thinking (e.g. Aizikovitsh-Udi, Kuntze & Clarke, 2013), but CT elements may also be so dominant, that they impede statistical thinking (ibid.).

In line with the theoretical background developed in more detail in Kuntze, Aizikovitsh-Udi and Clarke (2017), we choose a *scientific reasoning* perspective (e.g. Bullock & Ziegler, 1999; Klahr & Dunbar, 1989; Kuhn, 2010; Kuhn, Amsel & O'Loughlin, 1988; Kuntze, 2004) in order to have a base for a simultaneous focus on CT and ST. This perspective affords describing key requirements of data-based argumentation. The key issue of this perspective is a clear distinction between data (playing the role of *evidence* in scientific reasoning) and interpretations of data (playing the role of *hypotheses* or *theory* in scientific reasoning). If evidence contradicts the theory in scientific reasoning, the theory/hypothesis has to be rejected and new hypotheses have to be developed that are consistent with the evidence. For data-based argumentation, this means that a contradiction between a claim or an interpretation of data (theory) and the data (evidence) is an appropriate argument for a negative evaluation of the claim/the interpretation.

In the example given in Figure 2 below, the doctor's claim can be rejected as the data given in the lower part of the diagram shows that not all persons who have been cured with tablet 2 have recovered earlier than the persons in the tablet 1 subgroup.

Accordingly, data-based argumentation requires linking a claim/an interpretation of data with relevant available data and drawing a conclusion for the claim/for an interpretation. Following Toulmin's (2003) terminology, the link with the available data *warrants* this conclusion. Describing data-based argumentation in Toulmin's framework also helps to deal with the situation that when working with statistical data, contradicting but appropriate data-based argumentations can be developed even on the base of the same data set (see example in Figure 2).

Studies have shown that even primary students are already able to apply strategies of scientific reasoning and that improving such skills of primary students is possible (cf. Bullock & Ziegler, 1999; Sodian, 2008). Nevertheless, these and further studies also show that children often seek for confirming evidence only and tend to accept interpretations too hastily. It has also been observed that children rather use experimental evidence for illustrating and confirming existing theories than for challenging them (Kuhn et al., 1988; Bullock & Ziegler, 1999; Klahr & Dunbar, 1989). In contrast, as laid out above, an important skill for dealing with data is to challenge data-related claims actively by seeking for counter-evidence in data (Kuntze et al., 2013). When primary students have to generate arguments on the base of data, we thus expect that seeking for counter-evidence could be an obstacle for some children.

RESEARCH INTEREST

Empirical research about primary student's data-based argumentation is needed, in particular under the theoretical scope described in the previous section. Consequently, the core aim of this study is to provide answers to the following questions:
(1) *Is it possible for primary students to generate data-based arguments?*
(2) *Is it possible to detect specific difficulties primary students encounter when they develop data-based arguments?*

DESIGN AND SAMPLE

The reported study is based on a reanalysis of data from an earlier project (e.g. Kuntze, Martignon, Vargas & Engel, 2015). We analysed answers from N=385 German year four primary students (191 female, 193 male; average age *M*=10.0, *SD*=0.61). The task the students had to answer (see Figure 2) presents a diagram on recovery times of two sorts of tablets against headache. The students were asked to find arguments in favour and against a given claim by a doctor, who prefers the second tablet.

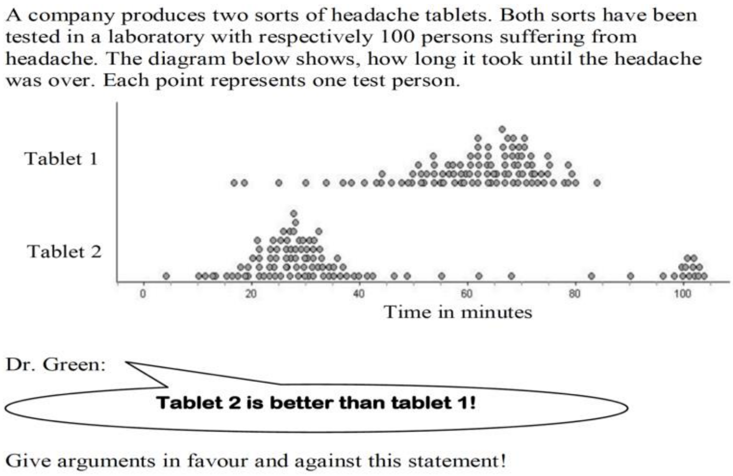


Figure 2: Task the students had to answer (Kuntze et al., 2015, p. 11).

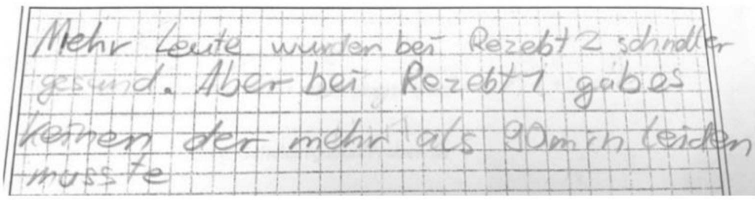
In a first step of analysis, we used a top-down coding derived from the theoretical background introduced above. According to the requirements expressed in the task, a successful answer (Code A) can be expected to contain at least one complete argument in favour and one argument against the given claim. Each argument has to refer to given data, which means, that a relevant and consistent connection with the given data has to be made supporting resp. contradicting the doctor's claim.

Moreover, we expected that there might be partial answers containing only one data-based argument in favour *or* against the claim. Such answers were coded as well (code B); it was coded in addition whether the student's argumentation was in favour or against the given statement. All answers, which did not fulfill the requirements of code A or B were assigned to code C. This code thus contains answers without successful arguments, including cases with blank response fields. The overall measured inter-rater reliability of the top-down coding was satisfying (κ=.853). In all cases of initial disagreement, agreement among the raters about the code assignment could be reached in subsequent criteria-based discussion.

For answering the second research question on potential difficulties, we subjected the answers, which were in the top-down coding assigned to code C, to a bottom-up analysis. Following Mayring's (2015) approach of qualitative content analysis, we developed a set of distinct categories. The reliability of the resulting coding was ensured by a follow-up top-down-rating of all answers: In this rating, the answers were assigned to the categories which had emerged from the bottom-up analysis. The coding by two raters reached an inter-rater reliability of κ_{Cohen}=.964.

RESULTS

The top-down coding shows that almost 17% of the primary students were able to generate data-based arguments in favour and against the given claim as required (code A). Figure 3 shows a sample answer. This answer first states that "more people recovered faster with tablet 2", which refers to a comparison of the two accumulations of cases shown in the diagram in Figure 2. The second statement in the answer is a consideration of single cases shown in the diagram ("there was nobody") which supports a negative evaluation of the doctor's statement, even if a direct reference to the doctor's statement is not made. With the word "but" the student indicates that now the perspective changes from supporting to challenging the doctor's claim.

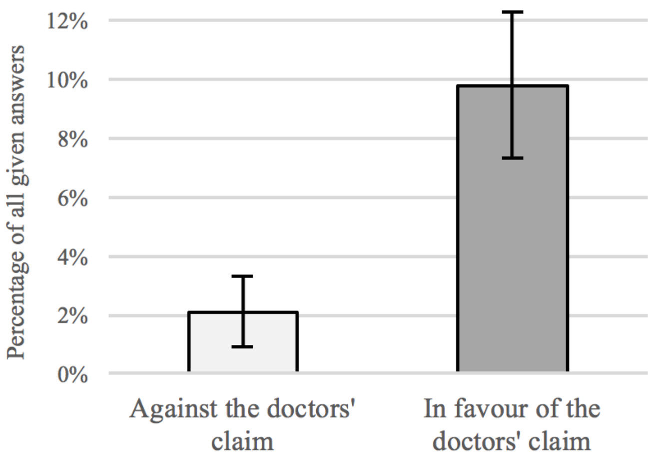


Translation:
"More people recovered faster with tablet 2. However, with tablet 1 there was nobody who had to suffer for more than 90min."

Figure 3: Answer assigned to code A.

Further 11,9% of the primary students were able to generate at least one reasonable argument based on data in favour *or* against the given claim (code B). The remaining cases (71.2%) did not fulfil the requirements of data-based arguments described above (code C).

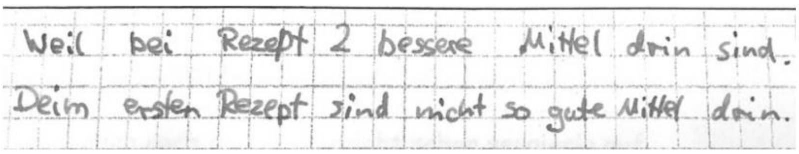
The *second research question* focused on evidence about student's difficulties. We start with the partial answers (code B). The data in Figure 4 suggest that more children generated arguments which support the given claim rather than identifying evidence against it (9.8% vs. 2.1%). This means that more students who generated only one argument referred to confirming evidence



than to counter-evidence in the given data.

The bottom-up analysis of the code C answers offers a more detailed view of different types of potential difficulties of the primary students. We identified several sub-categories of code C. 19.2% of the whole sample did not answer at all. Another 19.2% of the answers could not be evaluated as they were unreadable or the content of the student's answers could not be reconstructed in the context of the task. 10.4% just mentioned a claim but did not substantiate it in any way (e.g. "Tablet 2 is better than tablet 1."). 1.6% of the student's answers showed operations of counting or calculating which had to be coded as irrelevant for solving the task. These students, for example, counted the points of the given diagrams or divided any numbers. In 3.4% of the analysed answers data were mentioned, but there was no evidence of any implication being drawn by the student in favour or against the given claim. 11.5% of the answers formally contained data-based arguments, but these references to data were based on inconsistent interpretations of the given data. For example, such students interpreted the data points as the number of tablets which have to be taken depending on time.

In further 5.9% of the sample, views or knowledge about the context played a major role. We were also able to differentiate these answers into two distinct categories. In 3.6% of the answers, views or knowledge about the context was so dominant that data-based arguments were absent (see for example answer presented in Fig. 5).



Translation:
"Because tablet 2 contains better substances. The first Tablet doesn't contain such good substances."

Figure 5: Example of an unsuccessful answer based on views about the context.

The remaining of 2.3% generated elements of data-based argumentation which were however dominated or rendered invalid by additional contradicting statements based on views or knowledge about the context.

DISCUSSION AND CONCLUSIONS

The results show that already primary students can be able to generate data-based arguments. In our study, almost 17% of the primary students were able to generate arguments both in favour and against the given claim. This shows that these students were in particular able to evaluate the given claim negatively after successfully having found counter-evidence in the given data. Further 11.9% of the primary students developed at least one argument, which means that at least more than a quarter of the primary students showed abilities of entering in data-based argumentation.

Among the students who gave only one argument, a majority developed arguments, which were confirming to the given claim. These students hence did not mention evidence contradicting the given claim. These students should be fostered with respect to CT strategies with relevance for scientific reasoning, so that their awareness of seeking for counter-evidence may increase. The findings thus inform follow-up research: potential effects of corresponding learning environments will be examined in future studies.

The findings also indicate that knowledge and views about the context can play an ambivalent role for generating data-based arguments. On the one hand, it is basically needed to interpret the given data against the background of the context they refer to. It can be assumed that those students who were able to generate an answer rated with code A or B, would not have been able to do so, if they had not successfully used context knowledge about medicine and headache. On the other hand, some students restricted their data-based arguments by inconsistent context-related statements. Around 4% students also did not even mention the given data and instead used rather speculative ideas based on views about the context.

Also beyond these specific groups of students, learning environments focused in data-based argumentation could improve primary student's corresponding abilities: Knowledge about the status of data-related claims/interpretations on the one hand and data on the other according to a scientific reasoning perspective might help students to succeed in data-related argumentation tasks. We expect that such interventions will reduce the frequency of blank answers and at the same time help students to focus also on data even if they have strong knowledge and views about the context. In this sense, meta-knowledge about data-based argumentation could be a key for the learners.

References

Aizikovitsh-Udi, E. & Kuntze, S. (2014). Critical Thinking as an Impact Factor on Statistical Literacy – Theoretical Frameworks and Results from an Interview Study. In K. Makar, B. de Sousa & R. Gould (Eds.), *Sustainability in statistics education. Proceedings of ICOTS9.* Voorburg, The Netherlands: International Statistical Institute.

Aizikovitsh-Udi, E., Kuntze, S. & Clarke, D. (2013). Exploring the Relationship between Statistical Thinking and Critical Thinking. Paper presented in AERA 2013, USA. Retrieved [03.01.2018] from the AERA Online Paper Repository.

Bullock, M. & Ziegler, A. (1999). Scientific reasoning: Developmental and individual differences. In F. Weinert & W. Schneider (Eds.), *Individual Development from 3 to 12. Findings from the Munich Longitudinal Study* (pp. 38-60). Cambridge: University Press.

Ennis, R. H. (1987). A taxonomy of critical thinking dispositions and abilities. In J. B. Baron & R. J. Sternberg (Eds.), *Teaching thinking skills: Theory and practice* (pp. 9-26). New York: Freeman.

Ennis, R. H. (1989). Critical thinking and subject specificity: Clarification and needed research. *Educational Researcher, 18*, 4-10.

Gal, I. (2002). Adult's statistical literacy: Meanings, components, responsibilities. *International Statistical Review 70,* 1–51.

Klahr, D. & Dunbar, K. (1989). Developmental differences in scientific discovery process. In D. Klahr & K. Kotovsky (Eds.), *Complex information processing* (pp. 109-143). Hillsdale: Erlbaum.

Kuhn, D. (2010). What is scientific thinking and how does it develop? In U. Goswami (Hrsg.), *Handbook of Childhood Cogn. Development* (2nd ed.) (pp. 371-393). Oxford: Blackwell.

Kuhn, D., Amsel, E. & O'Loughlin, M. (1988). *The development of scientific thinking skills.* San Diego, California: Academic Press.

Kuntze, S. (2004). Wissenschaftliches Denken von Schülerinnen und Schülern bei der Beurteilung gegebener Beweisbeispiele aus der Geometrie. *Journal für Mathematik-Didaktik, 25(3/4),* 245-268.

Kuntze, S. (2016). Understanding statistics about society between statistical thinking and critical thinking – the role of individual context knowledge. In J. Engel (Ed.), *Promoting understanding of statistics about society. Proceedings of the Roundtable Conference of the International Association of Statistics Education (IASE),* July 2016, Berlin, Germany.

Kuntze, S., Aizikovitsh-Udi, E. & Clarke, D. (2013). Strategies for evaluating claims – an aspect that links critical thinking and statistical thinking. In Lindmeier, A. & Heinze, A. (Eds.). *Proceedings of the 37th Conf. of the IGPME,* Vol. 5 (p. 235). Kiel: PME.

Kuntze, S., Aizikovitsh-Udi, E. & Clarke, D. (2017). Hybrid task design: connecting learning opportunities related to critical thinking and statistical thinking, *ZDM 49*(6), 923-935.

Kuntze, S., Lindmeier, A. & Reiss, K. (2008). "Using models and representations in statistical contexts" as a sub-competency of statistical literacy – Results from three empirical studies. *Proceedings of ICME 11*. [http://tsg.icme11.org/document/get/474].

Kuntze, S., Martignon, L., Vargas, F. & Engel, J. (2015). Competencies in understanding statistical information in primary and secondary school levels: An inter-cultural empirical study with German and Colombian students. *AIEM*, 7, 5-25.

Lipman, M. (1991). *Thinking in education*. New York: Cambridge University Press.

Mayring, P. (2015). *Qualitative Inhaltsanalyse: Grundlagen und Techniken* (12th ed.). Basel: Beltz.

McPeck, J. (1981). *Critical Thinking and Education*. New York: St. Martin's Press.

Reading, C. (2002). Profile for statistical understanding. In B. Philips (Ed.), *Developing a statistically literate society. Proceedings of the 6th International Conference on Teaching Statistics,* Cape Town (South Africa). Voorburg: International Statistics Institute.

Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. K. Lester (Ed.), *The second handbook of research on mathematics teaching and learning* (pp. 957-1010). Charlotte, NC: Information Age Publishing.

Sodian, B. (2008). Entwicklung des Denkens. In R. Oerter & L. Montada, *Entwicklungspsychologie* (6th ed.) (pp. 434-479). Weinheim: Belz.

Toulmin, S. (2003): *The use of argument* (2nd ed.). Cambridge: University Press.

Watson, J. & Callingham, R. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal, 2*(2), 3-46.

Wild, C. & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review, 3,* 223-266.

FINNISH PRIMARY TEACHERS' INTERACTION WITH CURRICULUM MATERIALS – DIGITALISATION AS AN AUGMENTING ELEMENT

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This paper investigates how Finnish primary teachers talk about their interaction with curriculum materials, especially the additional facilities that digitalisation and technology provide to mathematics education. Digital curriculum materials are seen as part of available resources for teaching and learning mathematics. The data of this qualitative study consists of semi-structured interviews with seven primary teachers. Six thematic categories emerge in the data illustrating the elements that teachers consider crucial in evaluating and using the curriculum resources. The Finnish teachers prove to be critical and strategic consumers who understand the potential of the digital curriculum materials but make decisions about the use primarily in terms of enhancing student learning.

INTRODUCTION

Digital resources, theorizing the character of them and research on how they transform educational processes and practices have been recently under elaboration (Pepin, Choppin, Ruthven & Sinclair, 2017). While we know relatively much about teachers' interaction with printed curriculum resources (e.g. Brown, 2009; Remillard, 2005), research on the interaction with digital resources has yet to be fully explored. There has been a concern about how teachers manage to choose among the rapidly changing and easily available digital tools for mathematics learning (Hollebrands, 2017), and if they tend to seek for new resources in the first place (Tanhua-Piiroinen, Viteli, Syvänen, Vuorio, Hintikka & Sairanen, 2016). This paper reports an exploratory study that sets a ground for a larger scale cross-cultural research aiming to increase our understanding of the capacity required for teachers to use these resources well and the factors that influence it. We need to fill the gap in our knowledge about, on the one hand, how the growing supply of digital curriculum resources impact teachers' classroom practices and, on the other hand, how teachers perceive the ongoing change and expectations to be met.

Finnish teachers have great autonomy in making decisions about the supply of curriculum resources and the way they wish to utilise such materials in their mathematics classes. Still, the development of mathematics curriculum materials and teacher guides in particular have had an important role in enhancing new ways of teaching mathematics in Finland (Pehkonen, 2004). Finnish curriculum materials are commercially

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 259-266). Umeå, Sweden: PME.

produced with no national inspection of them. Information of upcoming curriculum reforms is available in public that enables publishers to produce materials that are in line with the current national core curriculum setting the outline for school education.

This paper focuses on teachers' stance towards digital curriculum materials as part of various resources available for teaching and learning mathematics. Earlier research has often focused on the use of either traditional or digital curriculum materials but instead, our approach is to consider the curriculum resources to comprise a whole package despite the source or the form of the material (Ruthven, 2014; cf. Pepin et al., 2017). Especially, the aspects characterising teachers' perception of the curriculum materials and thus serving the basis for choosing and using particular resources are at the core of the study. The research question is how the Finnish teachers perceive digital curriculum material in their mathematics teaching.

THEORETICAL FRAMEWORK

There is a need for understanding the foundations for change and potential when applying digital curriculum resources in mathematics classroom (e.g. Pepin et al., 2017). The globalization of the curriculum publishing industry and the fact that digital resources are available to teachers throughout much of the world generate a new setting for studies on curriculum use. Recently, it has been argued that the research field should focus on digitalization from a teacher's perspective, building on the knowledge of teachers' use of print resources, and taking into account features that are unique to digital resources. The demands placed on teachers and potential to support them should be considered in such research (e.g., Hoyles & Lagrange, 2010), particularly since there is evidence to suggest that particular characteristics of digital resources put different demands on the teacher (Remillard, 2016).

One theoretical perspective proposed by Remillard (2005) conceptualizes teachers' curriculum use as a dynamic interplay between the teacher and the curriculum resource, and thus, it views the curriculum use as a participatory process rather than a passive process of implementation. Along this line, a construct frequently referred to is Pedagogical Design Capacity (PDC) (Brown, 2009). PDC refers to "an individual teachers' capacity to perceive and mobilize existing resources in order to craft instructional episodes" (p. 29). This capacity includes the skill required to perceive and interpret the affordances of curricular resources and make decisions about how to deploy them to planning for instruction. Still needed is research on teachers' PDC in relation to digital resources.

Teachers seem to face a challenge when applying new digital resources in the classroom. Ruthven (2014) discusses the role of teaching expertise underpinning the successful use of digital technology in the mathematics classroom. In his framework, the tension arises from trying to apply new digital resources in line with existing elements, such as textbooks and traditional facilities. Hollebrands (2017) brings about the challenge of educating future teachers to be competent and willing to choose critically from the available curriculum resources in order to enhance student learning. For example, prospective teachers' stance towards digital curriculum resources are found to be characterised by the aspects related to surface features of the software and providing a motivational tool, for example, fun in mathematics classroom rather than deeper engagement with enhancing mathematical understanding (Johnson and Suh, 2009; Smith, Shin & Kim, 2017). Contrary to these findings, Pepin et al (2017) highlight three features that make the use of digital curriculum resources beneficial for teachers: 1) flexibility in terms of adaptation and redesign when applying the resource and potentially work in social and professional environment; 2) potential for differentiation and personalisation when addressing the needs of individual students; and 3) tools for assessment, namely access to pupil learning and potential for monitoring the progress.

METHOD

This qualitative case study (Bryman, 2012) is based on insights emerging in the interviews with seven Finnish primary teachers in autumn 2017. Since the aim was to understand various approaches into the use of curriculum resources and the way teachers evaluate mathematics curriculum materials as part of their work, we invited primary teachers representing different grade levels (1-6) and teaching experience, different schools, school regions and school size to participate in the study. The data consists of one-hour semi-structured interviews based on the themes related to 1) teacher background and school environment, 2) the curriculum resources in use, 3) views on curriculum material usage, and 4) views of teaching and learning mathematics. The interview took place in the classroom of each teacher that allowed the researcher to see the environment and look at the curriculum materials during the interview if needed.

The analysis started with transcribing the recorded data and identifying the three aspects that Pepin et al. (2017) associate with the beneficial use of digital curriculum resources. Three additional themes, i.e. supplementary facilities of realization, contribution to teaching and learning mathematics, and practical aspects, emerged from the data along the analysis. The trustworthiness of the study is strengthened by a pilot study for testing the original interview protocol in spring 2017. Furthermore, the analysis was carried out in several cycles parallel by two first authors that helped to ensure a consistent and trustworthy manner of the analysis. (cf. Bryman, 2012)

RESULTS

Teachers consider six emerging features when reflecting on their relation with digital resources as part of the available mathematics curriculum material and the use of them in teaching mathematics.

Flexibility in terms of adaption

The most usual way to utilize the flexibility of the digital materials is to modify the available tests that are included in teachers' curriculum material. The teachers stated that they select the test items in accordance with what they have taught and what students could possibly manage.

> …I actually try to select such tasks that I assume my students to understand. Not necessarily that easy but similar to assignments that we've done in the class (Teacher 3)

Teachers found that the flexibility of available digital materials varies. On the one hand, the conveyance tools of the curriculum resource (Dick & Hollebrands, 2011) are seen stiff, not flexible. If the content and the logic of animations are not in line with teachers own thinking, it is found as a hindrance for fully adapting the material into teaching.

> If you don't go through them [animations] well beforehand it's likely to be surprised what happens when you click the arrow forward [for the next step]] …then the timing of instructional speech is sometimes wrong. It's inconvenient. And sometimes it takes several rounds to understand the logic behind. (Teacher 4)

On the other hand, some teachers prefer the same resources particularly as it is time-consuming to develop flexible digital materials to suit one's own ideas. Teachers rely on traditional working methods and, for example, the use of concrete materials because they know well how to adapt such implementation smoothly in their teaching.

> the digital material of the textbook series is something like you still need to add a lot of elements yourself… if I need to invent something by myself I prefer to draw or use macarons or do arts and crafts… (Teacher 7)

Surprisingly, no teacher brought about the flexibility of the digital resources in terms of designing lessons collectively, creating professional development sessions or working distance (cf. Pepin et al., 2017).

Personalization and differentiation

All Finnish teachers in our study seem to seek for such tools that allow them to take account of different learners, for example, high-achievers, students with learning disabilities or the ones speaking Finnish as a second language. This overlaps with the previous category when designing tests suitable for different learners. Teachers appreciate the possibilities of personalization and differentiation in general when using the curriculum materials. The personalization can be obtained by a variety of digital tasks that the teacher can choose from or by an application that vary the difficulty of tasks according to prior performance.

> You don't need to indicate the same [tasks] for everyone as there're plenty of them, as many as you feel up to do… low-performing students had some tasks that repeated really the basics instead of doing average level tasks… (Teacher 1)

Teachers provide their students possibilities to choose from various additional activities after completing the basic level tasks of the textbook. Teachers appreciate also that the digital materials allow students to work at home online.

> Logging in with personal identification made it easy to continue working at home and it [assignment] was completed on the Internet (Teacher 1)

Yet, teachers reflected on the meaning of knowing the available material thoroughly in order to utilize it efficiently. Teachers highlighted the meaning of special introduction training when starting to use a new resource in order to understand the underlying idea and to picture up the supply of tasks to be used with students.

> I'd like to participate also myself if the training was available. The problem is to find time for becoming familiar with such a broad supply… that you'd know who benefits from which tasks (Teacher 5)

Assessment and monitoring student learning

Teachers hardly reflected on the possibility to develop assessment procedures and tools for summative or formative assessment in order to monitor student learning. Only one teacher mentioned the benefits provided by digital materials that allow easy access to witness student progress and direct the pathway that an individual student takes.

> It's easy for a teacher to monitor and download new assignments weekly and then check who had completed them all (Teacher 1)

Supplementary facilities of realization

Teachers paid attention to supplementary facilities that digital resources potentially provide if compared to printed ones, namely, ready-made exact drawings and illustration presented with animated digital manipulatives.

> The biggest change when digital materials appeared in the market… it was a huge thing to replace multi-links and manipulatives and such material… because it's really clear in my opinion that you can show them on the board and pause and go back and forth (Teacher 1)

One teacher highlighted the importance of making mathematical process visible. He found it easier to accomplish such demand with the traditional blackboard instead of digital presentations. The meaning of using concrete materials, for example, ten base manipulatives divided the teachers. On the one hand, possibility to work with concrete materials and laboratory work comprise the ground for learning mathematics, i.e. embodied activities and tactile experience serve the basis for the learning process of the students.

> I use a lot of laboratory work and I have certain materials available. At the moment, ten base manipulatives have served the ground for expanding the number area… it's the corner stone of the autumn term. (Teacher 6)

On the other hand, teachers discussed the expectation from digital curriculum material to provide additional facilities, namely something new.

> It seems that digital extra material is just like doing tasks similar to the ones our textbook includes but doing them without a pen… it'd be better to have different than the textbook tasks by nature (Teacher 5)

However, digital curriculum materials seem to provide poorly an overall package for mathematics classes, and thus, textbooks still play a central role in schoolwork. The printed material was found sometimes more convenient to access, for example, when flipping through the provided curriculum elements and picturing up the overall idea of a particular lesson. Still, teachers found single activities such as games and interactive tasks an important additional affordance in learning mathematics.

View of teaching and learning mathematics

Teachers appear to be critical consumers of all kind of curriculum material but especially of digital materials that are to open up new sceneries in mathematics classrooms. They evaluate the curriculum material in terms of whether they support student learning and achieving learning objectives. Hence, these teachers appreciate materials that include various kinds of tasks, not only training calculation skills. Mathematical thinking emerged as a core theme.

> It's about encouraging students to think, communicate and apply mathematics. The idea isn't to learn through repeating things but instead using own head (Teacher 7)

Curriculum material should be mathematically correct and clear in order to avoid confusing children by an unfamiliar task form or unclear assignment.

> The assignment is about which numbers you find between two given numbers [in the number line] but it says nothing about dealing with whole numbers... if you just use it straightforward, well-performing students are lost (Teacher 4)

Although the teachers strive to make mathematics meaningful for students, they stressed that the aim of using digital material is not just to entertain students or making mathematics fun. They understand their role to be responsible for choosing such curriculum resources that push towards reaching good learning outcomes. Teachers seem to work with curriculum materials in a way that it suits their views of teaching and learning mathematics and personal readiness for utilizing various resources.

Practical aspects

Various practical issues emerge especially when utilising digital materials. Technical problems make teachers frustrated when applying digital resources and technology.

> It's extremely frustrating to see that digital materials have worked poorly during the recent years, it's my opinion. It's the reason why I've kept some old [mathematics] textbooks in my cupboard. It makes it possible to find at least some types of tasks and use them by putting something together myself, even have photocopies (Teacher 4)

Starting to use new digital materials is seen demanding and many times the user interface seems to be unclear or too complicated for both students and a teacher. The prevailing habit to use traditional textbooks in mathematics classes is strong still nowadays. A challenge is to diversify the way mathematics curriculum materials are used.

> Some students questioned it also, like why they need to use computers all the time… we've done some other projects with them… I think we do all sort of things with computers and I felt that I don't need to promote digitalisation especially in mathematics if I don't feel like it (Teacher 6)

One teacher discussed about the challenges caused by students being unfamiliar with the user interface of a particular application. Thus, a great deal of valuable lesson time might be lost for solving practical problems. Moreover, teachers feel that practical arrangements take sometimes too much time and effort if compared to gained benefits.

> For example, last time when I'd booked the laptops for my class and I got them, then we couldn't log in. It took almost the whole lesson. I think we did some three assignments before starting the lesson break… and we're supposed to rehearse for the test and the whole session was a disaster (Teacher 3)

A practical hindrance is that it is time-consuming to find high-quality material on the Internet and getting familiar with the supply of digital curriculum material.

DISCUSSION

Our study shows that the Finnish teachers seem to be critical consumers of the digital curriculum materials. They choose carefully the resources and especially in which ways to utilize them in mathematics teaching. However, teachers seem to expect that the curriculum material provides augmenting facilities and the use of the material is worth the effort; for example, that the digital material enables them to work more efficiently than before or provides new approaches to mathematics teaching. Digital curriculum materials serve to be a purposeful resource only if the teachers recognize a clear contribution to student progress and a help in schoolwork (cf. Pepin et al, 2017). Teachers see the curriculum resources as an overall package and they utilize the resources in their classrooms firstly for enhancing student learning and improving the quality of their own work.

We found hardly evidence about teachers to prioritise either making mathematics fun or other issues related 'edutainment' when evaluating the potential curriculum material. The surface level features of the curriculum material hardly guide the decision-making and the use of the digital curriculum material (cf. Johnson and Suh, 2009; Smith, Shin & Kim, 2017). The novelty of digital curriculum materials and technology serve no additional value without a clear contribution to the quality of teaching and learning mathematics. Teachers have high expectations.

The Finnish teachers are principally willing to apply new resources in their classroom and see the potential of the modern resources. Recent concern has focused on the quantity of using digital and technological resources (e.g. Tanhua-Piiroinen et al., 2016). Instead of blaming the school system or reluctant teachers, the focus should be on developing such curriculum resources that provide a meaningful addition to existing supply and in which pedagogical aspects would be of a primary concern. The traditional approach to curriculum materials seems to outperform still in the beginning of the 21st century.

Acknowledgements: This work is part of the research project *Teachers' use of Mathematics Curriculum Resources in the 21st century: a cross-cultural project* supported by the Swedish Research Council (2016-04616).

References

Brown, M. W. (2009). The teacher-tool relationship. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.). (2011). *Mathematics teachers at work: Connecting curriculum materials and classroom instruction*, pp. 17–36. Routledge.

Bryman, A. (2012). *Social research methods*. New York, NY: Oxford University Press Inc.

Dick, T. P., & Hollebrands, K. F. (2011). *Focus in high school mathematics: Technology to support reasoning and sense making.* Reston, VA: National Council of Teachers of Mathematics.

Hoyles, C., & Lagrange, J. B. (2010). *Mathematics education and technology: Rethinking the terrain.* Berlin, Germany: Springer.

Hollebrands, K. F. (2017). A Framework to Guide the Development of a Teaching Mathematics with Technology Massive Open Online Course for Educators (MOOC-Ed). In E. Galindo, & J. Newton (Eds.). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Indianapolis, IN: Hoosier.

Males, L. M., Earnest, D., Dietiker, L. C., & Amador, J. M. (2015). Examining K-12 Prospective Teachers' Curricular Noticing. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 88–95)*. East Lansing, MI: Michigan State University.

Johnson, C., & Suh, J. (2009). Pre-service elementary teachers planning for math instruction: Use of technology tools. In I. Gibson et al. (Eds.), *Proceedings of Society for Information Technology & Teacher Education International Conference 2009 (pp. 3561–3566)*. Chesapeake, VA: Association for the Advancement of Computing in Education.

Pehkonen, L. (2004). The magic circle of the textbook–an option or an obstacle for teacher change. In M. Johnsen Høines & A. B. Fuglestad (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 513-520),* Bergen University College.

Pepin, B., Choppin, J., Ruthven, K., & Sinclair, N. (2017). Digital curriculum resources in mathematics education: foundations for change. *ZDM, 49*(5), 645-661.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of educational research, 75*(2), 211-246.

Remillard, J. T. (2016). Keeping an eye on the teacher in the digital curriculum race. In M. Bates & Z. Usiskin (Eds.), *Curricula in school mathematics* (pp. 195-204). Greenwich, CT: Information Age Publishing.

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In *The mathematics teacher in the digital era* (pp. 373-393). Springer Netherlands.

Smith, R. C., Shin, D., & Kim, S. (2017). Prospective and current secondary mathematics teachers' criteria for evaluating mathematical cognitive technologies. *International Journal of Mathematical Education in Science and Technology, 48*(5), 659-681.

Tanhua-Piiroinen, E., Viteli, J., Syvänen, A., Vuorio, J., Hintikka K., & Sairanen, H. (2016). *The current state of the digitalisation of learning environments in basic education and the readiness of teachers to utilise digital learning environments (in Finnish).* Publications of analysis, assessment and research activities, The Finnish Government 18/2016.

USING FINGERS TO DISCERN THE STRUCTURE OF PART-WHOLE RELATIONS OF NUMBERS IN PRESCHOOL

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In this paper we report on results from an eight-month intervention with preschool teachers aiming to enhance five-year-olds' learning of basic arithmetic skills. The purpose of this study is to investigate how the children's learning developed through participation in the theoretically driven intervention, which was based on the idea of experiencing numbers and their part-whole relationships. We report on an analysis of task-based interviews with 103 children before and after the intervention. Our findings show that the learning outcomes of the intervention group were significantly higher compared to those of the control group after the intervention, and that differences between the groups remained a year after the intervention.

INTRODUCTION

In Swedish preschools, Mathematics is not usually taught in a formalized way. While there are goals for mathematics learning in the curriculum, they are general rather than specific and are taught informally during the children's play. This means that, prior to formal schooling, preschool children show great differences in knowledge of numbers and arithmetic skills. International research from the past three decades has given us a comprehensive picture of general learning trajectories of arithmetic skills (e.g., Baroody, 2016; Clements & Sarama, 2009), but we still find it puzzling how different ways of perceiving numbers affect children's opportunities to benefit from teaching activities and develop their skills towards successful arithmetic strategies.

The aim of our project (FASETT) is to investigate preschool children's conceptions of numbers and early arithmetic skills, and how these can be developed through participation in a theoretically driven pedagogical approach. The purpose of this paper is to analyze the learning outcomes from the intervention. The current study is based on task-based interviews with 103 children, from both the intervention and control groups, to establish their knowledge of numbers and arithmetic tasks. Our research questions are: How did children's learning of arithmetic tasks in the intervention group develop? How does the development appear in comparison to a control group?

CHILDREN'S EXPERIENCES OF NUMBERS AND EARLY ARITHMETIC

There is an extensive body of research on children's development of number knowledge and early arithmetic skills (see Baroody, Lai, & Mix, 2006; Carpenter & Moser, 1984; Fuson, 1992, for overviews). Already in infancy children show an ability to discriminate small quantities, which develops into number concepts used in arith-

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 267-274). Umeå, Sweden: PME.

metic operations during childhood (Wynn, 1998). These abilities are found in all cultures, but how they are shaped and expressed is culturally influenced, not least by different base systems and linguistic structures. Nevertheless, there seem to be some basic fundaments that are necessary to learn in order to understand and use numbers in arithmetic tasks, such as ordinality in the counting sequence, numbers' cardinality, and structuring numbers as part-whole relations (Baroody, 2016; Fuson, 1992).

"Very young children typically see a quantity as an aggregate of single units, and thus they need to count when finding the total" (Murata & Fuson, 2006, p. 432). However, it has been suggested that children's numerical concepts can likely be developed through activities that do not necessarily involve sequential counting (Wright, 1994). An overemphasis on counting strategies may even delay children's development of more advanced mathematical skills, since "preschool children who receive continuous encouragement when using counting strategies are reluctant to try the new more advanced decomposition strategy" (Cheng, 2011, p. 30).

Moeller et al. (2011) summarized neuro-scientific studies and concluded that successful finger-based counting and arithmetic serve as building blocks for later numerical and arithmetical development. This is likely due to the structure of numbers that finger-pattern sets can give the child. Moreover, Neuman (1987) suggests that children who use their fingers to represent numbers' part-part-whole relationships are more likely to develop successful strategies for solving arithmetic problems. Neuman (1987, 2013) argues that a lack of recognizing numbers as part-part-whole relations may be a cause for not learning number facts, which leads to the need for cumbersome strategies, such as double-counting of single units. Similarly, Gray, Pitta and Tall (2000) found that children who are less able to solve arithmetic problems relied extensively on counting procedures.

Based on the findings described above, we adopted the use of fingers to represent numbers, and particularly to develop finger patterns as a means to discern numbers' part-part-whole relations. We conjecture that this facilitates a conceptual understanding of numbers rather than a procedural use of fingers as countables. Since counting has long been regarded as the dominating path to development, less is known about how five-year-olds learn arithmetic skills through experiences of numbers' part-part-whole relations.

METHOD AND THEORETICAL FRAMEWORK

The intervention program was designed in accordance with the variation theory of learning (Marton, 2015), and was built on Neuman's conjecture suggesting that in order to learn to solve arithmetic tasks one needs to discern and experience numbers' "manyness" and part-part-whole relationships. As previously stated, finger patterns can be used to do this. To experience "manyness" means to embrace the idea that a group of objects may be seen as a quantity (whose cardinality can be determined by counting or estimating), and experience this quantity as comprised of units that form a composite whole.

A total of 103 children participated in the study, of whom 65 attended five preschools involved in the intervention program and 38 attended four preschools that did not receive any particular guidance.

Written consent was given by all parents/legal guardians of the children before participation. All children were interviewed three times with the same tasks: at the beginning of their last year in preschool (as 4-5-year-olds), after the intervention period, and a year after the intervention. In the year after the intervention, no special treatment was given to either group. However, this year the children attended new schools with new teachers and the groups were mixed with other children.

The interviews

Each individual interview lasted about 15-20 minutes and was comprised of mathematical tasks within the number range 1-10. The tasks were given verbally. During the interviews no numerals were shown, and no manipulatives such as counters or similar objects were available to be used for calculation. The interviews were video-recorded, with the exception of a few cases in which we only had permission to observe and audiotape. In this paper we report on only eight arithmetic tasks in the interview. The tasks' context as well as type differed (see Figure 2); for example, "If you have ten candies and eat six of them, how many are left?" (10-6=_) and "You have three glasses, but are going to set the table for eight people; how many more glasses do you need?" (3+_=8). In another task, inspired by Neuman (1987), the child was asked to count seven marbles lined up on a table. The interviewer told the child she was going to hide the seven marbles in her two hands. Thereafter, the child was asked how many marbles there could be in the left and right hands (7=_+_). After this the interviewer opened one hand, and the child was asked to find the other part (7=4+_).

The interviews were coded for correct or incorrect answers, giving a maximum score of eight points. When no answer was given, either because the child did not provide an answer or because she had previously given two subsequent incorrect answers to easier tasks and thus the interviewer did not ask any new questions, this was coded as incorrect. We analyzed correct answers to tasks and used ANOVA to analyze differences in means between the intervention and control groups in order to study the effects of the intervention.

The intervention

The teachers worked in an iterative process in collaboration with the research group, whereby designed activities were enacted with the children, documented with video, and used as a basis for investigating the children's understanding and how to develop the activities even more. Four activities were designed and enacted during a period of eight months, October-May: a) the statement game, b) the five- and ten-snake game, c) finger patterns, and d) arithmetic context tasks. The activities were enacted repeatedly, and in line with the variation theory of learning. The primary goal was for the children to learn to use their fingers to represent numbers, and particularly develop finger patterns as a means to discern and make use of numbers' part-part-whole relationships.

The statement game (October-November): The aim of the statement game was to highlight that numbers can be represented differently with fingers (Sensevy, Quilio, & Mercier, 2015). When playing, the children were to show how the number on a die could be represented on two hands, and in different ways compared to the other children participating in the game. This would facilitate the children's development of a structural approach to arithmetic tasks by using their fingers as representations of numbers in a part-whole structure.

Five-snake and ten-snake (December-March): The five- and ten-snake game was conducted with a string of beads (five in one color and five in another color, grouped together). The teacher would hide some of the beads under her hand, leaving the rest of the beads visible. During the game the child modeled the whole (five or ten) with her fingers and thereafter modeled the visible part and was asked to figure out how many the teacher had hidden under her hand. The child was able to see the missing part, usually without counting. The game emphasized the part-part-whole structure of numbers as a fundament for later work with arithmetic tasks involving similar part-part-whole structures to find a missing part.

Finger patterns (April): In order to direct attention to the structural aspect of numbers, the teachers used finger patterns to extend the children's conceptual subitizing range (recognizing finger patterns that included an undivided five). The children were also asked, for instance, how many fingers should be put down to make a seven pattern from a nine pattern, and the reverse. In this manner, the relationship between seven, two and nine was brought to the foreground and visualized as finger patterns, whereby the whole was not static like in the snake game.

Arithmetic context tasks (March-May): Arithmetic tasks were developed that included change or comparison in quantities up to ten, such as "Five bears were walking in the woods and three ran off; how many were left?" and "Eight tired bears came to a cottage, but there were only six beds; how many bears did not get a bed to rest in?". The children were encouraged to model the task on their fingers in the same way as in the snake game.

RESULTS

We found that the learning outcomes for the intervention group (Figure 1) were higher compared to those of the control group after the intervention. The mean for the intervention group increased from 1.69 (max. 8) in the pre-interview to 4.97 (SD 2.27) in post-interview 1, and 6.32 (SD 1.84) in post-interview 2 (12 months after post-interview 1). The mean for the control group was 1.61 (SD 1.82) in the pre-interview, 3.18 (SD 2.09) in post-interview 1, and 5.42 (SD 2.26) in post-interview 2.

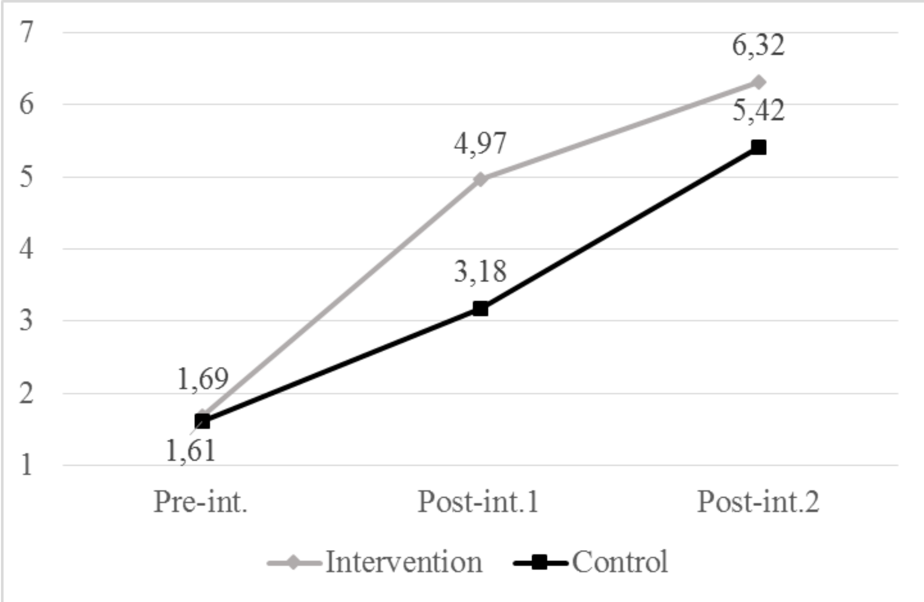

Figure 1. Mean results for the intervention and control group at pre-interview, post-interview 1, and post-interview 2.

A mixed design ANOVA was conducted, with Group (Intervention, Control) as a between-subjects factor and Test occasion (Pre-Interview, Post-Interview.1 and Post-Interview.2) as within-subjects factor. Mauchly's test indicated that the assumption of sphericity had not been violated ($\chi^2(2)$=1.147, p=.564), and Levene's tests revealed no violation of assumption of homogeneity of variance between groups on all three test occasions ($F_{\text{Pre-int.}}(1,101)$=.021, p=.885, $F_{\text{Post-int1}}(1,101)$=.769, p=.383, FPost-int2(1,101)=2.165, p=.144). As expected, the main effects for both Test occasion (F(2,202)=221.517, p=.000, ηp2=.687) and Group (F(1,101)=7.404, p=.008, η_p^2=.068) were significant, suggesting that, on average, the children's mean scores increased from one test occasion to the next, and that the intervention group had better overall performance than the control group.

Most importantly, confirming our main hypothesis, the analysis confirms significant interaction between Group and Test occasion (F(2,202)=8.890, p=.000, η_p^2=.081), suggesting that the increase in learning outcome was different for the intervention and control groups. We see in Figure 1 that, while the intervention and control groups started off with the same average results (pre-interview), the intervention group achieved higher scores at post-interview 1 and remained more successful after a year (post-interview 2). The difference between means in the intervention and control groups is almost two points after the intervention (post-interview 1, Figure 1). The difference between groups at post-interview 2 was smaller than immediately after the intervention, but still remained at almost 1 point.

Analysis on task-level

The results show that the intervention group scored higher on all items compared to the control group in both post-interviews 1 and 2 (see Figure 2). The increase shown in post-interview 1 compared to the pre-interview was higher on particular items, e.g.

tasks C (+47%), D (+55%), and F (+52%) for the intervention group compared to the control group (34%, 45%, and +24% points, respectively). These tasks have similar features that may explain their standing out as an outcome of the intervention: they all start with a known whole, and one or neither of the parts is known. Task A is of similar type. However, in this case the semantic nature of the task "You and your friend collected five shells together. You collected four of them; how many did your friend collect?" made it seemingly difficult to comprehend. We draw this conclusion as many children in both groups answered "five" on this task. Task C, about finding two unknown parts, shows a great difference between the groups, with an increase in correct answers from 19% to 66% (an increase of 47% points) for the children in the intervention group, and from 16% to 34% (an increase of 18%) for those in the control group. In this case, it seems as if the activities involving identifying one part of the part-part-whole relations during the intervention also help the children learn to identify two interrelated parts of a given whole. We can see that the children in the intervention group also improved more than those in the control group on tasks involving addition (tasks E and G). The children in the control group improved on all tasks, improving the most on tasks involving addition, e.g. task G (+27%).

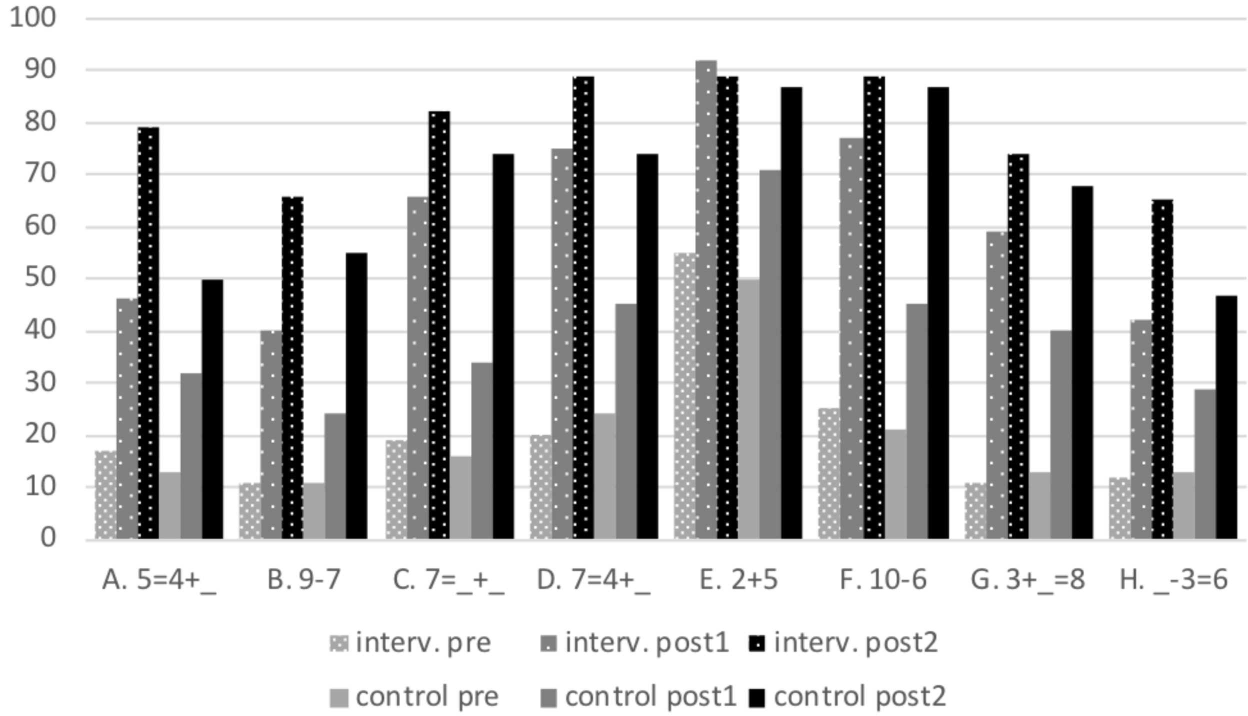

Figure 2. Comparison of correct answers in percent on tasks in pre-interview, post-interviews 1 and 2 in the intervention (interv.), and control group.

The results from post-interview 2, 12 months after the intervention, show that there was only a small difference between the groups on tasks E and F (2+5 and 10-6), two seemingly straightforward addition and subtraction tasks (intervention group 89% and 89%, control group 87% and 87% correct answers). A greater difference between the groups is found in the other types of tasks, for instance tasks A, C and D, in which the whole is known and one or neither of the parts is known. The greatest difference is on tasks A and H, which require a comprehensive understanding of the relational structure of the whole and the parts involved.

CONCLUSION AND DISCUSSION

This study adds to previous research by showing how the use of finger patterns to structure part-part-whole relations can help preschool children develop arithmetic skills without having to rely on strategies of counting single units (cf. Murata & Fuson, 2006). The results show that the intervention group, compared to a control group, increased their performance significantly, and on all tasks. They especially became better at tasks involving partitioning a whole into two parts or finding one part when the whole and the other part are known. The activities in the intervention program encouraging the children to use their fingers to structure numbers as part-whole relations, we suggest, benefitted the intervention group and influenced their performance, even in more complex arithmetic context tasks (see tasks A and H).

What does having discerned the structure of part-part-whole relations entail for future learning in school? We cannot say. However, our study shows that there was a significant difference between the intervention and control groups a year after the intervention, when the children had completed their first year of formal mathematics education in school (preschool class, post-interview 2). It has been argued that the counting strategies children use for early arithmetic have a tendency to be used later as well, even though more advanced methods have been introduced (Cheng, 2011). We thus believe it is important which strategies are taught in early arithmetic and suggest further research on how the strategies affect future learning. Our study focused on incorrect and correct answers only, not taking into account the methods the children used for calculation. The next step will be to analyze children's solution methods and reasoning in order to give a more fine-grained analysis.

References

Baroody, A. J. (2016). Curricular approaches to connecting subtraction to addition and fostering fluency with basic differences in grade 1. *PNA*, *10*(3), 161-190.

Baroody, A. J., Lai, M.-l., & Mix, K. S. (2006). The development of young children's early number and operation sense and its implications for early childhood education. In B. Spodek & O. N. Saracho (Eds.), *Handbook of Research on Education of Young Children*. Mahwah, N.J: Lawrence Erlbaum.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*, 179-202.

Cheng, Z.-J. (2011). Teaching young children decomposition strategies to solve addition problems: An experimental study. *The Journal of Mathematical Behavior, 31*(1), 29-47.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math. The learning trajectories approach*. New York: Routledge.

Fuson, K. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan Publishing.

Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge.

Moeller, K., Martignon, L., Wesselowski, S., Engel, J., & Nuerk, H.-C. (2011). Effects of finger counting on numerical development. the opposing view of neurocognition and mathematic education. *Frontiers in Psychology, 2,* 328-336.

Murata, A., & Fuson, K. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using ten. *Journal for Research in Mathematics Education, 37*(5), 421-456.

Neuman, D. (1987). *The origin of arithmetic skills: A phenomenographic approach.*, Göteborg: Acta Universitatis Gothoburgensis.

Neuman, D. (2013). Att ändra arbetssätt och kultur inom den inledande aritmetikundervisningen (Changing the culture and ways of working in early arithmetic teaching). *Nordic Studies in Mathematics Education, 18*(2), 3-46.

Sensevy, G., Quilio, S., & Mercier, A. (2015). *Arithmetic and comprehension at primary school.* Paper presented at the International Commission on Mathematical Instruction (ICMI) Study 23, Macau.

Wright, R. J. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics, 26*(1), 25-55.

Wynn, K. (1998). Numerical competence in infants. In C. Donlan (Ed.), *The development of mathematical skills*. Hove: Psychology Press.

TEACHERS' CRITERION AWARENESS AND THEIR ANALYSIS OF CLASSROOM SITUATIONS

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Mathematics teachers' noticing and their analysis of classroom situations is considered as a key component of teacher expertise in a growing body of empirical research. However, research into what dispositions may direct teachers' noticing and their criteria-based analysis is still scarce. In this study, we use the notion of teaches' criterion awareness for exploring interdependencies between teachers' analysis of classroom situations and their awareness. Building on our prior research, the study concentrates on awareness criteria related to dealing with representations in the mathematics classroom. The findings suggest interdependencies of the teachers' reported awareness with the teachers' analysis scores, and encourage the development of further indicators.

INTRODUCTION

The short formula "you can only see what you know" points to the influence of observers' cognitions when they notice or analyse aspects of context situations. It can be assumed that such an interaction takes also place when mathematics teachers are confronted with classroom situations. It is therefore highly probable that the awareness of specific criteria can make a difference for teachers' noticing and their analysing of classroom situations. We describe such interdependencies in a theoretical model by considering criterion awareness as a set of experience-based cognitions which facilitate access to specific professional knowledge when teachers are being faced with situation contexts: Through corresponding awareness teachers can analyse these situation contexts against the corresponding criteria. However, little is known empirically about the potential influence of criterion awareness on teachers' competence of analysing classroom situations, especially as far as criteria in the field of dealing with representations in the classroom are concerned. To our knowledge, this is consequently the first paper approaching this research need. Based on a definition of criterion awareness and on a corresponding model of its potential influence on teachers' analysis, the design of a first indicator instrument is described, which affords analysing profiles of criterion awareness within a set of different criteria. The empirical results show interdependencies between different awareness profiles and the competence of analysing.

In the following, we introduce the theoretical framework of the study (1), which leads to the paper's research interest (2). We then report on sample and methods (3), before presenting results (4). The results will then be discussed in the concluding section (5).

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 275-282). Umeå, Sweden: PME.

THEORETICAL BACKGROUND

Recent approaches to describing mathematics teachers' expertise focus less on whether teachers possess specific professional knowledge (Shulman, 1986; Kuntze, 2012) and more on how teachers make use of their professional knowledge in situation contexts (e.g. Kersting et al., 2012; Sherin et al., 2011). In these approaches, notions such as "Noticing" in the sense of "selective attention and knowledge-based reasoning" (Sherin et al., 2011) or "Usable Knowledge" (Kersting et al., 2012) are used to emphasise a phenomenological perspective which concentrates on describing what teachers actually notice or what knowledge they use when analysing a classroom situation. The approaches assume that the teachers' professional knowledge might be more extensive than the knowledge they use in the specific noticing or analysing process. A key question arising from this assumption is how processes of noticing or analysing, which connect observations to specific elements of professional knowledge, are started, and what guides and triggers these processes of noticing or analysing. Consequently, there is a need for models which can explain the use of professional knowledge in these processes.

For responding to this need, we concentrate on teachers' *analysing* (Kuntze, Dreher & Friesen, 2015; Friesen & Kuntze, 2014; cf. Seidel et al., 2011; Schneider et al., 2016). This notion encompasses core elements of the above-mentioned approaches (Kersting et al., 2012; Sherin et al., 2011; Berliner, 1991; Sherin et al., 2011; Dreher & Kuntze, 2015) and is understood as "an awareness-driven, knowledge-based process which connects the subject of analysis with relevant criterion knowledge and is marked by criteria-based explanation and argumentation" (Kuntze et al., 2015, p. 3214). Classroom situations can be subjects of analysis in the sense of this definition. In our model of the process of analysing classroom situations (see Fig. 1), a circular structure comparable to the modelling cycle (e.g., Blum & Leiss, 2005) is used to describe the process of generating (1) a situation model ("real model") of the classroom situation which can then be interpreted based on criteria (2): Professional knowledge (including teachers' views, see model described in Kuntze, 2012) is the background which provides the analysing teacher with models for describing observations—and on this base, it affords drawing conclusions (3). The explanatory power of these conclusions can then be validated against the situation model of the classroom situation (4). Comparable to findings for the modelling cycle (Borromeo-Ferri, 2006), it makes sense to assert that jumps between the phases shown in Figure 1 may occur. Parts of the process may take place unconsciously, before entering in a more intense, explicit knowledge-based analysis process (e.g., with repeated cycles). Finally, we consider the process as awareness-driven. This means that (possibly simultaneous) awareness for (possibly different) specific criteria continuously supports the possible criteria-based interpretation, connection with professional knowledge and validation. In short, criterion awareness keeps the cycle moving, comparable to a computer stand-by, which fully activates the corresponding explicit knowledge-based analysis cycle in case a criterion appears as useful for describing a relevant situation aspect.

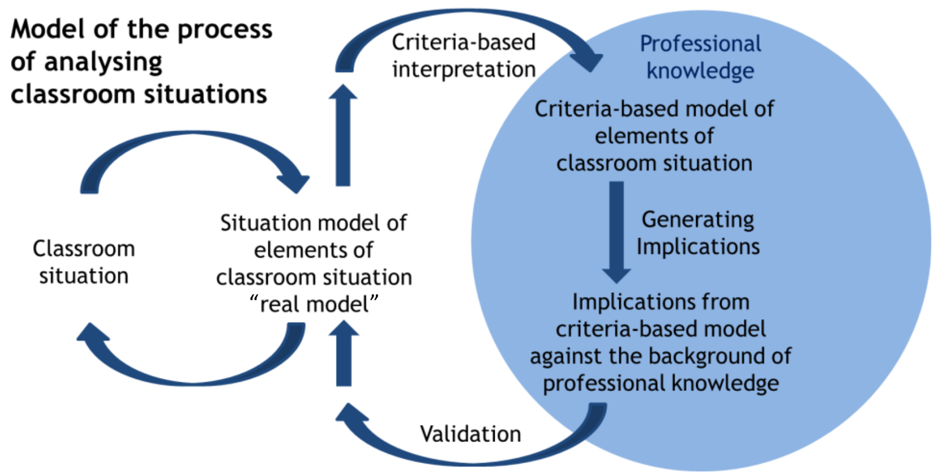

Figure 1: Model of the process of analysing (Kuntze & Friesen, 2016)

Criterion awareness (Kuntze & Dreher, 2015, p. 298) has been described as "a part of professional knowledge which influences the readiness and ability of teachers to use this professional knowledge element in instruction-related contexts". In this way, criterion awareness makes professional knowledge accessible, so that it can be used in the process of analysing classroom situations.

As multiple criteria (including individual criteria based on teachers' views) can be used for analysing classroom situations (e.g. Clausen, Reusser & Klieme, 2003), it is likely that criterion awareness related to different criteria coexist in a competing relationship. For example, it might happen that a teacher's criterion awareness for students' motivation turns out to be predominant over criterion awareness for students' understanding of mathematical representations and it might hence impede teachers' corresponding analysis (cf. qualitative findings in Kuntze & Dreher, 2015).

This competing relationship is a challenge, as research designs should be able to identify the relative predominance of specific awareness criteria within multiple criteria. For this reason, we selected a focus criterion domain (use of representations), which will be considered as competing with other reference awareness criteria.

Teachers' awareness and analysis related to dealing with representations

In prior research, we have investigated teachers' analysis of how representations of mathematical objects are dealt with in classroom situations (e.g., Friesen & Kuntze, 2016). As dealing with representations is a key aspect of learning and hence also a key quality aspect of learning opportunities in the mathematics classroom, teachers' corresponding criteria-based analysis is an important component of their expertise. Using a corresponding vignette-based instrument, we were able to measure teachers' competence of analysing in this domain and found that it can be described in a one-dimensional Rasch model (e.g., Friesen & Kuntze, 2016; Kuntze & Friesen, 2016). Teachers' awareness related to how representations are being dealt with has, however, not been in the focus of our research so far. In particular, the reported awareness for the use of representations in comparison with other, less specific criteria such as students' motivation, attention or prior knowledge has not been explored yet.

RESEARCH INTEREST

As discussed in the section above, there is a need for exploring teachers' criterion awareness empirically and its potential interrelatedness with their analysis of classroom situations. Since teachers' competence of analysing the use of representations in mathematics classrooms can be measured with an existing vignette-based instrument (e.g., Friesen & Kuntze, 2016), a first indicator instrument for teachers' criterion awareness had to be developed. As testing time is often restricted, the focus on the teachers' self-reported criterion awareness may be used as an indicator in this first approach. Profiles of criterion awareness with respect to a set of criteria might be of high interest, as different criteria might concur with each other.

Consequently, the study aims at answering the following research questions:

(1) Is it possible to implement reliable indicators for criterion awareness in a corresponding questionnaire?

(2) What profiles of criterion awareness can be observed?

(3) Do the indicator scales or the profiles interdepend with the analysis score (related to the competence of analysing the use of representations in classroom situations)?

DESIGN AND SAMPLE

The sample of this study consists of *N*=125 German mathematics teachers at the beginning of their induction phase at secondary schools (81 female, 44 male, mean age 26.9 years; *SD*=4.1 years). These teachers' competence of analysing the use of representations was assessed with a test comprising of eight vignettes, in which the teachers had to analyse the use of representations in eight classroom situations (cf. Friesen & Kuntze, 2016, with subtests specific for the content domains of fractions and functions). Core parts of the instrument (for the domain of fractions) have been presented in prior papers, including a PME research report, to which we would like to refer in order to meet space limitations (Friesen & Kuntze, 2016; cf. also Kuntze & Friesen, 2016).

For this study, in addition to the competence test, the beginning teachers were asked about their awareness related to four different criteria: students' motivation, students' attention, representations, students' prior knowledge. A corresponding rating-scale instrument used the indicator scales presented in Figure 2 (4-point Likert scales). The items express what criteria teachers might activate when observing that students have difficulties in the process of solving a task.

RESULTS

The *first research question* focused on the implementability in reliable indicator scales. Reliability values of the scales are displayed in Figure 2. Given the low number of items per scale, the reliability values range from good to still satisfactory. It was thus possible to implement sufficiently reliable indicators for self-reported criterion awareness in the questionnaire.

Indicator scale for awareness related to…	Sample item	Number of items	α (Cronbach)
…motivation	If a student does not advance with a task, I check whether s/he is currently not very motivated.	2	.67
…attention	If a student does not advance with a task, I check whether s/he is thinking of something which does not have to do with the mathematics classroom.	2	.85
…representations	If a student does not advance with a task, I check whether s/he is unable to link different representations.	2	.76
…prior knowledge	If a student does not advance with a task, I check whether s/he lacks prior knowledge about mathematical concepts.	2	.79

Figure 2: Indicator scales for awareness related to different criteria

There are significant correlations between the scales: awareness of students' motivation (M=2.51; SD=.64) correlates ($r_{pearson}$=.52; p< .01) with awareness of students' attention (M=2.24; SD= .70) and interdepends slightly ($r_{pearson}$= .18; p< .05) with awareness of students' prior knowledge (M=3.02; SD=.66); awareness of the use of representations (M=3.02; SD=.64) correlates ($r_{pearson}$=.42; p<0.01) with awareness of students' prior knowledge.

The *second research question* concentrates on profiles of criterion awareness. For exploring these, a cluster analysis (Ward method) was carried out on the base of the four variables shown in Figure 2. The cluster analysis yielded two clusters of teachers with comparable size. Figure 3 presents the mean scale values of the two clusters.

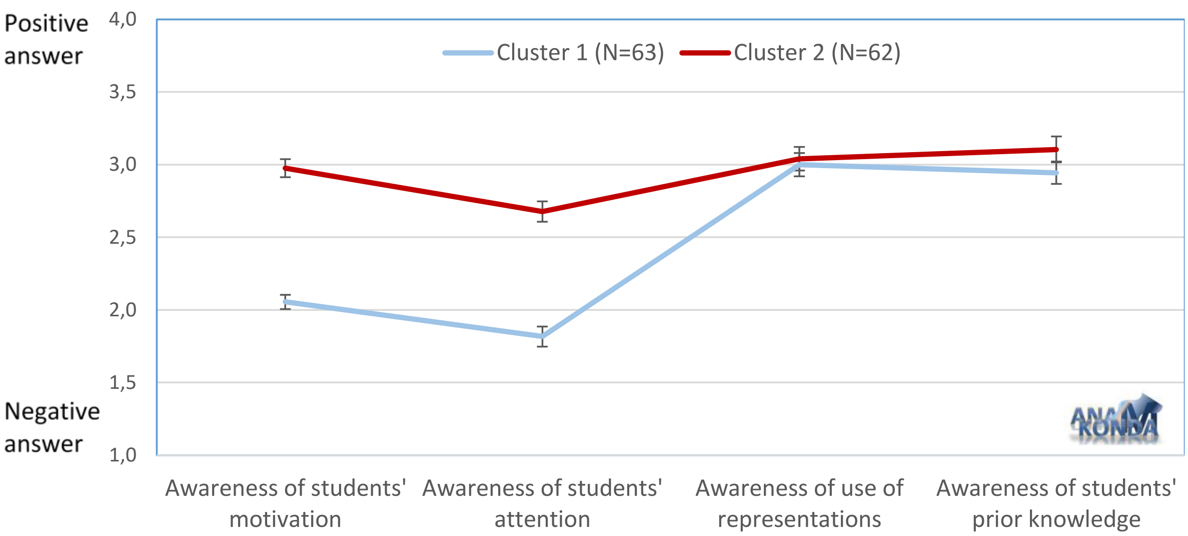

Figure 3: Results from cluster analysis (means and their standard errors)

The clusters shown in Figure 3 do not differ with respect to their criterion awareness related to the use of representations and prior knowledge, however the beginning teachers in the two clusters answered differently regarding their awareness of the students' motivation and attention. Whereas the teachers in cluster 1 evaluated their awareness regarding the criteria of motivation and attention on average negatively, their counterparts in cluster 2 gave rather positive ratings for all awareness criteria.

Both clusters' ratings related to the students' prior knowledge were close to each other. If the awareness of different criteria is seen as being in potential competition with each other, then the teachers in cluster 1 rather concentrate on criteria related to representations and prior knowledge according to their self-reports. In contrast, the teachers in cluster 2 reported simultaneous awareness of all criteria.

The *third research question* addresses potential interdependencies of the indicator scales or the profiles with the analysis score regarding the use of representations in classroom situations. The analysis score (Friesen & Kuntze, 2016) was calculated on the base of rating scale answers of the teachers for eight classroom situations (four situations each for the content domains of fractions and functions). The theoretical score maximum was 4 points, the average score was 1.26 (SD=0.63).

As far as interdependencies between single indicator scales for criterion awareness and the analysis score is concerned, we did not observe any significant correlations between these variables and the analysis score.

However, considering the mean analysis scores for the two clusters reveals differences: Figure 4 displays the mean competence scores for the two clusters, also as far as the distinction between the content domains (fractions and functions) is concerned. There is a significant difference between clusters for the over-all competence score (T=2.33; df=123; p= .02; d=0.42), which results above all from the significant difference in the competence score related to the content domain of functions (T=2.60; df=123; p= .01; d=0.47). Cluster 1 with the less concurring awareness criteria (with the awareness of representations) reached on average better analysing results.

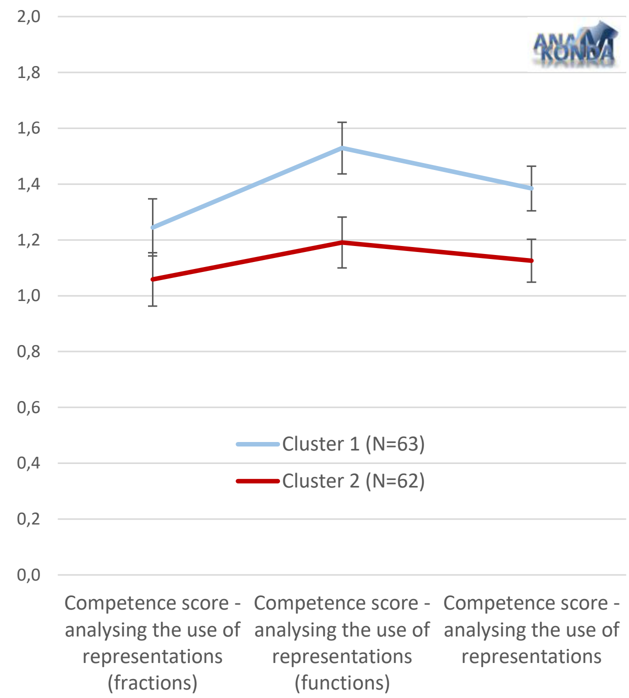

Figure 4: Scores for the competence of analysing the use of representations for the two clusters (means and their standard errors)

DISCUSSION AND CONCLUSIONS

The findings suggest that the short scales used in the questionnaire were sufficiently reliable. This means that the instrument can yield indicators for the teacher's awareness of different criterion domains in follow-up studies. The analysis related to the second research question suggests that there were two correlated pairs of awareness variables, namely motivation/attention and representation/prior knowledge. The further analysis revealed two clusters with similar mean ratings for representations and prior knowledge. The differences between clusters in reported awareness regarding students' motivation and attention—which are criteria less specific for mathematics instruction—may indicate that teachers in cluster 1 are more disposed to concentrate their analysis on criteria related to dealing with representations, in particular. In relation with the awareness of the set of the other criteria, the teachers in cluster 1 have on average the higher *relative* criterion awareness for representations.

This higher *relative* criterion awareness appears to make a difference for the teachers' competence of analysing: The teachers in cluster 1 scored higher. This was the case especially in the analysis subtest related to the domain of functions, as the subtest scores for the content domain of fractions showed only a non-significant tendency. We would like to recall that the cluster analysis had been carried out only on the base of the reported awareness questionnaire, so that the observation of competence differences supports the hypothesized role of awareness for the analysis cycle (Fig. 1).

The evidence should be interpreted with care, given that the findings should be replicated and that the sample is non-representative. However, the evidence encourages the further empirical examination of criterion awareness also with differrent instruments which may then rely less on teachers' self-reports. The relatively positive awareness self-reports in all the sample suggest that teachers' self-reported answers as captured by the present instrument might have a positive bias and should be complemented by other indicators which might then correlate directly with the analysis score.

There is a spectrum of follow-up questions: Can criterion awareness be fostered and how? Does criterion awareness develop with teachers' experience growth? Does the awareness of different criteria always impede each other or is it possible to observe also mutual support of different awareness criteria? How much professional knowledge is necessary for criterion awareness? How do teachers from different cultures/school cultures differ in their profiles of criterion awareness? Is criterion awareness interrelated with teachers' instruction-related views or beliefs? etc. In our next research steps, we aim to explore some of these follow-up questions.

Acknowledgements

This study (as a part of the projects ANAKONDA-M and EKoL) is supported by the Ministry of Science, Research and the Arts in Baden-Wuerttemberg.

References

Berliner, D. C. (1991). Perceptions of student behavior as a function of expertise. *Journal of Classroom Interaction*, 26(1), 1-8.

Blum, W. & Leiß, D. (2005). Modellieren im Unterricht mit der „Tanken"-Aufgabe. *Mathematik lehren*, *128*, 18–21.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. ZDM - *Zentralblatt für die Didaktik der Mathematik*, *38*(2), 86-93.

Clausen, M., Reusser, K., & Klieme, E. (2003). Unterrichtsqualität auf der Basis hoch-inferenter Unterrichtsbeurteilungen. *Unterrichtswissenschaft, 31*(2), 122-141.

Dreher, A. & Kuntze, S. (2015). Teachers' professional knowledge and noticing: The case of multiple representations in the mathematics classroom. *Educational Studies in Mathematics*, *88*(1), 89-114.

Friesen, M., Dreher, A., & Kuntze, S. (2015). Pre-service teachers' growth in analysing classroom videos. In K. Krainer & N. Vondrová (Eds.), *Proceedings of CERME 9* (pp. 2783–2789). Prague, Czech Republic: Charles University in Prague and ERME.

Friesen, M., & Kuntze, S. (2014). Pre-service teachers' competence of analysing the use of representations in mathematics classroom situations. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.) *Proceedings of PME 38/PME-NA 36*, Vol. 6 (p. 309). Vancouver: PME.

Friesen, M. & Kuntze, S. (2016). Teacher Students Analyse Texts, Comics and Video-Based Classroom Vignettes Regarding the Use of Representations - Does Format Matter? In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.), *Proc. of the 40th Conf. of the IGPME*, Vol. 2 (pp. 259-266). Szeged, Hungary: PME.

Kersting, N., Givvin, K., Thompson, B., Santagata, R., & Stigler, J. (2012). Measuring Usable Knowledge: Teachers' Analyses of Mathematics Classroom Videos Predict Teaching Quality and Student Learning. *Am. Educ. Research Journal*, *49*(3), 568–589.

Kuntze, S. (2012). Pedagogical content beliefs: global, content domain-related and situation-specific components. *Educational Studies in Mathematics*, *79*(2), 273-292.

Kuntze, S. & Friesen, M. (2016). Criterion awareness and professional knowledge as prerequisites for teacher noticing and analysis. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.), *Proc. of the 40th Conf. of the IGPME*, Vol. 1 (p. 310). Szeged, Hungary: PME.

Kuntze, S. & Dreher, A. (2015). PCK and the awareness of affective aspects reflected in teachers' views about learning opportunities – a conflict? In B.Pepin & B.Rösken-Winter (Eds.) *From beliefs and affect to dynamic systems: (exploring) a mosaic of relationships and interactions* (pp. 295-318). Springer.

Kuntze, S., Dreher, A., & Friesen, M. (2015). Teachers' resources in analysing mathematical content and classroom situations – The case of using multiple representations. In K. Krainer & N. Vondrová (Eds.), *Proc. of CERME 9* (pp. 3213–3219). Prague: ERME.

Schneider, J., Kleinknecht, M., Bohl, T., Kuntze, S., Rehm, M., & Syring, M. (2016). Unterricht analysieren und reflektieren mit unterschiedlichen Fallmedien: Ist Video wirklich besser als Text? [Analysing and reflecting on classrooms in different case media: Is video really better than text?]. *Unterrichtswissenschaft*, *44*(4), 474-490.

Seidel, T., Stürmer, K., Blomberg, G., Kobarg, M., & Schwindt, K. (2011). Teacher learning from analysis of videotaped classroom situations. *Teaching & Teach. Educ.*, *27*, 259-267.

Sherin, M., Jacobs, V., Philipp, R. (2011). *Mathematics Teacher Noticing. Seeing Through Teachers' Eyes*. New York: Routledge.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

PRIMARY GRADE STUDENTS' FUNDAMENTAL IDEAS OF GEOMETRY REVEALED VIA DRAWINGS

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Despite the importance of geometry in mathematics curriculum, the trend in reduction of geometry in school mathematics is ongoing. This raises the question concerning geometry competencies students acquire in school mathematics. The goal of this exploratory study was to analyze grade 3-6 students' understanding of geometry by using drawings, and through it to gain insight into school geometry nowadays. The results show that students have a rather narrow understanding of geometry. While fundamental idea of elementary geometric forms and their construction dominated in the students' drawings, fundamental ideas of geometric patterns, coordinates, and geometrization were minimally present. Based on the data, the results are discussed with regard to their theoretical and practical implications.

INTRODUCTION

In the past several decades, geometry seems to have lost its central position in mathematics teaching with the overall amount of geometry being reduced in many national curricula (e.g., Backe-Neuwald, 2000; Mammana & Villani, 1998). The reason for the reduction was the temptation to increase the coverage of other mathematical disciplines in school mathematics, such as algebra, and data analysis and probability (Jones, 2000). Due to ICME-7 resolution in Québec concerning geometry curricula, and reassessment of the role of geometry with respect to perspectives on the teaching of geometry for the 21st century, trends have begun to counteract this tendency (Mammana & Villani, 1998). Nevertheless, to what extent these trends found their way into the geometry classrooms, and what meanings students assign to geometry remain open. In order to gain insight into young students' understanding of geometry, and through it better understand how geometry is taught nowadays, viable and age-appropriate methods are paramount. Recent research (e.g., Halverscheid & Rolka, 2006; Laine et al., 2015; Pehkonen, Ahtee, & Laine, 2016; Rolka & Halverscheid, 2011) showed that the use of drawings provides a multi-dimensional and a holistic view of students' latent experiences in mathematics classroom. As such they allowed children – in a unique and holistic manner – to better recall, and express in more detail events and phenomena in focus.

In this sense, we used drawings to gain insight into grade 3-6 students' understanding of geometry through their lenses by using drawings. The following research questions guided the study: What fundamental ideas of geometry can be seen in the primary grade students' drawings? What similarities and differences exist between primary grade students' fundamental ideas of geometry?

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 283-290). Umeå, Sweden: PME.

THEORETICAL PERSPECTIVE

For many years, the geometry curriculum worldwide has been somewhat an eclectic mix of activities, which contributed to increased coverage of other mathematical ideas at the expense of geometry (Van de Walle & Lovin, 2006). One of trends to counteract the fading of geometry in school mathematics focuses on the construction of the geometry curriculum organized around *fundamental ideas* as a means for curriculum development. This term can be interpreted in many different ways (Rezat, Hattermann, & Peter-Koop, 2014). Winter (1976) defined fundamental ideas as ideas that have strong references to reality and can be used to create different aspects and approaches to mathematics. In addition, they are characterized by a high degree of inner richness of relationships, and by gradual and continuous development in every grade (Rezat et al., 2014; Van de Walle & Lovin, 2006). For instance, *Principle and Standards for School Mathematics* have provided a content framework for geometry organized around shapes and properties, transformation, location, and visualization (Van de

Fundamental idea	Description
geometric forms and their construction	Understanding the structural framework of elementary geometric forms as a three-dimensional space, which is composed of 0-dimensional points, 1-dimensional lines, 2-dimensional surfaces, and 3-dimensional solids. They can be constructed in a variety of ways (e.g., drawing tools, material) through which their properties are imprinted.
operations with forms	Understanding possible operations with geometric forms (e.g., translation, rotation, point symmetry, axial symmetry shearing, composing/decomposing), and how these influence the properties of the forms being operated on.
coordinates	Understanding how position of different geometric forms can be described using a coordinate system.
measurement	Understanding that geometric forms can be qualitatively and quantitatively described (e.g., length, perimeter, surface area, volume, angle measurement).
geometric patterns	Understanding that there are many possibilities to relate points, lines, surfaces, and solids in such a way that geometric patterns emerge (e.g., polyhedral formula).
geometric forms in the environment	Understanding that real objects, operations on and with them as well as relations between them can be described with the help of geometric forms.
geometrization	Understanding that plane and spatial geometric facts, properties, and problems can be translated into the language of geometry.

Table 1: Wittmann's fundamental ideas of geometry (1999).

Walle & Lovin, 2006). Similarly, Wittmann (1999) proposed that school geometry be organized around the following *seven fundamental ideas*: geometric forms and their construction, operations with forms, coordinates, measurement, geometric patterns, geometric forms in the environment, and geometrization (see Table 1).

Wittmann's (1999) fundamental ideas are aligned with ICMI study recommendations for the new geometry curricula (Mammana & Villani, 1998), which have been adopted by many national curricula. For instance, in the German curriculum (RLP, 2015) all seven fundamental ideas are present. Thus, the German curriculum reflects the multi-dimensional view of geometry, but the extent of this focus differs. The fundamental ideas of geometric forms and their construction, and measurement dominate the geometry content from early grades on, whilst the fundamental ideas of coordinates, and geometric patterns little attention is given. What influence this, however, may have on the meanings students assign to geometry, and if they recognize the multi-dimensionality of geometry and to what degree, remains open.

Research with young students normally uses observations, interviews, and/or questionnaires, which have shown not to be always reliable due to their young age (e.g., Einarsdóttir, 2007; Pehkonen et al., 2016). Drawings have been recognized as an alternative form of expression for children. Barlow, Jolley, and Hallam (2011) reported that free hand drawings tent to facilitate the recalling of events that are unique, interesting to students and can help students better recall and express more details about events they depicted. In that manner, drawings open a holistic way into children's lived experiences, conceptions of mathematics and on teaching (e.g., Einarsdóttir, 2007). In the recent years, researchers (e.g., Halverscheid & Rolka, 2006; Laine et al., 2015; Pehkonen et al., 2016; Rolka & Halverscheid, 2011) successfully used drawings to access and study students' views of mathematics. However, they focused on mathematics in general, and not on a specific mathematical content, such as geometry.

RESEARCH PROCESS

For this study, an explorative qualitative research design was chosen. The study **participants** were grade 3-6 students. This age group was optimal for the purposes of the study as this is an important period for the development of geometric thinking (Mamanna & Villani, 1998). In total 114 students from several urban schools in the federal state of Brandenburg (Germany) participated in the project (see Table 2). A purposeful sampling strategy was utilized as a way of collecting rich and in-depth data.

Grade level	Grade 3	Grade 4	Grade 5	Grade 6
Number of subjects	25	33	28	28

Table 2: Participant sample.

Main **sources of data** were student work and semi-structured interview. Student work was based on an adaptation of the instrument from the work of Rolka and Halverscheid (2011), and Halverscheid and Rolka (2006) focusing on Wittmann's fundamental ideas of geometry (1999). The research data were collected in one-to-one setting between a student and the first author of the paper. The students were given a piece of paper with the following assignment: “Imagine that you are an artist. A good friend asks you what geometry is. Draw a picture in which you explain to him/her what geometry is for you. Be creative in your ideas.” In addition, the students answered the following three questions:

- In what way is geometry included in your drawing?
- Why did you choose these elements in your drawing? Why did you choose this kind of representation?
- Is there anything you did not draw but still want to say about geometry?

Based on the age of the student, these questions were answered orally or in written. When this was done orally, the data were audio-taped, otherwise the students wrote down their answers. After the student was done drawing, the semi-structured interview started. In the semi-structured interview, the students were asked to describe what they have drawn. Multiple data sources were used to assess the consistency of the results, and to increase the validity of the instruments.

The analysis of drawings is understood as interpreting meanings that students have given to situations and objects which they presented. These meanings influence students’ actions (Blumer, 1986), and what they draw. Data analysis involved the first and the third author of the paper coding the data independently, and identifying themes, which were then validated through an iterative process, and constant comparison. The analysis contained the following steps: (1) analysis of drawings with respect to the framework of Wittmann (1999), (2) confirmation of the interpretation by content analysis of the three questions, (3) coding of other subconceptions included in the student oral/written data. Different representations of fundamental ideas of geometry were first assigned one of the Wittmann’s (1999) categories (see Table 1), before assigning a specific subcategory. If descriptor was not given, then both researchers discussed together the nature of the fundamental idea before developing a new subcode, and extended the coding manual. The fundamental ideas that were revealed through the questions only, were also included in the analysis. The interrater reliability was high (89% agreement). Afterwards we made adjustments to our coding and the coding manual, after which the interrater reliability was at 100%.

RESULTS

Table 3 shows the absolute and relative frequencies of grade 3-6 students’ fundamental ideas of geometry. The fundamental idea of *geometric forms and their construction* (F1) was the most often coded fundamental idea of geometry (73.8%), which was independent of the grade level, with almost all students’ drawings pertaining one aspect regarding this idea. The second most often coded fundamental idea was *geometric forms in the environment* (F6) with 10.2%. This was followed by the fundamental ideas of *measurement* (F4), and *operations with forms* (F2), with 6.2% and 5.8%, respectively. The fundamental ideas of *coordinates* (F3), *geometrization* (F7), and *geometric* *patterns* (F5) were the three least coded fundamental ideas with 2.2%, 1.1%, and 0.8%, respectively.

Grade	Absolute and relative frequencies of fundamental idea						
	F1	F2	F3	F4	F5	F6	F7
Gr. 3	110 (73.3%)	13 (8.7%)	12 (8%)	2 (1.3%)	2 (1.3%)	11 (7.3%)	0 (0%)
Gr. 4	224 (77.8%)	13 (4.5%)	3 (1.0%)	13 (4.5%)	2 (0.7%)	31 (10.8%)	2 (0.7%)
Gr. 5	174 (67.7%)	16 (8.2%)	2 (1.0%)	10 (5.1%)	2 (1.0%)	31 (16.4%)	1 (0.5%)
Gr. 6	181 (74.2%)	9 (3.7%)	2 (0.8%)	29 (11.9%)	1 (0.4%)	15 (6.2%)	7 (2.9%)
Total	647	51	19	54	7	89	10

Table 3: Absolute and relative frequencies of fundamental ideas of geometry.

The drawings of grade 3-6 students showed both similarities and differences (see Figure 1 and Table 3). With respect to F1 no increase in knowledge is discernible. However, there were some patterns in students' answers pertaining to different aspects of this fundamental idea. In all grades, different plane surfaces dominated in the drawings, ranging from 41% in both grades 5 and 6, to 59.1% and 55.8% in grades 4 and 3, respectively. The second most often depicted aspect were solids, which again were most often seen in grade 4 student drawings. In all other grades, the range was between 15.9% (grade 5) and 26.4% (grade 3). Different drawing tools (e.g., drawing stencil, ruler, protractor, compass) were the third most often coded aspect of F1 ranging from 6.4% in grade 3 to 17.1% in grade 6. Most notably students differed with respect to mentioned properties of geometric forms. This aspect was only seen in grade 4-6 student drawings. However, this aspect was coded in 3.1% (grade 4) and 7.7% (grade 6) of cases, whereas in grade 5 in 21.8% of cases. Thus, properties of geometric forms gain on importance as primary grade students progress into higher grades.

F2 does not show a linear increase from grades 3-6, as this fundamental idea was most often coded in grade 3 and least coded in grade 6. Interestingly, about 24% of the grade 3 and 4 students drew some aspect pertaining to this fundamental idea, whereas in grade 5 39% and in grade 6 only 25% of the students drew an aspect attributed to this fundamental idea. Not all operations with forms were present in all grades. For instance, translation and dilation were only present in grade 4, and point reflection in grade 5 student drawings. On the other hand, axial symmetry dominated in the drawing of grade 3-6 students, with the highest frequency in grade 3 (69.2%).

Figure 1: Examples of grade 3-6 students' drawings.

With respect to F3, a rapid decrease from lower into higher grades is observable. Moreover, lower grade students used prepositions (e.g., right, left, below) to describe the position of geometric forms, while upper grade students used coordinate system for it. On the other hand, student drawings portray an increase from lower grades (1.3%) into higher grades (11.9%) with respect to F4. Thus, students show an understanding of qualitative and quantitative attributes assigned to geometric forms at a progressive rate. While in grade 3 student drawings only length of segments is addressed, in grade 6 five different measurements were addressed, namely length, perimeter, surface area, volume, and angle measurement. Very few students think of patterns (F5), when thinking about geometry. Except from grade 5 where two student drawings revealed this aspect, only one student per grade level portrayed this understanding of geometry. F6 shows an increase from grade 3 to grade 5, but a decrease in grade 6. Concretely, this fundamental idea was drawn by every second grade 5 student and every fifth grade 6 student. F7 is the most abstract fundamental idea, which can explain few codes pertaining to this fundamental idea. However, an increase is visible, reaching its maximum in grade 6 with only 2.9% of drawing revealing an aspect aligned with F7.

CONCLUSIONS

The study results show that primary grade students have a rather narrow understanding of geometry, albeit all fundamental ideas being covered by the curriculum (RLP, 2015). Majority of the students drew either one or two fundamental ideas in their drawings. Rarely student drawings and interviews revealed an understanding of geometry containing three or four fundamental ideas of geometry. Independent of the grade level, the fundamental idea of geometric forms and their construction dominated in the student drawings. This focus is not surprising as this fundamental idea dominates in the German mathematics curriculum (RLP, 2015). However, it is astonishing that students mainly associated geometric forms with plane surfaces and solids, even though 0- and 1-dimensional objects are covered in every grade in the German curriculum (RLP, 2015). The results showed that these aspects decreased from lower into higher grades. This might mean, that with time students associate geometry with 2- and 3-dimensional forms, which may be due to the fact that surface area and volume calculations are added to measurement of distances in higher grades. In addition, it seems that properties of geometrical objects are not internalized, and students mainly operate on the level of visualization (van Hiele, 1985). Peculiarly, students associated geometry more with geometric forms in the environment than with measurement, though the former is addressed only once per grade level in the curriculum, whilst the latter dominates throughout the curriculum (RLP, 2015). Moreover, the fundamental idea of geometric forms in the environment showed an increase from grade 3 to grade 5, whilst it decreased in grade 6. This can possibly be attributed to contents of the respective grade level. Geometry curriculum in grades 3 to 5 deals explicitly with objects from the environment; they are searched for, described and sorted by properties (RLP, 2015), while in grade 6 this content is no longer primarily part of the curriculum, and the focus shifts onto a more deductive approach to geometry. Accordingly, an increase from grade 3 to grade 5 can be expected. This may also explain an increase of drawings addressing the fundamental idea of geometrization. The fundamental idea of measurement was hard for students to draw, instead, concepts pertaining to measurement (e.g., perimeter, surface area, volume) were presented in the drawings as words, formulae or added in the interview. The fundamental idea of coordinates has not been often found in the students' drawings, even though this topic and its different aspects are relevant from early grades on. Since the coordinate systems are part of the curriculum for grades 5 and 6 (RLP, 2015), it is, however, very surprising that this content has been primarily addressed by grade 3 students at a basic level by using prepositions, and that there is no increase in grades 5 and 6. The low results with respect to the fundamental idea of geometric patterns suggests that this content either does not seem to be directly linked to geometry lessons or it is rarely discussed (Backe-Neuwald, 2000). Although student drawings revealed all fundamental ideas, the fundamental ideas of geometric objects and their construction, and measurement prevailed. However, these are just two fundamental ideas of geometry, and its sole focus may resolve in students developing a rather narrow view of geometry. Thus, it would be necessary to re-question the curriculum requirements regarding the multi-dimensional nature of geometry.

Drawings opened a new way to gain insight into student understanding of geometry going beyond the purely cognitive, and it might have a potential as a starting point for a discussion in the classrooms (Halverscheid & Rolka, 2006). Nevertheless, drawbacks occurred; some children had difficulties drawing, some do not like to draw, some drew objects which they found easy to illustrate (e.g., forms). Other aspects (e.g., measurement, geometrization) have shown to be hard to draw. Here additional data sources (e.g., post-interviews) were necessary. In this sense, the search for alternative research methods, that would provide a holistic understanding of this multi-faceted phenomena, is an issue of concern, and remains an ongoing research area.

References

Backe-Neuwald, D. (2000). *Bedeutsame Geometrie in der Grundschule: Aus Sicht der Lehrerinnen und Lehrer, des Faches, des Bildungsauftrages und des Kindes.* Dissertation. Universität Paderborn.

Barlow, C. M., Jolley, R. P., & Hallam, J. L. (2011). Drawings as memory aids: optimising the drawing method to facilitate young children's recall. *Applied Cognitive Psychology, 25*(3), 480–487.

Blumer, H. (1986). *Symbolic interactionism. Perspective and method*. Berkeley, CA: University of California Press.

Einarsdóttir, J. (2007). Research with children: methodological and ethical challenges. *European Early Childhood Education Research Journal, 15*(2), 197–211.

Halverscheid, S., & Rolka, K. (2006). Student beliefs about mathematics encoded in pictures and words. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 233–240). Prague: PME.

Jones, K. (2000). Critical issues in the design of the geometry curriculum. In B. Barton (Ed.), *Readings in mathematics education* (pp. 75–90). Auckland, New Zealand: University of Auckland.

Laine, A., Ahtee, M., Näveri, L., Pehkonen, E., Portaankorva-Koivisto, P., & Tuohilampi, L. (2015). Collective emotional atmosphere in mathematics lessons based on Finnish fifth graders' drawings. *LUMAT – Research and Practice in Math, Science and Technology Education, 3*(1), 87–100.

Mammana, C., & Villani, V. (1998). *Perspectives on the teaching of geometry for the 21st century: an ICMI study*. Dodrecht, the Netherlands: Kluwer.

Pehkonen, E., Ahtee, M., & Laine, A. (2016). Pupils' drawings as a research tool in mathematical problem-solving lessons. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems. Advances and new perspectives* (pp. 167–188). New York, NY: Springer.

Rezat, S., Hattermann, M., & Peter-Koop, A. (Eds.). (2014). *Transformation – A fundamental idea of mathematics education*. New York, NY: Springer.

Rolka, K., & Halverscheid, S. (2011). Researching young students' mathematical world views. *ZDM Mathematics Education*, *43*(4), 521–533.

Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, Ministerium für Bildung, Jugend und Sport des Landes Brandenburg. (Eds.). (2015). *Rahmenlehrplan Jahrgangsstufen 1-10. Teil C, Mathematik*. Berlin, Potsdam.

van Hiele, P. (1985). *The child's thought and geometry*. Brooklyn, NY: City University of New York.

Van de Walle, J., & Lovin, L. H. (2006). *Teaching student-centered mathematics. Grades 5-8*. Boston, MA: Pearson.

Winter, H. (1976). Was soll Geometrie in der Grundschule. *Zentralblatt Didaktik der Mathematik, 8*, 14–18.

Wittmann, E. Ch. (1999). Konstruktion eines Geometriecurriculums ausgehend von Grundideen der Elementargeometrie. In H. Henning (Ed.), *Mathematik lernen durch Handeln und Erfahrung. Festschrift zum 75. Geburtstag von Heinrich Besuden* (pp. 205–223). Oldenburg: Bueltmann und Gerriets.

CONNECTED WORKING SPACES: DESIGNING AND EVALUATING MODELLING-BASED TEACHING SITUATIONS

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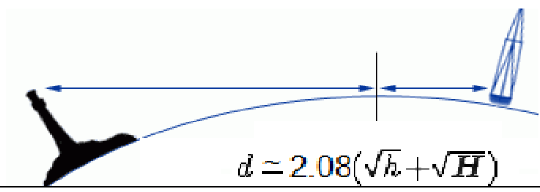
This contribution focuses on modelling at upper secondary level. Modelling is considered as a work on various models of a reality, belonging to different scientific fields, with varied mathematizations. The framework of Connected Working Spaces is chosen in order to describe the work on each model, and the connections made along the modelling process. The hypothesis is that these choices allow designing and evaluating situations that help students to understand comprehensively concepts taught at upper secondary level and enable them to appreciate how diverse fields contribute to a scientific perception of the sensible world. This hypothesis is tested by way of an experiment in realistic school settings.

MODELLING IN MATHEMATICS EDUCATION. WHAT FOR AND HOW?

Teaching/learning at upper secondary level should give meaning to mathematics as a tool for understanding the sensible world, and that is why modelling based teaching situations are often proposed. However, curricula and resources often reduce modelling to “translation” between reality and mathematics. Ideas brought by research like the “modelling cycle”, have the advantage of distinguishing models of different natures and of characterizing the type of activity involved in the transition from one to the other (English, Ärlebäck & Mousoulides 2016). This paper aims to build critically upon these ideas. A starting point is that the “modelling cycle” sharply separates “reality” and mathematics. I take an example from Blum & Ferri (2009, p. 48): it is asked to find the distance of a navigator to a lighthouse when the navigator perceives its light exactly on the horizon. The authors consider mainly two models: a "situation model" (*in reality*) and “*the* mathematical model”. Both models assume that the navigator is at sea level. I offer to consider here three alternative models (figure 1).

A “navigational science” model:
d the distance in nautical miles, h and H the heights respectively of the eye of the observer and of the light in meters.



A “geometrical-algebraic” model:
R the earth radius $d=\sqrt{h^2+2\cdot R\cdot h}+\sqrt{H^2+2\cdot R\cdot H}$

An “analytical model”: $d \simeq \sqrt{2R}(\sqrt{h}+\sqrt{H}) \simeq 1.93(\sqrt{h}+\sqrt{H})$

Figure 1: Three models for the distance to a lighthouse problem.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 291-298). Umeå, Sweden: PME.

The first model considers actual navigation conditions, the navigator above sea level, and exploits empiric observations. Then, on the one hand, this model is more realistic than Blum & Ferri's "situation model". On the other hand, it is a scientific model since it belongs to "navigation", a science taught in naval schools. The two other models are mathematical models; they differ in the way they take into account the "preponderance" of the earth radius over the heights of the objects.

This simple example illustrates multifaceted links between reality and mathematics in modelling: for a given phenomenon, different scientific fields bring different models, each involving some aspects of reality and some mathematization. Each model also involves a specific type of work. For instance, the "geometrical-algebraic" model involves the identification of relevant geometrical relationship and calculations based on the Pythagorean Theorem, whereas the "analytical" model implies reasoning on approximation, using equivalence in a neighborhood. As for the "navigational science" model, the work implies considering and generalizing empirical data. Additionally, the example shows the possibility of multiple to-ings and fro-ings between models. These to-ings and fro-ings are necessary for confronting models: in the example, discussing the discrepancy between the "navigational science" and the "analytical" models and identifying refraction of the air as the origin. They also give meaning to the concepts involved in the different models by the connections to which they lead.

CONNECTED WORKING SPACES

Considering modelling as an activity involving models in different scientific fields and branches of mathematics, a theoretical framework is necessary to describe the work in each model. The framework of the Mathematical Working Spaces (MWS) allows characterizing the way the concepts make sense in a given work context. According to Kuzniak & Richard (2013) a MWS is an abstract space organized to ensure the mathematical work in an educational setting, based on the articulation of two fundamental levels: an epistemological level related to mathematical organization and tasks to perform, and a cognitive level related to individuals' activity and task enactment. The epistemological level associates a "representamen", that is to say a representation of the object on which one works (here a model), the artefacts relevant for this work and a theoretical framework of reference. The cognitive level associates three processes: visualization making use of material and mental representations, construction using "instruments" (i.e. internalized artefacts), and a discursive process consisting in reasoning and proof. This epistemological plan is connected to a cognitive plane by three types of genesis: (1) the semiotic genesis that gives the representamen its mathematical status of representation for visualization, (2) the instrumental genesis transforming artifacts into instruments for the construction process and (3) the discursive genesis that gives meaning to the frame of reference by mobilizing it for the discursive process.

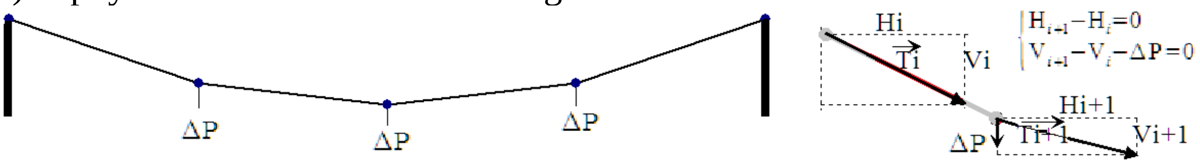
Modelling, as we understand it, involves several models, and for each of these models, a working space. Minh & Lagrange (2016) built on the MWS framework to introduce the idea of "connected working spaces" to give account of how connections between WMS bring meaning to the concepts involved. Here, this framework is expected to allow designing and evaluating teaching situations where students go back and forth between different models of a real world setting and make connections between working spaces, and help identifying geneses at work through these connections. In the following section, I will test this hypothesis by way of an experiment.

AN EXPERIMENT

I report here about the design and experimentation of a situation for 12th grade students. It deals with the modelling of suspension bridges, aiming to make students discover a function modelling the main cable by way of a study of tensions. Inspired by the framework defined in the previous section this section presents first briefly four models and then the associated working spaces and a classroom implementation.

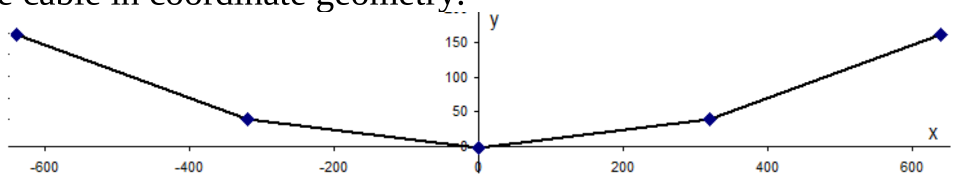
Suspension bridges: four models

a) A physical model of tensions along the cable.



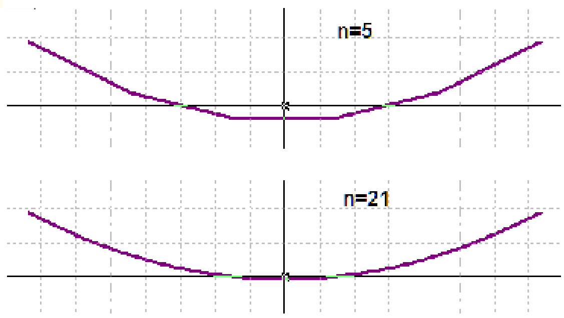
b) A model of the cable in coordinate geometry.

i	slope	x	y
0	-0,39	-640	163
1	-0,13	-320	38,2
2	0,13	0	-3,4
3	0,39	320	38,2
4		640	163



c) An algorithmic model of the cable

//coordinates of the anchoring point
$x \leftarrow -640$
$y \leftarrow 163$
//Tension on the anchoring point
$V \leftarrow -5$
Pour i allant de 1 à n
$x \leftarrow x+\frac{1280}{n}$
$y \leftarrow y+\frac{1280}{n}\cdot\left(\frac{V}{H}\right)$
$V \leftarrow V+\frac{10}{n-1}$
Fin Pour



d) A continuous model of the cable

$f'(x)=\frac{x}{128 \cdot H}$

$f(x)=\frac{x^2}{256 \cdot H}$

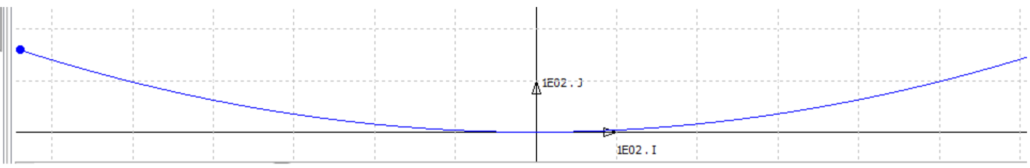


Figure 2: Models of a suspension bridge.

A suspension bridge is a type of bridge in which the deck (generally a roadway) is hung below main cables by vertical suspensors equally spaced. The weight of the deck applied via the suspensors results in a tension along the main cables. The central question is to find models of a main cable, allowing solving technical questions like the value of the tension for given data characterizing the bridge.

In a first model a main cable is represented by a physical mockup, i.e. weights equally spaced horizontally and suspended to a line. The static equilibrium law allows studying the sequence of tensions $\overrightarrow{Ti}$ in the line between the suspension points: the horizontal component H_i has the same value *H* in all segments and the sequence of values of the vertical component (V_i) is in arithmetic progression (figure 2a). The second model is a broken line in coordinate geometry. The slope of each segment is the ratio of the vertical by the horizontal components of the tension in this segment, and therefore is also in arithmetical progression. It is then possible to compute the sequence of the coordinates of the points (figure 2b). The third model systematizes the construction of the second model by way of an algorithm defining a piece wise linear function. Animating the global variables *n* (number of segments) and *H* (horizontal component of the tension) allows visualizing their influence (figure 2c). The fourth model is the curve of a mathematical function depending on a parameter *H* that can be also animated: the derivative is calculated as a limit of the slopes of segments in the broken line and the function is obtained by integration (figure 2d). The third and fourth models involve a software environment: functions are created by formula and domain, and also by an algorithm; graphs can be obtained and animated by way of sliders. The software Casyopée was used for the experiment (http://casyopee.eu).

Potentialities, constraints and general design

The brief presentation above shows that studying a suspension bridge implies considering data in the real world as well as a number of interrelated concepts in physics and calculus: tension, static equilibrium of forces, projection of vectors, slope of segments and gradient of curves, arithmetic progression and linear function, integration, discrete and continuous models, limits and integration… All these contents are taught in secondary curricula, thus the goal for students is not to "reinvent" each of them in isolation, but rather to recognize how modelling a real world situation involves understanding these concepts operationally and in interaction. In the French curriculum, the study of a suspension bridge can be carried out in the last year of the secondary scientific stream (12th grade, Terminale). The framework of connected working spaces was used to design a classroom situation exploiting the potential of the study of a suspension bridge. I consider four working spaces, each related to one of the four models. In each space, I characterize briefly the epistemological plane.

In the first working space, the representamen is the mockup with posts, line and weights, and the rules are the static equilibrium law and the properties of arithmetic progressions. Artefacts are concrete measurement devices used in physics and mathematics, dynamometers, protractor, and also more "abstract" tools like the decomposition of tensions in vertical and horizontal components. We name this space, the static systems working space, or shortly, the *statics working space*. In the second working space the representamen is a broken line. The main rule is the analytical definition of a segment: students have to compute the coordinates of the end point, knowing the coordinates of the other point, the slope and the difference between abscissas. We name this space, *the coordinate geometry working space*. In the third working space the representamen is the graph of a continuous piecewise function and an important artefact is the programming module in the software environment that allows computing the series of points defining the function by way of a simple iterative treatment and animating global variables. We name this space, *the algorithmics working space*. Finally, the representamen in the fourth space is a function governed by classical rules in calculus. This is the *mathematical functions working space*. An important artefact is the software environment that can be used to get a curve of this model, compare to a picture of the bridge and to the curve of the algorithmic model's piecewise function, and animate the parameter *H* in order that the three models fit.

IMPLEMENTATION

In order to be feasible in the context of a French 12th grade class a few weeks before the baccalaureate, the implementation is limited to three and a half hours and organized in four phases. The first phase is one hour long and has been prepared with the physics teacher. It aims first to introduce students to questions related to bridges, particularly suspension bridges. They are invited to consult a dedicated website (http://structurae.info/ouvrages/ponts-et-viaducs), to select and sketch four bridges of different types, to look at a video illustrating the idea of tension along a horizontal rope and the fact that, whatever the tension, the rope is no more a straight line, as soon as force is applied vertically on a point, and to answer questions about suspension bridges. Also in this first phase, the students have to build apparatus combining dynamometers and weights, compute the horizontal and the vertical components of tensions and verify the static equilibrium of forces.

The second phase is 50 min long. At the beginning, the data related to the Golden Gate Bridge is presented to the whole class. Students also look at a physical mockup where the cable is a succession of dynamometers allowing observing the evolution of tensions along the cable. Then students are split into groups of four. Each group has one task, A or B, C or D. Task A is related to the *statics working space*: inspired by the work in the first phase, students have to consider the sequence of horizontal and vertical components of tensions at the connection points, recognize that the horizontal component is constant and compute a formula for the series of vertical components. Task B is related to the *coordinate geometry working space*. A formula for the value of the slope of each segment in a discrete model of the main cable is given to the students, depending on a parameter *H*, and on the number *n* of segments. Students have to compute the series of *x* and *y*-coordinates of the suspension points for a small value of *n* and a given value of *H*. Task C is related to the *algorithmics working space*. An algorithm like in figure 2c is given to students; they have to enter and execute the algorithm, interpret the parameter *n*, and adjust the parameter *H* in order that the model given by the algorithm conforms to the shape of the cable. Task D is related to the *mathematical functions working space*. Students have to search for a function f whose curve models a main cable. They are informed that the horizontal component of the tension in the cable is a constant H and of a linear formula for the vertical component of the tension at a point of given x-coordinate. They have to find a formula for the derivative of f, taking into account that the tension is in the direction of the tangent to the curve. Then, using the software environment, they have to find a formula for f and adjust the parameter H in order that the curve of the function f conforms to the cable.

The third phase is also 50 min long. In this phase, the students are expected to develop connections between the working spaces of the second phase. They form new groups. Each of these new groups is made in order to bring together one or two students of each of the previous groups respectively doing task A, B, C and D. Students are invited to share their findings and to write a report emphasizing the important points of the study. The fourth phase (30 min long) is a collective synthesis led by the teacher.

Observation and evaluation

This implementation was observed in a class of 35 students by the end of March. The students were mostly average achievers. The contents at stake in physics and mathematics had been taught to students in previous lessons. The phases have been video recorded. In the first phase, most students sketched a suspension bridge without suspensors. They understood from the video that the weight of the deck "bends" the cable, but they did not link the shape of the cable with the uniform repartition of the weight, thanks to the suspensors. In the rest of the phase, the students correctly recorded angles and intensity of forces and recognized the static equilibrium law.

For the group work in phases 2 and 3, I report on a series of four groups observed doing each task in phase 2, and on one group in phase 3 bringing together students observed in phase 2. In phase 2, students doing task A mainly succeeded, while difficulties were observed for students doing other tasks. Students doing task B started by sketching a bridge with a lot of suspensors, not allowing to consider segments. They were prompted by the teacher to limit to 4 suspensors. They took time to find the coordinates of the anchoring point and had difficulties to use the formula given for the slope of the segments and the distance between suspensors, in order to calculate the coordinates of the next point. Students doing task C took time to enter the algorithm in the software. Nothing or a wrong display appeared on the screen, because of small mistakes. They corrected when the teacher helped them to analyze the algorithm. They identified the parameter n as related to the number of suspensors and proposed the value 83 (the number of suspensors in the Golden Gate Bridge). They considered that this value is "close to infinity" and that is why the curve did not appear as a broken line, in contrast to small values of n. Students doing task D found a formula for the vertical tension but had difficulty to interpret the fact that the tension is in the direction of the tangent.

In the group of phase 3, each student explained her task and her work in the preceding phase. Other students listened attentively and asked for further explanation. The parameter H was identified by students as playing a role in each task; for instance when a student who did task C did not remember the effect of increasing *H*, confusing with the "height of the cable", the student who did task A corrected him, saying that it is a tension and then increasing should "straighten" rather than "slacken" the cable. The same student helped to overcome the difficulty met by the student who did task D to find the direction of the tangent to the curve and then the derivative of the function, saying "you just integrate the quotient of *V* and *H*".

To further evaluate the connections students made between working spaces during the group work, I report on interviews with 3 students after phase 3. Each interview was 20 min long. The students were first questioned on their impression about the tasks and then they were invited to summarize their findings. They stressed that the situation was more complex than usually and that they were "not used to mix physic and mathematics". Commenting the first phase, they showed how their awareness of the structure of a bridge progressed, mentioning correctly the role of the suspensors. They still had difficulties in considering the slopes of the segments in task B in order to find the coordinates of the suspension points. They correctly interpreted the algorithm of task C, and were able to connect the evolution of *H*, and *x* and *y* respectively to task A and B. They did not show clear awareness that the function of task D was a limit of the continuous piecewise function of Task C. From graphical evidence they thought that it was more or less the same function for big values of *n*. The observer asked to explain why the gradient in a point of the curve is the quotient of *V* and *H*. The expected answer was that the tension has the direction of the tangent, but the students simply wrote $f'(x) = \Delta y/\Delta x = V(x)/H$. The first equality is common in the physics course, and the second derives from the definition of the components in task A. Thus, students made a connection between the statics space and the mathematical function space without explicit consideration of a limit.

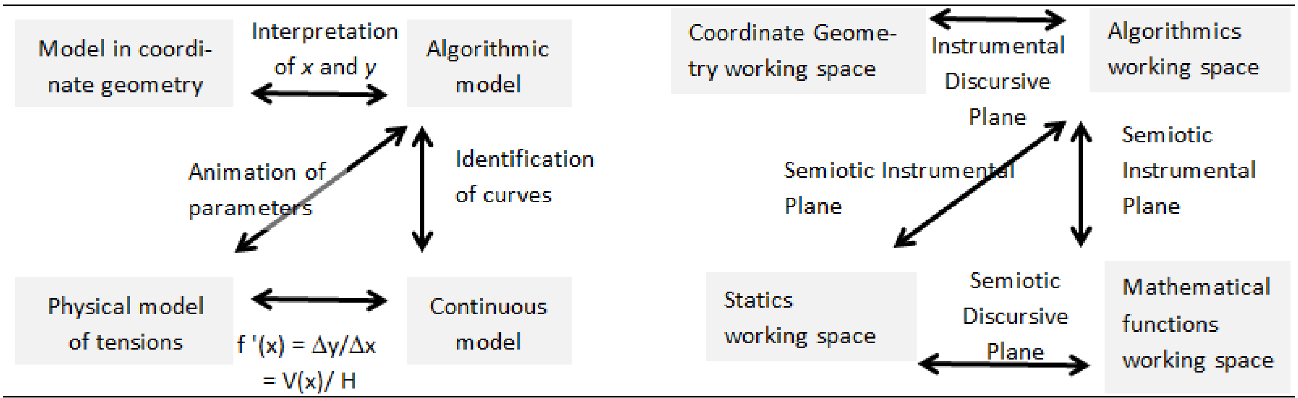


Figure 3: Connections made by students.

The figure 3 (left) summarizes the connections between models made by students. On the right, each connection is interpreted as connecting working spaces and participating in geneses. When students interpreted the evolution of the variables *x* and *y* in the algorithm, by connecting the body of the loop with the recurrence law of the coordinates in the geometrical model, it participates in a *discursive* validation of an *instrumental* object. The animation of parameters (an *instrumental* activity) within the software environment supports visualization (a *semiotic* activity). The *discursive* explanation that students gave of the value of the gradient of the curve of the mathematical function is inspired by the *semiotic* notations of derivatives in physics. Each connection sheds light on concepts at stake: recurrence and iterative treatment, parameters in a function, tension, discrete and continuous models, gradient and derivative. Kuzniak, Tanguay, & Elia (2016, p. 728) name “vertical planes” the two by two combinations of geneses and depict them as “valuable tools for describing the evolution of the mathematical work in the solving process”. They especially warn that “activities with a strong constructivist feature” have to be designed in order to “avoid to be imprisoned in the Semiotic Instrumental plane”. Figure 3 shows that, in students’ activity, this plane is complemented by the two other planes, involving validation.

CONCLUSION

The experiment indicates benefits of a design inspired by the connected working spaces framework. The framework allows thinking of modelling as involving several models, each of them making reality and mathematics interact in different ways. This allows seeing models close to reality as belonging to scientific domains (physical or nautical science…) rather than to common sense and modelling as active appropriation of models of varied natures for a given reality rather than a “translation” between reality and mathematics. By considering several models and characterizing activity on each model as a work in a specific working space the framework helped to organize effective group work in realistic classroom settings. Developing modelling interdisciplinary activities for students is encouraged by curricula, but rarely achieved; the framework and the organization proposed here could be means to overcome implementation difficulties. In spite of the complexity of the situation, students made sense of the main aspects of the models and connections between them. These connections enabled them, through specific geneses including validation as well as visualization, to understand more comprehensively concepts taught at upper secondary level. Moreover, they could appreciate how mathematics and experimental sciences contribute together to understanding the sensible world.

References

Blum, W., & Ferri, R. B. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application*, 1(1), 45–58.

Kuzniak, A., Tanguay, D. & Elia I. (2016). Mathematical Working Spaces in schooling: an introduction. *ZDM Mathematics Education*, 48-6, 721-737.

Minh, T. K., Lagrange, J.B. (2016). Connected functional working spaces: a framework for the teaching and learning of functions at upper secondary level. *ZDM Mathematics Education*, 48-6, 793-808.

English, L.D., Ärlebäck, J.B., Mousoulides, N. (2016) Reflections on Progress in Mathematical Modelling Research. In: Gutiérrez Á., Leder G.C., Boero P. (eds) *The Second Handbook of Research on the Psychology of Mathematics Education*. (pp. 383-413). Sense Publishers, Rotterdam.

THE INFLUENCE OF SALIENCY IN INTUITIVE REASONING

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Intuitions play an important role in mathematical reasoning. Stavy and Tirosh proposed the intuitive rules theory and showed how various tasks are incorrectly solved on the basis of intuitive rules triggered by salient task characteristics. In this study we wanted to replicate and extend the results of Stavy and Tirosh. Furthermore, we wanted to test two different ways of making intuitive reasoning more likely (textual and graphical saliency). We found that all tasks tested in this study showed similar patterns of accuracy rates and reaction times as in the studies of Stavy and colleagues. However, we were not able to replicate the result that more salient task elements lead to more intuitive reasoning. We propose explanations for these different results and discuss implications for further research and educational practice.

INTRODUCTION

Intuitions play an important role in mathematical reasoning (Fischbein, 1987; Stavy & Tirosh, 2000). It is argued that many errors that occur in a variety of mathematical tasks can be explained by the occurrence of intuitive reasoning processes. In this paper we apply the intuitive rules theory (Stavy & Tirosh, 2000) to five different tasks that were administered to secondary school students.

Stavy and Tirosh (2000) have formulated the intuitive rules theory, which describes three different intuitive rules that they argue are often applied to problems in the field of science and mathematics. First, the 'Same *A* – Same *B*' rule leads people to reason that when two objects are the same with respect to quantity *A*, they are also the same with respect to quantity *B*, even when quantity *A* and *B* are unrelated. Second, the 'More *A* – More *B*' rule means that a person reasons that a perceptual quantity *A* in a task is related to the queried quantity *B*, while in fact a larger quantity of *A* does not in all cases mean that the quantity of *B* also increases. Third, the 'everything can be divided' rule refers to the incorrect generalization of this rule, which may be correct for theoretical questions, to real-life situations wherein it no longer applies. The theory further states that the application of these rules is often based on irrelevant but salient characteristics of the task. When a person is confronted with these salient task characteristics, the use of an intuitive rule is triggered.

Various studies have provided evidence for the occurrence of the first two intuitive rules in different tasks. For example, the 'More *A* – More *B*' rule has been studied in geometry (Babai, Nattiv, & Stavy, 2016; Babai, Zilber, Stavy, & Tirosh, 2010; Stavy & Tirosh, 2000) and probabilistic reasoning (Babai, Brecher, Stavy, & Tirosh, 2006),

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 299-306). Umeå, Sweden: PME.

and the 'Same *A* – Same *B*' rule in trigonometry (Stavy & Tirosh, 2000) and physics (Van Dooren, De Bock, Weyers, & Verschaffel, 2004).

The methods used in studies on intuitive rules are largely based on methods used in research on the dual process theory, a framework from cognitive psychology that has in recent years also been applied to mathematics education (e.g., Gillard, Van Dooren, Schaeken, & Verschaffel, 2009). According to this theory (Evans, 2006), people can use two reasoning systems: an intuitive one (often also called heuristic) and an analytic one. Intuitive processing is initiated when confronted with a task, leading to a fast response. In many cases this intuitive response is correct, but this is not always the case, making it necessary for analytic reasoning to intervene. This is slower and more effortful, making it possible to empirically validate whether people use intuitive or analytic reasoning when solving a certain task by looking at accuracy rate and reaction time patterns on two different item types. In congruent items the intuition under study leads to the correct response, while incongruent items require analytic reasoning in order to achieve a correct answer. Congruent items are hence expected to be solved quickly and correctly, as only fast intuitive processing is necessary. Incongruent items, on the other hand, are expected to be solved less accurately, and when a correct response is given it is expected that this response is slower than a correct response to a congruent item as more time-consuming analytic reasoning is necessary to find this correct response.

A well-studied task in the intuitive rules literature is the so-called polygon task, which is presented in Figure 1. In this task two polygons are presented with a different area and the student has to compare the area of the polygons. It has been found that incongruent items are solved less accurate and slower than congruent items (Stavy & Tirosh, 2000). Furthermore, it was shown that when the area of the polygon is made more salient, fewer correct responses are given (Stavy, Goel, Critchley, & Dolan, 2006). This suggests that the reasoning of participants was indeed led by the area that gets bigger or smaller, without looking at other characteristics of the polygons.

The goal of the current study was three-fold. First, we wanted to replicate the results of Stavy and Tirosh (2000), Babai et al. (2006) and Stavy et al. (2006) with respect to the effect of congruency and saliency on two 'More *A* – More *B*' tasks. Second, we wanted to generalize the findings of Stavy et al. (2006), Babai et al. (2006) and Stavy and Tirosh (2000) to other tasks, which elicit other intuitions than the 'More *A* – More *B*' intuitive rule. Finally, we wanted to test whether saliency can be manipulated in a different way than Stavy et al. (2006) did. More specifically, Stavy et al. (2006) used a visual way to make the items more salient, namely by shading the polygons appearing in the task. This is a task-specific operationalization of saliency that cannot be applied to all tasks. Hence, instead of this visual manipulation, we used a textual way of making a task more salient, namely by adding part of the anticipated intuitive rule to the task.

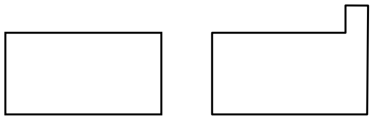
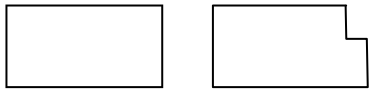
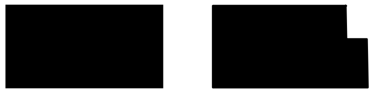
	Congruent	Incongruent
Non-salient	 Area gets larger, perimeter gets larger	 Area gets smaller, perimeter stays the same
Salient	 Area gets larger, perimeter gets larger	 Area gets smaller, perimeter stays the same

Figure 1: The polygon task: is the perimeter of the right polygon larger or smaller than that of the left polygon?

METHOD

Participants were 123 pupils from a secondary school in Flanders, Belgium. About half of the participants (n = 65) were in the fourth grade, while the other half (n = 58) were in the fifth grade. We used five tasks in which the application of different intuitive rules could be tested. We only present two of them here in figures because of space restrictions. First, there were two 'More *A* – More *B*' tasks: the polygon task used by Stavy et al. (2006; see Figure 1) and a probability task from Babai and colleagues (2006). These two tasks were included to replicate the results of Stavy and Tirosh (2000), Stavy et al. (2006), and Babai et al. (2006), which was the first goal of the study. The third task was a 'Same *A* – Same *B*' task involving rectangles (see Figure 2). This task was created by the authors and was included to be able to generalize previous findings to other tasks, which was the second goal of the study. Finally, there were two proportional reasoning tasks, one about triangles and one about cylinders. These tasks cannot be included in this paper due to length restrictions. As with the third task, these two tasks were included to generalize previous results to new tasks.

For each of the five tasks, three items were created: one congruent and two incongruent items. This means that there were five times three items to be solved by every participant (= 15; 5 congruent items and 10 incongruent items). Of each item, a salient and a non-salient version were created to constitute two conditions: saliency was varied between subjects. In the non-salient version, the item was administered without any special cues. In the salient version, however, a sentence was added that stressed the fact that one of the elements was the same or different, making it more likely that people would indeed use this element to (incorrectly) base their reasoning on. Participants were randomly assigned to either the salient or the non-salient condition. Of the polygon task, we also used a second type of saliency that was tested next to the textual type of saliency used for all tasks. We filled the polygons to draw more attention to the area of the polygons, just like Stavy et al. (2006) did. This way we could compare these two types of saliency, which was the third goal of the study. For the other four tasks it turned out impossible to make the tasks more salient in a similar, visual, way.

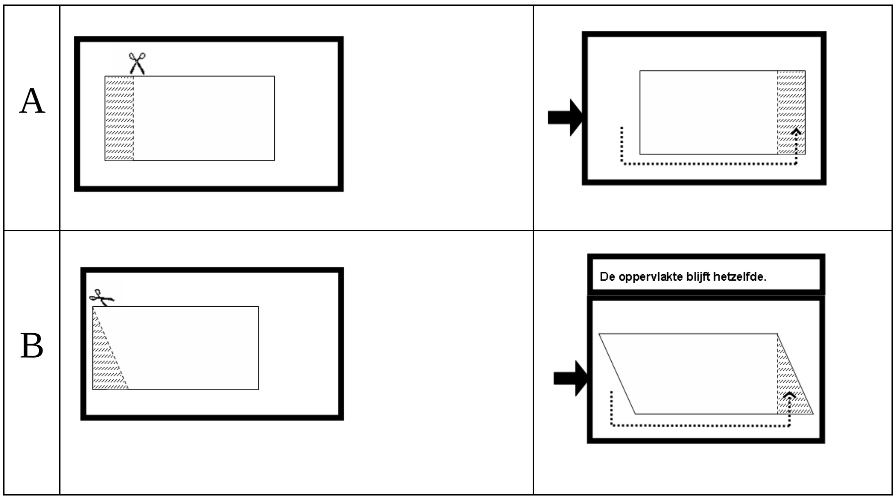


Figure 2: Rectangle task. Does the perimeter change when part of the figure is moved? A: Congruent item, non-salient. B: Incongruent item, salient (English translation: "The area remains the same"). The only difference between the non-salient and the salient items is the presence of the text in the second image

Participants were randomly assigned to one of two conditions: non-salient or salient. The experiment was done individually on a computer, in groups of five to ten pupils at a time. In each item, two situations had to be compared. Participants were asked to work at a steady pace, without working so fast that they would be prone to making errors. The first situation was first presented for two seconds, after which the second situation was presented next to the first situation. After another two seconds, three answer alternatives were presented simultaneously and the participant could answer by pressing the number on the keyboard corresponding to their answer. The reaction time measurement started only when the answer alternatives were presented and stopped as soon as a response was given.

RESULTS

Effect of congruency

For each of the tasks, we tested whether congruent items were solved more accurately than incongruent items. This was the case for all tasks (Table 1). These results suggest that intuitive reasoning lies at the basis of most participants' reasoning.

For the reaction times, we tested for each of the tasks whether accurate responses to incongruent items had longer reaction times than accurate responses to congruent items. As is shown in Table 2, this was always the case. This is a second type of evidence that intuitive reasoning lies on the basis of the responses, and needs to be overcome in incongruent items.

	Accuracy congruent items	*Accuracy incongruent items*	χ^2	df	*p*
Polygon task	96.67	57.08	34.46	1	< .001
Probability task	96.67	72.50	20.57	1	< .001
Rectangle task	92.50	20.42	102.90	1	< .001
Triangle task	95.00	22.92	86.85	1	< .001
Cylinder task	80.83	21.25	96.05	1	< .001

Table 1: Main effect of congruency on accuracy (in %), per task

	RT congruent items	*RT incongruent items*	*F*	df	*p*
Polygon task	3287	7302	23.83	1, 249	< .001
Probability task	9952	12376	10.56	1, 288	.001
Rectangle task	6949	13860	71.67	1, 158	< .001
Triangle task	8133	13066	40.57	1, 165	< .001
Cylinder task	10743	14225	10.11	1, 145	.002

Table 2: Main effect of congruency on reaction times (in *ms*) of accurate responses, per task

Effect of saliency

As explained above, for the polygon task we used two types of saliency: graphical (like in the study of Stavy et al., 2006), and textual (like in the other items). This was done to test whether both types of saliency would be equally effective in influencing the number of intuitive responses. Before analyzing the effect of saliency for the other items, we needed to know whether there was a difference between these two types of saliency. A logistic regression analysis did not show an interaction effect on accuracy of type of saliency and congruency, Wald $\chi^2(1) = 0.00$, $p = 1.00$. The same result was found when we used the number of intuitive responses as the dependent variable. There was also no effect of type of saliency on the reaction times of accurate responses, $F(1,56) = 0.91$, $p = .344$. These results suggest that both types of saliency for the polygon task had the same effect on the extent to which intuitive reasoning was triggered and applied. This also means that in the analyses that follow we did not need to distinguish both saliency types and coded both types of saliency as 'salient'.

To study the effect of saliency we looked at the number of intuitive responses to the incongruent items of all tasks separately for both the salient and the non-salient items (Table 3). For none of the tasks we found a significant main effect of saliency on the number of intuitive responses, which does not follow our hypothesis that salient items would lead to more intuitive reasoning than the non-salient items.

	NS	*S*	*Wald* χ^2	df	*p*
Polygon task	21.67	33.33	1.83	1	.177
Probability task	15.00	14.17	0.02	1	.900
Rectangle task	70.83	79.17	0.56	1	.456
Triangle task	70.00	74.17	0.14	1	.709
Cylinder task	71.67	66.67	.20	1	.654

Table 3: Main effect of saliency on the percentage of intuitive responses for the incongruent items, per task (NS = non-salient condition, S = salient condition, numbers are percentage of intuitive responses)

For the reaction times of intuitive responses to incongruent items we expected faster responses for the salient items than for the non-salient items: Saliency makes that the intuitive response is triggered. Table 4 shows the results of these analyses. Only for the polygon task a significant main effect of saliency on reaction time was found, but the effect was in the opposite direction than what we hypothesized.

	NS	*S*	*F*	df	*p*
Polygon task	5136	7628	4.47	1,236	.036
Probability task	14827	16554	0.03	1,238	.856
Rectangle task	9673	10875	2.34	1,237	.127
Triangle task	10465	11625	0.88	1,236	.350
Cylinder task	12123	12011	0.88	1,237	.349

Table 4: Main effect of saliency on reaction times for the intuitive responses to incongruent items, per task (NS = non-salient condition, S = salient condition, numbers are the reaction times in ms)

DISCUSSION AND CONCLUSION

In this study we wanted to replicate, generalize, and extend results found by Stavy and Tirosh (2000), Babai et al. (2006) and Stavy et al. (2006) concerning the use of intuitive rules. First, we looked for effects of congruency on accuracy and reaction times. For all five tasks included in the study we found that congruent items were solved significantly better than incongruent items, which suggests that participants indeed applied the anticipated intuitive rule. Also when looking at the reaction times of accurate responses we found evidence that points in this direction. With these results, we met our first and second research goal concerning replication of the effect of congruency on accuracy and reaction time, and the extension to other tasks than the 'More *A* – More *B*' tasks used so far. Second, we looked for effects of saliency. We did not find an effect of the type of saliency (graphical versus textual) for the polygon items. This means that we were able to do the rest of the analyses concerning saliency without making a distinction between both types of saliency. For none of the items we obtained a main effect of saliency on accuracy. This means that we were not able to replicate the results of Stavy et al. (2006) concerning saliency, and also that we were not able to generalize their results to other tasks. Furthermore, saliency only had an effect on the reaction times of intuitive responses for one task, namely the polygon task. The effect was, however, in the opposite direction of the effect that Stavy et al. (2006) found: We found that intuitive responses to salient items took longer than intuitive responses to non-salient items.

Various factors could play a role in explaining the different results concerning saliency. First, our participants were younger than those of Stavy et al. (2006). It has already been shown that the 'More *A* – More *B*' rule tends to become less influential with age (Stavy & Tirosh, 2000). It is hence possible that in our sample the rule was still very strongly present in the non-salient condition, meaning that the salient condition was not able to add to this effect. A second possibility lies in the long reaction times found in our study as compared to the study of Stavy et al. (2006). It is possible that our participants were reflecting on their responses to a much larger extent. This, in its turn, may be due to the fact that we used a variation of tasks, while Stavy et al. (2006) offered a longer series of identical items to their participants.

The results of this study have various implications for future research and theory. An important challenge for future research is to study where the differences in findings between our study and the studies of by Stavy and Tirosh (2000), Babai et al. (2006) and Stavy et al. (2006) come from. Possibilities to include are the age of the participants, as mentioned before, but also the previous education of participants, differences in the instructions given to the participants, etc. Another challenge is to create other ways than the two types of saliency included in the current study to change accuracy. This will not only increase our understanding of how intuitive reasoning works in the brain, but also opens possibilities to reduce the tendency towards inaccurate intuitive reasoning. A possibility that was recently tested to improve the intuitive interpretation of box plots was refutational text (Lem, Onghena, Verschaffel, & Van Dooren, 2016). Refutational text is a text that explicitly states and refutes a misconception or intuition. It was shown that this can indeed help to remediate intuitions. A final reflection we can make is that both dual process theory and the intuitive rules theory are very general theories about reasoning and that subject specific knowledge, for example on mathematics (education), is necessary to apply these theories to mathematical reasoning. We have tried doing this by looking at salient features of the tasks that are most likely to elicit incorrect intuitive reasoning. Future research could try to look more into the conceptual features of the tasks in order to find new approaches to both studying the occurrence of intuitive reasoning in specific tasks and finding ways of improving this reasoning.

The results of this study are important for teachers as they show that intuitive reasoning is very common and can occur in different tasks. Teachers should be aware that common mistakes can be caused by these intuitions and are hence not so easy to change. Teachers can use this knowledge to construct exercises and exam questions that can help them diagnose difficulties students have. Also, they can try to use techniques that can help their students to achieve conceptual change, like the earlier mentioned refutational text (Lem et al., 2016), in order to overcome their incorrect intuitions and replace them with correct intuitions.

References

Babai, R., Brecher, T., Stavy, R., & Tirosh, D. (2006). Intuitive interference in probabilistic reasoning. *International Journal of Science and Mathematics Education*, *4*, 627–639. http://doi.org/10.1007/s10763-006-9031-1

Babai, R., Nattiv, L., & Stavy, R. (2016). Comparison of perimeters: improving students' performance by increasing the salience of the relevant variable. *ZDM*, *48*(3), 367–378. http://doi.org/10.1007/s11858-016-0766-z

Babai, R., Zilber, H., Stavy, R., & Tirosh, D. (2010). The effect of intervention on accuracy of students' responses and reaction times to geometry problems. *International Journal of Science and Mathematics Education*, *8*(1), 185–201. http://doi.org/10.1007/s10763-009-9169-8

Evans, J. S. B. (2006). The heuristic-analytic theory of reasoning: Extension and evaluation. *Psychonomic Bulletin & Review, 13*(3), 378–395. http://doi.org/10.3758/BF03193858

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht: Reidel.

Gillard, E., Van Dooren, W., Schaeken, W., & Verschaffel, L. (2009). Dual processes in the psychology of mathematics education and cognitive psychology. *Human Development*, *52*(2), 95–108. http://doi.org/10.1159/000202728

Lem, S., Onghena, P., Verschaffel, L., & Van Dooren, W. (2016). The power of refutational text: changing intuitions about the interpretation of box plots. *European Journal of Psychology of Education*. http://doi.org/10.1007/s10212-016-0320-y

Stavy, R., Goel, V., Critchley, H., & Dolan, R. (2006). Intuitive interference in quantitative reasoning. *Brain Research*, *1073–1074*, 383–388. http://doi.org/10.1016/j.brainres.2005.12.011

Stavy, R., & Tirosh, D. (2000). *How students (mis-)understand science and mathematics*. New York: Teachers College Press.

Van Dooren, W., De Bock, D., Weyers, D., & Verschaffel, L. (2004). The predictive power of intuitive rules: A critical analysis of the impact of `more A–more B' and `same A–same B'. *Educational Studies in Mathematics*, *56*(3), 179–207. http://doi.org/10.1023/B:EDUC.0000040379.26033.0d

IS MATHEMATICAL CREATIVITY RELATED TO MATHEMATICAL EXCELLENCE? TEACHERS' BELIEFS

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This study investigates mathematics teachers' beliefs regarding the relationship between mathematical creativity and mathematical excellence. Written responses to an open question regarding this relationship led to six types of relationships. Findings indicated that most teachers believed that mathematical creativity can lead to excellence, with a few believing no relationship exists. Teachers' implicit beliefs regarding creativity were also analysed. It was found that the same implicit beliefs about creativity may be held by teachers with different beliefs regarding the relationship between mathematical creativity and excellence.

INTRODUCTION

Promoting mathematical creativity is seen as one of the aims of mathematics education. According to Silver (1997), mathematical creativity is "an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (p. 75). Several researchers agree and have found that hallmarks of creativity, such as fluency, flexibility, and originality can be observed in elementary school mathematics classrooms (e.g., Levenson, 2011), as well as in secondary mathematics classrooms (e.g., Tabach & Friedlander, 2017).

The teacher has a prominent role in promoting mathematical creativity. It is up to her or him to choose tasks that may occasion mathematical creativity, create a supportive environment, and adjust planned lessons according to student responses (Leikin & Dinur, 2007; Levenson, 2011). Yet, research has also found that some teachers confuse characteristics of giftedness with creativeness, claiming that a creative student is of high intelligence, verbal ability, and intrinsic motivation (Aljughaiman & Mowrer-Reynolds, 2005). Other teachers claim that only high achievers can be creative. Such beliefs may affect if, when, and how mathematical creativity is promoted in the classroom. For example, teachers who believed that only high achievers are capable of creativity, also believed that it was not their responsibility to foster creativity among all students (Aljughaiman & Mowrer-Reynolds, 2005). If we wish to encourage teachers to promote mathematical creativity among all students, not only among the highest mathematics achievers, then it is important to first investigate teachers' beliefs regarding this issue. Previous studies have investigated mathematics teachers' general conceptions of mathematical creativity (e.g., Bolden, Harries, & Newton, 2010). This study focuses on mathematics teachers' beliefs regarding the relationship between mathematical creativity and mathematical excellence.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 307-314). Umeå, Sweden: PME.

TEACHERS' BELIEFS RELATED TO CREATIVITY

Studies of teachers' beliefs related to creativity in general, and mathematical creativity specifically, have found several issues related to these beliefs. To begin with, there is the mathematics and how teachers view this content area. Some prospective teachers believe mathematics to be a closed field, with little room for independence and creativity (Bolden et al., 2010, Shriki, 2010). Those teachers claim that art, music, and language, and not necessarily mathematics, are contexts that occasion creativity (Bolden et al. 2010). In fact, Sheffield (2017) claimed that one of the most dangerous myths held by teachers and students alike is that mathematics is not a creative field. These beliefs may change with appropriate professional development, leading to a view of mathematics as a subject full of beauty and surprise, where students can develop their own creativity (Shriki, 2010).

Who can be creative and under what circumstances creativity arises, are also related beliefs. In one study, mathematics teachers from seven different countries agreed that a creative person is born that way (Leikin et al. 2013). In other studies, however, mathematics teachers, as well as other teachers, claimed that creativity can be developed (Bolden et al., 2010; de Souza Fleith, 2000), and that all students should have access to tasks that can promote creativity (Levenson, 2013). Some teachers claim that classroom environments, such as whether or not students share ideas or are given choices, may enhance or inhibit creativity (de Souza Fleith, 2000).

Regarding the characteristics of creative students, some teachers, not necessarily mathematics teachers, characterize a creative person as imaginative, willing to take risks, independent, a high-achiever, intelligent, and open-minded (Aljughaiman & Mowrer-Reynolds, 2005; Diakidoy & Phtiaka, 2002). To that list, mathematics teachers add that a creative mathematics student is motivated, curious, and enjoys mathematics (Leikin et al., 2013; Shriki, 2010). Some prospective mathematics teachers characterize creative students as those who ask challenging questions (Emre-Akdoğn & Yazagan-Sağ, 2015).

Also relevant is how teachers identify evidence of creativity and mathematical creativity. For example, coming up with unique solutions or ideas was mentioned by teachers of different ages and different subjects in several studies (e.g., Aljughaiman & Mowrer-Reynolds, 2005; Leikin et al., 2013). Specifically related to mathematics, some teachers claim that evidence of creativity may be seen when students propose different solutions and approaches to solving problems and when they associate mathematics with other subjects (Emre-Akdoğan & Yazgan-Sağ, 2015). Bolden et al. (2010) found that prospective elementary school teachers associate mathematical creativity with undertaking investigations and computational flexibility.

To summarize this section, some beliefs are related to what may be called creative processes, such as proposing different approaches to problem solving, while others relate to the product of creativity, such as unique ideas. A third set of beliefs is related to the nature of creativity, including who can be creative and what factors may influence creativity. Finally, affective aspects, such as enjoyment, were also mentioned. This study focuses on perceptions regarding the relationship between mathematical creativity and mathematical excellence and asks the following questions: (1) Do teachers believe that there is a relationship between mathematical creativity and mathematical excellence, and if so, what types of relationships do they believe exist? (2) What implicit beliefs regarding mathematical creativity surface, as teachers describe the relationship between mathematical creativity and excellence? (3) Are different implicit beliefs regarding creativity associated with different beliefs regarding the relationship between mathematical creativity and excellence?

METHODOLOGY

Participants and tool

Participants in this study were 45 graduate students in Israel working toward a Master's degree in Mathematics Education. Some of the graduate students had no teaching experience and were concurrently studying toward their teaching degree while others were experienced teachers, the most having 25 years teaching experience. None of the participants had previously taken a formal course related to creativity.

The research tool was an assignment given to each participant that began with the following request: Choose a task or activity from a mathematics text book that in your opinion promotes mathematical creativity, and explain why this task has the potential to promote mathematical creativity. Results of this part of the assignment were reported in (Levenson, 2013) and showed that most teachers associated creativity with being different and unusual. The second part of the assignment contained a controversial statement followed by a request for a response. The statement was: "There are those who say that mathematical creativity is related to excellence in mathematics. What is your opinion?" The word "excellence" was chosen, as opposed to "gifted", because teachers in this study did not teach in a program for gifted students, nor was it the intention of this study to focus on teachers' conceptions of gifted mathematics students. Rather, the term "excellent" is a commonly used term amongst teachers when describing students who have high grades in mathematics.

Data Analysis

To begin with, a grounded theory approach was used to analyse the data. The initial reading categorized participants' responses into "yes, excellence in mathematics is related to creativity," "no, there is no relationship," and "undecided." Further readings led to a finer categorization scheme based on the type of relationship teachers claimed to exist between mathematical creativity and mathematical excellence (e.g., that one was a prerequisite for the other). This led to six basic categories, described in the next section. The author and another mathematics education researcher independently categorized all participants' responses. Where there was disagreement, a discussion ensued until agreement was reached. A third mathematics education researcher then categorized 20% of the responses, ending in 90% agreement between researchers.

After completing the basic categorization, a second analysis investigated implicit beliefs related to creativity. Due to the interpretive nature of the data, a qualitative analysis was carried out based on creativity-related beliefs found in previous studies, and described in the background. Inferred beliefs were assigned to one of four categories: beliefs related to (1) creative processes, (2) the product of creativity, (3) the nature of creativity, including how creativity might be affected by the environment (e.g., opportunities afforded in some classroom), and (4) affective issues.

RESULTS

This section presents the six main categories that resulted from the data analysis. For each category examples from participants' responses are given, showing the reasoning behind those beliefs and shedding further light on participants' implicit beliefs. Examples were chosen to reflect the various implicit beliefs that came to light. The section ends with a summary and an overview of general trends and frequencies.

Category A: Mathematical excellence precedes mathematical creativity

In general, this category includes responses that claim mathematics excellence in some way promotes mathematical creativity. For some participants, it is the deep mathematical knowledge of an excellent student that allows creativity to emerge. For example, T22 wrote: "Creativity is the ability to break down a problem and re-build it with mathematical knowledge from different areas. In order to do this, you need to know mathematics, to think mathematically, and be fluent in various mathematical topics." Similarly, T1 wrote: "Excellence in mathematics promotes creativity because a deep understanding of mathematics causes you to think out of the box." Other participants hinted at affective issues related to being excellent in mathematics, that in turn may affect creativity. For example, T37 related to the motivation of excellent mathematics students: "Being excellent in mathematics drives you to being creative in mathematics." T21 related to self-confidence: "When a student is good at mathematics, his self-confidence rises, which causes him to dare more and to try different solution methods without fear of failure." Finally, some participants attributed environmental factors such as opportunities provided especially for excellent mathematics students. T9 wrote: "Excellent students are exposed to problems that require high-order reasoning and can be solved in multiple ways, thus promoting creativity."

In the above examples, we find implicit beliefs related to creative processes, such as being able to break down a problem (T22), thinking out of the box (T1), and trying different solutions (T21). Beliefs related to creative products were not mentioned. The nature of creativity was hinted at by mentioning that the environment, in the way of opportunities offered, can promote creativity (T9). Affective issues (e.g., not fearing failure) were raised by T21.

Category B: Mathematical creativity precedes mathematics excellence

Responses in this category inferred that mathematical creativity can promote excellence in mathematics. Some teachers attributed this to what they considered as characteristics of a creative person. For example, T3 wrote: “Creativity comes from having an open mind, solving problems in many different ways, which leads to excellence.” T38 wrote: “Creativity means using various thinking processes that lead to a meaningful product such as a problem solution, idea, or concept, which in turn leads to excellence.” T15 wrote: “Creativity means producing unconventional and valued products.” Affective issues were also raised in this category. For example, T13 wrote: “creativity in mathematics contributes to enjoyment, which contributes to excellence.” One teacher (T16) referred to the nature of creativity:

> Mathematical creativity promotes excellence in mathematics because it allows a student to think of unusual solutions, and then the student can excel in mathematics. However, excellence in mathematics does not promote creativity because creativity is genetic and cannot be acquired. By the way, weaker mathematics students can also be creative, but may have some problem such as a short attention span, which affects their grade in mathematics.

Implicit beliefs found in the above examples include those related to creative processes, such as solving problems in different ways (T3) and using various thinking processes (T38). Creative products mentioned were meaningful and unconventional (T38 and T15). The nature of creativity was referred to by T16 who claimed that creativity is an innate trait. Finally, there was an affective connection relating enjoyment to creativity (T13).

Category C: Creativity and excellence in mathematics are reciprocally related

Some participants believed that mathematical creativity can influence excellence and that mathematical excellence can contribute to creativity. T40 wrote the following:

> Mathematical creativity is about thinking out of the box. Creativity and excellence go hand in hand, but I cannot decide which leads to which. On the one hand, thinking out of the box leads to finding interesting … solutions, which can lead to excellence. On the other hand, excelling in mathematics leads to expertise that can broaden one’s thinking, leading to creativity.

In the above, “thinking out of the box” implies a creative process, while an “interesting” solution hints at the product of creativity. Another teacher (T33) specifically related to students who learn mathematics at different levels:

> In the weaker mathematics classes, students are only interested in learning procedures… In the stronger mathematics classes, there is discourse and the students … are not afraid of making mistakes. The relationship between creativity and excellence is two-way. While it may be that stronger mathematics students exhibit more mathematical creativity, if teachers promote creativity among the weaker students, they will become stronger in mathematics.

T33 relates to the nature of creativity. The statement above suggests a belief that creativity can be developed and that the environment, (e.g., opportunities provided in different classes), can have a strong impact on students’ mathematical creativity.

Category D: There is a non-influential relationship between mathematical creativity and excellence

Included in this category were two participants who believed that a relationship between creativity and excellence exists, but one does not preclude or influence the other. For example, T18 simply wrote: "There is a relationship between mathematical creativity and mathematical excellence, but one is not a sufficient condition for the other." The second participant (T42) wrote: "Mathematical creativity and excellence are related, ... Creativity is one characteristic of mathematical excellence, but it is not sufficient." Implicit beliefs related to creativity did not arise.

Category E: Mathematical creativity and excellence are not related

Three participants claimed that no relationship exists between mathematical creativity and excellence, basing their belief on their experience as mathematics teachers of all levels. T36 wrote:

> As a teacher, I see many students, some excellent in mathematics (when it comes to tests), but when I sit with them while solving a problem, they limit themselves to using formulas learned in class. But, I also teach weak mathematics students, and they think differently. If I give those students the same problem I gave high achieving students, I think I would get more creative solutions from the weaker students.

One participant (T12) focused on the teacher's role in developing mathematical creativity and claimed: "Creativity in mathematics can be developed and acquired, even among lower achieving mathematics students. It is dependent mostly on a supportive environment of which the teacher is responsible."

An implicit belief related to the creative process was hinted at by the term "thinking differently (T36). Regarding the nature of creativity. Both teachers imply that the environment has a role in supporting mathematical creativity, that the weaker student could solve the same problem given to stronger students, if only given the chance.

Category F: Undecided

One participant in this category wrote that she was undecided because it depends on how one defines excellence in mathematics (she did not mention defining creativity). Two additional participants were also categorized as undecided. For example, T41 wrote: "There is a relationship between mathematical creativity and mathematical excellence…. We see students who are mathematically creative but not mathematically excellent and the other way around." Although T41 initially claims there is a relationship, she goes on to describe a situation where mathematical creativity and excellence are unrelated.

Summarizing the quantitative data

Table 1 presents the frequencies of participants' responses according to the type of relationship they believed to exist or not exist. We first note that 84% of the participants believed that some relationship between mathematical creativity and mathematical excellence exists. In addition, considering all those who believed that mathematical creativity can promote mathematical excellence, including those that believed the relationship to be mutual, we find that half of the participants believed mathematical creativity to have some influence on mathematical excellence.

Category	A	B	C	D	E	F
Frequency (%)	13 (29)	19 (42)	4 (9)	2 (4)	4 (9)	3 (7)

Table 1: Frequencies (in %) of the number of participants' statements per category

DISCUSSION

The main finding of this study is that most mathematics teachers believed that mathematical creativity and mathematical excellence are related, with approximately half believing that creativity can lead to excellence in mathematics. This belief can be a first stepping stone when encouraging teachers to promote creativity in their classroom; if they think that creativity can lead to excellence, they may be more interested in learning how to promote mathematical creativity. In general, professional development should take into consideration, not only teachers' knowledge, but also their related beliefs (Beswick, 2012).

Yet, not all teachers who believed that creativity can lead to mathematical excellence expressed a belief that creativity can be promoted. As was shown above, one teacher believed that mathematical creativity leads to excellence, and explicitly stated that creativity is an innate trait. Interestingly, a different teacher also stated that creativity is innate, and yet expressed that mathematical excellence can lead to mathematical creativity. In general, answering the second research question, implicit beliefs found in this study were aligned with previous studies of teachers' conceptions of creativity, such as mathematically creative people are open-minded (Aljughaiman & Mowrer-Reynolds, 2005), come up with unique solutions (Leikin et al., 2013), and propose different approaches to solving problems (Emre-Akdoğan & Yazgan-Sağ, 2015). On a positive note, none of the participants in this study claimed that mathematics is a closed subject with no room for creativity.

As to the third research question, findings indicated that the same implicit beliefs about creativity were held by participants with opposing beliefs regarding the relationship between excellence and creativity. One possible reason for this, is that beliefs are not isolated entities, but part of a system of beliefs with complex relationships (Green, 1971). Thus, for example, a belief that the environment is a factor in promoting mathematical creativity, may lead one teacher to claim that excellence (because of opportunities given to excellent students) leads to mathematical creativity, while another teacher claims that since it is up to the environment, mathematical creativity and excellence are unrelated. A second possible reason is that participants' various teaching experiences might have affected the variety of belief interactions. Beswick (2012) found that teaching experience can have a great influence on one's beliefs, including beliefs about mathematics as a school subject. This in turn might affect teachers' beliefs regarding what it means to excel in mathematics. This needs further investigation. A future study might build on this study, by aiming to investigate not only specific beliefs related to teachers' conceptions of creativity, but the complex interactions that might underlie those beliefs.

References

Aljughaiman, A., & Mowrer-Reynolds, E. (2005). Teachers' conceptions of creativity and creative students. *Journal of Creative Behavior*, 39(1), 17–34.

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics, 79*(1), 127-147.

Bolden, D. S., Harries, T. V., & Newton, D. P. (2010). Pre-service primary teachers' conceptions of creativity in mathematics. *Educational Studies in Mathematics*, *73*(2), 143-157.

de Souza Fleith, D. (2000). Teacher and student perceptions of creativity in the classroom environment. *Roeper Review*, 22(3), 148-153.

Diakidoy, I. N., & Phtiaka, H. (2002). Teachers' beliefs about creativity. *Advances in Psychology Research*, *15*, 173-188.

Emre-Akdoğan, E., & Yazgan-Sağ, G. (2015). Prospective Teachers' Views of Creativity in School Mathematics, In F. M. Singer, F. Toader, & C. Voica (Eds.), *Proceedings of the 9th International Mathematical Creativity and Giftedness International Conference* (pp. 182–187). Sinaia, Romania: MCG.

Green, T.F. (1971). *The activities of teaching*. New York: McGraw-Hill.

Leikin, R., & Dinur, S. (2007). Teacher flexibility in mathematical discussion. *The Journal of Mathematical Behavior*, *26*(4), 328-347.

Leikin, R., Subotnik, R., Pitta-Pantazi, D., Singer, F. M., & Pelczer, I. (2013). Teachers' views on creativity in mathematics education: an international survey. *ZDM*, *45*(2), 309-324.

Levenson, E. (2011). Exploring collective mathematical creativity in elementary school. *Journal of Creative Behavior*, *45*(3), 215–234.

Levenson, E. (2013). Tasks that may occasion mathematical creativity: Teachers' choices. *Journal of Mathematics Teacher Education*, 16(4), 269-291.

Tabach, M., & Friedlander, A. (2017). Algebraic procedures and creative thinking. *ZDM*, *49*(1), 53-63.

Sheffield, L. J. (2017). Dangerous myths about "gifted" mathematics students. *ZDM*, *49*(1), 13-23.

Shriki, A. (2010). Working like real mathematicians: developing prospective teachers' awareness of mathematical creativity through generating new concepts. *Educational Studies in Mathematics*, 73(2), 159–179.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM*, 3, 75–80.

EARLY MATHEMATICAL REASONING – THEORETICAL FOUNDATIONS AND POSSIBLE ASSESSMENT

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Mathematical reasoning is a complex skill and as such requires coherent cumulative learning experiences. Although there is a strong research base on mathematical reasoning at the secondary level, it is hardly investigated in early mathematics education so far. There is a lack of theoretical conceptions of early mathematical reasoning as well as of empirical findings concerning prerequisites and forms of mathematical reasoning of young children. In this contribution we first discuss the nature of early mathematical reasoning and characterize it along the dimensions knowledge, representations and formulation from a theoretical perspective. This results in a description of facets of early mathematical reasoning processes. Second, we sketch the development of task to assess early mathematical reasoning and provide first empirical findings. The contribution hence provides an approach to further research on early mathematical reasoning with the aim of better understanding an allegedly important root of advanced mathematical thinking.

INTRODUCTION

Mathematical reasoning is an important aim of mathematics education and therefore a key standard of school mathematics (e.g. CCSSI, 2012; NCTM, 2000): Mathematics instruction should enable students to reason mathematically, including to investigate conjectures, develop, and evaluate arguments or proofs. They should distinguish correct logic from flawed, and use related skills to communicate mathematically. These goals are pursued in formal mathematics instruction that starts with preK-level (age 5, e.g., USA, Switzerland) or first grade (age 6, e.g., Germany, Italy).

Although these descriptions from curricular documents suggest that mathematical reasoning is well-understood, research still struggles to attain a shared understanding. The discourse is traditionally stronger in respect to the function and nature of mathematical proof, the rigorous form of mathematical reasoning especially relevant for secondary education (e.g., Hanna, 2007). Conceptions of mathematical reasoning that are aiming at elementary and early mathematics are recently emerging (Jeanotte, & Kieran, 2017; Nunes et al., 2012; Stylianides, 2007).

However, reasoning mathematically is unanimously understood as requiring complex abilities. As early skills are usually found to have a relevant influence on the later acquisition of mathematics (e.g., Watts et al., 2014), it would be reasonable to foster mathematical reasoning in an age-adequate way already before first grade. But so far, a

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 315-322). Umeå, Sweden: PME.

concise and shared understanding of early mathematical reasoning is missing. Accordingly, evidence about corresponding abilities of young children that could inform age-adequate instructional practices is lacking as well.

The aim of this paper is to present the rationale of an approach to early mathematical reasoning abilities of young children (age 5-6) that might not be exposed to formal mathematics instruction. After a review of theoretical foundations, we propose a framework, and exemplify the development of a standardized interview to assess the early mathematical reasoning abilities. Findings from an exploratory study provide first evidence for the validity of the approach and illustrate that mathematical reasoning occurs in a variety of different forms. The findings support that early mathematical reasoning can be understood as a distinct area of early mathematical.

MATHEMATICAL REASONING IN EARLY MATHEMATICS

Mathematical reasoning in a narrow sense can be understood as mathematical proving in a more or less rigorous way (Hanna, 2007). From an educational perspective, it is interesting to trace mathematical reasoning down to the start of organized mathematical learning. It is obvious that early mathematical reasoning processes, especially if they occur in informal settings, are less formal, less complex, and can only build on the available set of early mathematical knowledge. Nonetheless, early mathematical reasoning should ideally lay the ground for the later development of more formal and complex forms.

What is early mathematical reasoning?

Contemporary conceptions emphasize procedural aspects and the explanatory function of mathematical reasoning to a greater extent. Additionally, they take the social and discursive nature of mathematical reasoning into account (Hanna, 2007; Jeanotte & Kieran, 2017). Mathematical reasoning then refers to processes where relations between mathematical structures are used to change the epistemic value of a proposition, statement, or observation (cf. Duval, 2007). As such, mathematical reasoning might take different forms and cover different processes. Precursory skills and more experimental activities, like comprehending arguments and making connections, can be subsumed. Stylianides (2007) suggested four dimensions to conceptualize more specifically what can be understood as mathematical reasoning in certain contexts. They refer to (1) the mathematical foundation, (2) the representation, (3) the formulation, and (4) the social dimension. In essence, mathematical reasoning can be understood as reasoning in mathematical situations. It therefore requires the use of mathematical resources (1) and is related to typical structural characteristics or means of reasoning (3), what refers to the dimension of formulation. These structural aspects (e.g., is a reasoning deductive or inductive) and procedural aspects (i.e., what kind of reasoning processes occur) surface also in a broader analysis by Jeanotte and Kieran (2017) as central features of mathematical reasoning. As subject of communication, mathematical reasoning has to be further represented in a way that conforms to a common understanding (2). This social nature of mathematical reasoning is mirrored also in a need of interpersonal relevance (4) in the context it occurs. We finally step on and describe early mathematical reasoning along the first three of the four dimensions. The fourth dimension, as a social dimension, has to be postponed at the moment, as it requires first a deeper understanding of theoretical aspects of early mathematical reasoning.

First, early mathematical reasoning occurs on the ground of early (1) *mathematical knowledge*. Research showed, that the mathematical development before school age is impressive and an essential part of the cognitive development (Ginsburg et al., 2008). Children acquire a base of important mathematical concepts from different areas, including numerical and geometrical skills. As the body of knowledge is large and well documented, we refer to the literature here (e.g., Clements et al., 2004).

However, young children acquire these mathematical knowledge first and foremost in informal, everyday contexts. As a consequence, this knowledge is by nature often contextualized and bound to situations (e.g., Sophian, 1999). Early mathematical reasoning hence depends on apt (2) *representations* of this knowledge. Young children do not dispose of abstract, transferable mathematical objects and in order to (re-)structure their mathematical knowledge in reasoning processes, children need contexts and familiar representations. It is important to note, that the way knowledge is represented depends on the cultural practices children encounter as well as aspects of general cognitive development (Schliemann & Carraher, 2002). Specific characteristics of early cognitive development, like the development of executive functions and language skills limit the range of representations suited for early mathematical reasoning (cf. Nunes et al., 2007; Watts et al., 2005).

Further, characterizing early mathematical reasoning faces the challenge what are possible (3) *formulations* of early mathematical reasoning. As mentioned above we apply a broader understanding of mathematical reasoning, which includes for example experimental activities. But even in a broader understanding, there is still a need for rigor of thinking (but not of formalism). Hanna motivates this aspect of formulation by the social functions of reasoning: “Rigor is a question of degree in any case. In the classroom one need provide not absolute rigor, but enough rigor to achieve understanding and to convince” (Hanna, 1997). Following this line of argument, it needs to be delineated what formulations of reasoning processes are pertinent for early mathematical reasoning. Especially, there is the need to make explicit how mathematical reasoning differs from an everyday logic although it is expected to occur in familiar contexts (Schliemann & Carraher, 2002). Mathematical reasoning is in need of a logically consistent argumentation, cogent reasoning and uses reasoning processes specific for mathematics, even if it is represented in a less formal way, e.g., in an everyday language. Jeanotte and Kieran (2017) provide an extensive synthesis of such processes of mathematical reasoning, for example generalizing, or justifying. They propose that these processes fall into two categories that relate to (i) the search for similarities and differences and (ii) validating in the sense that the latter processes can lead to changes in epistemic values.

These categories can be understood being related to a more rigid understanding of mathematical reasoning (ii) and a more broad and experimental understanding of mathematical reasoning (i). They are to a certain extent hierarchical, as the processes of validation often use results from processes of category (i). This distinction can help to further differentiate facets of the formulation-dimension for early mathematical reasoning. We will explain this in detail, as it plays a central point in our endeavour to characterize early mathematical reasoning (see Tab. 1).

First, due to the informal nature of early mathematical knowledge, children need to be able to access relevant mathematical knowledge for reasoning processes. They have to *identify* mathematical structures that are relevant for the reasoning situation. Accessibility of knowledge is a question of identification as well as *representing*, as children have to use the relevant structures in the reasoning process. They have to find a way of externalizing their thinking, maybe even translate between modes of representation, for example through manipulating the objects in a situation or giving a verbal explanation. This reasoning process shows that the dimension of formulation has a strong connection to the dimension of representation. Compared to the categories according to Jeanotte and Kieran, these processes can be understood as belonging to the first category. We speak more specifically of an early mathematical reasoning facet that refers to accessing relevant mathematical structures (see Tab. 1).

Second, early mathematical reasoning can also occur as process relevant to a change in epistemic value. According to the literature, we see *the explanation of mathematical relations through mathematical structures* as the most important aspect of early mathematical reasoning that affects epistemic value (Stylianides, 2007). As explained above, early mathematical reasoning is not expected to be restricted to the construction of argumentations, but has to be related to discursive processes and its communicative functions. As such, processes like evaluating a given mathematical reasoning or justifying a claim can be seen as characteristics of early mathematical reasoning as well. In line with the informal nature of early learning processes, also reconstructive or reproductive processes can be seen as indicative of early mathematical reasoning as they can be a source for intuitive knowledge about mathematical reasoning. Comparing to the categories according to Jeanotte and Kieran, these processes can be understood as belonging to the second category, processes of validation. We speak more specifically of an early mathematical reasoning facet that refers to explaining mathematical relations through mathematical structures (see Tab. 1).

Assessing early mathematical reasoning

To sum up, we characterized early mathematical reasoning above along the dimensions (1) knowledge, (2) representations, and (3) formulation. The latter led to a more concise description of two facets of early mathematical reasoning through processes that make relevant mathematical structures accessible (i) and processes that contribute to the explanation of mathematical relations through mathematical structures (ii) as summarized in Tab. 1.

Early mathematical reasoning facets	Associated processes
(i) Accessing relevant mathematical structures (ii) Explaining mathematical relations through mathematical structures	• Recognizing relevant mathematical structures • Representing relevant mathematical structures • Constructing an argumentation • Evaluating and justifying given reasons • Reconstructing and reproducing mathematical reasoning

Table 1: Mathematical reasoning facets as clusters of associated mathematical reasoning processes

Having attained a first draft of early mathematical reasoning, the framework was in the following used to design a first set of 14 problems in order to empirically investigate the relevance of the framework. Figure 1 display an exemplary reasoning problem as part of a standardized interview with children of pre-school age.

Materials: Hand puppets Ben and Lisa, foam sheet chocolate bars (F1, F2, F3 was added for purpose of this publication)

This is about chocolate. Here, we have a whole bar (interviewer I. points to F1). *This is the left-over of Ben* (I. points to F2) *and this is Lisa's* (I. points to F3). *Ben* (I. plays hand puppet) *looks what's left for Lisa and says: "I have more left than you, Lisa!".*

(I. addresses the child) *What do you think? Why?*

Figure 1: Sample problem "Chocolate"

The sample problem illustrates the construction principles applied for the development of early mathematical reasoning processes. They are mostly direct applications of the theoretical considerations detailed above. First, the problem is situated in a context that refers to shared experiences of young children in our cultural context. The situation can be seen as personally relevant for children. Second, Ben's claim occurs as unsubstantiated within the context and refers to a comparison of the chocolate left-overs. But due to the structure of the material, the claim can be refuted by using mathematical knowledge, in this case knowledge of measuring in geometrical contexts. Therefore, a child has to identify (i) the knowledge, evaluate (ii) Ben's claim, construct an argument (ii) to support or refute the claim and – in the latter case – produce a new claim that is supported by the reasoning process. And third, the hand puppets in our interviews serve as discursive partners, mirroring the social nature of mathematical reasoning. In the sample item, two puppets interact, in other items the hand puppets even interact with the child. Finally, the context and materials do not only serve as a way to present the mathematical reasoning problem, but provide means for the children to represent their own mathematical reasoning, as they are allowed to manipulate the materials, for example to compare the figures directly.

EVIDENCE FOR THE ACCURACY OF OUR APPROACH

With the delineation of early mathematical reasoning as given above and the development of a test instrument as sketched, we set the ground for empirical investigations concerning the viability of the approach. As our considerations show, children have to dispose of mathematical knowledge, as well as general cognitive skills in order the reason mathematically. In a first quantitative study (cf. Lindmeier, Grüssing & Heinze, 2015), we investigated the relations between such prerequisites and early mathematical reasoning with a sample of $N = 120$ children (age $M = 5.2$ years; $SD = 0.5$). The results showed that mathematical reasoning (Cronbach's $\alpha = .69$, $M = .29$, $SD = .19$) and mathematical knowledge (aligned to the mathematical reasoning test, $\alpha = .68$, $M = .58$, $SD = .16$) are related, but empirically separable ($r = .58$, $p < .01$). What remains open in this first investigation is the question, if this is a meaningful differentiation. For example, the approach is blind to the range of mathematical reasoning processes that children apply. The aim of subsequent in-depth analyses is to substantiate the quantitative findings and get further insight to early mathematical reasoning processes. The research question was accordingly specified as follows: What kind of early mathematical reasoning can be observed?

Methods and results

We applied an inductive coding to find types of early mathematical reasoning in the children's responses. As this endeavour is very content-specific, the analysis was conducted on item level. Due to place limitations, the following report is restricted to the geometry sample task portrayed in Figure 1 and the available 111 responses. Tab. 2 provides an overview of the different types of early mathematical reasoning that could be distinguished.

Label	Mathematical reasoning
Half of the bar	Both kids' left-over are of the same size, as each ate/has left HALF of the whole bar (F1)
Decomposing the bar	The whole bar (F1) can be HALVED in two different ways, what results in the different left-over shapes (of the same size)
Doubling the bar	Both left-overs can be DOUBLED and then built a whole bar in each case, so they have the same size
Completing the bar through re-structuring	One left-over (e.g. F2) can be completed to a whole bar by RE-STRUCTURING (e.g. break apart) the other left-over (e.g. F3). So the left-overs can be seen as a fair share.
Re-structuring to show congruence	One left-over (e.g. F2) can be RE-STRUCTURED (e.g. break apart) to be congruent to the other left-over (e.g. F3)

Table 2: Types of observed early mathematical reasoning with the chocolate-task

The first three categories use the mathematical concepts of half, halving, or doubling and construct transitive arguments that relate the left-overs to the whole bar of chocolate without making a direct connection between the left-overs. The other two categories rely stronger on the restructuring of the given situation to relate the two different left-overs directly to each other. Of course, the argument also relies on the property of the left-overs being half of the whole bar, but this remains mostly implicit. All five different types could be observed in connection with a successful or unsuccessful reasoning process in terms of correctness. The observation of mathematical reasoning that relies on restructuring of the left-overs occurs dominantly (2/3 of classified answers), so that restructuring seems to be easier than the transitive reasoning processes that relate the left-overs to the whole bar.

The categorization of early mathematical reasoning processes for the sample item hence gives evidence that young children (age 5-6) already can engage in demanding reasoning problems. Some children are able to provide complex argumentations that use transitive arguments. More children use approaches of re-structuring, that can be interpreted as representing a reasoning that is more grounded in context and concrete operations. Not all children that use the classified approaches succeed in refuting the (wrong) claim or proposing an alternate correct claim.

DISCUSSION AND OUTLOOK

The aim of this paper was, first, to present the rationale of an approach to early mathematical reasoning abilities of young children that might not be exposed to formal mathematics instruction. The suggested characterization is specific for the targeted context, as it takes into account the very nature of early mathematics. Especially, it highlights the importance of identifying and representing relevant mathematical structures as basic processes of mathematical reasoning. The theoretical analysis led to a framework that proved to be applicable for the development of early reasoning task. First empirical results indicate that early mathematical reasoning might be distinct from mathematical knowledge. An in-depth analysis and classification of children's mathematical reasoning in the sample tasks shows, that various reasoning processes can indeed be identified. We want to stress, that the children in the sample were not involved in formal mathematical instruction on early mathematical reasoning, so that the richness of the occurring evidence of mathematical reasoning is rather astonishing, given that mathematical reasoning is usually considered to be complex by nature.

Nonetheless, the limitations of our study so far are obvious. The investigations reported can still be understood as exploratory in nature. In order to better understand early mathematical reasoning, its prerequisites and development, it is necessary to subject the preliminary findings to more rigid research. Especially, the development of cognitive functions, and the exposure to formal mathematical instruction should be investigated in greater detail in order to attain a sound understanding of early mathematical reasoning. Due to the restrictions to one sample item, our findings in this article are bound to the topic of this sample item (measurement in geometry). It is evident that early mathematical reasoning is highly content specific. Finally, we want to stress that our work at the moment does not reflect the social discursive dimension of mathematical reasoning in an appropriate way. This is partly due to the research approach taken, that starts from a standardized interview situation, where no co-constructive processes are pursued. However, through the operational trick to involve hand puppets as peers, communicative aspects of early mathematical reasoning are mirrored to a certain extent. To sum up, this research advances a theoretical foundation of early mathematical reasoning and lays the ground for subsequent research based on that understanding. For example, the given characterizations can serve as the theoretical framework for an intervention that investigates the effect of instruction on early mathematical reasoning.

References

CCSSI (2010). *Common Core State Standards for Mathematics.* Washington, DC: National Governors Association.

Clements, D. H., Sarama, J., & DiBiase, A.-M. (Eds.). (2004). *Engaging young children in mathematics: Standards for early childhood mathematics education.* Mahwah, NJ: Lawrence Erlbaum Associates.

Duval, R. (2007). Cognitive functioning and the understanding of mathematical processes of proof. In P. Boero (Ed.), *Theorems in School* (pp.137-161). Rotterdam: Sense.

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). Mathematics Education for Young Children: What It Is and How to Promote It. Social Policy Report. Volume 22, Number 1. *Society for Research in Child Development*.

Lindmeier, A. M., Grüßing, M., & Heinze, A. (2015). Why is it so? – Eliciting precursors of mathematical reasoning in kindergarten. In K. Beswick, T. Muir, & J. Wells (Hrsg.), *Proceedings of the 39th PME* (vol. 1, p. 236). Hobart (Tasmania): PME.

NCTM (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Hanna, G. (2007). The ongoing value of proof. In P. Boero (Ed.), *Theorems in School.* (pp. 3–16). Rotterdam: Sense.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics,* 1-16.

Nunes, T., Bryant, P., Barros, R., & Sylva, K. (2012). The relative importance of two different mathematical abilities to mathematical achievement. *British Journal of Educational Psychology, 82*(1), 136-156.

Schliemann, A. D., & Carraher, D. W. (2002). The Evolution of Mathematical Reasoning: Everyday versus Idealized Understandings. *Developmental Review, 22,* 242–266.

Sophian, C. (1999). Children's ways of knowing: Lessons from cognitive development research. *Mathematics in the early years*, 11-20.

Stylianides, A. J. (2007). Proof and Proving in School Mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.

Watts, T. W., Duncan, G. J., Siegler, R. S., & Davis-Kean, P. E. (2014). What's past is prologue: Relations between early mathematics knowledge and high school achievement. *Educational Researcher, 43*(7), 352-360.

HOW DRAGGING MEDIATES A DISCOURSE ABOUT FUNCTIONS

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Assuming that the dynamic features of dynamic algebra and geometry environments may provide a basic representation of both covariation and functional dependency and taking a commognitive perspective, a teaching experiment has been designed for introducing students to functions. This paper points to the crucial role that the Dragging tool can play as communicational mediator for discourse on functions. In particular, the episodes we are presenting here show that three different phases of dragging mediated discourse can occur when students are asked to work on activities involving both a dynamic and the traditional static environment.

INTRODUCTION AND THEORETICAL BACKGROUND

While many students still struggle with calculus, several studies have shown instances of positive effects on calculus learning made possible by technology. In particular, the introduction of dynamic geometry and algebra environments has given rise to new ways of teaching and representing calculus (Hitt & González-Martin, 2016).

Goldenberg et al. (1992) designed a new representation of functions, called dynagraph, which has also been implemented in other studies (Sinclair et al., 2009; Lisarelli, 2017). This representation is made up of two horizontal, parallel axes and two points, one for each line, which move according to two different types of motion: the point representing the independent variable can always be dragged and it causes the indirect motion of the other point, which cannot be directly dragged. The movement of the dependent variable, bounded to its line, depends on the choice of the function.

As is clear from the short description, dynagraphs need a dynamic environment to be realised and, in particular, the dragging tool offered by these software enables the user to interact with parameters in embodied ways and to observe changes dynamically, experiencing the covariation of the two variables. A rich literature investigates the role of dragging in students' cognitive processes involved in explorations within a dynamic geometry environment (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010). In the matter of the teaching and learning process of calculus, Falcade, Laborde and Mariotti (2007) showed how a dynamic environment can help high school students grasp the notion of function; they focused on the potentialities of the trace tool as semiotic mediator that can introduce the two-fold meaning of trajectory, both global and local.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 323-330). Umeå, Sweden: PME.

The current study examines students' discourse about functions, introduced through their dynagraph and then represented both in a dynamic and in the traditional static environment, focusing on the role played by the dragging tool. Although Sfard's communicational framework (2008) does not pay particular attention to the role of digital tools in mathematical thinking, we adopted her commognitive lens sharing the assumption that the use of symbolic tools and other artifacts in the process of learning has long-term effects on mathematical thinking, which can be observed in the characteristic discursive patterns produced through their use.

As the term *commognition* suggests, Sfard unifies cognition and communication by defining thinking as "an individualised version of interpersonal communication". Here the word communication is made to include all forms of communication, not just verbal. More precisely, it is defined as a collectively performed patterned activity in which one action of an individual is followed by a re-action of another individual. Within this approach, the term discourse is to be understood as a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions". This means that a discourse, being any act of communication, encompasses all forms of communication, verbal or not, with others or with oneself (Sfard & Lavie, 2005). According to this theory, communicational mediators are perceptually accessible objects with the help of which the actor performs her prompting action and the re-actor is being prompted. Mediators are often artifacts produced specially for the sake of communication and they can have auditory, visual, or even tactile effects on individuals.

Although Sfard considers both gestures and diagrams as forms of visual mediators, her view of visual mediation does not distinguish between the static and the dynamic. Significant work in this direction has been conducted by Ng (2016), who uses a commognitive approach to analyse calculus students' thinking in two environments. She extends Sfard's communicational theory by distinguishing between dynamic and static visual mediators, in an effort to highlight the importance of temporality in mathematical discourse. In particular, her analysis suggests that dragging actions may not only be selecting and moving objects on a computer screen, but also gestural communications, to communicate the dynamic features and properties of the sketch in the very moment of dragging. She defines *dragsturing* as an action subsuming both dragging and gesturing characteristics: it both induces movement on an object and it fulfils a communicational function.

THE STUDY

This study comes from a greater research project in which we investigate the role of dragging in students' discourse about covariation, both in terms of its semiotic potential (Bartolini Bussi & Mariotti, 2008) and its place for the student.

We designed nine lessons to introduce students to functions through dynagraphs and we implemented them in a 10th grade Italian class. Students worked in pairs on pre-designed interactive files with open tasks aimed at promoting their discourses. Each lesson has been video-recorded by three cameras and a screen capturing software.

From the analysis of these videos, carried out to study the role of dragging in students' discourse on covariation in this context, we noticed that students' discourse is, indeed, heavily mediated by dragging (here a mediator is intended according to the theory of commognition, as explained in the previous section). In particular, the videotapes were transcribed and for each transcript the following was coded: whether the dragging tool is physically used during the speaking, whether the dragging blends with a gesture becoming an act of dragsturing, whether the subject is a person or an object, whether mathematical objects are considered, which verb tense is used.

The analysis of students' discourse, led by this coding scheme, brought us to identify different phases in which dragging seems to mediate their discourse; in this paper we will characterise and provide examples of each phase. In order to obtain the same classification when giving a set of episodes to different people and asking them to decide which phase that examples belong to, it's important to be provided with a detailed description of each phase. This test has been done for several excerpts, all taken from the sequence of lessons just described, and then we selected for this paper the examples that resulted to be more appropriate as models.

DRAGGING MEDIATED DISCOURSE

During the analysis of the videos we focused on possible modification and extension of students' discourse and the first thing we noticed is that, when working with dynagraphs, students' discourse is rich in references to movement, time and space (Colacicco, Lisarelli & Antonini, 2017). Almost all the activities that we designed involve the use of the software GeoGebra and, especially thanks to the dragging tool, activated through the mouse, students can experience the dependence relation that links a point to the one that is directly dragged. Thanks to the possibility of dragging they can also visualise the movements of the two variables and the relation between these variations, that is the covariation. Indeed, this representation of the function brings the aspect of time into play, in contrast with the static Cartesian graph where it is completely removed.

Gestures and dragging can be used both repeatedly to define a discursive pattern and as a mediator to complement word use. Investigating their role as mediators, we identified three different types of what we call *dragging mediated discourse*: a passive phase, an active phase and a detached phase.

Passive Phase. The dragging tool is physically used. The discourse is about dragging, it is a description of the direct action of the dragging tool on an object, while there are no mathematical objects considered. In this first phase of dragging mediated discourse the subject is a person, the user, and the focus is on the objects upon which s/he acts through dragging. The user does not seem to be in control of all the movements happening on the screen, indeed s/he may be surprised by what s/he sees. The verb tense used within this discourse is the present simple and some typical expressions or words that characterise it are: “I can(not) move it”, “you can(not) move it”, “drag it”, “move it” [in Italian: “(non) lo posso muovere”, “(non) lo puoi muovere”, “trascinalo”, “spostalo”].

Active Phase. The dragging tool is physically used. The discourse is about the effects of dragging that are visible on the computer screen, it is a description of the perceived relations between the moving objects. The focus of this dragging mediated discourse is on the mathematical objects, while the action of dragging is not explicitly depicted and so the description looks as if it was independent from the person. The user seems to control over what happens on the screen as if s/he could decide what to move and how to move it. The verb tense used within this discourse is the present simple and some typical expressions or words that characterise it are: “if x… f(x)…”, “when x… f(x)…”, “as x… f(x)…” [in Italian: “se x… f(x)…”, “quando x… f(x)…”, “man mano che x… f(x)…”.

Detached Phase. The dragging tool is not physically used. The structure and the contents of the discourse are very similar to those in the active phase: the focus is on the relation between movements, but the dragging tool is not used to act upon any objects, so these movements are only envisioned/imagined. Within this discourse, verbs can be in the present tense, but also in the future tense or expressed in the “-ing” form; typical expressions that characterise it are: “I imagine to drag”, “by dragging x... f(x) will move...” [in Italian: “immagino di trascinare”, “trascinando x… f(x) si muoverà…”.

In order to get a better idea of the characteristics of each phase and to highlight the differences between them, now we are going to analyse three different episodes, all chosen from the sequence of lessons that we designed. They have been selected because we consider them as representative examples of the three phases of dragging mediated discourse just described.

Example of dragging mediated discourse from the Passive Phase

This dialogue is taken from the beginning of the third lesson. Andrea and Nico explore the dynagraph of a function, with Nico handling the mouse, and they describe which the possible or impossible movements are. The independent variable is labelled A, the dependent one is labelled B and the function is not defined in x=3, where it has a vertical asymptote.

1 Andrea: Drag A... This fact irritates me, move it to 3.

2 Nico: It does not go there (now he uses the arrows of the keyboard to drag).

3 Andrea: Yes it goes there, you have to move it and then you can drag it. Look now if you drag it, it goes by one.

4 Nico: (He drags A forward and backward close to 3)

5 [...]

6 Nico: (He tries to drag B)

7 Andrea: A is the only point that we can drag!

First of all, Nico uses the dragging tool in two different ways: through the mouse (line 1), through the arrows of the keyboard (line 2) and through the mouse again (line 4 and line 6). For this reason, he obtains two different qualities of motion of the independent variable, because the mouse causes a “continuous” movement (it depends on how they move it but, generally, it is quite uniform), while the arrows of the keyboard make the point jump by one. Andrea seems to notice this fact and he suggests Nico to use the mouse when he says “you have to move it” (line 3).

In general, the focus of this discourse is what students can or cannot move on the screen. Indeed, the dialogue starts with Andrea asking to Nico to drag A and move it (line 1), then the subject of the action becomes A for a while (line 2 and the beginning of line 3), until Andrea resumes the structure of the discourse with himself or Nico being the subject and expressing what and how they can move (line 3).

The dragging mediates their discourse so significantly that in two lines of this short excerpt Nico substitutes the dragging for words (lines 4 and 6), because he doesn’t speak aloud but by dragging he succeeds in communicating with Andrea, who replies. They do not seem to have the situation completely under control, especially Nico who, at the end, experiences impossible dragging (line 6) because he tries to drag B which cannot be directly moved; but immediately Andrea, irritated at him, stresses that A is the only object they can drag (line 7).

Example of dragging mediated discourse from the Active Phase

In the following excerpt from the seventh lesson a student explores the dynagraph (with perpendicular axes) of a function and he has to draw its Cartesian graph on a sheet of paper. He describes the file while he is manipulating it.

Andrea: From here, from minus one to minus two it moves more or less by one but from minus five to minus six…mm…….it moves, from minus five to minus six no let's do from minus four to minus five it moves by less than one, so as x decreases, that is, also the relationship existing between f(x) and x changes and so it cannot be like this (with gestures he simulates a peak) but it is a…that is to say it is not a broken line (then he draws a smooth curve).

Andrea refers mainly to motion, as we can see from his frequent use of the verb “to move”, but the focus of his discourse is the search for a possible relation between movements, more than the movement itself. The subject is always a variable and not Andrea himself; initially it is unexpressed but then Andrea says “as x...”. This happens at the end where he describes changing in ratio of f(x) to x in relation to the dragging of x. This is a very representative example of dragging mediated discourse from the active phase. Indeed, he acts on the file by dragging x to the left and at the same time his discourse focuses on the relationship between movements that he observes happening on the screen. So, it differs from the previous example because the object of this discourse is not the action of dragging, it is just mediated by it.

At the very beginning of this excerpt, there appears to be an example of dragsturing: when he says “from here” he uses the mediation of dragging also as a gesture, because he indicates what he intends for here but he does not directly describe this action.

Example of dragging mediated discourse from the Detached Phase

In this short excerpt from the eighth lesson, a student is working on a task within the static paper and pencil context. In particular, by looking at the trace left by the point (x, f(x)) (that is, a bit of the Cartesian graph of a function) she has to mark on the same Cartesian plane, drawn on a sheet of paper, the trace that f(x) would have left. Here she explores the graph just for positive abscissas.

Maria: But wait, no no...x you have to move it inevitably towards here, because you do like this, you do like this because it has to stay perpendicular... while over here, here you do like that, it comes back up, yes...so there are some parts that are more marked in double quotation marks, yet, not here but here since it does...did you understand?

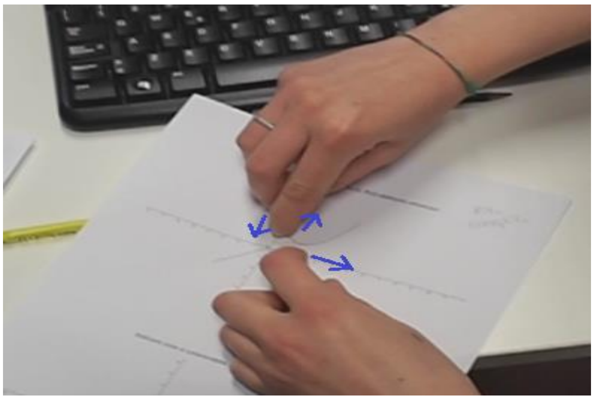

Figure 1: Maria’s gestures in her discourse.

By analysing the video, we notice that, while speaking aloud, Maria moves her fingers on the sheet of paper in this way: the left hand from the origin to the right along the x-axis, the right hand from up to down and then from down to up along the y-axis (see Figure 1). Moreover, her discourse is rich in references to space such as “towards here”, “over here”, “up”, which are understandable thanks to the dragging mediation, that is actually an example of dragsturing, and at the end of the excerpt this dragging mediation even replaces words, when she doesn’t explicit what “it does” by words but she moves her fingers on the graph to communicate it to her companion (during the pause indicated by the suspension dots).

On the one hand this excerpt could be seen as an example from the passive phase because the subject of her discourse is a person and she describes the direct action of dragging on the objects. On the other hand, we notice several differences with the first example illustrated above. First of all, in this case the dragging tool is not physically used but a dragging action of the two variables re-created within the static context plays the role of mediator in Maria’s discourse; she also seems to have covariation under control, since she is able to evoke their movements with her body. Finally, she closes with “there are some more marked parts” that suggests her focus to be on finding an answer to the task, that is, discovering how to mark the trace of f(x); so her discourse is about mathematical objects, too. These elements are all characterizing of the detached phase only.

DISCUSSION

The evolution in digital technology has influenced our thinking, learning and modes of interaction with mathematics. The invention of graphing and dynamic geometry software has offered new ways of doing mathematics and representing mathematical objects (Healy & Sinclair, 2007).

The one-dimensional representation of a function, called dynagraph, is made possible by the use of a dynamic environment and, in particular, by the Dragging tool. Indeed, the student can experience the asymmetric relation that links the two variables thanks to the possibility of dragging one of the two points and to the impossibility of dragging the other one. For this reason, dragging has a two-fold role: it allows to speak and it even becomes necessary to speak about covariation. Its close connection with the temporal aspect can enter the discourse within the static context as well. The last example shows this; moreover, we observed many other instances of students’ discourse still rich in references to movement, time and space also when they were asked to work on activities without the use of GeoGebra. This finding seems to be in contrast with that of Ng (2016), according to which participants communicated about the fundamental calculus ideas differently within different types of environments.

In this paper, we described three different phases of dragging mediated discourse and showed three representative examples. These examples come from a sequence of lessons that we designed to introduce students to functions. We used the term “phase” because it suggests a sort of temporal evolution from the first one to the third one. There is a need for further research in this regard but, with respect to our sequence of nine lessons, we found most instances of the passive phase from the first to the third lesson, of the active phase from the second to the ninth lesson and of the detached phase from the seventh to the ninth lesson. We are tempted to consider this temporal evolution also as a development of students’ discourse towards a discourse closer to that of an expert mathematician. For example, the students in the first dialogue substitute the dragging for words and Nico explores the file trying to discover possible and impossible movements, which he does not seem to be aware of. Differently, Maria in the third excerpt evokes temporal and dynamic aspects in the static representation of the function, showing a good control over the covariation of the two variables, bounded to the Cartesian axes. This seems to be a very interesting aspect with respect to the study of students’ cognitive processes that should be looked into more deeply.

The commognitive assumption that learning mathematics involves building mathematical communicative competence is important in this study because it establishes a strong link between mathematics learning and communication. In line with Falcade, Laborde and Mariotti (2007), students seem to have exploited the functionalities of the dragging tool to communicate covariation. The dynamic environment, the design of the sketches and the tasks promoting students' explorations and communication, all played a role in promoting dynamic and temporal thinking in calculus.

References

Arzarello, F., Olivero, F., Paola, D., & Robutti, O., (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM*, *34*(3), 66-72.

Baccaglini-Frank, A., & Mariotti, M. A., (2010). Generating Conjectures in Dynamic Geometry: the Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225-253.

Bartolini Bussi, M. G., & Mariotti, M. A., (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of International Research in Mathematics Education,* second edition, (pp. 746-783). New York and London: Routledge.

Colacicco, G., Lisarelli, G., & Antonini, S., (2017). Funzioni e grafici in ambienti digitali dinamici. *Didattica della Matematica: dalla ricerca alle pratiche d'aula*, *2*, 7-25.

Falcade, R., Laborde, C., & Mariotti, M. A., (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66*, 317-333.

Goldenberg, E. P., Lewis, P., & O'Keefe, J., (1992). Dynamic representation and the development of an understanding of functions. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*, *25*. MAA Notes.

Healy, L., & Sinclair, N., (2007). If this is your mathematics, what are your stories? *International Journal of Computers for Mathematics Learning*.

Hitt, F., & González-Martin, A. S., (2016). Generalization, Covariation, Functions and Calculus. In A. Gutierrez, G. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education,* (pp. 3-38). Sense Publishers.

Lisarelli, G., (2017). Exploiting potentials of dynamic representations of functions with parallel axes. In *Proc. 13th Int. Conf. on Technology in Mathematics Teaching* (Vol. 1, pp. 144-150). Lyon, France: ICTMT.

Ng, O., (2016). Comparing calculus communication across static and dynamic environments using a multimodal approach. Digital Experiences in Mathematics Education, *2*(2), 115-141.

Sfard, A., & Lavie, I., (2005). Why Cannot Children See as the Same What Grown-ups Cannot See as Different? Early Numerical Thinking Revisited. *Cognition and Instruction*, *23*(2), 237-309.

Sfard, A., (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge: Cambridge University Press.

Sinclair, N., Healy, L., & Reis Sales, C., (2009). Time for telling stories: Narrative thinking with Dynamic Geometry. *ZDM*, *41*, 441-452.

MATHEMATICS TEACHERS' IDENTITY DEVELOPMENT IN THE CONTEXT OF PROFESSIONAL MASTER'S DEGREES

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Considering that participation in teacher education initiatives usually involves negotiating new ways of being and projecting into the teaching profession, this article develops an interpretative case study focused on a mathematics teacher's identity development from his participation in a professional master's degree. Conceptualizing identity as a shifting entity that involves constructing and reconstructing meaning over multiple and conflicting discourses, the article analyzes how the mathematics teacher orchestrates voices and discourses coming from the professional master's degree and from his teaching practice to create self-understandings as a mathematics teacher.

INTRODUCTION

Currently, in-service teacher education initiatives are considered key opportunities for promoting teachers' change. Studies focused on teachers' professional development have reported on the impact of different initiatives, some of them leading to small changes while others lead to significant transformation in teaching practice (Chapman, 2017). We consider that a teacher's change, originated from participation educational initiatives, is a complex, fragile, and uncertainty-ridden process (Chronaki & Matos, 2014) which involves experimenting with tensions concerning herself and her practice. Therefore, participation in teacher education initiatives frequently involves identity development.

This article reports results from a wider research project focused on a specific initiative: Professional Master's Degrees (PMDs) directed to mathematics teachers. This special modality of Master's Degrees–that have gained momentum over the last years in Brazil–is intended to train qualified professionals, promoting opportunities for articulating knowledge, methodological approaches, and application to the professional field (Brazil, 2009). In this direction, the master's thesis is a central opportunity for connecting the PMD with teaching practice since it usually involves the elaboration and implementation of classroom activities based on the pedagogical perspectives introduced during the program. In this context, our research project aims at understanding how participation in PMDs contributes to the development of mathematics teachers' professional identities.

In the last years, identity has been used as a theoretical lens to address the relationships between mathematics teachers, the institutions where they work, and the societies where they live (Darragh, 2016; Losano & Cyrino, 2016). In this direction, several authors conceptualized professional identity as a shifting, unfixed, and unending entity

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 331-338). Umeå, Sweden: PME.

that involves the reconstruction of meaning over space and time (e.g., Chronaki & Matos, 2014; Neumayer-Depiper, 2013). Considering in-service teachers, this involves exploring how they negotiate professional identity as they struggle for meaning over multiple and sometimes conflicting discourses. Such discourses can come from different spaces (the schools where they work, PMDs, etc.) and times (present, past, and envisioned futures about teaching mathematics). This article approaches this topic by developing an interpretative case study centered on an in-service mathematics teacher who graduated from a PMD of national scope. On these bases, and using the work of Holland, Skinner, Lachicotte, and Cain (1998) as a theoretical perspective for conceptualizing identity, the paper addresses the following research question*: How did a mathematics teacher orchestrate discourses and voices coming from a PMD and from his teaching practice to create self-understandings as a mathematics teacher*?

THEORETICAL BACKGROUND

Drawing on Holland et al. (1998) we conceptualize a teacher's professional identity as:

> a set of self-understandings related to ways of being, living, and projecting into the teaching profession, facing the voices, demands, and social and political conditions of the teaching practice (Losano, Fiorentini, & Villarreal, 2017, p. 5).

These self-understandings are socially and historically constructed with other participants of the world of teaching. From this perspective, identity is a notion that articulates the personal and the social worlds (Holland et al., 1998). Thus, a mathematics teacher's professional identity is dialogically developed in an interface between her intimate terrain and the practices and discourses to which she is exposed in the present.

Two notions are relevant for capturing this interplay between the personal and the social dimensions of identity. The first is the notion of **figured worlds** (FWs), which refers to realms of interpretation and performance that are socially and culturally constructed. A PMD can be understood as a FW since it develops practices and discourses concerning mathematics education, offers a set of roles to its participants, and values some results more than others. In-service teachers enrolled in a PMD are also participants of the FW of teaching mathematics in the school, a world historically developed through the daily participation of students, principals, and themselves. As participants of these FWs, mathematics teachers become familiar with the practices and discourses created and allowed in these settings. In some cases, such practices and discourses become resources for them to understand their emerging sense of themselves as teachers (Horn et al., 2008). Thus, considered as FWs, the PMDs supply the context of meaning for some of the understandings that mathematics teachers come to make of themselves (Holland et al., 1998).

Second, the notion of **space of authoring** emphasizes that a person is continuously being addressed by different voices charged with social intentions and meanings about her identity. In the case of mathematics teachers, these voices can come from the PMD, from members of the school community, and from the teacher's own past experiences. A person is constantly involved in a dialogical process of responding to these voices and producing meanings for them. Whereas a novice surrenders to the authority voice, a more experienced person begins to re-orchestrate different voices, filling them with her own intentions and accents. Through this orchestration process, the teacher constructs her own voice; she authors herself. In this way, the space of authoring is defined by the interrelationship of different voices in the social world (Holland et al., 1998). This notion emphasizes that identity is multi-vocal, being tied up in the present and past discourses in which a person participates (Gutiérrez, 2010).

PROJECT OVERVIEW AND RESEARCH METHODOLOGY

This study reports results from a large ongoing research project aiming at describing the FWs constructed around four PMDs in Brazil as well as developing case studies focused on the identity development of mathematics teachers who graduated from such programs. In this article, we present an interpretative case study centered on one participant of our research, whom we will call Andrew, a mathematics teacher who graduated from a PMD that we will call *Redemat*.

Educational context and research participant

Redemat is a two-year large-scale program that combines face-to-face and distance learning. It is coordinated by a mathematicians' national association and is aimed at in-service mathematics teachers, especially the ones who teach in public schools. The program's main goal is to provide "a solid mathematics education, relevant for teaching mathematics in Secondary Education" (*Redemat*, 2014, § 2). To achieve this objective, the FW of *Redemat* offers subject matter courses centered on the fundamentals of the mathematics topics included in the secondary education national curriculum and on a revision of the advanced mathematics already studied during pre-service education (Algebra and Calculus courses). Thus, *Redemat* is organized around the hypothesis that teachers need to know whatever mathematics is in the curriculum, but «deeper». In this FW, most of the teacher educators are mathematicians, and the use of the history of mathematics in teaching is one of the pedagogical resources more discussed and valued. Therefore, this FW is focused mainly on disciplinary content knowledge structured and presented using the practices and discourses developed and legitimated by the mathematicians' community. Several reasons justify choosing *Redemat* for this study: its importance at the national level, its scope–at present, more than 3,500 teachers have graduated from the program all over the country–and the important public investments directed toward its conception and implementation.

In this article we present the case of Andrew, a mathematics teacher with 28 years of experience. He enrolled in *Redemat* in 2012 and won a fellowship that allowed him to reduce his teaching load to 20 hours per week during the two years the degree lasted. Before that he was a full-time teacher (teaching 30-40 hours per week) at public and private schools. After he graduated, Andrew continued teaching (30 hours per week) only at a public school. This professional trajectory justifies choosing Andrew's case for this study since it seemed to be an interesting case for analyzing how an experienced teacher orchestrated voices coming from *Redemat* with those coming from his teaching practice to author himself as a mathematics teacher.

Data collection and analytical procedures

Concerning Andrew's case, our source of data is an in-depth semi-structured interview carried out in November 2017, two years after his graduation. The interview revolved around his professional experience and teaching practice, the contributions of the PMD to his professional development (focusing mainly on the courses and the master's degree thesis), and on the possible links between *Redemat* and his teaching practice. Other important complementary data was his master's thesis.

We developed the data analysis in the form of a narrative analysis (Riessman, 2005). First, we transcribed the interview's audiotape, read the transcript, and selected a set of episodes in which Andrew resorted to discourses coming from *Redemat* for developing understandings of himself as a mathematics teacher. Second, we developed a performative narrative analysis (Riessman, 2005) of each episode, considering both its content and form (Kaasila, 2007).

According to Clandinin and Connelly (2000), a "good narrative inquiry" (p. 185) should satisfy criteria of authenticity, adequacy, and plausibility, as well as explanatory and invitational quality. To meet these criteria, we described each episode in detail, looking for the teacher's voice to be sufficiently «raised», we emphasized the ability of the narrative to explain (Kaasila, 2007), we used his master's thesis for complementing our analysis, and we shared a preliminary version of the analysis with Andrew to determine whether he recognized himself in the narratives.

NARRATIVE ANALYSIS

During the interview, Andrew began to narrate his trajectory as a teacher by positioning himself regarding two different mathematics: "There is some polarization between pure mathematics and mathematics from the pedagogical point of view, and I always was halfway between them" In his speech, Andrew understood himself as engaged in two FWs, the FW of mathematics and the FW of teaching mathematics in the school. He wove discourses coming from these FW for producing self-understandings as a teacher: he values mathematics as a field of study as well as mathematics as a subject to teach. He also resorted to discourses from these FWs to express his expectations about *Redemat*:

> I expected it to be directed to the things I can speak in the classroom but mathematics there is deeper. But you cannot just say, "Because of that the mathematics isn't oriented toward the classroom," because it is. Because it provides a basis for the teacher, so he can feel more confident about what he is going to speak, to transmit.

In this episode, Andrew seemed to take up a discourse that pervades the FW of mathematics at *Redemat* for authoring himself as a teacher: a sound understanding of the mathematical domain is the principal foundation for his speech in the classroom, reassuring him and making him feel more secure. Andrew also revealed that language plays an important role in his professional identity. In this episode he seemed to orchestrate, without major conflicts, voices and discourses coming from the FW of mathematics at *Redemat* and from the FW of teaching mathematics in the school for producing understandings about himself as a mathematics teacher.

According to Andrew, the most significant piece of all the "deep mathematics" he learned at *Redemat* was the knowledge of the history of mathematics. When we asked him to expand this idea, he told us that the history of mathematics had been "one of my interests for a long time," an interest he began to cultivate in his adolescence, reading books devoted to this topic. For Andrew, those opportunities were openings to the FW of mathematics and once he decided to become a teacher, he began to wonder, "What part of all this can I transport to the classroom?". With this previous experience, "it was natural that I identified myself with this area" of *Redemat*. Thus, the FW of mathematics at *Redemat* allowed Andrew to recover some previous interests that motivated him to join the teaching profession.

This identification was a key factor when Andrew formulated the main goal of his master's degree thesis: "to propose classroom activities that, in the light of the History of Mathematics, help the teacher to show the challenges confronted and the success achieved by ancient Greeks". The thesis is based on a valued book inside *Redemat*: *Euclid's Elements*. Resorting to this artifact, Andrew presented and formally demonstrated, throughout many pages, several geometrical constructions, from the squaring of different polygons to the estimation of π. This process involved a change in the way he positioned himself concerning Euclid's book:

> [Before] I was afraid of looking at it. During the Master's Degree I began tackling it in another way. I even began to think in the future […] to translate at least the first book that contains the basic notions to popular language, to people's language […] If I could translate it to the students' language and they could understand it it'd be great.

Andrew brought the experiences lived during *Redemat* to his narrative and used them for positioning and projecting himself into the future as a mathematics teacher who serves as a translator between Euclid's work and his students. The emphasis placed on the translation seemed to highlight the distance between the FW of mathematics at *Redemat* and the FW of teaching mathematics in the school–a distance that, in Andrew's speech, seemed to be linguistic: his intention is to translate Euclid's work to popular or students' language. In our opinion, the emphasis placed on the translation also reflected Andrew's efforts at filling the discourses from *Redemat* with his own intentions, strongly related to the teaching and learning of mathematics in the school.

The classroom activities Andrew designed for his thesis can be regarded as a result of his translation efforts. Each one of them includes a set of detailed procedures that students must follow step by step to carry out a geometrical construction and some questions focused on reflecting about the procedure performed, involving the development of "demonstrations" or "reasonings". The structure of the activities seems inspired by Euclid's work and follows a constructive logic: the geometric construction studied in an activity involves using knowledge and procedures acquired in previous ones. Andrew described the process of elaborating these activities as “painful” and highlighted:

> The process was painful because I had to consider the students, their difficulties, and transport them to the academics’ language. The academy is rigorous and is far away from the classroom. When I made this [he refers to his thesis] I was supervised by Professor Olive, who is well versed in this area, and she demanded that I use a rigorous vocabulary. And I had difficulties doing that [...] because sometimes I used a word that my students and I understand but that, from the formal mathematical point of view, was criticized.

In this episode, Andrew seemed immersed in an intense struggle, trying to orchestrate the contradictory voices and discourses coming from the two FWs. Inside the FW of teaching mathematics in the school, Andrew understood himself as a teacher concerned about presenting mathematical knowledge using a language understandable by his students. In his speech Andrew introduced the voice of Olive, his supervisor, to highlight that inside the FW of *Redemat* the legitimated practices and discourses are those of formal mathematics. In introducing his adviser’s voice, Andrew underlined that this FW did not seem overly flexible in terms of language. We consider that Olive’s voce functioned as an authoritative discourse, i.e., a discourse associated with the authority that “demands that we acknowledge it, that we make it our own; it binds us” (Bakhtin, 2011, p. 81). Thus, it is particularly difficult to introduce modifications in such authoritative discourses, to fill them with one’s own accents and intentions. In this way, the language he had to adopt for elaborating the classroom activities resonated as a foreign voice for Andrew, since his own teaching intentions and proposals were not completely considered by it. This process involves great conflict because it requires the development of classroom activities, setting aside important aspects of his professional identity to meet the requirements of the FW of *Redemat*. Andrew ended the interview stating that “a more effective dialogue between the academy and the teacher should be established”. With his words he highlighted that the possibility of approaching these two distant FWs depends, to a great extent, on the development of a common language in which the daily classroom concerns could be expressed, analyzed and attended.

DISCUSSION AND CONCLUSION

While being a program focused on mathematics, *Redemat* provides more than disciplinary knowledge. Concerning teacher education initiatives, there are always strong ties between the construction of mathematics and the construction and negotiation of identities for teachers (Ma & Singer-Gabella, 2011). Throughout its discourses and practices the FW of *Redemat* also constructs, legitimates, and affords understandings about what it is to *do*, to *teach*, and to *learn* mathematics. Orchestrating these discourses and the ones coming from the figured world of teaching mathematics in the school, Andrew developed self-understandings as a mathematics teacher.

Regarding Andrew’s case, some of the discourses coming from the FW of *Redemat* became self-understandings that helped him to organize part of his professional identity. In this way, he orchestrated discourses and voices coming from this FW to understand himself as a teacher with a solid and deep disciplinary knowledge, highlighting that this is the first and fundamental requirement for teaching mathematics. In participating in *Redemat*, Andrew could also retrieve part of his interests–particularly the knowledge of the history of mathematics–that have long motivated him. This fact allowed Andrew to identify himself with this FW. On other occasions, *Redemat* appeared in Andrew's speech as a FW populated by conflicting voices. In some of the episodes we reported, Andrew surrendered to authoritative discourses, trying to adapt his own practices and discourses to its demands.

The narrative analysis revealed that orchestrating voices and discourses coming from *Redemat* and from his teaching practice for developing self-understandings as a teacher was also a conflictive and complex process for Andrew. In our opinion, much of Andrew's effort in this direction is encapsulated in his attempt to develop translations between the *Redemat*'s language and the language of the FW of teaching mathematics in the school. Strong tensions emerged when *Redemat*, represented by his adviser's voice, demanded that its language should be adopted inside the figured world of teaching mathematics in the school, dismissing knowledge, norms and values Andrew learned in teaching practice. This demand engaged Andrew in a process full of risk and ambivalence, revealing that the differences in the languages adopted and afforded inside these FW were not only a matter of using a different vocabulary. Those languages encapsulated diverse and, in some cases, opposing norms and values related to mathematics, its teaching and learning. Throughout the episodes, Andrew seemed immersed in a struggle to establish an authorial stance, that is, "a voice that over time speaks categorically and/or orchestrates the different voices in roughly comparable ways" (Holland et al., 1998, p. 182). Those voices came from different figured worlds, figured worlds that he valued and in which he was considered a legitimate participant.

Many researchers in the mathematics education field have already stressed that teacher education initiatives organized around the needs and demands stemming from the teaching practice have more potential for promoting teachers' change (e.g., Kieran, Krainer, & Shaughnessy, 2013). The interpretative case study developed in this paper shows that a teacher education initiative organized around scholarly ways of understanding mathematics remote from the classroom could help teachers become more confident about their disciplinary knowledge, but it also engages teachers in intense struggles concerning their professional identity. Such struggles derive from the difficulty of building bridges between the identities promoted by the teacher education initiative and the ones teachers developed from their teaching practice in the schools.

References

Bakhtin, M. (2011). *Las fronteras del discurso*. Buenos Aires: Las Cuarenta.

Brazil. (2009). *Portaria normativa n^{0}17.* Brasília: MEC.

Chapman, O. (2017). Mathematics teachers' perspectives on turning points in their teaching. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proc. 41st Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 45-60). Singapore: PME.

Chronaki, A. & Matos, A. (2014). Technology use and mathematics teaching: teacher change as discursive identity work. *Learning, Media and Technology*, 39(1), 107-125.

Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research*. San Francisco: Jossey-Bass.

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, 93(1), 19-33.

Gutiérrez, R. (2010). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 44(1), 37–68.

Holland, D., Skinner, D., Lachicotte, W., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge: Harvard University Press.

Horn, I., Nolen, S., Ward, C., & Campbell, S. (2008). Developing practices in multiples worlds: the role of identity in learning to teach. *Teacher Education Quarterly*, 35(3), 61-72.

Kaasila, R. (2007). Using narrative inquiry for investigating the becoming of a mathematics teacher. *ZDM Mathematics Education*, 39, 205-2013.

Kieran, C., Krainer, K., & Shaughnessy, J. M. (2013). Linking research to practice: teachers as key stakeholders in mathematics education research. In M. A. Clements, A. Bishop, C. Keitel, J. Kilpatrick, and F. Leung (Eds.), *Third international handbook of mathematics education* (pp. 361–391). New York: Springer.

Losano, L., & Cyrino, M. (2016). Current research on prospective secondary mathematics teachers' professional identity. In M. Strutchens, R. Huang, L. Losano, D. Potari, J. P. da Ponte, M. C. C. T. Cyrino, and R. M. Zbiek (Eds.), *The mathematics education of prospective secondary teachers around the world – ICME 13 Topical Survey* (pp. 25-32). New York: Springer.

Losano, L., Fiorentini, D., & Villarreal, M. (2017). The development of a mathematics teacher's professional identity during her first year teaching. *Journal of Mathematics Teacher Education*, https://doi.org/10.1007/s10857-017-9364-4.

Ma, J., & Singer-Gabella, M. (2011). Learning to teach in the figured world of reform mathematics: negotiating new models of identity. *Journal of Teacher Education*, 62(1) 8–22.

Neumayer-Depiper, J. (2013). Teacher identity work in mathematics teacher education. *For the Learning of Mathematics*, 33(1), 9-15.

Redemat (2014). *Internal regulation*. Campinas: *Redemat*.

Riessman, C. K. (2005). Narrative analysis. In N. Kelly, C. Horrocks, K. Milnes, B. Roberts, and D. Robinson (Eds.) *Narrative, memory & everyday life* (pp. 1–7). Huddersfield: University of Huddersfield.

CHANGES IN ATTITUDES REVEALED THROUGH STUDENTS' WRITING ABOUT MATHEMATICS

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The ways in which a student relates to mathematics is known to affect how they learn and perform in mathematics: anxiety may be compensated with avoidance; enjoyment with engagement. Therefore, there is a need to understand students' relationships with mathematics and to see how these are affected by mathematics education. This paper presents results from the early stages of a mixed-methods study aimed at assessing changes in students' attitudes towards mathematics as revealed in their writings about mathematics. In contrast to existing survey instruments on attitudes towards mathematics, the methods and discussion presented here have the potential to inform the analysis of more idiosyncratic, personal, and diverse relationships with mathematics in authentic, large-scale educational settings.

INTRODUCTION

Learning and performing in mathematics is seldom strictly about knowledge of mathematics. In particular, a students' beliefs, attitudes, and emotions can affect the way the student (dis)engages with mathematics. However, what constitutes "beliefs, attitudes, and emotions", how best to conceptualize and operationalize these terms, and how these might interact with mathematics learning and performance are evolving, contemporary issues in the mathematics education research literature. A number of thorough literature reviews, monographs, and working groups have emerged over the recent years in an effort to coordinate and clarify the plethora of diverse research and perspectives on beliefs, attitudes, and emotions in relation to mathematics education (Hannula, 2012; Pepin and Roesken-Winter, 2015; Goldin, et al., 2016). The current work considers the issue of observing change in students' attitudes towards mathematics, conceptualized below, over the course of an educational program. The intention with this work is to highlight the need for educational practitioners to assess the effects education has on their students' attitudes towards mathematics and identify challenges in this endeavour.

Attitude towards mathematics

In an effort to clarify constructs in the literature on attitudes towards mathematics, Di Martino and Zan (2010) created the Three-dimensional Model for Attitude (TMA). In this model, the construct *attitude* exists along three dimensions: Emotional, Vision of Mathematics, and Perceived Competence. Each of these dimensions are rich, encompassing arrays of potential student/mathematics relationships. To improve parsimony of the model, the authors suggest the Emotional dimension be conceived as comprising

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 339-346). Umeå, Sweden: PME.

positive and negative emotions, the Vision dimension to be, following (Skemp, 1976), comprised of relational and instrumental views of mathematics, and the Competence dimension to be high/low. I follow these suggestions here in an application of the TMA model, being mindful that these dichotomous scales are one *possible* way of refining the TMA model. Even with this simple refinement, the TMA elaborates the attitude construct, realising its true multi-facetness.

Further research on attitude (Hannula, 2012) has related *affect* – a construct that subsumes beliefs, motivation, values, moods, etc., but also in particular, attitudes – to embodied and enactivist theories of learning. In so doing, a *metatheory* of affect is created with i) cognitive, motivational, emotional; ii) ephemeral and stable; and iii) social, psychological, and physiological dimensions. These dimensions emphasize that attitudes are not strictly individual, static traits, but maleable and socially emergent. An implication of this perspective is that attitudes can potentially be changed through education. Though the overall causal nature of attitudes on performance in mathematics has been reported as equivocal – see (Di Martino and Zan, 2010; Goldin, et al., 2016) for reviews of this literature – strongly negative attitudes likely result in poor engagement with mathematics (Maciejewski and Tortora, *under review*) and so "improving" these attitudes ought to be the focus of an education in mathematics, especially for at-risk populations.

The current study utilizes the TMA framework of Di Martino and Zan (2010) as a way of observing changes in students' attitudes towards mathematics over a Summer university preparatory course. Equally as important as the results reported here is the discussion of implementation and feasibility issues that follows the results section.

METHODS

The data for this study comes from students enrolled in a 5-week Summer pre-university preparation program at San José State University (SJSU). Students in this program were admitted to SJSU, but failed an entry-level mathematics test and required to enroll in developmental courses during their first year of university. Students were invited, based on financial and academic need (ie. deficient academic background), to attend the Summer program, which is intended to smooth the students' transitions to university and improve their overall chances of success.

The Summer program consisted primarily of courses in elementary mathematics and English, but also included a series of sessions conducted by the university's counseling services that targeted students' attitudes towards mathematics. Specifically, the sessions focused on mindfulness, fostering a positive attitude, self-esteem and confidence in relation to performance, academic skills, stereotype threat, and relaxation. The inclusion of these counseling sessions was intended to target and improve the developmental students' attitudes towards mathematics, as developmental students are known, in general, to have less favourable attitudes towards mathematics than their non-developmental counterparts and that this significantly hinders their progression through university (Maciejewski and Tortora, *under review*). The content of the

Summer program is not the focus of the current paper. Rather, I seek to observe changes in students' attitudes towards mathematics as revealed in their writing about mathematics.

Specifically, students in the Summer program were invited at the start and end of the program to write a short response to the prompt:

> Tell us about a personal experience you've had with math. Try to write at least 200 words.

This prompt was chosen to be as open as possible and to not narrow responses to be specifically about attitude or approaches to mathematics, etc. The intention here is for the student to recall a memory of their own interactions with mathematics; such memories are known to have associated emotional content, which is often articulated (Maciejewski, 2017).

Start-of-program essays (*N* = 134) were matched with end-of-program essays (*N* = 134) to form the dataset (*N* = 116 start/end matched essay pairs) for this study. Each essay was scored by the author according to the TMA framework of Di Martino and Zan (2010) according to the following chart:

TMA Dimension	Possible Score		
Emotional Disposition	N/A	Positive (+)	Negative (-)
Vision of Mathematics	N/A	Relational (r)	Instrumental (i)
Perceived Competence	N/A	High (h)	Low (l)

Table 1: TMA dimensions and accompanying scores, in parentheses.

Scores for each essay in each category were assigned by the author – acting as educator/researcher – with the specific criteria for the categories within the TMA framework emerging through the reading of the essays. The criteria are as follows.

- Emotional: explicit mention of emotional states or feelings towards mathematics or mathematical activity. Specific emotional words or phrases are:
  - Positive: feel good, love math, proud, favourite subject, enjoyment, etc.
  - Negative: upset, fear, nervous, scared, afraid, frustration, hate, etc.
- Vision of Mathematics:
  - Instrumental: equating understanding in mathematics with assessment outcome ("I understand math because I got a B"); memorization without understanding; practicing formulas.
  - Relational: more than one way to do a problem; interconnectedness; focus on understanding rather than correct answers.
- Competence: indication of the student's perceived ability to perform in mathematics. High/low competence was determined as:
  - High: explicit admission of high ability; attributing successful performance or ability in mathematics to ones' self; a recognition that performance and ability can improve through effort.

- Low: explicit admission of low ability; attributing performance in mathematics to a teacher or an external entity; mathematical ability as fixed.

Considering the essay prompt was general, a student's response may not contain writing related to any one of these dimensions, which would warrant an N/A score on that dimension.

By way of a sample scoring, consider the following essay.

> I've never been good with math. If I do learn something, I usually forget it not too long after. I almost always have a hard time understanding math or even just the point to all the extra formulas or ideas about it. Also since I get really mad and irritated easily when I don't understand a problem it doesn't help me or anyone else.

This was assigned a "negative" on the Emotional scale (for the text "...I get really mad and irritated..."), an "instrumental" on the Vision scale ("...all the extra formulas..."), and a "low" on the competence scale ("I've never been good with math").

After each essay was scored, the aggregate scores were assessed using χ^2 and z tests to test for statistically-significant differences between start- and end-of-term essays. As will be discussed, there is not a singular best way to analyse the aggregate score data. An analysis is then performed on individual-level essays.

RESULTS

In aggregate, end-of-program essays revealed more positive and less negative emotions associated with mathematics, a greater relational and lower instrumental understanding of mathematics, and higher student competence; see Table 2. Note that the N/A scores were higher in the start-of-program essays than those at the end, which left open the possibility of increased positive/relational/high attitude score counts without a corresponding decrease in negative/instrumental/low attitude scores. However, this was not observed.

At this stage of the analysis, a question emerges which deserves to be presented in itself as a result of this study:

> How best to determine if a change of attitudes occurred?

A naïve application of a statistical test – for example, a χ^2 test – on the start/end positive/relational/high or the negative/instrumental/low scores reveals no statistically significant differences at the 0.05 level for all categories.

However, as noted above, the N/A scores are different for start and end, so a more appropriate measure of aggregate attitude change may be to compare the proportion of positive/relational/high to total number of non-N/A scores using a z-test. This yields a statistically significant difference in start/end Competence ($p < 0.01$) and Emotional scores ($p = 0.04$), but non-significant results for the Vision category.

	Pos./rel./high		Neg./ins./low		N/A	
	Start	End	Start	End	Start	End
Emotional	30	44	49	41	37	31
Vision	10	12	63	53	43	51
Competence	32	53	76	56	8	7

Table 2: Results of a TMA analysis of the student essays.

Another approach is to consider changes in the proportion of positive/relational/high or negative/instrumental/low scores out of the total number of essays. Again, a z-test reveals a significant change for the Competence and Emotional scores ($p < 0.01$).

As noted in (Di Martino and Zan, 2010), a change in *any* category can be taken as a change in attitude. Therefore – and again at the program level – the net number of students whose essays scored as negative/instrumental/low at start-of-program then scored as positive/relational/high at end-of-term may be of interest. These are 6 for Emotional, 1 for Vision, and 16 for Competence. There were, however, shifts in attitudes from the positive/relational/high categories to negative/instrumental/low from start to end: 9 for Emotional, 3 for Vision, and 13 for Competence.

As a final perspective on how changes in attitude might be assessed, I shift focus to changes exhibited in the writing of individual students.

Changes in individual students' attitudes

Shifting to individual students, but still maintaining a focus on broader trends in the program, the number of essays that received the "least favourable" combination of scores, negative/instrumental/low, went from 30 at the start to 17 at the end. The corresponding numbers of essays receiving the "most favourable" combination of positive/relational/high scores went from 5 at the start to 6 at the end.

In terms of trends at the individual level, no one student had an essay categorized as negative/instrumental/low at the start and a corresponding positive/relational/high at the end. This is likely because relatively few essays had relational scores in the Vision category at either the start or end of program.

In terms of a particular student's change in attitude consider the following. At the start of the program, the student writes:

> My personal experience with math was great at the beginning I loved math but when senior year came around and I took trig it become my worst subject and I was not so great at it anymore.

This was rated negative on Emotion (for the past tense "loved"), N/A for Vision, and low for Competence ("not so great at it..."). At the end of the program, they write:

> The experience I have had with math went from being really good. Freshman year through junior year math was my favorite subject and I was so good at it. Once senior year hit math

> got really difficult and I started to become less interested in math. Coming into this program I got a good review and understanding of what I learned in my past years. I really liked it because I received help from my tutors which was very helpful and convenient. I learned how to solve problems I had a hard time with and finally I don't anymore. I also learned to find math fun again even though it can be challenging at times I can say I like math again.

This was rated positive, N/A, and high. The writing clearly indicates, to this researcher/educator, an "improvement" – a construct to be returned to in the discussion – in the student's attitude.

However, change is not always apparent. Consider the student who writes at the start of term (negative/instrumental/low):

> I usually prepare for a test by doing a pratice test with sample questions. However, I could never get a good grade on a test because when the test comes, my mind freezes. The problem feels completely different and more difficult. Even though sometimes the difference in the problem was just a few numbers. I try to get through the problem by thinking hard about the pratice test and writing all the formulas down.

And the student's essay at the end of term:

> Math is very entertaining! I love solving math problems only when I know it. If you have a good teacher then you will learn to understand math. Sometimes I get angry when I get stuck on a problem, everyone can solve. Overall, I love being good at math. It makes me feel smart when I solved for a problem.

The student's writing is contradictory in spots and is difficult to categorize. Ultimately, it was rated positive/instrumental/high, but the question emerges, did the student's attitude *improve*?

DISCUSSION AND FUTURE DIRECTIONS

The TMA framework was originally intended to aid in clarifying and strengthening theoretical constructs in the mathematics education literature around students' attitudes. The framework was used here to assess and categorise students' attitudes towards mathematics as revealed through their writing about mathematics. Generally, it was useful in helping parse the essays, but was limited in its ability to reveal changes in students' attitudes, for at least two reasons. The first stems from the elaboration of the three TMA categories in (di Martino and Zan, 2010). For example, emotions are not always conveniently placed in a dichotomy – a student may love proving and despise computation, or enjoy mathematical modelling and fear examinations. It would seem, then, that "emotional disposition" toward mathematics is too broad a category in which simple, dichotomous changes can be observed. As for "Views of Mathematics," the instrumental/relational dichotomy is only one of many possible ways of conceptualizing a view of mathematics; Sfard (1991) proposes, for example, the structural/operational conceptualization and argues that this is not a dichotomous view of mathematics.

Issues with the TMA framework aside, once a satisfactory – to the researcher/educator – categorization of the essays was reached a second major unresolved issue emerged: how might the application of the TMA framework reveal changes in students' attitudes towards mathematics? The analysis performed here explored this question using a variety of quantitative methods. These methods and the ensuing results should not be interpreted as authoritative. Rather, the message of this research is in the methods and analysis: even with a rigorous framework for the conceptualization of students' attitudes towards mathematics, the issue of how best to observe change – or even to define "change" – in these attitudes remains. This is especially true in larger-scale educational settings where individual interviews or essay analyses are impractical. For example, the program in this study had a total enrollment of 134, and these students fed into a program of enrollment approximately 970. There is a tension: we as researcher-educators desire to understand students' idiosynchratic relationships with mathematics and simultaneously desire they transform productively through the education we offer, yet there are often so many students that meaningful one-on-one interactions are impractical.

Another, inescapable issue related to the above is the notion of "improvement" in students' attitudes. It is a plausible expectation that educators desire improvements in students' attitudes towards mathematics, but what might constitute an "improvement", and who determines that? In the final sample essay given in the results section, the student writes that they "love being good at math" but also that they "get angry when [they] get stuck on a problem…". Having a "positive" emotion (love) associated with performance and a "negative" emotion (anger) associated with a natural state of being stuck on math is likely not the type of emotional relationship with mathematics an educator desires for their students. One approach to this issue, and working towards an operationalization of the notion of improvement, is to refine and narrow the categories in the TMA and scales, positive/negative or otherwise, to each. However, this is akin to the development of the surveys that initiated research in this domain, which are not without their shortcomings.

In terms of survey instruments designed to assess aspects of attitudes towards mathematics, Di Martino and Zan (2010) raise genuine concern over their development and continued use. In particular: how are survey items chosen, their scales assigned, and can they describe something as multidimensional as attitude? However, some researchers have considered these specific issues in the design, construction, and refinement of their Likert-based instruments. For example, the Mathematics Attitudes and Perceptions Survey (MAPS; Code, et al., 2016) traces the genesis of its survey items in students' authentic writing and talking about learning and performing in STEM fields. Moreover, the MAPS instrument avoids the issue of creating normalized scales for each category it reports on by assigning scores relative to expert consensus. This, in a sense, makes for a more versatile, holistic instrument – there isn't a universal definition of "interest", for example, but mathematics experts have a consensus on "interest" which can act as a relative, community-defined datum for the students' "interest". Further, studies with MAPS have analysed interactions between the MAPS categories (Code, et al., 2016). The criticisms of Di Martino and Zan (2010) remain quite valid, however, and a future study will compare a TMA analysis of students' attitudes towards mathematics present in their writings with their attitudes as revealed by a multi-dimensional survey instrument, such as the MAPS (Code, et al., 2016). This can further address the challenge of scale, mentioned above.

One of the important points of this current work is that observing changes in attitudes is not a straight-forward endeavor. Assessing individual students for attitudinal changes, through interviews or through their writing, can lend insights into that particular student's relationship with mathematics. However, educators are often faced with the task of evaluating the effectiveness of an entire program, which often enrolls large numbers of students, in transforming students' atitudes. The TMA framework (Di Martino and Zan, 2010) clarifies what is meant by "attitude" towards mathematics; the next step is to clarify "change" and "improvement" in attitudes.

References

Code, W., Merchant, S., Maciejewski, W., Thomas, M., & Lo, J. (2016). The mathematics attitudes and perceptions survey: an instrument to assess expert-like views and dispositions among undergraduate mathematics. *International Journal of Mathematical Education in Science and Technology, 47(6),* pp 917-937.

Di Martino, P., and Zan, R. (2010). 'Me and maths': towards a deffinition of attitude grounded on students' narratives. *Journal of Mathematics Teacher Education, 13*, 27-48.

Goldin, G.A., Hannula, M.S., Heyd-Meztuyanim, E., Jansen, A., Kaasila, R., Lutovac, S., Di Martino, P., Morselli, F., Middleton, J.A., Pantziara, M., & Zhang, Q. (2016). *Attitudes, Beliefs, Motivation and Identity in Mathematics Education: An Overview of the Field and Future Directions*. Hamburg, Germany: Springer Open.

Hannula, M. (2012). Exploring new dimensions of mathematics-related affect: embodied and social theories. *Research in Mathenatics Education, 14(2),* 137-161.

Maciejewski, W. (2017). Mathematical knowledge as memories of mathematics. In B. Kaur, W.K. Ho, T.L. Toh, & B.H. Choy (Eds.) *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education, Vol. 3,* (p. 209-216). Singapore: PME.

Maciejewski, W., and Tortora, C. (*under review*). Who are developmental mathematics students? Demographics and dispositions.

Pepin, B., and Roeskin-Winter, B. (2015) *From beliefs to dynamic affect systems in mathematics education.* Cham: Springer.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1-36.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77,* 20-26.

WHICH KEY MEMORABLE EVENTS ARE EXPERIENCED BY STUDENTS DURING CALCULUS TUTORIALS?

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The paper focuses on student learning experiences during large-group undergraduate Calculus tutorials. We identify eight types of Key Memorable Events – emotionally loaded events that are meaningful for the learning process in class from a student perspective. The findings are predominantly based on stimulated-recall interviews with 36 students, corresponding to 7 filmed lessons. Implications are drawn in relation to both the learning and teaching in the undergraduate mathematics classroom.

INTRODUCTION

In recent years, the teaching and learning of undergraduate mathematics has become an emerging topic in the field of mathematics education research (Nardi, Biza, González-Martín, Gueudet, & Winsløw, 2014). Nonetheless, the topic is still considered under-researched, and calls for further research have been made (e.g., Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, 2016). A survey of the literature reveals that most of the existing research has focused on the learning and teaching in *lectures*, while relatively little research has examined these aspects in context of *large-group tutorials* – lessons presenting problems accompanying the theoretical material of the lectures, usually taught in a frontal teaching style, and typically attended by several dozens of students. More specifically, while emotions have long been recognized to take an integral part in mathematical activity (McLeod & Adams, 1989), we have found virtually no existing published research on the role affect plays in student learning during large-group undergraduate tutorials. This paper focuses on student learning experiences in Calculus tutorials, and is situated within a growing body of research that addresses learning in light of the instructional context in which it occurs.

THEORETICAL BACKGROUND

In order to examine student learning experiences during undergraduate mathematics tutorials, a theoretical lens would need to address the changing and flowing cognitive and affective states students may go through during the course of a lesson. It is therefore appropriate to use Goldin's (2000) approach on local affect as contextually-rooted emotions that take part while engaging in mathematical problem-solving activity. Goldin defines *emotional states* as "the rapidly changing (and possibly very subtle) states of feeling that occur during problem solving" (p. 210). Furthermore, Goldin distinguishes between *traits* and *states*, where traits are relatively stable affective characteristics of individuals, and states (i.e. emotional states) relate to "in the moment" mathematical behavior. In context of undergraduate mathematics, Mar-

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 347-354). Umeå, Sweden: PME.

tínez-Sierra and García-González (2016) have recently classified different emotional experiences of students in Linear Algebra courses, while linking these emotions to the classroom situations that triggered them. They considered the reported emotions as trait emotions, and called for further research that would identify emotional states experienced in undergraduate courses by methodological means such as stimulated-recall interviews, a call we pursue in this paper.

However, not all emotional states are of equal significance, and some may be more meaningful than others. Rodd (2003) claimed that "undergraduate learning is frequently triggered by those unique events which contribute to an individual's agency or self-motivation" (p. 20). In line with this claim, Weber (2008) demonstrated how a single and strong positive experience of success may have a considerable effect on a student's success in a challenging real analysis course, by altering the student's attitude and type of engagement with the material for the continuation of the course.

As a lens focusing on those strong affective moments during a lesson, Marmur (2017; accepted) has proposed the theoretical construct of *Key Memorable Events* (KMEs) – classroom events that are perceived by many students as memorable and meaningful in support of their learning, and typically accompanied by strong emotions, either positive or negative. Marmur suggested that by focusing on those memorable "snapshots" of a lesson from a student perspective, it is possible to gain insight into the learning process and related affective aspects that occur, as well as utilize the gained understanding for the design of tutorials in support of a positive learning-experience. Accordingly, we have pursued the following research questions: *(a) What are tutorial events that serve as KMEs for students? (b) What are the possibilities for student learning afforded by these KMEs? (c) How do these KMEs relate to the teaching that takes place?* Due to the scope of this paper, the findings herein primarily address the first question, whereas the second and third questions serve as contextual background.

METHOD: DATA COLLECTION AND ANALYSIS

The research took place in first year undergraduate Calculus tutorials attended by computer science and electrical engineering students of a highly-ranked research university. The main corpus of data came from stimulated-recall interviews with 36 volunteering students in regard to 7 filmed lessons: three taught by the first author, and the others by four experienced instructors. The stimulated recall methodology was chosen as it can aid students in re-experiencing a lesson, and consequently supplies insight into the students' thought-process during its course (e.g., Calderhead, 1981). Additional data included observation notes, pre- and post interviews and communications with the instructors, and lesson-notes taken by the interviewed students.

At the beginning of the interview, the students were given explanations on its general goal as an inquiry into student learning during tutorials, how it would be conducted, and what lesson and problem it would focus on. This allowed the students an opportunity to *unsolicitedly* share their memories, thoughts, and emotions regarding the problem or entire lesson, if they so chose to (as was mostly the case). Subsequently, the students were presented with a 15-20 minute excerpt of the filmed lesson, and were asked to stop the playback whenever they had a particular recollection of what they thought or felt at that moment. When students stopped the video to share their memory of that moment in class, they were often asked clarifying questions, mainly in the form of: "can you explain *why* you thought/felt this way at that moment?" At the end of the interview, the students were asked several follow-up questions regarding themes that came up during earlier stages of the interview. The interviews were concluded with the following questions regarding the problem, lesson, and course: a) Was the problem memorable for you, and if so, in what way? b) What were you pleased and displeased with during the lesson? and c) What is your general attitude towards the course? The interviews were conducted in the immediate days following the lesson, and were 30-65 minutes long, though mostly around 50 minutes.

The analysis utilized a general inductive approach (Thomas, 2006), as this methodology supports the coordination of extensive data into a brief summary that addresses the "underlying structure of experiences or processes" (p. 238) most apparent in the data. The analysis consisted of the following stages: (a) Dividing each lesson into a list of consecutive events over time, and refining it based on the students' accounts; (b) Identifying memorable events for each student based on *unsolicited* supplied details of thoughts or feelings regarding the events; and (c) Identifying KMEs – for identifying the events with the potential of being memorable and meaningful to many, we concentrated on those events addressed by all 4-6 students interviewed on a given lesson (with the exception of one at most) around a common theme. Within these, we looked for those events where strong student emotion could be inferred. This included statements such as "I loved/hated this", "it was beautiful/horrible", etc. Additionally, we looked for students' use of *intensifying adverbs* (e.g., "I *really* loved it") and *repeated* unsolicited statements on the event throughout the interview (e.g., before, during, and after watching the event). Consequently, these KMEs were examined in relation to the learning and teaching in class. As validation we examined other collected data for further supporting evidence, as well as alternative interpretations of the findings.

FINDINGS

Eight KMEs were identified and categorized as a result of the analysis process: Surprise; Heuristic-didactic discourse; Suspense; Bridging; Engaging questions; Undigested symbols; Leaving theoretical loose ends; and Overarching connections. Each lesson included 2-3 KMEs, except for one lesson in which 5 KMEs were identified. In the following we explain and elaborate on the different KMEs.

Surprise

The evocation of surprise was identified as a KME in two lessons in relation to three instructor actions: (a) Reaching a dead-end in the solution (not declared in advance); (b) Presenting an unexpected non-routine solution; and (c) Reaching an unexpected result. The student interviews supplied further evidence to previous research claims on surprise serving as a factor that supports student learning by raising curiosity, interest, enjoyment, and mathematical aesthetic experience (Marmur & Koichu, 2016; Movshovits-Hadar, 1988). This is exemplified by the following excerpts:

> "I thought the instructor would simply make some kind of change and we would move on. That there will be some small nuance the instructor will use and we'll be able to continue. But here it was extremely dramatic. The instructor just erased it all! [the solution on the board] […] This raised my concentration because I became more interested to see what was going to happen." [regarding a surprising event of category (a)]

> "Wow! What a beautiful problem! […] It's kind of beautiful when I'm surprised that it's not what I thought at first glance. […] I personally really love it when the instructor brings problems that are unexpected. […] it's also the most memorable. […] I don't remember any of the other problems […] in that lesson, but this problem I remember till now. [...] What a lesson this was! It really excites me!" [regarding a surprising event of category (c)]

Heuristic-didactic discourse

In three lessons we identified KMEs that involved a meta-level discourse focused on how to approach a challenging problem – a type of discourse referred to by Marmur and Koichu (in press) as *heuristic-didactic discourse*. These events corresponded to any of the following: a) Discussing aspects of the problem before solving it; b) "Playing" with the problem (as opposed to directly solving it); c) Utilizing a student method for the solution and/or highlighting possible student mistakes; d) Presenting an expert (yet approachable) way to address the problem; and e) Including an undeclared teacher's goal (along with the declared goal of solving the problem). As illustrated by the following excerpts:

> "I really love it when they [instructors] write a problem and immediately afterwards talk to the class with the following approach: ok, I'm a student now, I got this problem, how do I approach it? […] Where do I begin? Like, let's start playing with it, let's see what's going on here. I really appreciate it when a teacher doesn't immediately 'toss' the solution on the board, […] but rather puts himself through our eyes while looking at the problem."

> "After the instructor spoke about 'first think, before you just start writing', […] the instructor showed two edge cases. It really helps to think about a problem in its two extremes, and then see where this [problem] lies. [...] It gives a certain direction and a certain line of thought, it helps in getting a feeling for the problem. And […] it really helps in knowing whether [I am] in the right direction or not."

Suspense

Events of suspense were identified as KMEs in two lessons in relation to three instructor actions: (a) Presenting a solution where students did not know whether it was correct or incorrect in advance, yet were made aware both options were possible; (b) Starting a solution after having allocated time for students to independently think about the problem; (c) Supplying conflicting information. The following excerpts illustrate the utility of suspense in evoking independent and critical thinking on a given solution:

> "It keeps you alert. Let's see, let's see if this time the instructor is tricking us or not. [...] This way everyone is involved in the lesson. [...] It also helped me understand better. Because then you put more attention on why this is correct, why that is incorrect. And through this process you get deeper into the material and understand it better." [regarding an event of category (a) of suspense]

> "I remember that here I was in suspense to see what the instructor was going to do. I was really nervous. I told myself: Did I succeed? Did I not succeed? Was I in the right direction? In the wrong direction?" [regarding an event of category (b) of suspense]

Bridging

By *bridging* we refer to the meaning ascribed by Ejersbo, Leron, and Arcavi (2014) as helping students bridge the gap between intuitive and analytic thinking. A bridging KME was identified in one lesson in which the bridging was stimulated by a three-dimensional visual aid that helped reconcile an unintuitive analytic result. The student excitement of the event, as well as its role in supporting their understanding, are illustrated in the following excerpts:

> "Already in the first time the instructor did this rotation, we could see the thing [the lines]. I remember I told myself: Wow! Like, I was convinced. I knew it was true because the instructor had solved the problem earlier and I believed him, but like – something in me was convinced."

> "Let's say that if the instructor hadn't brought this thing [model], I still wouldn't have believed him there are two lines. [...] No matter how much he would have tried to force me. [...] it also changed a bit my perception on how this shape really looks like, and not how we draw it approximately. [...] This lesson was really meaningful, because it changed my perception."

Engaging questions

By KMEs of *engaging questions* we refer to events where an instructor's question managed to actively engage the students in mathematical thought or activity. While the act of question posing by instructors was found in all the observed lessons, it is interesting to note that this was identified as a KME in one lesson only. A comparison between the different lessons suggests that in this particular lesson there was a smaller amount of teacher questions, though more selective and focused (see also Almeida, 2012). As such, the questions evoked engagement and participation, even in cases where students were uncertain of the correctness of their answer:

> "I love it that the instructor sometimes asks questions where you're not sure. [...] My first thought was: take a parameterization of a circle […] and find y from the intersection with the plane. And this is really what you do. But at that moment you suddenly think [...]: Could it be that she's asking this because this is not the answer? Because it's not the obvious thing?"

Undigested symbols

A KME of *undigested symbols* refers to a classroom event where students' symbol sense (Arcavi, 1994), or lack thereof, becomes the center of student attention. Such a

KME was identified in two lessons. In the first, the KME supported a positive learning experience around the different meanings of the dot symbol within the topic of vectors. In the second, however, the KME indicated an unresolved issue for student learning on the relation between index symbols and subsequences. In the first case, the KME seemed to additionally impact the students' note-taking actions during the lesson, which included formulating elaborated explanations that were not written on the board:

> "When we deliberated whether the solution was correct or incorrect, then the instructor really emphasized the issue that here there is a scalar that you multiply with a vector. And we all got confused because of the mess of 'when is it a multiplication between a scalar and a vector?', and 'when is it a scalar product?' […] this was really a very very [good] emphasis that I really took, and during the next solution I was still writing it to myself."

Leaving theoretical loose ends

This KME was identified in one lesson in which the instructor left a supporting claim to be independently proven by students at home. Even though the instructor had explained in class that this activity would be beneficial for the students' learning, the interviewed students claimed they did not understand the reasoning behind this, and referred to the event with emotions of frustration, which included statements such as "it's really stressing me out". The following excerpt supplies further illustration:

> "Something that I really don't like is when a claim is presented [...] and then there's a supporting claim. [...] this kind of hocus-pocus, 'there's a supporting claim' – do it by yourself. […] I feel like [...] information is being withheld from us. [...] To get to this, you have to prove this [supporting] claim. So regarding this [supporting] claim – I ask *why*?! [...] Why can't you tie up loose ends? [...] For me personally, it ruins the way of thinking."

Overarching connections

A KME of *overarching connections* was identified in three lessons, and refers to the use of solution methods or theoretical material presented earlier in the course. By making such connections, previous topics are put in new or generalized context, which subsequently may support a unified understanding of the course (see Bergsten, 2007, on 'connections' as a characteristic of high-quality undergraduate teaching). This KME was accompanied by two opposing types of responses. On the one hand, some students expressed enthusiasm evoked by the presented connections, as well as increased motivation to refresh their memory on past topics as a result of the event. On the other hand, other students were anxiously wondering how they were supposed to come up with such ideas on their own. As illustrated by the following:

> "It was beautiful. It's an eye opener. I mean, I loved it. [...] Even though I knew the one-variable [Riemann] function, I never thought of transferring it to two variables."

> "And then everything connects. It's really fun when it happens in the lesson."

> "After this lesson I decided – it's about time I prepare this summary page [of theorems]"

> "Why do people even remember this function? What did they do that I didn't?"

> "At this stage I thought to myself I would never in my life be able to come up with this."

DISCUSSION

The above identification and categorization of KMEs possess potential implications regarding both the learning and teaching in the undergraduate classroom. From the perspective of learning, the identified KMEs were associated with learning opportunities such as heuristics on how to approach a challenging problem; evaluating the correctness of a given solution; sense-making of mathematical symbols; connecting between analytic and intuitive modes of thinking; problem-solving behaviors including failed attempts; exposure to common mistakes; creating connections between different mathematical topics; and enhancing memorability of the material. Furthermore, the learning in the KMEs was typically *via* problem solving (see Schroeder & Lester, 1989), concentrated on thought-processes and knowledge that could be generalized for future problems, rather than merely focused on a solution-oriented end result.

From the perspective of teaching, lesson designs in the considered lessons generally went through 1-2 initial KMEs that created *tension*, followed by raised student engagement and anticipation, and concluded with a KME that generated *resolution*. This tension-resolution mechanism created two foci that emerged from the interviews: a first on affect and a subsequent on mathematical content. The KMEs that created tension evoked strong emotional responses (even “negative” ones, such as anger due to a surprising failed solution), whereas the resolution KME was centered around a mathematical goal (such as making overarching connections). As such, the tension-resolution combination of KMEs served as an effective method to promote student learning via affective routes.

Indeed, lessons following this KME combination were considered “successful” not only based on positive student evaluations, but also since the identified KMEs were characterized by heightened engagement, whether expressed by verbal participation, raised interest, elaborated self-formulated notes (cf. Lew et al., 2016), or independent work on a problem. Especially when considering the established norm of teacher-focused instruction in Calculus tutorials, these findings suggest that even within the “boundaries” of this norm, the utilization of KMEs by instructors can aid students in becoming actively engaged learners in class.

References

Almeida, P. A. (2012). Can I ask a question? The importance of classroom questioning. *Procedia - Social and Behavioral Sciences, 31*, 634–638.

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics, 14*(3), 24–35.

Bergsten, C. (2007). Investigating quality of undergraduate mathematics lectures. *Mathematics Education Research Journal, 19*(3), 48–72.

Calderhead, J. (1981). Stimulated recall: A method for research on teaching. *British Journal of Educational Psychology, 51*, 211–217.

Ejersbo, L. R., Leron, U., & Arcavi, A. (2014). Bridging intuitive and analytical thinking: Four looks at the 2-glass puzzle. *For the Learning of Mathematics*, *34*(3), 1–6.

Goldin, G. A. (2000). Affective pathways and representation in mathematical problem solving. *Mathematical Thinking and Learning*, *2*(3), 209–219.

Lew, K., Fukawa-Connelly, T. P., Mejia-Ramos, J. P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, *47*(2), 162–198.

Marmur, O. (2017). *Undergraduate student learning during calculus tutorials: Key memorable events* (Doctoral dissertation). Technion – Israel Institute of Technology.

Marmur, O. (accepted). Key memorable events during undergraduate classroom learning. In *The 21st Annual Conference on Research in Undergraduate Mathematics Education.* San Diego: USA.

Marmur, O., & Koichu, B. (2016). Surprise and the aesthetic experience of university students: A design experiment. *Journal of Humanistic Mathematics*, *6*(1), 127–151.

Marmur, O., & Koichu, B. (in press). What can calculus students like about and learn from a challenging problem they did not understand? In *The 10th Congress of European Research in Mathematics Education*. Dublin, Ireland, 1-5 February 2017.

Martínez-Sierra, G., & García-González, M. D. S. (2016). Undergraduate mathematics students' emotional experiences in Linear Algebra courses. *Educational Studies in Mathematics*, *91*(1), 87–106.

McLeod, D. B., & Adams, V. M. (Eds.). (1989). *Affect and mathematical problem solving: A new perspective*. New York: Springer-Verlag.

Movshovits-Hadar, N. (1988). School mathematics theorems: An endless source of surprise. *For the Learning of Mathematics*, *8*(3), 34–40.

Nardi, E., Biza, I., González-Martín, A. S., Gueudet, G., & Winsløw, C. (2014). Institutional, sociocultural and discursive approaches to research in university mathematics education. *Research in Mathematics Education*, *16*(2), 91–94.

Rodd, M. (2003). Witness as participation: The lecture theatre as site for mathematical awe and wonder. *For the Learning of Mathematics*, *23*(1), 15–21.

Schroeder, T. L., & Lester, F. K. (1989). Understanding mathematics via problem solving. In P. Trafton (Ed.), *New directions for elementary school mathematics* (pp. 31–42). Reston, VA: National Council of Teachers of Mathematics.

Thomas, D. R. (2006). A general inductive approach for analyzing qualitative evaluation data. *American Journal of Evaluation*, *27*(2), 237–246.

Weber, K. (2008). The role of affect in learning Real Analysis: A case study. *Research in Mathematics Education*, *10*(1), 71–85.

ANALYSIS OF THE MATHEMATICAL DISCOURSE OF UNIVERSITY STUDENTS WHEN DESCRIBING AND DEFINING GEOMETRICAL FIGURES

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This is part of a bigger and ongoing empirical research study that uses the commognitive framework in order to characterize how university students (that are also pre-service teachers) define 3D geometrical figures. We consider that the process of defining plays an important role in the generation of a mathematic knowledge specific for teaching, which future teachers must acquire. With the purpose of understanding this process, we designed a task with open questions and used as data sources audio recordings of one-hour sessions (and their transcripts) and written answers of four groups of students when they solved the task. We have identified different routines that appear during the process of defining as indicators of students' knowledge.

INTRODUCTION

Research on teaching and learning mathematics at university level is an important field in mathematics education that has been addressed from different approaches and in different contexts (Biza, Giraldo, Hochmuth, Khakbaz & Rasmussen, 2016; Holton, 2001). On the other hand, Heyd-Metzyuyanim et al. (2013) point out that

> in recent years educational researchers' interest in human communication has been steadily growing, and today, discourse is the main focus of many, if not most, of educational studies (p. 155).

Among the various theoretical frameworks that can be considered, here we will focus on the commognitive one, which has proved useful for studying mathematics education at university level (Nardi, Ryve, Stadler & Viirman, 2014; Tabach & Nachlieli, 2015).

According to the commognitive approach (Sfard, 2008), teaching and learning is a sociocultural activity. For Sfard (2008), commognition ('communication' and 'cognition') is not a mere replacement of cognition with another different theory. Indeed, Sfard (2008) states that "commognitive research differs from both its predecessors, behaviorism and cognitivism, in its epistemology, ontology, and methods" (p. 275) and that "the choice of discourse as the principal object of attention is what sets this approach apart from other types of participationists' research" (p. 275). The discourse that is the object of study in the commognitive framework has four discursive characteristics for Sfard (2008): word use, visual mediators, endorsed narratives and routines, which will be described in the following section.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 355-362). Umeå, Sweden: PME.

Several studies have used this commognitive approach to study learning and teaching of mathematics at university level. In particular, Nardi et al. (2014) focused on the discursive shifts of university students when they study calculus; Tabach and Nachlieli (2015) studied how university students (that are also prospective teachers) used definitions of functions; Thoma and Nardi (2016) investigated the discourse of closed-book examinations; and Biza (2017) focused on the substantiations of narratives about the tangent line in university mathematics students' discourses.

In this work, we will investigate the mathematical practice of defining among university students in the context of 3D geometry when they describe geometrical figures and when they construct mathematical definitions. These students are also primary pre-service teachers, so this study has a double implication. On the one hand, it informs the university teachers of these students about their actual knowledge, in order to help them construct their mathematical knowledge. On the other hand, this mathematical knowledge will be the base that allows students to construct the knowledge for their future teaching.

When we study this practice of defining, we consider it a process, in which the definition itself is the final product. In particular, we will consider the process of defining as the one that begins with the description of objects and ends with a formal mathematical definition, with the intermediate steps of proposing preliminary definitions and deciding how many details are needed. We focus on this process because it plays an important role in the generation of a mathematic knowledge specific for teaching, which future teachers must acquire.

THEORETICAL FRAMEWORK

Since we will use as theoretical framework the theory of commognition of Sfard (2008), we will present here the main characteristics of this approach that sustain our work. First of all, according to Sfard (2008), discourses are

> different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors (p. 93).

She also states that "seemingly the most natural way to distinguish discourses from one another is to specify their respective objects" (p. 129). Therefore, mathematics is a discourse that can be described by four characteristics: word use (which include both mathematical terms like 'polygon' and ordinary words used in a mathematical context, like 'edge' of a figure), visual mediators (like drawings of geometrical figures, graphs, symbols), endorsed narratives (description of objects, definitions, etc.) and routines (practices like 'defining' or 'proving', as well as any other regularly employed practices that are used).

Among the different discursive characteristics mentioned above, we will focus on the last one: the routines. Sfard (2008) characterizes the routines as "a set of metarules that describe a repetitive discursive action" (p. 208), where a metarule is a "rule that defines patterns in the activity of the discursants" (p. 299).

In particular, in this study we aim to provide information about the process of defining of pre-service teachers through the identification of different types of routines that appear during that process. Among other things, we have considered routines because they are the characteristic that best informs about the procedural actions of these students when they construct definitions.

METHODOLOGY

Participants and context

The data of this study come from a wider study concerned with identifying and classifying metarules and routines of university students. In the part of the study that we present here, we focus on students of the undergraduate degree of primary education of a big public university in Spain. As part of their undergraduate degree, these students have a compulsory course on mathematics on their first year. Once a week, they work in mixed-gender groups of four to six students while they strive to solve different proposed mathematical problems and tasks.

The participants of this study were 4 of these groups (called G1, G2, G3 and G4) during one-hour sessions. The great majority of the students were in the 18-21 years old range.

Instrument

We are interested in the discourse of students, thus we designed a task with a series of open questions with the aim of promoting discussions among them.

Since we did not want to alter the normal sessions of the mathematics course that the students were taking, we chose for the task one of the topics that they would study: 3-dimensional geometry. At the time of this session, all these students had already been taught about 2-dimensional geometry but not yet about 3-dimensional geometry.

The task had a brief explanation in the first page concerning the aim of the study and thanking the students for their help. Then there were pictures of three prisms: a cube, a parallelepiped that was not a cube, and a prism with a concave base. Then, some questions were asked. Examples of these questions were:

- Basic elements as faces, vertices, edges, etc. can be identified in these figures. What properties or characteristics of these elements can you observe in each figure?
- Can you identify any property among the previous ones that only two figures have in common?
- Can you identify any property that the three figures have in common?
- Define each of these figures.
- Can you give another definition of any of the figures?

- Is one of your definitions valid for another figure too? For example, is the definition of figure 1 also valid for figures 2 or 3?

Each of the groups of students were provided with a copy of the task and were instructed to verbalize their answers to each question as much as possible (which we audio recorded) and reach an agreement (which they had to write down).

Data collection

Audio recordings of one-hour sessions (and their transcripts) and written answers of four groups of students when they solve the task were our data sources. The first source is crucial because it permits us to access the discussion and negotiation in the process of solving the task. We consider the discourse of the whole group as a unit of analysis, without making distinctions among the students. The second source is important because students reflect in their written answers their agreements as to what constitutes a correct answer. In this way, they validate explicitly their achieved agreements.

ANALYSIS

Once we transcribed the recordings from the four groups into written text, we analyzed our data in two different steps. In the first one, we identified Sfard's (2008) four discursive characteristics using the transcripts and the written answers. In the part of the study presented here, in order to identify routines, we analyzed the mathematical words and narratives used by the students searching for patterns in the discourse, from which we inferred routines. In a second step, we analyzed the routines to determine if it is possible to identify some characteristics of the students' process of defining.

We will now show a small example of our analysis using group G1 (with students S1, S2, S3, S4).

First step

For instance, when the students answered the question "how would you define figure 1", they had the following discussion (translated from Spanish), where we show the narratives in italics and the mathematical words in bold:

219:	S1:	*first the name*
220:	S2:	a **cube**, right?
221:	S3:	yes
222:	S2:	*a **cube** is… a **solid***
223:	S4:	*which are all **prisms** because they are formed by several **polygons***
224:	S1:	***polygons** of 6 **faces***
225:	S2:	*of 6 **faces**… that are **equal*** [writing]
226:	S4:	*with a **square basis***
227:	S2:	[repeats while writing]
228:	S1:	*it is a **hexahedron**, that is the first thing* [they did not include this in the written answer to the task]

In line 223, "they are formed by polygons" is our way of translating from Spanish a mistake that the group commits, not a mistranslation.

The showed transcript begins and ends with narratives that express the same idea (lines 219 and 228): in order to construct a definition, students say that it is necessary to first give a name and then follow it by the properties of the figure. This idea is present several times in narratives of the students of this group (both in their written answers and their oral discussions), so we infer a routine that we will name "Defining is labelling and describing the properties of a figure".

Second step

In this step, we analyzed all the routines found in the first step looking for possible relations with the students' process of defining. For example, the routine "Defining is labelling and describing the properties of a figure" indicates an initial procedure during the process of defining that can affect its development.

RESULTS

We will now show types of routines that we have identified. In order to improve the validity of our study, we will focus only on the four routines that appear in more than one group. We will first label and describe the routines and then give examples of their use that have been translated from the transcripts in Spanish.

Counting when describing and, if possible, adding a relation or adjective. This routine is inferred from the narratives of students when they try to identify properties of an element and it means that students count how many times the element appears and, if possible, they add an adjective that describes it or they describe its relation with other elements. We differentiate between two types of counting: one by one, which we call *additive counting* (present in G1, G3, G4), or counting how many times a group appears, called *multiplicative counting* (G1, G3). Group G2 also counts the elements but not aloud, thus we do not have evidence of what type of counting they used.

An example of this routine can be identified when students of group G3 use both *additive counting* and *multiplicative counting* when describing faces, edges and vertices:

8:	S3:	Figure 1, faces. They are these, aren't they?
9:	S2:	1, 2, 3, 4, 5, 6, yes.
10:	S3:	1, 2, 3, 4, 5, 6. Let's see, vertices and edges.
11:	S2:	1, 2, 3, 4, 5, 6, 7, 8.
12:	S1:	Four times six, … no, wait.
13:	S2:	No because there are edges in common.
14:	S3:	1, 2, 3, 4, 5, 6, 7, 8. 8 vertices, right? And I don't know what edges are.
15:	S2:	Edges are these lines.
16:	S1:	So there are some here … no because they coincide….
17:	S3:	1, 2, 3, 4, right?

18: S2: No, 1, 2, 3, 4, 5, 6, 7… and 11 and 12

Defining is labelling and describing the properties of a figure. This second routine is identified in G1, G2, G3, G4 and is used by the students when defining a figure. It describes a very common pattern: the students first label the figure and then describe its mathematical properties, which are often the number of faces, edges and vertices that they had obtained before by using the previous routine.

A representative example of this routine can be seen when group G2 defines a figure:

61: S1: A cube, a figure with volume, right? That occupies a space.

62: S3: But that is true for all of them, isn't it?

63: S1: Yes but now we have to say that one is regular, another one has… everything that we said before. […]

Resorting to 2D to solve 3D problems (groups G2, G3, G4). This routine is not exclusive to the practice of defining, but rather reflects a return to simpler problems when facing an unknown situation. This way, students turn to their knowledge of 2D as a way to bolster their (scant) knowledge of 3D geometry. For instance, group G2 uses this routine when they discuss if the definition of a figure is valid for another one:

134: S3: This is like with the triangles and the rhombi and all those, the squares are inside the rhombus, so maybe this definition is inside this other one but not necessarily.

135: S1: Yes but it's like…we don't know how it is. I don't know which definition. And this is more complicated. Are all three of them prisms?

Finally, the last routine is called *Searching for information in external sources* and it appears in the groups G1, G2, G4 when the students ask the teacher for help, search in their class notes or on the internet. This routine is not exclusively mathematical (although this information is sometimes related to mathematical content) but has a social nature. For example, the students of group G1 use this routine several times:

28: S3: [asking the teacher] But…, we have to define in the first part, like..., what is a vertex, what is a face…Then, what do we have to say, like…, how many faces they have, how many vertices are there?

[…]

59: S2: Look that up on the internet.

Later, S1 searched her class notes to recall some definitions.

CONCLUSIONS

In our work, the routines that we have identified inform us about the characteristics of the mathematical process of defining of the university students that participated in this study.

The first routine, *Counting when describing and, if possible, adding a relation or adjective,* tells us about the first stage of the process: the description of objects. Of all the possibilities that exist when describing a solid, like describing the shape of its faces, the parallelism between them, etc., all the groups chose to describe a solid by first counting how many vertices, edges and faces it has. The fact that this routine is common to all groups is for us an indicator of its importance.

The second routine, *Defining is labelling and describing the properties of a figure,* reflects what all groups consider a definition to be. This characterizes a later stage of the process of construction of definitions.

These two routines contrast with the situation found in 2D geometry by previous researchers interested in the construction of definitions, like Gavilán-Izquierdo, Sánchez-Matamoros and Escudero (2014). Indeed, these authors found that when students define the square, rhombus or rectangle, they use primarily qualitative properties like the parallelism of their sides or if the sides are equal in length or not.

The last two routines, *Resorting to 2D to solve 3D problems* and *Searching for information in external sources,* can come from the school culture. In the Spanish curricula, 2D geometry is studied more in depth that 3D geometry, so this can be the reason why students resort to the more familiar 2D geometry. The last routine is not specific of mathematics and it may be shared by other subject matters.

This study has allowed us to identify routines that may affect the process of defining of groups of students. This way, we have enlarged the existing literature with the study of the mathematical process of defining from a commognitive approach. This approach has been useful because, through the routines, it has permitted us to identify procedures that may affect the process of defining. These procedures may have remained hidden when using other approaches. Of course, this study can be continued by considering other discursive characteristics, other students and other tasks. This can provide us with a more holistic vision of the mathematical practice of defining from a sociocultural approach.

We acknowledge that a limitation of the work presented here is the number of groups of participants that we have considered. However, this is part of a wider study with more tasks and more groups of participants (to whom we are very grateful for their participation), which we will present in future works.

Finally, we would like to point out that, with this qualitative-interpretative study, we have tried to initiate a way that allows us to obtain valuable information for us, both as researchers and teachers.

References

Biza, I. (2017, in preparation). "Points", "slopes" and "derivatives": Substantiations of narratives about tangent line in university mathematics students' discourses. In T. Dooley, & G. Gueudet (Eds.), *Proceedings of the 9th Conference of European Research in Mathematics Education* (Vol X, pp. XX-YY). Dublin: DCU Institute of Education and ERME.

Biza, I., Giraldo, V., Hochmuth, R., Khakbaz, A., & Rasmussen, C. (2016). *Research on teaching and learning mathematics at the tertiary level: State-of-the-art and looking ahead*. doi:10.1007/978-3-319-41814-8

Gavilán-Izquierdo, J.M., Sánchez-Matamoros, G., & Escudero, I. (2014). Aprender a definir en matemáticas: Estudio desde una perspectiva sociocultural. [Learning to define in mathematics: Study from a sociocultural approach]. *Enseñanza de las ciencias: Revista de investigación y experiencias didácticas*, *32*(3), 529-550.

Heyd-Metzyuyanim, E., Morgan, C., Tang, S., Nachlieli, T., Sfard, A., Sinclair, N., & Tabach, M. (2013). Development of mathematical discourse: Insights from "strong" discursive research. In A. M. Lindmeier, & A. Heinze (Eds.), *Proc. 37th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 155-179). Kiel, Germany: PME.

Holton, D. (Ed.) (2001). *The teaching and learning of mathematics at university level: An ICMI study*. Dordrecht: Kluwer Academic Publishers.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education*, *16*(2), 182-198.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing*. New York, NY: Cambridge University Press.

Tabach, M., & Nachlieli, T. (2015). Classroom engagement towards definition mediated identification: The case of functions. *Educational Studies in Mathematics, 90*(2), 163-187.

Thoma, A., & Nardi, E. (2016). A commognitive analysis of closed-book examination tasks and lecturers' perspectives. In E. Nardi, C. Winsløw, & T. Hausberger (Eds.), *Proceedings of the First Conference of the International Network for Didactic Research in University Mathematics* (pp. 411-420). Montpellier: University of Montpellier and INDRUM.

LINKING INFORMAL AND FORMAL MATHEMATICAL REASONING: TWO DIRECTIONS ACROSS THE SAME BRIDGE?

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What do students need to do in order to use their formal mathematical knowledge in everyday informal situations? How can informal everyday mathematical reasoning be used as a foundation for developing new mathematical knowledge? Are these two directions on the same bridge – that which lies between informal and formal mathematical reasoning? Herein we argue that connections between the informal mathematics of everyday life and formal mathematical instruction must be encouraged and supported throughout the mathematical curriculum, including also in late primary and lower secondary school. We argue that there are crucial mental processes that underlie both and that these processes may be keys to developing all students' abilities to connect their informal and formal mathematical reasoning.

BRIDGING INFORMAL AND FORMAL MATHEMATICS

Providing students with the skills needed to use their formal mathematical knowledge in everyday situations is a stated goal of many mathematical curricula around the world (Common Core State Standards Initiative, 2010). However, this may take too limited a perspective on the relation between informal and formal mathematics. Certainly, students need to be able to reason about their everyday worlds in mathematical ways. But just as important, instructional practices can also draw upon students' informal mathematical reasoning in order to develop formal mathematical knowledge. In fact, this bootstrapping of informal mathematical reasoning into formal mathematical knowledge is already apparent in much of the early-years of mathematical development (Purpura, Baroody, & Lonigan, 2013). Everyday intuitive understanding of concepts relating to the magnitude of natural numbers, arithmetic relations between whole numbers, and spatial-numerical connections are easily accessed and drawn upon in the first years of mathematical instruction. However, as mathematical topics begin to expand into more culturally constrained topics, such as non-natural numbers and algebra, the direct connection between everyday reasoning and formal mathematics seems to dissipate within the mathematics classroom.

It is crucial for instruction to help students construct a more stable bridge between their informal and formal mathematical reasoning, both in and out of the classroom. A pure mathematical form with no connections to reality only upholds part of the power of mathematics. Likewise, mathematics only framed as a practical enterprise inspires little exploration and will lead to stagnation and irrelevance in the field. Thus, it is critical that students be able to apply their mathematical knowledge in everyday situ-

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ations. As well, their informal reasoning about mathematical aspects of everyday situations should also be used to support learning complex mathematical topics. If these goals are explicitly addressed it is often through the use of word problems. However, traditional word problems often do not require any actual mathematical model of the situation, and if such a model is actually needed, many students fail to use one and instead rely only on superficial features to solve the task (Verschaffel, Greer, & De Corte, 2000). Moreover, there is little evidence of the widespread use of students' informal, intuitive mathematical reasoning in supporting their formal mathematical learning.

Students and learners of all types are constantly surrounded by opportunities and experiences to use informal and formal mathematical reasoning in their everyday life. Informal experiences with mathematical reasoning in everyday life can be also used to provide strong foundations for new formal mathematical development (Resnick, 1986) and applying formal mathematical knowledge in informal situations may support further matheatical development (Lehtinen & Hannula, 2006). The question then arises, are there mechanisms that could be employed in the mathematical curriculum for supporting (a) the bootstrapping of intuitive, informal mathematical reasoning in everyday situations into new formal mathematical knowledge and (b) the flexible and fluid application of formal mathematics in making meaning of everyday life. These two processes can be described as the anchors of the bridge between the informal and formal mathematics that students encounter. In the present manuscript, we argue that there are underlying processes that are core to crossing the bridge in either direction between the informal and formal mathematics, which may be crucial for supporting such connections among all students.

BRIDGING THE INFORMAL AND FORMAL: BOUNDARY EXAMPLES

We aim to explore the potential bridge between the informal reasoning that occurs in everyday life and formal mathematics by exploring the meaning making processes involved in two examples of mathematical reasoning in everyday situations.

The first example involves a young child using his informal reasoning to develop new formal mathematical knowledge in order to solve a problem puzzling him: how do you take larger numbers away from smaller numbers in subtraction (i.e. 3 minus 4)? In this case, the second author's young child (roughly age 8) began to discuss the "underground numbers" that were in the elevators in their French apartment building. These were, of course, negative numbers, which he identified as similar to whole positive numbers, but which also had novel features. In this case, the child used his intuitive reasoning about the nature of these underground numbers and recognized that they were a potential solution for the formal mathematical problem that had been puzzling him. Upon his informal model of what the underground numbers was, he was able to construct a formal mathematical model that solved the problem of subtraction a larger number from a smaller.

We draw from the literature for the second example, provided by Lave and colleagues

(Lave, Murtaugh, & de la Rocha, 1984), of a shopper considering the unit price of spaghetti in making their grocery choices. In this example, the shopper is able to negotiate a mathematical solution to an informal everyday problem (in contrast to the informal everyday solution to the mathematical problem in the *underground numbers* examples). The shopper is able to apply her formal mathematical knowledge to an informal situation, which does not necessarily require a mathematical solution. In this case, after considering their previous, less frugal choice of macaroni options (chosen because of space constraints), the shopper begins to discuss the best option for a spaghetti purchase, "But this one, you don't save a thing. Here's three pounds for a dollar 79, and there's one pound for 59…No, I'm sorry, that's 12 ounces. No, it's a savings" (Lave et al., pg. 87). Lave and colleagues, note the multiple calculations embedded within this short moment, which involve estimation and inverse proportional reasoning. For instance, once the shopper recognizes that 12 ounces is less than one pound, they immediately note that there is a savings (for the bigger bag), as the price per unit is higher for the smaller bag.

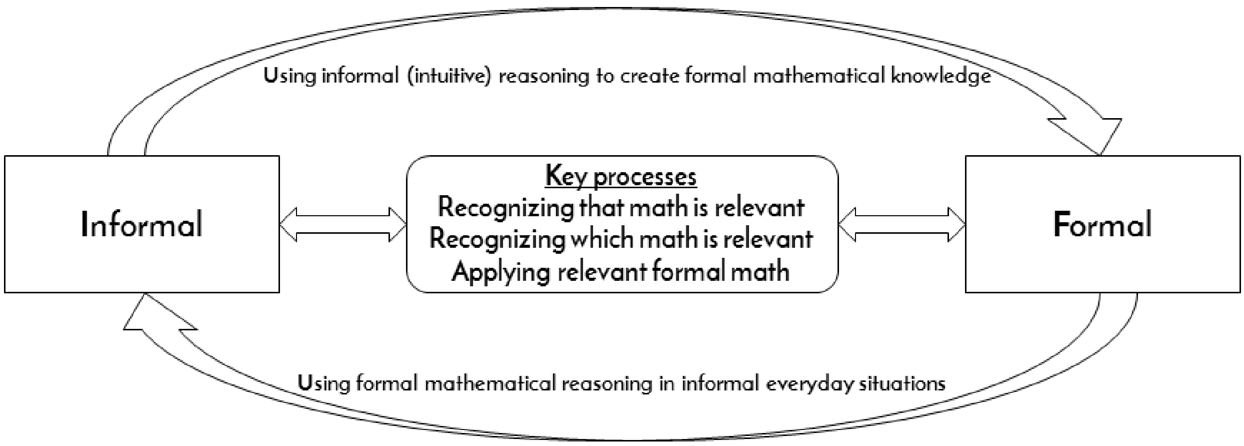

Figure 1: A model bridging informal and formal mathematical reasoning.

We argue that within both cases there are core mathematical processes that occur which support bridging between informal and formal mathematics within the situations, as outlined in Figure 1. First, in both instances the individual must be made aware of the mathematics embedded in the situations. Despite numerical qualities present in both situations, it is not guaranteed that all individuals will sufficiently recognize the mathematics embedded in these situations. As well, the individuals must also recognize what are the most relevant and useful mathematics. There are often competing features of the environment, including multiple mathematical features, and an individual must make a choice in determining which of these aspects their focus and reasoning will most closely consider. Additionally, the individual must also make a connection between the situation and their existing formal mathematical knowledge. This connection to existing mathematical knowledge may take the form of acting as the basis for new informal reasoning (as in the case of the underground numbers) or as the application of this formal knowledge in the informal situation (in the case of the spaghetti purchase).

When is math relevant?

The mathematics embedded in the everyday world is not similarly noticed by all individuals. Instead, there are substantial individual differences in the tendency to pay attention to mathematical aspects in and out of the classroom. These individual differences are positively related to mathematical development from early childhood through secondary school (Hannula-Sormunen, Lehtinen, & Räsänen, 2015; McMullen, Hannula-Sormunen, & Lehtinen, 2017). We argue that in both examples described in the present study, the individual needed to recognize on their own the relevance of mathematics in the situations.

For the young child examining and considering the different labels for the floors in the building, it is entirely possible that these labels (the negative numbers) could be seen as nominal markers that merely represent a way of distinguishing the floors, in particular that they differentiate between the above- and below-ground floors using a “dash”. However, this child made an explicit connection between the numbers and symbols used to denote the floors in the elevator and his knowledge of the formal counting system. Crucially, he extended this connection and recognition of the relevance of mathematical thinking to include new forms of numbers, in this case the negative numbers. In this way, he recognized that there was some explicit mathematical content in these notations that deserved consideration; something that many children of his age might not always do.

In the supermarket example, the mathematical nature of the situation may be more readily recognized due to practical importance of prices. However, as Lave and colleagues show, every individual does not always note the mathematics involved in making purchases as they often consider other factors. In fact, we often ignore the mathematical realities of shopping situations in making our purchasing choices, often for good reasons, such as space constraints. We argue that it was through the own individuals self-initiated actions that mathematics was introduced into their reasoning about the situation.

Our research has shown that those children and students who more readily spontaneously pay attention to mathematical aspects of everyday situations do better with learning formal mathematics (Hannula-Sormunen et al. 2015; McMullen et al., 2017). Crucially, this advantage is not entirely explained by skills with reasoning in informal situations, and appears to truly be about the student’s tendency to notice when mathematics is relevant. Importantly, such tendencies to spontaneously focus on mathematical aspects can be enhanced through social interaction and explicit instruction (Hannula, Mattinen, & Lehtinen, 2005; McMullen, Hannula-Sormunen, Kainulainen, Kiili, & Lehtinen, 2017), suggesting that supporting these tendency can be a positive step in building better bridges between the informal and formal mathematics in and out of the classroom.

Which math is relevant?

Not only do students need to be able to recognize that mathematical aspects are relevant for a situation, they also need to be able to determine which are the *most* relevant mathematical features. While the first process deals with separating the mathematical aspects from the non-mathematical aspects of everyday situations, once it is established that the is math involved in the situation an individual still often needs to negotiate between competing mathematical features. Importantly these different mathematical aspects may have (a) differing levels of relevance (Degrande, Verschaffel, & Van Dooren, 2017) and/or (b) conflicting roles, often involving maladaptive intuitions (Boyer, Levine, & Huttenlocher, 2008).

In the case of our young child exploring what the underground numbers represent, one crucial component of this invention was his recognition that this new number type was similar to, but still was novel from, the whole numbers he was already aware. In understanding that these underground numbers were connected to the continuum of the aboveground numbers in a meaningful way, he also needed to realize that he could not just treat them exactly the same. He needed to inhibit his potential reaction of treating these new numbers exactly the same as the old and recognize that, for example, with underground numbers bigger numbers do not mean a larger magnitude and that with them you can now subtract a larger number from a smaller. In effect, no one told him that the dashes before the numbers had special significance, or that the reverse ordering of the numbers was also important. These new features needed to be identified as core to the situation by himself.

The issue of conflicting mathematical features is even more clear when examining the use of the unit price, and the proportional reasoning that goes along with it, in the example of the shopper's spaghetti choices. There is a very obvious competing mathematical idea, namely the overall price. In order to find the most frugal and cost efficient choice of spaghetti, the individual needed to recognize that it was comparing the prices per unit, and not the overall prices, that was the correct approach to determining the best buy. While this may seem obvious to many frugal shoppers, an abundance of research suggests that many students do not recognize this, even after explicit instruction on relevant topics such as proportions (Van Dooren, De Bock, & Verschaffel, 2010).

Being able to distinguish between multiple competing mathematical features of a single situation may often be a core aspect of bridging the informal and formal mathematics in and out of the classroom. This suggests that mathematics instruction should not simply provide students with mathematical problems that contain features and require reasoning with only a single mathematical skill or type of knowledge, but also present students with tasks that require them to recognize the crucial and most relevant mathematical aspects of a situation.

How is math relevant?

Crucial to the whole endeavor of bridging the formal and informal mathematical worlds, is making an explicit connection with existing formal mathematics knowledge. This involves either drawing on prior knowledge that can be built upon when building up intuitive reasoning (as in the case of the underground numbers) or using the formal mathematical tools that are necessary to solve the informal situation and transcribing this formal mathematical knowledge into action and/or informal reasoning (as in the case of the supermarket). In order to successfully apply their formal mathematical knowledge in novel situations students must still identify which formal skills to apply, especially when formations do not align with traditional mathematical tasks.

In the case of the underground numbers, in order to develop his intuitive reasoning about the underground numbers and their role in subtraction there were two aspects of formal mathematics that the child needed to translate into the situations. First, he needed to explicitly attach his prior knowledge about whole numbers to the floor-level numbers, including knowledge such as the fixed order to the counting sequence and the relation between the order of numbers and their relative magnitudes. Most crucially, the child needed to represent the extant, but problematic, constraint imposed by positive whole numbers – one cannot subtract a larger number from a smaller one. In both cases, this prior knowledge was called upon to create the intuitive understanding that the underground numbers were potential solutions to these bigger minus smaller problems.

In the case of the shopping example, the transcription of formal mathematics onto the everyday situation is precisely the key feature of this activity. It was the act of applying their formal mathematical knowledge in an informal context that is core to this situation. However, it should be noted while the individual in this case was successful in their attempt to model this situation mathematically, this may not always be the case. The crucial feature of the proportional reasoning involved in the situation is that after the individual recognized that the overall size of the 12 ounce package was not one pound, but indeed less than that, the cost per unit actually went up and therefore made the smaller package less frugal. This inverse relation was recognized immediately, but was still founded in a the formal mathematics of inverse proportional relations.

This inverse proportional relation is a complex interaction that many students struggle with in the mathematical classroom context, but can be fluidly dealt with by many without much formal mathematical instruction when embedded in everyday contexts (Nunes, Schliemann, & Carraher, 1993). More explicit instruction involving mathematical word problems that include the messiness and realistic considerations involved in using formal mathematics in everyday life may help support the bridge between the informal and formal mathematics.

EDUCATIONAL IMPLICATIONS AND FUTURE DIRECTIONS

Instructional contexts that aim to bridge students' informal and intuitive understanding of mathematical concepts and formal mathematics are often limited to trivial word problems, which most of the time only superficially place the most recently taught mathematical procedures in highly stylized contexts (Verschaffel et al., 2000). In contrast, the early years of mathematical development offer much more robust opportunities for children to negotiate between formal and informal mathematical reasoning, as the topics are more in line with basic structures of human cognition (e.g. magnitude, natural numbers).

As an alternative to the highly structured and superficial approach of traditional world problems, students may be better served by more rich opportunities to bridge their informal and formal reasoning also in later grades. Mathematics instruction may be able to offer tools to support bootstrapping intuitive reasoning into formal mathematical reasoning and may also be developed to support taking formal mathematical knowledge back out into students' everyday lives, where they can gain the required experiences needed to build up strong conceptual structures. Putting students in situations where informal mathematical reasoning is needed in order identify and solve everyday problems is crucial for building these connections.

Any potential boon for supporting the bridge between informal and formal mathematical reasoning must be able to account for at least some of three aspects of this dynamic relation as described above: recognizing the relevance of mathematics in the situation, recognizing the *most* relevant mathematics, and applying the relevant formal mathematical knowledge in the situation. Others have found success in promoting the recognition of mathematics in non-explicitly mathematical situations (e.g. Hannula et al., 2005), and our research has shown that it is also possible to do so when there are conflicting mathematical features, such as recognizing multiplicative relations when exact number is a foil (McMullen et al., 2017a). Serious mathematical games involving novel, complex, and open problems may prove valuable to fostering the application of prior and relevant formal mathematical knowledge to informal and novel situations (Devlin, 2011).

Examining the connection between informal and formal mathematical reasoning is not a novel pursuit. However, the present study provides a new perspective on the potential for building an accumulative curriculum for mathematics that is strongly supported by this two-way connection between informal and formal mathematical reasoning. It is possible that by supporting applying formal mathematics in everyday informal contexts through explicit instruction, this also leads to better opportunities to bootstrap informal reasoning in the creation of new formal mathematical knowledge. Likewise, supporting the creation of new formal mathematical knowledge by means of informal exploration may lead to better applicability of formal mathematical knowledge in everyday life. Such a positive, iterative, feedback loop could have long-term positive outcomes for all students' mathematical development.

References

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: where young children go wrong. *Developmental Psychology, 44*(5), 1478–90. http://doi.org/10.1037/a0013110

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. National Governors Association Center for Best Practices and the Council of

Chief State School Officers. Washington, DC. Retrieved from http://www.corestandards.org/

Degrande, T., Verschaffel, L., & Van Dooren, W. (2017). Spontaneous Focusing On quantitative Relations: Towards a characterisation. *Mathematical Thinking and Learning, 19*, 260–275. http://doi.org/10.1080/10986065.2017.1365223

Devlin, K. J. (2011). Mathematics Education for a New Era: Video Games as a Medium for Learning. AK Peters Ltd.

Hannula-Sormunen, M. M., Lehtinen, E., & Räsänen, P. (2015). Preschool Children's Spontaneous Focusing on Numerosity, Subitizing, and Counting Skills as Predictors of Their Mathematical Performance Seven Years Later at School. *Mathematical Thinking and Learning, 17*(2–3), 155–177. http://doi.org/10.1080/10986065.2015.1016814

Hannula, M. M., Mattinen, A., & Lehtinen, E. (2005). Does social interaction influence 3-year-old children's tendency to focus on numerosity? A quasi-experimental study in day care. In L. Verschaffel, E. De Corte, G. Kanselaar, & M. Valcke (Eds.), Powerful environments for promoting deep conceptual and strategic learning. (pp. 63–80).

Lave, J., Murtaugh, M., & de la Rocha, O. (1984). The Dialectic of Arithmetic in Grocery Shopping. Everyday Cognition: Its Development in Social Context.

Lehtinen, E., & Hannula, M. M. (2006). Attentional processes, abstraction and transfer. In L. Verschaffel, F. Dochy, M. Boekaerts, & S. Vosniadou (Eds.), Instructional psychology: Past, present and future trends (pp. 39–55). Elsevier.

McMullen, J., Hannula-Sormunen, M. M., Kainulainen, M., Kiili, K., & Lehtinen, E. (2017). Moving mathematics out of the classroom: Using mobile technology to enhance spontaneous focusing on quantitative relations. *British Journal of Educational Technology*. http://doi.org/10.1111/bjet.12601

McMullen, J., Hannula-Sormunen, M. M. M., & Lehtinen, E. (2017). Spontaneous focusing on quantitative relations as a predictor of rational number and algebra knowledge. *Contemporary Educational Psychology, 51*, 356–365.

Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). Street mathematics and school mathematics Cambridge University Press

Purpura, D. J., Baroody, A. J., & Lonigan, C. J. (2013). The transition from informal to formal mathematical knowledge: Mediation by numeral knowledge. *Journal of Educational Psychology, 105*, 453–464. http://doi.org/10.1037/a0031753

Resnick, L. B. (1986). The development of mathematical intuition. In M. Perlmutter (Ed.), Perspectives on intellectual development: The Minnesota Symposia on child psychology (pp. 159–194). Minneapolis: University of Minnesota Press.

Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From Addition to Multiplication … and Back: The Development of Students' Additive and Multiplicative Reasoning Skills. *Cognition and Instruction, 28*, 360–381. http://doi.org/10.1080/07370008.2010.488306

Verschaffel, L., Greer, B., & De Corte, E. (2000). Making sense of word problems. Lisse, The Netherlands: Swets & Zeitlinger.

STUDENTS' SENSE OF BELONGING TO MATHEMATICS IN THE SECONDARY-TERTIARY TRANSITION

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A "sense of belonging to math" (SBM) scale has been shown to predict undergraduate mathematics students' intent to study mathematics in the future. In this study, we use the scale to examine the impact of the transition from secondary school to university on 33 first year undergraduate students' SBM. Using a cluster analysis, we identify three clusters: students in both Cluster 1 (n=21) and Cluster 2 (n=9) display a strong SBM at secondary school. Following the transition, those in Cluster 1 exhibit a decrease in SBM, while those in Cluster 2 show only a marginal decrease. Students in Cluster 3 (n=3) show a strong increase in their SBM, but they started with the lowest SBM initially. From an analysis of interviews with seven of the students, factors that might impact students' SBM during the transition are discussed.

INTRODUCTION

A "sense of belonging to math" (SBM) scale has been shown to predict undergraduate students' intent to study mathematics in the future (Good, Rattan, & Dweck, 2011). A person's SBM relates to whether one feels a member of a mathematical community, and feels valued and accepted by that community. Other factors influencing one's SBM are: affect - the feelings and emotions surrounding learning mathematics; trust that members of the community have one's best interests at heart; and, a willingness to actively participate in the community (Good et al., 2012).

In this paper, we focus on examining the impact that the transition from secondary school to university mathematics has on students' SBM. The students in this study are first-year undergraduates enrolled to a Science programme at a university in Ireland. They are "high-achievers" in mathematics in that most have taken higher level mathematics at school, and have chosen first year university subjects which make them eligible to pursue a mathematics (or related) degree. The structure of the Science programme means that they do not commit to a major until the end of second year - consequently, they can opt out of mathematics at the end of first or second year. Given the emphasis in Ireland on increasing capacity in the mathematics pipeline (Department of Education and Science, 2017), it is important to investigate the effect that the transition to university mathematics has on students' SBM, the factors which may impact it, and in turn, affect a student's decision to continue pursuing mathematics. We address the following research question:

- How does the transition from secondary mathematics to university mathematics, affect high-achieving students' SBM?

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 371-377). Umeå, Sweden: PME.

LITERATURE REVIEW

Good et al. (2011) conceptualise SBM as involving “one’s personal feelings of membership and acceptance in an academic community in which positive affect, trust levels, and willingness to engage remain high” (p. 3). They created and validated a 28-item SBM scale containing five subscales. Two of these relate to feelings of membership of, and acceptance by, one’s mathematical community. As positive emotions towards a subject are likely to be linked to a feeling of belonging, the third subscale relates to affect. The final two subscales relate to trust and a desire to fade. Trusting that peers and teachers/professors in the mathematical community have your best interests at heart, and wanting to actively participate in the community, are likely to contribute to a positive SBM. In a study of undergraduate mathematics students, Good et al. (2011) showed that SBM reliably predicted one’s intention to study mathematics in the future.

Given the ability of the SBM scale to predict one’s intention to study further mathematics, it is important to examine factors that build, or erode, one’s SBM. In a longitudinal study of Calculus students, Good et al. (2011) examined the effect that students’ perceptions of two messages from the environment had on their SBM. The message that mathematical ability is fixed, together with a stereotype message that men have higher ability than women in mathematics, was found to adversely affect the SBM of women over the semester but not men. However, female students who internally believed that mathematical ability could be improved with work, seemed immune to the stereotype message and their SBM remained high.

A person who believes that ability is something you cannot change, is said to have a fixed mindset, or an entity theory of intelligence, while someone who believes that ability is malleable is said to have a growth mindset, or an incremental theory of intelligence (Dweck, 2006). Implicit theories of intelligence and their impact on achievement, learning, motivation, and resilience have been studied extensively by Dweck and colleagues (see for example Dweck, 2006). One particular study examined how implicit theories of intelligence impacted seventh grade students’ mathematical transition to middle school (Blackwell, Trzesniewski, & Dweck, 2007). It is common for seventh grade students’ mathematics grades to drop, and this was observed in the control group. However, in the group of who received the incremental theory intervention, the decrease in results was reversed and by the end of the year had almost returned to the levels reported at the beginning. A growth mindset seemed to provide students with the resilience to navigate this mathematical transition.

SBM is a complex construct and implicit theories of intelligence are just one of many factors that may affect a student’s SBM at the secondary or university level. For example, Boaler (2002) argues that the mathematical practices students engage in, shape not only their mathematical identity, but the “disciplinary relationships” (p. 119) they develop. The mathematical transition from secondary to university mathematics is similarly complex and multi-faceted, involving transitions at the individual, socio-cultural, and institutional levels, with students facing difficulties in many areas from how they think about, and communicate, the subject, to grappling with the different didactical contracts of school and university (Gueudot, 2008).

Mathematical community on the SBM scale is described as "the broad group of people involved in that field, including the students in a math course" and participants are informed that they could consider themselves a member "by virtue of having taken many math courses, both in highschool and/or [university]" (Good et al., p.18). At university, one would expect to find a community of practice of mathematicians (Wenger, 1998) and ideally the undergraduate mathematics student should be a legitimate peripheral participant of this community (Lave and Wenger, 1992). In a study of twelve first year mathematics undergraduates, Solomon (2007) found that a student identity of apprentice to this community of practice was rare. Experiencing mathematics as rules to be followed without understanding, not feeling ownership over the mathematical knowledge, and feeling vulnerable to failure due to fixed-ability beliefs about mathematics, all contributed to feelings of not belonging.

METHODOLOGY

The students in this study were enrolled to the first year of a Science degree programme at a university in Ireland in 2014-15. In first year, students are free to pursue modules in their area(s) of interest, for example, in biology, chemistry, mathematical sciences, physics, and/or mathematics education. In second year, they choose more specialist modules, and at the end of this year they commit to one of twenty-six degree majors. Students registered to the first-year mathematics education module (n=40) were invited to take part in the study and 33 participated, of which 20 were female and 13 were male. All but three of these were enrolled to modules that made them eligible to continue the study of mathematics in second year if they wished.

In Ireland, for the final two years at secondary school (17-19 years), students study the Leaving Certificate Curriculum and sit the terminal state examination in six to eight subjects. These examinations are high-stakes as the total number of "points" received for a maximum of six subjects determines students' entry to university. Almost all students take mathematics which is offered at three levels: Foundation, Ordinary and Higher. In this study, 31 students had taken higher level mathematics, with the remaining two entering university via a different route. For this reason, the majority can be considered as "high-achievers" in mathematics.

A survey was administered in class towards the end of their first year. Students were asked to complete the SBM scale found in Good et al. (2012) twice – first as it related to when they were "in a maths setting at school", and secondly, as it related to when they "are in a maths setting at university", and state what subjects they intended to purse in Stage 2. They were also invited to participate in follow-up interviews. Seven students (four female and three male) agreed and these were conducted a few weeks later. The interviews were aimed at gaining more insight into the factors impacting a students' SBM. In relation to both school and university mathematics, students were asked about their experiences of studying mathematics, how confident they felt, and whether they felt a sense of belonging to the mathematics community.

The SBM scale contains both negatively and positively worded items on a 5-point Likert scale (1=strongly disagree; 5=strongly agree). For the purpose of quantitative analysis, the negative items were reversed and the internal consistency of the SBM scale was investigated for each factor (membership, acceptance, affect, desire to fade and trust) as well as the composite SBM. As in Good et al. (2012), the composite SBM was created through developing subscale averages for each of the five factors and then averaging them to achieve an overall SBM score at both levels. SBM achieved a Cronbach alpha of=0.83 for school and α=0.87 for university.

To analyze the change in students' SBM in the transition from secondary to university, students' subscale averages for each factor for both secondary school and university were clustered using mixed-model clustering (Scrucca et al., 2016). mclust is more flexible than K-means clustering as it allows for varying volume, orientation, and shape of clusters. Cluster analysis, based on the Bayesian Information Criterion, identified a three-cluster solution as the optimal solution.

Finally, the interview audio-recordings were transcribed and analyzed thematically (Braun & Clarke, 2006).

RESULTS

On analyzing the change in students' SBM from secondary school to university, cluster analysis identified a three-cluster solution as optimal. Each cluster can be discussed in terms of the change in students' SBM over the transition (see Figure 1).

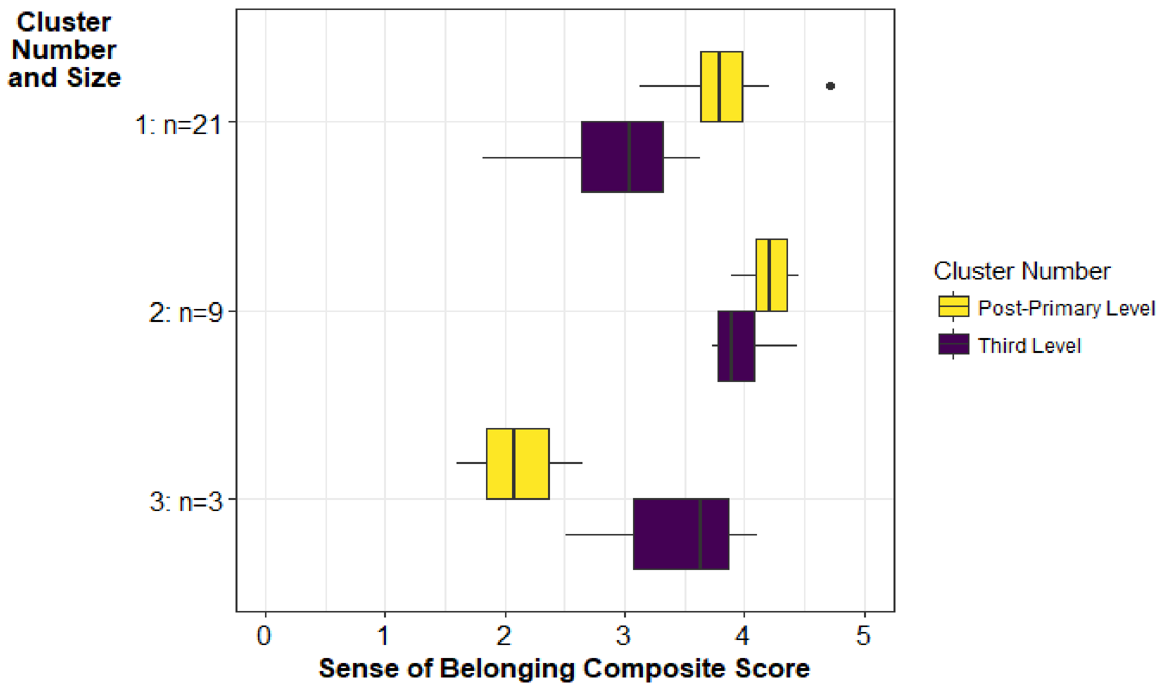

Figure 1: Boxplot of students' composite sense of belonging for each cluster and for secondary school and university

Students in the first cluster (n=21) display a strong SBM at secondary school that decreases following the transition. In comparison, students in the second cluster (n=9) have a strong SBM at school, however, this marginally decreases following the transition. Students in the third cluster (n=3) show a strong increase in their SBM, but these students started with the weakest SBM initially. In addition, two of these students had chosen subjects which made them ineligible to study mathematics in second year.

In the following academic year 2015-16, 8 of the 9 students in Cluster 2 and 10 of the 21 students in Cluster 1 were studying mathematics modules. No-one from Cluster 3 studied mathematics in Stage 2.

Of the seven students interviewed, four were from Cluster 1 (Grace, Joe, Julie and Lucy), two from Cluster 2 (Kate and Sean), and one was from Cluster 3 (Charlie). Grace, Lucy, Kate and Sean all continued to study mathematics in second year. (Pseudonames were assigned to the participants.)

DISCUSSION AND CONCLUSION

The secondary-tertiary transition is complex and involves many types of transitions. For this reason, Gueudot (2008) suggests that research dealing with issues faced by students in the last two years of school, and the first two years of university, may all contribute to our effort to better understand the transition. SBM is also complex, comprising of one's feelings of membership and acceptance by a mathematical community, trust that the community has your best interests at heart, affect, and one's willingness to activity participate in the community (Good et al., 2012). Even the notion of community is complex, as students may belong to several, and sometimes conflicting, local communities of practice (Solomon, 2002). It is with this backdrop that we discuss our findings, and attempt to gain some insight into factors that may affect mathematical high-achievers' SBM in the secondary-tertiary transition.

In terms of Cluster 3, two of the students were enrolled to two, core "general" mathematics modules which made them ineligible to study mathematics in second year. The third had not taken the Irish Leaving Certificate. None continued to study mathematics in second year. It is interesting to examine Charlie's experience of studying higher level mathematics at school, which ultimately turned him away from the subject. He described his teacher as "absolutely disastrous" and said: "he didn't know any of our names, even in sixth year". In his final year, he, along with "half my class", had to pay for private tuition. Therefore, while his SBM did exhibit the largest overall increase in going from school to university, he was starting from a very low base and at university, just wanted to do the core mathematics "to get it out of the way". It is not surprising that he did not wish to study mathematics at university.

In relation to Clusters 1 and 2, it is not surprising that students' SBM at secondary school was high. Most of these students had studied higher level mathematics, had gained entry to a highly competitive science programme, and 29 of the 30 had voluntarily chosen higher level mathematics modules in their first year at university, making them eligible to study mathematics in second year. From the interviews of the six students from Clusters 1 and 2, a picture of successful, top-set students emerges. They came from schools where teachers knew them well and knew what they were capable of, and on occasions motivated them to do better. Most described themselves as being in the top higher mathematics class, and being confident at mathematics.

When asked about their experiences of studying mathematics at university most mention: the impersonal large lecture setting and adapting to the resulting teaching style; the increased level of difficulty of mathematics; and, working independently. One could see how these factors might erode a student's SBM as they progress through first year, which makes the cases of Kate and Sean interesting. Despite experiencing these challenges their SBM decreased only marginally. Due to space constraints, we will briefly highlight two possible contributory factors in the case of Sean.

Sean was the only student, who when asked about a sense of community at university, said there "is definitely" a "maths community" and gave a description most closely resembling that of a community of practice of mathematicians (Wenger, 1998). He explained that there are "so many more lecturers" compared to only three or four mathematics teachers at school. He is a member of the student *Maths Society* and has been to some of their events. In terms of participation, he said he has been settling in and has been a "bit quiet" but is "determined to get more involved next year". He has also "chatted" to one of his lecturers a few times after class, and visited another in his office a few times. Sean's recognition of the community of mathematicians, and desire to become more involved, suggests an identity of legitimate peripheral participant, which can be rare among first year mathematics undergraduates (Solomon, 2002).

Secondly, Sean exhibits a growth mindset (Dweck, 2006). At school, he embraced challenging problems: "I actually love to sit there and like even if it took two hours, just to sit there and try and get my head around doing it". At university, despite struggling to understand some of the mathematical concepts, and feeling "confused" and "frustrated", his reaction is to acknowledge that it is all new, seek help, and put in the work. When speaking about Analysis he admitted "for the first time ever in a maths exam, I am genuinely one hundred percent frightened". However, he explains that he is "just not used to" Analysis and has actively sought help from the university *Maths Support Centre* and looked up videos on *Khan Academy*. He acknowledged the amount of independent work he was putting in: "There has been an awful lot more of my own work going in" but he felt that was "natural" at university. His growth mindset seems to have given him the resilience to persist through challenges (Yeager & Dweck, 2012) and help him navigate the difficult mathematical transition (Blackwell et al., 2007).

In conclusion, we have used the SBM scale (Good et al., 2011) to examine the impact of the secondary-tertiary transition on students' SBM. Further qualitative analysis and research is required to better understand the many factors that might erode, or protect, a students' SBM during this transition.

Acknowledgements

We would like to thank Dr Anthony Cronin for conducting the seven interviews.

References

Boaler, J. (2002a). The development of disciplinary relationships: Knowledge, practice, and identity in mathematics classrooms. In A. D. Cockburn & E. Nardi (Eds.), *Proc. 26th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 113-120). Norwich, UK: PME.

Blackwell, L. A., Trzesniewski, K. H., & Dweck, C. S. (2007). Theories of intelligence and achievement across the junior high school transition: A longitudinal study and an intervention. *Child Development, 78,* 246-263.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology, 3*(2), 77-101.

Department of Education and Skills (2017). *STEM education policy statement 2017-2026.* Retrieved from Department of Education and Skills website: https://www.education.ie/en/The-Education-System/STEM-Education-Policy/stem-education-policy-statement-2017-2026-.pdf

Dweck, C. S. (2006). *Mindset*. New York, NY: Random House.

Good, G., Rattan, A., & Dweck, C. S. (2011). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology, 102*(4), 700-717.

Gueudet, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics, 67,* 237-254.

Lave, J., & Wenger, E. (1992). *Situated learning: Legitimate peripheral participation.* Cambridge, UK: Cambridge University Press.

Scrucca, L., Fop, M., Murphy, T. N., & Raftery, A. (2016). mclust 5: Clustering, classification and density estimation using Gaussian finite mixture models. *The R Journal, 8*(1), 205-233.

Solomon, Y. (2002). Not belonging? What makes a functional learner identity in undergra-du-ate mathematics. *Studies in Higher Education, 32*(1), 79-96.

Wenger, E. (1998). *Communities of practice.* Cambridge, UK: Cambridge University Press.

Yeager, S. C., & Dweck, C. S. (2012). Mindsets that promote resilience: When students believe that personal characteristics can be developed. *Educational Psychologist, 47*(4), 302-314.

THE PROFESSIONAL, PEDAGOGICAL LANGUAGE OF MATHEMATICS TEACHERS: A CULTURAL ARTEFACT OF SIGNIFICANT VALUE TO THE MATHEMATICS COMMUNITY

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This paper draws on a project involving nine mathematics communities internationally that set out to identify the familiar, professional, pedagogical vocabulary in use by middle school mathematics teachers. The national research teams comprise both academic researchers and experienced teachers and each lexicon identified the actual terms that teachers use when describing the phenomenon of the middle school mathematics classroom. Each such lexicon can be thought of as a cultural artefact of the mathematics teaching community in which its practitioners name the valued, pedagogical practices in their respective world. The documentation of these lexicons has significant practical value to each participating community and can also be used for the study and promotion of reflective practice of teachers.

INTRODUCTION

Researchers and mathematics teachers in teaching communities of Australia, Chile, China, Czech Republic, Finland, France, Germany, Japan and the USA set out to document the lexicons employed in nine countries for the description of middle school mathematics as part of the International Classroom Lexicon Project. These are the actual terms by which students, teachers and researchers name the objects in their respective worlds and constitute empirical rather than theoretical frameworks through which phenomena of the mathematics classroom can be described. These assemblages of local terms (in the original language), used to identify classroom practices, reflect the well-established pedagogical traditions by which each of the participating communities describes the activities of the mathematical classroom. In this paper, we will give an overview of the English-language lexicon of Australian teachers, as an example, and reflect on its significance as a cultural artefact for its teaching community. We will also show, with illustrative examples from other non-English languages, that these lexicons can be viewed as indicative of forms of conventionalised practice; products of the pedagogical history of each community; and, tools for the study and promotion of reflective practice.

THE PROFESSIONAL VOCABULARY OF TEACHERS

The "English-language" professional lexicon available to teaching communities in the USA (and Australia) has been contrasted unfavourably with what is considered a well-articulated pedagogical naming system in China and with the strong traditions in

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 379-386). Umeå, Sweden: PME.

Japan of educators and teachers discussing research lessons (Lesson Study) (Lampert, 2000).

In their cross-professional study of clergy, teachers and clinical psychologists, Grossman and her colleagues found that

> "among our trio of professions, this language of practice seems particularly well-developed in clinical psychology but less developed in teaching" (Grossman, Compton, Igra, Ronfelt, Shahan & Williamson, 2009, p. 2075).

Lortie's much earlier observation, in his social portrait of the 'Schoolteacher' reported a lack of 'technical language' in teaching (Lortie, 1975). Whilst Lampert's later observation also concurred that "no professional language for describing and analysing [teaching] practice has developed in the United States" (Lampert, 2000, p. 90). The absence of a "grammar of practice" (Grossman et al., p. 2069) has implications for the preparation of professions. Without the existence of a language and structure for describing pedagogical practice the provision of learning opportunities for novices is limited (p. 2075). Yet teachers do talk about their practice and this project set out to document the lexicon in current use by each community and to compare them.

The lexicons documented for this international study were purposefully chosen to initiate cross-cultural dialogue in part to make amends for some of the less beneficial consequences of the internationalisation of English. The consequence of the function of English as the preferred speech of the international community includes the amplification of constructs and theories articulated in English to the exclusion of constructs only available in other languages. The International Classroom Lexicon Project seeks to correct this restriction of linguistic resources and the associated limiting of our capacity to identify, recognise and conceptualise pedagogical and didactical phenomena by expanding the lexical terms available to educators internationally, and thereby offering new possibilities for practice and access to ideas and approaches. One key point must be made: the intention was (and is) to document existing professional lexicons in use by different communities of mathematics teachers, not to construct a single hybrid lexicon. We would contest the viability of a single international composite lexicon as inevitably discarding much of the associative richness only available separately to each language group. But we would argue that much can be learned from the comparison of the emphases and structure of the various lexicons.

> Of all the tools for cultural and pedagogical intervention in human development and learning, talk is the most pervasive in its use and powerful in its possibilities (Alexander, p. 92).

THE RESEARCH DESIGN FOR DOCUMENTING THE LEXICONS

The research teams were tasked with addressing the question: *What are the terms that teachers use to describe the phenomena of the middle school mathematics classroom?* In order to do so a project-wide stimulus package of nine middle school mathematics lessons, a key catalyst for data generation, was compiled to include a lesson from each participating community (see Figure 1). This collection of lessons, presenting a variety of instructional approaches and classroom settings, served to stimulate thinking about the terms used by teachers to describe the phenomena of the middle school mathematics classroom.

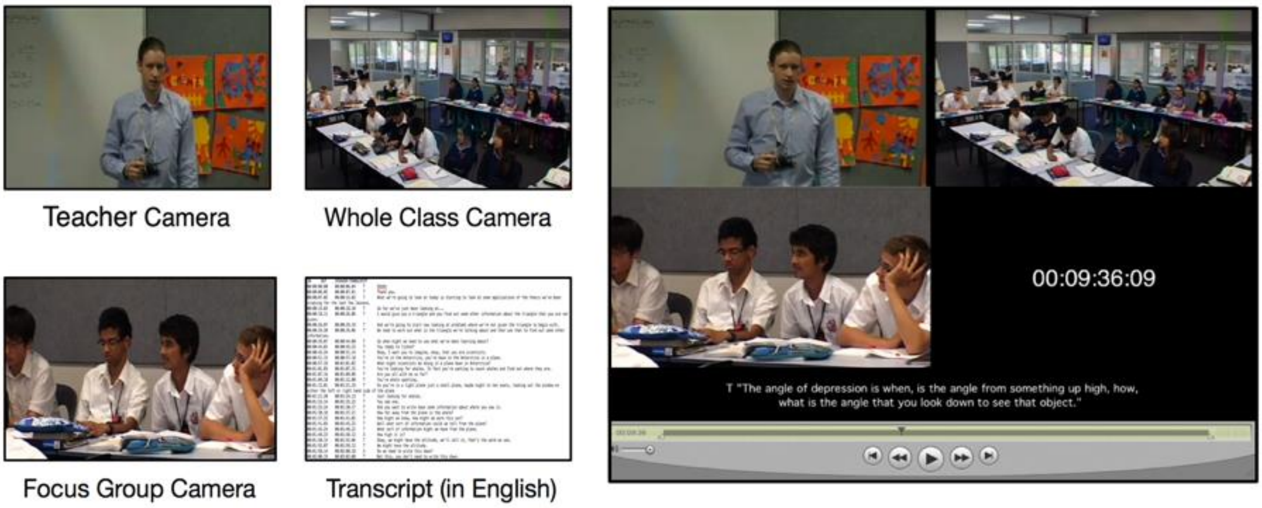

Figure 1. An example of the video material included in the stimulus package

The very general prompt *"What do you see that you can name?"* was crafted to help stimulate thought about the video and to not place restrictions on what could be named. The use of video excerpts of classroom practice in very different school settings in different countries was intended to stimulate thinking about possible terms for inclusion in the lexicon whether the term (activity, characteristic) was present in the video or not. The essential intention was to record words or short phrases that were familiar and in use by teachers within each participating community with a consistent and agreed meaning.

Operational definitions were developed for the initial set of candidate terms. The essential set of elements included: the named term (in the original language), a description, examples, and non-examples (in the original language and in English). The terms and operational definitions of the proposed lexicons were subjected to a local validation process by each national team. This process assisted in investigating the extent to which the local community of mathematics education researchers (and, in some cases, their educational colleagues) would endorse the listed terms and any emergent organisational structure, as well as the descriptions, examples and non-examples.

Following local validation, national surveys were subsequently developed to collect information about teachers' level of familiarity with each of the terms, the extent to which they endorsed the descriptions, examples and non-examples, and the frequency with which they used the terms (or phrases) in conversation with their colleagues. Opportunities for commenting on the clarity and appropriateness of the descriptions and the examples and non-examples for each term were also provided.

THE AUSTRALIAN LEXICON

The Australian National Lexicon consists of 61 terms that are considered familiar by teachers in the mathematics education community (See Figure 2 for a sample).

Guiding	The teacher offers advice or suggestions in the form of questions or comments to assist students in completing and solving tasks.	For example: • A teacher asks, "What else might you do?" *Non-example:* • *"Try multiplying."* • *The teacher gives step by step instructions.*
Justifying	An activity undertaken by the teacher or students that involves expressing why particular mathematical processes, solutions or theories work and providing evidence.	For example: • The teacher encourages students to explain why a mathematical generalisation holds true. *Non-example:* • *The teacher makes a statement about how a particular mathematical idea is true in all cases but gives no explanation related to how or why this is so.*
Practising	The activity of repeating a procedure for the purpose of improving efficiency or accuracy in its use.	For example: • A student solves ten consecutive tasks all involving the addition of fractions. • A student works through the problems on past exam papers. *Non-example:* • *A student attempts to make use of the property of similar triangles in a real-world context for the first time.*

Figure 2. A sample of terms and operational definitions from the Australian Lexicon

The terms are distributed across five categories as follows: Administration (8 terms); Assessment (10 terms); Classroom Management (5 terms); Learning Strategies (27 terms) and Teaching Strategies (49 terms). Some of the terms appeared in more than one category; 23 terms belong to both the Learning and Teaching Strategies categories (see Figure 3 for a sample).

Administration	Assessment	Classroom Management	Learning Strategies	Teaching Strategies	Learning Strategies	Teaching Strategies
Assigning Homework	Correcting	Giving Praise	Answering Questions	Guiding	Questioning	
Handing Out Equipment	Monitoring	Positive Reinforcement	Practising	Whole Class Discussion	Reflecting	

Figure 3. A sample of operational definitions organised by category

An interesting feature of the lexicon is that few terms reveal a singular pedagogical intention or purpose for engaging in the particular instructional practice. For example, the term *Questioning* might be used to review a homework task, elicit prior knowledge or collect student approaches for solving a worded problem. This attribute of many of the terms of the Australian Lexicon might be seen either as inclusiveness or as lack of precision. Another feature of the Australian Lexicon is the high prevalence of gerunds (noun/verbs) (*Correcting, Guiding, Practising, Justifying, Guiding, Questioning, Reflecting*). Such terms give a sort of dynamism to the Australian Lexicon.

The results of the national survey indicated high familiarity with the term names. However, there appear pairs of terms with similar meanings that teachers found more difficult to distinguish one from the other. These include, for example, the terms *Guiding* and *Scaffolding* as well as the terms *Recapping* and *Reviewing*. This insight into teachers' familiarity, use and understanding would assist the community in developing a more precise and practical understanding not only of the language but the practices themselves.

In documenting the Australian Lexicon public recognition is given to the professional vocabulary by which the community of mathematics teachers conceive, implement and reflect on their teaching. The structure of each lexicon offers insights into the way in which the teaching community conceptualizes and organizes its practice and is a significant and valuable resource for the promotion of reflective practice by teachers.

LEXICONS AS FRAMEWORKS

The lexicons documented in this project constitute empirical rather than theoretical frameworks through which phenomena of the mathematics classroom can be described. In the following section we will show how these lexicons can be viewed as: a) indicative of forms of conventionalised practice; b) products of the pedagogical history of each community; and c) tools for the study and promotion of reflective practice among teachers. As such, there is great potential for insight into the evolution of mathematics classroom practice in several distinct cultures (as articulated in the associated languages).

A. Indicative of forms of conventionalised practice

The nomenclature of classroom phenomena, documented by the lexicon, identifies practices that have become conventionalised. That is, the practice named by a particular lexical item is familiar to teachers and appears well-understood within that community without the need of a formalised description. However, what is considered a conventionalised and thus a named practice in one community might be unnamed in another. Consider the following examples:

i. Kikan-Shido

One of the most immediately familiar events in mathematics classrooms around the world is that moment when the teacher walks around the room after having set student work. The Japanese community refers to this classroom event as *Kikan-Shido*, literally translated as *Between Desks Instruction*.

> Our use of 'Kikan-Shido' honours the existence in one language of an established term that succinctly encapsulates an activity that could only be described in English by an extended phrase or lengthy definition. The utilisation of such terms conforms to a tradition that has seen 'déjà vu' and 'Schadenfreude' assimilated into English usage for precisely the same reasons. (O'Keefe, Clarke & Xu, 2006, p. 76)

Whilst the form of this practice is familiar, it remains unnamed in the Australian national lexicon, for example.

ii. Warm-up

In an earlier analysis of data from the Learner's Perspective Study (LPS), Mesiti and Clarke (2006) examined the beginnings of lessons with respect to form and function in eight classrooms from the USA, Australia, Japan and Sweden. This phase of the lesson involved the first ten minutes and was made up of a sequence of activities of which some were customary in all classrooms whilst others were particularly prevalent in one teaching community only. In this previous study, we found the focusing activity referred to as *warm-up* as particularly prevalent in the US classrooms. This named activity was familiar and the lexical term *warm-up* was used by both teachers and pupils, to the extent that if the activity were omitted, some students inquired "We're not doing our warm-up?" (US1-L02. Every such named activity is reflective of local pedagogical history, and can be thought of as having arisen as meeting some instructional need. For example, an evident consequence of the use of *warm up* was the students' immediate engagement in lesson-relevant activity. Minimum teacher direction was required for students to begin working: "Okay guys let's go ahead and get started on today's warm-up," (US1-L01).

iii. Bansho

Considered, by Japanese teachers, to be an essential and important technique was *Bansho*, or the *effective use of the chalkboard* (Shimizu, 2007). Underlying this activity, is the purpose of maintaining on the board the written progression of the entire lesson. That is, by minimising the need to erase anything that was recorded at the board, students and teachers alike have access to a visible record suitable for reflection concerning the mathematical progression of the lesson. Anticipated in teachers' lesson plans, the board is usually entirely filled with a structured array of problems, student solutions and teacher notes about their solutions.

B. Products of the pedagogical history of each community

The language by which teachers name the practices they orchestrate also reflects cultural-historical origins. Let's consider two terms, the Japanese term *Matome* and the Australian term *Differentiating*:

i. Matome

Matome refers to a teacher-orchestrated discussion that draws together the major conceptual threads of a lesson or extended activity, most commonly a summative activity at the end of the lesson (Shimizu, 2006). This term names an activity that has been refined and elaborated over time as an essential component of accomplished practice and is an essential element of a professional vocabulary of teachers. The term *Matome* is also the name given to the last line of a haiku poem, where it serves the same synthetic purpose: the bringing together of key points into a crystallised whole.

ii. Differentiating

The lexical term *Differentiating* (in this context, the instructional accommodation of student difference) was proposed and endorsed as a legitimate term for inclusion in the Australian lexicon. However, one of the team members remarked that this practice was once referred to as "catering for student individual differences." It appears that this term, in the Australian lexicon, has come to replace the previous term. It is interesting to speculate what difference in pedagogical orientation is encapsulated in the shift from "catering for difference" to *Differentiating*. While the more contemporary term may sound more concerned with distinguishing between students with different needs than with meeting those needs, it appears that the spirit of its use is highly similar to the original form. Noting how and when a term makes the leap from the teachers' staffroom to more formal curriculum and policy documents might help us identify mechanisms by which an educational reform might be more effectively and efficiently implemented.

C. Tools for the study and promotion of reflective practice

Consider the professional language of French teachers. Included are two constructs, *Mise en commun* and *Bilan* that both relate to the lesson phase of summative whole class discussion. *Mise en commun* can be described as "a whole-class activity in which the teacher elicits student solutions for the purpose of drawing on the contrasting approaches to synthesise and highlight targeted key concepts" (Clarke, 2010, p. 6) whilst *Bilan,* similarly, is also a whole-class discussion that identifies and synthesises the main points of the mathematical activity of the lesson. The distinguishing feature is whether the discussion synthesis was orchestrated by the students (*Mise en commun*) or by the teacher (*Bilan*). These two constructs in the French lexicon equip the teacher to reflect on these discussion practices because it makes the distinction between student summative synthesis from teacher summative synthesis. These constructs operate as reflective tools and are unavailable to teachers whose lexicon fail to recognise this distinction.

CONCLUSION

The primary intention of the International Classroom Lexicon Project was to document the national lexicons of middle school mathematics teachers from Australia, Chile, China, Czech Republic, Finland, France, Japan and the USA. These national lexicons represent the familiar constructs that make up the professional vocabulary used by teachers, in discussion with others in relation to their classroom practice.

One local intention of the Australian research team was to provide insight into the naming system employed by Australian mathematics teachers in relation to their classroom practice. From this foundation, we hope to inform efforts to better equip contemporary mathematics teachers with a sophisticated lexicon to shape their professional practice.

The focus of this paper has been on the implications of such lexicons as cultural artefacts and, potentially, as empirically-constituted bases for analytical frameworks. The detailed discussion of individual lexical terms is intended to illustrate how these lexicons can serve as representative of forms of conventionalised pedagogical practice with cultural and historical characteristics. The documentation of these lexicons has significant practical value to each participating community, to the international mathematics education community for the access then encapsulated wisdom of centuries of pedagogical history and can also be used for the study and promotion of reflective practice of teachers.

Acknowledgements

The project involves an international community of classroom researchers led by: David Clarke (project director), Carmel Mesiti (project manager), Valeska Grau-Cardenas, Cao Yiming, Jarmila Novotná, Alena Hošpesová, Markku Hannula, Fritjof Sahlström, Michèle Artigue, David Reid, Christine Knipping, Yoshinori Shimizu and Miriam Sherin. The project was established with a Discovery Grant from the Research Council of the Australian Government (ARC-DP140101361).

References

Alexander, R. (2008). *Essays on pedagogy*. NY: Routledge.

Clarke, D.J. (2010). The Cultural Specificity of Accomplished Practice: Contingent Conceptions of Excellence. In Y. Shimizu, Y. Sekiguchi, & K. Hino (Eds.). *In Search of Excellence in Mathematics Education - Proceedings of the 5th East Asia Regional Conference on Mathematics Education (EARCOME5)* (pp. 14-38). Tokyo: Japan Society of Mathematical Education.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P.W. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record, 111*(9), 2055-2100.

Lampert, M. (2000). Knowing teaching: The intersection of research on teaching and qualitative research. *Harvard Educational Review, 70*(1), 86-99.

Lortie, D. C. (1975). *Schoolteacher*. Chicago: University of Chicago Press.

Mesiti, C., & Clarke, D. J. (2006). Beginning the lesson: The first ten minutes. In D. J. Clarke, J. Emanuelsson, E. Jablonka & I. A. C. Mok (Eds.), *Making connections: Comparing mathematics classrooms around the world* (pp. 47-71). Rotterdam: Sense Publishers.

O'Keefe, C., Clarke, D. J., & Xu, L. (2006). Kikan-Shido: Between desks instruction. In D. J. Clarke, J. Emanuelsson, E. Jablonka & I. A. C. Mok (Eds.), *Making connections: Comparing mathematics classrooms around the world* (pp. 73-105). Rotterdam: Sense Publishers.

Shimizu, Y. (2006). How do you conclude today's lesson? The form and functions of 'matome' in mathematics lessons. In D. Clarke, J. Emanuelsson, E. Jablonka & I. A. C. Mok (Eds.), *Making connections: Comparing mathematics classrooms around the world* (pp. 127-145). Rotterdam: Sense Publishers.

Shimizu, Y. (2007). How do Japanese teachers plan and structuralize their lessons. In M. Isoda, M. Stephens, Y. Ohara & T. Miyakawa (Eds.), Japanese lesson study in mathema-tics: Its impact, diversity and potential for educational improvement (pp. 64-67). Singapore: World Scientific Publishing.

YOU SEE (MOSTLY) WHAT YOU PREDICT: THE POWER OF GEOMETRIC PREDICTION

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We consider geometric prediction (GP), as a mental process through which a figure is manipulated, and its change imagined, while certain properties are maintained invariant. In this report on a recent study, we concentrate: 1) on capturing processes of GP before explorations are carried out in a dynamic geometry environment (DGE), to gain insight into possible characteristics of such processes; 2) on possible implications it can have in a subsequent process of dynamic exploration of a DGE figure, in particular in the solver's interpretation of feedback from the DGE.

INTRODUCTION AND THEORETICAL GROUNDING

As mathematics students, it has probably happened to many of us to listen to our professors quickly reason about a geometric configuration, reaching an "obvious" conclusion that could "clearly be seen" on the paper or the board in front of them. With uneasiness, and some embarrassment, we would nod and run to our room to try to see what was supposedly so clear, and maybe then try and prove it.

In this paper, we focus on *geometric prediction (GP)*, a mental process through which a figure is manipulated, and its change imagined, while certain properties are maintained invariant (Miragliotta, Baccaglini-Frank & Tomasi, 2017; Mariotti & Baccaglini-Frank, in press). In the vignette above, the expert geometry professors (for the sake of this paper let us think of Euclidean Geometry) are so skillful in carrying out GPs that they conceive new configurations or geometrical objects that others cannot even see. We are interested in gaining insight into processes through which GPs are accomplished, and how these predictions might condition subsequent explorations.

Indeed, on the one hand, mathematics educators have recognized the importance of helping students learn to think like mathematicians; for example, Cuoco, Goldenberg and Mark have proposed to organize curriculum around *mathematical habits of mind*. Among these, we find "visualizing" and "tinkering" (Cuoco, Goldenberg & Mark, 1996). Within the types of visualization discussed, the researchers include:

> reasoning about simple subsets of plane or three-dimensional space with or without the aid of drawings and pictures. […] Visualizing change. Seeing how a phenomenon varies continuously is one of the most useful habits of classical mathematics. Sometimes the phenomenon simply moves between states […]. Other times one thing blends into another […]. This habit cuts across many of the others […] (ibid, pp. 382-383).

These seem to be well aligned with Presmeg's description (2006), which is

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 387-394). Umeå, Sweden: PME.

> taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics (ibid, p. 206).

Tinkering is described as being “at the heart of mathematical research” and it consists in “taking ideas apart and putting them back together”, and asking “what happens if…” (Cuoco et al., 1996, p. 379). Our notion of GP seems to have a lot in common both with the forms of visualization described and with the idea of tinkering, which justify its educational significance.

On the other hand, studying GP and its relationships with dynamic geometry environments (DGEs) seems especially important since this sort of technology can affect conceptualization and problem solving processes in Mathematics (e.g., Arcavi & Hadas, 2000). We believe GP to be key in geometrical problem solving, and we believe it can be trained, possibly using the support of a DGE (Mariotti & Baccaglini-Frank, in press). Moreover, we conjecture that once a GP is carried out, it can affect the subsequent process of dynamic exploration in a DGE, influencing what the solver can or cannot “see”. Exploring this conjecture has particular educational significance.

In the study we report on here, we concentrate firstly on capturing processes of GP before dynamic explorations are carried out, to gain insight into possible characteristics of such processes; and, secondly, on possible implications it can have in a subsequent process of dynamic exploration of a DGE figure, in particular on the solver’s interpretation of feedback from the DGE.

RESEARCH QUESTIONS AND METHODOLOGY

The data we present are part of a doctoral research project on geometric prediction for which 15 geometrical problems were designed and proposed to 18 Italian high school students (ages 14-18), undergraduates and graduate students majoring in mathematics (ages 19-33), during the months of November and December 2017. The problems were designed to elicit processes of GP and they were used within clinical interviews conducted by the first author of this paper. Although all data has not yet been thoroughly analyzed we wish to use some of it to present a preliminary report on the following questions: 1) When GP is used (either spontaneously or prompted by the interviewer), by what kinds of verbal or gestural descriptions is it accompanied? 2) Once a GP is advanced and the student is given the opportunity to interact with a DGE figure corresponding to the configuration reasoned upon, how does the student interpret the feedback from the DGE? Does such feedback lead him/her to change the GP?

All interviews were carried out in a quiet room and each student spent 60 minutes with the interviewer and worked through as many interview problems as they could.

The “locus of P” problem

The following task is a variation of a geometric problem described by De Finetti (1967). The task used in this study is composed of two parts. The first one is:

“Read and perform the following step-by-step construction: fix two points A and B; connect them with a segment AB; choose a point P on the plane; connect A and P with a segment AP; construct M as the midpoint of AP; construct the segment MB. Imagine moving the point P. If the length of the segment MB must always be constant, what can you say about the point P?”

The step-by-step construction could be accomplished with paper and pencil (obtaining a construction like that in Fig. 1a); the question about P one was proposed first mentally, then solvers were offered the possibility of drawing ideas on paper.

Once a construction was made in a DGE (*GeoGebra*) and the solver had proposed a solution or stated that s/he was not able to find one, part two of the task was given. The interviewer would ask the solver to move P in the DGE figure, consistently with her/his prediction, or else to explore the dynamic figure to help reach a solution.

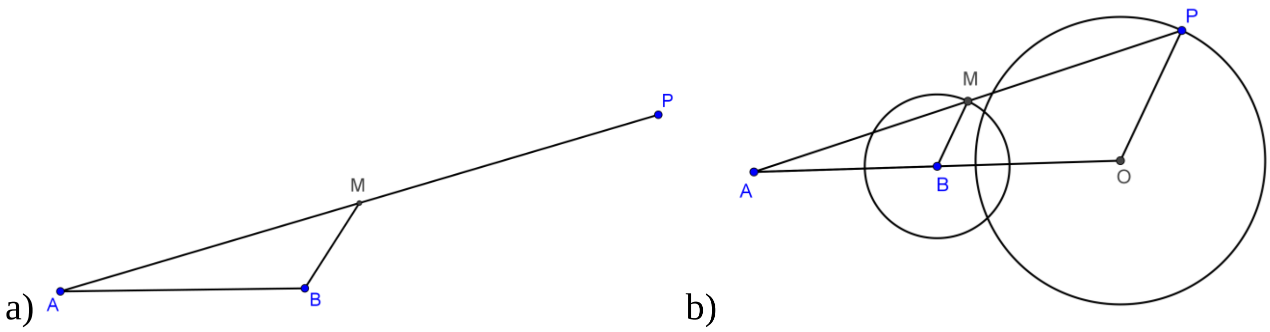


Figure 1: a) Configuration obtained from part 1; b) loci of M and P.

To solve the task, the following mathematical facts are important to note: 1) M is the midpoint of AP, so AM is always equal to 1/2 of AP; 2) MB must always have the same length, the locus of M is a circle with center in B and radius BM.

Solvers can reason in different ways; here, we describe a few possible steps leading to a solution. The discursive element “*MB must always be constant”* may foster recollection of the definition of circle, leading to immediate recognition of the locus of M. It is also possible to recognize such locus when looking for “good positions” of P, that is, positions for which the length of MB remains constant. The solver can imagine moving M along the circle, or draw it, and observe different positions of P, discovering that also P lies on a circle. Using the relationship AP = 2AM, s/he can view the locus of P as the circle corresponding to the locus of M through a homogenous dilation of factor 2. This theoretical consideration may also help the solver find the center and radius of the locus of P: the center is a point O on the line through A and B, satisfying the relationship AO = 2AB; the radius has length 2MB (Fig. 1b).

We conjectured that the task would foster various processes of GP, in particular for the loci of P and of M; indeed, recognizing the locus of M seemed a likely stepping stone. In the second part of the task, we expected the solver to use several dragging modalities (Arzarello, Olivero, Paola, & Robutti, 2002) and, in particular, maintaining dragging (MD) (Baccaglini-Frank & Mariotti, 2010) to maintain certain predicted properties.

STUDENTS' ANSWERS

In light of our research questions, we analyzed the videos and transcripts of all students' interviews in the following way. We searched for and coded all excerpts containing use of GP (spontaneous or in response to interviewer prompts), marking its being mathematically correct or not, and whether it was accompanied by explanations or not (whether they were mathematically correct or not). We also labelled all verbal or gestural descriptions by which these were accompanied. We then identified the excerpts in which the solvers explored the constructed DGE figure, and marked: whether the students spoke about new or contradictory properties (changing the product of their GP), whether the solvers seemed surprised by the feedback from the DGE, and whether they eventually reached a correct solution to the problem.

Here we report on the analyses of the interviews of the eight students who were assigned the "locus of P problem". Each of them uses different words and gestures. We will give more detail on two of the interviews in the next section. Table 1 summarizes results of the analyses of the outcomes of the most common of these students' GPs: P is fixed (predicted by 5 students); M on a circle (predicted by 2 students); P on a circle (predicted by 3 students).

GP	Student	Explained	New ideas	Surprise	Correct solution
P is fixed (GP1)	S2	Yes	No	No	No
	S3	Yes	Yes	Yes	No
	S4	Yes	No	No	No
	S5	Yes	No	Yes	No
	S6	No	Yes	No	No
	S8	Yes	Yes	No	Yes
M on a circle (GP2)	S1	No	Yes	No	Yes
	S2	Yes	No	No	Yes
P on a circle (GP3)	S1	No	Yes	No	Yes
	S2	No	Yes	No	No
	S7	Yes	Yes	Yes	Yes

Table 1: Analyses of the outcomes of 8 students' GPs on the "locus of P" problem

Only 2 of the 8 students (S1, S7) find that the locus of P is a circumference; S2 declares this possibility, but then suddenly discards it. No student succeeds in correctly predicting the center or radius of the circle: some predict the center to be B. The solution processes of S1 and S7 have in common that the locus of M is made explicit, either verbally or by a drawing. As predicted, a process of GP leading to the locus of M seems to be essential in reaching a GP of the locus of P. Other common aspects of these students' processes of GP are: the GP on the locus of M is supported by the discursive element "*constant/invariant length*" of MB; the GP on the locus of P is accompanied by visual-spatial considerations that do not seem to have strong theoretical grounding. Indeed, S1 does not explain why she imagines a circle; while S7 describes the motion of P very well, but he does not refer to any theoretical elements supporting his GP. This lack of theoretical grounding of the GPs could explain the difficulties of both students in defining the circle's center and radius.

Another interesting consideration is that in all cases in which the incorrect GP "P is fixed" (GP1) is made explicit, it becomes dominant in the problem solving process and it impedes to find other "good positions" for P. In particular, 4 out of the 5 students who make GP1 and never reach the correct solution of the problem try to explain why they see P as fixed. Although 2 of these get new ideas from the DGE exploration, leading them to partially change their mind, none are able to generate and interpret the feedback in a way that allows them to reach the correct locus of P. Moreover, only 2 of these students seem surprised by the DGE feedback. These data support the hypothesis that students who succeed in explaining (incorrectly) their GPs have more difficulty in grasping DGE feedback in contradiction with them.

On the other hand, S1, who fails to explain why the locus of P is a circle, and S7, who merely refers to spatial elements, seem more prepared to recognize and possibly modify some characteristics of the locus they had imagined. In particular, S1, in her dynamic exploration, looks for its center and radius; and S7 approximates the locus of P first with "a curve" then with "an ellipse" and finally with "a circle".

In general, it seems that all (correct and incorrect) GPs, produced before the DGE exploration – and more so when accompanied by explanations – strongly influence the subsequent exploration, conditioning how students interpret the DGE feedback. In particular, in the case of (mathematically) incorrect outcomes of processes of GP produced before a dynamic exploration, the DGE feedback in general does not help students to completely change their minds and recognizing a new contradictory geometrical property. In the case of (mathematically) correct outcomes of processes of GP produced before a dynamic exploration, the DGE feedback does help students refine the outcomes of their GPs. We now present to more detailed analyses of excerpts of two students' interviews.

S1: unexplained semi-correct GP and proper interpretation of DGE feedback

Shortly after reading the problem and having seen the configuration on the computer screen, but before dragging anything, the student S1 said:

1	S1:	The length of MB has to be constant…so I… so instinctively I would answer that P has to move on a circle […]
2	S1:	I'm not…I'm not that sure it is a circle. I mean, intuitively I imagine it, but I wouldn't actually know what center and what radius, I can't imagine it.

S1 immediately describes the product of a GP: the locus of P as a circle (GP3 in Table 1). GP3 is identified seemingly quite rapidly (line 1) without passing through the locus of M. However, GP3 is not completely clear to S1, who realizes she has not identified some of its characterizing features (line 2). The outcome of the GP is strong enough to allow S1 to trace the circle with her finger (Fig. 2a, 2b).

a) b) c)

Figure 2: a, b) S1 traces GP3 with her finger on the screen; c) S1 traces the locus of M (GP2) using her fingers as a compass.

Then S1 tries to predict the center and radius of GP3. To do this she predicts the locus of M (GP2 in Table 1), using her fingers as a compass (Fig. 2c):

3 S1: For this [MB] to be constant…

4 S1: [she places the thumb in B and index of her right hand on M and rotates her index around the thumb] Yes, P should…maybe moves along a circle…with center at B.

At this point S1 has produced a verbal and gestural description of GP3 and a gestural description of GP2. When she is asked to check her answers using the DGE figure.

Initially S1 moves P continuously but only along a short arc of a circle, the one she had predicted. Doing this, she focuses on M and she verbally describes its locus:

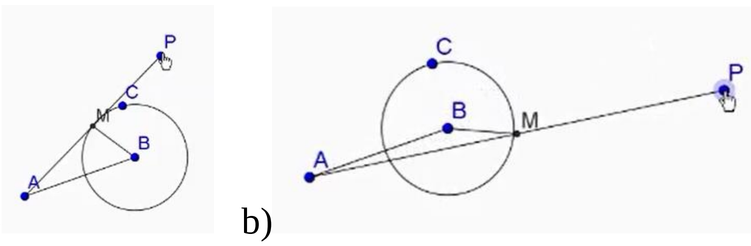
5 S1: So, yes, in order for MB to always have the same length, it is fundamental for M to move on a circle with center at B and…if M moves along a circle with center at B…also P will move on a circle.

S1 uses MD expressing her intention of having MB “always have the same length”. This seems to lead her to perceive a relationship between the two predicted circles, expressed in the form “if …, also…” which seems logically close to the conditional form “if…, then…”. The interviewer asks for additional information on the locus of P:

6 Int.: Can you say anything else about this circle?

7 S1: Eh, I am asking myself if its radius is…if its center is B, but I don’t think so.

As S1 answers (line 7) she keeps on using MD, making bigger movements along a predicted circle, which leads her to reject her initial conjecture for the center of the locus of P.



a) b)

Figure 3: a) initial configuration from which S1 starts using MD to maintain the length of MB constant; b) P describes a circle as it is dragged using MD.

Finally, in light of her new GP supported by the DGE feedback (lines 5, 7), S1 decides to use MD on P trying to maintain M (Fig. 3b) on a constructed circle centered in B with a given radius (in Fig. 3 she chose point C randomly on the plane).

This convinces S1 that P moves along a circle, but definitely not with center at B. So S1 seems to have a very constructive interaction with the DGE, using the feedback to support and refine her GPs.

S2: explained incorrect GP and improper interpretation of DGE feedback

Student S2 starts working in a paper and pencil environment, and quickly predicts that P has to be fixed:

1 S2: If I move this [her finger runs along BM] it [M] moves but this distance [length of BM] increases…I would say that P is univocally determined, once A, B and m are fixed.

This GP (GP1 in Table 1) seems to arise because of the possible movement imagined for M and P: whenever S2 speaks of moving P she moves her index or the tip of a pen in the direction of PA, as it is drawn, without ever changing its inclination. This GP seems closely related to the kind of movement imagined for M: it is only possible to stretch or shrink MB, but not to move M maintaining constant m, the length of BM. Moreover, at times, there seems to be ambiguity between "m" (the length of BM) and point "M", leading to the idea that M must be fixed. Therefore, S2 predicts that, at most, A and P can move "coming closer and farther, to M in a proportional manner". With this in mind, once she realizes that A is fixed, S2 is sure that P must also be fixed.

The strength of her GP seems to inhibit S2's ability to constructively interpret the feedback obtained from the DGE. Indeed, when asked to explore the dynamic figure, at first S2 tries to move P maintaining m = 2:

2 S2: Ok, let's put 2, even though I will never be able to make it 2. I don't know.

Although at least to positions in which m = 2 appear on the screen, S2 does not seem to notice them, and she puts P back in the original position, an instance in which m = 3.

3 S2: Ehm, let's put this one. I can move P, let's activate the trace, maybe.

As she moves P, S2 mentions the possibility of P moving on a circle, but she seems unsure and rapidly discards that possibility, even though a few good positions for M had appeared on the screen (Fig. 4). Instead, S2 continues to speak about the "fixed-ness" of m that necessarily determines a single good position for P.

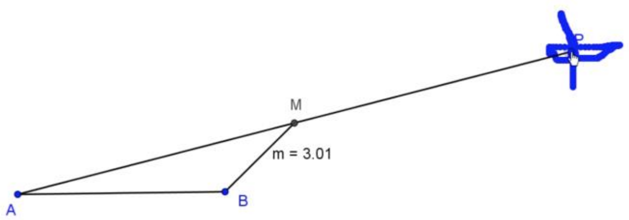


Figure 4: S2's attempts to drag P maintaing m = 3.

We see this as a case in which an incorrect GP with an explanation that seems to be very convincing for the student does not allow the student to generate and interpret constructively the feedback from the DGE. Indeed, although the DGE exploration makes use of MD, the student appears to be “blinded” by her original GP, to the extent that she cannot see any “good positions” for P other than the original configuration.

CONCLUSION

The analyses of students’ videos and transcripts revealed that processes of GP (or at least the description of their outcomes) tend to be accompanied by verbal or gestural explanations: for the GPs considered in this study 7 out of 11 times these explanations were present. The verbal and gestural forms varied from student to student. Moreover, our data suggests that once a GP is advanced and the student is given the opportunity to interact with a DGE figure corresponding to the configuration reasoned upon, the GP has influence on the DGE exploration. In particular, if the students are quite convinced by their (correct or incorrect) GP, the exploration only serves to refine or confirm the predictions, and there seem to rarely be instances of surprise in the exploration. This phenomenon, that we refer to as “power of the GP”, is particularly striking in cases in which the GP is incorrect and still drives the solver to see on the screen only what s/he has predicted and is, therefore, prepared to see.

References

Arcavi, A. & Hadas, N. (2000). Computer mediated learning: An example of an approach. *International Journal of Computers for Mathematical Learning, 5*, 25-45.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri environments, *ZDM*, *34*(3), 66-72.

Baccaglini-Frank, A., & Mariotti, M.A. (2010) Generating Conjectures in Dynamic Geometry: the Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225-253.

Cuoco, A., Paul Goldenberg, E., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. The *Journal of Mathematical Behavior, 15*(4), 375–402.

De Finetti, B. (1967). *Il “saper vedere” in Matematica*. Torino: Loescher.

Mariotti, M.A. & Baccaglini-Frank, A. (in press). Developing geometrical exploration skills through dynamic geometry. In S. Carreira, N. Amado and K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect.* Springer.

Miragliotta, E., Baccaglini-Frank, A., & Tomasi, L. (2017). Apprendimento della geometria e abilità visuo-spaziali: un possibile quadro teorico e un’esperienza didattica (prima parte). *L'Insegnamento della Matematica e delle Scienze Integrate*, *40B*(3), 339-360.

Presmeg, N.C. (2006). Research on visualization in learning and teaching mathematics, in A. Gutierrez, P. Borero (eds.) *Handbook of Research on Psychology of Mathematics Education: Past, Present and Future*, (pp. 205-235). Rotterdam/Taipei: Sense Publishers.

PROOF AND PROVING IN HIGH SCHOOL GEOMETRY: A TEACHING EXPERIMENT BASED ON TOULMIN'S SCHEME

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In this paper, we discuss a teaching intervention utilising a two-faceted didactical tool that draws upon Toulmin's scheme in order to introduce high school students to mathematical proof. An experimental research design was implemented. The results of the conducted analyses suggest that the proposed tool helped the students in discerning and differentiating data from claims in a geometry proving problem, whilst they obtained appropriate overview of the structure of a valid proof, including their developing a need for including only necessary arguments in a proof.

INTRODUCTION

Proof and proving have been at the focus of extensive mathematics education research project identifying the students' difficulties to experience the various functions of proof and, inescapably, an internal need for proof (Balachef, 1988; Hanna, Jahnke & Pulte, 2010; Harel, 2013; Herbst, 2002; Moutsios-Rentzos & Spyrou, 2014; Zaslavsky, Nickerson, Stylianides, Kidron & Winicki-Landman, 2012). These difficulties have been linked with the students' limited opportunities to be engaged with the proof process in the textbooks and everyday classroom teaching practices (Alibert & Thomas, 1991; Thompson, Senk, & Johnson, 2012). To address these issues, researchers have suggested ways of presenting and organising proofs —including Leron's (1983) structural approach, the two-column proof writing format (Herbst, 2002), self-explanations (Hodds, Alcock & Inglis, 2014), generic proofs (Leron & Zaslavsky, 2013). The plethora of proof-related research projects is in contrast with the lack of research-based interventions (stressed in the 2017 special issue of Educational Studies in Mathematics, edited by Gabriel Stylianides and Andreas Stylianides). Consequently, in this paper, we discuss aspects of a broader study about the utilisation of Toulmin's theory and scheme in teaching interventions in high school geometry proof and proving. Focussing on the triadic relationship 'data-warrant-claim', we present a two-faceted didactical tool designed to facilitate the students' gaining deeper understanding about the local and broader functions of these three elements, as implemented in a teaching intervention in a high school geometry class.

TOULMIN'S SCHEME IN MATHEMATICS EDUCATION

Toulmin's (2003) theory of argumentation and his scheme concerns to all argument types and by adopting his perspective we essentially adopt the perspective of proof as a special case of argument, or a series of arguments (cf. Aberdein & Dove, 2013). Toulmin (2003) considers the argument as an organism, which "has both a gross, an-

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 395-402). Umeå, Sweden: PME.

atomical structure and a finer, as-it-were physiological one" (p. 87). He schematically organises five functional elements in the argument structure (Data, Warrant, Backing, Qualifier, Rebuttal, Claim). Mathematics educators, starting with Krummheuer (1995) have implemented Toulmin's scheme initially in its restricted version (Data-Warrant-Claim; for example, Martinez & Pedemonte, 2014) and later in its full version (for example, Inglis, Mejia-Ramos, & Simpson, 2007) to analyse the students mathematical proof and proving. The case for the utilisation of the full Toulmin scheme goes beyond the scope of this paper, since in our intervention we employed a restricted version of this scheme, including the triad Data-Warrant-Claim.

It is argued that Toulmin's scheme may act as an accessible, attractive way of presenting and organising the mathematical proof, helping the students to avoid common mistakes and to gain deeper understanding in the various components of proof. Furthermore, Toulmin's scheme may be utilised to facilitate the students' realising that the links between data and claims are crucially dependent on the link itself: the warrant. For example, the same datum may be linked through different warrants with different claims, different data may be linked through different warrants with the same claim, whilst the same datum may be linked through different warrants with the same claim. Importantly, the scheme may be employed to reveal the local nature of these labels within an argument. For example, an intermediate claim is usually used as datum in a subsequent argument. In line with these ideas, Hein and Prediger (2017) drew upon Toulmin's ideas to design an educational material combining an expansion of Toulmin's scheme with structural scaffolding aiming to facilitate the students' formal reasoning through their explicit experiencing and becoming aware of the elements of the utilised arguments and their functions.

A TWO-FACETED TOOL FOR TEACHING PROOF IN GEOMETRY

Following these, Toulmin's scheme may be used as the basis of didactical tools that visually differentiate seemingly similar, yet logically distinct cases, which may be otherwise didactically conflated, thus supporting the students' gaining access to the inner structure of the proof construction. A didactical tool was designed comprised of two complementary parts and was implemented in a teaching intervention investigating the effects on the students' understanding of proof. We focussed on two aspects of the validity of a proof: a) *micro-local validity*; each argument needs to constitute an appropriate data-warrant-claim triad, b) *macro-local validity*; each proof needs to be founded on a series of such arguments. The two complementary parts of the didactical tool were the *DWC (Data-Warrant-Claim) Table tool* (Figure 1; left) and the *Proof bearing structure tool* (Figure 1; right). The DWC Table tool is designed to facilitate the students experiencing the local DWC relationships in the arguments that constitute the building blocks of the proof. It consists of three main (plus two auxiliary) columns entitled respectively 'Data', 'Why?', 'Conclusions' in line with the language established in everyday pedagogy and with Toulmin's respectively Data, Warrant, Claims. Each line of the table constitutes a mathematically valid argument. In the 'Data' column, the students fill in with the appropriate text (including mathematical formulae, natural language etc), characterised in an auxiliary column either as ‘H_i’ (referring to ‘Hypothesis’, in the sense that it was given to hold true in the particular problem or a known geometrical fact) or with ‘C_i’ (referring to an already proven ‘Claim’). In the ‘Conclusions’ labelled column, the students would fill in the particular claim argued in the specific line, characterised in an auxiliary column either as ‘C_i’ or as ‘A’ (what is asked to be proved in the particular problem). The ‘Why?’ column is filled with the warrant that links a particular ‘H_i’ with a ‘C_i’ (or ‘A’). The proof would be concluded in the case that a C_i is the same with the statement asked to be proved, thus denoted with ‘A’. Subsequently, the Proof Bearing Structure tool was utilised to create a DWC-framed structure of proof, following a building metaphor. The foundations of the proof frame should be H_i. Note that, ‘H_i’ are denoted with boxes, ‘C_i’ (or ‘A’) with ellipses and warrants with arrows. The proof is ‘vertically’ developed in levels, with each ‘higher’ level (claim) requiring ‘pillars’ (warrants) founding them on the ‘lower’ levels (data). A level may be at the same time be founded on another lever or act as the base of another level, thus embodying the potential synchronous data-claim nature of a statement. The highest level of the structure is ‘A’. These ideas are schematically outlined in Figure 1 and exemplified in Figure 2.

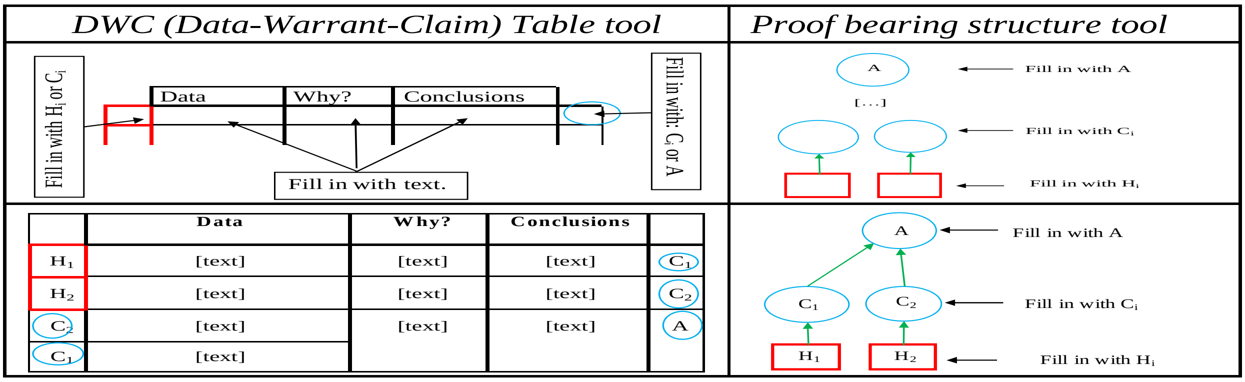


Figure 1: The two-faceted didactical tool

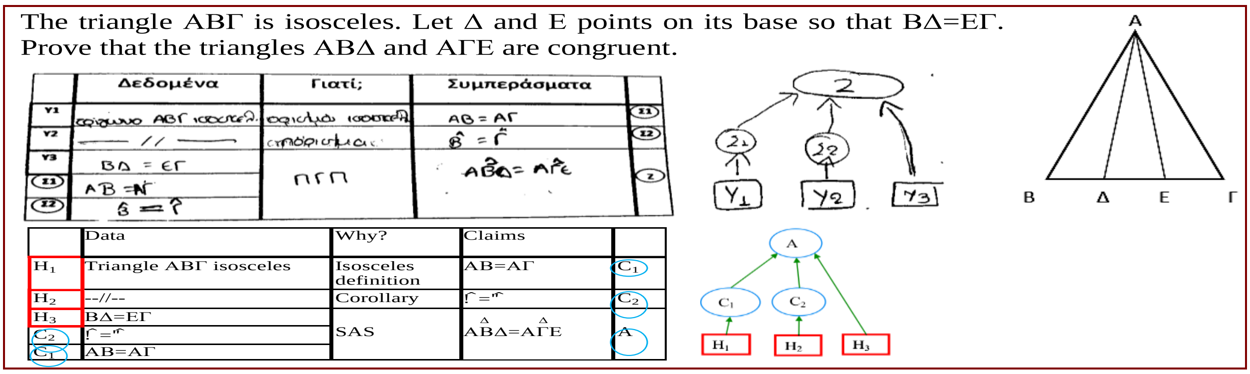


Figure 2: An example of the implementation of the tool

Following these, a study was conducted to investigate the educational implications of a teaching design incorporating the abovementioned two-faceted proof construction tool. In this paper, two aspects of the broader study are discussed the students’: a) *understanding and differentiation of what is ‘given’ and what is ‘asked’ in a geometrical proving problem,* and b) *ability to evaluate proof structures.*

METHODS AND PROCEDURES

The participating system: the students, the teacher and the curriculum

The students attending two classes in the first grade of Lykeio (16 years old) in Greece participated in the experimental design of the study: the class that the intervention was implemented ('intervention' class; N_{INT}=23) and the class taught as usual ('non-intervention' class; N_{NINT}=24). The grade was purposefully chosen, as in Greece the students are introduced to deductive proof in the Geometry course taught in this grade (two 40-minute 'class hours' per week). The curriculum explicitly mentions that the teaching of Geometry at this grade should focus on mathematical proof, facilitating the transition from the empirical and inductive reasoning to the formal and deductive reasoning, as means for securing the mathematical ideas and claims. Hence, the students participating in the study are expected to be in their introduction to mathematical proof stage. Both classes were taught by the same teacher: an experienced mathematician with a masters degree in mathematics education. Informal interviews were conducted to identify her usual proof teaching practices. Though in line with the Greek curriculum, she would choose different approaches when her experience and education would advise her otherwise. When presenting a geometrical proving problem, she emphasises the distinction between what is 'Given' and what is 'Asked', using a two-column table, filled in with parts of the text included in the problem wording, thus transforming it to a more functional format for a proof construction (according to the teacher). Considering the students that participated in the study, she argued that the majority held negative views of the course, not being able to see its usefulness, whilst many students could not differentiate proof from observation. She stressed that her students had insufficient knowledge of terminology and definitions and they faced difficulties with the Data-Claims distinction.

The research design

A five-phase research design was adopted, consisting of the intervention (audio recorded, spread in three class hours) and the pre-/post-intervention testing (paper-and-pencil, two class hours): a) *Pre-intervention test* (both classes, 40 min.; including data-claim differentiation and proof evaluation), b) *Introduction of the two-faceted tool* (intervention class, 30 min.), c) *Proof structure evaluation* (intervention class, 20 min.), d) *Proof construction* (intervention class, 40 minutes), e) *Post-intervention test* (both classes, 40 min.; similar to the Pre-intervention test, including Proof bearing structure questions only for the intervention class).

Considering the content and the procedures of the intervention, the teacher was informed about the two-faceted tool after the aforementioned interview and sufficient time was dedicated in her 'training' to appropriately utilise the tool, emphasising the need to consistently use specific terms and phrases, as well the way she would respond to various scenarios. This process was a crucial part of our design, as we wished for our intervention to be the least intervening in the pedagogical contract established in the class system. For example, the choice of the words 'Why?' and 'Conclusion' were selected as best fit with her everyday teaching. This was especially important in the 'Introduction of the two-faceted tool' phase, where the students first filled in a 'DWC Table' for a given proof and, subsequently they constructed the respective 'Proof bearing structure'. Emphasis was given in making explicit of the non-bijective relationship between a datum and a conclusion, thus requiring appropriate linking to specify the specific bond that secures the specific data-claim relationship as a valid mathematical argument. Furthermore, the building-construction metaphor was evident in various wordings and representations, in an effort to mobilise appropriate embodied metaphors, which have been found to be appropriate in the teaching of geometry (Moutsios-Rentzos et al, 2013). In the 'Proof evaluation phase', the students were asked to utilise these experiences in their evaluation of given proof bearing structures. Finally, in the 'Proof construction phase', the students were asked to fill in an empty 'DWC Table' using proof excerpts given to them and to appropriately signify them (by using 'C_i', 'H_i' and 'A'). Subsequently, they were asked to construct the respective 'Proof bearing structure'. In the last part of this phase, the students fill in the 'DWC Table' and to construct the respective 'Proof bearing structure' for a given proof.

Finally, all the audio recordings and the digitised tests were submitted to both quantitative and qualitative analysis. In this study, we focus on two questions of post-intervention test, drawing upon comparisons between intervention and non-intervention class in a 'Given'-'Asked' differentiation question (Figure 3), complemented with qualitative analysis of the intervention class responses in proof bearing structures evaluation questions (Figure 4).

RESULTS

In this paper, we discuss only two aspects of the study: the distinctness between 'Given' and 'Asked' in the wording of a geometrical proving problem and the evaluation of proof bearing structures. In the pre-intervention testing, no statistically significant differences were found between the two classes with respect to their ability to identify and differentiate 'Given' from 'Asked'. Subsequently, we compared the students' responses in the 'Given'-'Asked' distinction post-intervention testing question (Figure 3). The Fisher's exact test revealed that the intervention class scored statistically significantly higher than the non-intervention class only in the first item in the 'Given' column "AΔ bisector of the angles A and Δ" P=0.020; respectively 7/23 and 1/24 correct), implying that the students of the intervention class were more successful in the 'Given'-'Asked' distinction. In order to gain deeper understanding of this finding, we considered the fact that the same statement was used in the sixth item in the 'Asked' column to investigate whether it would be classified as being both 'Given' and 'Asked'. No students of the intervention class made this mistake. That is, even the students who were unsuccessful in correctly identifying this statement as 'Given' or 'Asked' did not assign to it a dual, logically impossible, status. In contrast, in the non-intervention class 4 out of 24 students made this mistake. Thus, it is argued that the intervention design helped the students to overcome this difficulty more successfully than those taught in the usual approach of the particular teacher.

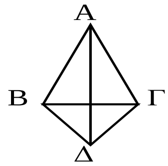
For each of the following statements, choose ‘T’ (true) when they contain all the Given or all the Asked of the following statements. Otherwise choose ‘F’ (false).					
In two isosceles triangles with common base ΒΓ, ΑΔ is the common bisector of the angles Α and Δ. 					
Given			Asked		
1. ΑΔ bisector of the angles Α and Δ	T	F	4. ΑΒ=ΑΓ και ΒΔ=ΓΔ	T	F
2. Triangles ΑΒΓ and ΒΓΔ isosceles ΒΓ base of ΑΒΓ and ΒΓΔ	T	F	5. Triangles ΑΒΓ, ΒΓΔ isosceles	T	F
3. ΑΒ=ΑΓ, ΒΔ=ΓΔ and ΒΓ common	T	F	6. ΑΔ bisector of the angles Α and Δ	T	F

Figure 3: ‘Given’ – ‘Asked’ distinction (both classes)

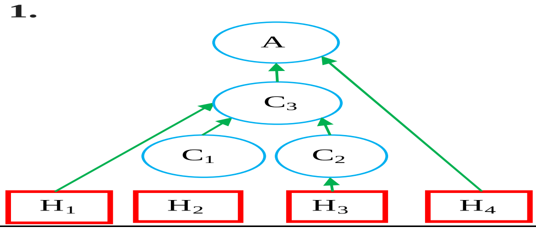
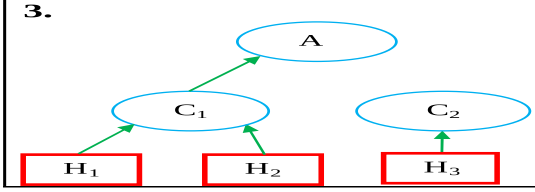
Study carefully the following proof bearing structures. Choose ‘T’ if you think the structure is correct and ‘F’ if not. In the case you think it is incorrect, please note on the structure which part(s) of the structure is/are incorrect and provide a short explanation about the mistakes you noticed.		
	T	**F (explanation)**
1. 		
3. 		

Figure 4: Two Proof bearing structure evaluations (intervention class only)

With respect to the ability of the students of the intervention class to evaluate proof bearing structures, the analysis of the students’ evaluations and justifications in a series of seven proof bearing structures (PBS) questions, two of which are discussed in this paper (PBS1, PBS3; Figure 4) revealed that most of the students realised the necessity that each claim should stem from another, already founded claim (‘given’ or proved). For example, in PBS1 (Figure 4, ‘up’) 19 students (out of 23) correctly identified that this structure as impossible. 17 out of the 19 provided explanations, all of which also correctly noted that C_1 is not supported by a valid claim. For example, the students wrote “C_1 is not supported”, “C_1 is not linked with any hypothesis” and “C_1 does not lean on anything”. Thus, it seems that the students build on the building metaphor to justify their decision, which is evident by their adopting the corresponding wordings. Nevertheless, it was unclear only from these responses whether their expressed need for support was local –in the sense of the existence or not of immediate supporting pillars– or was extended to cover the whole proof bearing structure.

Furthermore, an interesting finding was revealed in the students’ justifications. In PBS3 (Figure 4) 8 out of the 23 students characterised the structure as impossible. This structure has elements that though not micro-locally invalid, they are not necessary for the macro-local validity of the proof. 6 out of the 8 students who provided justifications stressed that PBS3 was not correct as “C_1 is not linked with A” or “C_2 does not support anything”. It appears that the two-faceted tool helped the students to develop a need for *economy* in proving, in the sense that a mathematical proof should contain only necessary building blocks. Such a view may be linked with the mathematicians’ perspective of proof who would choose to produce a proof containing only logically necessary parts. A proof containing logically abundant parts may be characterised as not ‘elegant’ or maybe ‘didactically helpful’, but would not be classified as an invalid proof. Importantly, it is argued that these answers may be didactically utilised to discuss delicate matters such as logical necessity and mathematical validity.

CONCLUDING REMARKS

In this study, we discussed aspects of a broader study about the implementation of a two-faceted didactical tool, designed to facilitate the students’ producing and evaluating proofs in high school geometry. We drew upon Toulmin’s restricted scheme to design a tool that explicitly links two distinct proof validity levels: the micro-local level of the arguments employed and the macro-local level of a proof bearing structure. We discussed the short-term effects on the students’ ability to discern and differentiate what is given and what is asked in a geometrical proving problem, as well as their proof bearing structures evaluations. The conducted analyses revealed that the intervention class statistically significantly outperformed the non-intervention class in the ‘Given’-‘Asked’ task, whilst no student of the intervention class characterised a statement both as ‘given’ and as ‘asked’. It is argued that the proposed two-faceted tool seems to enjoy the benefits reported in similar designs (such as the two-column proof format). This may be also linked with the students’ active engagement with the multiple re-organisations of a proof (Hein & Prediger, 2007), which facilitated their gaining a deeper understanding of the triadic link amongst data-warrant-claim in a proof. Though further research is being conducted to investigate the temporal stability and the long-term effects of the utilisation of the proposed tool in everyday teaching, it is stressed that the building representation metaphor accompanied with embodied verbal metaphors in the teaching (both in the representation and the wording choices) seemed to have a broader effect to some students that transformed the need for appropriate support to a need for appropriate necessary support. It is posited that such an effect may be didactically utilised to help the student’s view of proof as an anthropological, accessible to them, construction (cf. Hersh, 1997), thus facilitating their experiencing internal needs for proof (cf. Moutsios-Rentzos & Spyrou, 2013).

References

Aberdein, A., & Dove, I. J. (Eds.) (2013). *The argument of mathematics*. Dordrecht: Springer.

Alibert, D., & Thomas, M. (1991). Research on mathematical proof. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 215-230). Dordrecht: Kluwer.

Balacheff, N. (1988). Aspects of proof in pupils’ practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London: Hodder & Stoughton.

Hanna, G., Jahnke, H. N., & Pulte, H. (Eds.). (2010). *Explanation and proof in mathematics: philosophical and educational perspectives*. New York: Springer.

Harel, G. (2013). Intellectual need. In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 119-151). New York: Springer.

Hein, K., & Prediger, S. (2017, February 1-5). *Fostering and investigating students' pathways to formal reasoning: A design research project on structural scaffolding for 9th graders*. Paper presented in CERME 10, Dublin.

Herbst, P. G. (2002). Establishing a custom of proving in American school geometry: Evolution of the two-column proof in the early twentieth century. *Educational Studies in Mathematics, 49*(3), 283-312.

Hersh, R. (1997). *What is mathematics really?* New York: Oxford University Press.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education, 45*(1), 62-101.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3-21.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauershfeld (Eds.), *The emergence of mathematical meaning*: *Interaction in classroom cultures* (pp. 220-269). Hillsdale, NJ: Lawrence Erlbaum.

Leron, U. (1983). Structuring mathematical proofs. *The American Mathematical Monthly, 90*(3), 174-185.

Leron, U., & Zaslavsky, O. (2013). Generic proving: Reflections on scope and method. *For the Learning of Mathematics, 33*(3), 24-30.

Martinez, M. V., & Pedemonte, B. (2014). Relationship between inductive arithmetic argumentation and deductive algebraic proof. *Educational studies in mathematics, 86*(1), 125-149.

Moutsios-Rentzos, A., Spyrou, P., & Peteinara, A. (2014). The objectification of the right-angled triangle in the teaching of the Pythagorean Theorem: an empirical investigation. *Educational Studies in Mathematics, 85*(1), 29-51.

Moutsios-Rentzos, A., & Spyrou, P. (2013). The need for proof in geometry: A theoretical investigation through Husserl's phenomenology. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 329-336). Kiel, Germany: PME.

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education, 43*(3), 253-295.

Toulmin, S. E. (2003). *The uses of argument* (updated edition). New York: CUP.

Zaslavsky, O., Nickerson, S. D., Stylianides, A.J., Kidron, I., & Winicki-Landman, G. (2012). The Need for Proof and Proving: Mathematical and Pedagogical Perspectives. *In G. Hanna & M. de Villiers (Eds.), Proof and Proving in Mathematics Education (Vol.15, pp. 215-229).* Netherlands: Springer.

STUDENTS' PATHWAYS FOR SOLVING PROBABILITY PROBLEMS

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To produce mathematics knowledge for teaching probability, this study explored students' representations and associated errors as they solved probability tasks. Sfard (2007)'s argument of mathematics as discourse and, Chaput, Girard and Henry (2011)'s three probability learning modelling processes guide the study. Data from sixteen grade 10 students aged between 15 and 17 years showed that students regarded representations as thoroughfares that mediate task solution, but most students could not construct correct representations. Further, interviews showed that some students had little faith in their constructed representations and suggested better task solutions independent of their representations. Probing of students' visual representations in interviews helped them to get better insight for solving the probability problems.

INTRODUCTION

This article aims to contribute to mathematical knowledge for teaching probability through the study of the errors and misconceptions grade 10 students have when solving probability problems through their visual representations of the probability problems. Polya (2014) suggested that to be able to solve a problem, the first thing needed is to understand the problem. The second stage is to devise a plan or strategy of how to tackle the problem, before actually implementing and executing the plan. One of the problem solving strategies that Polya proposed among others was to draw a picture; that could be a visual or a model to represent known information so that relationships between variables can be analysed in greater detail. Written down representations give rise to visually mediated cognition. As probability problems are often abstract to students, Borovcnik (2014) argues that students' use of probability models is essential to conceptualise the inner-working of probability and inference. In the same way Pfannkuch and Ziedins (2014) suggest that "…although they [models] are only approximations to what happens in the real world, these approximations can help us better understand the behaviour in the real world" (p. 103). Once models are understood, they can be generalised to a class of problems (Lee & Doerr, 2016), so that new knowledge in the real world problems is gained.

THEORETICAL FRAMEWORK

Engaging mathematics tasks is seen as participating in a mathematics discourse (Sfard, 2007). According to Sfard (2007, p.571) "visual mediators are means with which participants of discourses identify the object of their talk and co-ordinate their communication". It is envisaged that when students engage with new mathematical

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 403-410). Umeå, Sweden: PME.

tasks such as probability, they begin that through the platform of old discourses, which discourses may not be sufficient. As they work with more knowledgeable others, they learn new discourses and thus gain knowledge. Errors and misconceptions in probability mean that the students will still be using old discourses incommensurate with new knowledge. Further, Sfard regards all mathematical activity as visually mediated in some way or another.

Brodie and Berger (2010) have suggested a class of errors that researchers may see in students' work. The first is the error due to procedure; the "halting signal" error (described by Sfard 2008, p. 214). This occurs when a learner gets an answer to a question and believes that it is the final answer not realizing that they need to process it further as they are only 'half-way' through. Thus, a halting signal triggers premature closure of a routine. In this case an incomplete answer, a pre-mature one is taken as final. The second type of error proposed by Brodie and Berger (2010) is a 'keyword trigger'. A particular word in a probability question may evoke a learner to use a particular routine over and above others. The third is the error of signifiers. A learner might have used certain terminology successfully in the past and when they encounter new situations, they fail detach themselves from past meanings. The fourth error is that of 'familiarity'. The fifth is the 'visual mediation error'. This occurs when students fail to represent information given in a probability task using representations such as tree diagram or contingency table. This also occurs if a learner, having drawn a good representation fails to encode it, to re-interpret it so that they can obtain the answer befitting to the original probability question.

Chaput, Girard and Henry (2011) suggest that a probability learning modelling process has three levels. The first is translating the contextual problem presented into a pseudo-concrete/visual working model. This comes through decoding what the question says and representing it visually by a sort of representation most often through paper and pencil. They argue that the second stage is mathematising the model and working with the model to investigate the nature of the original contextual problem. The third stage involves checking whether the model fits with the empirical data and re-interpreting the model with the original probability task if needed. This inductive work is very important as it usually results in students discovering patterns for answering particular types of probability tasks. This results in probability laws such as the binomial random variable, the uniform random variables and so on. It is within this framework that we studied the student errors and misconceptions when answering probability questions with the mediations of visual representations.

In this article the research questions are:

- What errors and misconceptions exhibit in student's visual representations as they solve probability problems? and;
- What explanations can we propose for the errors and misconceptions as a result of the students' visual representations?

METHOD

Kvale (1996) argues that qualitative interviews seek to factually describe participants' thinking *in vivo*; and also fathom participants' meanings on the unit(s) of analysis of the study. In this study the unit of analysis of the study concern the errors and misconceptions associated with learners' probability solutions. The study was carried out on a class of 16 grade 10 boys and girls in a Johannesburg, Soweto Township School, South Africa. These are students who are in their third year in high school. The students were aged between 15 and 17 years. The tasks were constructed in accordance to the current curriculum. This was done mainly for ethical reasons so that students would engage in learning activities that would also help them in their current studies and not only for research purposes. At first, a set of grade 10 level probability tasks were collected from the mathematics curriculum assessments. The tasks were validated by other three grade 10 mathematics teachers and piloted with five grade 10 students from a different school in Soweto and adjustments made.

RESULTS AND DISCUSSION

The unit of analysis was students' representations of probability problems which act as visual mediators of mathematics discourse (Sfard, 2007). Students used these to communicate with themselves as they solved the probability tasks.

Item 1

What is the probability of getting two 3s if a six-sided die is tossed two times? Show all your work with a diagram.

Most learners who attempted it failed to realise that the tree diagram ought to have six branches, one for each number in the dice or respectively two branches one for a 3 and the other for a not 3. Two students who got the correct answer for the item drew incorrect tree diagrams.

Prisca's tree diagram which was a typical example of the tree diagrams drawn by other students. This representation shows learners' counterintuitive ideas and paradoxes in solving probability problems. We called Prisca to an interview in which we ensured that she like all other interviewees was relaxed and felt secure. Below is an excerpt of the interview with Prisca:

Researcher: Prisca, tell me how you used this diagram to find the probability of obtaining two 3s when a six sided die is tossed two times?

Prisca: when the die is tossed, getting a 3 I wrote here (pointing to the top 3)

Researcher: … and the other 3s you wrote… what do they represent?

Prisca: the second 3 (pointing to the bottom 3) is the second 3 that is obtained

Researcher: What about that other 3 and the $\frac{1}{3}$

Prisca: The $\frac{1}{3}$ is the probability of getting the first 3

Researcher: What then is the final answer to the question?

Prisca: I think $\frac{1}{3}$ and $\frac{1}{3}$ which gives $\frac{2}{3}$

Researcher: Do you think your answer is correct Prisca?

Prisca: I am not sure it is correct

Researcher: Why do you have two branches for each toss?

Prisca: When we did tree diagrams for tossing a coin we had two branches, one for head, the other for tail…

Researcher: Is that the reason why you had two?

Prisca: Yeah

We were not sure that the student could notice that there were two outcomes in a coin, so we got further.

Researcher: In tossing a coin what are the possibilities the coin can fall?

Prisca: A head or a tail

Researcher: Which is, how many possibilities?

Prisca: Two

Researcher: In a die?

Prisca: Six

Researcher: Which are those?

Prisca: 1, 2, 3, 4, 5 and 6

Researcher: So how many branches must you have on each throw

Prisca: Six ma'am. I realize my mistake!

Researcher: … and the 3x3?

Prisca: I just multiplied

Researcher: Why?

Prisca: I don't know

We had expected 6 branches on each toss given the student was not advanced enough to analyse at that level. The student's response confirmed our assumption that the two branches for each toss did not mean a "3"' and "not a 3" which could have been an acceptable representation for the problem. This signified a fixation on a familiar type of representation; of tossing a coin that has two outcomes, a head and a tail, and therefore the corresponding tree diagram also has two branches. We interpret this as an error of familiarity (Brodie & Berger, 2010), of wrongly assimilating unknown into the known. This may also be seen as the natural error of using old discourses in new situations where new discourses must be learnt (Sfard, 2007). Although the tree diagram is one of the most helpful representations, though not the most economic for this problem, with this example we realize that often a representation can become a problem if it is not carefully constructed and interpreted as in Prisca's case. This was a 'visual mediation error'; failure to properly encode variables in the question in a representation. From another angle we presume that the first stage of Polya (2014) problem solving approach was not understood. If a student does not understand the problem, then any success in producing a model to solve it is greatly compromised as in this Prisca's case.

A very common incorrect response to item was $\frac{1}{6}$, the chance of getting a 3 on just one roll. This is an example of a 'halting signal error'. Students felt no further need to do any more working. The fact that some students did not attempt the item at all was intriguing. Janine was one such learner. The authors became interested to know why she did not attempt the item.

This interview suggested that Janine did not attempt the question because she had no plan where or how to start. In other words she did not know what discourse to engage (Sfard, 2007). Janine's response shows that she did not comprehend the task (Zahner & Corter, 2010), the first stage to solve a probability task. After a researcher had explained the problem in simpler terms for her, Janine said "Oh, ok. I was supposed to use that tree diagram" whereupon she drew a tree with two branches and got stuck in assigning outcomes. She went on to draw a tree diagram with two branches meant for tossing a coin. This suggests that she chose a familiar representation (Brodie & Berger, 2010) but was unable to adapt it to the new problem. So this was an error of familiarity.

Item 2

In a class of 33 students, 6 of the 15 boys are left-handed and 5 of the 18 girls are left-handed.

a. Draw a suitable diagram to represent the given information

b. Find the probability that a girl chosen at random will be left-handed

Most students answered this item correctly and attempted to solve both parts of the item. Ten students constructed contingency tables, three drew other type of tables and three drew Venn diagrams.

We next analyse Tongai's answer (Figure 1).

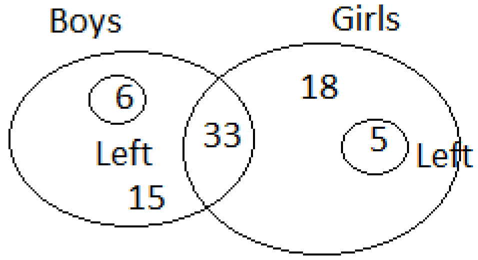

Figure 1. Tongai's Initial Venn diagram

Tongai drew a 'Venn diagram' rather than a contingency table. The following excerpt captures the interview with him.

Researcher: Tell me Tongai, did you use this diagram to answer these questions (pointing to the item)?

Tongai: Yes maám.

Researcher: Show me how.

Tongai: Maám, the totals for boys and girls are shown here and those who are left-handed are also shown. I could have drawn a normal graph but I chose to draw a Venn diagram because there are 33 students in the class… (silence)…Eish, I don't know why I put the 33 in the middle there. Eish maám, can I start over. (Evidence that he could have started off using the diagram but abandoned it later).

Researcher: Yes, feel free (offering pen and paper).

Through the inducement of the interview, Tongai felt he could re-do his representation. His response shows that he was aware that a representation can be refined to make it more accurate.

Although he claimed to have used the diagram, the interview helped him realise that a better Euler diagram could be more helpful in solving the problem. He was one not to give up. This excerpt shows that students can learn through their misconceptions. This example suggests that students do not always stick to their representations if they think they are not helpful. Therefore we may assert that, if students encounter problems with their visual constructions they abandon them to pursue other strategies to help them solve the problem.

This study focused on exploring students' errors when they solve probability problems using visual representations. Sfard (2008) argues that students' misconceptions are a result of using old discourses in cases where new discourses must be learnt by modifying old ones. Similarly, Michelet, Adam and Luengo (2007) see misconceptions as conceptions having a domain of validity. For example Sara was using a two branch tree for a tossing a die problem reminiscence of tossing a coin. She needed to slightly modify the discourse of tossing an item with two to one with six equi-probable events. Sometimes students did not have any representations to probability problems, for example Janine. This means that they could not carry out any mathematical discourse, even on the platform of an old inappropriate one. It could be a case that they did not understand the problem in first place (Polya, 2014). An important finding is that often students disregarded the representations they initially drew to mediate their solutions. Whenever they felt these were not helpful enough they overrode them. For example, two students obtained correct answers for Item 1 despite their wrong representations, while Prisca openly said she did not use their representations in their answers. One error that seemed common was the "halting signal" error (Sfard 2008, p. 214), in that once most obtained an answer of $\frac{1}{6}$ on Item 1, they felt that was sufficient.

For Prisca, even though they avoided their representations, their other methods did not yield correct answers. This could be so as Nascimento, Morais and Martins (2016) argued that internal representations are aided through external representations (Goldin, 2002). Since some students often disregarded their external representations they could not be expected to come up with correct answers. However, others managed to get correct answers despite their incorrect representations (see Tongai transcript). Some like Sara managed also without a diagram.

CONCLUSION

A main finding is that in the most, the representations constructed were not accurate. These were modelling errors (Chaput, Girard & Henry, 2011). The tree diagram representations were very popular but when these were made they often had two branches (even if the task was on tossing a six sided die, though two branches are also correct for someone experienced on the topic). This was in reminiscence to tossing coins; the first probability experiments they met in an earlier grade. To Sfard they were using the discourse of tossing a coin on that of a die. What was required was to slightly modify the discourse. In the interviews most students did not have faith in their representations and used other reasoning independent of the representations to come up with their answers. It was clear that they felt representations such as tree diagrams can help them to make sense of the tasks (Sedlmeier & Gigerenzer, 2001) but they felt that they were not yet good at constructing them. In other cases students made reasonable progress when they used partially completed tree diagrams. But they often failed to take advantage of the partially filled and completely filled tree diagrams to obtain correct answers. In this case, students were often not sure of what the questions required them to do despite their good representations.

References

Borovcnik, M. (2014). Modelling and experiments–An interactive approach towards probability and statistics. In T. Wassong, D. Frischemeier, P. R. Fischer, R. Hochmuth, & P. Bender (Eds.) *Mit Werkzeugen Mathematik und Stochastik lernen–Using Tools for Learning Mathematics and Statistics* (pp. 267-281). Fachmedien Wiesbaden: .Springer

Brodie, K., & Berger, M. (2010). Toward a discursive framework for student errors in mathematics. In V. Mudaly (Ed.) *Proceedings of the 18th meeting of SAARMSTE.* (pp. 196-181) Natal: University of KwaZulu-Natal,.

Chaput, B., Girard, J.-C., & Henry, M. (2011). Frequentist approach: Modelling and simulation in statistics and probability teaching. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching statistics in school mathematics-Challenges for teaching and teacher education* (pp. 85-95). The Netherlands: Springer.

Goldin, G. (2002). Perspectives on representation in mathematical learning and problem solving. In L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 178–203). New York, NY: Routledge.

Kvale, S. (1996). *Interviews: An Introduction to Qualitative Research Interviewing*, New York: Sage Publications.

Lee, H. S. & Doerr, H. M. (2016). A framework of probability concepts needed for teaching repeated sampling approaches to inference *Proceedings of the International Congress on Mathematical Education; Topic Study Group 14*. University of Hamburg, Hamburg, 24-31 July.

Michelet, S., Adam, J. M., & Luengo, V. (2007). Adaptive learning scenarios for detection of misconceptions about electricity and remediation. *International Journal of Emerging Technologies in Learning*, *2*(1), 1-5.

Nascimento, M., Morais, E. & Martins, J. (2016). Representations in probability problems. *Proceedings of the International Congress on Mathematical Education; Topic Study Group 14*. University of Hamburg, Hamburg, 24-31 July.

Pfannkuch, M., & Ziedins, I. (2014). A modelling perspective on probability. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic Thinking* (pp. 101-116). Dordrecht, The Netherlands: Springer.

Polya, G. (2014). *How to solve it: A new aspect of mathematical method*. NY: Princeton university press.

Sfard, A. (2007). When the rules of discourse change but nobody tells you: making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences*, *16*(4), 567-613.

Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing.* Cambridge: Cambridge University Press.

Sedlmeier, P., & Gigerenzer, G. (2001). Teaching bayesian reasoning in less than two hours. *Journal of Experimental Psychology: General, 130(3)*, 380-400.

Zahner, D., & Corter, J. E. (2010). The process of probability problem solving: Use of External Visual Representations. *Mathematical Thinking and Learning*, *12*(2), 177 - 204.

THE MATHEMATICS TEXTBOOK FOR RURAL POPULATION IN BRAZIL: LEARNING TO BE A MODERNIZED FARMER

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The Brazilian National Textbook Program has evaluated and distributed textbooks for rural population in mathematics. From a Foucaultian perspective, textbooks are conceived as a technology that governs. Through a discourse analysis, statements about the modernization of peasants' practices with and through mathematics are identified. The results show that textbooks use images and texts of good traditional rural lifestyle to contextualize mathematical activities. But at the same time, the idea that mathematics is necessary for the modernization of rural work to become effective and industrialized is constructed. School mathematics, as articulated in the textbooks, plays an important role in peasant subjectivation processes, being a powerful validation for the need to adopt modern and economically effective production.

INTRODUCTION

In 1985 the Brazilian government created the Nacional Textbook Program (PNLD), which monitors the elaboration of authorized textbooks for different compulsory school subjects to be used in all public schools. In 2013, the government opened for the creation of textbooks designed for rural primary schools. The textbooks should fulfill a pedagogical function adjusting information and concepts to the rural population; and a social function preserving a conception of rural forms of life as cultural spaces where knowledge is produced and where development can take place sustainably (Brasil, 2013, p. 27; Brasil, 2016, p. 41). The formulation of textbooks particularly designed for rural population is the result of the struggles of social movements such as The Landless Movement, fighting for regaining the right to ownership of land by poor rural population who have been displaced from small farms by large landowners. They understand education as "a key element for the social justice project they are attempting to build" (Knijnik, 2012). The political and educational principles in this movement are linked to the notions on how to improve peasants' life and work conditions, where collectivity, social justice, traditional farming techniques, familiar agriculture, agroecology, land reform, and others are important.

Two textbook collections were approved and produced. Each collection has five textbooks, since the primary school in Brazil goes from the first grade to fifth grade. The textbooks are very important for education in the country side since access to knowledge resources is limited. The textbooks are massively distributed, and each child gets a book for the school year. Therefore, these books play a central role in building ideas of the mathematical content and its purpose in the life of peasants.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 411-418). Umeå, Sweden: PME.

Following the socio-cultural-political axes developed in PME (Planas & Valero, 2016), we examine how mathematics textbooks for the rural population produce *effects of subjectivation* on how to be a competent farmer. The analysis interrogates the discursive formations that emerge in textbooks were notions of competence in mathematics intertwine with notions of what characterizes a good, desirable peasant.

MATHEMATICS TEXTBOOKS AND SUBJECTIVATION

Since World War II, mathematics plays an important role in the development of a narrative of economic progress and development for the individual and society. Yolcu (2017) and Valero (2017) have showed that the development of mathematical skills in children has been an important technology of government for modernity and modernization: "Math skills are proven to be fundamental to a person not only as a skilled workforce, but also as a citizen", to achieve "social progress, economic growth, and citizenship" (Valero, 2017, p. 123). Research on the cultural politics of mathematics education adopts the view that the teaching and learning of mathematics are not only cognitive or pedagogical processes. They are significant cultural spaces for making types of people through the objectivation of knowledge and the subjectivation of the individual - a key assumption of socio - cultural research (Radford, 2008). Researchers investigate the entanglement of school mathematics as part of the school curriculum and the processes of governing of the population to become good, competent citizens (Walkerdine, 1995). In other words, school mathematics, its teaching and learning are fundamental for the creation and maintenance of modern forms of life" (Valero & Knijnik, 2016, p. 1). This means that in classrooms and schools "we do not only teach mathematical concepts. The school disciplines people in very peculiar ways" (Silva, 2016, p. 51), and the textbook is an important technology of governmentality in this process.

Textbooks are a powerful technology to conduct the conduct of individuals because they mobilize "practices of representation" (Hall, 1997, p. 10) such as knowledge together with behaviors, patterns, habits and notions about what is appropriated in society: "Few instruments shape children's and young people's minds more powerfully than the teaching and learning materials used in schools. Textbooks convey not only knowledge but also social values and political identities, and an understanding of history and the world". (UNESCO, 2016, p. 1). In this way, when used as part of pedagogical processes, textbooks contribute to the creation of the learner's subjectivities and their sense of who they are meant to be and how the appropriation of school knowledge is meant to transform them into desired types of people.

Despite the assumptions of neutrality linked with school mathematics and science (Valero & Orlander, 2017), the process of subjectivation in particular directions is evident in textbooks. Spinik (2005) shows how in textbooks in Afghanistan between 1986 to 1992 "some Mujahideen groups developed maths exercises with examples of how to divide ammunition to maximize Soviet fatalities". In this case, the exercises both confirm and naturalize war practices, to create a sense of national identity by installing national values and ideals. The question about the effects of subjectivation produced in rural mathematics textbooks remains. Which images and text about what is better and necessary to be a good citizen and a competent farmer are present in rural mathematics textbooks?

THEORICAL-METHODOLOGICAL TOOLS

In order to preserve the work, we chose to nominate de two collections as "Collection A" and "Collection B". We conducted a discourse analysis of the ten textbooks and we also included the guidelines for teachers, which gives access to the orientations that publishers give to improve the teaching objectives. For Foucault, "discourses are more than ways of giving meaning to the world; they imply forms of social organization and social practices which structure institutions and constitute individuals as thinking, feeling and acting subjects" (Walshaw, 2016, p. 47). In this sense, we have identified the statements that constitute the discourse. Such statements are formed in discursive and non-discursive practices. They are mirror of a time and a place, and they relate to other statements to portrait desirable practices (Foucault, 1972).

For the data analysis mathematics tasks, exercises, examples, images, texts, and other strata are used to help us in the understanding of the ideas presented on how to be a competent farmer. The textbooks are made for primary schools, so images, characters and cartoons are very common in this textbook with the purpose of illustrating both the activities and contents. Therefore, the images are very important in our analysis process because they send messages about the practices and truths in the society, especially for the target audience of these textbooks. Collange, Almeida and Amorim (2014) argue that the materiality in the images allow to explore the systems of qualification set in operation in texts and thus create ideas of the content and how it should act in culture.

ANALYSIS

The images below belong to different textbooks and they illustrate mathematical exercises on estimation, counting and spatial localization as they are the mathematical content for primary school level (Brasil, 1997). In addition, these drawings show a rural context that appears to be a harmonic, simple and happy rural lifestyle. The drawings show the rural lifestyle like a good and healthy lifestyle: "To be a child is very good. To play in the countryside is very fun" (Figure 3). Teachers let students colour and count the objects so that they see mathematics in their every day.

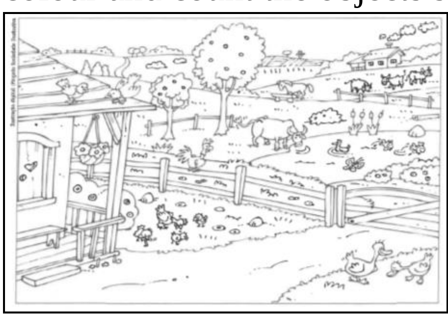

Figure 1

Figure 2

Figure 3

The figures 1 and 2 are from “Collection A” while figure 3 is from “Collection B”. In five textbooks of one of the collections, the aims of the collection and its ideological approaches were made explicit: “As part of current debates, the books strengthen peoples’ identity in their land, through production, meaning-making and systematization of school basic knowledge, in dialogue with the knowledge of the community where they are” (Collection A). In excerpts like this, the relation between rural lifestyle defended by The Landless Movement and the textbook is evidenced. In the three images above the farms are small, the territory is occupied by several houses located close to each other; different children with different skin colour live in these places. There are different types of crops and there are many animals scattered freely in large areas. These images promote the idea that the rural work with its traditional practices are good, a trend that has been commercially exploited to sell products through concepts such as “organic produce”, “happy meat” and meet from “animal welfare” (see Cole, 2011). The images in the textbooks are about rural lifestyle and work practices related to familiar agriculture, in other words, the drawings show us the notions about traditional farming techniques, agroecological and organic practices, land reform, and other very important concepts for the Brazilian Landless Movement. In these mathematics is an element present in the peasants’ everyday life. These images show us one specific type of spatial occupation that is totally opposite of the monocultural practices, often linked to the agrobusiness practices. In such a way, the textbooks are apparently in line with the principles of public policy in Brazil “related to the guarantee of rights and citizenship of the rural peoples understood in their identities and ways of life, as opposed to other projects related to the rural world or agribusiness” (Collection A p. 206).

However, throughout the textbooks other notions also appear. For example, in the first grade it is necessary to work with notions of grouping and estimation. In the curricular orientations in Brazil there is an explicit mention to this: “Organization in groupings to facilitate counting and comparison between large collections” (Brasil, 1997, p. 50). In one of the textbooks analysed this idea appears as follows:

Figure 4

Figure 5

The figures 4 and 5 are from “Collection A”. In the forty figure the activity invites to estimate how many animals there are in this farm without counting. The cattle are scattered, and all animals are free in the farm. The animals seem to live harmoniously with other species, such as shown above in figures 1, 2 and 3. However, in the 5th situation (Figure 5), the animals are counted, classified according to their species, separated in different fencing, and they are aligned. The activity leaves the children to understand that after estimation, counting means and optimization that organizes and separates the species. The optimization of estimation in counting is best because it can be done in less time and more accurately, as question B in figure 5 suggests: “Was it easier to count the animals in the farm São João [in figure 4] or in the farm São Pedro? We interpret that the figure 5 shows us the model of “factory farming” aligned with an economic rationality of efficiency and agro-industry. This can be observed in the objectives found in one of the textbooks, as an expression of new forms of knowing and acting that peasants needs now:

> It happens that social life and productive organization have been changing and demanding workers who, in addition to knowing how to perform their tasks, also plan and be creative. These changes are due to the economic reorganization of the capitalist countries, the dissemination of information and technological advances. (Collection A)

In other words, the work practices of peasants need to be modernized. And this idea is linked with increasing productivity, lower costs, optimizing work practices, space, time and human work force. That is, the peasants need to understand and to practice a different rationality of work. The necessity of modernization in rural forms of life is illustrated in Brown’s (2015) discussion of neoliberalism as a rationality of government in rural areas: “with the promise of giant crop yields and an end to struggling with pests, the agribusiness giants aim to convert farmers across the developing world from “traditional” to “modern” techniques, materials, *and* markets (p. 142). Brown show how in 2003, Iraqi farmers were “lured into the new agricultural techniques” (p. 145) by big corporations that stopped traditional practices and brought new elements with the argument of increasing Iraqi agricultural production. However, “the problem is that farming in general is uniquely vulnerable to fluctuations in nature, such as draughts and floods, and farming for export is also vulnerable to fluctuations in world markets” (p. 146). The new elements disregarded local specificities so, after a period of much losses because of production and the world market, the consequences were dramatic and ended in “an epidemic of farmer suicides” (p. 146). This example is extreme, it to illustrates the effects of the ideas mobilized in the order of discourse of which these mathematics textbooks are part.

In the textbooks, there is also an explicit valorization of the peasants’ practices. The goal of this exercise is to teach “percentages” in “problem solving”, in the of egg production. In the text that introduces an exercise, the “organic chicken” is constructed as healthier than the chicken that spend its life in big agrobusiness farms.

Criação de galinhas para produção de ovos e carne em sistema caipira

A "velha" galinha conhecida como "pé-duro" ou "caipira" dos terreiros e quintais pode botar de 50 a 80 ovos por ano. Mais de 80% das propriedades rurais possuem essas aves que têm contribuído para melhorar a alimentação das família, muitas vezes auxiliando como parte da renda na economia familiar. A galinha caipira está sendo cada vez mais valorizada por ser mais nutritiva e saudável que os frangos criados com ração em grandes granjas.

No sistema de produção caipira, a escolha do tipo de ave a ser criada é de fundamental importância; portanto, as aves passam por uma seleção.

São selecionadas aves que podem botar de 270 a 300 ovos ao ano e também aves especializadas para a produção de carne. Para que a ave possa ter uma ótima capacidade produtiva, deve-se dar atenção à sua nutrição, ao ambiente de criação, à higiene e ao manejo.

Disponível em: <www.ceplac.gov.br/radar/Artigos/artigo15.htm>. Acesso em: 30 jun. 2014. (Adaptado.)

The traditional hen "caipira" produces 50 to 80 eggs a year and is in more than 80% of farms. The hens have improved families' nutrition and economy. Therefore, they are valued because they are more nutritious than chicken raised in big farms. Birds that can produce 270-300 eggs per year are selected. Nutrition, the environment for their growth and hygiene are important.

Figure 6

The questions of the problem below:

De acordo com a reportagem, uma galinha caipira pode botar de 50 a 80 ovos por ano. Complete as frases com base nessa informação.

a) Uma galinha caipira produz de 50 a 80 ovos por ano.

b) Duas galinhas caipiras produzem de 100 a 160 ovos por ano.

c) Três galinhas caipiras produzem de 150 a 240 ovos por ano.

d) Quatro galinhas caipiras produzem de 200 a 320 ovos por ano.

In a report, a caipira hen can produce 50 to 80 eggs a year. Complete the sentences based on the information:

Two hens produce from ___ to ___ eggs a year.

Three hens produce ___

Four hens produce ____…

Figure 7

This exercise creates the idea that the eggs are produced as an industrial product, even though the hens are of the "organic" type. The estimative about the egg production increase with in same proportion as the amount of chickens. The production increases in a linear relationship to the number of hens. This is in line with the notion of "animal machines" (see Colle, 2011) and against notions of organic production. In sum, in addition to the estimative ignore that the animals cannot produce as machines, that is, the linear proportion cannot be used in this case, the notion about increasing production and profits remains around mathematic exercise.

The conflict between the appeal to the goodness of traditional countryside lifestyle, and the need of modernization and profitable optimization of production were illustrated above. The way in which mathematics appears in traditional settings but allows the peasant to act to better transform life into a modernized form of life is also recurrent in the textbooks. Both these ideas are intertwined in the discourse of the rural mathematics textbooks, and appear systematically in the illustrations as well as in the explanations and problems. In the textbook, the statement about the necessity of learning mathematics for the modernization of peasants' practices is mobilized in the discourse not only through their repetition, but also through other elements in the texts. The analysis of the images, the contents and the mathematics explores the uniqueness of the discourse mobilized in the textbooks, and give us the possibility of recognizing “the general form of a sentence, a meaning, a proposition” (Foucault, 1972, p. 101), through which notions of the desired mathematically competent peasant are put forward for learners.

CONCLUSION

The aim in this paper was not to identify a fixed or stable truth about what is means to be a mathematically competent peasant. After all, what is better or worse, what are the desirable identities of peasants are always in change in historically and situationally articulated discourses. These may sometimes be contradictory, as shown in the analysis above.

Despite the apparent contradiction between the four first figures and the fifth, we can claim that there is no contradiction. Instead the misfit evidences that the mathematics textbooks for rural population embody a project of people formation, in this case, a project of modernization of peasants’ practices. Mathematics education, with the technologies that make part of its practices, fabricate types of people. Popkewtiz (2004) has pointed to the governing effect of mathematics in the making of children’s subjectivity. This project is in full development, sending ideas about the competent, productive, and modernized peasant. The analysis helps us to highlight the regularities about peasants’ practices and the role of school mathematics in the validation, reproduction and propagation of a type of mathematically competent child that will turn into the subject who can change production in more profitable ways; that is, becoming neoliberal in themselves. As Foucault reminds us, discourse and subjectivity are entangled:

I shall abandon any attempt, therefore, to see discourse as a phenomenon of expression - the verbal translation of a previously established synthesis; instead, I shall look for a field of regularity for various positions of subjectivity. Thus conceived, discourse is not the majestically unfolding manifestation of a thinking, knowing, speaking subject, but, on the contrary, a totality, in which the dispersion of the subject and his discontinuity with himself may be determined. (Foucault, 1972, p. 55)

Therefore, we can claim that mathematics education through textbooks have an important role in the subjectivation process, because it is used as a tool to validate and to produce the statements that we had identified in this article.

References

BRASIL (2013). Edital de Convocação (05/2011 – CGPLI) para o Processo de Inscrição e Avaliação de Obras Didáticas para o Programa Nacional do Livro Didático do Campo PNLD Campo.

BRASIL (2016). Edital de Convocação (04/2014 – CGPLI) para o Processo de Inscrição e Avaliação de Obras Didáticas para o Programa Nacional do Livro Didático do Campo PNLD Campo.

Brown, W. (2015) *Undoing the demos*: neoliberalism's stealth revolution.Vanderbilt Street, Brooklyn, New York 11218

Collection A. (2014) *Coleção Campo Aberto*. São Paulo. Global Editora.

Collection B (2014) *Novo girassol*: saberes e fazeres do campo. São Paulo: FTD.

Collange, M. S.; Almeida, C. A. & Amorim, A. C. (2014) Natureza em imagens de livros didáticos de Biologia do Ensino Médio. *Revista da SBEnBio*. Número 7.

Foucault, M. (1972) *The archaeology of knowledge.* (World of man). Translation of archeologie du savoir. Includes the author's The Discourse on Language, translation of ordre du discours. Manufactured in the United States of America.

Hall, S. (1997) *Representation*: Cultural Representations and Signifying Practices.

Knijnik, G. (2009) Mathematics Education and The Brazilian Landless Movement: Three Different Mathematics in The Context of The Struggle for Social Justice. *Critical Issues in Mathematics Education* / edited by Paul Ernest, Brian Greer, Bharath Sriraman

Planas, N., & Valero, P. (2016). Tracing the Socio-Cultural-Political Axis in Understanding Mathematics Education. In A. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education. The Journey Continues* (pp. 447-479). Rotterdam: Sense Publishers.

Popkewitz, T. S. (2004). The Alchemy of the mathematics curriculum: Inscriptions and the fabrication of the child. *American Educational Research Journal*, (pp. 3-34), *41*(1).

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in Mathematics Education: Epistemology, History, Classroom, and Culture* (pp. 215-234). Rotterdam: Sense.

Silva, M. A (2016). Investigações Envolvendo Livros Didáticos de Matemática do Ensino Médio: a trajetória de um grupo de pesquisa. *Jornal Internacional de Estudos em Educação Matemática* (pp 36-54), v. 9, n.3 São Paulo.

UNESCO. (2016) *Textbooks pave the way to sustainable development*. Global Education Monitoring Report.

Valero, P. (2017) Mathematics for all, economic growth, and the making of the citizen-worker. In T. S. Popkewitz, J. Diaz, & C. Kirchgasler (Eds.), *A political sociology of educational knowledge:* Studies of exclusions and difference (pp. 117- 132). New York: Routledge.

Valero, P., & Knijnik, G. (2016). Mathematics education as a matter of policy. In M. A. Peters (Ed.), *Encyclopedia of Educational Philosophy and Theory* (pp. 1-6). Singapore: Springer Singapore.

Valero, P. & Orlander, A.A. (2017) Democracy and Justice in Math and Science Curriculum. *Oxford Research Encyclopedia of Education***.** New York: Oxford University Press.

Walkerdine, V. (1995) O raciocínio em tempos pós-modernos. *Revista Educação & Realidade* (pp. 207-226), v. 20, n.2, v. 20, n.2, jul./dez jul./dez Porto Alegre.

Walshaw, M. (2016). Michel Foucault. In: E. De Freitas & M. Walshaw. *Alternative theoretical frameworks for mathematics education research: Theory meets data*. Springer.

COGNITIVE ABILITIES AND MATHEMATICAL REASONING IN PRACTICE AND TEST SITUATIONS

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Research studies have shown that to develop conceptual understanding of mathematics, practice needs to that focus this skill. In this study, the aim is to examine how different practice tasks, which promotes either imitative or creative mathematical reasoning, can influence which variables (i.e., cognitive abilities, mathematics grade, and gender) that are important for task completion. Two earlier studies show that cognitive abilities are more important in the test situation when students have practiced with imitative tasks. The result from this study indicate that although cognitive abilities are important when practicing with creative tasks, the influence of cognition is only implicit during the test. Since students often practice imitatively with given solution methods, this study suggests that a substantial part of what we test in school could be cognitive abilities rather than mathematics.

INTRODUCTION

Many studies have shown the inefficiency of rote-learning that transpires without understanding (e.g., Hiebert, 2003). Hiebert and Grouws (2007) argue that students need to struggle with important mathematical concepts or properties in order to get a deeper understanding of mathematics. This positive productive struggle could occur if the task involves some desirable difficulties that forces the student to regard the mathematical properties of the task. Bjork and Bjork (2011) argue that desirable difficulties are important whatever you are trying to learn, and can, while not as efficient at first, be more efficient in the long run (Fyfe & Rittle-Johnson, 2016). It is however important that the imposed difficulty should be surmountable and relate to the subject or skill you are about to learn (Bjork & Bjork, 2011). Otherwise it would be enough to just turn out the lights to create difficulty in the mathematics classroom. But since this has nothing to do with mathematics it would be an obstruction rather than a desirable difficulty. Desirable difficulties can however induce some amount of failure during task solving, but this might not necessarily be a bad thing. In a number of studies Kapur explored productive failure as an instructional design, and found that it can be effective for developing conceptual understanding of mathematics (e.g., Kapur, 2010; Kapur, 2015).

Studies of mathematics textbooks have shown that most textbook tasks lack the difficulties and struggle that Bjork and Bjork (2011) and Hiebert and Grouws (2007) are arguing for. Most textbook tasks are procedural and can mostly be solved by provided

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 419-426). Umeå, Sweden: PME.

solution methods or by looking at worked examples (e.g., Jäder, Lithner, & Sidenvall, 2015; Newton & Newton, 2007; Shield & Dole, 2013). Jäder et al. (2015) concluded in their cross-national study of textbooks from twelve countries that only 9% of the tasks required more extensive conceptual knowledge and justification, while 79% of the tasks could be solved completely by imitating or following given instructions. The textbooks also contain more tasks than what is reasonable for a student to solve during a course, and the more demanding tasks are commonly located to the last part of each section. This implies that many students have to select which tasks to solve. Since they tend to choose the basic tasks first they will often not reach the more demanding tasks at the end of the section (Sidenvall, Lithner, & Jäder, 2015). Bergqvist and Lithner (2012) observed that teacher presentations also are dominated by procedure and that most presentations consider how tasks should be solved rather than the concepts or properties behind the procedures. Conceptual instruction has proven to be more beneficial than procedural instruction when trying to promote a more thorough understanding of procedures and concepts (Rittle-Johnson, Fyfe, & Loehr 2016). Hence, more conceptual tasks seem to be needed in both teacher instruction and textbooks.

FRAMEWORK

Lithner (2008) proposed a research framework for mathematical reasoning where he concludes that there are two main types of reasoning that can occur while solving mathematical tasks, imitative and creative. Imitative algorithmic reasoning (AR) occurs where a solution method is already known or presented in close proximity to the task, so that the student can imitate or recall a solution method. An understanding of the concepts or mathematical properties is not imperative for this type of reasoning to solve the task at hand. The second type, creative mathematically founded reasoning (CMR), concerns student reasoning where no solution method is available. Not giving a solution method in advance force the students to consider mathematical properties when constructing a valid solution method. There is of course an effort involved in this process which is not necessary when solving a task imitatively and this effort, or struggle if you will, is close to what Hiebert and Grouws (2007) argued for. However, as most textbooks provide procedural solution methods, imitative reasoning is normal practice for most students in school.

From a theoretical perspective, Brousseau (1997) states that it is imperative for students to take responsibility for their own solution process. Brousseau argues that for this to happen, the teacher has to hand over responsibility to the students after constructing a well-designed task where the students can, with some work and arguments, construct the solution by themselves. When students work with tasks where they can imitate a given solution method, this argumentation and construction will not take place. Brousseau denotes the activity where the students work alone to solve the task an adidactical situation. In a study where students practice solution methods by either imitative or creative reasoning, Jonsson, Norqvist, Liljekvist, and Lithner (2014) utilized this adidactical situation to study the effectiveness of the different practice tasks (AR or CMR). The study showed that practicing with tasks that promote CMR is more effective with regard to test scores than practicing with AR-tasks. Two additional studies have confirmed this result (Norqvist, 2018; Wirebring et al., 2015).

Studies have also shown that students' cognitive abilities is important for mathematical achievement (e.g., Primi, Ferrão, & Almeida, 2010; Swanson & Alloway, 2012). For example, working memory (i.e., the ability to store information whilst processing other information) is highly correlated to mathematical achievement (Swanson & Alloway, 2012) and has been shown to be predictive of mathematics learning in the early school years (Passolunghi, Vercelloni & Schadee, 2007). Another cognitive ability that is closely related to mathematics is fluid reasoning, which is related to faster mathematical learning (Primi et al., 2010).

Jonsson et al. (2014) also showed that cognitive abilities (i.e., working memory and fluid reasoning) are important for test scores, especially for students that practice by AR. To rule out that this was not contributed to similarities between CMR-practice tasks and the test tasks that were used, a follow-up study was made. Here transfer-appropriate processing was contrasted to productive struggle to see if the higher test performance could be attributed to similarities in task design (Jonsson, Kulaksiz, & Lithner, 2016). The results showed that transfer appropriate processing accounted for only a minor part of the efficiency of the CMR-group. However, since Jonsson et al. (2014) and Norqvist (2018) did focus on test scores and the efficiency of AR and CMR, neither of the studies gave much notice to the practice scores and which variables that were important for the two practice conditions. A study of this could help us understand why CMR seems to be more efficient and at the same time give us a clue to why AR appears to be efficient during practice but not when it comes to the post-test.

AIM AND RESEARCH QUESTIONS

In the previous studies (Jonsson et al., 2014; Norqvist, 2018), there are strong indications that cognition play an important role for solving test tasks, especially for students that have practiced by AR. Jonsson et al. (2016) also showed that the difference in test-scores between the two practice groups (AR and CMR) was not attributed to similarities between practice- and test-task, so called transfer appropriate processing. However, the practice session could also help us understand why CMR-practice has proven to be more efficient as measured by test scores. The aim of this study is therefore to examine which of the measured variables (i.e., mathematics grade, gender, and cognitive abilities) influence students task solving during practice, depending on what type of reasoning the students utilize during practice. It is also interesting to examine if there is any difference between practice and test, regarding taxation on cognitive abilities, since this could be of importance for teaching.

1. How will the combined sample affect earlier results of the importance of cognitive abilities for the to test-scores?

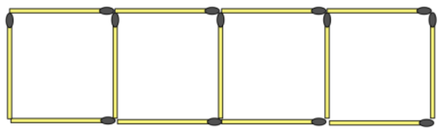
2. How does the practice condition, AR or CMR, affect which variables (i.e., mathematics grade, gender, and cognitive abilities) that are most influential on students' completion of the given practice tasks?

METHOD

Participants

The present study utilize data gathered in two earlier studies where the efficiency of different mathematics tasks was in focus. The samples in (Jonsson et al., 2014) and (Norqvist, 2018) comprised a total of 252 students in the natural science program in Swedish upper secondary school (16-17 y.o.). 44 students were excluded due to attrition. Before the analysis the practice and test data was scanned for outliers, and to compensate for eventual ceiling or floor effects participants that had the maximum score, or that scored lower than 10%, during practice where removed from the sample. This control excluded 13 participants from the sample. Also, in Norqvist (2018) 38 students were practicing with a third task type, and these students were also excluded from this study. Finally, 157 participants remained in the sample. In both studies, the participating students were divided into matched practice groups based on cognitive abilities (i.e., working memory and fluid reasoning), mathematics grade, and gender.

When squares are put in a row it looks like the figure to the right. 13 matches are needed for four squares.



If x is the number of squares, then the number of matches y can be calculated by the function

$$y=3x+1$$

Example: If 4 squares are put in a row, then $y=3x+1=3\cdot 4+1=13$ matches are needed.

How many matches are needed to get 20 squares in a row?

Figure 1: Example of an AR practice-task. The text written in italics is absent in the corresponding CMR-task.

Practice and test

The data collection was partly designed to mimic a common situation in school, where students often practice by solving textbook tasks of an AR-type, and often meet more complex tasks in a test. The students practiced 14 solution methods (i.e., formulas) by either AR- or CMR-tasks depending on the group (see example in Figure 1). During practice the teacher did not intervene and the students were instructed not to talk to each other. Since each AR-task is faster to complete than the corresponding CMR-task, the AR-group did more tasks for each solution method to get a comparable practice time. A computer software recorded solution frequencies and the time the students spent on each task.

One week later the students took a test where each solution method was tested with three tasks. The first asked for the formula used during practice while the second and third task asked for a numerical answer. The first two test-tasks were limited to 30 seconds each so that no (re)construction could take place, while the last task was limited to 5 minutes. In this study, cognitive data, mathematics grade, gender, practice-scores (i.e., the proportion of correct answers to the practice-tasks), and test-scores (i.e., the proportion of correct answers to the test-tasks) from the mentioned studies are analyzed to find answers to the research questions.

Analysis method

Results from the cognitive tests (i.e., Operation span that tests working memory and Raven's progressive matrices which is a test for fluid reasoning) were standardized and used to calculate a cognitive index. The practice-score was transformed to compensate for skewness (-3.66) and kurtosis (18.3) of the practice-score for the AR-group. This transformed practice-score was later used in the following analyses.

Two separate analyses were conducted on each of the practice-groups. First, a regression analysis, with test-score as the dependent variable and practice-score, CPI, gender, and mathematics grade as independent variables, was performed. This was done to control that the results presented in Jonsson et al. (2014) and Norqvist (2018) were valid for the combined sample as well (i.e., that cognitive abilities influenced the test-scores to a higher extent in the AR-group than in the CMR-group). Secondly, another regression analysis was performed, with practice-score as the dependent variable, to see to what degree the independent variables (i.e., cognitive index, mathematics grade, and gender) did influence students' completion of the given practice-tasks. All statistical analyses were made in SPSS 24.

RESULT

	AR			CMR		
Variables	*B*	*SE B*	*ß*	*B*	*SE B*	*ß*
Cognitive index	.504	.359	$.359^{**}$	.117	.099	.104
Mathematics grade	.115	.308	$.308^{**}$	.021	.027	.074
Practice score	.411	.175	.175	.836	.127	$.665^{*}$
Gender	.067	.046	.046	.048	.143	.027
F total		8.057^{*}			22.681^{*}	
Adjusted R^2		.284			.543	

$^{*}p < .001$, $^{**}p < .01$.

Table 1: Regression Analysis Summary for Variables Predicting Test Score.

The first regression analysis showed that the previous results, regarding the difference in relation between cognitive abilities and test-scores for the different practice groups, were valid for the combined sample. Cognitive index and mathematics grade were highly predictive of test-score in the AR-group while practice-score did predict the test-score in the CMR-group (see Table 1).

The second regression analysis showed that none of the included variables were predictive of the practice-score for the AR-group, while the CMR-practice score is highly dependent on cognitive index and mathematics grade (see Table 2).

	AR			CMR		
Variables	*B*	*SE B*	*ß*	*B*	*SE B*	*ß*
Cognitive index	.201	.119	.208	.301	.086	.366*
Mathematics grade	.043	.032	.167	.107	.022	.468*
Gender	-.022	.188	-.014	.044	.134	.031
F total		2.432			14.555*	
Adjusted R^2		.057			.358	

*p < .001.

Table 2: Regression Analysis Summary for Variables Predicting Practice Score.

DISCUSSION AND CONCLUSION

Introducing CMR-practice as an alternative has been shown to be more efficient than the procedural AR-tasks that constitutes the main part of textbook tasks (Jonsson et al., 2014; Norqvist, 2018). The results from the regression analyses confirm, as indicated in (Jonsson et al., 2014), that students that practice by AR are more dependent on cognitive abilities during the test situation than students practicing by CMR. This does not seem to be attributed to transfer-appropriate processing but rather to the productive struggle that the CMR-participants meet during practice (Jonsson et al., 2016). The results from the regression analysis on practice-scores show that CMR-practice is more taxing on cognitive abilities than AR-practice. This confirms that CMR-tasks are creating some struggle for the students and, as Hiebert and Grouws (2007) argued, this would then yield higher test scores. The desirable difficulties that CMR-tasks provides will not only help students to focus the important mathematical properties, but could actually force them to take these properties into account. AR-practice lacks the desirable difficulty that CMR-practice provides and the difficulty will therefore emerge during the test instead. Hence, the higher strain on cognitive abilities.

The importance of the practice-score for later test-performance in the CMR-group would indicate that successful CMR-practice leads to a deeper processing of the learning material, which in turn leads to easier retrieval during the test. The importance of a good practice-score for the CMR-students could of course be problematic for students with lower cognitive abilities or lesser mathematical skills, but this is where the teacher comes in. If the teacher (or maybe the author of the textbook) has designed the tasks well, so that the task will help students to consider the important mathematical properties, the teacher can support student learning by asking questions aimed at these properties. The difficulty could hereby be reduced but not removed completely, and it would still concern the important mathematics.

During AR-practice there are less difficulties for the students since they can use the given solution method to solve the task. Also, if a student would have difficulties with an AR-task, the focus would most likely be on how to use the given information (e.g., arithmetic difficulties when calculating the answer or how to exchange a variable with a number), and not mainly on the intrinsic mathematical properties in the task.

If the ability to solve novel problems or solve tasks that require transfer of knowledge is something that students are supposed to show during a test, they should have had the opportunity to practice these skills during lessons. Practicing with mainly AR-tasks gives an impression that tasks are standardized and short where the solution method is known (e.g., all tasks in the section can be solved by setting up and solving a linear equation). This can be preferable if the aim is to develop computational fluency but not if the aim is to develop conceptual understanding (Hiebert & Grouws, 2007). Students that only meet computational difficulties and never have to give any thought on why a solution method works will not be prepared to find new methods or solve novel tasks, neither during the upcoming test nor in their future working life. It would therefore be important to have textbooks and teacher presentations that include more tasks that compel students to consider mathematical properties to overcome the desirable difficulties of the task. Otherwise, there is a risk that the upcoming test will measure cognitive abilities rather than mathematical knowledge.

References

Bergqvist, T., & Lithner, J. (2012). Mathematical Reasoning in Teachers' Presentations. *Journal of Mathematical Behavior*, 31(2), 252-269.

Bjork, E. L., & Bjork, R. A. (2011). Making things hard on yourself, but in a good way: Creating desirable difficulties to enhance learning. In M. A. Gernsbacher, R. W. Pew, L. M. Hough, & J. R. Pomerantz (Eds.), *Psychology and the real world: Essays illustrating fundamental contributions to society*. (pp. 56-64). New York, NY, US: Worth Publishers.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht; Boston: Kluwer Academic Publishers.

Fyfe, E. R., & Rittle-Johnson, B. (2016). Mathematics practice without feedback: A desirable difficulty in a classroom setting. *Instructional Science*, 1-18.

Hiebert, J. (2003). What research says about the NCTM standards. In J. Kilpatrick, W. G. Martin, D. Schifter, & National Council of Teachers of Mathematics. (Eds.), *A research companion to Principles and standards for school mathematics* (pp. x, 413 p.). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 371–404). Greenwich, CT: Information Age.

Jonsson, B., Kulaksiz, Y. C., & Lithner, J. (2016). Creative and algorithmic mathematical reasoning: effects of transfer-appropriate processing and effortful struggle. *International Journal of Mathematical Education in Science and Technology*, 1-20.

Jonsson, B., Norqvist, M., Liljekvist, Y., & Lithner, J. (2014). Learning mathematics through algorithmic and creative reasoning. *The Journal of Mathematical Behavior, 36*, 20-32.

Jäder, J., Lithner, J., & Sidenvall, J. (2015). Reasoning requirements in school mathematics textbooks: an analysis of books from 12 countries. In J. Jäder (Ed.), *Elevers möjligheter till lärande av matematiska resonemang,* Licenciate thesis. Linköping: Linköping University.

Kapur, M. (2010). Productive failure in mathematical problem solving. *Instructional Science, 38*(6), 523-550. doi:10.1007/s11251-009-9093-x

Kapur, M. (2015). Learning from productive failure. *Learning: Research and Practice, 1*(1), 51-65. doi:10.1080/23735082.2015.1002195

Lithner, J. (2008). A Research Framework for Creative and Imitative Reasoning. *Educational Studies in Mathematics, 67*(3), 255-276.

Newton, D., & Newton, L. (2007). Could Elementary Mathematics Textbooks Help Give Attention to Reasons in the Classroom? *Educational Studies in Mathematics, 64*(1), 69-84.

Norqvist, M. (2018). The effect of explanations on mathematical reasoning tasks. *International Journal of Mathematical Education in Science and Technology*, *49*(1), 1-16.

Passolunghi, M. C., Vercelloni, B., & Schadee, H. (2007). The Precursors of Mathematics Learning: Working Memory, Phonological Ability and Numerical Competence. *Cognitive Development, 22*(2), 165-184.

Primi, R., Ferrão, M. E., & Almeida, L. S. (2010). Fluid intelligence as a predictor of learning: A longitudinal multilevel approach applied to math. *Learning and Individual Differences*, *20*(5), 446-451.

Rittle-Johnson, B., Fyfe, E. R., & Loehr, A. M. (2016). Improving conceptual and procedural knowledge: The impact of instructional content within a mathematics lesson. *British Journal of Educational Psychology, 86*(4), 576-591.

Shield, M., & Dole, S. (2013). Assessing the potential of mathematics textbooks to promote deep learning. *Educational Studies in Mathematics, 82*(2), 183-199.

Sidenvall, J., Lithner, J., & Jäder, J. (2015). Students' Reasoning in Mathematics Textbook Task-Solving. *International Journal of Mathematical Education in Science and Technology, 46*(4), 533-552.

Swanson, H. L., & Alloway, T. P. (2012). Working memory, learning, and academic achievement. In K. R. Harris, S. Graham, T. Urdan, C. B. McCormick, G. M. Sinatra, & J. Sweller (Eds.), *APA educational psychology handbook, Vol 1: Theories, constructs, and critical issues* (pp. 327-366). Washington, DC, US: American Psychological Association.

Wirebring, L. K., Lithner, J., Jonsson, B., Liljekvist, Y., Norqvist, M., & Nyberg, L. (2015). Learning mathematics without a suggested solution method: Durable effects on performance and brain activity. *Trends in Neuroscience and Education, 4*(1-2), 6-14.

ARE ADULTS BIASED IN COMPLEX FRACTION COMPARISON, AND CAN BENCHMARKS HELP?

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When people compare simple fractions, the natural number components can interfere with processing of fraction magnitudes ("natural number bias"). There is conflicting evidence about whether this also occurs for complex fraction comparisons. We asked 107 university students to solve complex fraction comparison problems. Fractions varied in their relative positions to benchmarks (i.e., reference numbers such as 0, ¼, ½, ¾, or 1), which may help people activate fraction magnitudes. We found a "smaller components—larger fraction" bias in participants with lower mathematical experience and a reduced bias in participants with more experience. The benchmarks 0 and 1 facilitated performance and reduced the bias; effects of other benchmarks were small. The study highlights the variability of the natural number bias.

THEORETICAL BACKGROUND

Many people struggle with fractions (Lortie-Forgues, Tian, & Siegler, 2015; Van Dooren et al., 2016). One source of difficulties with fractions is people's overreliance on natural number reasoning in fraction problems. When people have to choose the larger of two fractions, the fractions' natural number components can interfere with their reasoning about the overall fraction magnitudes, resulting in "natural number bias" (Ni & Zhou, 2005). Studies document that students and adults are more accurate and faster on simple comparison problems that are *congruent* (larger fraction has larger component, e.g., 3/5 > 2/5) rather than *incongruent* (larger fraction has smaller component, e.g., 1/3 > 1/4) with natural number reasoning (Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013).

To account for the finding that even older students and adults who have developed sound concepts of fractions still show a natural number bias, researchers used dual-process theories to describe the cognitive mechanisms that may underlie the bias (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009; Vamvakoussi et al., 2012, Van Hoof et al., 2013). According to dual-process theories, fraction processing may trigger fast and intuitive System 1 processes, namely the automatic activation of natural number magnitudes. In fact, cognitive research shows that people process magnitudes of natural number symbols automatically even when doing so is irrelevant for the task (Hubbard, Piazza, Pinel, & Dehaene, 2005). These System 1 processes may interfere with slower and analytic System 2 processes, namely the more effortful activation of overall (holistic) fraction magnitudes.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 427-434). Umeå, Sweden: PME.

Although previous empirical evidence is in line with the dual-process account, an important limitation is that many studies on the natural number bias have used fraction pairs with common numerators or common denominators (see examples above), or pairs with very familiar fractions (e.g., ¾). Studying these special cases of fraction comparison problems limits our understanding of the natural number bias because in comparisons with common components, there is no need to actually reason about the overall fraction magnitudes. Thus, performance differences may be due to the different strategies people use for different problem types, rather than due to the interference of natural number magnitudes with fraction magnitudes. When comparing fractions with common components, people rely largely on comparing the unequal components rather than the fraction magnitudes (Obersteiner & Tumpek, 2016; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013). For familiar fractions, on the other hand, activating magnitudes may be strongly automated. It is, therefore, unclear how component magnitudes may interfere with fraction magnitudes in fraction comparison problems that are more complex.

Few studies have explored the natural number bias in more complex comparisons of unfamiliar fractions without common components (e.g., 19/24 vs. 25/36), and the results of these studies are inconclusive. Obersteiner, Van Hoof, and Verschaffel (2013) found no natural number bias in academic mathematicians. However, DeWolf and Vosniadou (2011) did find a natural number bias (better performance on congruent rather than incongruent comparison problems) in a sample of university students. Yet other studies with university students found a "reverse" bias (better performance on incongruent rather than congruent problems) (Barraza, Avaria, & Leiva, 2017; DeWolf & Vosniadou, 2015). One explanation for these conflicting findings could be that the occurrence and strength of the natural number bias depends on the interaction of a number of factors, such as problem types, mathematical ability or experience, and strategy use. People may be less biased if they focus more strongly on fraction magnitudes rather than on fraction components, and whether they activate fraction magnitudes may depend on their mathematical ability or experience (Alibali & Sidney, 2015). Following this assumption, supporting people to reason about fraction magnitudes in complex problems (those in which relying on component strategies only is not efficient) may help them overcome a potential bias.

One potential way to activate fraction magnitudes is to use "benchmarks" (Liu, 2017). Benchmarks are common numbers that serve as references. They allow easy access to approximate fraction magnitudes, which are often sufficient for solving fraction comparison problems. For example, to decide that 19/24 > 25/36, one can use 3/4 as a benchmark: 19/24 > 3/4 while 25/36 < 3/4, hence 19/24 > 25/36 (Clarke & Roche, 2009; Fazio, DeWolf, & Siegler, 2016). Because benchmark strategies include reasoning about (approximate) overall fraction magnitudes, using such strategies may allow people to avoid comparing fractions componentially. Therefore, problems that can be solved by such benchmark strategies may be less prone to the natural number bias than problems in which no common benchmarks are available.

AIMS AND RESEARCH QUESTIONS

The aim of this study is to investigate how various factors influence the occurrence of the natural number bias in complex fraction comparison. We address three questions: 1) Do university students with lower and higher mathematical experience show a natural number bias in complex fraction comparison? 2) Does the occurrence of the natural number bias depend on whether problems can be solved with benchmark strategies? 3) Does encouraging people to use benchmarks enhance their performance and reduce the natural number bias?

We expected to find a natural number bias, particularly in participants with less mathematical experience, and we expected that this bias would be stronger for problems that could not be solved by benchmark strategies than for problems that afforded benchmark strategies. We also expected that encouraging people to use benchmarks would lead to better performance and a reduced bias, and that this effect would depend on how well people were able to spontaneously adapt their strategies to the affordances of the problems.

METHODS

Participants

Participants were 107 university students (48 male, 59 female; mean age = 20.0 years) in the United States. Participants were divided into two groups on the basis of their mathematics course work, a lower-experience group (less than 2 semesters of calculus; $n = 53$) and a higher-experience group (two semesters of calculus or more; $n = 54$). As part of the instructions for the experiment, about half of the participants ($n = 57$) received a tip that using benchmarks such as ½, ¼, or ¾ can be helpful in solving fraction comparison problems. The example “5/8 vs. 3/7” with 5/8 > 1/2 and 3/7 < 1/2 was provided to illustrate the benchmark strategy.

Procedure

Participants were asked to solve 56 fraction comparison problems on a computer as quickly and accurately as possible. On each trial, two fractions appeared on the screen next to each other, and participants indicated the larger fraction by pressing the left (“f”) or right (“j”) key on a regular keyboard. Response times and accuracy were recorded using E-Prime software.

Comparison Problems

None of the fraction pairs included common components. All fractions were smaller than 1, and most fraction components were two digits. Half of the problems were congruent and half were incongruent. Within the congruent and incongruent problem subsets, there were three categories depending on the fraction magnitudes relative to the potential benchmarks 0, ¼, ½, ¾, and 1: In “straddling” problems, one fraction was smaller and the other larger than ¼, ½, or ¾ (thus “straddling” a benchmark). In “in-between” problems, both fractions were in between two adjacent benchmarks. A special case of "in-between" problems, and therefore a separate category, were problems in which fractions were both smaller than ¼ or both larger than ¾, because in these problems, one fraction was close to 0 or 1 ("0-1" problems), and 0 and 1 may be especially salient benchmarks. A number of factors were controlled across problem types, most notably the average numerical difference between fractions. Across problem types, the average differences ranged from 0.12 to 0.15. The problems were presented to the participants in random order.

RESULTS

We analysed the data using a general estimation equation (GEE) procedure to account for correlated measures within participants (Nussbaum, 2015). Accuracy data was analysed using binary logistic regression, and response time data was analysed using a linear regression model with a logarithmic link function. For the analysis of response times, we excluded incorrectly solved problems and problems for which response times were more than two standard deviations from the individual participant's mean.

In each analysis, the within-subject factors were *congruency* (congruent/incongruent) and *benchmark* (0-1/straddling/in-between); between-subject factors were *tip* (yes/no) and *mathematical experience* (lower/higher). Table 1 presents Wald statistics for analyses of accuracy and response times.

Participants with higher mathematical experience scored higher overall than participants with lower mathematical experience (Estimated Marginal Means and Standard Errors for accuracy: M_{EM} = 88%, SE = 1.3, vs. M_{EM} = 82%, SE = 1.6), but participants in both experience groups had similar response times (M_{EM} = 3.81 sec, SE = 0.24, vs. M_{EM} = 3.61 sec, SE = 0.20). Participants who received the tip were less accurate, on average, than those who did not receive the tip (M_{EM} = 83%, SE = 1.8, vs. M_{EM} = 87%, SE = 1.2), but participants in both conditions had similar response times (M_{EM} = 3.74 sec, SE = 0.24, vs. M_{EM} = 3.67 sec, SE = 0.20).

For both accuracy and response times, there was a significant main effect of *congruency*, with better overall performance on incongruent rather than congruent problems (M_{EM} = 89%, SE = 1.1, vs. M_{EM} = 81%, SE = 1.4 for accuracy; M_{EM} = 3.59 sec, SE = 0.15, vs. M_{EM} = 3.82 sec, SE = 0.17 for response times), indicating a "smaller components—larger fraction" bias. There was also a main effect of *benchmark*, such that participants were most accurate and fastest on "0-1" problems (M_{EM} = 87%, SE = 1.2, and M_{EM} = 3.32 sec, SE = 0.15), followed by "straddling" problems (M_{EM} = 85%, SE = 1.3, and M_{EM} = 3.91 sec, SE = 0.17) and "in-between" problems (M_{EM} = 83%, SE = 1.2, and M_{EM} = 3.93 sec, SE = 0.17); the difference between the latter two problem types was not significant, either for accuracy or response times. For response times, these main effects were qualified by a *congruency* x *benchmark* interaction, such that the (reverse) natural number bias was observed for "in-between" problems (M_{EM} = 3.74 sec, SE = 0.16 for incongruent, vs. M_{EM} = 4.12 sec, SE = 0.20 for congruent) and "straddling" problems (M_{EM} = 3.76 sec, SE = 0.17 vs. M_{EM} = 4.07 sec, SE = 0.18) but not for "0–1" problems (M_{EM} = 3.30 sec, SE = 0.15 vs. M_{EM} = 3.33 sec, *SE* = 0.16). This two-way interaction in turn was qualified by a three-way *congruency* x *benchmark* x *math experience* interaction, such that the *congruency* x *benchmark* interaction was present only for participants with lower mathematical experience and not for those with higher mathematical experience (see Figure 1). Post hoc comparisons revealed that for participants with higher mathematical experience, the differences between congruent and incongruent problems were not significant for any of the benchmark types. No other main effects or interactions were significant.

	Accuracy			Response Time		
Source	Wald X^2	df	*p*	Wald X^2	df	*p*
Math Experience	**6.83**	**1**	**.009**	0.41	1	.520
Tip	**4.16**	**1**	**.041**	0.05	1	.819
Benchmark	**11.68**	**2**	**.003**	**80.83**	**2**	**<.001**
Congruency	**26.34**	**1**	**<.001**	**7.87**	**1**	**.005**
Math Experience x Tip	0.20	1	.652	2.38	1	.123
Math Experience x Benchmark	0.49	2	.783	4.88	2	.087
Math Experience x Congruency	0.00	1	.988	0.87	1	.351
Tip x Benchmark	0.34	2	.842	1.42	2	.492
Tip x Congruency	0.22	1	.637	0.13	1	.716
Benchmark x Congruency	1.78	2	.410	**7.44**	**2**	**.024**
Math Experience x Tip x Benchmark	2.66	2	.265	3.51	2	.173
Math Experience x Tip x Congruency	0.01	1	.940	0.17	1	.680
Math Experience x Benchmark x Congruency	1.44	2	.487	**6.51**	**2**	**.039**
Tip x Benchmark x Congruency	1.53	2	.466	0.04	2	.981
Math Experience x Tip x Benchmark x Congruency	5.16	2	.076	0.19	2	.910

Table 1: Effects of the GEE estimation procedure (significant effects in bold).

DISCUSSION

This study shows that for complex fraction comparison problems that cannot be solved by using 0 and 1 as benchmarks, adults show a “smaller components—larger fractions” bias, and this reverse natural number bias is more pronounced in adults with less mathematical experience (at least for response times). One explanation is that people with less mathematical experience “overgeneralize” their understanding that large fractions can be composed of small natural numbers. Rinne, Ye, and Jordan (2017) identified such an understanding as a typical intermediate step in children’s learning, which occurs before they reach a full understanding of fraction magnitudes. Considering that percentage correct was very high overall, however, it seems unlikely that the participants in this study had a limited understanding of fraction concepts comparable to young children.

Another potential explanation is that participants, particularly those with less mathematical experience, may rely on specific strategies that are more successful in incongruent than in congruent comparison problems, such as “gap thinking” (Clarke & Roche, 2009, Fazio et al., 2016). Gap thinking involves reasoning about the differences (rather than the quotients) between the numerator and the denominator of each fraction, and choosing the fraction with the smaller difference as the larger fraction (e.g., 19/24 > 25/36 because 24 – 19 = 5, which is smaller than 36 – 25 = 11). Although this strategy is mathematically incorrect, it leads—by definition of our problem types—to correct responses for all incongruent problems but not for all congruent problems. Participants’ individual preferences to use specific strategies may also explain the negative effect of providing a tip about using benchmarks on accuracy. To learn more about the effects of individual strategies on performance, we are currently analyzing verbal reports of strategy use on the same fraction comparison problems in another sample of participants.

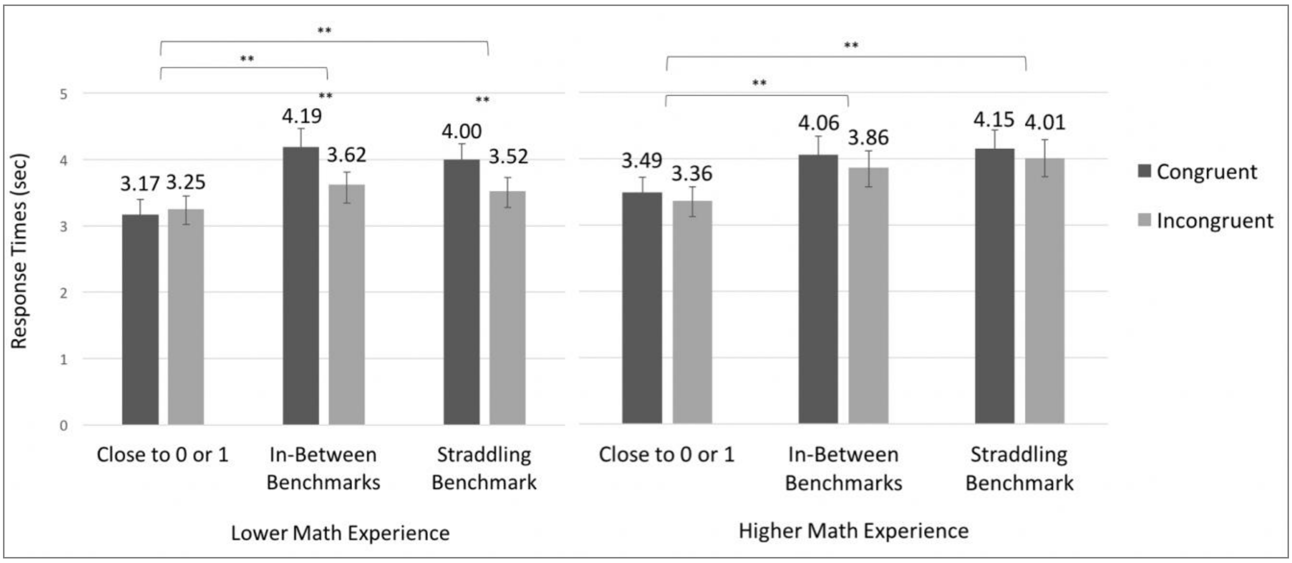

Figure 1: Response times for the lower (left) and higher (right) math experience groups, for congruent and incongruent problems, as a function of benchmark type. Note: **p < .001

This study further suggests that 0 and 1 are especially salient benchmarks that people use to solve fraction comparison problems when they are available. Activating these magnitudes enables people to overcome the natural number bias. This conclusion is in line with the more general claim that holistic reasoning about fraction magnitudes is an important aspect of understanding fraction concepts, as it can reduce the influence of quickly available magnitudes of the fraction natural number components (Siegler & Lortie-Forgues, 2014).

In conclusion, this study highlights variability in the natural number bias in fraction comparison problems. Our findings suggest that the occurrence of the bias depends on problem type and individual mathematical experience. Together with previous research, our study suggests that the fractions' natural number components can bias people in different directions (towards "larger components—larger fraction", or the opposite), depending on their experience and on their available strategies.

This conclusion is relevant for mathematics education: When teaching fractions, teachers may wish to make students aware of their potential implicit biases (Van Hoof, Vamvakoussi, Van Dooren, & Verschaffel, 2017). Furthermore, they may offer ways to "escape" the natural number bias. One way to do so could be to encourage students to reason about fraction magnitudes rather than their components by using benchmarks. For example, a productive exercise could be sorting comparison problems into categories depending on whether the two fractions are close to benchmarks or straddle a benchmark that may help in comparing their magnitudes. Another productive exercise could be discussing various strategies that can be used to compare fraction magnitudes. These exercises require reasoning about core concepts of fraction magnitude rather than only about procedures. As such, they may enhance students' fraction number sense and reduce natural number bias.

Acknowledgements

This research was supported by a Feodor Lynen research grant from the Alexander von Humboldt Foundation.

References

Alibali, M. W., & Sidney, P. G. (2015). Variability in the natural number bias: Who, when, how, and why. *Learning and Instruction, 37*, 56-61.

Barraza, P., Avaria, R., & Leiva, I. (2017). The role of attentional networks in the access to the numerical magnitude of fractions in adults / El rol de las redes atencionales en el acceso a la magnitud numérica de fracciones en adultos. *Estudios de Psicología, 38*(2), 495-522.

Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics, 72*, 127–138.

DeWolf, M., & Vosniadou, S. (2011). The whole number bias in fraction magnitude comparisons with adults. In L. Carlson, C. Hoelscher, & T. F. Shipley (Eds.), *Proceedings of the 33rd Annual Meeting of the Congnitive Science Society* (pp. 1751–1756). Austin, TX: Cognitive Science Society.

DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction, 37*, 39-49.

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 42*, 1–16.

Gillard, E., Van Dooren, W., Schaeken, W., & Verschaffel, L. (2009). Dual processes in the psychology of mathematics education and cognitive psychology. *Human Development, 52–108*.

Hubbard, E. M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience, 6*(6), 435-448.

Liu, F. (2017). Mental representation of fractions: It all depends on whether they are common or uncommon. *The Quarterly Journal of Experimental Psychology*, 1-38.

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning of fractions and decimal arithmetic so difficult? *Developmental Review, 38*, 201–221.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: the origins and implications of whole number bias. *Educational Psychologist, 40*, 27–52.

Nussbaum, E. M. (2015). Categorical and nonparametric data analysis. Choosing the best statistical technique. New York: Taylor and Francis.

Obersteiner, A., & Tumpek, C. (2016). Measuring fraction comparison strategies with eye-tracking. *ZDM Mathematics Education, 48*(3), 255-266.

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction, 28*, 64-72.

Obersteiner, A., Van Hoof, J., & Verschaffel, L. (2013). Expert mathematicians' natural number bias in fraction comparison. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 393–400). Kiel, Germany: PME.

Rinne, L. F., Ye, A., & Jordan, N. C. (2017). Development of fraction comparison strategies: a latent transition analysis. *Developmental Psychology, 53*, 713–730.

Siegler, R. S., & Lortie-Forgues, H. (2014). An integrative theory of numerical development. *Child Development Perspectives, 8*(3), 144–150.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *Journal of Mathematical Behavior, 31*, 344–355.

Van Dooren, W., Van Hoof, J., Verschaffel, L., Gómez, D. M., Dartnell, P., A., O., . . . Gabriel, F. (2016). Understanding obstacles in the development of the rational number concept—searching for common ground. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 383–387). Szeged, Hungary: PME.

Van Hoof, J., Lijnen, T., Verschaffel, L., & Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research in Mathematics Education, 15*(2), 154-164.

Van Hoof, J., Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2017). The transition from natural to rational number knowledge. In D. C. Geary, D. B. Berch, R. J. Ochsendorf, & K. Mann Koepke (Eds.), *Acquisition of complex arithmetic skills and higher-order mathematics concepts* (pp. 101-123). San Diego: Academic Press.

SECOND-GRADERS' PREDICTIVE REASONING STRATEGIES

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This paper reports predictive reasoning strategies used by ten second-graders in a classroom design study. A modelling activity based upon real data required students to predict maximum monthly temperatures for the current year using the natural variation provided by readings from the previous six years. The development of reasoning strategies was documented throughout the lesson sequence by analysis of responses to written prompts, videos of interviews and student drawn graphs. Student predictions reflected an emerging understanding variability, clusters and mean. Reasoning strategies became increasingly sophisticated using TinkerPlots, and with repeated opportunities for students to observe, represent, reflect upon trends in data.

STATISTICAL UNDERSTANDING IN THE ELEMENTARY YEARS

Recognising variability, grouping data according to attributes, and creation and representation of data sets are key competencies required for the development of statistical literacy in young students (English, 2012). However, it is widely understood that the development of statistical reasoning is a complex and not necessarily linear process, which is heavily reliant upon students' real-life experiences. Recent research suggests that young students draw upon personal context as well as the actual data values (Ben-Zvi & Aridor-Berger, 2016). Using data sets as evidence from which to make inferences proves a challenging process for young students, with many typically over generalising, or relying upon data such as very small sample sizes (Makar, 2016). Studies conducted in inquiry-based classrooms, where students have the opportunity to develop statistical investigations in a low stakes environment, allows exposure to data analysis strategies prior to formal instruction in statistics (English, 2012). By intentionally providing ambigious predictive tasks with multiple possible solutions, students can engage in meaningful statistical investigations and form predictions through a process of making sense of the information provided (Makar, Bakker, & Ben-Zvi , 2011).

Traditionally, predictive tasks for young students have centred on random response generators such as dice and spinners (Falk, Yedilevich-Assouline, & Elstein, 2012). However, prediction in 'real life' usually encompass both a degree of randomness alongside predictable, causal variation. Exposure to complex modelling activities including both these elements provides opportunities for developing reasoning skills in young students, alongside challenging their deterministic thinking and encouraging their observations of long-term trends. Examples of such tasks include modelling plant growth (Lehrer & Schauble, 2004; Mulligan, 2015) making predictions from picture books (English, 2012), examining first-graders' shoe size (Makar, 2016), and using

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 435-442). Umeå, Sweden: PME.

children's self-portraits to predict the age of the artist (Oslington, Mulligan, & Van Bergen, 2018). Natural phenomena such as weather and tides also provide rich opportunities for predicting and modelling variability (English, Fox, & Watters, 2005).

Konold and Pollatsek (2002) argue that data analysis from a student's perspective is fundamentally one of recognising a signal in a noisy environment, and this concept is accessible to students perhaps as early as the age of eight. The visual representations provided by TinkerPlots (Konold & Miller, 2005) software—an analytical tool designed to support students statistical reasoning —allows students to recognise both signal and noise for a given population. Through physically creating representations through drag and pull manipulations the software supports the development of informal statistical inference in the student. In the present study TinkerPlots was utilised with young children.

PURPOSE OF STUDY

The study reported is drawn from a larger design study incorporating four, multi-lesson instructional sequences examining students' development of reasoning skills and generalisation using TinkerPlots. The analysis presented here examines students' interpretation of a two-way table of data, and the strategies used for predicting missing values. The research questions were:

1. How do students use variability in given values to predict an unknown value?

2. What strategies are students using to justify their data choice?

3. How are students' interpretation of data reflected through their use of probabilistic language?

METHOD

Student participants

High achievers were selected from a cohort of 42 mixed ability second-graders attending an independent school in Sydney, Australia, through an assessment-based interview focused on pattern and structure (Mulligan, Mitchelmore, & Stephanou, 2015). Ten students (mean age 7 years 10 months) were withdrawn from the classroom for four weekly lessons using a design-based approach (Gravemeijer & van Eerde, 2009) with the researcher acting as teacher. The students had had previously exposure to TinkerPlots and were familiar with saving files, entering data and creating plots.

Design and Procedure

For the four-lesson sequence described here, students used variable natural data in the form of maximum temperature readings sourced from the Australian Bureau of Meteorology as a scaffold for predicting, representing and explaining their understanding of variability. During Lesson 1 students were provided with a data table containing maximum monthly Sydney temperatures for 2010-2016. A final row, labelled 2017 was left blank. Students worked in teacher-selected pairs to determine preferred values, with one pair of students videoed during the process. Once completed, they responded to the prompt “Write down anything you noticed about the numbers” and then individually represented the table data through tables or drawings. After completion of the representations, eight students participated in semi-structured video interviews explaining their representations. In Lesson 2, students responded to three written prompts “Is there anything you noticed about the data?” “How did you decide to choose these numbers?” and “How certain are you that the numbers you have chosen are accurate?” Students were then shown a graph of the data in TinkerPlots, and responded to the prompt “What patterns do you notice?” In Lesson 3, students added their predicted values to the Lesson 2 TinkerPlots graph. In Lesson 4 students used their own TinkerPlots graphs to respond to the following prompts “What do you notice about your data?”, “Do you want to change any of your numbers?” and “Why or why not?” Nine students then participated in individual semi-structured interviews. At all points within the lesson cycle, students were free to reflect upon and refine their data choices, thus providing the opportunity for the development of their own explicit mathematical interpretation (Mulligan, 2015).

Data sources and analysis

There were three data sources: student-constructed data sets and responses to written prompts (**written)**, transcripts (**oral)** from Lesson 1 and 4 semi-structured interviews and the transcripts of two videoed interviews in Lesson 1, and **representational** (hand-drawn graphs from Lesson 1 and TinkerPlots graphs from Lesson 3). In this preliminary report, an overview of emerging reasoning for each student is described as three levels of statistical reasoning: idiosyncratic, transitional and quantitative (Leavy, 2008).

Student-constructed data sets were coded on a three-point scale according to awareness of seasonal patterns and proximity of predictions to existing data set (no awareness, some awareness, close approximation). Student responses to written prompts (Lessons 1, 2a 2b and 4) were coded on a five-point scale of increasing generalised thinking, (nil/idiosyncratic response, individual scores observations only, observing differences between months, generalised comment about shape of data, generalised comment combined with awareness of mean, range or outliers). Oral responses from Lesson 1 were coded as for written prompts. Thematic coding was developed from the transcript of the videos from Lesson 1 and the semi-structured interviews in Lesson 4 (Flick, 2014). Hand-drawn graphs were coded according to levels of structural development, namely pre-structural, emergent, partial, structural and advanced structural (Mulligan, Hodge, Mitchelmore, & English, 2013). TinkerPlots graphs mirrored the data tables, and the same coding applied. Given the diversity of data sources, a range of analysis was used for trial purposes, with preliminary results only presented here.

RESULTS

Student predictions

The monthly predictions created by the five student pairs all showed at least some awareness of seasonality and the idea of range (Table 1). For pairs 1, 3 and 5, the seasonal pattern was distinct and the predicted values fell within or very close to the range of the data. Pair 2 showed a slight seasonal dip, however the predictions were well outside the range, by as much as 10°C in the case of July. Similarly, Pair 4's winter dip was only apparent in June and August with other values outside the range.

Pair	Jan	Feb	Mar	April	May	Jun	July	Aug	Sept	Oct	Nov	Dec
1	43	40	36	32	29	25	27	28	31	35	38	40
2	45	40	39	35	32	31	36	42	42	43	43	45
3	38	36	29	26	23	21	24	27	30	36	39	41
4	47	41	31	30	27	24	29	20	32	39	41	42
5	35	41	30	31	26	22	21	22	24	27	32	39

Table 1: Student pairs predict monthly maximum temperatures for 2017

In their responses to "Write down anything you notice" (Lesson 1) six students referred to data shape (Pairs 1, 5) and identification of seasons (Pair 3) thus interpreting the table as representing one unit of information. In contrast, Pair 2 focused on individual data points. For example, "The hottest numbers were 46, 41, 42, 39 and 38" (Ashley) and "The hottest temperatures were all forties and thirties" (Julian). This difference in understanding was apparent in their representations: Fritz (Pair 1) and Lanni and Rhys (Pair 2) produced structural or advanced structural representations, showing awareness of equal spacing, partitioning, structured counting, shape and alignment and sequencing as described by Mulligan et. al. (2013). Rhys and Fritz's representations were presented as line graphs, showing their understanding of the data as a continuous sequence. Although these students were experienced in reading two-way tables, this task required interpretation of 72 data points, which could be viewed in rows (years) or columns (months), as well as a global set of repeating cycles. Most student pairs 'read' the table from top to bottom, with emphasis upon seasonality, using range, proximity to other values and seasonal knowledge to assist. Julian and Ashley, in contrast, highlighted individual data points with Julian tabulating each temperature, with the number of occurrences written below it and Ashley selecting and colour coding temperatures for each year. When prompted at interview, Julian was unable to identify seasonal patterns. Ashley focused upon rows (i.e. calendar years) rather than columns (months) of data, stating that "in 2016 there were two hot temperatures".

Students' reasoning behind predictions

All three data sources show students used personal experience, knowledge of seasons and the data table to make predictions. Although not all students could correctly name the months belonging to each season, these could be deduced from the table. Joseph for example, explained "Well, I found out that were (sic) the season are, so in the middle it is winter and then it heats up on the sides". Students' drew on memories of hot days of the year, weather reports and specific events, such as birthdays to link months and temperatures. Completing the table on a rather cold day at the end of August was complicated by students experiencing a relative heatwave the previous week. Rose wrote: "2017 was a very hot year. Winter felt like summer."

The most powerful predictor for most students was the data itself. Students' used the table to demonstrate an understanding of variability and emerging generalisations. Joseph and Stuart (Pair 3) reflected explicitly about their process: "I started with the middle and ceep (sic) going up by 2-3 degrees and on the left side going down by 3-2 degree" (Joseph) and "I started at June which divides the year in half, because I knew it would be cold then I did January and December which would be the hottest and I went in" (Stuart). Caden and Aaron (Pair 4) both describe selecting numbers around the other numbers. The transcript from Pair 5 reveals explicit discussion of range (March and April), months being "all in the 20s" (May), November as "like summer", and June being cold. This pair then observed an annual cycle by adjusting their December figure in line with the one they had produced for January.

Use of TinkerPlots

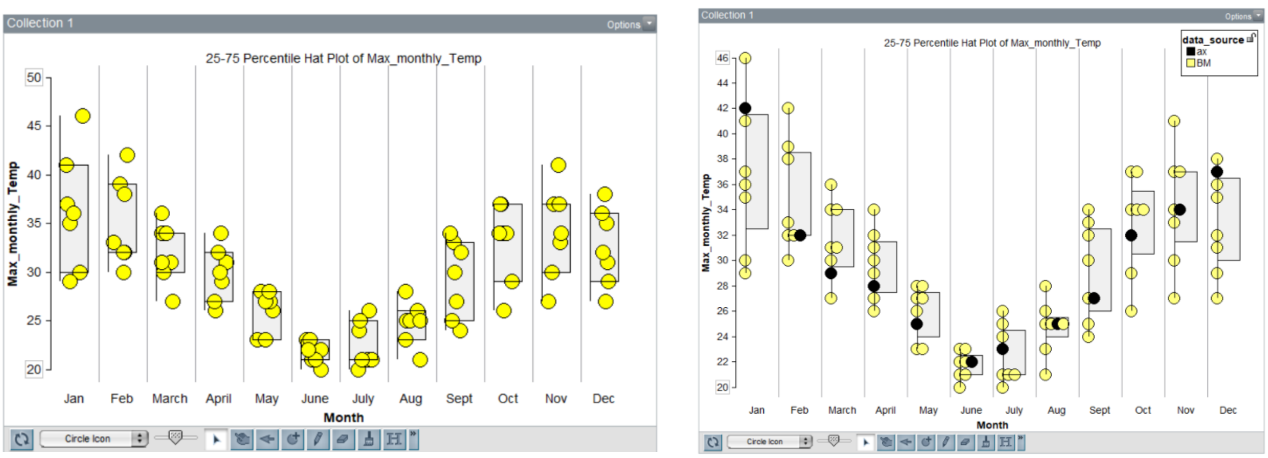

Figure 1 Maximum Sydney temperatures plotted using TinkerPlots software (left) and values including a student's estimations (right)

Transferring the data from a table to TinkerPlots format (Figure 1) assisted all students to recognise seasonality, and to display increasingly sophisticated statistical reasoning. Eight students interpreted the data spread as greater for warmer months: for example, Joseph responded, "It looks like the hotter the temperature is, the more spread out they are, like January or November, and the colder it is the more together they are like June". Caden described January as having a "big range, while June has a small, tight range". When working only from the table, Aaron wrote: "I notice no patterns", but after viewing the data graphically (Figure 3a) his response was "I see January is high and it gets lower and higher again." When compared with Leavy's (2008) three levels of statistical reasoning, eight of the ten students' explanations moved to a higher level after using TinkerPlots and five were at the third (quantitative) level, usually characteristic of students in grades 3 and above. The students' familiarity with the software potentially gave them access to concepts such as range and average, previously inaccessible for grade 2 students (Table 2).

Lesson	Fritz	Rose	Julian	Ashley	Joe	Stuart	Aaron	Cadel	Rhys	Lanni
1	trans	trans	trans	idio	trans	quan	trans	trans	trans	trans
2a	idio	idio	trans	idio	idio	idio	idio	trans	#	#
2b	quan	trans	quan	trans	quan	trans	quan	quan	trans	idio
4	quan	quan	quan	idio	quan	trans	trans	trans	trans	#

Table 2: Levels of second-graders statistical reasoning over four lesson sequence: idiosyncratic, transitional and quantitative # missing data

After plotting their own data against the table temperature (see sample Figure 2), students justified their data choice. Student explanations included "reasonableness" (Fritz), global warming (Julian), match to TinkerPlots data (six students), data located in the 'hat" (Fritz, Rose, Joseph), similarity to table figures (Rose, Caden) and seasonality (Stuart). When discussing their data, students either directly used or implied the following terms: outliers, clusters, spread, range, hats and quartiles, data shape, levels of certainty of prediction and discrepancies between their chosen values and the provided data table (Table 3).

Lesson	Fritz	Rose	Julian	Joe	Stuart	Aaron	Cadel	Rhys
Outliers	*					*		
Clusters	*	*				*		
Spread	*						*	*
Range	*						*	*
Hats	*	*	*	*	*			*
Data shape					*			*
Uncertainty	*		*	*	*			*
Certainty		*		*	*	*		
Discrepancies			*				*	

Table 3: Terms expressed by second-grade students when describing data variability

DISCUSSION

The task described here required students to predict temperatures for current and future dates by interpreting a two-way table data set. By using relevant data which involves natural variation, students could use their developing data-analysis strategies while simultaneously attending to other features relevant to them. These included recent experiences, knowledge of the seasons and weather reports. As in other studies (English, 2012), the data table itself formed the foundation for predictions, specifically student observations of maximums, minimums and range. When justifying their temperature figures, the TinkerPlots representation featured prominently in explanations, with students describing their data as “reasonable”, “looking right” or located in the TinkerPlots hat. Previous studies (Ben-Zvi & Aridor-Berger, 2016) show that young students move between the content i.e. the provided data and their personal context when developing inferential reasoning skills. Early efforts at data modelling typically show a disconnect between the two worlds, with integration emerging over time and modelling experiences. In the study presented here, students integrated the content and the context appropriately. However, this process is still in development, as some students over generalised about climate change or linked a whole month from a single remembered day. Makar (2016) argues that repeated exposure to such predicting and organising activities provides a positive experience with informal statistics and a growing awareness of both variability and generalisation.

The language used by the students included specific statistical concepts such as hats, quartiles, spread, range, clusters and outliers. The complexity and ambiguity of the task along with visual accessibility provided by TinkerPlots, supported student reasoning at level unexpected for such young students. According to Makar et al. (2011) inferential reasoning requires a statement of generalisation beyond the data, using data to support the generalisation and recognition of uncertainty. Through their generalisations about temperature cycles and the use of a data table as a principal resource for predicting, and through balancing certainty and uncertainty, these students are well on the way to meeting these criteria. Prolonged and focused activities, to which these students had had previous exposure, greatly supported their statistical reasoning. Multiple studies have reported on the development of students’ data analysis strategies in the early school years (English, 2012; Leavy, 2008; Mulligan, 2015, Oslington et al., 2018). This study extends this research by demonstrating the potential of TinkerPlots as a viable tool for developing data analysis strategies by students as young as the second-grade.

CONCLUSION

This study affirms previous research with young students demonstrating their capacity to utilise a data table to predict missing values and to use context to support their predictions. TinkerPlots allowed students to increase their level of statistical reasoning. This is a small, specialised study conducted with capable students, and thus the results cannot be generalised to all second-grade students. Nevertheless, they suggest that TinkerPlots can be used as a viable tool for data analysis by second-grade students, and the study confirms the feasibility of providing rich modelling tasks for developing statistical reasoning in young students.

References

Ben-Zvi, D., & Aridor-Berger, K. (2016). Children's wonder how to wander between data and context. In D. Ben-Zvi, & K. Makar, *The teaching and learning of statistics: International perspectives* (pp. 25-36). Springer .

English, L. (2012). Data modelling with first-grade students. *Educational Studies in Mathematics, 81*(1), 15-30.

English, L., Fox, J., & Watters, J. (2005). Problem posing and solving with mathematical modeling. *Teaching Children Mathematics, 12*(3), 156-163.

Falk, R., Yedilevich-Assouline, P., & Elstein, A. (2012). Children's concept of probability. *Educational Studies in Mathematics, 81*, 207-233.

Flick, U. (2014). *An introduction to qualitative research* (5 ed.). London, UK: Sage.

Gravemeijer, K., & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *The Elementary School Journal, 109*(5), 510-524.

Konold, C., & Miller, C. D. (2005). *TinkerPlots: Dynamic data exploration.* Emeryville, CA: Key Curriculum Press.

Konold, C., & Pollatsek, A. (2002). Data anaylsis as the search for signals in a noisy process. *Journal for Research in Mathematics Education, 33*(4), 259-289.

Leavy, A. (2008). An examination of the role of statistical investigation in supporting the development of young children's statistical reasoning. In O. N. Saracho, & B. Spodek, *Contemporary perspectives on mathematics in early childhood education* (pp. 215-232). Charlotte, NC: Information Age Publishing.

Lehrer, R., & Schauble, L. (2004). Modeling natural variation through distribution. *American Educational Research Journal, 41*(3), 635-679.

Makar, K. (2016). Developing young children’s emergent inferential practices in statistics. *Mathematical Thinking and Learning, 18*(1), 1-24.

Makar, K., Bakker, A., & Ben-Zvi , D. (2011). The reasoning behind informal statistical inference. *Mathematical Thinking and Learning, 13*(1-2), 152-173.

Mulligan, J. (2015). Moving beyond basic numeracy: data modeling in the early years of schooling. *ZDM Mathematics Education, 47*, 653-663.

Mulligan, J., Hodge, K., Mitchelmore, M., & English, L. (2013). Tracking structural development through data modelling in highly able grade 1 students. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Proceeding of the 36th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 530-536). Melbourne, Vic: MERGA.

Mulligan, J., Mitchelmore, M., & Stephanou, A. (2015). *Pattern and Structure Assessment: an assessment program for early mathematics.* Camberwell, Victoria: ACER Press.

Oslington, G., Mulligan, J., & Van Bergen, P. (2018). Young children's reasoning through data exploration. In V. Kinnear, M. Y. Lai, & T. Muir, *Forging connections in early mathematics teaching and learning.* Springer.

PRODUCTIVE WAYS OF ORGANISING PRACTICUM – WHAT DO WE KNOW? A SYSTEMATIC REVIEW

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The starting point for this review are questions on the empirical base for the organization of practicum. Selecting peer reviewed, empirically based articles for 2001-2017, with a focus on mathematics teacher education and the practicum, resulted in the inclusion of 51 articles for review. Exploring the outcomes and student teachers' experiences of practicum suggested that responsibility for teaching together with support from mentors, university lecturers, university coursework, peers or prompts to use a theoretical framework improves learning outcomes in practicum, and the length of time in a school context does not do so on its own.

INTRODUCTION

This paper presents a systematic review of empirical studies about mathematics student teachers' practicum. There is widespread agreement that practicum is an important part of teacher education (cf. Cochran-Smith & Zeichner, 2005; Grossman & McDonald, 2008), for future teachers of all levels and disciplines. However, a review of empirical research concludes that the reviewed articles give a "cloudy view of student teaching's contribution" (Anderson & Stillman, 2013, p. 36) in relation to desired outcomes from practicum. As teacher educators and researchers, we seek a better understanding of what existing research can, and cannot, tell us about the role and contribution of the practicum in teacher education. The purpose of this paper is to systematically synthesize and map findings from the empirically based knowledge related to practicum for pre-service mathematics teachers on what has been found to be productive ways of organizing practicum.

An earlier review of research on teacher education found more research within *mathematics* teacher education compared to other content areas of teacher education (Cochran-Smith & Zeichner, 2005). A systematic review on research on mathematics teacher education revealed a large number of studies focusing researchers' own practice, including efforts to demonstrate that a particular program works (Adler, Ball, Krainer, Lin & Novotna, 2005). They also found the majority of studies to have a narrow scope, and that few studies addressed student teachers' learning from experience beside the different reform contexts. Furthermore, few discussed the possibility of scaling up locally developed programs to multiple sites in new contexts.

We are deeply engaged in the practice of mathematics teacher education, and thus are interested in results which can inform practice. What comes to count as a productive organization of practicum must relate to the desired outcomes. In earlier research, it has been described how different kinds of knowledge is seen as important in the two

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 443-450). Umeå, Sweden: PME.

contexts, sometimes imposing a gap between theoretical and practical knowledge (see for instance Lampert, 2010, Zeichner, 2010). Our position is that a productive practicum allows transfers and integration of knowledge from both contexts. The term reasoned judgement is used by Rusznyak and Bertram (2015) to describe the specialised knowledge, content knowledge as well as general pedagogic and contextual knowledge teachers draw on when motivating decisions in the classroom. We have found a similar position on the image of the desired teacher knowledge can be traced in several of the reviewed articles (see Christiansen & Österling, 2018). We assume that the position taken on what counts as a desired learning outcome will also affect what counts as a productive way of organizing practicum.

It is currently challenging to get an overview of what research findings can tell us about the contribution of practicum to mathematics teacher education, and we make this review in order to learn from existing research how the organisation of practicum is related to desired outcomes, and the specificity, if it exists, with respect to mathematics student teachers. This gives rise to the following research questions:

- What are productive ways of organizing mathematics teacher education around practicum?
- Are there specific elements or characteristics of practicum in Mathematics, and if so, what are they?

The organization and role of practicum differ internationally, and go by different names. In this paper, we will use *practicum* to describe the phenomena of teacher education taking place in a school context in all its forms.

There are also various terms used for the prospective teachers, the practicing teachers who mentor them in schools, and the university staff who engage with the practicum element. We have chosen to use the terms *student teacher*, or *student* for short, for the prospective teachers during their education; *learner* for the pupils or school students, *mentor* for the practicing teachers, and *lecturer* for the academic university staff.

METHOD

Our first decision was to include only peer reviewed journal articles. We limited the search to 15 years, searching articles published 2001-June 2017. The journal *Pythagoras* was however only electronically available to us from 2004. The majority of papers were published in the last ten years. Once the potentially relevant articles were selected within each journal, resulting in a total of 107 articles, we checked that the article did indeed concern mathematics teacher education and practicum; that it reported on empirical research; and that it had to do with mathematics teaching; hence, these were our inclusion criteria. In addition, we decided not to include single case studies (exclusion criterion). This process resulted in a dataset of 51 articles. Using each article as the unit of analysis, we summarized it according to country of the data, aspect of practicum in focus, scope, type of participants, methods, and theoretical perspectives. In the present paper, only results concerning the posed research questions are summoned.

RESULTS

Productive ways of organising practicum

Researching the **effects of duration of practicum**, it appears that it is quality, not quantity that matters. Chinese student teachers' MKT was affected by the completion of courses and responsibility for teaching during practicum, not the length of student teaching or number of mathematics courses in university (Youngs and Quian, 2013). No relations between the length of practicum and the outcome was found in a study in an analysis of secondary data from 1044 respondents (Jacobson, 2017); instead, providing early practicum with possibilities to teach, together with corresponding campus courses, had effects on students' mathematical knowledge, and on their beliefs about active learning and maths-as-inquiry. Several other outcomes were found to be related to support from lecturers or mentors, as a higher perceived ability to carry out instructional tasks, attention and noticing in teaching, an interconnectedness between efficacy in teaching mathematics and managing the classroom, related to an ability to engage learners' thinking more and using less whole class instruction.

Several qualitative studies aimed at providing an understanding of **when and how learning in practicum takes place**. These studies all concluded that mentoring or other prompting support is important for the different learning outcomes from practicum. A number of studies from the USA revealed how student teacher developed their engagement with learners' participation or mathematical thinking. In a Turkish case study, the use of number patterns in school algebra was used to explore the contribution of practicum to student teachers' pedagogical content knowledge (PCK) (Yeşildere İmre & Akkoç, 2012). They found that students develop PCK conditional to the cooperating teacher displaying the necessary PCK. Positive changes were found in 142 elementary student teachers' attitudes towards mathematics after their student teaching, however, the previous mathematics methods course was the most important reason for students reporting negative attitude (Jong & Hodges, 2015). Four studies used a lesson-study intervention, and all reached the conclusion that lesson study supported a development of students' attention to learners' mathematical thinking. Five studies investigated interventions where students analysed the teaching of others. All such interventions were found to result in improved reflections, in terms of theorizing, attention to learners, questioning, use of research, or analysis of teaching.

Eight studies used interventions where **students engaged in specified techniques**, such as inquiry based mathematics, statistical investigations, integration of literature in mathematics, pedagogical difficulties, a bulletin board community, action research or Learning Bridges thematic practicum. Most studies reported positive results, but some unwanted outcomes were found. Heaton and Mickelson (2002) found US students unable to transfer their statistical PCK to teaching; Karp (2010) reported remaining difficulties with insufficient knowledge of curriculum materials after the intervention; and students found it difficult to define their own "tender spot" as a starting point for action research (Amir, Mandler, Hauptman & Gorev; 2017). Using concept mappings, interviews and journal writing of 51 secondary mathematics and physics student teachers, the teacher centered approach was found to be regarded as easier to perform and difficult to challenge (Özgün-Koca & Sen, 2006).

We found few studies on perspectives of teacher **professional identity and positioning.** These studies demonstrate how timing of theory and practice, good examples of collaborating practice and possibilities to participate in a professional conversation with peers, mentors or lecturers was important to student teachers' learning and identity formation (Kaasila & Laurila, 2011; Hodges & Hodge, 2015).

Reflections in relation to teaching could be regarded as a means for learning to teach, however, in the reviewed studies, reflections are regarded as a learning outcome, and it has not been explored how reflections assists students to develop their teaching. What constitutes reflections vary, so does the context of reflecting, and therefore different results are reported. Studies focusing refections as a learning outcome categorises reflections according to quality, breadth or depth. Another focus is the content and argument in reflections, including the use of theoretical or professional language. **Contextual factors** found to improve reflections are mentoring conversations, working with peers or integrating coursework in reflections on practicum, with an increased use of theories in reflections. Some results were more complex, where an evaluative approach in mentoring conversations was found to be in conflict with the reflective approach (Johnsson & Højnes, 2009), or a lesson-study intervention found that reflections on mathematics lessons did not follow the pattern of other subjects, focusing less on mathematics and more on learners after the intervention (Helgevold, Næsheim-Bjørkvik & Østrem, 2015). Studies also combine the **contextual explanations with theoretical aspects**, as when Kaasila and Lauriala (2012) found that the depth of reflections to a large extent depended on the experience of the Finnish student teachers, but could be improved through the reading of research articles. Improved areas were level of reflection, tendency to ground reflections in evidence, the way students analyzed learner thinking, and their use of pedagogy and learner thinking as bases for analyzing teaching. Also, Bieda, Sela and Chazan (2015) demonstrated a difference in reflections between early and late practicums, where mentors focusing on the obligations in the classroom were an important factor in the change of students' justifications of their teaching. An opposite result was found when Simpson, Vondrová and Žalská (2017) investigated whether students' attention on aspects of mathematics teaching increased after one, two or three blocks of practicum, and found that it did not. Others looked for reflections that include a **specified theoretical content,** most often PCK or mathematical knowledge for teaching (MKT), which we will return to below.

The included **intervention studies** engage different approaches to improve learning from practicum. Lesson study-interventions was found, in four studies, to have a potential for developing knowledge of planning and teaching. The studies of students observing others were all found to result in improved reflections, in terms of theorizing, attention to learners, questioning, use of research or evidence in analysis of teaching, whereas only one study reported improvements in reflections on students' teaching. Other successful interventions were mentoring conversations with both mentor and lecturer, an online bulletin board, literature integration in teaching, and a course which integrated elements from practicum, allowing students to try out course content in practice. In a few studies, unwanted outcomes were mentioned, but overall the interventions appeared to be considered successful by the authors.

The synthesis of all studies that explored the learning outcomes from practicum suggests that responsibility for teaching together with support from mentors, coursework or other resources is more important for the outcome of practicum than the length or number of periods spent in a school context. These studies reveal how spending time in school is not enough to achieve a learning outcome for mathematics teachers. From this we learn, not surprisingly, that students need preparation in terms of content knowledge, PCK, theoretical models etc., but also that it takes systematic and focused prompts from mentors, peers or teacher educators to learn to transfer this theoretical knowledge to reflections on classroom teaching.

Studies on what characterizes practicum in mathematics in particular

Out of the 51 reviewed articles, only eight explored the specificity, if it exists, of learning to teach mathematics in particular from practicum. Helgevold et al. (2015) saw that **reflections on mathematics teaching seems to differ** compared to other subjects. Before a lesson study intervention, mathematics mentoring conversations focused the subject and the student teachers' actions more than in other subjects. After the intervention, the focus on learners was increased. The mathematics mentoring conversations had very little focus on general concerns, compared to other subjects.

Two studies engaged **attitudes and beliefs of prospective teachers in relation to mathematics teaching**. Jong and Hoges (2015) found that several factors needed to be taken into account, where previous attitudes were the strongest predictor, but also mathematics methods environment and experiences from school, while only 3% was explained by student teaching experiences. The relationship between the instructional practices and the student teachers' beliefs about their efficacy to teach was investigated in another US study (Lee, Walkowiak, & Nietfeld, 2017). Their findings indicate that prospective teachers with higher levels of mathematics teaching efficacy beliefs taught lessons characterized by higher cognitive demand, extended learner explanations, learner-to-learner discourse and explicit connection between representations, whereas the lessons by those with lower levels were characterized by whole-class instructions.

A few studies explored the **impact of practicum on the development of specific mathematics PCK** or specifically MKT. The development of MKT was found to improve by instructional responsibility during practicum, together with the completion of and exposure to certain topics and learning experiences in mathematics courses and in general pedagogy courses. Several studies analysed the different aspects of MKT revealed in written reflections, and found that specific frameworks or tools could improve the presence of MKT in reflections. As an example, van den Kieboom (2013) found several examples of 'common content knowledge', fewer of 'specialised content knowledge', and very few of 'knowledge of content and students [learners]', leading the author to conclude that content knowledge is essential, yet finding that providing students with an analytic tool facilitated the development of MKT from practicum.

The studies focusing the **specificity for teaching mathematics** highlights the importance of content knowledge together with PCK. However, students have been found to improve their PCK during practicum when provided an analytical tool. In addition, two studies addressed students' beliefs or attitudes in relation to teaching mathematics, and one study found that the focus on learners was more challenging in learning to teach mathematics compared to learning to teach other subjects.

DISCUSSION

Despite the fact that practicum is differently organised, and the learning focus vary, we have found some consistencies. The strongest synthesis from this review is that the learning from practicum is improved by feedback, prompts or guidance from teacher educators, mentors or peers. We have been able to demonstrate some convergence in results when it comes to the importance of prompting students to make use of theories in relation to practice, where for example video analysis, mentoring or written reflections turned out to be successful contexts. The desired learning outcomes ranges over a large number of perspectives, as increased mathematical knowledge or PCK, constructivist teaching approaches with active learners, often focusing improved use of theory in reflections, and the student teachers implementation of instructional tasks. In this review, we found some studies focusing attitudes and identity development, whereas no study took a position of learning as completely contextually situated. This result is encouraging in relation to our own position, where we see teachers as engaging in reasoned judgement (Rusznyak & Bertram, 2015), providing theoretical as well as subject specific arguments for their choices and actions in teaching.

Yet, the reliability and transferability of this result are affected since so many of the studies are reports on the researchers' own practice, and may be impacted by the researchers' positions. We see a risk of research reinforcing existing practices, without problematising the purpose or rationale in relation to the different images of a desired teacher. Few studies reported negative or even surprising results. Also, this study reports on empirical studies predominantly from the Anglophone countries, a majority from USA. In addition, the intervention studies would more often use students reported learning, for example written reflections, than classrooms observations of performed teaching. Therefore, we limit our conclusions to find it demonstrated that when students are engaged in lesson study, a theoretical analysis of teaching or a specified teaching technique, that is what they will do and learn. The different studies generally appear to us as isolated islands of research; more studies aligning with the same project or model was rare, and most literature reviews focuses the chosen theoretical/methodological perspective rather than previous knowledge on outcomes from practicum. Our sense is that the focus of many studies is not on the construction of cumulative knowledge, but rather on developing own practices.

Such practices most likely rely on the pre-existing image of the desired teacher, and our review indeed does demonstrate different perspectives on the desired teacher (Christiansen & Österling, 2018). In our view, it is important for teacher education to engage more critically with its practices, wherefore we suggest initiatives of researching each others' practicum; developing more coherent research programs; using secondary data to compare and contrast; and testing results through follow-up studies on the same participants. Such initiatives would improve reliability, further generalisability/transferability, and challenge any taken-for-granted assumptions or values in researchers' own practice. Furthermore, we recommend putting more assumptions to the test, rather than reporting on how successful one's preferred practices are. This would also mean sharing results of interventions that did not have the expected or desired outcomes.

References

(Due to space restrictions, only cited papers of the 51 reviewed papers are listed here. A full list is available upon request.)

Adler, J., Ball, D., Krainer, K., Lin, F.-L., & Novotna, J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics, 60*(3), 359–381.

Amir, A., Mandler, D., Hauptman, S., & Gorev, D. (2017). Discomfort as a means of pre-service teachers' professional development - an action research as part of the 'Research Literacy' course. *European Journal of Teacher Education, 40*(2), 231–245.

Anderson, L. M., & Stillman, J. A. (2013). Student teaching's contribution to preservice teacher development: A review of research focused on the preparation of teachers for urban and high-needs contexts. *Review of Educational Research, 83*(1), 3–69.

Bieda, K. N., Sela, H., & Chazan, D. (2015). "You are learning well my dear" Shifts in novice teachers' talk about teaching during their internship. *Journal of Teacher Education, 66*(2), 150–169.

Christiansen, I. M. & Österling, L. (2018). The desired teacher reflected in research articles on practicum. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 259-266). Umeå, Sweden: PME.

Cochran-Smith, M., & Zeicher, K. M. (2005). *Studying teacher education: The report of the AERA panel on research and teacher education*. Mahwah, NJ: Lawrence Erlbaum.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184–205.

Heaton, R. M., & Mickelson, W. T. (2002). The learning and teaching of statistical investigation in teaching and teacher education. *Journal of Mathematics Teacher Education*, 5(1), 35–59.

Helgevold, N., Næsheim-Bjørkvik, G., & Østrem, S. (2015). Key focus areas and use of tools in mentoring conversations during internship in initial teacher education. *Teaching and*

Teacher Education, *49*, 128–137.

Hodges, T. E., & Hodge, L. L. (2015). Unpacking personal identities for teaching mathematics within the context of prospective teacher education. *Journal of Mathematics Teacher Education. 20*(2), 101-118.

Jacobson. (2017). Field Experience and prospective teachers' mathematical knowledge and beliefs. *Journal for Research in Mathematics Education*, *48*(2), 148.

Jong, C., & Hodges, T. E. (2015). Assessing attitudes toward mathematics across teacher education contexts. *Journal of Mathematics Teacher Education*, *18*(5), 407–425.

Kaasila, R., & Lauriala, A. (2011). Towards a collaborative, interactionist model of teacher change. *Teaching and Teacher Education, 26*(4), 854–862.

Kaasila, R., & Lauriala, A. (2012). How do pre-service teachers' reflective processes differ in relation to different contexts? *European Journal of Teacher Education*. *35*(1), 77-89.

Karp, A. (2010). Analyzing and attempting to overcome prospective teachers' difficulties during problem-solving instruction. *Journal of Mathematics Teacher Educ.*, *13*(2), 121–139.

van den Kieboom, L. A. (2013). Examining the mathematical knowledge for teaching involved in pre-service teachers' reflections. *Teaching and Teacher Education, 35*, 146–156.

Lampert, M. (2010). Learning teaching in, from, and for practice: what do we mean? *Journal of Teacher Education*, *61*(1–2), 21–34.

Lee, C. W., Walkowiak, T. A., & Nietfeld, J. L. (2017). Characterization of mathematics instructional practises for prospective elementary teachers with varying levels of self-efficacy in classroom management and mathematics teaching. *Mathematics Education Research Journal*, *29*(1), 45–72.

Özgün-Koca, S., & Şen, A. (2006). The beliefs and perceptions of pre-service teachers enrolled in a subject-area dominant teacher education program about "Effective Education". *Teaching and Teacher Education, 22*(7), 946–960.

Rusznyak, L. & Bertram, C. (2015). Knowledge and judgement for assessing student teaching: A cross-institutional analysis of teaching practicum assessment. *Journal of Education, 60*, 31–61.

Simpson, A., Vondrová, N., & Žalská, J. (2017). Sources of shifts in pre-service teachers' patterns of attention: the roles of teaching experience and of observational experience. *Journal of Mathematics Teacher Education*, 1-24.

Yeşildere İmre, S., & Akkoç, H. (2012). Investigating the development of prospective mathematics teachers' pedagogical content knowledge of generalising number patterns through school practicum. *Journal of Mathematics Teacher Education*, *15*(3), 207–226.

Youngs, P., & Qian, H. (2013). The influence of university courses and field experiences on Chinese Elementary Candidates' Mathematical Knowledge for Teaching. *Journal of Teacher Education*, *64*(3), 244–261.

Zeichner, K. (2010). Rethinking the connections between campus courses and field experiences in college- and university-based teacher education. *Journal of Teacher Education, 61*(1–2), 89–99.

THE USE OF 'MENTAL' BRACKETS WHEN CALCULATING ARITHMETIC EXPRESSIONS

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In this paper, the influence of the written format of an arithmetical expression on the way the students evaluate this expression, as well as a possible connection between this way of evaluation and an understanding of structure, are examined. Students from two countries evaluated a small number of rational expressions. The findings show that the rational form guided the students in their evaluation, temporarily leaving aside the rules for the order of operation. Instead, they used 'mental' brackets that mask a possible, or actual, structure sense.

INTRODUCTION

Brackets in arithmetic can be used by students in a manner that is either procedural or conceptual. The former is related to the rules for the order of operations in an arithmetic expression, indicating priority in the order of operations. For example in 12/(4+2) brackets are a signal to "do this first". The latter considers brackets as structural elements, which determine the relation between the different parts of an expression. For example, in $\frac{6+4}{5+3} = (6+4) \div (5+3)$ brackets are used to preserve the structure of the rational expression and determine the relation between the two terms of the fraction. Many studies show that students exhibit both a poor procedural knowledge and a lack of an understanding of structure (Kieran, 1989; Linchevski & Livneh, 1999). For structural understanding, it is important for the student to be able to parse the expression correctly and identify the relation between the constituent parts as well as between the parts and the whole. In this paper, we examine the possible connection between the way students evaluate a rational expression and an understanding of structure given.

THEORETICAL BACKGROUND

Typically, brackets are introduced to the students alongside with the rules for the order of operations, suggesting what should be calculated first. However, they can also be necessary to preserve an expression's mathematical structure (for example when a rational expression is rewritten horizontally) since they show how terms are grouped. Hence, students' use of brackets can reveal their understanding of mathematical structure (Linchevski & Livneh, 1999). Hoch and Dreyfus (2004), working with students in the context of solving equations, found that students react differently to the presence or absence of brackets. More specifically, they found that the presence of

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 451-458). Umeå, Sweden: PME.

brackets, by giving a clue of where to look and by focusing the students' attention, affected positively the students' structure sense. Brackets help students 'looking' before 'doing', which is a feature of using structure sense. Brackets focus the students' attention to recognize relationships between the parts of the expression as well as to consider a compound term as a single entity. They 'close' an expression by indicating its total, and therefore certain parts of the expression are considered as a whole, which is important for obtaining structure sense (Marchini & Papadopoulos, 2011). Despite the importance of understanding brackets in structure of arithmetic expressions, students seem to face difficulties in comprehending their role. Sometimes they ignore them, thus violating the priority of the involved operations. In their study, Blando, Kelly, Schneider, and Sleeman (1989), working with grade 7 students from a middle school, found that in the item $8-(2+4)$ some students calculated this as $6+4=10$ which means that they ignored the set of brackets and calculated first the subtraction $8-2$. Hewitt (2005) found that students, when reading written mathematical expressions with brackets, ignored the mathematical structure and the intended meaning of the expressions. Linchevski and Livneh (1999) claim that this lack of structure sense could result in that students focus on the numbers rather than on the structure or the operations. They explain that when the students have to deal with expressions of the form $a \pm b \times c$, it is necessary to make them detach the middle number (*b*) from the preceding addition/subtraction. They suggest that the use of brackets can resolve this issue, i.e., $\pm(b \times c)$. Finally, another way to make students focus on the structure of an expression is the use of 'useless' brackets to help students see algebraic structure (Hoch & Dreyfus, 2004) and to increase success rates in arithmetic expressions (Marchini & Papadopoulos, 2011). However, there are instances where useless brackets could cause impediment for the learning of the order of operations (Gunnarsson, Sönnerhed & Hernell, 2016).

In this study, our interest lies on the use of 'mental' brackets when rational expressions are written horizontally. 'Mental' brackets were introduced in the work of Linchevski and Livneh (1999). They noticed that some of their students, in their effort to solve the equation $926-167+167=?$, put 'mental' brackets around $167+167$. It seems that students imagined these brackets (not physically present) and view the equation as $926-(167+167)$. Additionally, 37% of their students put 'mental' brackets around the multiplicative terms in the expression $24 \div 3 \times 2$ in contradiction to the order of operations. Hence, Linchevski and Livneh (1999) successfully could understand students' behaviour by introducing the concept of 'mental' brackets. Therefore, we intend to apply this concept to study a different but adjacent topic. The influence of the written form of an expression upon how the same expression is evaluated, is a topic that so far has been less explored, but where research is needed. So, in this setting, our research questions are: How do the written form of an expression govern the way students evaluate arithmetic expressions? Is the interpretation of the written form related to the structural understanding of these expressions?

SETTING OF THE STUDY

The study took place in Sweden and Greece. The participants (11-12 years old) were 112 grade-6 students from Greece and 123 grade-5 students from Sweden. All the students had been taught the rules for the order of operations. A collection of groups of activities was designed aiming to reveal how the students understand the role and use of brackets while evaluating arithmetic expressions. In this paper, we examine the results from one group of activities. All the activities in this group invite students to initially re-write a rational (fractional) expression in horizontal form and then to evaluate this horizontal expression (Fig. 1). The aim of this group is to shed light on whether there is a connection between the format of the written expression and the way the students evaluate them.

You know that $\frac{8}{4}$ *can be written on a single line as 8/4.First re-write each fraction below on a single line, and then evaluate each arithmetical expression.* *Example:* $\frac{8}{4} = 8/4 = 2$

1) $\frac{12}{4} + 2$ = ________________ = ________

2) $\frac{12}{4+2}$ = ________________ = ________

3) $\frac{8+12}{3+2}$ = ________________ = ________

4) $\frac{20}{\frac{4}{2}}$ = ________________ = ________

5) $\frac{12+2\cdot 3}{3}$ = ________________ = ________

Figure 1: Rational expressions used in the study

A pilot study was conducted, and the findings were used to refine and decide the final form of the activities. The process was the same in both countries: sufficient time (no time limit) and same instructions. The students' worksheets constituted our data and the analysis took place in two levels: qualitative and quantitative. The qualitative part was based on content analysis (Mayring, 2014), aiming at organizing the students' answers in categories based on the solution strategies used for the expressions' evaluation. Each activity was examined separately, and the data were post-coded independently by the two authors. The coding results were compared, codes were clarified, and some data were recoded until agreement. The quantitative part is limited to the frequencies of the answers that belong in each solution strategy.

RESULTS

The re-writing of the rational expressions horizontally needs the use of brackets to preserve the structure of the rational expression and ensure that both the numerator and the denominator will be evaluated separately. For example, the 3rd activity must be written as $(8+12)\div(3+2)$ to be considered mathematically correct. The analysis resulted in nine categories of answers. Four of them correspond to correct answers and they are (i) *correct result using brackets* (C-brackets), (ii) *correct result without brackets* (C-No brackets), (iii) *correct result in the second step* since the students make the necessary calculations on the fraction's terms before the horizontal re-writing (C-2nd step), and (iv) *correct result based on the knowledge of operations of fractions* (C-fraction operations). The distribution of the students' answers into these four categories can be seen in Table 1. (Data represent absolute number of answers and the sums in each column do not add up to 123 (Swe) and 112 (Gre), because wrong or blank answers are not included.)

	Activity-1		Activity-2		Activity-3		Activity-4		Activity-5	
	Swe	Gre	Swe	Gre	Swe	Gre	Swe	Gre	Swe	Gre
C-brackets	0	3	2	5	3	7	3	10	1	6
C-No brack	62	52	33	82	32	76	28	54	31	56
C-2nd step	15	8	73	6	73	9	59	15	69	12
C-fr. oper.	0	9	0	0	0	1	0	1	0	0

Table 1: Arithmetical data of correct answers across the four categories

The wrong answers (not included in Table 1) have been divided into three categories, but it is out of the scope of this paper to present them in detail. In brief, the reasons for these wrong answers were miscalculations, left-to-right calculations, and lack of knowledge of the concept of fraction. There was also one category for the unanswered items and one for items that were not codable.

The 'C-brackets' strategy

This strategy refers to the correct use of the necessary brackets in the horizontal expression to preserve the structure of the rational expression. An example of students' answers based on this strategy can be seen in Fig. 2. It is interesting that the specific student used brackets for all the items except for the first one.

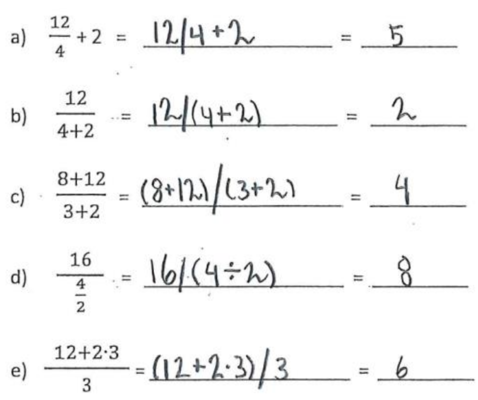

Figure 2: Answers that use brackets and preserve structure

The reason was that any pair of brackets around the division $12 \div 4$ would be 'useless' since the rules for the order of operation do not violate the structure of the given expression. Division must be done first, and this is in alignment with the structure of the rational expression. However, as it can be seen in Table 1, very few students found the correct answer using the necessary brackets (C-brackets row) in the horizontal form. An average (over all items) of 1.8 Swedish students and 6.2 Greek students used necessary brackets, thus preserving the structure of the given rational expression.

The 'C-No brackets' strategy

The students who applied this strategy found the correct arithmetic result without the use of brackets. Their evaluation of the horizontal expression is mathematically incorrect since it violates the order of operations. However, they manage to get the correct result (Fig. 3). A large number of students in both countries found the correct answers without the use of brackets. An average of 37 and 64 Swedish and Greek students, respectively, used this strategy.

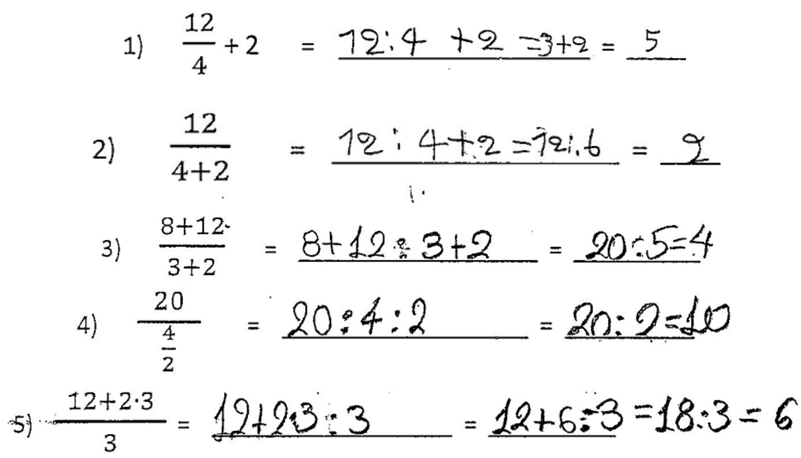

Figure 3: Correct result, but incorrect process

More specifically, the students wrote the horizontal expression without brackets. Therefore, it seems that structure is not preserved in the written form, but the result is correct. The first interesting thing to notice is that the first two items in their horizontal form are identical. However, the first one is evaluated as 5 while the second one as 2. As it is evident from the specific student's calculations, in the first item, the expression $12 \div 4 + 2$ was evaluated as $3 + 2$ whereas in the second as $12 \div 6$. The student seems to feel comfortable with these two expressions that look the same but give different results. Both final answers show an alignment with the structure of the initial rational expression of these items. In the third item, if one follows the rules for the order of operations, the result is $8 + 12 \div 3 + 2 = 8 + 4 + 2 = 14$. However, it seems that the student calculated separately the sums $8 + 12$ and $3 + 2$ (i.e., the terms of the fraction), before making the division $20 \div 5 = 4$. In a similar way, the fourth item included operations of the same priority. As it is written, the conventions are that the expression should be evaluated from left to right, resulting in $20 \div 4 \div 2 = 5 \div 2 = 2.5$, instead of 10. The last item is the most demanding, since its numerator included an arithmetic expression which needs the knowledge of the precedence rules for its evaluation. The result is again correct (6 and not 14, as it should be according to a formally correct evaluation of the horizontal expression, i.e., $12+2\times 3\div 3=12+6\div 3=12+2=14$).

The 'C-2nd step' strategy

This strategy, as well as the next one, is not fully aligned with the task's instruction. A large number of the Swedish students (an average of 57.8 students over all items) initially made the necessary calculation either on numerator or denominator before writing the fraction horizontally (Fig. 4). The corresponding average for the Greek students was 10 students.

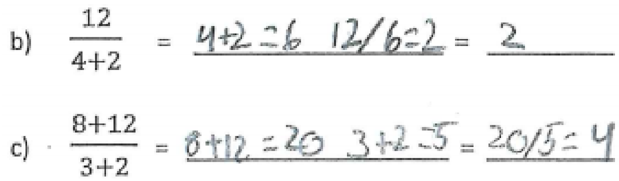

Figure 4: Correct answer with an intermediate calculation in the 2nd step

The fact is that by initially doing the calculation for either one of the terms or both, the horizontal form does not pose a dilemma to the students. For example, in Figure 4, second item, the student calculated initially the sum $4+2=6$ and therefore the horizontal form was simply asking for the division $12\div 6$. Similarly, in the third item, the student made initially the calculation for each one of the terms and therefore the horizontal form is a simple division.

The 'C-fraction operations' strategy

This strategy was applied only by a very small number of Greek student who ignored the instructions of the tasks and worked out the tasks using their knowledge about the operations of fractions (Fig. 5). So, in the first example (Fig.5, left), the student made equivalent fractions with common denominator to perform the addition. In the second example (Fig. 5, right), the student follows a rule about the simplification of a whole number over a fraction (complex fractions) that is usually taught in Greek classrooms (and this rather explains why only Greek students applied this strategy).

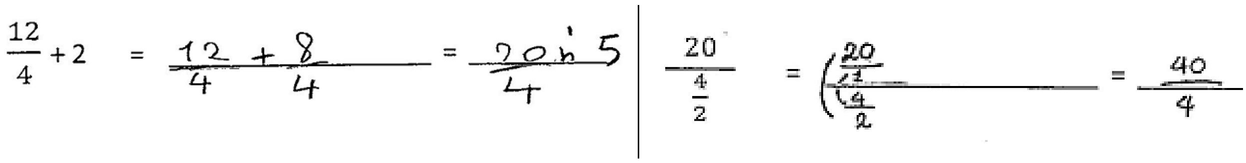

Figure 5: Correct answers based on operations of fractions

DISCUSSION

Our main interest lies in the 'C-No Brackets' category. The students' preference to this strategy is connected rather to the fact that the expressions were presented in rational form and this guided their evaluation. The students' conceptual understanding of fractions makes their evaluation of the expression straightforward as they seem to translate $\frac{a}{b}$ is translated into $a \div b$. So, in this case, the students write $8+12 \div 3+2$ (third item), but they evaluate the expression in a way that reflects an implicit presence of brackets that preserve the structure of the initial expression. We believe that in this case, the students use 'mental' brackets around the two sums as a kind of grouping mechanism. The same can be said for all the items of the group. The first two items were designed intentionally to contrast similar expressions in their horizontal form, $12 \div 4+2$. So, it can be said that in the first case the 'mental' brackets are used around the division $12 \div 4$ whereas in the second around the sum $4+2$.

This raises the question: Is the use of the 'C-No Brackets' strategy a mere consequence of the influence of the expressions' written format, thus indicating a lack of knowledge about the precedence rules? The knowledge of fractions is sufficient to correctly evaluate the first four items without necessarily knowing the precedence rules. Indeed, the rational form of the expression imposes the separate calculation of each term of the fraction, and given that these terms in the first four items include only one operation, it is easy to obtain the correct result. However, this knowledge is not sufficient for the last item. Its numerator includes an expression that demands an understanding of the rules for the order of operations. The fact that 31 and 56 Swedish and Greek students, respectively, used this strategy and found the correct result, is an indication that this cannot be attributed to the sole impact of the expression's written format. Therefore, it would useful to follow the students who did not use brackets in the first four items of the group, and correlate that with their answers in the fifth one. So, from the 31 Swedish students who solved correctly the fifth item, 28 did not use brackets in the first four items. The corresponding numbers for the Greek students were 51 students out of 56. But, if these students know the rules for the order of operations, how could their unorthodox behavior of the use of the 'C-No Brackets' strategy then be explained?

We argue that the rational form of the arithmetic expression imposes the way it is perceived and evaluated. The students respect the form and do not check the mathematical accuracy of their evaluation. Therefore, in the 'C-No brackets' strategy they are guided by the fractional form. So, by using mental brackets, they perceive the expressions as they should be perceived, but their writing does not preserve the structure of the initial expression. The students' understanding of the precedence rules are of minor importance in the first four tasks, because their format seem to trigger a correct evaluation accompanied by a rewriting that is not in agreement with the conventions. The students, however, who rely only on the fractional form of the expression, fail to succeed when the written format include more complex terms, such as the fifth item does, and that requires an understanding of the order of operations. Our data shows that there are students who used the 'C-No Brackets' strategy in the first four activities but were able to turn to the precedence rules when necessary (i.e., in the fifth expression). For these students, the written expressions were apparently violating the order of operations, but we believe that they mentally put brackets in the expressions when they evaluated them. Hence, the mental brackets made the students find the results of the calculations before figuring out how to write the horizontal expression. We conjecture, that in some way they see the writing as obsolete and therefore do not reflect on the structure of the expression in relation to the results of the calculation. However, even if they do so, they can turn to the precedence rules when the knowledge provoked by the format (fractions) is not sufficient for evaluating the whole expression.

CONCLUSIONS

The use of brackets is considered important for evaluating arithmetic expressions and exhibiting a structure sense (Linchevski & Livneh, 1999). We argue that this does not necessarily mean that the absence of these necessary brackets shows lack of structural understanding. Our data give evidence that when students write rational expressions in horizontal form, they do not use the brackets in the written form (to preserve the formal structure of the expression) but add the brackets mentally, in their own evaluation of the expression. Indeed, this can be interpreted as lack of structure sense. But, we argue that the way the students evaluate these horizontal expressions, even though they follow a seemingly unorthodox process that violates the order of operations, show that the structure is preserved through the use of 'mental' brackets.

References

Blando, J. A., Kelly, A. E., Schneider, B. R., & Sleeman, D. (1989). Analyzing and modeling arithmetic errors. *Journal for Research in Mathematics Education*, *20*(3), 301-308.

Gunnarsson, R., Sönnerhed, W. W., & Hernell, B. (2016). Does it help to use mathematically superfluous brackets when teaching the rules for the order of operations? *Educational Studies in Mathematics*, *92*(1), 91-105.

Hewitt, D. (2005). Chinese whispers - algebra style: Grammatical, notational, mathematical and activity tensions. In H. L. Chick & J. L. Vincent (Eds.), *Proc. 29th Conf. of the Int. Group for Psychology of Mathematics Education* (Vol. 3, pp. 129–136). Melbourne: PME.

Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In M. J. Høines & A. B. Fuglestad (Eds.) *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 49-56). Bergen, Norway: PME.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*, Reston, VA: NCTM.

Linchevski, L. & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, *40*(2), 173-196.

Marchini, C., & Papadopoulos, I. (2011). Are useless brackets useful for teaching? In B. Ubuz (Ed.), *Proc. 35th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 185-192). Ankara, Turkey: PME.

Mayring, P. (2014). *Qualitative Content Analysis: Theoretical foundation, Basic procedures and software solution.* Klangenfurt: Beltz.

MAKING MATHEMATICAL LEARNING LONG-TERMED AND EFFECTIVE USING INTERLEAVED PRACTICES

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While most educational approaches focus on improving learning through making things easier, the approach of desirable difficulties is to make the learning process more difficult, but long-termed. Within cognitive psychology several of those desirable difficulties could be identified, for example the interleaved practices. In the presented empirical classroom study the focus lies on the investigation of effects of interleaving practices in contrast to blocked learning with seventh Graders. Most students learn mathematics in the blocked way: they deal first with one topic and after it is completed, they start to learn the next one. Learning in this way is easier than learning several topics at the same time, which is called interleaved practice. The study and some of its results will be presented in this report.

THEORETICAL BACKGROUND

There are different techniques to make learning successful. But even if the learning success is reached, many students forget the contents that they have learned very quickly. As soon as they start to learn a new topic, most of the contents that they have learned before move into the background. How can the learning be made not only successful but also long-termed? This question has always been of great importance for most of the teachers. As it is known from cognitive psychology, the sustainability of the learning success can be reached if learning becomes more difficult (Bjork, 2011). This recognition sounds paradoxical and is in contrast to most didactical concepts that purposefully strive a simplification of learning. However, many studies showed that making learning more complicated in a certain way leads to a long-termed success.

Desirable difficulties of learning

There are some techniques, that make the learning more difficult and long-termed successful. They are known as "desirable difficulties" (Bjork, 1994). The term "desirable difficulties" comes from cognitive psychology and includes such popular techniques as the testing effect, the distributed learning, the generating effect, and the interleaving practice.

The testing effect, also known as retrieval practice, uses testing as an opportunity to recall the information, which has to be learned (e.g., Karpicke, & Roediger, 2008).

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 459-464). Umeå, Sweden: PME.

The distributed learning is used, when learners divide their learning time into shorter learning units with a time delay in between (e.g., Cepeda, Pashler, Vul, Wixted, & Rohrer, 2006).

The generating effect appears, when the information is generated from own knowledge (e.g., McDaniel, Waddill, & Einstein, 1988).

Finally, the interleaving practice means that several related topics or skills are learned simultaneously (e.g., Rohrer, & Taylor, 2007).

Common to all of these techniques is that they might slow down the learning process due to a higher cognitive effort, but they enhance remembering learning contents in the long term.

Interleaved practice

Our working group deals particularly with the effects of the interleaved practice. To give more insight into this type of desirable difficulties, one can explain it through the comparison to the so-called blocked learning, which is often used at many learning institutions and which is often suggested in many textbooks.

We learn in the blocked way, when we learn each topic until the end, before we start with a next one. For instance, if we divide a topic, that has to be learned, into smaller subtopics A, B, and C, then we start to learn the topic A, then, after the topic A is finished, we begin to learn the topic B and so on. The learning contents are structured according to the model “AAA BBB CCC”. This learning technique is very popular because the concentration of the learners is completely focused on a particular topic.

The interleaving practice is in contrast to the blocked learning. When we learn in the interleaved mode, we learn all the subtopics A, B, and C at the same time. The arrangement of the learning contents can be illustrated by the model “ABC ABC ABC” (e.g., Rohrer & Taylor, 2007). In this case, the learners have to concentrate on different topics at the same time. Because of this fact, their cognitive effort increases, and learning becomes more difficult. But the learners have generally a long-lasting remembrance of the learned contents in contrast to persons that learn in the blocked mode (e.g., Rohrer & Taylor, 2007).

The longer memorizing of the learned contents after the interleaving learning can be explained with the help of the New Theory of Disuse by Bjork. He describes two types of processing the incoming information in the human brain. The first type of processing is the “saving” of the information like a medium. In this case the storage strength is responsible for the amount of information, which the brain can notice and “record” additionally to the information, which already exists in the memory. If the “saved” information is not used for a long time, the memorization of it becomes more difficult. The second type of information processing is the recalling of it from the memory. In this case the retrieval strength is responsible for the quantity and completeness of the recalling information (Bjork & Bjork, 1992). The activation of the retrieval strength increases during the interleaved practice because the learners have to recall the learned information of the interleaved topics more often than the learners that learn the topics in the blocked way. This leads to the long-termed memorizing of the learned contents.

State of research on the interleaved practice

Compared to the other types of desirable difficulties, the interleaved practice is still little researched (Dunlosky et al., 2013). There were some empirical studies that showed benefits of the interleaving learning, particularly that the interleaved practice led to a long-termed learning success (Dunlosky et al., 2013; Rohrer & Taylor, 2007). Most of them were conducted with adults, but not with younger students. Furthermore, many studies were carried out in the laboratory (Rau, Aleven, & Rummel, 2013; Dobson, 2011; Rohrer & Taylor, 2007). So far, the investigation of the effects of the interleaved practice in the classroom is lacking.

Research questions

As mentioned above, the effects of the interleaved practice in the classroom remain a research gap. It is necessary to investigate, if the benefits of the interleaved practice also appear at schools. If that is confirmed, the education at schools could become more effective. Therefore, two main research questions are deduced for this report:

1. Does the interleaved practice lead to a higher learning success in the classroom than the blocked learning?
2. Does the learning become long-termed in the classroom, if the students learn in the interleaved manner?

Methodology

In September 2015 – February 2016 our working group carried out a study at German schools to compare the learning performance of 7th grade students in math, who learned in the interleaved way, to the performance of those, who learned in the blocked manner. The study consisted of an eight-hour lesson unit in both the blocked and the interleaved way, and of four math tests a 45 minutes: pre-test, post-test, follow-up test 1 and follow-up test 2, which were the same for all students. The post-test was performed immediately after the treatment. The follow-up test 1 took place three weeks and the follow-up test 2 ten weeks after the post-test. Additionally, there were two questionnaires of 45 minutes before and after the treatment. The topic of the lessons and of the tests was "Direct and inverse proportionality".

Altogether 124 students participated in the study. They were divided into two groups of equal size. The sample was randomized in both groups.

There was a difference between the arrangements of the learning contents in each group. One group learned in the blocked way: first, the students had to learn about direct proportional relationships until the end, before the second topic about the inverse proportionality started. After the end of the second topic they had to learn about other relationships that are neither directly proportional nor inversely proportional. The other group learned in the interleaved way: the students had to learn about all types of relationships simultaneously. But the learning contents and the tasks were the same in both groups.

Furthermore, the lesson unit was carried out by two instructed teachers from the working group. To avoid the influence of the teacher on the performance of the students, both teachers changed the group during the treatment after half of the lessons.

To examine the performance development of both groups between pre-test, post-test, and follow-up tests, it was determined, how well the learners solved the anchoring tasks, which were included in all the tests. The anchoring tasks that connected the pre-test and the post-test, were set up in such a way, that they could be processed without knowledge about proportionality and anti-proportionality. The tasks required the learners to calculate unknown variables or answer comprehension questions from a context that did not contain explicit specifications of direct or inverse proportionality. Therefore, these tasks could also be used in the pre-test. In the anchoring tasks of the post-test and follow-up tests, the learners were required to decide on the basis of given graphs and value tables, what kind of the relationship is shown, and to justify their decision. For each correctly solved task 1 point was awarded, incorrectly solved or incompletely processed tasks were evaluated with 0 points. The analysis of the test data was performed with the help of the repeated measures ANOVA (analysis of variance) with class attendance as covariate.

Results and discussion

The descriptive analysis of the development of the performance between pre-test and post-test is shown in Table 1.

Learning condition	Pre-test (max. 18 points)	Post-test (max. 18 points)
blocked	M = 5.28, SD = 2.86	M = 6.28, SD = 3.46
interleaved	M = 4.82, SD = 2.48	M = 5.56, SD = 2.75

Table 1: Descriptive statistics of the performance development of students between the pre-test and post-test.

The repeated measures ANOVA did not result in a statistically significant interaction effect between time and learning condition regarding the development of the performance between pre-test and post-test, i.e. the both groups developed in a comparable manner during this period ($F(1,109) = .380$, $p = .539$, $\eta^2 = .003$).

As shown in Table 2, in the period between the post-test and follow-up-test 1 the performance of the students who learned in the interleaved mode increased slightly, while the performance of the students who learned in the blocked way decreased.

Learning condition	Post-test (max. 26 points)	Follow-up 1 (max. 26 points)
blocked	$M = 6.76$, $SD = 4.43$	$M = 5.35$, $SD = 4.57$
interleaved	$M = 5.35$, $SD = 3.18$	$M = 5.72$, $SD = 4.64$

Table 2: Descriptive statistics of the performance development of students between the post-test and follow-up test 1.

There was a significant interaction effect between time and learning condition with $F(1,97) = 4.228$, $p = .042$, $\eta^2 = .042$.

Table 3 illustrates the performance development between the post-test and follow-up-test 2.

Learning condition	Post-test (max. 19 points)	Follow-up 2 (max. 26 points)
blocked	$M = 5.32$, $SD = 2.92$	$M = 4.37$, $SD = 3.31$
interleaved	$M = 4.89$, $SD = 2.70$	$M = 4.22$, $SD = 3.02$

Table 3: Descriptive statistics of the performance development of students between the post-test and follow-up test 2.

As it shown in Table 3, in the time period of ten weeks the performance of the students in both groups decreased slightly.

There was no significant interaction effect between time and learning condition, i.e. the performance of both groups decreased in a similar way ($F(1,108) = .267$, $p = .606$, $\eta^2 = .002$).

In conclusion, we could only find a significant interaction effect between post-test and follow-up-test 1, i.e. the long-termed learning effect of the interleaved practice in the time period of three weeks could be also noticed in the classroom. However, there was no significant interaction effect between time and learning condition directly after the lesson unit. This result was similar to the findings of other studies, which also showed that the advantage of the interleaved practice is often not noticeable immediately after the treatment. The long-lasting effect of the interleaved learning in the time period of ten weeks was also not identified. It is important to mention that many participants didn´t take tasks of the last test seriously. This resulted in many missing data of each group. Would the evaluation of the test data without missing information show the long-termed effect of the interleaved practice even ten weeks after the post-test? This question remains open.

References

Bjork, R. A. (1994). Memory and metamemory considerations in the training of human beings. In Metcalfe, J., & Shimamura, A. (Eds.) *Metakognition: Knowing about Knowing.* Cambridge, MA: MIT Press, 189-192.

Bjork, E. L., & Bjork, R. A. (2011). Making things hard on yourself, but in a good way: Creating desirable difficulties to enhance learning. *Psychology and the real world: Essays illustrating fundamental contributions to society*, 56-64.

Cepeda, N. J., Pashler, H., Vul, E., Wixted, J. T., & Rohrer, D. (2006). Distributed practice in verbal recall tasks: A review and quantitative synthesis. *Psychological Bulletin, 132*(3), 354-380.

Dobson, J. L. (2011). Effect of selected “desirable difficulty” learning strategies on the retention of physiology information. *Advances in Physiology Education, 35*(4), 378-383.

Dunlosky, J., Rawson, A., Marsh, E. J., Nathan, M. J., & Willingham, D. T. (2013). Improving students’ learning with effective learning techniques: Promising directions from Cognitive and Educational Psychology. *Psychological Science in the Public Interest, 14*(1), 4-58.

Karpicke, J. D., & Roediger, H. L. (2008). The critical importance of retrieval for Learning. *Science, 319*, 966-968.

McDaniel, M. A., Waddill, P. J., & Einstein, G. O. (1988). A contextual account of the generation effect: A three-factor theory. *Journal of Memory and Language, 27*(5), 521-536.

Rau, M. A., Aleven, V., & Rummel, N. (2013). Interleaved practice in multi-dimensional learning tasks: which dimension should we interleave? *Learning and Instruction, 23*, 98-114.

Rohrer, D., & Taylor, K. (2007). The shuffling of mathematics problems improves learning. *Instructional Science, 35,* 481-498.

PROSPECTIVE PRIMARY TEACHERS' CONCEPTUAL UNDERSTANDING OF MATHEMATICAL PROBLEMS AND PROBLEM SOLVING

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Prospective primary teachers' understanding of problem solving and the concept of what constitutes a mathematical problem were analysed. The exercise involved defining three fundamentals: characterisation of what a problem is, the problem-solving process and the willingness to undertake problem solution. That served as the basis for formulating a questionnaire responded to by 51 future teachers near the end of their pre-service training. The findings were uneven, for whereas participants exhibited knowledge in accordance with the literature on problem solving, contradictions were detected. For instance, while attaching importance to solvers' consideration, that notion was not taken into attention in practical examples.

TEACHERS' PROBLEM-SOLVING KNOWLEDGE

Problem solving (PS) has evolved into an essential tool for full participation in today's society. Despite the importance of PS proficiency, students have experienced difficulty in developing that skill, and teaching it to them has proven complex and elusive (Lester, 2013). One of the factors contributing to that situation has been the emphasis on the problem solver and the PS process to the detriment of the teacher's role. The knowledge teachers need to teach PS needs to be explored (Lester, 2013). Carpenter, Fennema, Peterson, and Carey (1988), deepen the pedagogical knowledge of school teachers on arithmetic word problems. Their results shows that teachers know the different types of arithmetic problems, but they have troubles in explaining how these differences affect the difficulty of the problem. Likewise, they have difficulties to identify differences between resolution strategies and, therefore, they are not able to predict the possible paths that students could use. The work of these authors highlights the existence of a particular knowledge for teachers about PS, closer to the nature of the process than the mathematical concepts involved in it.

It is generally agreed that teacher quality is a key to student performance. One indicator of such quality is their knowledge, for their classroom delivery depends largely on such professional acquis (Kilpatrick, Swafford, & Findell, 2001). Lester (2013) notes that to teach PS, teachers need to know what to do, when to do it and the implications of their actions. In this same vein, Chapman (2015) proposed a model for the knowledge required to teach PS. Her model addresses teachers' PS proficiency, affects and beliefs, content knowledge (of problems, PS and problem posing) and pedagogical knowledge (of students as problem solvers and instructional practices). While im-

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 465-472). Umeå, Sweden: PME.

portant, like the skills proposed by Lester (2013), it is recent and insufficiently researched to determine its utility in identifying teachers' PS knowledge in detail.

The present study aims to shed some light on that complex scenario, specifically by analyzing the elements that comprise the teachers' knowledge about PS proficiency.

PS PROFICIENCY AND TEACHER KNOWLEDGE

Based on theories on mathematical (e.g. Kilpatrick, et al, 2001; Rico, 2007) and PS (Chapman, 2015; OECD, 2014) proficiency, this construct is here understood to mean the actions adopted by a subject who identifies a situation as problematic, is favorably disposed to solve the problem and proceeds to do so by deploying a strategy in a series of not necessarily linear steps. Three theoretical elements that should form part of teachers' content knowledge can be identified in that definition. The first is related to problem characterization (the notion of what a problem is) and the second to PS, while the third element is non-cognitive.

In problem characterization, identification is imperative to the existence of the concept (Agre, 1982). This is not necessarily the intended context in references to school mathematical problems, however. Teachers who know their students can assign them tasks which while not regarded as problems per se, are in that context. While it is the problem solver who labels problems as such (Mason, 2016), classroom mathematical problems can be read at two levels, the student's and the teacher's. Teachers are the first to realize when their students are confronted with what they regard as a problem. That perception, which may be based on structural elements, i.e., formulation, context, the set of acceptable solutions or the methods for broaching them (Borasi, 1986), in itself characterizes/differentiates tasks regarded as problematic from those that are not. The problematic task is subsequently assigned to students, who must devise a formula to solve it. It is up to them to formulate and successively reformulate to perform the task by mobilizing a series of cognitive (knowledge and metacognition) and non-cognitive (affects and beliefs) aspects that are not predetermined by a prior knowledge of the process (Mayer & Wittrock, 2006). Such engagement generally stems from the absence of a known procedure to solve the problem. A problem is therefore understood to mean tasks which problem solvers feel committed to solving but for which they have no predetermined PS procedure.

Another factor to be borne in mind in this regard is the differentiation/characterization postulated by Borasi (1986). From their perspective the existence of different tasks that can be called problems can be inferred. While no full consensus has been reached around any of the several classifications in place, researchers concur on the acceptability of certain dichotomies, such as: exercises/ problems, routine/non-routine, and open/closed. Dichotomized problem types by Holmes (1985), was adopted for this paper. It establishes four categories (routine, non-routine, applied and not applied) that give rise to six types of problems—because non-routine problems can be open or closed.

In this approach to what is meant by a problem, the personal, guided and procedural process is conducted by stages. Historically, problem solvers have been seen to proceed as described by Pólya (1945). Similar ideas have been suggested in later research as variations on that scheme with more or less detailed descriptions and from a number of perspectives. The common denominator in all is their assumption that the process is cognitive, personal, not directly observable (Mayer & Wittrock, 2006) and (importantly) non-linear. As Wilson et al. (1993) note, the activity is flexible, accommodating motion both back and forth. Moreover, elements such as basic knowledge, metacognition and affects and beliefs play an essential role, governing and controlling the process (Schoenfeld, 1992). PS is consequently understood here as the process implemented in flexible, non-linear stages, in which the heuristic knowledge and strategies deployed are governed by non-cognitive factors and metacognition.

Lastly, disposition is vital, knowing how to solve problems is important, but wanting to is essential. The importance of non-cognitive factors in PS has been widely studied and it is generally agreed that depending on the suitability of the challenge posed, students become emotionally engaged (affect), with the concomitant impact on the mobilization of their intellect (Mason, 2016). That engagement is imperative in PS proficiency, for it drives the entire process. Table 1 synthesizes the descriptors associated with each component of teacher's professional PS knowledge.

Component	Knowledge
Problem characterization	Task with no known solving procedure
	Problem solver's consideration
	Type of tasks posed as problems
Problem solving	PS stages and their characterization
	Strategies
	Metacognition
	Non-cognitive factors
Disposition	Acceptance of the PS challenge

Table 1: Components of teachers' content knowledge on PS

This paper draws on three key aspects for the analysis of the teachers knowledge on a nature of mathematical PS (Table 1). Our goal is to describe the knowledge of prospective primary teachers about PS, based on the identified components.

METHOD

Participants

The 51 participants were fourth-year pre-service teacher trainees enrolled at the University of Granada's Faculty of Education. All subjects had taken the elective class 'Mathematical Skills in Primary Education' which contained a lesson on PS in which students were introduced to strategies and heuristics, problem posing and PS teaching strategies.

Instrument

The data were collected by means of a two-part questionnaire, one with 24 and the other with 42 closed, dichotomous questions. That procedure was chosen in pursuit of answers that would denote the presence or absence of subjects' knowledge of PS but not the meaning they attributed to the notion. The first part referred to the concept per se of what constitutes a problem, and the second to PS-related issues.

The questionnaire was designed to the type of knowledge involved. The first, conceptual part was divided into three sections addressing: a) task with no known solving procedure, b) problem solver's consideration, and c) the type of tasks that constitute problems.

The second part, consisted in two sections, one on PS process, further sub-divided into three units: a) the identification and characterisation of PS stages covering, for instance, PS is a dynamic process in which the solver may go back to find the solution; b) metacognition and the extent to which awareness of one's knowledge helps choose the most suitable PS strategy; and c) non-cognitive factors, such as whether a problem can be successfully solved in the absence of motivation. The other section targeted specific strategies. Both main parts of the questionnaire included questions on willingness.

Analytical procedure

Two analyses were conducted of future teachers' replies. In the first, the answers were grouped further to dimensional scaling multivariate analysis with ALSCAL (SPSS) software, defining the dimensions to be agreement, disagreement and contradictions. The second was a descriptive exercise, in which responses were reviewed in terms of the ideas defended in the literature. For reasons of brevity, only the latter is addressed here.

RESULTS

The sections below discuss the replies organised in accordance with the theoretical pre-analysis and the dimensional scaling findings, which clustered around two key considerations: future teachers' majority agreement or disagreement with item wording.

Task with no known solving procedure

Agreement was greatest in the answers to the items on conceptualisation of classroom tasks with no known solving procedure as mathematical problems: 74.5% of the future teachers disagreed with the premise that for a problem to be regarded as such, the solving procedure should not be known. Nonetheless, 86.3% deemed that tasks solvable only with previously learnt procedures should not be regarded as problems, while 98% replied that students must have acquired mathematical concepts to be able to establish a solving procedure.

Problem solver's consideration

Agreement, disagreement and even contradictions were identified in the responses to this set of questions. Subjects' tendency to regard only their own labelling of what constitutes a problem, disregarding students' labelling, may explain those discrepancies.

A direct question about the importance of the problem solver's role, for instance, was answered affirmatively by 50% of the respondents, whereas only one-third were able to identify such importance in a specific classroom situation. Moreover, 90.2% of the future teachers agreed that a given problem is not necessarily such for all students, for year of schooling and age must also be taken into consideration. There was no common understanding, however, about whether labelling a problem as such should depend on the solver's experience, with a 51% / 49% split between agreement and disagreement with that premise.

Type of tasks posed as problems

Tasks were regarded by participants as problems when they were: routine and applied (98%), non-routine, applied and closed (100%), non-routine, non-applied and closed (80.4%), or non-routine, non-applied and open (70.6%). Agreement was lowest in connection with non-routine, applied and open (54.0%) and routine, non-applied (21.6%) tasks.

These future teachers' knowledge of what constitutes a good problem concurred with the definition found in the literature, namely that it should accommodate more than one solution or more than one procedure to find the solution (Lester, 2013). For instance, 68.6% of the future teachers replied that word problems requiring no more than arithmetic calculation are not problems, whilst 100% agreed that a problem should be solvable in more than one way and 76.5% that it should have more than one solution. Certain contradictions were observed, however, when the items on types of tasks were analysed jointly with possible characteristics of problems. The inclusion of all the information required in the wording was defined as a characteristic by 59%. Of these, half replied that mathematical investigation-type open situations are problems. In contrast, one-fourth of the respondents who deemed that a problem need not contain all the necessary information did not regard open problems as such.

PS stages and their characterization

PS process and stages characterisation elicited high levels of agreement. Around 95% of the teachers in training could identify the stages involved in a specific solved problem and 98% characterised the process as flexible. The utility of representations to understand problems was acknowledged by 100% and of diagrams by 98%. Respondents also agreed (98%) that problems should not be solved unless they are understood. Review of and reflection on the tasks performed was likewise acknowledged to be recommended: by inventing similar problems (92%), seeking alternative pathways (91%) or generalising problem structure (98%). Ninety-eight per cent deemed that PS involves more than finding the correct answer.

Metacognition

Although most of the responses to the metacognitive questions were manifested in agreements, a number of interesting disagreements were expressed. The question at issue dealt with the reasons underlying calculation errors in solutions to problems. Whilst 87% recognised the error, nearly half attributed it to a lack of comprehension.

Non-cognitive factors

Most of the subjects agreed that if the challenge to solve a problem is assumed, the intellect will be induced by affective factors to find the solution. Some disagreement was also expressed, however. Although 96% of the future teachers deemed that motivation is important to tackling a problem, only 50% believed it to be instrumental to a successful outcome.

Strategies

The findings showed that the strategies with which the future teachers were most familiar included: operating (93.6%), draw a diagram (87.2%), work backwards (83%), look for a pattern (80.9%), building a table (72.3%), and guess and check (40.4%). Interestingly, a fairly large percentage of the teachers in training confounded guess and check with look for a pattern (29.8%) or building a table (21.3%).

Disposition

A total of 94.1% of the future teachers agreed that the PS challenge must be accepted for a problem to be so regarded. In addition, 96% deemed that successful PS depends not only on an understanding of mathematical concepts, but also on engagement.

DISCUSSION AND CONCLUSION

Teachers' PS knowledge has been scantly researched. Most studies exploring their role in PS instruction are confined primarily to their beliefs with little or no reference to their knowledge. The subject knowledge associated with PS as reinterpreted here identified three components (Table 1) related to the nature of PS that proved to be useful to study that knowledge.

One component builds on three elements. In the first, the solving procedure, respondents' ideas were observed to be aligned with the present conceptualisation of what constitutes a problem (Lester & Cai, 2016). However, the replies around the problem solver's consideration were heterogeneous and even contradictory. That may be because in classroom contexts, problem labelling is a two-stage process in which teachers first decide whether a given problem is suitable for some of their students, who then label the task as a problem or otherwise in accordance with their own experience. Whilst some of the responses around the knowledge of tasks posed as problems were aligned with the notions set out in the literature (Lester & Cai, 2016), more participants agreed that the tasks commonly found in textbooks were problems (Zhu & Fan, 2006). Unfortunately, such problems tend to be very routine. Participants' understanding of the characteristics that determine the existence of a problem was not consistent with their choice of sample problems. Although they knew, for instance, that problems can be solved by more than one pathway, they lacked the knowledge needed to recognise problems that would afford students that choice.

The second component, related to a knowledge of PS process, is structured around the understanding of four conceits: identification and characterisation of PS stages, metacognition, non-cognitive factors, and strategies. Some responses respecting PS stages and characterisation were consistent with a dynamic, cyclical and genuine interpretation of the phases defined by Pólya (Wilson et al., 1993). Nonetheless, further to the replies to the questions dealing with strategies, future teachers' classroom practice still proved to be quite linear. That inference was drawn from the fact that trial and error, the strategy they were least familiar with, is the one best suited to a cyclical, dynamic process.

In connection with the third component, willingness, future teachers agreed that where the PS challenge is assumed, affective factors induce the intellect to find the solution.

Futures teachers' knowledge was found to be complex, globally speaking, with contradictions that would have an adverse effect on students' learning. That disparity in the conceptual understanding of problems and their solution may be the result of a number of factors, including conceptions and beliefs (Lester, 2013). One promising finding was these trainees' realisation that inherent in the assumption of the challenge to solve a problem is the role of affective factors in inducing the intellect to seek the solution (e.g. Mason, 2016).

Acknowledgements

This study forms part of National R&D Project EDU2015-70565-P (MICINN); one of the authors benefitted from a PhD. scholarship granted by CONICYT (folio 72170314).

References

Agre, G. P. (1982). The concept of problem. *Educational Studies*, *13*(2), 121-142.

Borasi, R. (1986). On the nature of problems. *Educational Studies in Mathematics*, *17*(2), 125-141.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education, 19*(5), 385-401.

Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT, 3*(1), 19-36.

Holmes, E. E. (1985). *Children learning mathematics: A cognitive approach to teaching*. Englewood Cliffs, NJ: Prentice-Hall.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Lester, F. K. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast*, *10*(1&2), 245-278.

Lester, F. K., & Cai, J. (2016). Can mathematical problem solving be taught? Preliminary answers from 30 years of research. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems* (pp. 117-135). New York, NY: Springer.

Mason, J. (2016). When is a problem…? "When" is actually the problem! In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems* (pp. 263-285). New York, NY: Springer.

Mayer, R. E., & Wittrock, M. C. (2006). Problem solving. In P. A. Alexander & P. H. Winne (Eds.), *Handbook of Educational Psychology* (pp. 287-303). New York, NY: Routledge.

OECD. (2014). *PISA 2012 results: What students know and can do*. Paris, France: Autor.

Pólya, G. (1945). *How to solve it*. New Jersey: NY: Princeton University Press.

Rico, L. (2007). La competencia matemática en PISA [Mathematics Competence in PISA]. *PNA*, *1*(2), 47-66.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. Grows (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York, NY: Macmillan.

Wilson, J. W., Fernández, M. L., & Hadaway, N. (1993). Mathematical problem solving. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 57-78). New York, NY: MacMillan.

Zhu, Y., & Fan, L. (2006). Focus on the representation of problem types in intended curriculum: A comparison of selected mathematics textbooks from mainland China and the United States. *International Journal of Science and Mathematics Education, 4*(4), 609-626.

"I ALWAYS WISHED THAT I HAD A MATHEMATICAL MIND": MATHEMATICAL ABILITY AND OTHER STORIES

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Research on mathematics and gender suggests that mathematics, far from being a neutral discipline, is actually a masculine field. This perspective has significant implications in understanding girls' relationships with mathematics. In this report, I use a Foucauldian discourse analysis methodology to examine the mathematical identities constructed by two female high school students in relation to the discourse about mathematical ability. Although the discourses around mathematics, ability and gender prevent the students from identifying as "good at maths", the students actively negotiate their identities by challenging the discourses within which they act.

GENDER AND MATHEMATICS

During the last decades, gender has been a recurring theme in mathematics education literature, studied from various epistemological and ontological standpoints (Chronaki & Pechtelidis, 2012). A big body of research has focused on identifying the gender differences that affect girls' and boys' mathematical skills and abilities (Walkerdine, 1998). This perspective has been strongly criticized in that it treats girls as the problem and reinforces the "truth" that girls are deficient in mathematics (Walshaw, 2007). Following a different path, other researchers turn their attention to the ways that students construct their mathematical identities within discourses about mathematics and gender. The idea that the female (or male) identity entails essential features is rejected and it is instead proposed that gender identities are constructed within various socio-political contexts and power relations (Chronaki & Pechtelidis, 2012). At the same time, mathematics is viewed as a discursively constructed, masculine discipline and its consideration as a neutral, socially important and rational field is challenged (Mendick, 2006). The association of mathematics with intelligence and the idea that only some students can do well in mathematics are rejected, but there is limited research addressing the effects of this discourse on students' subjectifications. In this report, I examine how two female students construct their mathematical identities in relation to discourses about mathematical ability.

DISCOURSES AND IDENTITIES

The starting point of all discourse theories is that our access to reality is mediated by language, in a way that through language not only is a pre-existing reality reflected, but also reality itself is constructed (Jørgensen & Phillips, 2011). Discourses specify what is possible to be said, done, and thought, at a particular time; they have real, material effects on people's lives, both by formulating institutions and by constituting subjec-

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 473-480). Umeå, Sweden: PME.

tivities (Chronaki & Pechtelidis, 2012). Therefore, talking about “talented”, “charismatic” or “struggling” students is not just a usage of terms or words; it actually forms the limits within which students are allowed to experience learning (Walshaw, 2007).

Foucault (1972) proposes that *discourses* should be treated as practices that form the objects to which they refer. Working on Foucault's conceptualization of discourse, Doxiadis (2011) identifies four fundamental properties that every discourse has. Every discourse has the property of *referentiality* which means that the discourse points to something outside itself, which would exist even if the discourse did not exist; the property of *subjectivity* which is related to the discourse's conditions of enunciation; the property of *knowledge* which has to do with the concepts produced by the discourse; and lastly the property of *ideology* which emphasizes the political dimension of the discourse and its relation to issues of power.

Contrary to the standard Western understanding of subjects as autonomous entities, the Foucauldian subject is constituted within discourse. The subject is then decentered (Jørgensen & Phillips, 2011) and identities are multiple, fluid and unstable (Walshaw, 2013). This idea does not imply that identities are in constant flux. On the contrary, in specific situations, they can be quite inflexible (Jørgensen & Phillips, 2011). However, the Foucauldian idea that people construct themselves as subjects in-between many different and contradictory discourses embraces the possibility for the coexistence of contradictory identities and it provides a framework through which we can understand changes in one's identities. In order to study students' subjectification in relation to mathematics under this perspective, we should focus on the negotiation of discourses and the instability of mathematical identities.

RESEARCH QUESTIONS, METHODS AND METHODOLOGY

The data presented here is part of a larger study in which I interviewed eight students about their perceptions of the notion of mathematical ability and their mathematical identities as constructed in relation to this notion (Pitsili-Chatzi, 2015). In this report, I present the analysis of two semi-structured interviews with two female students, Ismini and Matina, who are in a process of negotiating their mathematical identities. Ismini and Matina challenge the very same discourses within which they act, thus opening up new possibilities for subjectifying themselves. In order to explore how Ismini and Matina form their mathematical identities in relation to the discourse of mathematical ability, I attempt to answer the following research questions: What discourses do the students form around mathematics and mathematical ability? How do they construct their identities through these discourses?

With regards to the data analysis, I have used a Foucault inspired methodology introduced by Doxiadis (2011). This tool orients the discourse analysis task along four axes, which correspond to the four fundamental properties of a discourse (presented in the previous section): a) *The axis of objects:* the discourse’s relation to what is outside of it is investigated. The focus is on the objects to which the discourse refers and which it constructs. For this axis, I focused on students' references to mathematics, discourses about mathematics, and students' past experiences. b) *The axis of enunciative modes:* the discourse's relation to itself is investigated and two parts are identified. The first part is concerned with the *external conditions of enunciation*, the conditions within which the discourse is produced, such as its origin and address. In this part, my main focus was the students' positionality. The second part of this axis is the *internal conditions of enunciation*, which refer to the ways in which the subjects are involved in the discourse. For this part, I was interested in the ways in which the students talk about themselves, their teachers and other individuals who have played a role in forming their current relationships with mathematics. c) *The axis of concepts:* the discourse's relation to other discourses is investigated. The focus is on the construction of concepts, which is the main product of the discursive practice. Within this axis, I focused on the ways that mathematics and mathematical ability were constructed as concepts. d) *The axis of thematics:* the discourse's relation to power and the antagonism expressed within the discourse and as a result of the discourse are investigated. For this axis, I examined the ways in which the students capture their own as well as other individuals' possibilities for action. My focus was on understanding how students act as agents who exercise or resist power.

THE CASE OF ISMINI

With regards to the external conditions of enunciation, Ismini was a student in a high school in Egypt, which follows the curriculum of the Greek Ministry of Education. She was born in Greece and her mother tongue is Greek. At the time of the interview, Ismini had been living in Egypt for eight years, but her interactions with Egyptian society were limited. She was 14 years old and had just graduated from the 3rd year of Gymnasium (9th Grade). During the school-year, I had been her mathematics teacher.

As far as the first research question is concerned, Ismini refers to everyday situations, including economic obligations and quantities described by numbers (axis of objects). She maintains that "mathematics is everywhere" and, thus her discourse captures mathematics as extremely important (axis of concepts). Mathematics is described as brief, accurate and simple in the presentation of its results. Ismini also refers to (axis of objects) and at the same time creates (axis of concepts) a series of dichotomies: "difficult maths" - "simple maths", "understand" - "use", "theory" - "practice", "mathematical thinking" - "theoretical thinking", "mathematical mind" - "hard work". Being strongly associated with these dipoles, mathematical ability is captured as a person's natural characteristic which can be enhanced through work:

Dionysia: Do you believe that everyone can do well in maths?

Ismini: Well, I think if they have help and attention, the way everybody needs it [yes], because I don't think that someone could be totally incapable of doing maths. Just with a certain amount of help and if somebody practices on the areas they need. Because somebody might be very good, have a mathematical mind, whereas somebody else might not. I believe that even the person who isn't very good, can get better and learn some things. Well,

> I don't believe that they're going to be exactly like someone who's gonna work and have a mathematical mind, but they're going to improve.

The role of the mathematics teacher is crucial in enabling what students learn (internal conditions of enunciation). In this sense, the learning process is mostly captured as a dependent process, in which the students as subjects are not considered autonomous. Their learning is guided by the teacher who communicates the knowledge, ideally in a way that would not require much more effort on the part of the students. In Ismini's words, for mathematics teaching to be a successful process:

> Ismini: I think that if the teacher can engage all students and if he could transmit it in a way that all students can -in that way- understand it. Not in a very difficult way so that they'd need a lot of effort to understand it. But if generally they understand it in class, so that they can practice afterwards.

With regards to the axis of thematics, the dichotomies drawn by Ismini frame the possible actions of a mathematics classroom's protagonists. The student's role is to try and study hard, while the teacher's role is to effectively transfer *the* knowledge. These stem from the neoliberal perspective that an individual's success in mathematics mostly depends on their genes, although hard work can enhance the outcome.

With regards to the second research question and the discourse's internal conditions of enunciation, Ismini refers to individuals who have played an important role in the construction of her mathematical identities, through a series of experiences to which she refers (axis of objects). These individuals have played a crucial role in identifying her twin brother Gerasimos as an exemplar of a mathematical mind.

> Ismini: So, by comparing and saying "Gerasimos is very good at maths", "Gerasimos is a star in maths"; they didn't say something about me. Let's say: "Well done Ismini", "Ismini is very good at maths". So, I believed that I wasn't good at maths. [...] It was always there. The "Ah! Gerasimos is very good at maths. He got that from your grandfather" for instance, as they always told me. And generally, they made this comparison. Although not in a bad way. They didn't say "You are not good at maths". They just said "Gerasimos is very good at maths. Ismini, you are very good at writing, at the way you talk, your vocabulary.".

Ismini constructs her mathematical identities through a durable process of comparing and contrasting herself to her brother. Her ability in mathematics and humanities, can be summarised in her own words as follows:

> Ismini: Since primary school, I have been better in humanities. That is to say, I was one of the first kids who were able to do certain things. […] And maths… it's not that I didn't understand it, I did, it's just that I've always been better at theory in maths. The theory, I read it once or twice and then I knew it. And I generally believe that mathematical thinking has more to do with practice, with solving exercises. So, when I saw that I wasn't so good at the exercises we had as homework and that I needed more help and more

practice, it was then that I understood that I wasn't… that I didn't have a mathematical way of thinking.

In the above quotation, Ismini's ability in both theoretical subjects and mathematics are captured as natural (axis of concepts). Ismini is constructing her identities, while these are to some extent already predetermined by her environment and by discourses which maintain that mathematical ability is transferred from grandfather to grandson. Although she believes that she is good in understanding the theory in mathematics, she conceptualizes this skill as not valuable in mathematical practice.

However, Ismini also negotiates her mathematical identities by participating in discursive practices that are contrary to the discourses within which she acts. Indeed, Ismini's preference for independent studying, can be understood as an effort to be the person in control of her own learning. She says: "When I hear somebody telling me all this [a lesson of any subject], unless I somehow do it practically or unless I study it alone, I don't learn it." Moreover, Ismini refers to (axis of objects) her engagement with mathematics in a way through which, she aims to prove something to herself (Mendick, 2006). As she states, re-solving an exercise makes her feel important and proves to her that she has the ability to do it.

Ismini: Sometimes I sit and I solve the same exercise over and over again and [when I do that] I feel that I am important in a way. That there is something in maths which I can do. Generally, I would like it if I understood maths. And I always wished that I had a mathematical mind. And somehow my brother's mind, so that I could solve exercises more easily. Because I like maths. I just do not think that I am good enough or that I am self-confident enough to solve it.

Although Ismini acts within discourses which prevent her from identifying as good at mathematics, she desires the position of "having a mathematical mind" and being successful in mathematics makes her feel important. By re-solving exercises or trying to be in control of her own learning, Ismini shows a form of agency which aims at her occupying an advantaged position in relation to mathematics (Walshaw, 2007).

THE CASE OF MATINA

Regarding the external conditions of enunciation, Matina was a student in the same school as Ismini. Matina came to the school in the middle of the school-year. She did not speak Arabic and, thus, her socialization was mainly within the Greek community. At the time of the interview, Matina was 16 years old and had graduated from the 2nd year of Lyceum (11th Grade), following the social sciences direction. During the school-year, I had not been her teacher.

Matina refers to the mathematics needed on an everyday basis and the mathematics taught at school (axis of objects). She constructs these categories as opposing and she maintains that not everybody needs to learn the mathematics taught at school (axis of concepts). With regards to the internal conditions of enunciation, Matina talks about the role of mathematics teachers. She proposes that a teacher needs to be able to engage their students so that the students can "love and feel mathematics". In the axis of thematics, she describes an ideal mathematics education, which is concerned with the knowledge needed in "real life". In Matina's words:

Matina: [Maths] is an everyday thing. Because you come across it everywhere. When you go to a supermarket to buy something, when you go anywhere. […] I, for example, everyday, come across subtraction, division, multiplication and addition, all of those. But I don't come across equations, except for at school. […] It's not necessary, with regards to maths, to learn everything. For example, it would be more practical for students to learn life's lessons, like taxes, one thing or the other.

In the axis of concepts, two binary oppositions are central: good at maths/bad at maths and smart/dumb. Matina discusses both of these oppositions and describes them as related within dominant discourses. However, she maintains the belief that these two skills are independently developed.

Matina: If from a young age I teach you something, and then you don't give it up, I believe you're always going to become better. And this is not just for maths; it's everywhere. […] I believe that being smart and being good at maths are two different paths. But maths helps you practice your mind, that is, it helps you to constantly practice your mind. But it doesn't prove that you are smart.

Matina rejects the correlation between intelligence and mathematical ability, as expressed in dominant discourses. Instead, she captures mathematical ability as a matter of mere practice. Introducing this discursive distinction is important for Matina's sense of agency, since she can position herself in different ways within the two oppositions.

Regarding the second research question and the internal conditions of enunciation, Matina argues that she always loved maths, but she could never understand it. She describes herself as a constant outsider in mathematics class and as a problem for her teachers.

Matina: Everyday [maths] was a problem. I was going inside [the classroom] and even if I wanted I couldn't understand what the teacher was saying. No matter how much I tried, I couldn't. Because I didn't have the previous knowledge. […] My teacher left me, he used to say "Ok, you're in Grade 6, you're in Grade 5, you'll go to the Gymnasium and they'll deal with you". He passed a problem over to another school. When I went to the Gymnasium, the teacher went crazy. He says "She knows nothing, what can I do with her?" Essentially, I was always a problem for every teacher.

Matina also refers to her past experiences (axis of objects) in which mathematics was an element of her socialization. She recalls that some of her peers were doing mathematical calculations as a mental game. Not being able to do well at this game, Matina experienced discrimination, since she was considered to be dumb.

Matina: For example, when a student said "Tell me! How much is this plus that?" and I needed to think, because indeed I hadn't practiced maths a lot, so I had to think "this plus that, minus this, plus the other". And before I found the answer, he has found it himself. And essentially *this*... it's as if it shows that she knows nothing, she's the dumb one, let's say. They have created this picture, that someone who doesn't know maths is dumb.

Although Matina accepts the identity of "not good at mathematics", she rejects the association of this identity with that of not being smart. This process of negotiation is a form of agency which aims at a more advantaged position: that of being clever.

Matina: [E]very day, for some years, I was thinking. I was saying, I might indeed be dumb. I might indeed have a problem, let's say, in my brain and indeed I *cannot* understand maths. But when somebody was sitting with me […] and gave me an exercise and I was working on it, I could solve it. And I was saying: Look! Since I can solve it if I think, if I work on it, then why do I believe that I'm dumb? And this picture came to my mind: that I do not need to know maths to be smart.

Matina perceives herself as smart, albeit not good at mathematics. Interestingly, however, she managed to occupy the position of being smart through her success in maths. Although this may seem contradictory, this tension highlights the deep discursive connection between mathematical capacity and intelligence.

DISCUSSION AND CONCLUSIONS

From a poststructural perspective, subjects are not pregiven, but are rather constituted within discourse (Walshaw, 2007). Given the dominance of discourses which associate mathematical ability with cleverness and masculinity, self-identifying as "good at math" is a rare positioning, especially among girls (Mendick, 2006). In this sense, it becomes important to challenge the discourses about mathematics which exclude students on the basis of gender or "ability". Exploring the ways in which students negotiate discourses about mathematics and their identities can be a powerful tool towards making mathematics education a more equitable sphere.

Although doing mathematics can be a form of doing masculinity (Mendick, 2006), girls are not completely trapped in the discourses about mathematics and gender, since they can have either an advantaged or a disadvantaged position in the emerging power relations (Walshaw, 2007). This paper aims to add on the existing literature by offering the examples of two female students who actively negotiate their identities, aiming to occupy an advantaged position.

The discourses about mathematics and ability within which Matina and Ismini act both constrain and enable their identities. Ismini accepts the tenets of the mathematics education which she experiences, but she occupies positions through which she learns independently from the school process. Although she does not consider herself as having a mathematical mind, she uses mathematics in order to achieve the advantaged position of "being important in a way". In other words, Ismini negotiates the tension between her belief that she is not good at maths and her desire to have had a mathematical mind by negotiating her mathematical identities. Matina, on the other hand, defends her identity of being smart, although not good at mathematics. She acts within the discourse that mathematics is important, as is evident by the fact that she accounts for the coexistence of her two identities, by stating that she has been successful at maths. However, Matina challenges the regime of mathematics education, by doubting the importance of what is taught and by neglecting the dominant interrelation between being good at maths and being smart.

Finally, the Foucauldian perspective has been instrumental in providing an interpretive framework for the seemingly contradictory identities occupied by the two students. Doxiadis' (2011) four axes methodology was a useful tool for implementing Foucauldian ideas. Being both flexible and simple, it can be used in mathematics education for the analysis of any discourse. Its focus on the social aspect rather than the linguistic makes the tool explicitly helpful for examining students' and teachers' discourses. Having tried to make my analysis as transparent as possible, I hope that this paper serves as a useful example of the way that the four axes methodology can be used in mathematics education research.

Acknowledgements

I thank Prof. Panagiotis Spirou for his contributions to an earlier version of this work and Prof. Richard Barwell for his feedback on this report. I also wish to acknowledge the support of the Onassis Scholarship Foundation for my PhD studies.

References

Chronaki, A., & Pechtelidis, Y. (2012). "Being good" at maths: Fabricating gendered subjectivity. *Journal of Research in Mathematics Education, 1*(3), 246-277.

Doxiadis, K. (2011). *Discourse analysis: A sociophilosophical grounding.* Champaign, IL: Common Ground Publishing.

Foucault, M. (1972). *The archaeology of knowledge and the discourse on knowledge*. New York: Pantheon.

Jørgensen, M., & Phillips, L. (2011). *Discourse analysis as theory and method*. Los Angeles, Calif: Sage.

Mendick, H. (2006). *Masculinities in mathematics.* Maidenhead: Open University Press.

Pitsili-Chatzi, D. (2015). *The discourse of mathematical ability under a Foucauldian perspective* (Master's thesis, University of Athens, Athens, Greece). Retrieved from: http://www.math.uoa.gr/me/dipl/

Walkerdine, V. (1998). *Counting girls out: Girls and mathematics.* London, UK: Falmer Press.

Walshaw, M. (2007). *Working with Foucault in education.* Rotterdam: Sense Publishers.

Walshaw, M. (2013). Post-structuralism and ethical practical action: Issues of identity and power. *Journal for Research in Mathematics Education, 44*(1), 110-118.

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